# On Interpretability between some Weak Essentially Undecidable Theories

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**Abstract.** We introduce two essentially undecidable first-order theories $\mathsf{WT}$ and $\mathsf{T}$. The intended model for the theories is a term model. We prove that $\mathsf{WT}$ is mutually interpretable with Robinson's $\mathsf{R}$. Moreover, we prove that Robinson's $\mathsf{Q}$ is interpretable in $\mathsf{T}$.

## 1 Introduction

A first-order theory $T$ is *undecidable* if there is no algorithm for deciding if $T \vdash \phi$. If every consistent extension of an undecidable theory $T$ also is undecidable, then $T$ is *essentially undecidable*.

We introduce two first-order theories, $\mathsf{WT}$ and $\mathsf{T}$, over the language $\mathcal{L}_{\mathsf{T}} = \{\perp, \langle \cdot, \cdot \rangle, \sqsubseteq\}$ where $\perp$ is a constant symbol, $\langle \cdot, \cdot \rangle$ is a binary function symbol and $\sqsubseteq$ is a binary relation symbol. The intended model for these theories is a term model: The universe is the set of all variable-free $\mathcal{L}_{\mathsf{T}}$-terms. Each term is interpreted as itself, and $\sqsubseteq$ is interpreted as the subterm relation ($s$ is a subterm of $t$ iff $s = t$ or $t = \langle t_1, t_2 \rangle$ and $s$ is a subterm of $t_1$ or $t_2$).

The non-logical axioms of $\mathsf{WT}$ are given by the two axiom schemes:

$$(\mathsf{WT}_1) \qquad s \neq t$$

where $s$ and $t$ are distinct variable-free terms.

$$(\mathsf{WT}_2) \qquad \forall x[\ x \sqsubseteq t \ \leftrightarrow \ \bigvee_{s \in \mathcal{S}(t)} x = s\ ]$$

where $t$ is a variable-free term and $\mathcal{S}(t)$ is the set of all subterms of $t$. There are no other non-logical axioms except those given by these two simple schemes, and at a first glance $\mathsf{WT}$ seems to be a very weak theory. Still it turns out that Robinson's essentially undecidable theory $\mathsf{R}$ is intepretable in $\mathsf{WT}$, and thus it follows that also $\mathsf{WT}$ is essentially undecidable. The theory $\mathsf{T}$ is given by the four axioms:

$\mathsf{T}_1$ $\forall xy[\ \langle x, y \rangle \neq \perp\ ]$
$\mathsf{T}_2$ $\forall x_1 x_2 y_1 y_2[\ \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \ \rightarrow\ (\ x_1 = y_1 \ \wedge\ x_2 = y_2\ )\ ]$
$\mathsf{T}_3$ $\forall x[\ x \sqsubseteq \perp \ \leftrightarrow\ x = \perp\ ]$
$\mathsf{T}_4$ $\forall xyz[\ x \sqsubseteq \langle y, z \rangle \ \leftrightarrow\ (\ x = \langle y, z \rangle \vee x \sqsubseteq y \vee x \sqsubseteq z\ )\ ]$ .

The Axioms of $\mathsf{R}$

$$\mathsf{R}_1\ \ \overline{n}+\overline{m}=\overline{n+m}\ ; \qquad \mathsf{R}_2\ \ \overline{n}\times\overline{m}=\overline{nm}\ ; \qquad \mathsf{R}_3\ \ \overline{n}\neq\overline{m} \text{ for } n\neq m\ ;$$
$$\mathsf{R}_4\ \ \forall x[\ x\leq\overline{n}\ \rightarrow\ x=0\ \vee\ \ldots\ \vee\ x=\overline{n}\ ]\ ; \qquad \mathsf{R}_5\ \ \forall x[\ x\leq\overline{n}\ \vee\ \overline{n}\leq x\ ]$$

The Axioms of $\mathsf{Q}$

$$\mathsf{Q}_1\ \ \forall xy[\ Sx=Sy\ \rightarrow\ x=y\ ]\ ; \qquad \mathsf{Q}_2\ \ \forall x[\ Sx\neq 0\ ]\ ; \qquad \mathsf{Q}_3\ \ \forall x[\ x\neq 0\ \rightarrow\ \exists y[\ x=Sy\ ]\ ]\ ;$$
$$\mathsf{Q}_4\ \ \forall x[\ x+0=x\ ]\ ; \qquad \mathsf{Q}_5\ \ \forall xy[\ x+Sy=S(x+y)\ ]\ ; \qquad \mathsf{Q}_6\ \ \forall x[\ x\times 0=0\ ]\ ;$$
$$\mathsf{Q}_7\ \ \forall xy[\ x\times Sy=(x\times y)+x\ ]\ ; \qquad \mathsf{Q}_8\ \ \forall xy[\ x\leq y\ \leftrightarrow\ \exists z[\ x+z=y\ ]\ ]$$

**Fig. 1.** The axioms of R are given by axiom schemes where $n, m \in \mathbb{N}$ and $\overline{n}$ denotes the $n^{\text{th}}$ numeral, that is, $\overline{0} \equiv 0$ and $\overline{n+1} \equiv S\overline{n}$.

It is not difficult to see that T is a consistent extension of WT. Thus, since WT is essentially undecidable, we can conclude right away that also T is essentially undecidable. Furthermore, since every model of the finitely axiomatizable theory T is infinite, T cannot be interpretable in WT, and the obvious conjecture would be that T is mutually interpretable with Robinson's Q.

The seminal theories R and Q are theories of arithmetic. The theory R is given by axiom schemes, and Q is a finitely axiomatizable extension of R, see Figure 1 (Q is also known as Robinson arithmetic and is more or less Peano arithmetic without the induction scheme). It was proved in Tarski et al. [9] that R and Q are essentially undecidable. Another seminal essentially undecidable first-order theory is Grzegorcyk's TC. This is a theory of concatenation. The language is $\{*, \alpha, \beta\}$ where $\alpha$ and $\beta$ are constant symbols and $*$ is a binary function symbol. The standard TC model is the structure where the universe is $\{a, b\}^+$ (all finite nonempty strings over the alphabet $\{a, b\}$), $*$ is concatenation, $\alpha$ is the string $a$ and $\beta$ is the string $b$. It was proved in Grzegorzyk & Zdanowski [3] that TC is essentially undecidable. It was later proved that TC is mutually interpretable with Q, see Visser [10] for further references. The theory $\mathsf{WTC}^{-\epsilon}$ is a weaker variant of TC that has been shown to be mutually interpretable with R, see Higuchi & Horihata [4] for more details and further references. The axioms of TC and $\mathsf{WTC}^{-\epsilon}$ can be found in Figure 2.

The overall picture shows three finitely axiomatizable and essentially undecidable first-order theories of different character and nature: Q is a theory of arithmetic, TC is a theory of concatenation, and T is a theory of terms (it may also be viewed as a theory of binary trees). All three theories are mutually interpretable with each other, and each of them come with a weaker variant given by axiom schemes. These weaker variants are also essentially undecidable and mutually interpretable with each other.

The theory T has, in contrast to Q and TC, a purely universal axiomatization, that is, there are no occurrences of existential quantifiers in the axioms. Moreover, its weaker variant WT has a neat and very compact axiomatization compared to R and $\mathsf{WTC}^{-\epsilon}$.

The Axioms of $\mathsf{WTC}^{-\epsilon}$

$$\begin{array}{ll}
\mathsf{WTC}_1^{-\epsilon} & \forall xyz\big[\ \big(\ x*(y*z) \sqsubseteq \underline{t} \lor (x*y)*z \sqsubseteq \underline{t}\ \big)\ \rightarrow\ x*(y*z) = (x*y)*z\ \big]\ ; \\
\mathsf{WTC}_2^{-\epsilon} & \forall xyzu\big[\ x*y = z*u\ \land\ x*y \sqsubseteq \underline{t}\ \rightarrow\ \big(\ (x = z \land y = u) \lor \\
& \qquad \exists w[\ (x*w = z \land w*u = y) \lor (z*w = x \land w*y = u\ ]\ \big)\ \big]\ ; \\
\mathsf{WTC}_3^{-\epsilon} & \forall xy[\ \alpha \neq x*y\ ]\ ; \qquad \mathsf{WTC}_4^{-\epsilon}\ \ \forall xy[\ \beta \neq x*y\ ]\ ; \qquad \mathsf{WTC}_5^{-\epsilon}\ \ \alpha \neq \beta
\end{array}$$

where $x \sqsubseteq y$ is defined by

$$x = y\ \lor\ \exists z_1 z_2[\ z_1 * x = y\ \lor\ x * z_2 = y\ \lor\ (z_1 * x) * z_2 = y\ \lor\ z_1 * (x * z_2) = y\ ]\ .$$

The Axioms of $\mathsf{TC}$

$$\begin{array}{ll}
\mathsf{TC}_1 & \forall xyz[\ x*(y*z) = (x*y)*z\ ]\ ; \\
\mathsf{TC}_2 & \forall xyzu\big[\ x*y = z*u \rightarrow \big(\ (x = z \land y = u) \lor \\
& \qquad \exists w[\ (x*w = z \land w*u = y) \lor (z*w = x \land w*y = u\ ]\ \big)\ \big]\ ; \\
\mathsf{TC}_3 & \forall xy[\ \alpha \neq x*y\ ]\ ; \qquad \mathsf{TC}_4\ \ \forall xy[\ \beta \neq x*y\ ]\ ; \qquad \mathsf{TC}_5\ \ \alpha \neq \beta
\end{array}$$

**Fig. 2.** $\mathsf{WTC}_1^{-\epsilon}$ and $\mathsf{WTC}_2^{-\epsilon}$ are axiom schemes where $t \in \{a,b\}^+$ and $\underline{t}$ is a term inductively defined by: $\underline{a} \equiv \alpha$, $\underline{b} \equiv \beta$, $\underline{au} \equiv \alpha * \underline{u}$ and $\underline{bu} \equiv \beta * \underline{u}$.

Another interesting theory which is known to be mutually interpretable with $\mathsf{Q}$, and thus also with $\mathsf{TC}$ and $\mathsf{T}$, is the adjunctive set theory $\mathsf{AST}$. More on $\mathsf{AST}$ and adjunctive set theory can found in Damnjanovic [2]. For recent results related to the work in the present paper, we refer the reader to Jerabek [5], Cheng [1] and Kristiansen & Murwanashyaka [7].

The rest of this paper is fairly technical, and we will assume that the reader is familiar with first-order theories and the interpretation techniques introduced in Tarski et al. [9]. In Section 2 we prove that $\mathsf{R}$ and $\mathsf{WT}$ are mutually interpretable. In Section 3 we prove that $\mathsf{Q}$ is interpretable in $\mathsf{T}$. We expect that $\mathsf{T}$ can be interpreted in $\mathsf{Q}$ by standard techniques available in the literature.

## 2 R and WT are Mutually Interpretable

The theory $\mathsf{R}^-$ over the language of Robinson arithmetic is given by the axiom schemes

$$\begin{array}{l}
\mathsf{R}_1^-\ \ \overline{n} + \overline{m} = \overline{n+m}\ ; \qquad \mathsf{R}_2^-\ \ \overline{n} \times \overline{m} = \overline{nm}\ ; \qquad \mathsf{R}_3^-\ \ \overline{n} \neq \overline{m}\ \ \text{for } n \neq m\ ; \\
\mathsf{R}_4^-\ \ \forall x[\ x \leq \overline{n}\ \leftrightarrow\ x = 0\ \lor\ \ldots\ \lor\ x = \overline{n}\ ]
\end{array}$$

where $n, m \in \mathbb{N}$. Recall that $\overline{n}$ denotes the $n^{\text{th}}$ numeral, that is, $\overline{0} \equiv 0$ and $\overline{n+1} \equiv S\overline{n}$.

We now proceed to interpret $\mathsf{R}^-$ in $\mathsf{WT}$. We choose the domain $I(x) \equiv x = x$ (thus we can just ignore the domain). Furthermore, we translate the successor function $S(x)$ as the function given by $\lambda x.\langle x, \perp\rangle$, and we translate the constant $0$ as $\langle \perp, \perp\rangle$. Let $\overline{n}^\star$ denote the translation of the numeral $\overline{n}$. Then we have $\overline{n+1}^\star \equiv \langle \overline{n}^\star, \perp\rangle$. It follows from $\mathsf{WT}_1$ that the translation of each instance of $\mathsf{R}_3^-$ is a theorem of $\mathsf{WT}$ since $\overline{m}^\star$ and $\overline{n}^\star$ are different terms whenever $m \neq n$.

We translate $x \leq y$ as $x \sqsubseteq y \wedge x \neq \perp$. It is easy to see that

$$\mathsf{WT} \vdash \forall x [\ x \sqsubseteq \overline{n}^{\star} \wedge x \neq \perp \ \ \leftrightarrow \bigvee_{s \in \mathcal{T}(n)} x = s\ ] \tag{1}$$

where $\mathcal{T}(n) = \mathcal{S}(\overline{n}^{\star}) \setminus \{\perp\}$ and $\mathcal{S}(\overline{n}^{\star})$ denotes the set of all subterms of $\overline{n}^{\star}$. We observe that $\mathcal{T}(n) = \{\overline{k}^{\star} \mid k \leq n\}$ and that (1) indeed is the translation of the axiom scheme $\mathsf{R}_4^-$. Hence we conclude that the translation of each instance of $\mathsf{R}_4^-$ is a theorem of $\mathsf{WT}$.

Next we discuss the translation of $+$. The idea is to obtain $n + i$ through a formation sequence of length $i$. Such a sequence will be represented by a term of the form

$$\langle \ldots \langle \langle \langle \overline{n}^{\star}, \overline{0}^{\star} \rangle, \langle \overline{n+1}^{\star}, \overline{1}^{\star} \rangle \rangle, \langle \overline{n+2}^{\star}, \overline{2}^{\star} \rangle \rangle \ldots, \langle \overline{n+i}^{\star}, \overline{i}^{\star} \rangle \rangle\ . \tag{2}$$

Accordingly we translate $x + y = z$ by the predicate $add(x, y, z)$ given by the formula

$$\begin{aligned} (\ y = \overline{0}^{\star} \wedge z = x\ ) \ \vee\ \Big\{\ & y \neq \overline{0}^{\star} \ \wedge\ \exists W [\ \langle x, \overline{0}^{\star} \rangle \sqsubseteq W \ \wedge \\ & \forall X\ \forall Y \sqsubseteq y [\ \langle X, Y \rangle \sqsubseteq W \ \wedge\ Y \neq y \ \wedge\ Y \neq \perp \quad \rightarrow \\ & \quad (\ \langle \langle X, \perp \rangle, \langle Y, \perp \rangle \rangle \sqsubseteq W \ \wedge\ (\ \langle Y, \perp \rangle = y \rightarrow \langle X, \perp \rangle = z\ )\ )\ ]\ ]\ \Big\}\ . \end{aligned}$$

**Lemma 1.** *For any $m, n \in \mathbb{N}$, we have*

$$\mathsf{WT} \vdash \forall z [\ add(\overline{n}^{\star}, \overline{m}^{\star}, z) \leftrightarrow z = \overline{n+m}^{\star}\ ]\ .$$

*Proof.* First we prove that $\mathsf{WT} \vdash add(\overline{n}^{\star}, \overline{m}^{\star}, \overline{n+m}^{\star})$. This is obvious if $m = 0$. Assume $m > 0$. Let

$$S_0^n \equiv \langle \overline{n}^{\star}, \overline{0}^{\star} \rangle \qquad \text{and} \qquad S_{i+1}^n \equiv \langle S_i^n, \langle \overline{n+i+1}^{\star}, \overline{i+1}^{\star} \rangle \rangle$$

and observe that $S_i^n$ is of the form (2). We will argue that we can choose the $W$ in the definition of $add(x, y, z)$ to be the term $S_m^n$.

So let $W = S_m^n$. By the axioms of $\mathsf{WT}$, we have $\langle \overline{n}^{\star}, \overline{0}^{\star} \rangle \sqsubseteq W$. Assume

$$\langle X, Y \rangle \sqsubseteq W \text{ and } Y \neq y = \overline{m}^{\star} \text{ and } Y \sqsubseteq y = \overline{m}^{\star} \text{ and } Y \neq \perp\ .$$

By the axioms of $\mathsf{WT}$, we have that $Y \sqsubseteq \overline{m}^{\star}$, $Y \neq \overline{m}^{\star}$ and $Y \neq \perp$ imply $Y = \overline{k}^{\star}$ for some $k < m$. Since $\langle X, Y \rangle \sqsubseteq W$, we know by $\mathsf{WT}_2$ that $\langle X, Y \rangle$ is one of the subterms of $W$. By $\mathsf{WT}_1$ and the form of $S_m^n$, we conclude that $X = \overline{n+k}^{\star}$. Furthermore, the form of $S_m^n$ and $\mathsf{WT}_2$ then ensures that $\langle \langle X, \perp \rangle, \langle Y, \perp \rangle \rangle \sqsubseteq W = S_m^n$. Moreover, if $\langle Y, \perp \rangle = \overline{m}^{\star}$, then by $\mathsf{WT}_1$, we must have $k = m - 1$, and thus, $\langle X, \perp \rangle = \langle \overline{n + (m-1)}^{\star}, \perp \rangle = \overline{n+m}^{\star}$. This proves that we can deduce $add(\overline{n}^{\star}, \overline{m}^{\star}, \overline{n+m}^{\star})$ from the axioms of $\mathsf{WT}$, and thus we also have

$$\mathsf{WT} \vdash \forall z [\ z = \overline{n+m}^{\star} \rightarrow\ add(\overline{n}^{\star}, \overline{m}^{\star}, z) ]\ .$$

Next we prove that the converse implication $add(\overline{n}^\star, \overline{m}^\star, z) \rightarrow z = \overline{n+m}^\star$ follows from the axioms of $\mathsf{WT}$ (and thus the lemma follows). This is obvious when $m = 0$. Assume $m \neq 0$ and $add(\overline{n}^\star, \overline{m}^\star, z)$. Then we have $W$ such that $\langle \overline{n}^\star, \overline{0}^\star \rangle \sqsubseteq W$ and

$$\forall X \; \forall Y \sqsubseteq \overline{m}^\star \big[ \; \langle X, Y \rangle \sqsubseteq W \;\wedge\; Y \neq \overline{m}^\star \;\wedge\; Y \neq \bot \quad \rightarrow \\ \big( \; \langle \langle X, \bot \rangle, \langle Y, \bot \rangle \rangle \sqsubseteq W \;\wedge\; ( \; \langle Y, \bot \rangle = \overline{m}^\star \rightarrow \langle X, \bot \rangle = z \; ) \; \big) \; \big] \; . \qquad (3)$$

Since $\langle n, \overline{0}^\star \rangle \sqsubseteq W$ and (3) hold, we have $\langle \overline{n+k+1}^\star, \overline{k+1}^\star \rangle \sqsubseteq W$ for any $k < m$. It also follows from (3) that $z = \overline{n+k+1}^\star$ when $m = k+1$. □

It follows from the preceding lemma that there for any $n, m \in \mathbb{N}$ exists a unique $k \in \mathbb{N}$ such that $\mathsf{WT} \vdash add(\overline{n}^\star, \overline{m}^\star, \overline{k}^\star)$. We translate $x + y = z$ by the predicate $\phi_+$ where $\phi_+(x, y, z)$ is the formula

$$\big( \; \exists! u[add(x, y, u)] \;\wedge\; add(x, y, z) \; \big) \;\vee\; \big( \; \neg\exists! u[add(x, y, u)] \;\wedge\; z = \bot \; \big) \, . \, (4)$$

The second disjunct of (4) ensures the functionality of our translation, that is, it ensures that $\mathsf{WT} \vdash \forall xy \exists! x \phi_+(x, y, z)$ (the same technique is used in [6]). By Lemma 1, we have $\mathsf{WT} \vdash \phi_+(\overline{n}^\star, \overline{m}^\star, \overline{n+m}^\star)$. This shows that the translation of any instance of the axiom scheme $\mathsf{R}_1^-$ can be deduced from the axioms of $\mathsf{WT}$.

We can also achieve a translation of $x \times y = z$ such that the translation of each instance of $\mathsf{R}_2^-$ can be deduced from the axioms of $\mathsf{WT}$. Such a translation claims the existence of a term $S_m^n$ where

$$S_1^n \equiv \langle \overline{n}^\star, \overline{1}^\star \rangle \qquad \text{and} \qquad S_{i+1}^n \equiv \langle S_i^n, \langle \overline{(i+1)n}^\star, \overline{i+1}^\star \rangle \rangle$$

and will more or less be based on the same ideas as our translation of $x + y = z$. We omit the details.

**Theorem 2.** *$\mathsf{R}$ and $\mathsf{WT}$ are mutually interpretable.*

*Proof.* We have seen how to interpret $\mathsf{R}^-$ in $\mathsf{WT}$. It follows straightforwardly from results proved in Jones & Shepherdson [6] that $\mathsf{R}^-$ and $\mathsf{R}$ are mutually interpretable. Thus $\mathsf{R}$ is interpretable in $\mathsf{WT}$. A result of Visser [11] states that a theory is interpretable in $\mathsf{R}$ if and only if it is locally finitely satisfiable, that is, each finite subset of the non-logical axioms has a finite model. Since $\mathsf{WT}$ clearly is locally finitely satisfiable, $\mathsf{WT}$ is interpretable in $\mathsf{R}$. □

## 3 Q is Interpretable in T

The language of the arithmetical theory $\mathsf{Q}^-$ is $\{0, S, M, A\}$ where 0 is a constant symbol, $S$ is a unary function symbol, and $A$ and $M$ are ternary predicate symbols. The non-logical axioms of the first-order theory $\mathsf{Q}^-$ are the the following:

$\mathsf{A}$ $\forall x y z_1 z_2[ \; A(x, y, z_1) \wedge A(x, y, z_2) \rightarrow z_1 = z_2 \; ]$ ;
$\mathsf{M}$ $\forall x y z_1 z_2[ \; M(x, y, z_1) \wedge M(x, y, z_2) \rightarrow z_1 = z_2 \; ]$ ;
$\mathsf{Q}_1$ $\forall x y[ \; x \neq y \rightarrow Sx \neq Sy \; ]$ ; $\mathsf{Q}_2$ $\forall x[ \; Sx \neq 0 \; ]$ ; $\mathsf{Q}_3$ $\forall x[ \; x = 0 \vee \exists y[ \; x = Sy \; ] \; ]$ ;
$\mathsf{G}_4$ $\forall x[ \; A(x, 0, x) \; ]$ ; $\mathsf{G}_5$ $\forall x y u[ \; \exists z[ \; A(x, y, z) \wedge u = Sz \; ] \rightarrow A(x, Sy, u) \; ]$ ;
$\mathsf{G}_6$ $\forall x[ \; M(x, 0, 0) \; ]$ ; $\mathsf{G}_7$ $\forall x y u[ \; \exists z[ \; M(x, y, z) \wedge A(z, x, u) \; ] \rightarrow M(x, Sy, u) \; ]$ .

Svejdar [8] proved that $\mathsf{Q}^-$ and $\mathsf{Q}$ are mutually interpretable. We will prove that $\mathsf{Q}^-$ is interpretable in $\mathsf{T}$.

The first-order theory $\mathsf{T}^+$ is $\mathsf{T}$ extended by the two non-logical axioms

$$\mathsf{T}_5 \;\; \forall x[\; x \sqsubseteq x \;] \quad \text{and} \quad \mathsf{T}_6 \;\; \forall xyz[\; x \sqsubseteq y \,\wedge\, y \sqsubseteq z \;\rightarrow\; x \sqsubseteq z \;] \;.$$

**Lemma 3.** *$\mathsf{T}^+$ is interpretable in $\mathsf{T}$.*

*Proof.* We simply relativize quantification to the domain

$$I \;\;=\;\; \{\; x \mid \;\; x \sqsubseteq x \;\wedge\; \forall uv[\; u \sqsubseteq v \wedge v \sqsubseteq x \;\rightarrow\; u \sqsubseteq x \;] \;\} \;.$$

Suppose $x_1, x_2 \in I$. We show that $\langle x_1, x_2\rangle \in I$. Since $\langle x_1, x_2\rangle = \langle x_1, x_2\rangle$, we have $\langle x_1, x_2\rangle \sqsubseteq \langle x_1, x_2\rangle$ by $\mathsf{T}_4$. Suppose now that $u \sqsubseteq v \wedge v \sqsubseteq \langle x_1, x_2\rangle$. We need to show that $u \sqsubseteq \langle x_1, x_2\rangle$. By $\mathsf{T}_4$ and $v \sqsubseteq \langle x_1, x_2\rangle$, at least one of the following three cases holds: (a) $v = \langle x_1, x_2\rangle$, (b) $v \sqsubseteq x_1$, (c) $v \sqsubseteq x_2$. *Case (a):* Since $u \sqsubseteq v$ and $v = \langle x_1, x_2\rangle$, we have $u \sqsubseteq \langle x_1, x_2\rangle$ by our logical axioms. *Case (b):* $u \sqsubseteq v \wedge v \sqsubseteq x_1$ implies $u \sqsubseteq x_1$ since $x_1 \in I$. By $\mathsf{T}_4$, we have $u \sqsubseteq \langle x_1, x_2\rangle$. *Case (c):* We have $u \sqsubseteq \langle x_1, x_2\rangle$ by an argument symmetric to the one used in Case (b). Hence, $\forall uv[\; u \sqsubseteq v \wedge v \sqsubseteq \langle x_1, x_2\rangle \rightarrow u \sqsubseteq \langle x_1, x_2\rangle \;]$.

This proves that $I$ is closed under $\langle\cdot,\cdot\rangle$. It follows from $\mathsf{T}_3$ that $\perp \in I$, and thus $I$ satisfies the domain condition. Clearly, the translation of each non-logical axiom of $\mathsf{T}^+$ is a theorem of $\mathsf{T}$. □

We now proceed to interpret $\mathsf{Q}^-$ in $\mathsf{T}^+$. We choose the domain $N$ given by

$$N(x) \;\equiv\; x \neq \perp \;\wedge\; \forall y \sqsubseteq x[\; y = \perp \;\vee\; \exists z[\; y = \langle z, \perp\rangle \;] \;] \;.$$

**Lemma 4.** *We have (i)* $\mathsf{T}^+ \vdash N(\langle\perp,\perp\rangle)$*, (ii)* $\mathsf{T}^+ \vdash \forall x[N(x) \rightarrow N(\langle x,\perp\rangle)]$ *and (iii)* $\mathsf{T}^+ \vdash \forall yz[\; N(y) \;\wedge\; z \sqsubseteq y \;\rightarrow\; (\; z = \perp \;\vee\; N(z) \;) \;]$*.*

*Proof.* It follows from $\mathsf{T}_1$, $\mathsf{T}_3$ and $\mathsf{T}_4$ that (i) holds. In order to see that (ii) holds, assume $N(x)$ (we will argue that $N(\langle x,\perp\rangle)$ holds). Suppose $y \sqsubseteq \langle x,\perp\rangle$. Now, $N(\langle x,\perp\rangle)$ follows from

$$y = \perp \;\vee\; \exists z[\; y = \langle z,\perp\rangle \;] \;. \tag{5}$$

Thus it is sufficient to argue that (5) holds. By $\mathsf{T}_4$, we know that $y \sqsubseteq \langle x,\perp\rangle$ implies $y = \langle x,\perp\rangle \vee y \sqsubseteq x \vee y \sqsubseteq \perp$. The case $y = \langle x,\perp\rangle$: We obviously have $\exists z[\; y = \langle z,\perp\rangle \;]$ and thus (5) holds. The case $y \sqsubseteq x$: (5) holds since $N(x)$ holds. The case $y \sqsubseteq \perp$: We have $y = \perp$ by $\mathsf{T}_3$, and thus (5) holds. This proves (ii).

We turn to the proof of (iii). Suppose $N(y) \wedge z \sqsubseteq y$ (we show $z = \perp \vee N(z)$). Assume $w \sqsubseteq z$. By $\mathsf{T}_6$, we have $w \sqsubseteq y$, moreover, since $N(y)$ holds, we have $w = \perp \vee \exists u[w = \langle u,\perp\rangle]$. Thus, we conclude that

$$\forall w \sqsubseteq z[\; w = \perp \;\vee\; \exists u[\; w = \langle u,\perp\rangle \;] \;] \;. \tag{6}$$

Now

$$z = \perp \;\vee\; \underbrace{(\; z \neq \perp \;\wedge\; \forall w \sqsubseteq z[\; w = \perp \;\vee\; \exists u[\; w = \langle u,\perp\rangle \;] \;] \;)}_{N(z)}$$

follows tautologically from (6). □

We interpret 0 as $\langle \perp, \perp \rangle$. We interpret the successor function $Sx$ as $\lambda x.\langle x, \perp \rangle$. To improve the readability we will occasionally write $\dot{0}$ in place of $\langle \perp, \perp \rangle$, $\dot{\mathrm{S}}t$ in place of $\langle t, \perp \rangle$ and $t \in N$ in place of $N(t)$. We will also write $\exists x \in N[\ \eta\ ]$ and $\forall x \in N[\ \eta\ ]$ in place of, respectively, $\exists x[\ N(x) \wedge \eta\ ]$ and $\forall x[\ N(x) \rightarrow \eta\ ]$. Furthermore, $\mathbf{Q}x_1, \ldots, x_n \in N$ is shorthand for $\mathbf{Q}x_1 \in N \ldots \mathbf{Q}x_n \in N$ where $\mathbf{Q}$ is either $\forall$ or $\exists$.

**Lemma 5.** *The translations of* $\mathsf{Q}_1$*,* $\mathsf{Q}_2$ *and* $\mathsf{Q}_3$ *are theorems of* $\mathsf{T}^+$*.*

*Proof.* The translation of $\mathsf{Q}_1$ is $\forall x, y \in N[\ x \neq y \rightarrow \dot{\mathrm{S}}x \neq \dot{\mathrm{S}}y\ ]$. By $\mathsf{T}_2$, we have $x \neq y \rightarrow \dot{\mathrm{S}}x \neq \dot{\mathrm{S}}y$ for any $x, y$, and thus, the translation of $\mathsf{Q}_1$ is a theorem of $\mathsf{T}^+$.

The translation of $\mathsf{Q}_2$ is $\forall x \in N[\ \dot{\mathrm{S}}x \neq \dot{0}\ ]$. Assume $x \in N$. Then we have $x \neq \perp$, and by $\mathsf{T}_2$, we have $\dot{\mathrm{S}}n \equiv \langle x, \perp \rangle \neq \langle \perp, \perp \rangle \equiv \dot{0}$.

The translation of $\mathsf{Q}_3$ is $\forall x \in N[\ x = \dot{0} \vee \exists y \in N[\ x = \dot{\mathrm{S}}y\ ]\ ]$. Assume $x \in N$, that is, assume

$$x \neq \perp\ \wedge\ \forall y \sqsubseteq x[\ y = \perp\ \vee\ \exists z[\ y = \langle z, \perp \rangle\ ]\ ]\ . \tag{7}$$

By $\mathsf{T}_5$, we have $x \sqsubseteq x$. By (7) and $x \sqsubseteq x$, we have

$$x \neq \perp\ \wedge\ (\ x = \perp\ \vee\ \exists z[\ x = \langle z, \perp \rangle\ ]\ )$$

and then, by a tautological inference, we also have $\exists z[x = \langle z, \perp \rangle]$. Thus, we have $z$ such that $\langle z, \perp \rangle \equiv \dot{\mathrm{S}}z = x \in N$. By Lemma 4 (iii), we have $z = \perp \vee z \in N$. If $z = \perp$, we have $x = \langle \perp, \perp \rangle \equiv \dot{0}$. If $z \in N$, we have $z \in N$ such that $x = \dot{\mathrm{S}}z$. Thus, $\mathsf{T}^+ \vdash \forall x \in N[x = \dot{0} \vee \exists y \in N[x = \dot{\mathrm{S}}y]]$. □

Before we give the translation of $A$, we will provide some intuition. The predicate $A(a, b, c)$ holds in the standard model for $\mathsf{Q}^-$ iff $a + b = c$. Let $\widetilde{0} \equiv \dot{0}$ and $\widetilde{n+1} \equiv \dot{\mathrm{S}}\widetilde{n}$, and observe that $a + b = c$ iff there exists an $\mathcal{L}_{\mathsf{T}}$-term of the form

$$\langle \ldots \langle \langle \langle \perp, \langle \widetilde{a}, \widetilde{0} \rangle \rangle, \langle \widetilde{a+1}, \widetilde{1} \rangle \rangle, \langle \widetilde{a+2}, \widetilde{2} \rangle \rangle \ldots, \langle \widetilde{a+b}, \widetilde{b} \rangle \rangle \tag{8}$$

where $c = a + b$. We will give a predicate $\phi_A$ such that $\phi_A(\widetilde{a}, \widetilde{b}, w)$ holds in $\mathsf{T}^+$ iff $w$ is of the form (8). Thereafter we will use $\phi_A$ to give the translation $\Psi_A$ of $A$.

Let $\phi_A(x, y, w) \equiv$

$$\begin{aligned} &(\ y = \dot{0} \rightarrow w = \langle \perp, \langle x, \dot{0} \rangle \rangle\ )\ \wedge\ \exists w' \exists z \in N[\ w = \langle w', \langle z, y \rangle \rangle\ ] \wedge \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall u \forall Y, Z \in N[\ \theta_A(u, w, Y, Z)\ ] \end{aligned}$$

where $\theta_A(u, w, Y, Z) \equiv$

$$\begin{aligned} &\langle u, \langle Z, Y \rangle \rangle \sqsubseteq w\ \wedge\ Y \neq \dot{0} \rightarrow \\ &\qquad \exists v\ \exists Y' Z' \in N\big[\ Z = \dot{\mathrm{S}}Z'\ \wedge\ Y = \dot{\mathrm{S}}Y'\ \wedge\ u = \langle v, \langle Z', Y' \rangle \rangle \wedge \\ &\qquad\qquad\qquad\qquad\qquad (\ Y' = \dot{0} \rightarrow (\ Z' = x\ \wedge\ v = \perp\ )\ )\ \big]\ . \end{aligned}$$

The translation $\Psi_A$ of $A$ is $\Psi_A(x,y,z) \equiv$

$$\exists w[\ \phi_A(x,y,w)\ \wedge\ \exists w'[\ w = \langle w', \langle z, y\rangle\rangle\ ]\ \wedge\ \forall u[\ \phi_A(x,y,u) \rightarrow u = w\ ]\ ]\ .$$

**Lemma 6.**

$$\mathsf{T}^+ \vdash \forall x \in N \forall w[\ \phi_A(x,\dot{0},w)\ \leftrightarrow\ w = \langle \bot, \langle x, \dot{0}\rangle\rangle\ ]\ .$$

*Proof.* We assume $x \in N$ and prove the equivalence

$$\phi_A(x,\dot{0},w)\ \leftrightarrow\ w = \langle \bot, \langle x, \dot{0}\rangle\rangle \tag{9}$$

The left-right direction of (9) follows straightforwardly from the definition of $\phi_A$. To prove the right-left implication of (9), we need to prove $\phi_A(x,\dot{0},\langle \bot, \langle x,\dot{0}\rangle\rangle)$. It is easy to see that $\phi_A(x,\dot{0},\langle \bot, \langle x,\dot{0}\rangle\rangle)$ holds if

$$\forall u \forall Y, Z \in N[\ \theta_A(u, \langle \bot, \langle x, \dot{0}\rangle\rangle, Y, Z)\ ] \tag{10}$$

holds, and to show (10), it suffices to show that

$$x, Y, Z \in N \ \text{ and }\ \langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle \bot, \langle x, \dot{0}\rangle\rangle \ \text{ and }\ Y \neq \dot{0} \tag{11}$$

is a contradiction. (If (11) is a contradiction, then (10) will hold as the antecedent of $\theta_A$ will be false for all $x, Y, Z \in N$ and all $u$.)

By $\mathsf{T}_4$ and $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle \bot, \langle x, \dot{0}\rangle\rangle$ we have to deal with the following three cases: (a) $\langle u, \langle Z, Y\rangle\rangle = \langle \bot, \langle x, \dot{0}\rangle\rangle$, (b) $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \bot$ and (c) $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle x, \dot{0}\rangle$. *Case: (a):* We have $Y = \dot{0}$ by $\mathsf{T}_2$, but we have $Y \neq \dot{0}$ in (11). *Case (b):* We have $\langle u, \langle Z, Y\rangle\rangle = \bot$ by $\mathsf{T}_3$, and this contradicts $\mathsf{T}_1$. *Case (c):*. By $\mathsf{T}_4$, this case splits into the three subcases: (a') $\langle u, \langle Z, Y\rangle\rangle = \langle x, \dot{0}\rangle$, (b') $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq x$ and (c') $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \dot{0}$. *Case (a'):* We have $\langle u, \langle Z, Y\rangle\rangle = \langle x, \langle \bot, \bot\rangle\rangle$ since $\dot{0}$ is shorthand for $\langle \bot, \bot\rangle$. Thus, by $\mathsf{T}_2$, we have $Z = \bot$ and $Y = \bot$. This contradicts $Y, Z \in N$. *Case (b'):* We have $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq x$ and $x \in N$. By Lemma 4 (iii), we have $\langle u, \langle Z, Y\rangle\rangle = \bot$ or $\langle u, \langle Z, Y\rangle\rangle \in N$. Now, $\langle u, \langle Z, Y\rangle\rangle = \bot$ contradicts $\mathsf{T}_1$. Furthermore, by our definitions, $\langle u, \langle Z, Y\rangle\rangle \in N$ implies that

$$\forall y_0 \sqsubseteq \langle u, \langle Z, Y\rangle\rangle[\ y_0 = \bot\ \vee\ \exists z_0[\ y_0 = \langle z_0, \bot\rangle\ ]\ ]\ .$$

By $\mathsf{T}_5$, we have $\langle u, \langle Z, Y\rangle\rangle = \bot\ \vee\ \exists z_0[\ \langle u, \langle Z, Y\rangle\rangle = \langle z_0, \bot\rangle\ ]$, and this yields a contradiction together with $\mathsf{T}_1$ and $\mathsf{T}_2$. Case (c') is similar to Case (a'), but a bit simpler. This completes the proof of the lemma. $\Box$

**Lemma 7.**

$$\mathsf{T}^+ \vdash \forall x, y \in N \forall z w w'[\ w = \langle w', \langle z, y\rangle\rangle\ \wedge\ \phi_A(x,y,w)\ \ \rightarrow \\ \phi_A(x, \dot{\mathrm{S}}y, \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle)\ ]\ .$$

*Proof.* We assume

$$x, y \in N \text{ and } w = \langle w', \langle z, y\rangle\rangle \text{ and } \phi_A(x,y,w)\ . \tag{12}$$

We need to prove $\phi_A(x, \dot{\mathrm{S}}y, \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle) \equiv$

$$
\begin{aligned}
&(\ \dot{\mathrm{S}}y = \dot{0} \to w = \langle \bot, \langle x, \dot{0}\rangle\rangle\ ) \land \\
&\qquad \exists w_0 \exists z_0 \in N[\ \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle = \langle w_0, \langle z_0, \dot{\mathrm{S}}y\rangle\rangle\ ] \land \\
&\qquad\qquad \forall u \forall Y, Z \in N[\ \theta_A(u, \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle, Y, Z)\ ]
\end{aligned} \tag{13}
$$

First we prove

$$
z \in N \quad \text{and} \quad \dot{\mathrm{S}}z \in N \tag{14}
$$

Since $\phi_A(x, y, w)$ holds by our assumptions (12), we have $z_1 \in N$ and $w_1$ such that $w = \langle w_1, \langle z_1, y\rangle\rangle$. We have also assumed $w = \langle w', \langle z, y\rangle\rangle$. By $\mathsf{T}_2$, we have $z = z_1$, and thus $z \in N$. By Lemma 4 (ii), we have $\dot{\mathrm{S}}z \in N$. This proves (14).

The second conjunct of (13) follows straightforwardly from (14). (simply let $z_0$ be $\dot{\mathrm{S}}z$ and let $w_0$ be $w$). The first conjunct follows easily from $\mathsf{T}_2$ and the assumption $y \in N$. Thus, we are left to prove the third conjunct of (13), namely

$$
\begin{aligned}
&\forall u \forall Y, Z \in N\big[\ \langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle\ \land\ Y \neq \dot{0} \to \\
&\qquad \exists v\ \exists Y' Z' \in N\big[\ Z = \dot{\mathrm{S}}Z'\ \land\ Y = \dot{\mathrm{S}}Y'\ \land\ u = \langle v, \langle Z', Y'\rangle\rangle \land \\
&\qquad\qquad (\ Y' = \dot{0} \to (\ Z' = x\ \land\ v = \bot\ )\ )\ \big]\ \big]
\end{aligned} \tag{15}
$$

In order to do so, we assume

$$
Y, Z \in N \text{ and } \langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle \text{ and } Y \neq \dot{0} \tag{16}
$$

and prove

$$
\begin{aligned}
&\exists v\ \exists Y' Z' \in N\big[\ Z = \dot{\mathrm{S}}Z'\ \land\ Y = \dot{\mathrm{S}}Y'\ \land\ u = \langle v, \langle Z', Y'\rangle\rangle \land \\
&\qquad (\ Y' = \dot{0} \to (\ Z' = x\ \land\ v = \bot\ )\ )\ \big]\ .
\end{aligned} \tag{17}
$$

By our assumptions (16), we have $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle$, and then $\mathsf{T}_4$ yields three cases: (a) $\langle u, \langle Z, Y\rangle\rangle = \langle w, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle$, (b) $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq w$ and (c) $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle$. We prove that that (17) holds in each of these three cases.

*Case (a):* By $\mathsf{T}_2$, we have $u = w$, $Z = \dot{\mathrm{S}}z$ and $Y = \dot{\mathrm{S}}y$. By (14), we have $z \in N$. By (12), we have $y \in N$. Moreover, by (12), we also have $u = w = \langle w', \langle z, y\rangle\rangle$. Thus there exist $v$ and $Y', Z' \in N$ such that

$$
Z = \dot{\mathrm{S}}Z'\ \land\ Y = \dot{\mathrm{S}}Y'\ \land\ u = \langle v, \langle Z', Y'\rangle\rangle\ .
$$

If $y = \dot{0}$, we must have $\langle v, \langle z, y\rangle\rangle = w = \langle \bot, \langle x, \dot{0}\rangle\rangle$ since $\phi_A(x, y, w)$ holds by our assumptions (12). By $\mathsf{T}_2$, this implies $z = x$ and $v = \bot$. This proves that (17) holds in Case (a).

*Case (b):* By our assumptions (12), we have $\phi_A(x, y, w)$, and thus we also have $\theta_A(u, w, Y, Z) \equiv$

$$
\begin{aligned}
&\langle u, \langle Z, Y\rangle\rangle \sqsubseteq w\ \land\ Y \neq \dot{0} \to \\
&\qquad \exists v\ \exists Y' Z' \in N\big[\ Z = \dot{\mathrm{S}}Z'\ \land\ Y = \dot{\mathrm{S}}Y'\ \land\ u = \langle v, \langle Z', Y'\rangle\rangle) \land \\
&\qquad\qquad (\ Y' = \dot{0} \to (\ Z' = x\ \land\ v = \bot\ )\ )\ \big]\ .
\end{aligned} \tag{18}
$$

We are dealing with a case where the antecedent of (18) holds, and thus (17) holds.

*Case (c):* This case is not possible. By $\mathsf{T}_4$, this case splits into the subcases: (a') $\langle u, \langle Z, Y\rangle\rangle = \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle$, (b') $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \dot{\mathrm{S}}z$ and (c') $\langle u, \langle Z, Y\rangle\rangle \sqsubseteq \dot{\mathrm{S}}y$. We prove that each of these subcases contradicts our axioms. *Case (a'):* Recall that $\dot{\mathrm{S}}y$ is shorthand for $\langle y, \perp\rangle$. Thus, by $\mathsf{T}_2$, we have $Y = \perp$. This contradicts the assumption (12) that $Y \in N$. *Case (b'):* By Lemma 4 (iii), we have $\langle u, \langle Z, Y\rangle\rangle = \perp \vee N(\langle u, \langle Z, Y\rangle\rangle)$. Now, $\langle u, \langle Z, Y\rangle\rangle = \perp$ contradicts $\mathsf{T}_1$. Furthermore, $N(\langle u, \langle Z, Y\rangle\rangle)$ implies that there is $z_0$ such that $\langle u, \langle Z, Y\rangle\rangle = \langle z_0, \perp\rangle$. By $\mathsf{T}_2$, we have $\langle Z, Y\rangle = \perp$. This contradicts $\mathsf{T}_1$. Case (c') is similar to Case (b'). This proves that (17) holds, and thus we conclude that the lemma holds. □

**Lemma 8.**

$$\mathsf{T}^+ \vdash \forall xy \in N \forall w[\ \phi_A(x, \dot{\mathrm{S}}y, w)\ \rightarrow\ \exists u \in N \exists w'[\ w = \langle w', \langle u, \dot{\mathrm{S}}y\rangle\rangle\ \wedge\ \phi_A(x, y, w')\ ]\ ]\ .$$

*Proof.* Let $x, y \in N$ and assume $\phi_A(x, \dot{\mathrm{S}}y, w)$. Thus, we have $w'$ and $z \in N$ such that

$$w = \langle w', \langle z, \dot{\mathrm{S}}y\rangle\rangle \quad \text{and} \quad \forall u \forall Y, Z \in N[\ \theta_A(u, w, Y, Z)\ ] \tag{19}$$

Use the assumptions (19) to prove that $\phi_A(x, y, w') \equiv$

$$(\ y = \dot{0} \rightarrow w' = \langle \perp, \langle x, \dot{0}\rangle\rangle\ )\ \wedge\ \exists w'' \exists z \in N[\ w' = \langle w'', \langle z, y\rangle\rangle\ ] \wedge \forall u \forall Y, Z \in N[\ \theta_A(u, w', Y, Z)\ ] \tag{20}$$

holds. We omit the details. □

**Lemma 9.** *The translations of* $\mathsf{A}$, $\mathsf{G}_4$ *and* $\mathsf{G}_5$ *are theorems of* $\mathsf{T}^+$.

*Proof.* The translation of the axiom $\mathsf{A}$ is

$$\forall x, y, z_1, z_2 \in N[\ \Psi_A(x, y, z_1) \wedge \Psi_A(x, y, z_2)\ \rightarrow\ z_1 = z_2\ ]\ .$$

Assume $\Psi_A(x, y, z_1)$ and $\Psi_A(x, y, z_2)$. Then it follows straightforwardly from the definition of $\Psi_A$ and $\mathsf{T}_2$ that $z_1 = z_2$. Hence the translation is a theorem of $\mathsf{T}^+$.

The translation of $\mathsf{G}_4$ is $\forall x \in N[\Psi_A(x, \dot{0}, x)]$, that is

$$\forall x \in N \exists w\big[\ \phi_A(x, \dot{0}, w)\ \wedge\ \exists w'[\ w = \langle w', \langle x, \dot{0}\rangle\rangle\ ] \wedge \forall u[\ \phi_A(x, \dot{0}, u) \rightarrow u = w\ ]\ \big]\ .$$

We have

$$\mathsf{T}^+ \vdash \phi_A(x, \dot{0}, \langle \perp, \langle x, \dot{0}\rangle\rangle) \text{ and } \mathsf{T}^+ \vdash \forall u[\phi_A(x, \dot{0}, u) \rightarrow u = \langle \perp, \langle x, \dot{0}\rangle\rangle$$

by Lemma 6, and it easy to see that the translation of $\mathsf{G}_4$ is a theorem of $\mathsf{T}^+$.

The translation of $\mathsf{G}_5$ is

$$\forall x, y, u \in N[\ \exists z \in N[\ \varPsi_A(x,y,z) \wedge u = \dot{\mathrm{S}}z\ ] \rightarrow \varPsi_A(x, \dot{\mathrm{S}}y, u)\ ]\ . \tag{21}$$

In order to prove that (21) can be deduced from the axioms of $\mathsf{T}^+$, we assume $\varPsi_A(x,y,z) \wedge u = \dot{\mathrm{S}}z$. Then we need to prove $\varPsi_A(x, \dot{\mathrm{S}}y, \dot{\mathrm{S}}z) \equiv$

$$\exists w\big[\ \phi_A(x, \dot{\mathrm{S}}y, w)\ \wedge\ \exists w'[\ w = \langle w', \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle\ ] \wedge \\ \forall u[\ \phi_A(x, \dot{\mathrm{S}}y, u) \rightarrow u = w\ ]\ \big]\ . \tag{22}$$

By our assumption $\varPsi_A(x,y,z)$ there is a unique $w_1$ such that $\phi_A(x,y,w_1)$ and $w_1 = \langle w_0, \langle z, y\rangle\rangle$ for some $w_0$. By Lemma 7, we have $\phi_A(x, \dot{\mathrm{S}}y, \langle w_1, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle)$. Thus, we have $w_2$ such that $\phi_A(x, \dot{\mathrm{S}}y, w_2)$ and $w_2 = \langle w_1, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle$. It is easy to see that (22) holds if $w_2$ is unique. Thus we are left to prove the uniqueness of $w_2$, more precisely, we need to prove that

$$\forall W_2[\ \phi_A(x, \dot{\mathrm{S}}y, W_2)\ \rightarrow\ W_2 = w_2\ ]\ . \tag{23}$$

In order to prove (23), we assume $\phi_A(x, \dot{\mathrm{S}}y, W_2)$ (we will prove $W_2 = w_2 = \langle w_1, \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\rangle$). By our assumption $\phi_A(x, \dot{\mathrm{S}}y, W_2)$ and Lemma 8, we have $u_0 \in N$ and $W_1$ such that $W_2 = \langle W_1, \langle u_0, \dot{\mathrm{S}}y\rangle\rangle$ and $\phi_A(x,y,W_1)$. We have argued that there is a unique $w_1 = \langle w_0, \langle z, y\rangle\rangle$ such that $\phi_A(x,y,w_1)$ holds. By this uniqueness, we have $W_1 = w_1 = \langle w_0, \langle z, y\rangle\rangle$. So far we have proved

$$w_2\ =\ \langle\ \overbrace{\langle w_0, \langle z, y\rangle\rangle}^{w_1},\ \langle \dot{\mathrm{S}}z, \dot{\mathrm{S}}y\rangle\ \rangle \text{ and } W_2\ =\ \langle\ \overbrace{\langle w_0, \langle z, y\rangle\rangle}^{W_1},\ \langle u_0, \dot{\mathrm{S}}y\rangle\ \rangle$$

and then we are left to prove that $u_0 = \dot{\mathrm{S}}z$. By our assumption $\phi_A(x, \dot{\mathrm{S}}y, W_2)$, we have $v$ and $Z', Y' \in N$ such that $u_0 = \dot{\mathrm{S}}Z'$, $\dot{\mathrm{S}}y = \dot{\mathrm{S}}Y'$ and $W_1 = \langle v, \langle Z', Y'\rangle\rangle$. Thus, $\langle v, \langle Z', Y'\rangle\rangle = \langle w_0, \langle z, y\rangle\rangle$. By $\mathsf{T}_2$, we have $z = Z'$, and thus, $u_0 = \dot{\mathrm{S}}Z' = \dot{\mathrm{S}}z$. This proves that (23) holds. ☐

We will now give the translation $\varPsi_M$ of $M$. Let $\phi_M(x,y,w) \equiv$

$$(\ y = \dot{0} \rightarrow w = \langle \bot, \langle \dot{0}, \dot{0}\rangle\rangle\ )\ \wedge\ \exists w' \exists z \in N[\ w = \langle w', \langle z, y\rangle\rangle\ ]\ \wedge \\ \forall u \forall Y, Z \in N\ \theta_M(u, w, Y, Z)$$

where $\theta_M(u, w, Y, Z) \equiv$

$$\langle u, \langle Z, Y\rangle\rangle \sqsubseteq w \wedge Y \neq \dot{0}\ \ \rightarrow\ \ \exists v\ \exists Y', Z' \in N\big[\ \varPsi_A(Z', x, Z) \wedge \\ Y = \dot{\mathrm{S}}Y'\ \wedge\ u = \langle v, \langle Z', Y'\rangle\rangle\ \wedge\ (\ Y' = \dot{0} \rightarrow Z' = \dot{0} \wedge v = \bot\ )\ \big]\ .$$

We let $\varPsi_M(x,y,z) \equiv$

$$\exists w\big[\ \phi_M(x,y,w)\ \wedge\ \exists w'[\ w = \langle w', \langle z, y\rangle\rangle\ \wedge\ \forall u[\ \phi_M(x,y,u) \rightarrow u = w\ ]\ \big]\ .$$

The translations of $\mathsf{M}$, $\mathsf{G}_6$ and $\mathsf{G}_7$ are

$$\begin{array}{ll}
\mathsf{M} & \forall x, y, z_1, z_2 \in N[\ \Psi_M(x,y,z_1) \wedge \Psi_M(x,y,z_2) \rightarrow z_1 = z_2\ ] \\
\mathsf{G}_6 & \forall x \in N[\ M(x, \dot{0}, \dot{0})\ ] \\
\mathsf{G}_7 & \forall x, y, u \in N[\ \exists z \in N[\ \Psi_M(x,y,z) \wedge \Psi_A(z,x,u)\ ] \rightarrow \Psi_M(x, \dot{\mathrm{S}}y, u)\ ]\ .
\end{array}$$

The proof of the next lemma follows the lines of the proof of Lemma 9. We omit the details.

**Lemma 10.** *The translations of* $\mathsf{M}$, $\mathsf{G}_6$ *and* $\mathsf{G}_7$ *are theorems of* $\mathsf{T}^+$.

**Theorem 11.** $\mathsf{Q}$ *is interpretable in* $\mathsf{T}$.

*Proof.* It is proved in Svejdar [8] that $\mathsf{Q}$ is interpretable in $\mathsf{Q}^-$. It follows from the lemmas above that $\mathsf{Q}^-$ is interpretable in $\mathsf{T}^+$ which again is interpretable in $\mathsf{T}$. Hence the theorem holds. ☐

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