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Carleman approximation by holomorphic automorphisms of $\mathbb{C}^n$

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Abstract. We approximate smooth maps defined on non-compact totally real manifolds by holomorphic automorphisms of $\mathbb{C}^n$.

1. Introduction

The aim of the present paper is to prove a version of the Andersén–Lempert theorem with control on *non-compact* totally real submanifolds of $\mathbb{C}^n$. We use coordinates $z_j = x_{2j-1} + i x_{2j}$ on $\mathbb{C}^n$, and by $\mathbb{R}^s \subset \mathbb{C}^n$ we mean $\{z \in \mathbb{C}^n; x_{2j-1} = 0 \text{ for } j > s,\ x_{2j} = 0 \text{ for } j \geq 1\}$. The following is our main result (see also Theorem 6.3 for a more general statement).

Theorem 1.1. *Let $K \subset \mathbb{C}^n$ be a compact set, let Ω be an open set containing K, and let*

$$\phi : [0,1] \times (\Omega \cup \mathbb{R}^s) \to \mathbb{C}^n, \quad s < n,$$

be an isotopy of smooth embeddings, $\phi_0 = \phi(0,\cdot) = \mathrm{id}$, such that the following hold:

(1) *$\phi_t|_\Omega$ is holomorphic for all t,*

(2) *$\phi_t(K \cup \mathbb{R}^s)$ is polynomially convex, and $\phi_t(\mathbb{R}^s \setminus K)$ is totally real, for all t, and*

(3) *there is some compact set $C \subset \mathbb{R}^s$ such that $\phi_t|_{\mathbb{R}^s \setminus C} = \mathrm{id}$ for all t.*

Then for any $k \in \mathbb{N}$ we have that ϕ_1 is $\mathcal{C}^k$-approximable on $K \cup \mathbb{R}^s$, in the sense of Carleman by holomorphic automorphisms of $\mathbb{C}^n$.

Recall that ϕ is $\mathcal{C}^k$-approximable in the sense of Carleman by holomorphic automorphisms if for any strictly positive continuous function δ on $\mathbb{R}^s$, there exists a holomorphic automorphism ψ of $\mathbb{C}^n$ such that

$$\left\| \frac{d}{dx^\alpha}(\psi|_{\mathbb{R}^s} - \phi)(x) \right\| < \delta(x),$$

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for all multi indices $\alpha = (\alpha_1, \dots, \alpha_{2n})$ with $\alpha_{2j-1} = 0$ for all j, $\alpha_j = 0$ for $j > 2s$, and $|\alpha| \le k$.

As a first application of this result we also prove the following:

Theorem 1.2. *Let $\phi : \mathbb{R}^s \to \mathbb{R}^s$ be a smooth automorphism, and assume that $s < n$. Then ϕ can be approximated in the sense of Carleman by holomorphic automorphisms of $\mathbb{C}^n$.*

Our main Theorem 1.1 generalizes work of Forstnerič–Rosay [5], Forstnerič [3] and Forstnerič–Løw–Øvrelid [4], where similar results were proved for compact totally real manifolds. The proof of the main theorem depends on the Andersén–Lempert theory and also results on Carleman approximation by entire *functions*. Carleman approximation by entire functions on $\mathbb{R}^s \subset \mathbb{C}^n$ was proved by Scheinberg [13] and Hoischen [8]. On smooth curves in $\mathbb{C}^n$ it was proved by Alexander [1], and more generally, for dendrites, it was proved by Gauthier and Zeron [7].

Controlling non-compact *one-dimensional* totally real manifolds has been extremely useful. In particular, it is one of the key ingredients in recent developments on proper holomorphic embeddings of Riemann surfaces (see, e.g., [6] and references therein). The one-dimensional case, however, depends on a certain "precomposition with a shear trick" which was introduced by Buzzard and Forstnerič [2], but this brakes down in higher dimensions due to topological problems. Therefore, passing to higher dimensions, as in the current paper, requires a different approach.

As a first application of our main theorem, we give an approximation result, Theorem 7.1, of smooth automorphisms on $\mathbb{R}^k \subset \mathbb{C}^n$. This can be seen as a first step towards approximating smooth symplectomorphisms/automorphisms on $\mathbb{R}^k \subset \mathbb{C}^n$ by holomorphic automorphisms *fixing* $\mathbb{R}^k$. It has been suggested by experts in dynamics that this could be very useful in the study of real symplectic dynamical systems. This will be pursued further in future work. In a different direction, our main result was recently a key ingredient in the study of Oka-properties of $X := \mathbb{C}^n \setminus \mathbb{R}^k$, i.e., flexibility properties of X as a target for maps from Stein manifolds.

The article is organized as follows. After a section with preliminaries, we include in Section 3 a slightly new version of the fundamental result of Andersén and Lempert, which is needed for our purposes. The point is to write any polynomial vector field on $\mathbb{C}^n$ as a sum of *complete* holomorphic vector fields with some additional geometric properties (see Lemma 3.1) which will allow us to use known results on Carleman approximation of *functions* as developed in [12]. In Section 4 we prove that isotopies of totally real manifolds of real codimension one generically are polynomially convex. This is very important for applications of Theorems 1.1 and 6.3, where convexity of the isotopy is included as an assumption, whereas for applications we are initially given a totally real isotopy which might not be polynomially convex. In Section 5 we prove that the isotopies initially given in Theorems 1.1 and 6.3, may be extended to isotopies on the ambient space $\mathbb{C}^n$ which are of full rank and $\overline{\partial}$-flat along M. The rank condition is necessary for being able to approximate by injective maps, and the $\overline{\partial}$-flat condition is necessary for being able to approximate by holomorphic maps. In Section 6 we state and prove our main result, and since the proof is quite technical, a brief sketch is included. Finally, in Section 7, we give the application mentioned above.

2. Preliminaries

Definition 2.1. For a function $f \in \mathcal{C}^k(\mathbb{C}^n)$ we let $|f|_{k,z}$ denote the pointwise semi-norm

$$|f|_{k,z} := \sum_{|\alpha| \leq k} \left| \frac{d^\alpha f}{dx^\alpha}(z) \right|.$$

If $\phi = (f_1, \dots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ is a $\mathcal{C}^k$-smooth map, we define

$$|\phi|_{k,z} := \sum_{1 \leq j \leq m} |f_j|_{k,z}.$$

In this article we will be interested in approximating $\mathcal{C}^\infty$-smooth embeddings

$$\phi : M \subset \mathbb{C}^n \to \mathbb{C}^n,$$

where M is a non-compact totally real submanifold, by holomorphic automorphisms. It will be convenient to consider instead smooth maps

$$\phi : \mathbb{C}^n \to \mathbb{C}^n$$

with some additional assumptions on the map ϕ along M, and approximate with respect to the norm $|\cdot|_{k,x}$. A necessary condition is clearly that ϕ has full rank along M, and since holomorphic functions satisfy the Cauchy–Riemann equations, we will have to work in the following class of maps:

Definition 2.2. Let A be a subset of $\mathbb{C}^n$ and let $\phi : \mathbb{C}^n \to \mathbb{C}^m$ be a smooth mapping of class $\mathcal{C}^k$. We will write $\phi \in \mathcal{H}_k(\mathbb{C}^n, A)$ if any component function f of ϕ has the property that $\overline{\partial} f$ vanishes to order $k-1$ on A, i.e., $\overline{\partial} \frac{d^\alpha f}{dx^\alpha}(z) = 0$ for all $\|\alpha\| < k$ and all $z \in A$, $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + i \cdot x_{2j}$.

Definition 2.3. Let $K \subset \mathbb{C}^n$ be compact. As usual we let

$$\widehat{K} := \left\{ z \in \mathbb{C}^n : |f(z)| \leq \|f\|_K \text{ for all } f \in \mathcal{O}(\mathbb{C}^n) \right\}$$

denote the *polynomially convex hull of* K. If $K \subset \mathbb{C}^n$ is any closed subset, we write K as an increasing union $K = \bigcup_{j \in \mathbb{N}} K_j$ of compact sets K_j with $K_j \subset K_j^\circ$, and define the *holomorphically convex hull* of K by

$$\widehat{K} := \bigcup_{j \in \mathbb{N}} \widehat{K}_j.$$

We define a set function h defined on closed subsets K of $\mathbb{C}^n$ by

$$h(K) := \widehat{K} \setminus K.$$

We say that a closed set $K \subset \mathbb{C}^n$ has *bounded E-hulls* (E = exhaustion) if for any compact set $Y \subset \mathbb{C}^n$ the set $h(K \cup Y)$ is bounded.

3. The fundamental result of Andersén and Lempert

The following is a slightly new version of the fundamental result of Andersén and Lempert. For a non-zero vector $v \in \mathbb{C}^n$ let π_v denote the projection of $\mathbb{C}^n$ along v to the orthogonal complement of $\mathbb{C}_v$.

Lemma 3.1. *Let X be a polynomial vector field on $\mathbb{C}^n$ and let $\mathrm{v} \in \mathbb{C}^n$ be a non-zero vector. Then for any $\epsilon > 0$, X can be written as sum $X = \sum_{j=1}^{N_1} X_j + \sum_{j=1}^{N_2} Y_j$ such that the following holds.*

(a) $X_j(z) = f_j(\pi_{v_j}(z))v_j$ *with* $f_j \in (\mathbb{C}^{n-1})$, $\|v_j - \mathrm{v}\| < \epsilon$, *and*

(b) $Y_j(z) = g_j(\pi_{w_j}(z)) \cdot \langle z, w_j \rangle w_j$ *with* $g_j \in (\mathbb{C}^{n-1})$, $\|w_j - \mathrm{v}\| < \epsilon$.

Here we identify $\mathbb{C}^n$ with its tangent space at each point as usual. If $\mathrm{div} X = 0$, *the terms on the form* (b) *are not needed.*

We will say that a decomposition of X like this is a decomposition *respecting* (v, ϵ).

Proof. Writing X as a sum of its homogeneous parts, it is sufficient to prove the lemma for $X = X_k$ homogeneous of degree k. For a homogeneous polynomial vector field on $\mathbb{C}^n$, we can apply the following lemma; see [10, Lemma 7.6].

Lemma 3.2. *There exist*

- $n \cdot \binom{n+k-2}{n-1} - \binom{n+k-2}{n-1}$ *linear forms* $\lambda_i \in (\mathbb{C}^n)^*$ *and vectors* $v_i \in \mathbb{C}^n$ *with* $\lambda_i(v_i) = 0$ *and* $\|v_i\| = 1$, *and*
- $\binom{n+k-2}{n-1}$ *linear forms* $\tilde{\lambda}_j \in (\mathbb{C}^n)^*$ *and vectors* $w_j \in \mathbb{C}^n$ *with* $\tilde{\lambda}_j(w_j) = 0$ *and* $\|w_j\| = 1$,

such that the homogenous polynomial maps

$$z \mapsto (\lambda_i(z))^k v_i, \quad i = 1, 2, \ldots, n \cdot \binom{n+k-1}{n-1} - \binom{n+k-2}{n-1}$$

of degree k together with the homogenous polynomial maps

$$z \mapsto (\tilde{\lambda}_j(z))^{k-1} \langle z, w_j \rangle w_j, \quad j = 1, 2, \ldots, \binom{n+k-2}{n-1}$$

of degree k form a basis of the vector space $V_k \cong S^k((\mathbb{C}^n)^*) \otimes \mathbb{C}^n$ of homogenous polynomial maps of degree k. Moreover, if $v_0 \in \mathbb{C}^n$ and a non-zero functional $\lambda_0 \in (\mathbb{C}^n)^*$ with $\lambda_0(v_0) = 0$ and $\|v_0\| = 1$ and a number $\epsilon > 0$ are given, then the vectors v_i, w_j together with the functionals $\lambda_i, \tilde{\lambda}_j$ can be chosen with $\|v_0 - v_i\| < \epsilon$, $\|v_0 - w_j\| < \epsilon$ and $\|\lambda_0 - \lambda_i\| < \epsilon$, $\|\lambda_0 - \tilde{\lambda}_j\| < \epsilon$.

Remark that the normalization for the vectors v_i and v_0 is not important since constants can be moved over to the linear functionals. Now finally remark that if λ is a linear functional with a non-zero vector v in its kernel, then it factors over the projection π_v.Thus $(\lambda_i(z))^k = f_i(\pi_{v_i})$. □

4. Perturbations of families of totally real manifolds

Without the compact set K the following perturbation result was proved by Forstnerič and Rosay in the case that M is a real analytic surface, and by Forstnerič assuming

$$\dim(M) \leq \frac{2n}{3}.$$

It was proved without parameters by the second author and Løw in [11]. The proof we give here is very similar to the non-parametric case, building on the original idea of Forstnerič and Rosay.

Proposition 4.1. *Let $K \subset \mathbb{C}^n$ be a compact set, and let $M \subset \mathbb{C}^n$ be a smooth manifold with $\dim_{\mathbb{R}}(M) < n$. Let $f : [0,1] \times (K \cup M) \to \mathbb{C}^n$ be an isotopy of continuous injective maps, let $K \subset U' \subset\subset U$ be open sets, and let $G_t : U \to U_t \subset \mathbb{C}^n$ be a continuous family of homeomorphisms such that $K_t = f_t(K) \subset U_t = G_t(U)$ for all t. Assume that the following hold for all t:*

(1) $\widehat{K}_t \subset U'_t := G_t(U')$, *and*

(2) $f_t : \overline{M \setminus K} \to \mathbb{C}^n$ *is a $\mathcal{C}^k$-smooth isotopy which is an embedding onto a totally real manifold.*

Then for any $\epsilon > 0$ there exist $g_t : [0,1] \times (K \cup M) \to \mathbb{C}^n$ and an open set $U'' \subset U$ with $K \subset U''$ such that

(1) $g_t|_{(K \cup M) \cap U''} = f_t$ *for all t,*

(2) $|g_t - f_t|_{x,k} < \epsilon$ *for all $x \in M \setminus K$, and*

(3) $h(g_t(K \cup M)) \subset U'_t$ *for all t.*

Assuming also that $f_0(K \cup M)$ and $f_1(K \cup M)$ are polynomially convex, we may achieve that $g_0 = f_0$ and $g_1 = f_1$.

Proposition 4.2. *Let $\Psi : \overline{B} \to \mathbb{C}^k$ be a $\mathcal{C}^1$-smooth map of the form $\Psi(x) = (x, \psi(x))$, where B is the unit ball in $\mathbb{R}^k$, and we write $\mathbb{C}^k = \mathbb{R}^k \oplus i\mathbb{R}^k$. Assume that ψ is Lipschitz-α with $\alpha < 1$. Then there exist ϵ, δ such that the following hold: for any $\tilde{\Psi}$ with $\|\tilde{\Psi} - \Psi\|_{1,x} < \epsilon$ for all $x \in \overline{B}$ and any point $z_0 \in \mathbb{C}^k \setminus \tilde{\Psi}(\overline{B})$ with $\pi_x(z_0) \in \pi_x(\tilde{\Psi}(\overline{B}_{1/2}))$, there exists an entire function g such that*

(1) $g(z_0) = 1$,

(2) $\|g\|_{\tilde{\Psi}(\overline{B})} < 1$, *and*

(3) $|g(z)| < 1$ *for all z such that* $\operatorname{dist}(z, \tilde{\Psi}(bB)) \leq \delta$.

Proof. We will give the argument considering only the map Ψ, and it will be clear that it is stable under small perturbations. Write $x_0 = \pi_x(z_0)$, $z' = x_0 + i\psi(x_0)$, and consider the function $h(z) = (z - z')^2$ on $\Psi(\overline{B})$. We have

$$\begin{aligned}\operatorname{Re}(h(z)) &= \operatorname{Re}\big(\big((x - x_0) + i(\psi(x) - \psi(x_0))\big)^2\big) \\ &= |x - x_0|^2 - |\psi(x) - \psi(x_0)|^2 \geq (1 - \alpha^2)|x - x_0|^2.\end{aligned}$$

Clearly $\mathrm{Re}(h(z_0)) < 0$ and so defining $g(z) := ce^{-h(z)}$ takes care of (1) and (2) for a suitable constant $c > 0$. And if δ is chosen small enough and $\mathrm{dist}(z, \Psi(b(B))) \leq \delta$, we have $|\pi_x(z) - x_0| > 0$ and $|\pi_y(z)| < |\pi_x(z)|$, and so by the same calculation as above we have $|g(z)| < 1$. Clearly these estimates can be made to hold under small perturbations. □

Corollary 4.3. *Let $(\Psi, \Phi) : \overline{B} \to \mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ be a $\mathcal{C}^1$-smooth map, where B is the unit ball in $\mathbb{R}^k$, and Ψ is as in the previous proposition. Let ϵ be as above. Then if $|\tilde{\Psi} - \Psi|_{1,x} < \epsilon$ for all $x \in \overline{B}$, we have that $K := (\tilde{\Psi}, \tilde{\Phi})(\overline{B}_{1/2})$ is polynomially convex, where $\tilde{\Phi}$ is any continuous map.*

Proof. Let π_k denote the projection onto $\mathbb{C}^k$. Now π_k is an entire map which maps K onto a polynomially convex totally real manifold $S \subset \mathbb{C}^k$. Since each point of S is then a peak point for the algebra $P(S)$, it follows that K is polynomially convex (see [12]). □

Corollary 4.4. *Let $K \subset \mathbb{C}^n$ be a compact set, and let $M \subset \mathbb{C}^n$ be a compact totally real set. Then for any open neighborhood U of K there exist $\beta, \epsilon > 0$ such that the following hold:*

(i) *If $S \subset M$ is any closed set, and if $h(K \cup S) \subset (K \cup S)(\beta)$, then $h(K \cup S) \subset U$, and*

(ii) *if M_ϵ is a $\mathcal{C}^1$-ϵ-perturbation of M, and if $S \subset M_\epsilon$ is closed, then* (i) *still holds.*

Proof. To show (i) it suffices by compactness to show that for any point $x \in M \setminus K$ there exists an $r > 0$ such that if α is small enough, then if $S \subset M$ is any closed set with $x \in S$, and if $h(K \cup S) \subset (K \cup S)(\alpha)$, then $h(K \cup S) \cap B_r(x) = \emptyset$. The argument we give will make it clear that this is stable under small perturbations of M.

Fix $x \in M \setminus K$. By scaling there exist $0 < r_1 < r_2 \ll 1$ and $\delta > 0$ such that for any $z_0 \in B_{r_1}(x) \setminus M$ there exists an entire function f with the following properties:

(a) $f(z_0) = 1$,

(b) $\|f(z)\| < 1$ for all $z \in M \cap B_{r_2}(x)$, and

(c) $\|f(z)\| < 1$ for all z with $\mathrm{dist}(z, bB_{r_2}(x) \cap M) < \delta$.

Now choose β so small that if we define

(d) $U_1 := (K \cup M)(\beta) \cap B_{r_2}(x)$, and

(e) $U_2 := [(K \cup M)(\beta) \setminus B_{r_2}(x)] \cup [(K \cup M)(\beta) \cap (bB_{r_2} \cap M)(\delta)]$

then $\{U_1, U_2\}$ is an open cover of $(K \cup M)(\beta)$ with $U_1 \cap U_2 \subset (bB_{r_2} \cap M)(\delta)$. Note that a function f as above will satisfy

(f) $\|f(z)\| < 1$ for all $z \in U_1 \cap U_2$.

Now let $S \subset M$ be any closed set with $x \in S$. Then $(K \cup S)(\beta) \subset (K \cup M)(\beta)$. If $h(K \cup S) \subset (K \cup S)(\beta)$, there exists a Runge and Stein neighborhood $\Omega \subset (K \cup S)(\beta)$ of $K \cup S$. We define $\tilde{U}_1 = U_1 \cap \Omega$ and $\tilde{U}_2 = U_2 \cap \Omega$. Then $\{\tilde{U}_1, \tilde{U}_2\}$ is an open cover of Ω. For any point $x \in B_{r_1}(x) \setminus M$ we let f be a function as above. Regarding f^m as a cocycle on $\tilde{U}_1 \cap \tilde{U}_2$ we solve Cousin problems with sup-norm estimates and get by (f) holomorphic functions on Ω that separate x from $K \cup S$.

We now have that $h(K \cup S) \cap B_{r_1}(x) \subset M$ which implies our claim, since totally real points on polynomially convex compact sets are peak points.

The result is stable of small perturbations of M since the sizes of r_1, r_2 and δ are stable. □

Fix an open neighborhood $W \subset \mathbb{C}$ of $I = [0, 1]$. We will now consider subvarieties Σ and Z of $W \times \mathbb{C}^n$, and by Σ_t and Z_t respectively, we will mean the fibers over a points $t \in W$.

Lemma 4.5. *Let $M \subset \mathbb{C}^n$ be a compact $\mathcal{C}^k$-smooth manifold (possibly with boundary) of real dimension $m < n$, and let $f : [0, 1] \times M \to \mathbb{C}^n$ be a $\mathcal{C}^k$-smooth isotopy of embeddings such that $f_0 = \mathrm{id}$ and assume that $f_t(M)$ is totally real for each $t \in [0, 1]$. Then there exists a $\delta > 0$ such that the following hold: for any point $x \in M$ and (relatively) open sets $U \subset\subset V \subset B_\delta(x) \cap M$, any variety $\Sigma \subset W \times \mathbb{C}^n$, and any $\epsilon > 0$, there exists a hypersurface $Z \subset W \times \mathbb{C}^n$, and a $\mathcal{C}^k$-smooth isotopy $g : [0, 1] \times M \to \mathbb{C}^n$ such that*

(i) $|g_t(x) - f_t(x)|_{k,x} < \epsilon$ *for all* $x \in M$, $t \in [0, 1]$,

(ii) $g_t(y) = f_t(y)$ *for all* $y \in M \setminus V$, $t \in [0, 1]$,

(iii) $g_t(U \cap M) \subset Z_t$ *for all* $t \in [0, 1]$, *and*

(iv) $\dim(Z \cap \Sigma) < \dim(\Sigma)$.

Proof. We may assume that f_t is defined on $\tilde{M} \subset \mathbb{C}^n$ with $M \subset\subset \tilde{M}$ with $f_t(\tilde{M})$ totally real for each t. If δ is small enough, we deduce from Corollary 4.3 that the following hold: for each $x \in M$ there exists a parametrization $\phi : \overline{B} \to \tilde{M}$ of $\tilde{M}$ near x, where B is the unit ball in $\mathbb{R}^k$, such that $f_t(\phi(\overline{B}))$ is polynomially convex for all t, and such that $B_\delta(x) \cap \tilde{M} \subset \phi(B)$. Fix x and choose a cutoff function χ which is identically one on $\overline{U}$ and compactly supported in V. We now consider $\mathbb{R}^k$ to be contained in $\mathbb{C}^{n-1} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. By [3] we see that $f_t \circ \phi$ is uniformly approximable in $\mathcal{C}^k$-norm on $\overline{B} \times I$ by a family F_t of holomorphic automorphisms of $\mathbb{C}^n$ also holomorphic in $t \in W$ (see also Lemma 6.4). Set $Z := F_t((\mathbb{C}^{n-1} \times \{0\}) \times W)$. By genericity we may assume that (iv) holds. Write $\tilde{g}_t := F_t \circ \phi^{-1}$, and finally set $g_t(x) := f_t(x) + \chi(x)(\tilde{g}_t(x) - f_t(x))$. □

Proof of Proposition 4.1. We give the proof first in the case of M being an embedded cube $f_0 : [0, 1]^k \to \mathbb{C}^n$. The general case follows by covering M by a locally finite family of cubes, and successively using the result for cubes and gluing (see [11] for details). We will prove the result by induction on k, and we note that the result is obvious for $k = 0$. Assume that the result holds for some $k \geq 0$. To avoid working with too many indices we give the argument for passing from $k = 1$ to $k = 2$, the passing from $k = 0$ to $k = 1$ is simpler, and the other cases are completely similar.

Choose an open set $K \subset V \subset U'$ such that

$$\widehat{[(K_t \cup M_t) \cap \overline{V}_t]} \subset U'_t$$

and choose an open set $K \subset U'' \subset\subset V$. For $m \in \mathbb{N}$ we let Γ_m denote the grid

$$\Gamma_m = \{x \in [0, 1]^2 : x_1 = j/m \text{ or } x_2 = j/m,\ 0 \leq j \leq m\}.$$

We let Q_{ij} denote the cube

$$Q_{ij} = [i/m, (i+1)/m] \times [j/m + (j+1)/m].$$

For small $\beta > 0$ we define

$$Q_{ij}^{\beta} = [i/m + \beta, (i+1)/m - \beta] \times [j/m + \beta, (j+1)/m - \beta],$$
$$\Gamma_m(\beta) = [0,1]^2 \setminus \bigcup_{ij} Q_{ij}^{\beta}.$$

If m is large enough then if $f_t(Q_{ij})$ is not contained in V_t then $f_t(Q_{ij})$ does not intersect U_t''. If m is large enough then if S is any collection of $n+1$ cubes Q_{ij} then

$$h\big(((K_t \cup M_t) \cap \overline{V}_t) \cup S_t\big) \subset U_t'.$$

Note that this still holds if we replace f_t by a small $\mathcal{C}^1$-perturbation. Now by the induction hypothesis we may (by possibly having to perturb f_t slightly) assume that for any collection of $n+1$ cubes we also have that

$$h\big(((K_t \cup M_t) \cap \overline{V}_t \cup \Gamma_{m,t}) \cup S_t\big) \subset U_t'.$$

For this we successively use the induction hypothesis and create the grid by attaching a 1-cube to collections of cubes Q_{ij}, perturbing the isotopy each time. Finally, by choosing a small enough β, we may assume that

$$h\big(((K_t \cup M_t) \cap \overline{V}_t \cup \Gamma_{m,t}(\beta)) \cup S_t\big) \subset U_t'$$

(use Corollary 4.4). Finally, by Lemma 4.5 we may assume that there are parametrized subvarieties $Z_{i,j,t} = Z(h_{i,j,t})$ of $\mathbb{C}^n$ with $Q_{i,j,t}^{\beta} \subset Z_{i,j,t}$ for all cubes not completely contained in V_t, and such that for a fixed t, any collection of $n+2$ subvarieties with distinct indices intersects empty.

Now fix $t \in [0,1]$, $x \in h(K_t \cup M_t)$, and let μ be a representative Jensen measure for evaluation at x. Then

$$\log|h_{i,j,t}(x)| \leq \int_{K_t \cup M} \log|h_{i,j,t}|\, d\mu,$$

and so if μ has mass on $Q_{i,j,t}^{\beta}$ then $x \in Z_{i,j,t}$. So μ has mass on at most $n+1$ cubes together with $\Gamma_{m,t}(\beta)$ and $(K_t \cup M_t) \cap \overline{V}_t$. So $x \in U_t'$. □

5. Extensions of maps from totally real manifolds

In this section we show how to obtain the conditions of Theorem 6.3 starting with embeddings defined only *on* the manifold M. Our approach is the same as that of Forstnerič and Rosay in [5] and Forstnerič [3] but the presence of an additional compact set K complicates things.

Theorem 5.1. *Let $M \subset \mathbb{C}^n$ be a compact totally real manifold of class $\mathcal{C}^\infty$ (possibly with boundary) and let $K \subset \mathbb{C}^n$ be a polynomially convex compact set. Let U be an open neighborhood of K and let $\varphi : [0,1] \times (U \cup M) \to \mathbb{C}^n$ be a $\mathcal{C}^\infty$-smooth map such that $\varphi_t|_M$ is a totally real embedding and $\varphi_t|_U$ is injective holomorphic for each fixed t. Assume also that $\varphi_t(K)$ is polynomially convex for each t. Then there exists an (arbitrarily small) open neighborhood U' of K such that for any $\epsilon > 0$ and any $k \in \mathbb{N}$ there exists a $\mathcal{C}^\infty$-smooth map $\Phi : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ such that for all $t \in [0,1]$ the following hold:*

(a) $\|\Phi_t - \varphi_t\|_{U'} < \epsilon$,

(b) $|\Phi_t - \varphi_t|_{x,k} < \epsilon$, $x \in M \cap U$,

(c) $\Phi_t|_{U'}$ *is holomorphic,*

(d) $\Phi_t \in \mathcal{H}_k(\mathbb{C}^n, M)$,

(e) $\Phi_t(x) = \varphi_t(x)$ *for all* $x \in M \setminus U$, *and*

(f) Φ_t *is of maximal rank along* M.

The content of the following lemma is essentially to be found in [5] – the difference is the claim that the identity extends to the identity.

Lemma 5.2. *Suppose that $M \subset \mathbb{C}^n$ is a totally real manifold of class $\mathcal{C}^\infty$, and let $\varphi : [0,1] \times M \to \mathbb{C}^n$ be a $\mathcal{C}^\infty$-smooth isotopy such that $M_t = \varphi_t(M)$ is totally real for each $t \in [0,1]$. Let $M' \subset M$ be compact and assume that $\varphi \equiv \mathrm{id}$ near M'. Then for any $k \in \mathbb{N}$ there exists a $\mathcal{C}^\infty$-smooth map $\tilde{\varphi} : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ such that $\tilde{\varphi}|_M = \varphi$, $\tilde{\varphi} \equiv \mathrm{id}$ near M', and $\tilde{\varphi}_t \in \mathcal{H}_k(\mathbb{C}^n, M)$.*

Proof. The maps φ_t already have maximal rank along M and the Cauchy–Riemann equations determine (at the level of jets) the extension in the complex tangent directions. We need to extend the maps in the complex normal directions, and the extensions should be $\bar{\partial}$-flat.

For each $t \in [0,1]$ let N_t denote the complex normal bundle of the embedded manifold M_t. Let N denote the total bundle $N = \bigcup_{t \in [0,1]} N_t$, and let $\tilde{N}$ denote the bundle $N_0 \times [0,1]$ over $M \times [0,1]$.

We let $N_e \subset \mathbb{C}^{n+1}$ and $\tilde{N}_e \subset \mathbb{C}^{n+1}$ denote embedded neighborhoods of the zero sections. These are both generic CR-manifolds. Let U be an open neighborhood of M' such that $\varphi \equiv \mathrm{id}$ on U. There is a natural bundle injection $f : \tilde{N}_0 \cup \pi^{-1}(U \times [0,1]) \to N$ since the two bundles are identically defined over $M_0 \cup (U \times [0,1])$. By Lemma 5.3 the map f extends to a bundle isomorphism $\tilde{f} : \tilde{N} \to N$, and $\tilde{f}$ induces CR-isomorphism $\tilde{f}_e : \tilde{N}_e \to N_e$. Note that $\tilde{f}_e$ is the identity near M'. The map $\tilde{f}_e$ determines a jet along $M \times [0,1]$ which has maximal rank and is $\bar{\partial}$-flat to order $k-1$. By Whitney's extension theorem the jet extends to a map $\tilde{\varphi}_t$. □

Let M be a $\mathcal{C}^k$-smooth manifold and let $\pi : N \to M \times I$ be a complex vector bundle of class $\mathcal{C}^k$ with fiber $\mathbb{C}^m$. Let N_0 denote the bundle $\pi^{-1}(M \times \{0\})$ over M, and form the bundle $\tilde{N} = N_0 \times I$. Denote the projection by $\tilde{\pi}$.

Lemma 5.3. *Let $M' \subset M$ be a compact subset, let $U \subset M$ be an open neighborhood of M', and let $f : \tilde{N}_0 \cup \tilde{\pi}^{-1}(U \times I) \to N$ be a $\mathcal{C}^k$-smooth bundle map ($\tilde{\pi} = \pi \circ f$) giving an isomorphism from $\tilde{N}_0 \cup \tilde{\pi}^{-1}(U \times I)$ onto the restriction of N to $M \times 0 \cup U \times I$. Then there exists a $\mathcal{C}^k$-smooth bundle isomorphism $\tilde{f} : \tilde{N} \to N$ ($\tilde{\pi} = \pi \circ \tilde{f}$) extending f on $\tilde{N}_0 \cup \tilde{\pi}^{-1}(M' \times I)$.*

Proof. If $\{U_j\}$ is an open cover of $M \times I$ over which the bundles are trivial then $\tilde{N}$ is represented by a family $g_{ij} : U_j \to \mathrm{GL}_m(\mathbb{C})$ of $\mathcal{C}^k$-smooth maps (transitions from $\pi^{-1}(U_j)$ to $\pi^{-1}(U_i)$), and N is likewise represented by a family $h_{ij} : U_{ij} \to \mathrm{GL}_m(\mathbb{C})$. Finding an isomorphism $\tilde{f} : \tilde{N} \to N$ amounts to finding local maps $\tilde{f}_j : U_j \to \mathrm{GL}_m(\mathbb{C})$ such that

$$\tilde{f}_i = h_{ij} \circ \tilde{f}_j \circ g_{ji} \quad \text{for all } U_i \cap U_j \neq \emptyset.$$

We may think of such an $\tilde{f}$ as a section of the $\mathcal{C}^k$-smooth $\mathrm{GL}_m(\mathbb{C})$ fiber bundle $\pi_x : X \to M \times I$ where the matrices transform according to the rule $A \mapsto h_{ij} \circ A \circ g_{ji}$. Our given bundle injection f is then interpreted as a section of $\pi_x^{-1}(M_0 \cup (U \times I))$. Choose a closed set $Y \subset U$ such that $M' \subset Y^\circ$ and such that (M, Y) is a relative CW-complex. According to [9, Theorem 7.1] the section f extends to a section $\tilde{f}$ of the total bundle X. By smoothing we may assume that $\tilde{f}$ is actually a $\mathcal{C}^k$-smooth section. □

Proof of Theorem 5.1. Choose open subsets U_j in $\mathbb{C}^n$ for $j = 1, 2, 3$ such that

$$K \subset U_3 \subset\subset U_2 \subset\subset U_1 \subset\subset U.$$

The set U_3 will play the role of U' in the theorem.

Note that if $\varphi \equiv \mathrm{id}$ on U_2 then the theorem follows immediately from Lemma 5.2 by defining $M' = M \cap \overline{U_3}$. To prove the theorem we will use a global holomorphic change of coordinates so that we are approximately in this situation.

By possibly having to choose a smaller U we may assume that φ is the uniform limit of one-parameter families $\psi_t \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$, i.e., we may assume that $\psi_t^\delta \to \varphi_t$ uniformly on $[0, 1] \times \overline{U}$ as $\delta \to 0$. Note that the Cauchy estimates imply the following: If ϑ_t^δ is close enough to the identity in $\mathcal{C}^k$-norm on $\overline{U_3} \cup (M \cap \overline{U_1})$ and if ψ_t^δ is close enough to φ_t on $\overline{U}$, then $\psi_t^\delta \circ \vartheta_t^\delta$ is close to φ_t in $\mathcal{C}^k$-norm on $\overline{U_3} \cup (M \cap \overline{U_1})$.

Define $\theta_t^\delta := (\psi_t^\delta)^{-1} \circ \varphi_t$. Then θ_t^δ converges to the identity uniformly in $\mathcal{C}^k$-norm as $\delta \to 0$ on $\overline{U_1}$. Let $\chi \in \mathcal{C}_0^k(U_1)$ such that $\chi_1 \equiv 1$ near $\overline{U_2}$. Write $\theta_t^\delta = \mathrm{id} + \sigma_t^\delta$ and define

$$\tilde{\theta}_t^\delta := \mathrm{id} + (1 - \chi_1) \cdot \sigma_t^\delta.$$

Then $\tilde{\theta}_t^\delta \to \mathrm{id}$ uniformly in $\mathcal{C}^k$-norm on $M \cap \overline{U}_1$ as $\delta \to 0$, where $\tilde{\theta}_t^\delta$ is the identity on $M \cap \overline{U_2}$ and $\tilde{\theta}_t^\delta = \theta_t^\delta$ outside U_1. Let $M' := M \cap \overline{U_3}$ and let ϑ_t^δ be the extensions of $\tilde{\theta}_t^\delta$ according to Lemma 5.2 which now can be extended to the identity near K. We set $\Phi_t = \psi_t^\delta \circ \vartheta_t^\delta$ for small enough δ. □

6. A Carleman version of a result by Forstnerič and Rosay

6.1. The nice projection property. Let $v \in \mathbb{C}^n$ be a nonzero vector and let $\epsilon > 0$. By v_ϵ we will mean an arbitrary vector satisfying $\|v_\epsilon - v\| \leq \epsilon$. We let π_{v_ϵ} denote the orthogonal projection to the orthogonal complement of the vector v_ϵ. To simplify notation we always write $\mathbb{C}^{n-1}$ for these orthogonal complements, and by $R\mathbb{B}^{n-1}$ we mean $R\mathbb{B}^n$ intersected with the orthogonal complements.

Let M be a smooth submanifold of $\mathbb{C}^n$. We will assume that M satisfies the following properties:

(A_1) The family $\pi_{v_\epsilon} : M \to \mathbb{C}^{n-1}$ is uniformly proper, i.e., for any compact set $K \subset \mathbb{C}^{n-1}$, the set $\bigcup_{v_\epsilon} \pi_{v_\epsilon}^{-1}(K)$ is compact.

(A_2) There exists a compact set $C \subset M$ such that $\pi_{v_\epsilon} : M \setminus C \to \mathbb{C}^{n-1}$ is an embedding onto a totally real manifold.

(A_3) The family $\pi_{v_\epsilon}(M)$ has uniformly bounded E-hulls in $\mathbb{C}^{n-1}$, i.e., for any compact subset $K \subset \mathbb{C}^{n-1}$ there exists an $R > 0$ such that $h(K \cup \pi_{v_\epsilon}(M)) \subset R\mathbb{B}^{n-1}$.

(A_4) For any compact set $K \subset M$ we have that $x \mapsto \langle x, v_{\epsilon_0}\rangle$ is uniformly bounded away from zero on K provided ϵ_0 is small enough (depending on K).

Remark 6.1. It follows from (A_3) that M has bounded E-hulls in $\mathbb{C}^n$.

Definition 6.2. Let $M' \subset \mathbb{C}^n$ be a smooth manifold. If there exists a holomorphic automorphism $\alpha \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ and a pair (v, ϵ) such that the manifold $M = \alpha(M')$ satisfies (A_1)–(A_4), we say that M' has the *nice projection property*.

Theorem 6.3. *Let $M \subset \mathbb{C}^n$ be a totally real manifold of class $\mathcal{C}^\infty$, fix a $\mathcal{C}^k$-norm on M, and assume that M has the nice projection property. Let $K \subset \mathbb{C}^n$ be compact, and assume that $K \cup M$ is polynomially convex. Let Ω be an open neighborhood of K. Let $\phi : [0,1] \times (\Omega \cup M) \to \mathbb{C}^n$ be a $\mathcal{C}^\infty$-smooth map with the following properties:*

(a) *ϕ_0 is the identity map,*

(b) *$\phi_t|_\Omega$ is injective for all t,*

(c) *$\phi_t|_\Omega$ is holomorphic for all t,*

(d) *$\phi_t|_M$ is an embedding for all t,*

(e) *there exists a compact set $S \subset M$ such that $\phi_t(z) = z$ for all $z \in M \setminus S$ for all t,*

(f) *$\phi_t(K \cup M)$ is polynomially convex, and $\phi_t(M \setminus K)$ is totally real, for all t.*

Then for any strictly positive continuous function $\delta \in \mathcal{C}(K \cup M)$ there exists a map $\psi \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$, such that

$$|\psi - \phi_1|_{k,x} < \delta(x)$$

for all $x \in K \cup M$.

Preparing for the proof, we start with a lemma.

Lemma 6.4. *Let $M \subset \mathbb{C}^n$ be a compact totally real manifold (possibly with boundary) of class $\mathcal{C}^\infty$, and let $K \subset \mathbb{C}^n$ be a compact set such that $K \cup M$ is polynomially convex. Let $A_1 \subset A_2 \subset\subset M \setminus K$ be closed subsets with $A_1 \subset \mathrm{int}(A_2)$. Let Ω be an open neighborhood of K, and let $\phi : \Omega \cup M \to \mathbb{C}^n$ satisfy* (a)–(d) *and* (f) *of Theorem* 6.3 *and also $\phi_t|_{A_2} = \mathrm{id}$*

for each t. Then there exist open neighborhoods $U' \subset U \subset \Omega$ of A_1, such that for any $\epsilon > 0$, $\delta > 0$ sufficiently small, and $k \in \mathbb{N}$, there exists $\psi : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ with $\psi(t, \cdot)$ holomorphic for each t and real analytic in t, such that for all $t \in [0,1]$ the following hold:

(i) $\|\psi_t - \phi_t\|_{\overline{K(\delta)}} < \epsilon$,

(ii) $|\psi_t - \phi_t|_{k,x} < \epsilon$ *for all* $x \in M$,

(iii) $\|\psi_t - \mathrm{id}\|_{\overline{U}} < \epsilon$,

(iv) ψ_t *has rank* n *on* M, *and*

(v) $h(\psi_t(K \cup M \cup \overline{U}')) \subset U \cup \psi_t(K(\delta))$.

Proof. By Corollary 4.4 and the assumption (f) there exists a δ small enough such that $h(\overline{K(\delta)} \cup M) \subset \Omega$, so for the purpose of approximating ϕ on $\overline{K(\delta)} \cup M$ we may assume that $\overline{K(\delta)} \cup M$ is polynomially convex. If $U' \subset\subset U \subset\subset U''$ are small enough neighborhoods of A_1, we may extend ϕ to be the identity on U'', and by the same corollary get that

(iv) $h(\overline{K(\delta)} \cup M \cup \overline{U}) \subset U''$,

(v) $h(\psi_t(K \cup M \cup \overline{U}')) \subset U \cup \psi_t(K(\delta))$, for any ψ which is a sufficiently small perturbation of ϕ on $\overline{K(\delta)} \cup M \cup \overline{U}$.

By Theorem 5.1 we may also assume that ϕ is $\overline{\partial}$-flat to order k along M, and has rank n along M.

It remains to show (i)–(iii) and for this we will transform the problem into an approximation problem without a parameter t. Set

$$K' := \mathrm{closure}\big(\widehat{[\overline{K(\delta)} \cup M \cup \overline{U}]} \setminus M\big),$$

define $\tilde{M} = M \times [0,1] \subset \mathbb{C}^n \times \mathbb{C}$ and $\tilde{K} = K' \times [0,1] \subset \mathbb{C}^n \times \mathbb{C}$. Note that $\tilde{M}$ is a totally real manifold and that $\tilde{K} \cup \tilde{M}$ is a polynomially convex. For $N \in \mathbb{N}$ we define a covering of the interval $I = [0,1]$ as follows: let

$$I_0 = \Big[0, \frac{1}{N}\Big), \quad I_N = \Big(1 - \frac{1}{N}, 1\Big], \quad I_j = \Big(\frac{2j-1}{2N}, \frac{2j+1}{2N}\Big), \ j = 1, 2, \ldots, N-1.$$

Let $\{\alpha_j\}$ be a partition of unity with respect to the cover $\{I_j\}$. Define

$$\tilde{\phi}_w(z) := \sum_{0 \le j \le N} \alpha_j(w) \cdot \phi_{j/N}(z),$$

on $\tilde{K} \cup \tilde{M}$, with coordinates (z, w) on $\mathbb{C}^n \times \mathbb{C}$. Each of the functions α_j may be approximated arbitrarily well on I by entire functions $\tilde{\alpha}_j$ on $\mathbb{C}$. So the mapping

$$\widehat{\phi}_w(z) := \sum_{0 \le j \le N} \tilde{\alpha}_j(w) \cdot \phi_{j/N}(z)$$

is in $\mathcal{H}_k(\mathbb{C}^{n+1}, \tilde{M}) \cap \mathcal{O}(\tilde{K})$. If N was chosen big enough, and if the approximation of the partition of unity was good enough, then

(a) $\|\widehat{\phi}_w - \phi_w\|_{K_w} < \frac{\epsilon}{2}$, and

(b) $|\widehat{\phi}_w - \phi_w|_{k,x} < \frac{\epsilon}{2}$ for all $x \in M_w$.

It follows directly from [12] that we can approximate $\widehat{\phi}$ on $\tilde{K} \cup \tilde{M}$. □

6.2. Proof of Theorem 6.3. Choose $r_1 > 0$ such that $S \subset r_1\mathbb{B}^n$. By possibly having to increase r_1 we may assume that $\phi_t((M \cap r_1\mathbb{B}^n) \cup K) \subset r_1\mathbb{B}^n$ for all t. It follows from (A_3) above that there exists an $R > 0$ such that

$$\overline{h\big(\pi_{v_\epsilon}(\overline{r_1\mathbb{B}^n} \cup M)\big)} \subset R\mathbb{B}^{n-1}$$

for all $\|v_\epsilon - v\| \leq \epsilon$. By possibly having to increase R we may assume that $\pi_{v_\epsilon}(C) \subset R\mathbb{B}^{n-1}$ for all $\|v_\epsilon - v\| \leq \epsilon$. Given $r_2 < r_3$ we let A denote the annular set

$$A = A(r_2, r_3) := \big\{z \in \mathbb{C}^n : r_2 \leq \|z\| \leq r_3\big\}.$$

Fix $r_2 < r_3$ such that

$$\pi_{v_\epsilon}(A \cap M) \subset \mathbb{C}^{n-1} \setminus \overline{R\mathbb{B}^{n-1}}$$

for all $\|v_\epsilon - v\| \leq \epsilon$. Denote $M_j := M \cap \overline{r_j\mathbb{B}^n}$ for $j = 2, 3$. Let $\chi \in \mathscr{H}_k(\mathbb{C}^n, M)$ be such that $0 \leq \chi \leq 1$, $\chi|_{r_1\overline{\mathbb{B}^n} \cup M_2} \equiv 1$, and $\chi|_{M \setminus M_3} \equiv 0$. Let $\tilde{A} := M \cap A$. Set

$$T := \sup_{x \in A}\{|\chi|_{k,x}\}.$$

By [11] we have that if M' is a sufficiently small $\mathscr{C}^1$-perturbation of $\pi_v(\overline{M \setminus C})$ which is equal to $\pi_v(M)$ on $\pi_v(M \setminus M_3)$, then

$$h\big(\pi_v(r_1\overline{\mathbb{B}^n}) \cup M'\big) \subset R\mathbb{B}^{n-1}.$$

It follows that there exists a constant $\epsilon_1 > 0$ such that, by possibly having to decrease ϵ, if M' is a $\mathscr{C}^1$-ϵ_1-perturbation of $M \setminus C$ which is equal to M outside M_3, then $\pi_{v_\epsilon}(M')$ is a totally real manifold with $h(r_1\overline{\mathbb{B}^n} \cup M') \subset R\mathbb{B}^{n-1}$.

Plan of proof. The theorem will be proved in several steps; the first steps are the same as in the usual A-L procedure.

(i) First we will approximate the whole isotopy $\phi_t(z)$ on $K \cup M_3$ by an isotopy $g_t(z)$ which is holomorphic on $\mathbb{C}^n \times [0, 1]$. In addition to being a good approximation on $K \cup M_3$ we need $g_t(z)$ to be uniformly small on a full open neighborhood U of A.

(ii) We interpret $g_t(z)$ as the flow of a time-dependent vector field $X_t(z)$, which we, by approximation, will assume is polynomial, and then approximate $g_1(z)$ by a composition of flows $h_t^j(z)$ of time-independent polynomial vector fields $X^j(z)$, $j = 1, \ldots, m$, all of them being uniformly small on U.

(iii) Each X^j may be written as a sum of shear and over-shear fields. We will then approximate each flow h_t^j by a composition of shear and over-shear flows.

(iv) Each shear and over-shear flow from step (iii) will be modified on $r_1\overline{\mathbb{B}^n} \cup M_3$ by multiplying with smooth cutoff functions along (images of) A, thereby obtaining good (smooth) shear and over-shear like maps defined on (images of) $r_1\overline{\mathbb{B}^n} \cup M$, being the identity outside M_3.

(v) Finally, the modified maps will be interpreted as shears and over-shears defined by using the projections of $r_1\overline{\mathbb{B}^n} \cup M$ along the vector v_ϵ (using the good projection property), and approximated by holomorphic shears and over-shears using Carleman approximation by entire functions.

Approximation by an isotopy of holomorphic injections. Let $U' \subset U$ be neighborhoods of A as in Lemma 6.4, and let δ be as in the same lemma. Let $0 < \epsilon_2 < \epsilon_1$ be a small constant to be determined later. According to Lemma 6.4 there exists an isotopy ψ_t of entire maps such that for all t we have

(i) $\|\psi_t - \phi_t\|_{\overline{K(\delta)}} < \epsilon_2$,

(ii) $|\psi_t - \phi_t|_{k,x} < \epsilon_2$ for all $x \in M_3$, and

(iii) $\|\psi_t - \mathrm{id}\|_{\overline{U}} < \epsilon_2$.

Note that

$$h\big(\psi_t(K \cup M_3 \cup \overline{U'})\big) \subset \psi_t(K(\delta) \cup U).$$

We now proceed to approximate the map ψ_1 on $K \cup M_3$ and the identity on $M \setminus M_3$. We may assume that $\psi_0 = \mathrm{id}$.

The reduction to flows of shear and over-shear fields. Let W be a neighborhood of $\overline{K(\delta)} \cup M_3 \cup \overline{U}$ such that $\psi_t : \overline{W} \to \mathbb{C}^n$ is a family of injections. We define a time-dependent vector field $X(t,z)$ by

$$X(t_0, z_0) := \frac{d}{dt}\Big|_{t=t_0} \psi_t(\psi_{t_0}^{-1}(z_0))$$

for all $z_0 \in \psi_{t_0}(W)$. Then ψ is the flow of X. We let X_t denote the autonomous vector field we get by fixing t. Note that

$$h\big(\psi_t(K \cup M_3 \cup \overline{U'})\big) \subset \psi_t(W)$$

and so each X_t is the uniform limit of polynomial fields near $\psi_t(K \cup M_3 \cup \overline{U'})$.

Thus Lemma 3.1 and the usual Andersén–Lempert construction allow us to find automorphisms $\Theta_k(z)$ of $\mathbb{C}^n$, $k = 1, \dots, N$, such that $\Theta(N) = \Theta_N \circ \Theta_{N-1} \circ \cdots \circ \Theta_1$ approximates ψ_1 as well as we want near $K \cup M_3 \cup \overline{U'}$. Moreover, each Θ_k is of one of the two forms

$$\Theta_k(z) = z + \tau_k(\pi_k(z)) \cdot v_k, \tag{6.1}$$

$$\Theta_k(z) = z + (e^{\tau_k(\pi_k(z))} - 1)\langle z, v_k\rangle \cdot v_k, \tag{6.2}$$

where the quantities $\langle z, v_k\rangle$ are bounded away from zero, and the functions $\tau_k(\pi_k(z))$ are as small as we like. Also by (iii) above we may assume that each map Θ_k is as small as we like on $\overline{U'}$.

Approximation by smooth maps. We will now describe an inductive procedure how to modify the maps Θ_k. We have (6.1) or (6.2) for entire functions τ_k, depending on wether Θ_k is a shear or an over-shear. At any rate, we may write

$$\Theta_k(x) = x + g_k(x) \cdot v_k,$$

defined for $x \in \Theta(k-1)(A)$. Let $\Theta_0 := \mathrm{id}$, and define inductively

$$\begin{aligned}\tilde{\Theta}_k(x) &:= x + \chi\big(\tilde{\Theta}(k-1)^{-1}(x)\big) \cdot g_k\big(\Theta(k-1) \circ \tilde{\Theta}(k-1)^{-1}(x)\big) \cdot v_k \\ &=: x + \tilde{g}_k(x) \cdot v_k \quad \text{for all } x \in \tilde{\Theta}(k-1)(A)\end{aligned}$$

and

$$\tilde{\Theta}_k(x) := \Theta_k(x) \quad \text{for all } x \in K \cup M_3 \setminus A,$$
$$\tilde{\Theta}_k(x) := \text{id} \qquad \text{for all } x \in M \setminus M_3.$$

Writing $\Theta(k)(x) = x + h_k(x) \cdot w_k$ on A, it is clear that

$$\tilde{\Theta}(k)(x) = x + \chi(x) \cdot h_k(x) \cdot w_k,$$

and so this is well defined if each composition $\Theta(k)$ is small enough on $\overline{U'}$. Note that $\Theta(N) = x + h_N(x) \cdot w_N$, and that $|h_N \cdot w_N|$ is as small as we like depending on the choice of ϵ_2 above. The choice of ϵ_2 is made after fixing χ, so we choose it depending on the constant T above, and so we may assume that $\tilde{\Theta}(N)$ is as close to the identity as we like on A.

Finally, we want to rewrite the $\tilde{\Theta}_k$ as shears and over-shears, i.e., defined via the projections π_k. First assume that the original Θ_k was a shear map: if ϵ_2 was chosen small enough, we have that $\pi_k(\tilde{\Theta}(k-1)(A))$ is a totally real manifold contained in $\mathbb{C}^{n-1} \setminus R\mathbb{B}^{n-1}$, so we may write $\tilde{g}_k(x) = \tilde{\tau}_k(\pi_k(x))$ on $\Theta(k-1)(A)$. Since $\tilde{\tau}_k$ will agree with τ_k near

$$\pi_k\big(\tilde{\Theta}(k-1)(A \cap \overline{(M_3 \setminus A)})\big),$$

we may extend $\tilde{\tau}_k$ to be equal to the original τ_k on $R\overline{\mathbb{B}^{n-1}} \cup \pi_k(\Theta(k-1)(M_3 \setminus A))$. We may also extend $\tilde{\tau}_k$ to be zero on $\pi_k(\Theta(k-1)(M \setminus M_3))$.

If Θ_k was an over-shear, we write first $\tilde{g}_k(x) = \psi_k(\pi_k(x))$, we extend g_k as we just did with $\tilde{\tau}_k$, but now we want to solve

$$(e^{\tilde{\tau}_k(\pi_k(x))} - 1)\langle x, v_k \rangle = \psi_k(\pi_k(x))$$

on $\pi_k(\Theta(k-1)(A))$. This is doable since $\langle x, v_k \rangle$ is uniformly bounded away from zero (independent of k) by (A_4) in the definition of the nice projection property, and we may assume that $\psi_k(\pi_k(x))$ is arbitrarily small compared to this.

Approximation by holomorphic automorphisms. We finally show by induction on k that the compositions $\tilde{\Theta}(k)$ may be approximated in the sense of Carleman on $K \cup M$.

More generally than showing this first for $k = 1$ we show first that for any k we have that $\tilde{\Theta}_k$ may be approximated in the sense of Carleman on $\tilde{\Theta}(k-1)(K \cup M)$. Note that

$$h\big(\pi_1(\tilde{\Theta}(k-1)(K \cup M))\big) \subset R\mathbb{B}^{n-1}$$

and so it follows by [12] that the function $\tilde{\tau}_k$ may be approximated in the sense of Carleman on $\pi_k(\tilde{\Theta}(k-1)(K \cup M))$ by entire functions.

Now the induction step is clear: since $\tilde{\Theta}_{k+1}$ may be approximated on $\tilde{\Theta}(k)(K \cup M)$ and since $\tilde{\Theta}(k)$ may be approximated on $K \cup M$, we get that $\tilde{\Theta}(k+1)$ may be approximated on $K \cup M$.

7. Approximation of smooth automorphisms of $\mathbb{R}^k \subset \mathbb{C}^n$

Theorem 7.1. *Let $\phi : \mathbb{R}^s \to \mathbb{R}^s$ be a $\mathcal{C}^\infty$-smooth automorphism, and assume that $s < n$. Then ϕ can be approximated in the sense of Carleman by holomorphic automorphisms of $\mathbb{C}^n$, i.e., given $\epsilon \in \mathcal{C}(\mathbb{R}^s)$ and $k \in \mathbb{N}$, there exists $\Psi \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ such that $|\Psi - \phi|_{k,x} < \epsilon(x)$ for all $x \in \mathbb{R}^s$.*

We start by describing a gluing procedure that will be used in an induction argument to prove Theorem 7.1. Let $M \hookrightarrow \mathbb{C}^n$ be a smooth embedded submanifold, and let $\pi : N \to M$ be an embedded neighborhood of the zero section of the normal bundle. Then any sufficiently small $\mathcal{C}^k$-perturbation M' of M can be thought of as $\mathcal{C}^k$-small section $s \in \Gamma(M, N)$. Fix a normal exhaustion $K_j \subset K_{j+1}^\circ$ of M, and fix functions $\chi_j \in \mathcal{C}_0^\infty(K_{j+1}^\circ)$ with $\chi_j \equiv 1$ near K_j.

Lemma 7.2. *Let ψ be a smooth diffeomorphism of M, and let $\epsilon \in \mathcal{C}(M)$ be a strictly positive function. Then there exists a strictly positive $\delta \in \mathcal{C}(M)$ such that the following hold: For any $m \in \mathbb{N}$ and any smooth embedding $\phi : M \to \mathbb{C}^n$ such that $|\phi - \psi|_{k,x} < \delta(x)$ for all $x \in K_{m+1}$, and such that $\phi(M)$ is a δ-perturbation of M in the sense that $\phi(M)$ can be written as a section $s \in \Gamma(M, N)$ where $|s - \mathrm{id}|_{k,x} < \delta(x)$ for all $x \in M$, the map*

$$\tilde{\phi} := \phi(x) + (1 - \chi_m(x))\big(s(\psi(x)) - \phi(x)\big)$$

satisfies $|\tilde{\phi} - \psi|_{k,x} < \epsilon(x)$ for all $x \in M$.

Proof. We check different parts of M. Since on K_j we have that $\tilde{\phi} = \phi$, it suffices that $\delta(x) < \epsilon(x)$ for all $x \in M$. On $M \setminus K_{m+1}$ we have that $\tilde{\phi}(x) = s(\psi(x))$, and given any $\tilde{\epsilon} < \epsilon$, it is clear that if δ decreases rapidly towards zero, then $|\psi - s \circ \psi| < \tilde{\epsilon}(x)$ for all $x \in M$. Finally, there exist constants C_m, $m \in \mathbb{N}$, such that we have

$$|\tilde{\phi} - \psi|_{k,x} \leq |\phi - \psi|_{k,x} + C_m|s(\psi(x) - \phi(x))|_{k,x} < \delta(x) + C_m\tilde{\epsilon}(x)$$

for all $x \in K_{m+1} \setminus K_m$, and so it is clear that the claim holds if δ decreases rapidly as x tends to infinity. □

Theorem 7.1 will be proved by an inductive argument where the main step will be covered by the following lemma.

Lemma 7.3. *Let $\psi \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ such that $M = \psi(\mathbb{R}^s)$ is a sufficiently small $\mathcal{C}^1$-perturbation (in the sense of Carleman) of $\mathbb{R}^s \subset \mathbb{C}^n$, $s < n$, and let $K \subset \mathbb{C}^n$ be a compact set such that $K \cup M$ is holomorphically convex. Assume given $R > 0$ such that $K \subset R\mathbb{B}^n$ and a $\phi \in \mathrm{Diff}(M)$ such that ϕ is orientation preserving and $\phi = \mathrm{id}$ near $K \cap M$. Then for any $k \in \mathbb{N}$, $\mu > 0$ and strictly positive $\delta \in \mathcal{C}(M)$ there exist (arbitrarily large) $l \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ such that the following hold:*

(1) $|\sigma - \phi|_{k,x} < \delta(x)$ *for all* $x \in \psi(\mathbb{R}^s \cap (l+1)\overline{\mathbb{B}^n})$,

(2) $\|\sigma - \mathrm{id}\| < \mu$ *near* K,

(3) $R\mathbb{B}^n \subset \sigma \circ \psi(l\mathbb{B}^n)$, *and*

(4) $\sigma(M)$ *is a* δ-$\mathcal{C}^k$*-small perturbation of* M.

Proof. Note that by [11] we have that if M is a sufficiently small $\mathcal{C}^1$-perturbation of $\mathbb{R}^s$ then $h(R\overline{\mathbb{B}^n} \cup M) \subset (R+1)\mathbb{B}^n$ for any $R > 0$.

Choose a compact set $C \subset M$ such that $\phi(M \setminus C) \subset M \setminus (R+1)\mathbb{B}^n$. Let $X(t,x)$, $t \in [0,1]$, be a non-autonomous smooth vector field such that ϕ is the time one map of X, and such that $X(t,x) = 0$ on $M \cap K$. Denote this flow by ϕ_t. Choose a compact set C' such that C'

contains the complete ϕ_t-orbit of C. Choose a smooth cutoff function $\chi \in \mathcal{C}_0^\infty(M)$ such that $\chi \equiv 1$ near C'. Define $\tilde{X}(t,x) := \chi(x)\cdot X(t,x)$, and let $\tilde{\phi}_t$ denote the flow of $\tilde{X}$. By Theorem 6.3 there exists $\Phi_1 \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ such that Φ_1 approximates $\tilde{\phi}_1$ on M in the sense of Carleman, hence also ϕ on C, and Φ_1 approximates the identity near K. We set $M_1 = \Phi_1(M)$.

Now choose $l \gg 0$ such that $R\mathbb{B}^n \subset \Phi_1 \circ \psi(l\mathbb{B}^n)$. Let $\tilde{\sigma} \in \mathrm{Diff}(M_1)$ be defined by $\tilde{\sigma} := \pi_1 \circ \phi \circ \Phi_1^{-1}$. Note that $\tilde{\sigma}$ is close to the identity on $\Phi_1(C)$, so after a small perturbation we may assume that σ is the identity on $\Phi_1(C)$ and furthermore σ can then be extended to the identity on $(R+1)\mathbb{B}^n$ which contains $h(R\overline{\mathbb{B}^n} \cup \Phi_1(M))$.

By an argument similar to the one above there exists $\Phi_2 \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ that approximates $\tilde{\sigma}$ on $\Phi_1 \circ \psi(M \cap (r+1)\overline{\mathbb{B}^n})$ and is near the identity on $R\mathbb{B}^n$. Now the composition $\psi := \Phi_2 \circ \Phi_1$ furnishes a desired map. □

Proof of Theorem 7.1. After possibly having to compose with the map

$$(z_1, z_2, \ldots, z_n) \mapsto (-z_1, z_2, \ldots, z_n)$$

we may assume that ϕ is orientation preserving, and we may also assume that $\phi(0) = 0$.

For $i \in \mathbb{N}$, we will inductively construct sequences of automorphisms $\psi_i \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$, real numbers $R_i \le r_i$, $R_i \to \infty$ as $i \to \infty$, and diffeomorphisms $\phi_i \in \mathrm{Diff}(\psi_i(\mathbb{R}^k))$ such that the following hold for $i \ge 1$:

(1_i) $|\psi_i - \phi|_{k,x} < \frac{1}{2}\epsilon(x)$ for all $x \in \mathbb{R}^s \cap r_i\overline{\mathbb{B}^n}$,

(2_i) $|\phi_i \circ \psi_i - \phi|_{k,x} < \frac{1}{2}\epsilon(x)$ for all $x \in \mathbb{R}^s$,

(3_i) $\|\psi_i - \psi_{i-1}\|_{\psi_{i-1}(r_{i-1}\overline{\mathbb{B}^n})} < (\frac{1}{2})^i$,

(4_i) $R_i\mathbb{B}^n \subset \psi_i(r_i\mathbb{B}^n)$, and

(5_i) $\phi_i = \mathrm{id}$ near $\psi_i(\mathbb{R}^s \cap r_i\overline{\mathbb{B}^n})$.

In addition, each $\psi_i(\mathbb{R}^s)$ will be a sufficiently small perturbation of $\mathbb{R}^s$ such that Lemma 7.3 applies.

If we set $r_0 = r_1 = R_0 = R_1 = 0$ and $\psi_0 = \psi_1 = \phi_0 = \mathrm{id}$, we get ($1_1$)–($4_1$), and we perturb ϕ slightly near the origin to get a ϕ_1 such that (5_1) also holds.

To complete the induction step we now assume that (1_i)–(5_i) hold for some $i \ge 1$. Choose $R_{i+1} \ge R_i + 1$ such that $\psi_i(r_i\mathbb{B}^n) \subset R_{i+1}\mathbb{B}^n$. For any strictly positive $\delta \in \mathcal{C}(\psi_i(\mathbb{R}^s))$ there exists by Lemma 7.3 an $r_{i+1} > R_{i+1}$ and a $\sigma \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$ approximating ϕ_i δ-well on $\psi_i(\mathbb{R}^s \cap (r_{m+1}+1)\mathbb{B}^n)$, and so that setting $\psi_{i+1} := \sigma \circ \psi_i$ we get (1_{i+1}), (3_{i+1}) and (4_{i+1}). Using Lemma 7.2, we also get a map

$$\tilde{\phi}_i : \psi_i(\mathbb{R}^s) \to \psi_{i+1}(\mathbb{R}^s)$$

such that setting $\phi_{i+1} := \tilde{\phi}_m \circ \sigma^{-1}$ gives us (2_{i+1}) and (5_{i+1}).

It now follows from (3_i) that the sequence $\Psi := \lim_{j\to\infty} \psi_j$ converges uniformly on $\mathbb{C}^n$, and we may assume that the limit is injective holomorphic. Moreover, it follows from (4_i) that the sequence ψ_i^{-1} also converges on $\mathbb{C}^n$, hence $\Psi \in \mathrm{Aut}_{\mathrm{hol}}\,\mathbb{C}^n$. By ($1_i$) we have that Ψ is a good enough approximation on $\mathbb{R}^s$. □

References

[1] *H. Alexander*, A Carleman theorem for curves in $\mathbb{C}^n$, Math. Scand. **45** (1979), no. 1, 70–76.
[2] *G. Buzzard* and *F. Forstnerič*, A Carleman type theorem for proper holomorphic embeddings, Ark. Mat. **35** (1997), no. 1, 157–169.
[3] *F. Forstnerič*, Approximation by automorphisms on smooth submanifolds of $\mathbb{C}^n$, Math. Ann. **300** (1994), no. 4, 719–738.
[4] *F. Forstnerič*, *E. Løw* and *N. Øvrelid*, Solving the d- and $\bar{\partial}$-equations in thin tubes and applications to mappings, Michigan Math. J. **49** (2001), no. 2, 369–416.
[5] *F. Forstnerič* and *J.-P. Rosay*, Approximation of biholomorphic mappings by automorphisms of $\mathbb{C}^n$, Invent. Math. **112** (1993), no. 2, 323–349.
[6] *F. Forstnerič* and *E. F. Wold*, Embeddings of infinitely connected planar domains into $\mathbb{C}^2$, Anal. PDE **6** (2013), no. 2, 499–514.
[7] *P. M. Gauthier* and *E. S. Zeron*, Approximation on arcs and dendrites going to infinity in $\mathbb{C}^n$, Canad. Math. Bull. **45** (2002), no. 1, 80–85.
[8] *A. Hoischen*, Eine Verschärfung eines Approximationssatzes von Carleman, J. Approximation Theory **9** (1973), 272–277.
[9] *D. Husemoller*, Fibre bundles, 3rd ed., Grad. Texts in Math. **20**, Springer, New York 1994.
[10] *F. Kutzschebauch* and *S. Lodin*, Holomorphic families of nonequivalent embeddings and of holomorphic group actions on affine space, Duke Math. J. **162** (2013), no. 1, 49–94.
[11] *E. Løw* and *E. F. Wold*, Polynomial convexity and totally real manifolds, Complex Var. Elliptic Equ. **54** (2009), no. 3–4, 265–281.
[12] *P. E. Manne*, *E. F. Wold* and *N. Øvrelid*, Carleman approximation by entire functions on Stein manifolds, Math. Ann. **351** (2011), no. 3, 571–585.
[13] *S. Scheinberg*, Uniform approximation by entire functions, J. Anal. Math. **29** (1976), 16–18.

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