

RELATIONS IN QUANTIZED FUNCTION ALGEBRAS

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ABSTRACT. We develop a method to give presentations of quantized function algebras of complex reductive groups. In particular, we give presentations of quantized function algebras of automorphism groups of finite dimensional simple complex Lie algebras.

1. INTRODUCTION

A reductive group can often be conveniently defined as the group of symmetries of a given collection of tensors on a finite dimensional vector space V . For example, the orthogonal, respectively symplectic, group is the group of symmetries of a symmetric, respectively skew-symmetric, nondegenerate inner product, and the special orthogonal group is the joint symmetry group of the inner product and the volume form. More interesting examples are the group G_2 , which may be defined as the group of symmetries of the inclusion $V \rightarrow V \otimes V$, where V is the 7-dimensional irreducible representation, and E_6 , which is the group of symmetries of the invariant cubic form on the 27-dimensional irreducible representation V . Finally, if \mathfrak{g} is a simple Lie algebra, then $\text{Aut}(\mathfrak{g})$ is the group of symmetries of the bracket $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. This gives rise to presentations of the function algebras $\mathcal{O}(G)$ of such groups G by generators and relations, where the generators are the matrix coefficients of V or of V and V^* , and the relations come from the condition that the given tensors are preserved.

The goal of this paper is to extend such presentations to the q -deformed setting. If $q = e^{\hbar}$, where \hbar is a formal parameter, and we consider formal (\hbar -adically complete) quantum deformations $\mathcal{O}_q(G)$ of $\mathcal{O}(G)$, then such an extension is rather straightforward: we just need to deform the relations so that they are satisfied in the q -deformed case, and this will give a presentation of $\mathcal{O}_q(G)$ by a standard formal deformation argument. However, if we work over $\mathbb{C}(q)$ or with numerical q , this argument does not work and it is not clear why the deformed relations are sufficient to define $\mathcal{O}_q(G)$. Showing the sufficiency of the relations, or, in other words, that the algebra they define has at most the expected size, is normally called the “easy part of the PBW theorem”, but in absence of suitable filtrations, as in our case, it is actually not so easy.

We propose a general method of proving the sufficiency of the deformed relations for generic q , i.e., outside of countably many complex values of q or over $\mathbb{C}(q)$. This method is based on an idea from the theory of \mathbb{C}^* -tensor categories. Namely, we replace the reductive group G with the corresponding compact real form K . Then we show that if K is defined as the group of unitary transformations of V preserving a collection of tensors T_i , then the category $\text{Rep } G = \text{Rep } K$ is generated by T_i and their (Hermitian) adjoints T_i^* , the duality morphisms and their adjoints, and the braiding, i.e., any morphism in this category can be expressed through them. The main idea of proving this is that the Karoubian category generated by all these morphisms is in fact semisimple abelian because of unitarity, and hence by Tannakian formalism must be the category of representations of some closed subgroup $K' \subset U(V)$ containing K , and then it is not hard to show that in fact $K' = K$.

This allows us to argue that for generic q the representation category $\text{Rep } G_q$ of the corresponding quantum group G_q is still generated by appropriate q -deformations of the classical morphisms. We then get a presentation of $\mathcal{O}_q(G)$ as the Hopf algebra of quantum symmetries of the q -deformations of T_i and T_i^* for generic q (and even over $\mathbb{Q}(q)$, if T_i are defined over \mathbb{Q}), which is a generalization of the description of quantum groups of classical series due to Faddeev, Reshetikhin and Takhtadzhyan [FRT]. This presentation gives a somewhat larger set of relations than one wanted, but one can hope that with additional work some of these relations turn out to be redundant.

We implement this strategy in full detail in the case of $G = \text{Aut}(\mathfrak{g})$, where \mathfrak{g} is a simple complex Lie algebra. Moreover, we use a certain noncommutative version of Grothendieck's generic freeness theorem, due to Artin, Small and Zhang [ASZ], to show that our presentations of quantized function algebras are valid outside of finitely many values of q . This, in particular, implies that they are valid for almost all roots of unity.

We expect that these results can be extended without significant changes to many other examples, but leave this outside the scope of the paper.

The organization of the paper is as follows. In Section 2 we outline the strategy. In Section 3 we prove the main theorem. In Section 4 we strengthen it so that it applies to all values of q except finitely many, and give an application to classifying tensor autoequivalences.

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2. THE STRATEGY

Our starting point is the following simple consequence of the Tannaka–Krein duality.

Theorem 2.1. *Let V be a finite dimensional complex Hilbert space. Consider the standard representation of the unitary group $U(V)$ on V and the contragredient representation on the dual Hilbert space V^* . Assume we are given a collection of tensors $T_i \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$, $i \in I$, and let $K \subset U(V)$ be the closed subgroup stabilizing all these tensors. Then the monoidal category $\text{Rep } K$ of finite dimensional complex representations of K is generated by V and V^* and the following morphisms:*

- (a) *the flip $\sigma: V \otimes V \rightarrow V \otimes V$;*
- (b) *the morphisms $r: \mathbb{C} \rightarrow V^* \otimes V$, $r(1) = \sum_k e^k \otimes e_k$, and $\bar{r}: \mathbb{C} \rightarrow V \otimes V^*$, $\bar{r}(1) = \sum_k e_k \otimes e^k$, and their adjoints, where $\{e_k\}_k$ is a basis in V and $\{e^k\}_k$ is the dual basis in V^* ;*
- (c) *the tensors T_i , viewed as morphisms $\mathbb{C} \rightarrow (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$, and their adjoints.*

In other words, any irreducible representation of K appears as a subrepresentation of a tensor product of the representations on V and V^ , and any K -intertwiner between such tensor products can be written as a linear combination of compositions of morphisms of the form $\iota \otimes \cdots \otimes \iota \otimes f \otimes \iota \otimes \cdots \otimes \iota$, where f is a morphism from the above list.*

We stress that we consider Hermitian adjoints here. Recall also that the Hermitian scalar product on the dual space V^* is defined so that the dual basis for an orthonormal basis in V is orthonormal.

Proof. Since any finite dimensional representation of K is unitarizable, we can equally well prove a similar statement in the setting of C^* -tensor categories (see, e.g., [NT2, Chapter 2]).

Consider the monoidal subcategory of the category Hilb_f of finite dimensional Hilbert spaces with objects the tensor products of the spaces V and V^* (in all possible orders) and morphism spaces generated by the morphisms in the formulation of the theorem. Completing this category with respect to subobjects and finite direct sums we get a C^* -tensor category \mathcal{C} . Since r and \bar{r} are morphisms in \mathcal{C} , V^* is dual to V in \mathcal{C} . Hence \mathcal{C} is rigid. Next, observe that the flip maps $V^* \otimes V \rightarrow V \otimes V^*$, $V \otimes V^* \rightarrow V^* \otimes V$ and $V^* \otimes V^* \rightarrow V^* \otimes V^*$ are morphisms in \mathcal{C} , since they can be expressed in terms of σ , r , \bar{r} and their adjoints. It follows that the flip maps on tensor products of Hilbert spaces define a unitary symmetry on \mathcal{C} . We have an obvious symmetric unitary fiber functor $\mathcal{C} \rightarrow \text{Hilb}_f$. Hence, by the Tannaka–Krein duality, \mathcal{C} is the category of finite dimensional unitary representations of a compact group H .

Since \mathcal{C} is generated as a C^* -tensor category by the object V , the representation of H on V is faithful, so H can be considered as a subgroup of the unitary group $U(V)$. Since the morphisms T_i are H -intertwiners, we then get $H \subset K$. This implies that

$$\text{Hom}_K((V^*)^{\otimes m} \otimes V^{\otimes n}, (V^*)^{\otimes k} \otimes V^{\otimes l}) \subset \text{Hom}_H((V^*)^{\otimes m} \otimes V^{\otimes n}, (V^*)^{\otimes k} \otimes V^{\otimes l}).$$

But by the definition of $\mathcal{C} = \text{Rep } H$ we also have the opposite inclusion, so

$$\text{Hom}_K((V^*)^{\otimes m} \otimes V^{\otimes n}, (V^*)^{\otimes k} \otimes V^{\otimes l}) = \text{Hom}_H((V^*)^{\otimes m} \otimes V^{\otimes n}, (V^*)^{\otimes k} \otimes V^{\otimes l}).$$

Finally, since the representation of K on $V \oplus V^*$ is faithful and self-dual, any irreducible representation of K appears as a subrepresentation of $(V \oplus V^*)^{\otimes n}$. \square

Remark 2.2. 1. It is not really necessary to work with C^* -categories. The main point is that any $*$ -algebra of operators on a finite dimensional Hilbert space is semisimple. This implies that the Karoubi envelope of the category with objects the tensor products of the spaces V and V^* and morphisms as in the proof of the theorem is a rigid semisimple monoidal category.

2. The tensors T_i can be viewed as morphisms $V^{\otimes m_i} \rightarrow V^{\otimes n_i}$, and we could equally well use this interpretation in part (c) of the theorem, since the two ways of considering T_i as morphisms are related by the duality morphisms r and \bar{r} and their adjoints.

3. While this paper was being written, N. Snyder informed us that a similar result to Theorem 2.1 appears independently in the work in progress by S. Morrison, N. Snyder and D. Thurston.

A similar result holds for subgroups of compact orthogonal and symplectic groups, or, in other words, for representation categories generated by real and quaternionic representations. Namely, we have the following.

Theorem 2.3. *Assume V is a finite dimensional complex Hilbert space with a real or quaternionic structure, so we are given an antilinear isometry $J: V \rightarrow V$ such that $J^2 = 1$ or $J^2 = -1$. Assume also that we are given a collection of tensors $T_i \in (V^*)^{\otimes m_i} \otimes V^{\otimes n_i}$, $i \in I$. Let K be the compact group of unitary transformations of V stabilizing all these tensors and commuting with J . Then the monoidal category $\text{Rep } K$ of finite dimensional complex representations of K is generated by V and the following morphisms:*

- (a) *the flip $\sigma: V \otimes V \rightarrow V \otimes V$;*
- (b) *the morphism $s: \mathbb{C} \rightarrow V \otimes V$, $s(1) = \sum_k e_k \otimes J e_k$, and its adjoint, where $\{e_k\}_k$ is an orthonormal basis in V ;*
- (c) *the tensors T_i , viewed as morphisms $V^{\otimes m_i} \rightarrow V^{\otimes n_i}$, and their adjoints.*

In other words, any irreducible representation of K appears as a subrepresentation of $V^{\otimes n}$, and any K -intertwiner $V^{\otimes m} \rightarrow V^{\otimes n}$ can be written as a linear combination of compositions of morphisms of the form $\iota \otimes \cdots \otimes \iota \otimes f \otimes \iota \otimes \cdots \otimes \iota$, where f is a morphism from the above list.

Proof. The proof is similar to that in the unitary case. Briefly, the Karoubi envelope of the category with objects $V^{\otimes n}$ and morphism spaces generated by the morphisms in the formulation of the theorem is the representation category of a closed subgroup $H \subset U(V)$. Since s is an H -intertwiner, the elements of H commute with J . As in the unitary case, we then conclude that the spaces of K - and H -intertwiners coincide. \square

Theorems 2.1 and 2.3 allow us, in principle, to find different generators of representation categories of compact Lie groups, or, equivalently, of reductive complex Lie groups. In fact, it follows from [GG, Theorem 15.1] that if we start with a compact connected simple Lie group K and a faithful irreducible unitary representation of K on V , then, apart from a few cases, we can find a tensor $T \in S^k V^*$ such that K coincides with the connected component of the identity of the stabilizer of T in $U(V)$. For some exceptional Lie groups, their realizations as stabilizers of homogeneous polynomials on representations of small dimensions can be found in [SV]. For example, the compact simply connected group of type E_6 has dimension 78, while its smallest nontrivial irreducible representation V has dimension 27. It is known that there is a unique up to a scalar factor homogeneous cubic polynomial on V invariant under E_6 , and then E_6 is exactly the group of unitary transformations stabilizing this polynomial.

Similar descriptions of representation categories can then be obtained for the q -deformations of reductive complex Lie groups G , at least for generic values of the deformation parameter or when q is an indeterminate. For this we have to deform morphisms of G -modules. Let us explain, or rather remind, in detail how to do this.

In order to simplify the discussion, let us consider only the simply connected semisimple case. So, let G be a simply connected semisimple complex Lie group with Lie algebra \mathfrak{g} . Fix a maximal torus and a system of simple roots. Consider the quantized universal enveloping algebra $U_q \mathfrak{g}$ over $\mathbb{Q}(q)$, where q is an indeterminate. Denote by $U_q^{\text{res}} \mathfrak{g} \subset U_q \mathfrak{g}$ Lusztig's restricted integral form, but with the scalars extended from $\mathbb{Z}[q, q^{-1}]$ to $\mathbb{Q}[q, q^{-1}]$, see [L1] or [CP, Section 9.3].

Recall that a representation of $U_q\mathfrak{g}$ over $\mathbb{Q}(q)$ is called admissible, or of type 1, if it decomposes into a direct sum of weight spaces. Every finite dimensional admissible $U_q\mathfrak{g}$ -module admits a $U_q^{\text{res}}\mathfrak{g}$ -stable $\mathbb{Q}[q, q^{-1}]$ -lattice M . Then on the specialization $M_1 = M/(q-1)M$ of M to $q=1$ we get a representation of the rational form $\mathfrak{g}_{\mathbb{Q}}$ of \mathfrak{g} generated over \mathbb{Q} by the Chevalley generators of \mathfrak{g} . Up to isomorphism this module does not depend on the choice of M . If V is simple, with a highest weight vector ξ of weight λ , then taking $M = (U_q^{\text{res}}\mathfrak{g})\xi$ we get that M_1 is also irreducible, with highest weight λ . It follows that for arbitrary V and M , the $U_q\mathfrak{g}$ -module V decomposes into simple highest weight modules in the same way as the $\mathfrak{g}_{\mathbb{Q}}$ -module M_1 does.

Lemma 2.4. *Let V' and V'' be finite dimensional admissible $U_q\mathfrak{g}$ -modules, $M' \subset V'$ and $M'' \subset V''$ be $U_q^{\text{res}}\mathfrak{g}$ -stable $\mathbb{Q}[q, q^{-1}]$ -lattices. Then any morphism $M'_1 \rightarrow M''_1$ of $\mathfrak{g}_{\mathbb{Q}}$ -modules lifts to a morphism $M' \rightarrow M''$ of $U_q^{\text{res}}\mathfrak{g}$ -modules.*

Proof. The $\mathbb{Q}[q, q^{-1}]$ -module $\text{Hom}_{U_q^{\text{res}}\mathfrak{g}}(M', M'')$, being a submodule of a free finite rank module, is itself free and of finite rank. Since any morphism $V' \rightarrow V''$ of $U_q\mathfrak{g}$ -modules can be multiplied by a nonzero element of $\mathbb{Q}[q, q^{-1}]$ to define a morphism $M' \rightarrow M''$ of $U_q^{\text{res}}\mathfrak{g}$ -modules, the rank is equal to $\dim_{\mathbb{Q}(q)} \text{Hom}_{U_q\mathfrak{g}}(V', V'') = \dim_{\mathbb{Q}} \text{Hom}_{\mathfrak{g}_{\mathbb{Q}}}(M'_1, M''_1)$. Therefore it suffices to show that if morphisms T_i , $1 \leq i \leq n$, form a $\mathbb{Q}[q, q^{-1}]$ -basis in $\text{Hom}_{U_q^{\text{res}}\mathfrak{g}}(M', M'')$, then their specializations $T_{i1}: M'_1 \rightarrow M''_1$ are linearly independent over \mathbb{Q} .

Assume $\sum_i a_i T_{i1} = 0$ for some numbers $a_i \in \mathbb{Q}$ not all of which are zero. Then the specialization of $T = \sum_i a_i T_i$ to $q=1$ is zero, so that $TM' \subset (q-1)M''$. But this implies that $(q-1)^{-1}T$ defines a morphism $M' \rightarrow M''$, which contradicts the assumption that the morphisms T_i form a $\mathbb{Q}[q, q^{-1}]$ -basis in $\text{Hom}_{U_q^{\text{res}}\mathfrak{g}}(M', M'')$. \square

Assume now that M_1 is a $\mathfrak{g}_{\mathbb{Q}}$ -module which integrates to a faithful representation of G . Assume also that a compact form of G is the group of unitary transformations of $\mathbb{C} \otimes_{\mathbb{Q}} M_1$ (with respect to a scalar product which is rational on M_1) commuting with operators $1 \otimes T_i: \mathbb{C} \otimes_{\mathbb{Q}} M_1^{\otimes m_i} \rightarrow \mathbb{C} \otimes_{\mathbb{Q}} M_1^{\otimes n_i}$. Take any admissible $U_q\mathfrak{g}$ -module V with a $U_q^{\text{res}}\mathfrak{g}$ -stable $\mathbb{Q}[q, q^{-1}]$ -lattice M such that its specialization to $q=1$ gives M_1 . Then, using Lemma 2.4, we can lift the morphisms T_i to morphisms $T_{iq}: M^{\otimes m_i} \rightarrow M^{\otimes n_i}$ of $U_q^{\text{res}}\mathfrak{g}$ -modules. Similarly for their adjoints, as well as for the morphisms r, \bar{r} and σ (or more precisely, their analogues over \mathbb{Q} for the space M_1) and their adjoints.

Theorem 2.1 implies that each space $\text{Hom}_{\mathfrak{g}_{\mathbb{Q}}}(M_1^{\otimes m}, M_1^{\otimes n})$ has a basis over \mathbb{Q} consisting of compositions of morphisms $\iota \otimes \dots \otimes \iota \otimes f \otimes \iota \otimes \dots \otimes \iota$, where f is one of the morphisms T_i, r, \bar{r}, σ or their adjoints. The same compositions of morphisms with f replaced by the corresponding lifts f_q to morphisms of $U_q^{\text{res}}\mathfrak{g}$ -modules are easily seen to be linearly independent over $\mathbb{Q}(q)$. Since the admissible $U_q\mathfrak{g}$ -modules have the same fusion rules as in the classical case, we conclude that these compositions span $\text{Hom}_{U_q\mathfrak{g}}(V^{\otimes m}, V^{\otimes n})$ over $\mathbb{Q}(q)$. We thus see that the $\mathbb{Q}(q)$ -linear monoidal category of finite dimensional admissible $U_q\mathfrak{g}$ -modules is generated by V and V^* and the lifts of T_i, r, \bar{r}, σ and their adjoints.

We can also specialize the lifts of morphisms to $q \neq 0$. At least for transcendental q we then get generators of the category of complex finite dimensional admissible representations of quantized universal enveloping algebras.

If we know generators of the category of finite dimensional admissible $U_q\mathfrak{g}$ -modules, then, in turn, we can get generators and a complete set of relations for the quantized algebra of regular functions on G , since this algebra is the coend of the forgetful fiber functor, and therefore it is generated by the matrix coefficients of any generating set of modules and the relations are given by any generating set of morphisms, see, e.g., [S, Corollary 2.3.13].

3. THE MAIN THEOREM

In this section, following the strategy described above, we will find a particular presentation of quantized function algebras of automorphism groups of simple complex Lie algebras.

Let G be a connected simple complex Lie group of adjoint type and \mathfrak{g} be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system $(\alpha_k)_k$ of simple roots. Let $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ be the standard ad-invariant symmetric form on \mathfrak{g} , so that $B(h_\alpha, h_\alpha) = 2$ for every short root α , where $h_\alpha \in \mathfrak{h}$ is defined by $B(h_\alpha, h) = \alpha(h)$ for $h \in \mathfrak{h}$. Denote by $t \in \mathfrak{g} \otimes \mathfrak{g}$ the corresponding invariant tensor.

As a compact form of \mathfrak{g} we take the real Lie algebra \mathfrak{k} generated by the elements $ih_k, i(e_k + f_k), e_k - f_k$, where h_k, e_k, f_k are the Chevalley generators of \mathfrak{g} , and denote by $K \subset G$ the corresponding compact form of G . The form B is negative definite on \mathfrak{k} , so $-B|_{\mathfrak{k}}$ extends to a positive definite Hermitian scalar product on \mathfrak{g} . Then the adjoint of the morphism $B: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ is given by $B^*(1) = t$.

Next, consider the group Γ of automorphisms of the Dynkin diagram of \mathfrak{g} . It acts by B -preserving automorphisms on \mathfrak{g} . Together with the adjoint representation of G this gives us a faithful representation of the semidirect product $G \rtimes \Gamma$ on \mathfrak{g} . Since $\text{Out}(K) \cong \Gamma$, it follows that we can identify the compact form $K \rtimes \Gamma$ of $G \rtimes \Gamma$ with the subgroup of the orthogonal group $O(\mathfrak{k}, -B|_{\mathfrak{k}})$ stabilizing the Lie bracket $L \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. By Theorem 2.3 we therefore get the following result.

Proposition 3.1. *The monoidal category of finite dimensional complex representations of $G \rtimes \Gamma$ is generated by \mathfrak{g} and the following morphisms: the flip $\sigma: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $B: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$, $t: \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$, $L: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and $L^*: \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$.*

Let V be a finite dimensional admissible $U_q\mathfrak{g}$ -module which is a deformation of the adjoint representation of $\mathfrak{g}_{\mathbb{Q}}$, that is, it is a simple module with highest weight equal to the maximal root α_{\max} . Fix a highest weight vector ξ and put $M = (U_q^{\text{res}}\mathfrak{g})\xi$. We identify M_1 with $\mathfrak{g}_{\mathbb{Q}}$.

The group Γ acts on $U_q\mathfrak{g}$ by automorphisms α_γ , $\gamma \in \Gamma$, permuting the generators. We have a unique representation of Γ on V which respects this action and such that its specialization to $q = 1$ gives the action of Γ on $\mathfrak{g}_{\mathbb{Q}}$. Namely, this representation is given by $\gamma(X\xi) = \chi(\gamma)\alpha_\gamma(X)\xi$ for $X \in U_q\mathfrak{g}$, where $\chi: \Gamma \rightarrow \{-1, 1\}$ is the same character as in the classical case.

Now we can use Lemma 2.4 to lift the Lie bracket $L|_{\mathfrak{g}_{\mathbb{Q}}}: \mathfrak{g}_{\mathbb{Q}} \otimes \mathfrak{g}_{\mathbb{Q}} \rightarrow \mathfrak{g}_{\mathbb{Q}}$ to a morphism

$$L_q: M \otimes_{\mathbb{Q}[q, q^{-1}]} M \rightarrow M$$

of $U_q^{\text{res}}\mathfrak{g}$ -modules and then, by averaging over Γ , assume that L_q is Γ -equivariant. The following shows that this determines L_q in an essentially unique way.

Lemma 3.2. *The Lie bracket is the unique up to a scalar factor Γ -equivariant morphism $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ of \mathfrak{g} -modules.*

Proof. Recall that we assume that \mathfrak{g} is simple. It is known then, see, e.g., [KW] for a more general statement, that if $\mathfrak{g} \not\cong \mathfrak{sl}_n(\mathbb{C})$ for $n \geq 3$, then $\mathfrak{g} \otimes \mathfrak{g}$ contains a single copy of \mathfrak{g} , so in this case we do not even need the Γ -equivariance to get uniqueness. On the other hand, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for some $n \geq 3$, then the multiplicity of \mathfrak{g} in $\mathfrak{g} \otimes \mathfrak{g}$ equals 2: there is an extra copy of \mathfrak{g} in $S^2\mathfrak{g}$. Namely, a morphism $S^2\mathfrak{g} \rightarrow \mathfrak{g}$, or equivalently, a pairing between $S^2\mathfrak{g}$ and \mathfrak{g} , comes from the nonzero invariant cubic form $\text{Tr}(X^3)$ on $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Up to an inner automorphism, the only nontrivial outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$ is given by $X \mapsto -X^t$. Since the cubic form is anti-invariant with respect to this automorphism, we see that the corresponding morphism $S^2\mathfrak{g} \rightarrow \mathfrak{g}$ is not Γ -equivariant. \square

It follows that the space of Γ -equivariant morphisms $V \otimes_{\mathbb{Q}(q)} V \rightarrow V$ of $U_q\mathfrak{g}$ -modules is one-dimensional. Hence to get L_q we just have to take any such nonzero morphism and multiply it by an element of $\mathbb{Q}(q)$ so that it maps $M \otimes_{\mathbb{Q}[q, q^{-1}]} M$ into M and specializes to a nonzero morphism at $q = 1$, and then rescale it by a rational number to get the right specialization at $q = 1$. Furthermore, we may require L_q to be indivisible in the free rank one $\mathbb{Q}[q, q^{-1}]$ -module $\text{Hom}_{(U_q^{\text{res}}\mathfrak{g}) \rtimes \Gamma}(M \otimes_{\mathbb{Q}[q, q^{-1}]} M, M)$. This determines L_q uniquely up to a factor q^n , $n \in \mathbb{Z}$.

Remark 3.3. Instead of the Γ -equivariance we could equivalently require L_q to be zero on the quantum symmetric tensors. Namely, assume $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for some $n \geq 3$, which is the only case we have to take care of. Then on the isotypic component of $V \otimes_{\mathbb{Q}(q)} V$ corresponding to V the braiding defined by the R -matrix has eigenvalues $\pm q^{-n}$ or $\pm q^n$, depending on which of the two standard R -matrices we take, see [KS, Corollary 23 in Section 8.4.3]. Then up to a scalar factor the morphism L_q of $U_q\mathfrak{g}$ -modules is specified by requiring it to kill the eigenvectors of the braiding with eigenvalue q^{-n} or q^n . Note also that explicit formulas for L_q are known, see, e.g., [DHGZ].

Similarly, we lift $B|_{\mathfrak{g}_{\mathbb{Q}}}: \mathfrak{g}_{\mathbb{Q}} \otimes \mathfrak{g}_{\mathbb{Q}} \rightarrow \mathbb{Q}$ to a morphism

$$B_q: M \otimes_{\mathbb{Q}[q, q^{-1}]} M \rightarrow \mathbb{Q}[q, q^{-1}]$$

of $U_q^{\text{res}}\mathfrak{g}$ -modules. Since V is simple and self-dual, the space of morphisms $V \otimes_{\mathbb{Q}(q)} V \rightarrow \mathbb{Q}(q)$ is one-dimensional. Hence, again, B_q is unique up to a scalar factor, and if we require B_q to be indivisible in the free rank one $\mathbb{Q}[q, q^{-1}]$ -module $\text{Hom}_{U_q^{\text{res}}\mathfrak{g}}(M \otimes_{\mathbb{Q}[q, q^{-1}]} M, \mathbb{Q}[q, q^{-1}])$, then B_q is unique up to a factor q^n , $n \in \mathbb{Z}$. Clearly, B_q is also Γ -equivariant.

In a similar way we can define Γ -equivariant lifts

$$B'_q: \mathbb{Q}[q, q^{-1}] \rightarrow M \otimes_{\mathbb{Q}[q, q^{-1}]} M \quad \text{and} \quad L'_q: M \rightarrow M \otimes_{\mathbb{Q}[q, q^{-1}]} M$$

of $B^*|_{\mathbb{Q}}$ and $L^*|_{\mathfrak{g}_{\mathbb{Q}}}$. Note that up to a factor, the morphism B'_q is given by $1 \mapsto \sum_k x^k \otimes x_k$, where $(x_k)_k$ is a basis in V and $(x^k)_k$ is the dual basis: $B_q(x_k, x^l) = \delta_{kl}$.

Next, consider the quantized algebra $\mathcal{O}_q(G)$ of regular functions on G over $\mathbb{Q}(q)$. By this we mean the algebra of matrix coefficients of finite dimensional admissible $U_q\mathfrak{g}$ -modules with weights in the root lattice. Put

$$\mathcal{O}_q(G \rtimes \Gamma) = \mathcal{O}_q(G) \otimes_{\mathbb{Q}} \mathcal{O}(\Gamma; \mathbb{Q}),$$

where $\mathcal{O}(\Gamma; \mathbb{Q})$ is the algebra of \mathbb{Q} -valued functions on Γ . Equivalently, $\mathcal{O}_q(G \rtimes \Gamma)$ is the algebra of matrix coefficients of finite dimensional representations of $U_q\mathfrak{g} \rtimes \Gamma$ over $\mathbb{Q}(q)$ such that their restrictions to $U_q\mathfrak{g}$ have only weights in the root lattice. Denote by \mathcal{C}_q the $\mathbb{Q}(q)$ -linear monoidal category of such representations.

Proposition 3.4. *The monoidal category \mathcal{C}_q is semisimple and braided, with braiding defined by the universal R -matrix \mathcal{R}_q of $U_q\mathfrak{g}$. Denote by R_q the image of \mathcal{R}_q under the representation on $V \otimes_{\mathbb{Q}(q)} V$. Then \mathcal{C}_q is generated by V and the following morphisms: $\sigma R_q: V \otimes_{\mathbb{Q}(q)} V \rightarrow V \otimes_{\mathbb{Q}(q)} V$, $B_q: V \otimes_{\mathbb{Q}(q)} V \rightarrow \mathbb{Q}(q)$, $B'_q: \mathbb{Q}(q) \rightarrow V \otimes_{\mathbb{Q}(q)} V$, $L_q: V \otimes_{\mathbb{Q}(q)} V \rightarrow V$, and $L'_q: V \rightarrow V \otimes_{\mathbb{Q}(q)} V$.*

Proof. The first statement follows from semisimplicity of the category of finite dimensional admissible $U_q\mathfrak{g}$ -modules using the standard Clifford–Mackey type analysis of representations of crossed products. Furthermore, the irreducible finite dimensional admissible representations of $U_q\mathfrak{g} \rtimes \Gamma$ are classified by the Γ -orbits of pairs (λ, π) , where λ is a dominant integral weight and $\pi: \Gamma_\lambda \rightarrow GL(H_\pi)$ is an irreducible representation (over $\mathbb{Q}(q)$) of the stabilizer Γ_λ of λ in Γ . The representation corresponding to (λ, π) is defined as follows. Consider the irreducible $U_q\mathfrak{g}$ -module V_λ with a highest weight vector ξ_λ of weight λ . We have a representation ρ_λ of Γ_λ on V_λ defined by $\rho_\lambda(\gamma)X\xi_\lambda = \alpha_\gamma(X)\xi_\lambda$ for $X \in U_q\mathfrak{g}$. The obvious representation of $U_q\mathfrak{g}$ on $V_\lambda \otimes_{\mathbb{Q}(q)} H_\pi$ together with the representation $\rho_\lambda \otimes \pi$ of Γ_λ define an irreducible representation of $U_q\mathfrak{g} \rtimes \Gamma_\lambda$. Finally, we induce this representation to a representation of $U_q\mathfrak{g} \rtimes \Gamma$.

The statement about the braiding follows from the fact that the universal R -matrix \mathcal{R}_q is Γ -invariant. This, in turn, can be checked either by looking at the explicit formula for \mathcal{R}_q or by using the characterization of \mathcal{R}_q by its action on the tensor product of lowest and highest weight vectors.

Observe next that any irreducible representation of Γ_λ over $\mathbb{Q}(q)$ has a unique up to isomorphism rational form. This can be seen either by noticing that the only nontrivial stabilizer groups we can get are S_2 and S_3 , or by using the general fact that any finite dimensional division algebra over \mathbb{Q} remains a division algebra after extending the scalars to $\mathbb{Q}(q)$. Therefore the simple objects of \mathcal{C}_q are parametrized by the Γ -orbits of pairs (λ, π) , where λ is a dominant weight in the root lattice and π is an irreducible representation of Γ_λ over \mathbb{Q} . The same set parametrizes the irreducible finite dimensional representations of $U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma$ over \mathbb{Q} with weights in the root lattice. It is clear that if we take the simple $U_q\mathfrak{g} \rtimes \Gamma$ -module $V' \in \mathcal{C}_q$ corresponding to (λ, π) and a $U_q^{\text{res}}\mathfrak{g} \rtimes \Gamma$ -stable $\mathbb{Q}[q, q^{-1}]$ -lattice $M' \subset V'$, then the specialization of M' to $q = 1$ gives us a simple $U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma$ -module corresponding to (λ, π) . This implies that the fusion rules in \mathcal{C}_q are the same as in the representation category of $U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma$. We can also conclude that if $V', V'' \in \mathcal{C}_q$ and $M' \subset V', M'' \subset V''$ are $U_q^{\text{res}}\mathfrak{g} \rtimes \Gamma$ -stable $\mathbb{Q}[q, q^{-1}]$ -lattices, then

$$\dim_{\mathbb{Q}(q)} \text{Hom}_{U_q\mathfrak{g} \rtimes \Gamma}(V', V'') = \dim_{\mathbb{Q}} \text{Hom}_{U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma}(M'_1, M''_1).$$

Now, if W is a simple finite dimensional $U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma$ module with weights in the root lattice, then by Proposition 3.1 we know that the $U\mathfrak{g} \rtimes \Gamma$ -module $\mathbb{C} \otimes_{\mathbb{Q}} W$ has a nonzero morphism into $\mathfrak{g}^{\otimes n} = \mathbb{C} \otimes_{\mathbb{Q}} (M_1^{\otimes n})$ for some n . Hence W embeds into $M_1^{\otimes n}$. By the above discussion it follows that any simple object in \mathcal{C}_q embeds into $V^{\otimes n}$ for some n .

Repeating the discussion at the end of the previous section, Proposition 3.1 implies that the space $\text{Hom}_{U(\mathfrak{g}_{\mathbb{Q}}) \rtimes \Gamma}(M_1^{\otimes m}, M_1^{\otimes n})$ has a basis over \mathbb{Q} consisting of compositions of morphisms $\iota \otimes \cdots \otimes \iota \otimes f \otimes \iota \otimes \cdots \otimes \iota$, where f is the restriction to the appropriate rational forms of one of the morphisms σ, B, B^*, L, L^* . The

same compositions of morphisms with f replaced, resp., by $\sigma R_q, B_q, B'_q, L_q, L'_q$, are then linearly independent over $\mathbb{Q}(q)$. Hence they span $\text{Hom}_{U_q \mathfrak{g} \rtimes \Gamma}(V^{\otimes m}, V^{\otimes n})$. \square

Fix a basis $(x_k)_k$ of the $\mathbb{Q}[q, q^{-1}]$ -lattice $M \subset V$. Let $t_{kl} \in \mathcal{O}_q(G \rtimes \Gamma)$ be the matrix coefficients of the representation of $U_q \mathfrak{g} \rtimes \Gamma$ on V in this basis. In the next theorem, which is our main result, we view R_q and L_q as matrices over $\mathbb{Q}[q, q^{-1}]$ with respect to the bases $(x_k \otimes x_l)_{k,l}$ of $M \otimes_{\mathbb{Q}[q, q^{-1}]} M$ and $(x_k)_k$ of M .

Theorem 3.5. *The $\mathbb{Q}(q)$ -algebra $\mathcal{O}_q(G \rtimes \Gamma)$ is generated by the coefficients of the matrix $T = (t_{kl})_{k,l}$, and the following is a complete set of relations:*

$$R_q T_1 T_2 = T_2 T_1 R_q, \quad (\text{R1})$$

$$A_q^{-1} T^t A_q T = 1 = T A_q^{-1} T^t A_q, \quad (\text{R2})$$

$$L_q T_1 T_2 = T L_q, \quad (\text{R3})$$

where A_q is the matrix $(B_q(x_k \otimes x_l))_{k,l}$.

Proof. As we already discussed in the previous section, since the category \mathcal{C}_q is semisimple, the algebra of matrix coefficients of $U_q \mathfrak{g} \rtimes \Gamma$ -modules in \mathcal{C}_q is nothing but the coend of the forgetful fiber functor on \mathcal{C}_q . Hence, by Proposition 3.4, we conclude that $\mathcal{O}_q(G \rtimes \Gamma)$ is generated by the elements t_{kl} and a complete set of relations is given by (R1), (R3) and

$$B_q T_1 T_2 = B_q, \quad T_1 T_2 B'_q = B'_q, \quad (\text{R2}')$$

$$T_1 T_2 L'_q = L'_q T, \quad (\text{R4})$$

where we again view B'_q and L'_q as matrices.

In terms of the matrix A_q the first relation in (R2') reads as $T^t A_q T = A_q$. Since up to a factor from $\mathbb{Q}(q)$ the morphism B'_q is given by

$$1 \mapsto \sum_k x^k \otimes x_k = \sum_{k,l} (A_q^{-1})_{kl} x_k \otimes x_l,$$

the second relation in (R2') reads as $T A_q^{-1} T^t = A_q^{-1}$. Thus relations (R2) and (R2') are equivalent.

It remains to show that (R4) follows from relations (R1)-(R3). For this we use the following description of L'_q . Equip $V \otimes_{\mathbb{Q}(q)} V$ with the nondegenerate invariant form

$$B_q^{(2)}(x \otimes y, z \otimes w) = B_q(x \otimes w) B_q(y \otimes z).$$

Then consider the adjoint L_q^\dagger of L_q with respect to the invariant forms $B_q^{(2)}$ and B_q on $V \otimes_{\mathbb{Q}(q)} V$ and V , so $B_q^{(2)}(L_q^\dagger x, y \otimes z) = B_q(x \otimes L_q(y \otimes z))$. In the classical case this would give us exactly the Hermitian adjoint L^* we considered before. Hence by multiplying L_q^\dagger by a polynomial $f \in \mathbb{Q}[q, q^{-1}]$ with value 1 at $q = 1$, we can get a morphism $L'_q: M \rightarrow M \otimes_{\mathbb{Q}[q, q^{-1}]} M$ such that its specialization to $q = 1$ equals $L^*|_{\mathfrak{g}\mathbb{Q}}$. Since B_q is Γ -equivariant, L'_q is Γ -equivariant as well, so this is the required morphism.

Now, by taking adjoints (with respect to the forms $B_q^{(2)}$ and B_q) of both sides of (R3) we get

$$(A^{-1} T^t A)_2 (A^{-1} T^t A)_1 L'_q = L'_q A^{-1} T^t A,$$

where we used that the adjoint of an operator C on V is $A^{-1} C^t A$ in our basis, and then that the adjoints of T_1 and T_2 are $(A^{-1} T^t A)_2$ and $(A^{-1} T^t A)_1$, resp. In view of (R2), the above identity is equivalent to (R4). \square

Next we want to relax relation (R2) to invertibility of T . This is possible by the following general result.

Proposition 3.6. *Assume \mathcal{U} is a quasitriangular Hopf algebra with invertible antipode and R -matrix \mathcal{R} , and V is a finite dimensional \mathcal{U} -module. Fix a basis v_1, \dots, v_n in V and consider a universal unital algebra \mathcal{A} with generators t_{ij} and \tilde{t}_{ij} and relations*

$$R T_1 T_2 = T_2 T_1 R, \quad T \tilde{T} = 1 = \tilde{T} T,$$

where $T = (t_{ij})_{i,j}$, $\tilde{T} = (\tilde{t}_{ij})_{i,j}$, and R is the matrix of the operator \mathcal{R} on $V \otimes V$ in the basis $(v_i \otimes v_k)_{i,k}$. Then \mathcal{A} is a Hopf algebra with coproduct $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$, $\Delta(\tilde{t}_{ij}) = \sum_k \tilde{t}_{kj} \otimes \tilde{t}_{ik}$ and antipode

$$S(T) = \tilde{T}, \quad S(\tilde{T}) = u T u^{-1},$$

where u is the Drinfeld element $m(S \otimes \iota)(\mathcal{R}_{21})$ acting on V .

Proof. It is easy to see that the coproduct is well-defined. The opposite algebra is defined by the relations $RT_2T_1 = T_1T_2R$ and $\tilde{T}^tT^t = 1 = T^t\tilde{T}^t$. From this we see that in order to prove that S is a well-defined anti-homomorphism, it suffices to check that $(uTu^{-1})^t$ and \tilde{T}^t are inverse to each other in \mathcal{A} . This will also imply that S is indeed an antipode, since both T and \tilde{T}^t are corepresentations of (\mathcal{A}, Δ) and their entries generate \mathcal{A} .

Viewing T and $T^{-1} = \tilde{T}$ as elements of $\text{End}(V) \otimes \mathcal{A}$ rather than as matrices, say, $T = \sum_k c_k \otimes t_k$, $T^{-1} = \sum_i c'_i \otimes t'_i$, we thus have to check that

$$\sum_{i,k} c'_i u c_k u^{-1} \otimes t_k t'_i = 1 = \sum_{i,k} u c_k u^{-1} c'_i \otimes t'_i t_k. \quad (3.1)$$

For this purpose take the relation $T_{23}^{-1}R_{12}T_{13} = T_{13}R_{12}T_{23}^{-1}$. Thus, if $\mathcal{R} = \sum_j a_j \otimes b_j$, we have

$$\sum_{i,j,k} a_j c_k \otimes c'_i b_j \otimes t'_i t_k = \sum_{i,j,k} c_k a_j \otimes b_j c'_i \otimes t_k t'_i \text{ in } \text{End}(V) \otimes \text{End}(V) \otimes \mathcal{A}.$$

Since $\mathcal{R}^{-1} = \sum_j S(a_j) \otimes b_j$, we have $\sum_{l,j} S^{-1}(a_l) a_j \otimes b_j b_l = 1$, hence we get

$$\sum_{i,k} c_k \otimes c'_i \otimes t'_i t_k = \sum_{i,j,k,l} S^{-1}(a_l) c_k a_j \otimes b_j c'_i b_l \otimes t_k t'_i.$$

Let us apply the operation m_{21} to both sides of this identity. We get

$$1 = \sum_{i,j,k,l} b_j c'_i b_l S^{-1}(a_l) c_k a_j \otimes t_k t'_i,$$

i.e., since $u = m(\iota \otimes S^{-1})(\mathcal{R}_{21})$,

$$1 = \sum_{i,j,k} b_j c'_i u c_k a_j \otimes t_k t'_i.$$

Therefore, using that $\sum_{l,j} a_j S^{-1}(a_l) \otimes b_l b_j = 1$, we get

$$\sum_{i,k} c'_i u c_k \otimes t_k t'_i = \sum_l b_l S^{-1}(a_l) \otimes 1 = u \otimes 1,$$

which gives the first identity in (3.1). The second identity is proved similarly, by starting with the relation $T_{23}R_{12}^{-1}T_{13}^{-1} = T_{13}^{-1}R_{12}^{-1}T_{23}$ and using that $u^{-1} = m(S^{-1} \otimes \iota)(\mathcal{R}_{21}^{-1})$. \square

Remark 3.7. The proof shows that the only properties of the invertible operator R that we need are that both R and R^{-1} are invertible in $\text{End}(V)^{\text{op}} \otimes \text{End}(V)$, and if $\sum_i p_i \otimes q_i$ and $\sum_j r_j \otimes s_j$ are these inverses, then $u = \sum_i q_i p_i$ and $v = \sum_j s_j r_j$ are invertible in $\text{End}(V)$. Then the antipode is given by $S(T) = T^{-1}$, $S(T^{-1}) = uTu^{-1} = v^{-1}Tv$.

We are now ready to prove the second version of our main result.

Theorem 3.8. *In the notation of Theorem 3.5, the $\mathbb{Q}(q)$ -algebra $\mathcal{O}_q(G \rtimes \Gamma)$ is generated by the coefficients of T and T^{-1} , and a complete set of relations is given by*

$$R_q T_1 T_2 = T_2 T_1 R_q, \quad L_q T_1 T_2 = T L_q.$$

Proof. Consider a universal algebra \mathcal{O} with generators and relations as in the formulation. In other words, \mathcal{O} is the quotient of the Hopf algebra \mathcal{A} from Proposition 3.6 by the additional relation $L_q T_1 T_2 = T L_q$. It is immediate that both the comultiplication and the antipode pass to the quotient, so \mathcal{O} is a Hopf algebra with antipode $S(T) = T^{-1}$, $S(T^{-1}) = uTu^{-1}$, where u is the Drinfeld element of $U_q \mathfrak{g}$ acting on V . (Note that $U_q \mathfrak{g}$ is not, strictly speaking, quasitriangular, as its R -matrix lives only in a completion of $U_q \mathfrak{g} \otimes_{\mathbb{Q}(q)} U_q \mathfrak{g}$, but the proof of Proposition 3.6 goes through without any change; see also Remark 3.7.)

The matrix T defines a right \mathcal{O} -comodule structure on V . To prove the theorem, it suffices to show that $B_q: V \otimes_{\mathbb{Q}(q)} V \rightarrow \mathbb{Q}(q)$ is a morphism of \mathcal{O} -comodules, since, as we already used in the proof of Theorem 3.5, this is equivalent to the identity $T^t A_q T = A_q$.

Consider the dual \mathcal{O} -comodule V^* . We have the following standard morphisms of comodules:

$$\begin{aligned} e: V \otimes_{\mathbb{Q}(q)} V^* &\rightarrow \mathbb{Q}(q), & x \otimes f &\mapsto f(ux), \\ i: \mathbb{Q}(q) &\rightarrow V \otimes_{\mathbb{Q}(q)} V^*, & 1 &\mapsto \sum_i x_i \otimes x^i, \end{aligned}$$

where $(x_i)_i$ is a basis in V and $(x^i)_i$ is the dual basis. We can then define a morphism of \mathcal{O} -comodules

$$\tilde{B}_q = e(L_q \otimes \iota)(\iota \otimes L_q \otimes \iota)(\iota \otimes \iota \otimes i): V \otimes_{\mathbb{Q}(q)} V \rightarrow \mathbb{Q}(q).$$

Since $\mathcal{O}_q(G \rtimes \Gamma)$ is a quotient of \mathcal{O} , this is also a morphism of $(U_q \mathfrak{g} \rtimes \Gamma)$ -modules. Hence it coincides with B_q up to a scalar factor. Therefore we only need to check that $\tilde{B}_q \neq 0$.

For this we consider the specialization of \tilde{B}_q to $q = 1$. More precisely, recall that we fixed a $\mathbb{Q}[q, q^{-1}]$ -lattice M in V and identified M_1 with $\mathfrak{g}_{\mathbb{Q}}$. As a lattice in V^* we take $M^* = \text{Hom}_{\mathbb{Q}[q, q^{-1}]}(M, \mathbb{Q}[q, q^{-1}])$. Its specialization to $q = 1$ is $\mathfrak{g}_{\mathbb{Q}}^* = \text{Hom}_{\mathbb{Q}}(\mathfrak{g}_{\mathbb{Q}}, \mathbb{Q})$. The morphism \tilde{B}_q restricts to a morphism $M \otimes_{\mathbb{Q}[q, q^{-1}]} M \rightarrow \mathbb{Q}[q, q^{-1}]$ and this restriction is defined in exactly the same way as \tilde{B}_q , but using M and M^* instead of V and V^* . Specializing this restriction to $q = 1$ and using that then L_q becomes the Lie bracket, while u becomes equal to 1, we see that the restriction of \tilde{B}_q becomes the Killing form $X \otimes Y \mapsto \text{Tr}((\text{ad } X)(\text{ad } Y))$ on $\mathfrak{g}_{\mathbb{Q}}$. \square

4. EXTENSION TO ALL q EXCEPT FINITELY MANY POINTS

As in the previous section, assume that G is a connected simple complex Lie group of adjoint type. Denote the algebra $\mathbb{Q}[q, q^{-1}]$ by A and consider the quantized algebra of functions $\mathcal{O}_q^A(G) \subset \mathcal{O}_q(G)$ over A defined by Lusztig [L2]. We can then consider the subalgebra

$$\mathcal{O}_q^A(G \rtimes \Gamma) = \mathcal{O}_q^A(G) \otimes_{\mathbb{Q}} \mathcal{O}(\Gamma; \mathbb{Q}) \subset \mathcal{O}_q(G \rtimes \Gamma)$$

and specialize it to any nonzero $q_0 \in \mathbb{C}$. We consider these specialized algebras over \mathbb{C} . Thus, we take the homomorphism $A \rightarrow \mathbb{C}$, $q \mapsto q_0$, and put

$$\mathcal{O}_{q_0}(G \rtimes \Gamma) = \mathbb{C} \otimes_A \mathcal{O}_q^A(G \rtimes \Gamma).$$

Note that it follows from the definition of $\mathcal{O}_q^A(G)$ in [L2] that for q_0 not a nontrivial root of unity the algebra $\mathcal{O}_{q_0}(G \rtimes \Gamma)$ can be defined in the same way as in the previous section but working over \mathbb{C} instead of $\mathbb{Q}(q)$.

Theorem 4.1. *Theorem 3.8 holds over \mathbb{C} for all numerical values of q except finitely many algebraic numbers. In fact, the following stronger statement holds: there is a polynomial $f \in \mathbb{Q}[q]$ with $f(1) \neq 0$ such that for the ring $B = \mathbb{Q}[q, q^{-1}, 1/f]$ the B -algebra $\mathcal{O}_q^B(G \rtimes \Gamma) \subset \mathcal{O}_q(G \rtimes \Gamma)$ is described by the generators and relations as in Theorem 3.8.*

Recall that when q is an indeterminate, then, by the discussion before Theorem 3.5, we view R_q and L_q as matrices over $\mathbb{Q}[q, q^{-1}]$ with respect to bases $(x_k \otimes x_l)_{k,l}$ of $M \otimes_{\mathbb{Q}[q, q^{-1}]} M$ and $(x_k)_k$ of M . These matrices can be specialized to any nonzero complex number, and it is these specializations that we use in Theorem 4.1. For finitely many roots of unity we might even get zero matrices, but this is allowed by the formulation of the theorem. When we take values different from nontrivial roots of unity, then the specializations are never zero, and we conjecture that Theorem 4.1 is true for all such values.

Proof. It is clear that Theorem 3.8 holds for all transcendental values of q , since we then have an inclusion $\mathbb{Q}(q) \hookrightarrow \mathbb{C}$. To extend the result to all but finitely many algebraic numbers, we argue as follows.

The generators and relations of Theorem 3.8 make sense over the ring A . Let \mathcal{O}^A be the algebra defined by these generators and relations. We have a natural algebra homomorphism $\psi: \mathcal{O}^A \rightarrow \mathcal{O}_q^A(G \rtimes \Gamma)$. Since the algebra $\mathcal{O}_q^A(G \rtimes \Gamma)$ is finitely generated by [L2, Proposition 3.3], the homomorphism ψ becomes surjective after inverting a polynomial $f_1(q)$. Using the fact that the specialization of $\mathcal{O}_q^A(G \rtimes \Gamma)$ to $q = 1$ maps an A -basis into a basis of $\mathcal{O}^{\mathbb{Q}}(G \rtimes \Gamma)$ it is easy to see that we may assume that $f_1(1) \neq 0$. (For similar reasons we can actually assume that the roots of f_1 are nontrivial roots of unity.)

Let $\mathcal{K} = \text{Ker } \psi$. By the arguments in the proof of [BG, Proposition I.8.17], the algebra \mathcal{O}^A is \mathbb{N} -filtered, and $\text{gr}(\mathcal{O}^A)$ is locally finite and strongly left Noetherian (more precisely, that proposition is proved when q is specialized to a numerical value, but it generalizes verbatim to our setting). Hence \mathcal{K} is a finitely generated

\mathcal{O}^A -module. Thus, it follows from [ASZ, Theorem 0.3] that \mathcal{K} is generically free over $\text{Spec } A$. In other words, \mathcal{K} becomes free after inverting some polynomial $f_2(q)$. But Theorem 3.8 implies that $\mathcal{K} \otimes_A \mathbb{Q}(q) = 0$. Hence \mathcal{K} becomes zero after inverting $f_2(q)$.

This means that the map ψ becomes an isomorphism after inverting $f = f_1 f_2$. In particular, we get an isomorphism if we specialize q to any nonzero value except the roots of f .

It remains to show that f can be chosen so that $f(1) \neq 0$, or equivalently, that the localization of \mathcal{K} to some Zariski neighbourhood of $q = 1$ is zero. Clearly, we can choose a polynomial f such that $f(1) \neq 0$, ψ is surjective after inverting f , and $(q-1)^N \mathcal{K}[1/f] = 0$ for some $N \geq 0$. Pick the smallest such N .

Assume that $N > 0$. Pick $v \in \mathcal{K}[1/f]$ such that $(q-1)^{N-1} v \neq 0$ (it exists since N was chosen the smallest possible). Then $v \notin (q-1)\mathcal{K}[1/f]$, i.e., $\mathcal{K}[1/f]/(q-1)\mathcal{K}[1/f] = \mathcal{K}/(q-1)\mathcal{K} \neq 0$. On the other hand, let us reduce the short exact sequence

$$0 \rightarrow \mathcal{K}[1/f] \rightarrow \mathcal{O}^A[1/f] \rightarrow \mathcal{O}_q^A(G \rtimes \Gamma)[1/f] \rightarrow 0$$

modulo $q-1$. Since $\mathcal{O}_q^A(G \rtimes \Gamma)$ is a free A -module, we have $\text{Tor}_1^A(A/(q-1)A, \mathcal{O}_q^A(G \rtimes \Gamma)[1/f]) = 0$. Hence we get a short exact sequence

$$0 \rightarrow \mathcal{K}/(q-1)\mathcal{K} \rightarrow \mathcal{O}^A/(q-1) \rightarrow \mathcal{O}^{\mathbb{Q}}(G \rtimes \Gamma) \rightarrow 0.$$

But $\mathcal{O}^{\mathbb{Q}}(G \rtimes \Gamma)$ is defined by the specialization of the relations of Theorem 3.8 to $q = 1$, hence $\mathcal{K}/(q-1)\mathcal{K} = 0$. This is a contradiction. Thus $N = 0$ and $\mathcal{K}[1/f] = 0$, as desired. \square

Corollary 4.2. *Outside of finitely many values of q (not including 1), every braided tensor autoequivalence of the category \mathcal{C} of comodules over $\mathcal{O}_q(G \rtimes \Gamma)$ is trivial.*

Proof. Denote as before by V a quantum deformation of the \mathfrak{g} -module \mathfrak{g} . It is not difficult to check using the arguments of the proof [DEN, Theorem 4.4(ii)] that for every autoequivalence $F: \mathcal{C} \rightarrow \mathcal{C}$ one has $F(V) \cong V$. Also, by [DEN, Proposition 3.12], Theorem 4.1 implies that outside of finitely many values of q , every braided tensor functor out of \mathcal{C} is determined by the image of V and of the quantum Lie bracket $L_q: V \otimes V \rightarrow V$. Thus, every braided tensor functor $F: \mathcal{C} \rightarrow \mathcal{C}$ is determined by $\lambda \in \mathbb{C}^*$ such that $F(L_q) = \lambda L_q$. But by rescaling the isomorphism $V \cong F(V)$, we can make $\lambda = 1$. This implies the corollary. \square

This provides a new proof of [NT1, Theorem 2.1] (for all but finitely many q not a nontrivial root of unity) and of [DEN, Theorem 4.4(ii)] for the groups $G_2, F_4, E_7^{\text{ad}}, E_8$, for which $\Gamma = 1$. For E_6 there is a similar proof based on an analogue of Theorem 3.8 representing quantum E_6 as the symmetries of the unique (up to scaling) cubic form on the 27-dimensional representation. Note that for the classical groups $SL(n), SO(n), Sp(2n)$, there is a direct argument in [DEN, Theorem 4.6] giving an even stronger statement, with a description of the set of forbidden values of q .

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