Tchebycheffian B-Splines Revisited

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Abstract Tchebycheffian splines are smooth piecewise functions where the different pieces are drawn from extended Tchebycheff spaces. They are a natural generalization of polynomial splines and can be represented in terms of an interesting set of basis functions, the so-called *Tchebycheffian B-splines*, which generalize the standard polynomial B-splines. We provide an accessible and self-contained exposition of Tchebycheffian B-splines and their main properties. Our construction is based on an integral recurrence relation and allows for the use of different extended Tchebycheff spaces on different intervals. The special class of *generalized B-splines* is also discussed in detail.

1 Introduction

Extended Tchebycheff (ET-) spaces are natural generalizations of algebraic polynomial spaces [13, 27]. Any nontrivial element of an ET-space of dimension $p+1$ has at most p zeros counting multiplicity. Extended complete Tchebycheff (ECT-) spaces are an important subclass that can be generated through a set of positive weight functions [23, 27] and allow for defining generalized power functions [17]. Relevant examples are nullspaces of linear differential operators on suitable intervals [9, 27].

Similarly to the polynomial spline case, Tchebycheffian splines are smooth piecewise functions whose pieces are drawn from ET-spaces [24, 27]. They share many properties with the classical polynomial splines but also offer a more flexible frame-

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work, due to the wide variety of ET-spaces. As it is difficult to trace all the works on Tchebycheffian splines, we refer the reader to [27] for an extended bibliography on the topic. Multivariate extensions of Tchebycheffian splines can be easily obtained via (local) tensor-product structures [5, 6]. Besides their theoretical interest, Tchebycheffian splines have application in several branches of the sciences, including geometric modeling and numerical simulations; see, e.g., [19, 20, 21, 28].

Most of the results in the polynomial case extend in a natural way to the Tchebycheffian setting. In particular, Tchebycheffian splines admit a representation in terms of basis functions, called Tchebycheffian B-splines, with similar properties to polynomial B-splines. Tchebycheffian B-splines were introduced in 1968 by Karlin [12] using generalized divided differences. We refer the reader to the historical notes in [27, Chapters 9 and 11] for further details. There are several other ways to define them, including Hermite interpolation [7, 26], de Boor-like recurrence relations [10, 17], integral recurrence relations [4], and blossoming [24]. Each of these definitions has advantages according to the problem one has to face or to the properties to be proved. All these constructions lead to the same Tchebycheffian B-splines, up to a proper scaling.

This paper aims to provide a self-contained exposition of Tchebycheffian B-splines and their main properties, which are often scattered, and sometimes hidden, in the literature. Our construction of Tchebycheffian B-splines is based on an integral recurrence relation and allows for the use of different ET-spaces on different intervals in order to be able to completely exploit the rich variety of ET-spaces, as often required in applications. This general piecewise structure, under certain constraints on the different ET-spaces and/or on the length of the considered intervals, is sometimes referred to in the literature as piecewise Tchebycheffian B-splines [25]. Although the construction and the properties we present are already known, the corresponding proofs –just based on elementary calculus– are largely new, resulting in an accessible, homogeneous, and original overview. On the other hand, due to space limitation, some properties of Tchebycheffian B-splines are not treated in this short overview. In particular, we do not discuss the Marsden identity, dual functionals and construction of quasi-interpolants; see, e.g., [1, 27].

Our presentation of the Tchebycheffian B-spline setting strongly relies on properties of ECT-spaces related to the generating weight functions. On the other hand, any ET-space on a bounded and closed interval is an ECT-space (see [22, Corollary 2.12] and [24]), and therefore it is also equipped with weights functions. In view of this important result, since only closed and bounded intervals are of interest to define spline spaces, we could avoid mentioning the concept of ECT-spaces for constructing Tchebycheffian B-splines, as it is sometimes the case in the literature (see, e.g., [24]). However, for the sake of completeness and clarity, we prefer to present the material in terms of ECT-spaces (similar to, e.g., [3, 27]).

The remainder of the paper is divided in three sections. Section 2 introduces ET-spaces and ECT-spaces. It also summarizes some of their properties to be used in the construction of Tchebycheffian B-splines. Section 3 contains the core of the paper: it defines Tchebycheffian B-splines through an integral recurrence relation and proves some of their main properties including non-negativity, smoothness, and knot insertion. Section 4 concludes the paper by discussing an interesting special class of Tchebycheffian B-splines, the so-called generalized B-splines.

2 Extended Tchebycheff Spaces

In this section we introduce spaces that are a natural generalization of algebraic polynomial spaces, the so-called extended Tchebycheff spaces. In particular, we mainly focus on the subclass of extended complete Tchebycheff spaces. Such spaces can be spanned by a set of basis functions that are a natural generalization of the polynomial power basis.

2.1 Definition and Basic Properties

Suppose we have a $(p+1)$-dimensional subspace $\mathbb{U}_p(I)$ of $C^p(I)$ where I is a real interval. A **Hermite interpolation problem** in $\mathbb{U}_p(I)$ consists of finding an element $g \in \mathbb{U}_p(I)$ satisfying the following conditions:

$$D^j g(z_i) = f_{i,j}, \quad j = 0, \ldots, m_i - 1, \quad i = 0, \ldots, \ell, \tag{1}$$

where $z_0, \ldots, z_\ell$ are distinct points in I, and m_i are integers such that $\sum_{i=0}^{\ell} m_i = p+1$, and $f_{i,j} \in \mathbb{R}$. We now define extended Tchebycheff spaces [1] on a real interval I.

Definition 1. *Let I be an interval of the real line. Given an integer $p \geq 0$, a space $\mathbb{T}_p(I) \subset C^p(I)$ of dimension $p+1$ is an* **extended Tchebycheff (ET-) space** *on I if any Hermite interpolation problem with $p+1$ data on I has a unique solution in $\mathbb{T}_p(I)$.*

The definition immediately implies that a $(p+1)$-dimensional subspace of $C^p(I)$ is an ET-space on I if and only if any nontrivial element of the space has at most p zeros in I counting multiplicity. Moreover, any ET-space on I is an ET-space of the same dimension on any nontrivial subinterval of I.

Example 1. The space $\mathbb{P}_p := \langle 1, x, \ldots, x^p \rangle$ of algebraic polynomials of degree less than or equal to p is an ET-space on the real line.

Example 2. The space $\langle \cos(x), \sin(x) \rangle$ is an ET-space on any interval $[a, a+\pi)$ with $a \in \mathbb{R}$. Indeed, the equation $c_1 \cos(x) + c_2 \sin(x) = 0$ has exactly one solution in the considered interval for any fixed c_1, c_2 not both equal to zero. On the other hand, on any interval $[a, a+\pi]$ or larger, this space is not an ET-space anymore.

We now focus on a special subclass of ET-spaces.

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[1] The space $\mathbb{T}_p(I)$ is called a **Tchebycheff (T-) space** if any solution of (1) with $m_0 = \cdots = m_p = 1$ is unique in $\mathbb{T}_p(I)$. In such a case, (1) is a **Lagrange interpolation problem**.

Definition 2. *Let I be an interval of the real line. Given an integer $p \geq 0$, the space $\mathbb{T}_p(I) \subset C^p(I)$ of dimension $p+1$ is an* **extended complete Tchebycheff (ECT-) space** *if there exists a basis $\{u_0, \ldots, u_p\}$ of $\mathbb{T}_p(I)$ such that every subspace $\langle u_0, \ldots, u_k \rangle$ is an ET-space on I for $k = 0, \ldots, p$. The basis $\{u_0, \ldots, u_p\}$ is called an* **ECT-system**.

Example 3. Taking $u_k(x) = x^k$, $k = 0, \ldots, p$, we see from Example 1 that the space $\mathbb{P}_p$ is an ECT-space on any interval of the real line.

Example 4. An ECT-space is clearly an ET-space, but the converse is not always true. It is sufficient to consider the space $\langle \cos(x), \sin(x) \rangle$. This is an ET-space on $[0, \pi)$, see Example 2, but not an ECT-space on $[0, \pi)$. However, the space is an ECT-space on $(0, \pi)$.

The next theorem shows that the classes of ECT-spaces and ET-spaces coincide in a very important case; see [22, Corollary 2.12] and [24] for details.

Theorem 1. *If I is a bounded closed interval, then any ET-space on I is an ECT-space on I.*

In the following we will provide a characterization of an ECT-space in terms of Wronskians. The **Wronskian** of $k+1$ functions $\{u_0, \ldots, u_k\}$ of class $C^k(I)$ is given by the determinant

$$W[u_0, \ldots, u_k](x) := \det\bigl(D^i u_j(x)\bigr)_{i,j=0}^{k}.$$

If there exists a point $\bar{x} \in I$ such that $W[u_0, \ldots, u_k](\bar{x}) \neq 0$, then the functions $\{u_0, \ldots, u_k\}$ are linearly independent. Wronskians can be used to characterize an ECT-space as follows; see [27, Theorem 9.1] for a proof.

Theorem 2. *A $(p+1)$-dimensional subspace of $C^p(I)$ is an ECT-space on I if and only if there exists a basis $\{u_0, \ldots, u_p\}$ such that all the Wronskians are positive; more precisely,*

$$W[u_0, \ldots, u_k](x) > 0, \quad k = 0, \ldots, p, \quad x \in I.$$

Note that the basis $\{u_0, \ldots, u_p\}$ in Theorem 2 is an ECT-system.

Example 5. In Example 3 we have shown that the space $\mathbb{P}_p$ is an ECT-space using the set $\{1, x, \ldots, x^p\}$. More generally, $\mathbb{P}_p$ can be seen as the span of the **power basis**

$$\left\{1, x-y, \frac{(x-y)^2}{2}, \ldots, \frac{(x-y)^p}{p!}\right\}, \tag{2}$$

for any fixed $y \in \mathbb{R}$. Indeed, the Wronskians of this set of functions are all equal to one.

Example 6. Theorem 2 gives a characterization of ECT-spaces in terms of Wronskians, but there is no similar characterization for ET-spaces. If $\langle u_0, \ldots, u_p \rangle$ is an ET-space on I, then the Wronskian $W[u_0, \ldots, u_p](x)$ does not change sign on I. However, the converse does not hold. It is sufficient to consider the space $\langle \cos(x), \sin(x) \rangle$. This is an ET-space only on intervals of the form $[a, a+\pi)$ or subintervals (see Example 2), but $W[\cos, \sin](x) = 1$ for all $x \in \mathbb{R}$.

2.2 Generalized Powers

In this section we introduce special functions that can be regarded as a generalization of the power basis in (2).

Definition 3. *Let $(v_1,\dots,v_p)$ be a vector of continuous functions on an interval I. For a nonnegative integer p and the points x,y in I, we define* **repeated integrals** *by*

$$G_p[v_1,\dots,v_p](x,y) := \int_y^x v_1(t)G_{p-1}[v_2,\dots,v_p](t,y)\,\mathrm{d}t, \quad p\geq 1, \tag{3}$$

starting with $G_0(x,y) := 1$.

From the definition we obtain

$$G_1[v_1](x,y) = \int_y^x v_1(t)\,\mathrm{d}t,$$

and for $p>1$,

$$G_p[v_1,\dots,v_p](x,y) = \int_y^x v_1(t_1)\int_y^{t_1} v_2(t_2)\cdots\int_y^{t_{p-1}} v_p(t_p)\,\mathrm{d}t_p\cdots\mathrm{d}t_1.$$

We only list two basic properties of repeated integrals; for proofs and further properties we refer the reader to [17].

- **Diagonal Property.** Let $v_j\in C^{p-1-j}(I)$, $j=1,\dots,p$. For any $y\in I$ we have

$$\frac{\partial^r}{\partial x^r}G_p[v_1,\dots,v_p](x,y)_{|x=y} = 0, \quad r=0,\dots,p-1. \tag{4}$$

- **Generalized Binomial Formula.** For any $x,y,c\in I$ we have

$$G_p[v_1,\dots,v_p](x,y) = \sum_{j=0}^{p}(-1)^{p-j}G_j[v_1,\dots,v_j](x,c)G_{p-j}[v_p,\dots,v_{j+1}](y,c). \tag{5}$$

We are now ready to define a generalization of the classical power basis in (2).

Definition 4. *Let $\boldsymbol{w} := (w_0,\dots,w_q)$ be a vector of continuous functions on an interval I. For a nonnegative integer $p\leq q$ and a fixed point y in I, we define the* **generalized powers** *by*

$$\begin{aligned}
u^{\boldsymbol{w}}_{0,p}(x,y) &:= w_p(x),\\
u^{\boldsymbol{w}}_{1,p}(x,y) &:= w_p(x)G_1[w_{p-1}](x,y),\\
u^{\boldsymbol{w}}_{2,p}(x,y) &:= w_p(x)G_2[w_{p-1},w_{p-2}](x,y),\\
&\;\;\vdots\\
u^{\boldsymbol{w}}_{p,p}(x,y) &:= w_p(x)G_p[w_{p-1},\dots,w_0](x,y),
\end{aligned} \tag{6}$$

with G_j the repeated integrals in (3).

We immediately obtain the following properties.

- **Recurrence Formula.** From the definition (3) of repeated integrals we deduce

$$u^{\boldsymbol{w}}_{j,p}(x,y) = w_p(x)\int_y^x u^{\boldsymbol{w}}_{j-1,p-1}(t,y)\,\mathrm{d}t, \quad j=1,\dots,p, \quad x,y\in I. \tag{7}$$

- **Smoothness.** If $w_j \in C^j(I)$, $j=0,\dots,p$, then

$$u^{\boldsymbol{w}}_{j,p} \in C^p(I), \quad j=0,\dots,p. \tag{8}$$

This follows from (7).

Example 7. If $w_{p-j} = \cdots = w_p = 1$ then

$$u^{\boldsymbol{w}}_{j,p}(x,y) = w_p(x)G_j[w_{p-1},\dots,w_{p-j}](x,y) = \frac{(x-y)^j}{j!}. \tag{9}$$

In this case, (5) takes the form

$$\frac{(x-y)^p}{p!} = \sum_{j=0}^p (-1)^{p-j}\frac{(x-c)^j}{j!}\frac{(y-c)^{p-j}}{(p-j)!} = \frac{\big((x-c)-(y-c)\big)^p}{p!}.$$

The next theorem shows that the Wronskians of generalized powers can be expressed in a simple form; see [12, page 278].

Theorem 3. *Let $\boldsymbol{w} := (w_0,\dots,w_p)$ be a vector of weight functions on an interval I such that $w_j \in C^j(I)$, $j=0,\dots,p$. For any $x,y\in I$ and $0\le k\le p$ we have*

$$W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y)](x) = w_p^{k+1}(x)w_{p-1}^k(x)\cdots w_{p-k}(x).$$

Theorem 3 leads to the following properties.

- **Linear Independence.** Suppose there exists a point $\bar{x}\in I$ such that

$$w_p(\bar{x})\cdots w_{p-k}(\bar{x}) \neq 0.$$

Then, $W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y)](\bar{x}) \neq 0$ and, as a consequence, the generalized powers $\{u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y)\}$ are linearly independent for any $y\in I$.

- **Weight Functions.** Let $\boldsymbol{w} := (w_0,\dots,w_p)$ be a vector of positive weight functions on an interval I such that $w_j\in C^j(I)$, $j=0,\dots,p$. Then, for any $x,y\in I$ the Wronskians of the generalized powers are positive, and we have

$$w_p(x) = u^{\boldsymbol{w}}_{0,p}(x,y), \quad w_{p-1}(x) = \frac{W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),u^{\boldsymbol{w}}_{1,p}(\cdot,y)](x)}{\big(u^{\boldsymbol{w}}_{0,p}(x,y)\big)^2},$$

and for $2\le k\le p$,

$$w_{p-k}(x) = \frac{W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y)](x)\,W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k-2,p}(\cdot,y)](x)}{\big(W[u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{k-1,p}(\cdot,y)](x)\big)^2}.$$

In the following, we discuss spaces spanned by generalized powers. The generalized binomial formula (5) and definition (6) imply

$$\langle u^{\boldsymbol{w}}_{0,p}(\cdot,y_1),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y_1)\rangle = \langle u^{\boldsymbol{w}}_{0,p}(\cdot,y_2),\dots,u^{\boldsymbol{w}}_{k,p}(\cdot,y_2)\rangle, \quad y_1,y_2\in I,$$

for $k=0,\dots,p$. This observation leads to the following well-posed definition.

Definition 5. *Let $\boldsymbol{w} := (w_0,\dots,w_q)$ be a vector of positive weight functions on an interval I such that $w_j\in C^j(I)$, $j=0,\dots,q$. For a nonnegative integer $p\le q$, we define the space $\mathbb{T}^{\boldsymbol{w}}_p(I)$ on the interval I generated by the weight vector $\boldsymbol{w}$ by*

$$\mathbb{T}^{\boldsymbol{w}}_p(I) := \langle u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{p,p}(\cdot,y)\rangle, \tag{10}$$

where $u^{\boldsymbol{w}}_j(\cdot,y)$, $j=0,\dots,p$ are given in (6) *and y is any fixed point in I.*

It is clear that the space $\mathbb{T}^{\boldsymbol{w}}_p(I)$ in (10) only depends on $w_0,\dots,w_p$.

Example 8. From Example 7 we see that if $w_0=\dots=w_p=1$ then

$$\mathbb{T}^{\boldsymbol{w}}_p(\mathbb{R}) = \left\langle 1, x-y, \frac{(x-y)^2}{2},\dots,\frac{(x-y)^p}{p!}\right\rangle = \mathbb{P}_p,$$

for any fixed $y\in\mathbb{R}$.

We now show that generalized powers defined in terms of positive weight functions span an ECT-space.

Theorem 4. *Let $\boldsymbol{w} := (w_0,\dots,w_p)$ be a vector of positive weight functions on an interval I such that $w_j\in C^j(I)$, $j=0,\dots,p$. The space $\mathbb{T}^{\boldsymbol{w}}_p(I)$ is an ECT-space of dimension $p+1$ on the interval I. In particular, the generalized powers $\{u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{p,p}(\cdot,y)\}$ are linearly independent.*

Proof. By recalling (8) we see that $\mathbb{T}^{\boldsymbol{w}}_p(I)\subset C^p(I)$. From Theorem 3 it follows that all the Wronskians of the generalized powers $\{u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{p,p}(\cdot,y)\}$ are positive on I, and in particular that $\{u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{p,p}(\cdot,y)\}$ forms a basis for $\mathbb{T}^{\boldsymbol{w}}_p(I)$. Theorem 2 completes the proof. □

From Theorem 2 we can assume that any ECT-space $\mathbb{T}_p(I)$ is spanned by an ECT-system with positive Wronskians. The next theorem shows that $\mathbb{T}_p(I)$ is spanned by generalized powers associated with certain positive weight functions. The proof can be deduced from [12, proof of Theorem 1.1, page 276].

Theorem 5. *Let $\mathbb{T}_p(I)$ be any ECT-space of dimension $p+1$ on the interval I, and let $\{u_0,\dots,u_p\}$ be an ECT-system with positive Wronskians spanning $\mathbb{T}_p(I)$. We define the positive weight functions*

$$w_p(x) := u_0(x), \quad w_{p-1}(x) := \frac{W[u_0,u_1](x)}{(u_0(x))^2},$$
$$w_{p-k}(x) := \frac{W[u_0,\dots,u_k](x)W[u_0,\dots,u_{k-2}](x)}{\big(W[u_0,\dots,u_{k-1}](x)\big)^2}, \quad k=2,\dots,p.$$

Then, the corresponding generalized powers $\{u^{\boldsymbol{w}}_{0,p}(\cdot,y),\dots,u^{\boldsymbol{w}}_{p,p}(\cdot,y)\}$ form a basis for $\mathbb{T}_p(I)$ for any $y\in I$.

Example 9. There exist different weight vectors generating the same ECT-space. Obviously, the weight vector $(c_0w_0,\dots,c_pw_p)$ generates the same space as the weight vector $(w_0,\dots,w_p)$ for any positive constants c_j, $j=0,\dots,p$. A less trivial case is illustrated in Example 10. A nice construction of all possible weight vectors generating the same ECT-space is given in [23].

Example 10. The space $\langle 1,\cos(x),\sin(x)\rangle$ is an ECT-space of dimension 3 on $(-\pi/2,\pi/2)$. This can be shown by considering the weight vector $\boldsymbol{w}=(w_0,w_1,w_2)$ with

$$w_0(x)=\frac{1}{\cos^2(x)},\quad w_1(x)=\cos(x),\quad w_2(x)=1.$$

Then, the corresponding generalized powers with $y=0$ are

$$\begin{aligned}
u^{\boldsymbol{w}}_{0,2}(x,0)&=w_2(x)=1,\\
u^{\boldsymbol{w}}_{1,2}(x,0)&=\int_0^x\cos(t)\,\mathrm{d}t=\sin(x),\\
u^{\boldsymbol{w}}_{2,2}(x,0)&=\int_0^x\cos(t_1)\int_0^{t_1}\frac{1}{\cos^2(t_2)}\,\mathrm{d}t_2\mathrm{d}t_1=\int_0^x\sin(t_1)\,\mathrm{d}t_1=1-\cos(x).
\end{aligned}$$

Actually, with some additional effort we can prove that $\langle 1,\cos(x),\sin(x)\rangle$ is an ECT-space of dimension 3 on $(-\pi,\pi)$. To this end, consider the weight vector $\boldsymbol{w}=(w_0,w_1,w_2)$ with

$$w_0(x)=w_1(x)=1\Big/\cos^2\Big(\frac{x}{2}\Big),\quad w_2(x)=\cos^2\Big(\frac{x}{2}\Big).$$

For $x\in(-\pi,\pi)$ and any fixed $y\in(-\pi,\pi)$ we find the following generalized powers:

$$\begin{aligned}
u^{\boldsymbol{w}}_{0,2}(x,y)&=w_2(x)=\cos^2\Big(\frac{x}{2}\Big)=\frac{1+\cos(x)}{2},\\
u^{\boldsymbol{w}}_{1,2}(x,y)&=\sin(x)-(1+\cos(x))\tan\Big(\frac{y}{2}\Big),\\
u^{\boldsymbol{w}}_{2,2}(x,y)&=2\sin^2\Big(\frac{x-y}{2}\Big)\Big/\cos^2\Big(\frac{y}{2}\Big)=\frac{1-\cos(x)\cos(y)-\sin(x)\sin(y)}{\cos^2(y/2)}.
\end{aligned}$$

Example 11. Let $\langle u,v\rangle$ be an ET-space of dimension 2 on $[a,b]$. We can assume without loss of generality that $u(a)=u(b)=1$ and $v(a)=0$, $v(b)\neq 0$. It turns out that u is positive on $[a,b]$ because otherwise it would have two zeros (counting multiplicity) in (a,b). Set

$$w_1(x):=u(x),\quad w_0(x):=D\Big(\frac{v}{u}\Big)(x)=\frac{u(x)Dv(x)-v(x)Du(x)}{(u(x))^2}=\frac{W[u,v](x)}{(u(x))^2}.$$

Since $\langle u,v\rangle$ is an ET-space, the Wronskian $W[u,v]$ does not change sign on $[a,b]$. Hence, we can assume it is positive on $[a,b]$; if this is not the case, we change the sign of v. This implies that w_0 is positive on $[a,b]$. Moreover,

$$w_1(x)\int_a^x w_0(t)\,\mathrm{d}t=w_1(x)\Big(\frac{v(x)}{u(x)}-\frac{v(a)}{u(a)}\Big)=v(x).\tag{11}$$

Therefore, $\langle u,v\rangle$ is the space $\mathbb{T}^{\boldsymbol{w}}_2([a,b])$ with $\boldsymbol{w}=(w_0,w_1)$, and so from Theorem 4 we know it is an ECT-space on $[a,b]$. We conclude that any ET-space of dimension 2 on a closed interval $[a,b]$ is an ECT-space of dimension 2 on $[a,b]$. This is in agreement with Theorem 1. Note that the statement does not hold anymore if the interval is not closed; see Example 4.

Example 12. Let $U,V \in C^p([a,b])$ be given such that $\langle D^{p-1}U, D^{p-1}V\rangle$ is an ET-space on $[a,b]$. Then, the space $\langle 1,x,\dots,x^{p-2},U(x),V(x)\rangle$ for $p \geq 2$ is an ECT-space of dimension $p+1$ on $[a,b]$. Indeed, it is the space $\mathbb{T}_p^{\boldsymbol{w}}([a,b])$ generated by the weight functions

$$w_0(x) = \frac{W[u,v](x)}{(u(x))^2}, \quad w_1(x) = u(x), \quad w_2(x) = \cdots = w_p(x) = 1,$$

where

$$u(x) := c_{0,u}D^{p-1}U(x) + c_{1,u}D^{p-1}V(x), \quad v(x) := c_{0,v}D^{p-1}U(x) + c_{1,v}D^{p-1}V(x),$$

such that $u(a) = u(b) = 1$ and $v(a) = 0$, $v(b) \neq 0$ (see Example 11).

Example 13. Let $\mathcal{L}_p$ be the linear differential operator defined by

$$\mathcal{L}_p f := D^{p+1}f + \sum_{j=0}^{p} a_j D^j f, \quad f \in C^{p+1}(I), \tag{12}$$

where $a_j \in C(I)$ and I is a real interval. Any operator of the form (12) is uniquely identified by its nullspace, denoted by $\mathbb{L}_p$. More details on linear differential operators can be found in [11, Chapter 5]. The nullspace $\mathbb{L}_p$ is an ECT-space on I if and only if there exist positive weight functions $w_j \in C^{j+1}(I)$, $j = 0,\dots,p$ such that

$$\mathcal{L}_p f = w_0 \cdots w_p D_0 \dots D_p f, \tag{13}$$

where

$$D_j f := D\Big(\frac{f}{w_j}\Big), \quad j = 0,\dots,p;$$

see [9, Theorem 2, page 91].

- If the coefficients a_j are equal to zero, then $\mathbb{L}_p = \mathbb{P}_p$ is an ECT-space on the real line; see Example 3.
- If the coefficients a_j are constants and the characteristic polynomial $\lambda^{p+1} + \sum_{j=0}^{p} a_j\lambda^j$ has only real roots, then $\mathbb{L}_p$ is an ECT-space on the real line; see [9, Proposition 16, page 124]. For example, given distinct real numbers $\alpha_0 < \alpha_1 < \cdots < \alpha_p$, the space $\langle \mathrm{e}^{\alpha_0 x}, \mathrm{e}^{\alpha_1 x}, \dots, \mathrm{e}^{\alpha_p x}\rangle$ is the nullspace of $\mathcal{L}_p f = (D-\alpha_0)\cdots(D-\alpha_p)f$, implying it is an ECT-space on the real line.
- If the coefficients a_j are constants and the characteristic polynomial has complex roots, then $\mathbb{L}_p$ is an ECT-space on a suitable interval. For example, the space $\langle 1,x,\dots,x^{p-2},\cos(x),\sin(x)\rangle$ with $p \geq 2$ is the nullspace of $\mathcal{L}_p f = D^{p+1}f + D^{p-1}f$; it is also an ECT-space on the interval $(-\pi/2, \pi/2)$ that can be generated by the weight functions

  $$w_0(x) = \frac{1}{\cos^2(x)}, \quad w_1(x) = \cos(x), \quad w_2(x) = \cdots = w_p(x) = 1;$$

  see Example 10 for the special case $p = 2$. The factorization (13) becomes

  $$\mathcal{L}_p f = \frac{1}{\cos} D\bigg(\cos^2\bigg(D\Big(\frac{1}{\cos}D^{p-1}f\Big)\bigg)\bigg).$$

  Actually, the space $\langle 1,x,\dots,x^{p-2},\cos(x),\sin(x)\rangle$ is an ECT-space on larger intervals whose maximum lengths increase with p; see Example 10 and [8].
- In the general case of nonconstant coefficients $a_j(x)$, $\mathbb{L}_p$ is an ECT-space on a suitable interval. For example, if I is compact, then $\mathbb{L}_p$ is an ECT-space on any subinterval of I of length less than $\min\big(1, \frac{1}{(p+1)M}\big)$, where $M \geq \max_{0\leq j\leq p}\max_{x\in I}|a_j(x)|$; see [9, Proposition 1, page 81].

3 Tchebycheffian B-Splines

In Section 2 we showed that ECT-spaces extend in a very natural way the space of algebraic polynomials. Now, we focus on smooth functions that are defined piecewise in ECT-spaces, and we define the so-called Tchebycheffian B-splines, which are a natural generalization of polynomial B-splines. Since we are interested in bounded and closed intervals, in view of Theorem 1, ET-spaces are ECT-spaces and so they are equipped with weight functions. Therefore, Tchebycheffian B-splines are actually defined piecewise in ET-spaces.

We start by introducing some preliminary notations. A function is called piecewise continuous on a finite interval I if it is bounded and continuous except at a finite number of points, where the value is obtained by taking the limit either from the left or the right. We denote the space of these functions by $C^{-1}(I)$. The right and left limits of a real number x are denoted by

$$x_+ := \lim_{\substack{t\to x\\ t>x}} t, \quad x_- := \lim_{\substack{t\to x\\ t<x}} t, \quad x \in \mathbb{R}.$$

Similarly, we denote right and left derivatives of a function f by

$$D_+ f(x) := \lim_{\substack{h\to 0\\ h>0}} \frac{f(x+h)-f(x)}{h}, \quad D_- f(x) := \lim_{\substack{h\to 0\\ h<0}} \frac{f(x+h)-f(x)}{h},$$

provided that the limits exist at the point $x \in \mathbb{R}$.

3.1 Definition and Basic Properties

In order to define Tchebycheffian B-splines we use the concept of knot sequences. Suppose for integers $n > p \geq 0$ that a knot sequence

$$\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1} = \{\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{n+p+1}\}, \quad n \in \mathbb{N}, \quad p \in \mathbb{N}_0,$$

is given. This allows us to define a set of n Tchebycheffian B-splines of degree p.

Definition 6. *Given a knot sequence $\boldsymbol{\xi}$, the functions $w_0, \ldots, w_p$ are called* **Tchebycheffian B-spline weights** *with respect to $\boldsymbol{\xi}$ if they are positive on $[\xi_1, \xi_{n+p+1}]$ and* [2] *for $j = 0, \ldots, p$,*

$$w_j \in C^j([\xi_i^+, \xi_{i+1}^-]), \quad \xi_i < \xi_{i+1}, \quad i = 1, \ldots, n+p, \tag{14}$$

and

[2] Let f be defined on $[a,b] \supsetneq [c,d]$. The notation $f \in C^j([c^+, d^-])$ means that f is a function of class C^j on the interval $[c,d]$ when considering the right/left limit in the left/right endpoint. Note that, in general, $D_+^r f(c) \neq D_-^r f(c)$ and $D_+^r f(d) \neq D_-^r f(d)$, $r = 0, \ldots, j$.

$$w_j \in C^{\max(j-\mu_i,-1)}(\xi_i), \quad i = 2,\ldots,n+p, \tag{15}$$

where μ_i is the multiplicity of ξ_i in $\boldsymbol{\xi}$.

Theorem 4 implies that the Tchebycheffian B-spline weights $\boldsymbol{w} := (w_0,\ldots,w_p)$ define an E(C)T-space $\mathbb{T}_p^{\boldsymbol{w}}([\xi_i^+,\xi_{i+1}^-])$ of dimension $p+1$ for each $\xi_i < \xi_{i+1}$. We will see that the smoothness at the knots suffices to define Tchebycheffian B-splines with smoothness properties similar to the polynomial B-spline case.

Example 14. For $i = 1,\ldots,n+1$, let $\langle u_i, v_i\rangle$ be ET-spaces of dimension 2 on $[\xi_i,\xi_{i+1}]$, where

$$u_i, v_i \in C^1([\xi_i,\xi_{i+1}]), \quad u_i(\xi_i) = u_i(\xi_{i+1}) = 1, \quad v_i(\xi_i) = 0,\ v_i(\xi_{i+1}) = 1,$$

and, according to Example 11, we can define the local weights

$$w_{0,i}(x) := \frac{W[u_i,v_i](x)}{(u_i(x))^2}, \quad w_{1,i}(x) := u_i(x), \quad x \in [\xi_i,\xi_{i+1}].$$

The global weights

$$w_0(x) := w_{0,i}(x), \quad w_1(x) := w_{1,i}(x), \quad x \in [\xi_i,\xi_{i+1}), \quad i = 1,\ldots,n+1$$

satisfy (14) and (15) for $p = 1$. Hence, w_0, w_1 are Tchebycheffian B-spline weights with respect to $\boldsymbol{\xi}$, and they generate the given ET-spaces on each (nontrivial) interval $[\xi_i^+,\xi_{i+1}^-]$.

Definition 7. *Let $\boldsymbol{w} := (w_0,\ldots,w_q)$ be a vector of Tchebycheffian B-spline weights with respect to a knot sequence $\boldsymbol{\xi}$. Suppose for a nonnegative integer $p \le q$ and some integer j that $\xi_j \le \xi_{j+1} \le \cdots \le \xi_{j+p+1}$ are $p+2$ real numbers taken from $\boldsymbol{\xi}$. The j-th* **Tchebycheffian B-spline** *$B_{j,p,\boldsymbol{\xi}}^{\boldsymbol{w}} : \mathbb{R} \to \mathbb{R}$ of degree p is identically zero if $\xi_{j+p+1} = \xi_j$ and otherwise defined recursively by*

$$B_{j,p,\boldsymbol{\xi}}^{\boldsymbol{w}}(x) := w_p(x)\left(\int_{\xi_j}^x \frac{B_{j,p-1,\boldsymbol{\xi}}^{\boldsymbol{w}}(y)}{\gamma_{j,p-1,\boldsymbol{\xi}}^{\boldsymbol{w}}}\,\mathrm{d}y - \int_{\xi_{j+1}}^x \frac{B_{j+1,p-1,\boldsymbol{\xi}}^{\boldsymbol{w}}(y)}{\gamma_{j+1,p-1,\boldsymbol{\xi}}^{\boldsymbol{w}}}\,\mathrm{d}y\right), \tag{16}$$

starting with

$$B_{i,0,\boldsymbol{\xi}}^{\boldsymbol{w}}(x) := \begin{cases} w_0(x), & \text{if } x \in [\xi_i,\xi_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Here, $\gamma_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}}$ is defined as the integral of $B_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}}$,

$$\gamma_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}} := \int_{\xi_i}^{\xi_{i+k+1}} B_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}}(y)\,\mathrm{d}y, \tag{18}$$

and we used the convention that if $\gamma_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}} = 0$ then

$$\int_{\xi_i}^x \frac{B_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}}(y)}{\gamma_{i,k,\boldsymbol{\xi}}^{\boldsymbol{w}}}\,\mathrm{d}y := \begin{cases} 1, & \text{if } x \ge \xi_{i+k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

The Tchebycheffian B-spline $B_{j,p,\boldsymbol{\xi}}^{\boldsymbol{w}}$ is called **normalized** *when $w_p = 1$.*

In order to stress the similarity with the polynomial B-spline case,[3] the term **degree** p is used in Definition 7 to refer to the dimension $p+1$ of the underlying E(C)T-space. We also use the terms **linear** in case of $p=1$, **quadratic** in case of $p=2$, and so on. Furthermore, we use the notation

$$B[\xi_j,\ldots,\xi_{j+p+1};w_0,\ldots,w_p] := B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}},$$

showing explicitly on which knots and weight functions the Tchebycheffian B-spline depends.

Definition 7 allows for the construction of Tchebycheffian B-splines where the different pieces are drawn from different ET-spaces. This can be done provided that we are able to construct Tchebycheffian B-spline weights as in Definition 6 which identify on each interval the desired ET-space. Example 14 shows that this is always the case if we consider ET-spaces of dimension two. In view of Example 12, this paves the path for the construction of the so-called generalized B-splines which will be discussed in Section 4. A more general setting can be addressed by relying on the elegant constructive procedure for finding all weight vectors associated with a given ET-space in a bounded closed interval presented in [23]. In particular, this procedure, which is based on the properties of Tchebycheffian Bernstein functions, has been exploited in [25] to construct Tchebycheffian B-splines with pieces belonging to ET-spaces of dimension four (see also [2]).

Example 15. The **linear Tchebycheffian B-spline** is given by

$$B^{\boldsymbol{w}}_{j,1,\boldsymbol{\xi}}(x) = B[\xi_j,\xi_{j+1},\xi_{j+2};w_0,w_1](x) = \begin{cases} w_1(x)\dfrac{\int_{\xi_j}^{x} w_0(y)\,\mathrm{d}y}{\int_{\xi_j}^{\xi_{j+1}} w_0(y)\,\mathrm{d}y}, & \text{if } x\in[\xi_j,\xi_{j+1}), \\ w_1(x)\dfrac{\int_{x}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}{\int_{\xi_{j+1}}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}, & \text{if } x\in[\xi_{j+1},\xi_{j+2}), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, when $w_0(x)=1/\cos^2(x)$, $w_1(x)=\cos(x)$ and $-\pi/2<\xi_j<\xi_{j+2}<\pi/2$, we have

$$B[\xi_j,\xi_{j+1},\xi_{j+2};w_0,w_1](x) = \begin{cases} \dfrac{\sin(x-\xi_j)\cos(\xi_{j+1})}{\sin(\xi_{j+1}-\xi_j)}, & \text{if } x\in[\xi_j,\xi_{j+1}), \\ \dfrac{\sin(\xi_{j+2}-x)\cos(\xi_{j+1})}{\sin(\xi_{j+2}-\xi_{j+1})}, & \text{if } x\in[\xi_{j+1},\xi_{j+2}), \\ 0, & \text{otherwise.} \end{cases}$$

All spline pieces belong to the trigonometric space $\langle\cos(x),\sin(x)\rangle$. This is in agreement with Example 10. This function is discontinuous at a double knot and continuous at a simple knot.

Example 16. The **quadratic (normalized) Tchebycheffian B-spline** defined on the uniform knot sequence $\{i\omega\}_{i=0}^{3}$ with $\omega<\pi$, and generated by the weight functions

[3] Our Tchebycheffian B-spline construction follows the approach of [3, 4], while it differs from [24] in two ways: the indexing of the weight functions and the positioning of the weight functions with respect to the integration. This provides a more intuitive notation.

$$w_0(x) = \frac{1}{(w_1(x))^2}, \quad w_1(x) = \frac{\cos(x-(i+1)\omega/2)}{\cos(\omega/2)}, \quad w_2(x) = 1, \quad x \in [i\omega, (i+1)\omega), \tag{20}$$

is given by

$$B[0,\omega,2\omega,3\omega;w_0,w_1,1](x) = \begin{cases} \dfrac{1-\cos(x)}{2(1-\cos(\omega))}, & \text{if } x \in [0,\omega), \\ \dfrac{\cos(2\omega-x)+\cos(x-\omega)-2\cos(\omega)}{2(1-\cos(\omega))}, & \text{if } x \in [\omega,2\omega), \\ \dfrac{1-\cos(3\omega-x)}{2(1-\cos(\omega))}, & \text{if } x \in [2\omega,3\omega), \\ 0, & \text{otherwise.} \end{cases}$$

All spline pieces belong to the trigonometric space $\langle 1,\cos(x),\sin(x)\rangle$. The knots are simple and it can be verified that the function is continuous with a continuous first derivative for all $x \in \mathbb{R}$. Observe that, to obtain quadratic Tchebycheffian B-splines with pieces belonging to $\langle 1,\cos(x),\sin(x)\rangle$, we could have used the simpler weight functions $w_0(x) = 1/\cos^2(x)$, $w_1(x) = \cos(x)$, $w_2(x) = 1$ instead of (20). However, this choice results in the restriction $\omega < \pi/6$ to ensure positivity of the weight functions on the interval $[0,3\omega]$.

Let χ_i denote the characteristic function on the interval $[\xi_i, \xi_{i+1})$. The general explicit expression for a Tchebycheffian B-spline is quite complicated. Applying the recurrence relation in Definition 7 repeatedly we find

$$B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = \sum_{i=j}^{j+p} B^{\boldsymbol{w},\{i\}}_{j,p,\boldsymbol{\xi}}(x)\chi_i(x), \quad p \geq 0, \tag{21}$$

where $B^{\boldsymbol{w},\{i\}}_{j,p,\boldsymbol{\xi}}$ is defined on the interval $[\xi_i, \xi_{i+1})$ as the restriction of $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ to that interval, and it is assumed to be zero if $\xi_i = \xi_{i+1}$. In particular, for the nontrivial cases we have

$$B^{\boldsymbol{w},\{j\}}_{j,0,\boldsymbol{\xi}}(x) = w_0(x), \quad B^{\boldsymbol{w},\{j\}}_{j,1,\boldsymbol{\xi}}(x) = w_1(x)\frac{\int_{\xi_j}^{x} w_0(y)\,\mathrm{d}y}{\int_{\xi_j}^{\xi_{j+1}} w_0(y)\,\mathrm{d}y},$$
$$B^{\boldsymbol{w},\{j+1\}}_{j,1,\boldsymbol{\xi}}(x) = w_1(x)\frac{\int_{x}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}{\int_{\xi_{j+1}}^{\xi_{j+2}} w_0(y)\,\mathrm{d}y}.$$

For $p \geq 1$, in the nontrivial cases, it follows that the first and last piece are given by

$$\begin{aligned} B^{\boldsymbol{w},\{j\}}_{j,p,\boldsymbol{\xi}}(x) &= w_p(x)G_p[w_{p-1},\dots,w_0](x,\xi_j) \Big/ \prod_{i=1}^{p} \gamma^{\boldsymbol{w}}_{j,i-1,\boldsymbol{\xi}}, \\ B^{\boldsymbol{w},\{j+p\}}_{j,p,\boldsymbol{\xi}}(x) &= w_p(x)G_p[w_{p-1},\dots,w_0](\xi_{j+p+1},x) \Big/ \prod_{i=1}^{p} \gamma^{\boldsymbol{w}}_{j+p-i+1,i-1,\boldsymbol{\xi}}, \end{aligned} \tag{22}$$

where G_p is defined in (3). If $\xi_j < \xi_{j+1} = \xi_{j+p+1}$ then (22) simplifies to

$$B^{\boldsymbol{w},\{j\}}_{j,p,\boldsymbol{\xi}}(x) = w_p(x)\frac{G_p[w_{p-1},\dots,w_0](x,\xi_j)}{G_p[w_{p-1},\dots,w_0](\xi_{j+1},\xi_j)}, \tag{23}$$

and if $\xi_j = \xi_{j+p} < \xi_{j+p+1}$ then

$$B^{\boldsymbol{w},\{j+p\}}_{j,p,\boldsymbol{\xi}}(x) = w_p(x)\frac{G_p[w_{p-1},\dots,w_0](\xi_{j+p+1},x)}{G_p[w_{p-1},\dots,w_0](\xi_{j+p+1},\xi_{j+p})}. \tag{24}$$

In the following, we list some basic properties of Tchebycheffian B-splines that can be directly derived from Definition 7.

- **Local Support.** A Tchebycheffian B-spline is locally supported on the interval given by the extreme knots used in its definition. More precisely,

$$B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = 0, \quad x \notin [\xi_j, \xi_{j+p+1}). \tag{25}$$

  This can be proved using induction on the recurrence relation (16).

- **Piecewise Structure.** A Tchebycheffian B-spline has a piecewise Tchebycheff structure, i.e.,

$$B^{\boldsymbol{w},\{m\}}_{j,p,\boldsymbol{\xi}} \in \mathbb{T}^{\boldsymbol{w}}_p([\xi_m, \xi_{m+1})), \quad m = j,\dots,j+p. \tag{26}$$

  Proof. We proceed by induction on p. Clearly, the case $p = 0$ holds by the definition in (17). Suppose (26) holds for degree $p-1$. By (16) the function $B^{\boldsymbol{w},\{m\}}_{j,p,\boldsymbol{\xi}}(x)$ for $x \in [\xi_m, \xi_{m+1})$ is a linear combination of

$$w_p(x)\int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y, \quad w_p(x)\int_{\xi_{j+1}}^{x} \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y,$$

  where we recall

$$\int_{\xi_i}^{x} \frac{B^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y = 1, \quad x \geq \xi_{i+p}.$$

  The result immediately follows from the induction hypothesis, the recurrence relation (7) and Definition 5. □

- **Local Partition of Unity.** The sum of the Tchebycheffian B-splines of degree p is given by

$$\sum_{j=m-p}^{m} B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = w_p(x), \quad x \in [\xi_m, \xi_{m+1}), \quad p+1 \leq m \leq n. \tag{27}$$

  In particular, for normalized Tchebycheffian B-splines this relation simplifies to

$$\sum_{j=m-p}^{m} B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = 1, \quad x \in [\xi_m, \xi_{m+1}), \quad p+1 \leq m \leq n. \tag{28}$$

Proof. For $p = 0$, the relation (27) follows from (17). For $p \geq 1$, we obtain from (16), (25) and (19) that

$$\begin{aligned}
\sum_{j=m-p}^{m} B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) &= w_p(x) \sum_{j=m-p}^{m} \left(\int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y - \int_{\xi_{j+1}}^{x} \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y \right) \\
&= w_p(x) \left(1 + \sum_{j=m-p+1}^{m} \int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y - \sum_{j=m-p}^{m-1} \int_{\xi_{j+1}}^{x} \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y \right) \\
&= w_p(x).
\end{aligned}$$

In case of normalized Tchebycheffian B-splines we have $w_p = 1$. □

- **Differentiation.** The derivative of a Tchebycheffian B-spline can be simply expressed in terms of two consecutive Tchebycheffian B-splines of lower degree as

$$D_+ \left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x)}{w_p(x)} \right) = \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}, \quad p \geq 1, \tag{29}$$

where fractions with zero denominator have value zero. In particular, for normalized Tchebycheffian B-splines the relation simplifies to

$$D_+ B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}, \quad p \geq 1. \tag{30}$$

Example 17. The standard **polynomial B-splines** of degree p (see, e.g., [18, Definition 2]) are normalized Tchebycheffian B-splines of degree p generated by the weights $w_0 = \cdots = w_p = 1$ and defined on the same knot sequence $\boldsymbol{\xi}$. This is in agreement with Example 8 showing that $\mathbb{T}^{\boldsymbol{w}}_p([a,b]) = \mathbb{P}_p([a,b])$ when $w_0 = \cdots = w_p = 1$.

Example 18. Let $\boldsymbol{w} := (w_0, \ldots, w_p)$ be a vector of positive functions on the interval $[a,b]$ such that $w_j \in C^j([a,b])$, $j = 0, \ldots, p$, and consider the knot sequence

$$\boldsymbol{\xi} := \{a =: \xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} = \cdots = \xi_{2p+2} := b\},$$

which consists of only two different knots (a and b) but both of multiplicity $p+1$. Then, for $p \geq 1$ the functions in Definition 7 are given by

$$\begin{aligned}
B^{\boldsymbol{w}}_{1,p,\boldsymbol{\xi}}(x) &= w_p(x) \left(1 - \int_a^x \frac{B^{\boldsymbol{w}}_{2,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{2,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y \right), \\
B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) &= w_p(x) \left(\int_a^x \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y), \boldsymbol{\xi}}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y - \int_a^x \frac{B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y \right), \quad 2 \leq j \leq p, \\
B^{\boldsymbol{w}}_{p+1,p,\boldsymbol{\xi}}(x) &= w_p(x) \int_a^x \frac{B^{\boldsymbol{w}}_{p+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{p+1,p-1,\boldsymbol{\xi}}} \,\mathrm{d}y.
\end{aligned}$$

These functions are called **Tchebycheffian Bernstein functions** of degree p and span the ECT-space $\mathbb{T}^{\boldsymbol{w}}_p([a,b])$ of dimension $p+1$. They reduce to the standard **Bernstein polynomials** of degree p when $w_0 = \cdots = w_p = 1$.

3.2 Further properties of Tchebycheffian B-Splines

In this section we prove several properties of Tchebycheffian B-splines, in particular nonnegativity, smoothness and local linear independence. The most technical part is to prove that $\gamma^{\boldsymbol{w}}_{i,\ell,\boldsymbol{\xi}} > 0$ whenever the support of the corresponding Tchebycheffian B-spline is nontrivial, i.e., $\xi_i < \xi_{i+\ell+1}$. The construction of $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ requires all the Tchebycheffian B-splines $B^{\boldsymbol{w}}_{i,\ell,\boldsymbol{\xi}}$ for $i = j,\dots,j+p-\ell$ and $\ell = p-1,p-2,\dots,0$; this involves the corresponding $\gamma^{\boldsymbol{w}}_{i,\ell,\boldsymbol{\xi}}$.

We first note that from Definition 7 the function

$$\int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y, \quad p \ge 1$$

is of class $C^0([\xi_1,\xi_{n+p+1}])$ if $\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}} \neq 0$, and of class $C^{-1}([\xi_1,\xi_{n+p+1}])$ otherwise. The next lemma discusses the behavior of Tchebycheffian B-splines at the endpoints of their support.

Lemma 1. *Suppose $\gamma^{\boldsymbol{w}}_{i,\ell,\boldsymbol{\xi}} > 0$ whenever $\xi_i < \xi_{i+\ell+1}$ for $i = j,\dots,j+p-\ell$ and $\ell = 0,\dots,p-1$.*

(i) Let $1 \le \mu_j \le p+1$ such that $\xi_j = \cdots = \xi_{j+\mu_j-1} < \xi_{j+\mu_j}$. We have

$$D^r_+ B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = 0, \quad r = 0,\dots,p-\mu_j,$$

and

$$D^{p+1-\mu_j}_+ B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = w_p(\xi_j) \prod_{k=\mu_j-1}^{p-1} \frac{w_k(\xi_j)}{\gamma^{\boldsymbol{w}}_{j,k,\boldsymbol{\xi}}}.$$

(ii) Let $1 \le \mu_j \le p+1$ such that $\xi_{j+p+1-\mu_j} < \xi_{j+p+2-\mu_j} = \cdots = \xi_{j+p+1}$. We have

$$D^r_- B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_{j+p+1}) = 0, \quad r = 0,\dots,p-\mu_j,$$

and

$$D^{p+1-\mu_j}_- B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_{j+p+1}) = (-1)^{p+1-\mu_j} w_p(\xi_{j+p+1}) \prod_{k=\mu_j-1}^{p-1} \frac{w_k(\xi_{j+p+1})}{\gamma^{\boldsymbol{w}}_{j+p-k,k,\boldsymbol{\xi}}}.$$

Proof. We focus on statement (i). For $\mu_j = p+1$ the result follows from the explicit expression in (24), and in particular the result holds for $p = 0$. Suppose now $1 \le \mu_j \le p$. It follows from the definition that $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = 0$, and for $r \ge 1$ the differentiation formula (29) implies

$$D^r_+\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x)}{w_p(x)}\right) = \frac{D^{r-1}_+ B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{D^{r-1}_+ B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}}. \tag{31}$$

We proceed by induction on p. The case $p=0$ was already shown before. Since $B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}$ and $B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}$ have a knot of multiplicity μ_j and μ_j-1 at ξ_j, respectively, we deduce from the induction hypothesis that

$$D^r_+ B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(\xi_j) = 0, \quad r = 0,\dots,p-\mu_j-1,$$
$$D^r_+ B^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}}(\xi_j) = 0, \quad r = 0,\dots,p-\mu_j,$$

and

$$D^{p-\mu_j}_+ B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(\xi_j) = w_{p-1}(\xi_j) \prod_{k=\mu_j-1}^{p-2} \frac{w_k(\xi_j)}{\gamma^{\boldsymbol{w}}_{j,k,\boldsymbol{\xi}}}.$$

Therefore, from (31) we obtain

$$D^r_+\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right)(\xi_j) = 0, \quad r = 1,\dots,p-\mu_j,$$

and

$$D^{p-\mu_j+1}_+\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right)(\xi_j) = \prod_{k=\mu_j-1}^{p-1} \frac{w_k(\xi_j)}{\gamma^{\boldsymbol{w}}_{j,k,\boldsymbol{\xi}}}.$$

Recall that $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_j)=0$, and so $\big(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\big)(\xi_j)=0$. Finally, taking into account the smoothness of w_p in (14), the Leibniz rule gives

$$D^r_+ B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = \sum_{k=0}^{r} \binom{r}{k} D^{r-k}_+ w_p(\xi_j)\, D^k_+\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right)(\xi_j), \quad r = 0,\dots,p-\mu_j+1,$$

which completes the proof of statement (i). The proof of statement (ii) is similar. □

In the following, we investigate the number of sign changes of linear combinations of Tchebycheffian B-splines. We first define what we mean by sign changes of a function.

Definition 8. *The number of sign changes of a function $f:[a,b]\to\mathbb{R}$ is defined by*

$$S^-(f) := \sup_{k\ge 2} \sup_{a\le x_1<\cdots<x_k\le b} S^-(f(x_1),\dots,f(x_k)),$$

where $S^-(c_1,\dots,c_k)$ denotes the number of (strict) sign changes in the sequence of real numbers $c_1,\dots,c_k$.

Lemma 2. *For a given function $g\in C^{-1}([a,b])$, we set*

$$f(x) := f(a) + \int_a^x g(y)\,\mathrm{d}y, \quad x\in[a,b].$$

Suppose the function f has a finite number of sign changes on $[a,b]$, as defined in Definition 8. Then,

(i) $S^-(g) \geq S^-(f)+1$ *if* $f(a)=f(b)=0$ *and* $f \not\equiv 0$*;*
(ii) $S^-(g) \geq S^-(f)$ *if* $f(a)f(b)=0$*;*
(iii) $S^-(g) \geq S^-(f)-1$.

Proof. From its definition it follows that f is continuous on $[a,b]$. We start by proving statement (i). Suppose $S^-(f)=k$. Since $f(a)=f(b)=0$ and $f \not\equiv 0$, there exists a sequence of points

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b,$$

and $z_i \in (x_i, x_{i+1})$, $i=0,\ldots,k$, such that

$$f(x_i)=0, \quad i=0,\ldots,k+1, \quad f(z_i) \neq 0, \quad i=0,\ldots,k.$$

Since

$$\int_{x_i}^{z_i} g(y)\,\mathrm{d}y = f(z_i)-f(x_i) \neq 0, \quad \int_{x_i}^{x_{i+1}} g(y)\,\mathrm{d}y = f(x_{i+1})-f(x_i)=0,$$

the function g changes sign at least once in the interval (x_i,x_{i+1}). This implies that $S^-(g) \geq k+1 = S^-(f)+1$. With a similar line of arguments we can prove the statements (ii) and (iii). □

The proof of the next lemma is inspired by [3, Lemma 2.11].

Lemma 3. *Let* $k \geq 0$*. Suppose* $\gamma^{\boldsymbol{w}}_{i,\ell,\boldsymbol{\xi}} > 0$ *whenever* $\xi_i < \xi_{i+\ell+1}$ *for* $i=j,\ldots,j+k+p-\ell$ *and* $\ell=0,\ldots,p-1$*. For* $c_i \in \mathbb{R}$*,* $i=j,\ldots,j+k$*, the function*

$$s(x) := \sum_{i=j}^{j+k} c_i B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}(x)$$

has at most k *sign changes on the interval* $[\xi_j, \xi_{j+k+p+1}]$*.*

Proof. If $s \equiv 0$ there is nothing to prove. Otherwise we use induction on p. For $p=0$ the result follows from the definition (17) of $B^{\boldsymbol{w}}_{i,0,\boldsymbol{\xi}}$ and the positivity of the weight function w_0. Assuming $p \geq 1$ and using (16), we can write

$$\begin{aligned} s(x) &= c_j w_p(x) \int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y - c_{j+k} w_p(x) \int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{j+k+1,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{j+k+1,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y \\ &\quad + w_p(x) \sum_{i=j+1}^{j+k} (c_i - c_{i-1}) \int_{\xi_j}^{x} \frac{B^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}(y)}{\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}}\,\mathrm{d}y. \end{aligned} \tag{32}$$

Suppose now that for any k and any j it holds that $\sum_{i=j}^{j+k} c_i B^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}$ has at most k sign changes. If one of the knots has multiplicity $p+1$ in the knot sequence $\{\xi_{j+1} \leq \cdots \leq \xi_{j+k+p}\}$, say $\xi_{j+\ell} = \xi_{j+\ell+p}$ for some $\ell \in \{1,\ldots,k\}$, then

$$s(x) = \sum_{i=j}^{j+\ell-1} c_i B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}(x) + \sum_{i=j+\ell}^{j+k} c_i B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}(x) =: s_1(x) + s_2(x).$$

From (25) we know that $s_1(x) = 0$ for $x \notin [\xi_j, \xi_{j+\ell})$ and $s_2(x) = 0$ for $x \notin [\xi_{j+\ell}, \xi_{j+k+p+1})$. Hence,

$$S^-(s) \leq S^-(s_1) + S^-(s_2) + 1. \tag{33}$$

Therefore, to show that $S^-(s) \leq k$ it suffices to show that $S^-(s_1) \leq \ell - 1$ and $S^-(s_2) \leq k - \ell$. These are two subproblems of the same structure. A repeated application of this argument allows us to remove all knots of multiplicity $p+1$ in the knot sequence $\{\xi_{j+1} \leq \cdots \leq \xi_{j+k+p}\}$. Therefore, it is enough to prove the result in the case

$$\xi_i < \xi_{i+p}, \quad i = j+1, \ldots, j+k. \tag{34}$$

Assuming that (34) holds, we consider four cases.

First case: $\xi_j < \xi_{j+p}$ and $\xi_{j+k+1} < \xi_{j+k+p+1}$. For $x \in [\xi_j, \xi_{j+k+p+1}]$, let

$$f(x) := \int_{\xi_j}^{x} g(y)\,\mathrm{d}y, \quad g(x) := \sum_{i=j}^{j+k+1} \left(\frac{c_i - c_{i-1}}{\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}} \right) B^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}(x),$$

with $c_{j-1} := 0$ and $c_{j+k+1} := 0$. Since $\xi_i < \xi_{i+p}$ for $i = j, \ldots, j+k+1$, the hypothesis ensures that $\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}} > 0$ for $i = j, \ldots, j+k+1$. Hence, g is well defined and belongs to $C^{-1}([\xi_j, \xi_{j+k+p+1}])$ because of (26). By the induction hypothesis, we also know that g has at most $k+1$ sign changes. As a consequence, f is a continuous function with a finite number of sign changes and $f(\xi_j) = 0$. From (32) it is clear that $s(x) = w_p(x) f(x)$, and the positivity of w_p implies $S^-(s) = S^-(f)$. Moreover, from the local support of the B-splines it follows that $f(\xi_{j+k+p+1}) = 0$. Thus, from statement (i) of Lemma 2 we get $S^-(s) = S^-(f) \leq S^-(g) - 1 \leq k$.

Second case: $\xi_j = \xi_{j+p}$ and $\xi_{j+k+1} = \xi_{j+k+p+1}$. For $x \in [\xi_j, \xi_{j+k+p+1}]$, let

$$f(x) := c_j + \int_{\xi_j}^{x} g(y)\,\mathrm{d}y, \quad g(x) := \sum_{i=j+1}^{j+k} \left(\frac{c_i - c_{i-1}}{\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}} \right) B^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}}(x).$$

Since $\xi_i < \xi_{i+p}$ for $i = j+1, \ldots, j+k$, the hypothesis ensures that $\gamma^{\boldsymbol{w}}_{i,p-1,\boldsymbol{\xi}} > 0$ for $i = j+1, \ldots, j+k$. Hence, g is well defined and belongs to $C^{-1}([\xi_j, \xi_{j+k+p+1}])$. Since $\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}} = \gamma^{\boldsymbol{w}}_{j+k+1,p-1,\boldsymbol{\xi}} = 0$, and taking into account (19), we have $s(x) = w_p(x) f(x)$ for $x \in [\xi_j, \xi_{j+k+p+1})$. From (25) we see that $s(\xi_{j+k+p+1}) = 0$, and so the positivity of w_p implies $S^-(s) = S^-(f)$. With the same line of arguments as in the previous case, from statement (iii) of Lemma 2 and the induction hypothesis, we get $S^-(s) = S^-(f) \leq S^-(g) + 1 \leq k$ on $[\xi_j, \xi_{j+k+p+1}]$.

In the two remaining cases $\xi_j < \xi_{j+p}$ and $\xi_{j+k+1} = \xi_{j+k+p+1}$ or $\xi_j = \xi_{j+p}$ and $\xi_{j+k+1} < \xi_{j+k+p+1}$, the result follows in a similar way by using statement (ii) of Lemma 2. $\square$

We are now ready to show nonnegativity of Tchebycheffian B-splines.

Lemma 4. *If* $\xi_j < \xi_{j+p+1}$ *then* $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) \geq 0$ *for* $x \in [\xi_j, \xi_{j+p+1}]$ *and moreover* $\gamma^{\mathbf{w}}_{j,p,\boldsymbol{\xi}} > 0$.

Proof. We proceed by induction on p. For $p = 0$ the result follows from the definition (17) of $B^{\mathbf{w}}_{i,0,\boldsymbol{\xi}}$ and the positivity of the weight function w_0. Suppose now that $\gamma^{\mathbf{w}}_{i,\ell,\boldsymbol{\xi}}(x) > 0$ whenever $\xi_i < \xi_{i+\ell+1}$ for $i = j, \dots, j+p-\ell$ and $\ell = 0, \dots, p-1$. Let $1 \leq \mu_j \leq p+1$ such that $\xi_j = \cdots = \xi_{j+\mu_j-1} < \xi_{j+\mu_j}$. From Lemma 1 and the induction hypothesis we get

$$D^r_+ B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = 0, \quad r = 0, \dots, p-\mu_j, \quad D^{p+1-\mu_j}_+ B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) > 0.$$

Therefore, $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) > 0$ for $x \in (\xi_j, \xi_j + \varepsilon)$ and some $\varepsilon > 0$. Moreover, from Lemma 3 (with $k = 0$) it follows that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ has no sign changes on $[\xi_j, \xi_{j+p+1}]$. This means that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ is nonnegative on $[\xi_j, \xi_{j+p+1}]$ and it is a nontrivial function on an open subset of $[\xi_j, \xi_{j+p+1}]$. As a consequence, $\gamma^{\mathbf{w}}_{j,p,\boldsymbol{\xi}} > 0$. $\square$

The positivity of $\gamma^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ whenever $\xi_j < \xi_{j+p+1}$ shown in Lemma 4 implies that the assumptions of Lemma 1 and Lemma 3 are always satisfied.

Theorem 6 (Nonnegativity). *A Tchebycheffian B-spline is nonnegative everywhere, and positive inside its support, i.e.,*

$$B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) \geq 0, \quad x \in \mathbb{R}, \quad \textit{and} \quad B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) > 0, \quad x \in (\xi_j, \xi_{j+p+1}). \tag{35}$$

Proof. It suffices to prove that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) > 0$ for $\xi_j < x < \xi_{j+p+1}$. Indeed, nonnegativity of $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ on $\mathbb{R}$ follows from the local support (25) and Lemma 4.

If $\xi_j < \xi_{j+1} = \xi_{j+p+1}$ or $\xi_j = \xi_{j+p} < \xi_{j+p+1}$ the result follows immediately from the expression of the first and last piece in (23) and (24).

Now, suppose $\xi_j < \xi_{j+p}$ and $\xi_{j+1} < \xi_{j+p+1}$. From Lemma 4 we obtain $\gamma^{\mathbf{w}}_{j,p-1,\boldsymbol{\xi}} > 0$ and $\gamma^{\mathbf{w}}_{j+1,p-1,\boldsymbol{\xi}} > 0$, so that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(\xi_j) = B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(\xi_{j+p+1}) = 0$. Moreover, from the proof of the same lemma, we know that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) > 0$ for $x \in (\xi_j, \xi_j + \varepsilon)$ and some $\varepsilon > 0$. In a similar way, we can also prove that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) > 0$ for $x \in (\xi_{j+p+1} - \varepsilon, \xi_{j+p+1})$ and some $\varepsilon > 0$. Assume now that there exists a point $\bar{x} \in (\xi_j, \xi_{j+p+1})$ such that $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(\bar{x}) = 0$. We will show that this assumption leads to a contradiction. Since

$$B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) = w_p(x) \int_{\xi_j}^{x} g(y)\,\mathrm{d}y, \quad g(x) := \frac{B^{\mathbf{w}}_{j,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\mathbf{w}}_{j,p-1,\boldsymbol{\xi}}} - \frac{B^{\mathbf{w}}_{j+1,p-1,\boldsymbol{\xi}}(x)}{\gamma^{\mathbf{w}}_{j+1,p-1,\boldsymbol{\xi}}},$$

and $g \not\equiv 0$ on $(\xi_j, \bar{x})$ and $(\bar{x}, \xi_{j+p+1})$, we have

$$\int_{\xi_j}^{\bar{x}} g(y)\,\mathrm{d}y = 0, \quad \int_{\bar{x}}^{\xi_{j+p+1}} g(y)\,\mathrm{d}y = 0.$$

From statement (i) of Lemma 2 we deduce that g must have at least one sign change on $[\xi_j, \bar{x}]$ and at least another sign change on $[\bar{x}, \xi_{j+p+1}]$. On the other hand, from Lemma 3 it follows that g can have at most one sign change on $[\xi_j, \xi_{j+p+1}]$. This contradiction concludes the proof. $\square$

We now describe the smoothness behavior of Tchebycheffian B-splines at the knots in their support.

Theorem 7 (Smoothness). *If ξ is a knot of $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ of multiplicity $\mu \leq p+1$ then*

$$B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi), \tag{36}$$

i.e., its derivatives of order $0,1,\ldots,p-\mu$ are continuous at ξ. Moreover, if $\mu = p+1$ then $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi)$ is bounded.

Proof. By Lemma 1 (and Lemma 4) the result holds if $\xi = \xi_j$ or $\xi = \xi_{j+p+1}$. Suppose now $\xi_j < \xi < \xi_{j+p+1}$. Observe that $\mu \leq p$, $\xi_j < \xi_{j+p}$ and $\xi_{j+1} < \xi_{j+p+1}$, and therefore by Lemma 4 we have $\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}} > 0$ and $\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}} > 0$.

We first prove that $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ is continuous at ξ whenever $\mu \leq p$. Indeed, by (15) we have $w_p \in C^0(\xi)$ and by (26) the integrands in (16) are bounded, and so $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ and $\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}$ are continuous at ξ.

In order to show (36) for $\mu < p$, we proceed by induction on p. Both terms in the differentiation formula (29) have a knot of multiplicity at most μ at ξ, and from the induction hypothesis we obtain $D\left(\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}\right) \in C^{p-1-\mu}(\xi)$. Moreover, since $\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p}$ is continuous at ξ, we can conclude that $\frac{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}}{w_p} \in C^{p-\mu}(\xi)$ for $\mu < p$. Since $\xi_j < \xi < \xi_{j+p+1}$ then by (15) we have $w_p \in C^{p-\mu}(\xi)$, and so $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}} \in C^{p-\mu}(\xi)$. $\square$

Finally, we show that Tchebycheffian B-splines are (locally) linearly independent on each knot interval and span the local E(C)T-space defined on such interval.

Theorem 8 (Local Linear Independence). *The set $\{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}\}_{j=m-p}^{m}$ forms a basis for the E(C)T-space $\mathbb{T}^{\boldsymbol{w}}_p$ on $[\xi_m, \xi_{m+1})$ for any $p+1 \leq m \leq n$.*

Proof. By the piecewise Tchebycheff structure (26) of Tchebycheffian B-splines, it suffices to prove that the functions $\{B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}\}_{j=m-p}^{m}$ are linearly independent on $[\xi_m, \xi_{m+1})$ for any $p+1 \leq m \leq n$.

We use induction on p. The case $p = 0$ follows from (17). Now, let $p \geq 1$. Fix m such that $\xi_m < \xi_{m+1}$, and suppose that for all $x \in [\xi_m, \xi_{m+1})$,

$$\frac{1}{w_p(x)} \sum_{j=m-p}^{m} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = 0. \tag{37}$$

After differentiating (37), it follows from (29) and (25) that

$$\sum_{j=m-p+1}^{m}\left(\frac{c_j-c_{j-1}}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}\right)B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(x)=0.$$

Since $\xi_m<\xi_{m+1}$, Lemma 4 implies that $\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}>0$, $j=m-p+1,\dots,m$. Then, the induction hypothesis gives us that $c_{m-p}=\cdots=c_m$, so

$$\frac{c_m}{w_p(x)}\sum_{j=m-p}^{m}B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x)=0.$$

By relation (27) we get $c_m=0$. As a consequence, all functions $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$ in (37) are linearly independent on $[\xi_m,\xi_{m+1})$. □

3.3 The Tchebycheffian Spline Space

In this section we focus on the span of the Tchebycheffian B-splines of degree p specified by the knot sequence $\boldsymbol{\xi}:=\{\xi_i\}_{i=1}^{n+p+1}$ and the vector of Tchebycheffian B-spline weights $\boldsymbol{w}$, i.e.,

$$\mathbb{S}^{\boldsymbol{w}}_{p,\boldsymbol{\xi}}:=\left\{s:[\xi_{p+1},\xi_{n+1}]\to\mathbb{R}:s=\sum_{j=1}^{n}c_jB^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}},\ c_j\in\mathbb{R}\right\}. \tag{38}$$

This is the space of **Tchebycheffian splines** spanned by the Tchebycheffian B-splines $\{B^{\boldsymbol{w}}_{1,p,\boldsymbol{\xi}},\dots,B^{\boldsymbol{w}}_{n,p,\boldsymbol{\xi}}\}$ over the interval $[\xi_{p+1},\xi_{n+1}]$, which is called the **basic interval**. We define the Tchebycheffian B-splines to be left continuous at the right endpoint ξ_{n+1}, so as to avoid asymmetry in the construction of the space.

We now introduce some terminology to identify certain properties of knot sequences which are crucial in the study of the space (38).

- A knot sequence $\boldsymbol{\xi}$ is called $(p+1)$**-regular** if $\xi_j<\xi_{j+p+1}$ for $j=1,\dots,n$. By the local support (25) such a knot sequence ensures that all the Tchebycheffian B-splines in (38) are not identically zero.
- A knot sequence $\boldsymbol{\xi}$ is called $(p+1)$**-basic** if it is $(p+1)$-regular with $\xi_{p+1}<\xi_{p+2}$ and $\xi_n<\xi_{n+1}$. As we will show later, the Tchebycheffian B-splines in (38) defined on a $(p+1)$-basic knot sequence are linearly independent on the basic interval $[\xi_{p+1},\xi_{n+1}]$.

From the results in the previous section, we can immediately conclude the following list of properties of Tchebycheffian splines in the B-spline representation.

- **Smoothness.** If ξ is a knot of multiplicity μ then $s\in C^r(\xi)$ for any $s\in\mathbb{S}^{\boldsymbol{w}}_{p,\boldsymbol{\xi}}$, where $r+\mu=p$. This follows from the smoothness property of the Tchebycheffian B-splines (Theorem 7). Therefore, the relation between smoothness, multiplicity and degree is the same as in the polynomial B-spline case:

$$\text{“smoothness} + \text{multiplicity} = \text{degree”}. \tag{39}$$

- **Local Support.** The local support (25) of the Tchebycheffian B-splines implies

$$\sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = \sum_{j=m-p}^{m} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x), \quad x \in [\xi_m, \xi_{m+1}), \quad p+1 \leq m \leq n, \tag{40}$$

and if $\xi_m < \xi_{m+p}$ then

$$\sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_m) = \sum_{j=m-p}^{m-1} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(\xi_m), \quad p+1 \leq m \leq n+1. \tag{41}$$

- **Minimal Support.** From the smoothness properties it can be proved that if the support of $s \in \mathbb{S}^{\boldsymbol{w}}_{p,\boldsymbol{\xi}}$ is a proper subset of $[\xi_j, \xi_{j+p+1}]$ for some j then $s = 0$. Therefore, the Tchebycheffian B-splines have minimal support.
- **Partition of Unity.** By (27) we have

$$\sum_{j=1}^{n} B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = w_p(x), \quad x \in [\xi_{p+1}, \xi_{n+1}]. \tag{42}$$

In particular, for normalized Tchebycheffian B-splines this relation simplifies to

$$\sum_{j=1}^{n} B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = 1, \quad x \in [\xi_{p+1}, \xi_{n+1}]. \tag{43}$$

Since these splines are nonnegative it follows that they form a **nonnegative partition of unity** on $[\xi_{p+1}, \xi_{n+1}]$.
- **Differentiation.** By (29) we have for $p \geq 1$,

$$D_+\left(\frac{1}{w_p(x)} \sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x)\right) = \sum_{j=2}^{n} c_j^{(1)} B^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{44}$$

where

$$c_j^{(1)} := \frac{c_j - c_{j-1}}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}}}, \tag{45}$$

and fractions with zero denominator have value zero.
- **Linear Independence.** If $\boldsymbol{\xi}$ is $(p+1)$-basic, then the Tchebycheffian B-splines $\{B^{\boldsymbol{w}}_{1,p,\boldsymbol{\xi}}, \ldots, B^{\boldsymbol{w}}_{n,p,\boldsymbol{\xi}}\}$ are linearly independent on the basic interval. Thus, the spline space $\mathbb{S}^{\boldsymbol{w}}_{p,\boldsymbol{\xi}}$ is a vector space of dimension n.

Proof. We must show that if

$$s(x) = \sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = 0, \quad x \in [\xi_{p+1}, \xi_{n+1}],$$

then $c_j = 0$ for all j. Let us fix $1 \le j \le n$. Since $\boldsymbol{\xi}$ is $(p+1)$-regular, there is an integer m_j with $j \le m_j \le j+p$ such that $\xi_{m_j} < \xi_{m_j+1}$. Moreover, the assumptions $\xi_{p+1} < \xi_{p+2}$ and $\xi_n < \xi_{n+1}$ guarantee that $[\xi_{m_j}, \xi_{m_j+1})$ can be chosen in the basic interval. From the local support property (40) we know

$$s(x) = \sum_{i=m_j-p}^{m_j} c_i B^{\boldsymbol{w}}_{i,p,\boldsymbol{\xi}}(x) = 0, \quad x \in [\xi_{m_j}, \xi_{m_j+1}).$$

Theorem 8 implies $c_{m_j-p} = \cdots = c_{m_j} = 0$, and in particular $c_j = 0$. $\square$

In the following, we are looking for a characterization of the Tchebycheffian spline space $\mathbb{S}^{\boldsymbol{w}}_{p,\boldsymbol{\xi}}$ in terms of piecewise Tchebycheff functions with a certain smoothness.

Definition 9. *Let Δ be a sequence of break points,*

$$\Delta := \{\eta_0 < \eta_1 < \cdots < \eta_{\ell+1}\}, \quad a := \eta_0, \quad b := \eta_{\ell+1}, \tag{46}$$

and let $\boldsymbol{r} := (r_1, \ldots, r_\ell)$ be a vector of integers such that $-1 \le r_i \le p$ for $i = 1, \ldots, \ell$. Furthermore, let $\boldsymbol{w} := (w_0, \ldots, w_p)$ be a vector of positive weight functions on $[a,b]$ such that for $j = 0, \ldots, p$,

$$\begin{aligned} &w_j \in C^j([\eta_i^+, \eta_{i+1}^-]), \quad i = 0, \ldots, \ell, \\ &w_j \in C^{\max(j-p+r_i,-1)}(\eta_i), \quad i = 1, \ldots, \ell. \end{aligned} \tag{47}$$

The space $\mathbb{S}^{\boldsymbol{r},\boldsymbol{w}}_p(\Delta)$ of **piecewise Tchebycheff functions** *of degree p with smoothness $\boldsymbol{r}$ over the partition Δ is defined by*

$$\begin{aligned} \mathbb{S}^{\boldsymbol{r},\boldsymbol{w}}_p(\Delta) := \big\{ s : [\eta_0, \eta_{\ell+1}] \to \mathbb{R} :\ & s \in \mathbb{T}^{\boldsymbol{w}}_p([\eta_i, \eta_{i+1})),\ i = 0, \ldots, \ell-1, \\ & s \in \mathbb{T}^{\boldsymbol{w}}_p([\eta_\ell, \eta_{\ell+1}]),\ s \in C^{r_i}(\eta_i),\ i = 1, \ldots, \ell \big\}. \end{aligned} \tag{48}$$

Any element $s \in \mathbb{S}^{\boldsymbol{r},\boldsymbol{w}}_p(\Delta)$ can be written in the form

$$s(x) = \sum_{j=0}^{p} c_{0,j} u^{\boldsymbol{w}}_{j,p}(x, \eta_0) + \sum_{i=1}^{\ell} \sum_{j=r_i+1}^{p} c_{i,j} u^{\boldsymbol{w}}_{j,p}(x, \eta_i)_+, \quad x \in [a,b], \tag{49}$$

where $u^{\boldsymbol{w}}_{j,p}(x,y)$ are generalized powers (see Definition 4) and

$$u^{\boldsymbol{w}}_{j,p}(x,y)_+ := \begin{cases} u^{\boldsymbol{w}}_{j,p}(x,y), & x > y, \\ 0, & x < y, \end{cases} \tag{50}$$

where the value at y is defined by taking the right limit. The functions in (50) are called **generalized truncated powers**. From the smoothness conditions in (47) we see that $w_{p-k} \in C^{\max(r_i-k,-1)}(\eta_i)$, $k = 0, \ldots, p$, and from Definition 4 we immediately get

$$u^{\boldsymbol{w}}_{j,p}(\eta_k, \eta_i) \in C^{r_i}(\eta_k), \quad k = i+1, \ldots, \ell, \quad j = 0, \ldots, p.$$

Moreover, since $w_{p-k} \in C^{p-k}([\eta_i^+, \eta_{i+1}^-])$, $k = 0, \dots, p$, by combining (4) and (6) we have

$$D_+^l u_{j,p}^{\boldsymbol{w}}(\eta_i, \eta_i) = 0, \quad l = 0, \dots, r_i, \quad j = r_i + 1, \dots, p.$$

This shows that the function in (49) belongs to the space (48). The representation (49) implies

$$\dim(\mathbb{S}_p^{\boldsymbol{r},\boldsymbol{w}}(\Delta)) \leq p + 1 + \sum_{i=1}^{\ell} (p - r_i). \tag{51}$$

The next theorem states that the Tchebycheffian spline space $\mathbb{S}_{p,\boldsymbol{\xi}}^{\boldsymbol{w}}$ is equal to the space $\mathbb{S}_p^{\boldsymbol{r},\boldsymbol{w}}(\Delta)$ with a prescribed partition Δ and smoothness $\boldsymbol{r}$.

Theorem 9 (Characterization of Spline Space). *Let $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ be a $(p+1)$-basic knot sequence. The space $\mathbb{S}_{p,\boldsymbol{\xi}}^{\boldsymbol{w}}$ spanned by Tchebycheffian B-splines of degree p defined over the knot sequence $\boldsymbol{\xi}$ is characterized by*

$$\mathbb{S}_{p,\boldsymbol{\xi}}^{\boldsymbol{w}} = \mathbb{S}_p^{\boldsymbol{r},\boldsymbol{w}}(\Delta),$$

where Δ is a partition as in (46) *defined from the knot sequence as follows,*

$$\xi_{p+1} =: \eta_0, \quad \xi_{p+2}, \dots, \xi_n =: \overbrace{\eta_1, \dots, \eta_1}^{\mu_1}, \dots, \overbrace{\eta_\ell, \dots, \eta_\ell}^{\mu_\ell}, \quad \xi_{n+1} =: \eta_{\ell+1},$$

and the smoothness $\boldsymbol{r}$ is defined by

$$r_i := p - \mu_i, \quad i = 1, \dots, \ell.$$

Proof. Since we are dealing with a $(p+1)$-basic knot sequence $\boldsymbol{\xi}$, we have $\eta_0 < \eta_1$ and $\eta_\ell < \eta_{\ell+1}$. The Tchebycheffian B-spline weights $\boldsymbol{w}$ satisfy the smoothness conditions in (47); see Definition 6. From the piecewise structure (26) and the smoothness (36) of Tchebycheffian B-splines it follows that the space $\mathbb{S}_{p,\boldsymbol{\xi}}^{\boldsymbol{w}}$ is a subspace of $\mathbb{S}_p^{\boldsymbol{r},\boldsymbol{w}}(\Delta)$. Moreover, using (51) we arrive at

$$\dim(\mathbb{S}_{p,\boldsymbol{\xi}}^{\boldsymbol{w}}) = n = p + 1 + \sum_{i=1}^{\ell} (p - r_i) \geq \dim(\mathbb{S}_p^{\boldsymbol{r},\boldsymbol{w}}(\Delta)).$$

This concludes the proof. ☐

3.4 Knot Insertion

In this section we are addressing the problem of representing a given Tchebycheffian spline on a refined knot sequence. In particular, we focus on the special case where only a single knot is inserted. Since any refined knot sequence can be reached by repeatedly inserting one knot at a time, it suffices to deal with this case.

Without loss of generality, we assume that the spline $s = \sum_{j=1}^{n} c_j B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ is given on a $(p+1)$-basic knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$. We want to insert a knot ξ in some subinterval $[\xi_m, \xi_{m+1})$ of $[\xi_{p+1}, \xi_{n+1})$, resulting in a new $(p+1)$-basic knot sequence $\tilde{\boldsymbol{\xi}} := \{\tilde{\xi}_i\}_{i=1}^{n+p+2}$ defined by

$$\tilde{\xi}_i := \begin{cases} \xi_i, & \text{if } 1 \le i \le m, \\ \xi, & \text{if } i = m+1, \\ \xi_{i-1}, & \text{if } m+2 \le i \le n+p+2. \end{cases} \tag{52}$$

We are interested in the Tchebycheffian B-spline form of s on the new knot sequence.

Lemma 5. *Let the $(p+1)$-basic knot sequence $\tilde{\boldsymbol{\xi}} := \{\tilde{\xi}_i\}_{i=1}^{n+p+2}$ be obtained from the $(p+1)$-basic knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ by inserting just one knot ξ in $[\xi_{p+1}, \xi_{n+1})$. Then,*

$$B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}} = \alpha_{j,p,\boldsymbol{\xi}} B^{\mathbf{w}}_{j,p,\tilde{\boldsymbol{\xi}}} + \beta_{j+1,p,\boldsymbol{\xi}} B^{\mathbf{w}}_{j+1,p,\tilde{\boldsymbol{\xi}}}, \tag{53}$$

where

(i) $\alpha_{j,p,\boldsymbol{\xi}} = 1$ *and* $\beta_{j+1,p,\boldsymbol{\xi}} = 0$ *if* $\xi \ge \xi_{j+p+1}$*;*
(ii) $\alpha_{j,p,\boldsymbol{\xi}} > 0$ *and* $\beta_{j+1,p,\boldsymbol{\xi}} > 0$ *if* $\xi_j < \xi < \xi_{j+p+1}$*;*
(iii) $\alpha_{j,p,\boldsymbol{\xi}} = 0$ *and* $\beta_{j+1,p,\boldsymbol{\xi}} = 1$ *if* $\xi \le \xi_j$*.*

Proof. From Theorem 9 it follows that $\mathbb{S}^{\mathbf{w}}_{p,\boldsymbol{\xi}} \subseteq \mathbb{S}^{\mathbf{w}}_{p,\tilde{\boldsymbol{\xi}}}$, so every $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$ can be written as a linear combination of the Tchebycheffian B-splines defined over $\tilde{\boldsymbol{\xi}}$. If $\xi \ge \xi_{j+p+1}$ then $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}} = B^{\mathbf{w}}_{j,p,\tilde{\boldsymbol{\xi}}}$, which shows (53) in case (i). If $\xi \le \xi_j$ then $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}} = B^{\mathbf{w}}_{j+1,p,\tilde{\boldsymbol{\xi}}}$, which shows (53) in case (iii). In the remainder, we focus on the last case (ii) and assume $\xi_j < \xi < \xi_{j+p+1}$.

Fix j. We can write

$$B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) = \sum_{i=1}^{n+1} c_{j,i} B^{\mathbf{w}}_{i,p,\tilde{\boldsymbol{\xi}}}(x).$$

If $x \in [\tilde{\xi}_k, \tilde{\xi}_{k+1})$ with $\tilde{\xi}_k \ge \xi_{j+p+1} = \tilde{\xi}_{j+p+2}$, then

$$0 = B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}(x) = \sum_{i=k-p}^{k} c_{j,i} B^{\mathbf{w}}_{i,p,\tilde{\boldsymbol{\xi}}}(x),$$

and by local linear independence we get $c_{j,i} = 0$ for any $i \ge j+2$ since $k-p \ge j+2$. Similarly, $c_{j,i} = 0$ for any $i \le j-1$. This implies (53) for some $\alpha_{j,p,\boldsymbol{\xi}}$ and $\beta_{j,p,\boldsymbol{\xi}}$.

Next, we show the positivity of $\alpha_{j,p,\boldsymbol{\xi}}$. Let μ_j be the multiplicity of ξ_j as a knot of $B^{\mathbf{w}}_{j,p,\boldsymbol{\xi}}$. Then, ξ_j appears μ_j times as a knot of $B^{\mathbf{w}}_{j,p,\tilde{\boldsymbol{\xi}}}$ and $\mu_j - 1$ times as a knot of $B^{\mathbf{w}}_{j+1,p,\tilde{\boldsymbol{\xi}}}$. We consider the $(p+1-\mu_j)$-th derivative at ξ_j of the two sides in (53).

By using the expression in statement (i) of Lemma 1 and recalling that the weight functions are positive, we get

$$\alpha_{j,p,\boldsymbol{\xi}} = \begin{cases} 1, & \mu_j = p+1, \\ \frac{\gamma^{\boldsymbol{w}}_{j,p-1,\tilde{\boldsymbol{\xi}}} \cdots \gamma^{\boldsymbol{w}}_{j,\mu_j-1,\tilde{\boldsymbol{\xi}}}}{\gamma^{\boldsymbol{w}}_{j,p-1,\boldsymbol{\xi}} \cdots \gamma^{\boldsymbol{w}}_{j,\mu_j-1,\boldsymbol{\xi}}}, & \mu_j \leq p. \end{cases} \tag{54}$$

Note that by Lemma 4 all γ's involved in (54) are positive because $\xi_j < \xi_{j+\mu_j}$. The positivity of $\beta_{j+1,p,\boldsymbol{\xi}}$ can be proved in a similar way. Let μ_{j+p+1} be the multiplicity of ξ_{j+p+1} as a knot of $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}$. From statement (ii) of Lemma 1 we get

$$\beta_{j+1,p,\boldsymbol{\xi}} = \begin{cases} 1, & \mu_{j+p+1} = p+1, \\ \frac{\gamma^{\boldsymbol{w}}_{j+2,p-1,\tilde{\boldsymbol{\xi}}} \cdots \gamma^{\boldsymbol{w}}_{j+p+2-\mu_{j+p+1},\mu_{j+p+1}-1,\tilde{\boldsymbol{\xi}}}}{\gamma^{\boldsymbol{w}}_{j+1,p-1,\boldsymbol{\xi}} \cdots \gamma^{\boldsymbol{w}}_{j+p+1-\mu_{j+p+1},\mu_{j+p+1}-1,\boldsymbol{\xi}}}, & \mu_{j+p+1} \leq p. \end{cases} \tag{55}$$

This completes the proof. □

Theorem 10 (Knot Insertion). *Let the $(p+1)$-basic knot sequence $\tilde{\boldsymbol{\xi}} := \{\tilde{\xi}_i\}_{i=1}^{n+p+2}$ be obtained from the $(p+1)$-basic knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ by inserting just one knot ξ, such that $\xi_m \leq \xi < \xi_{m+1}$ with $p+1 \leq m \leq n$ as in* (52)*. Then,*

$$s(x) = \sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = \sum_{i=1}^{n+1} \tilde{c}_i B^{\boldsymbol{w}}_{i,p,\tilde{\boldsymbol{\xi}}}(x), \quad x \in [\xi_{p+1}, \xi_{n+1}], \tag{56}$$

where

$$\tilde{c}_i = \begin{cases} c_i, & \text{if } i \leq m-p, \\ \alpha_{i,p,\boldsymbol{\xi}} c_i + \beta_{i,p,\boldsymbol{\xi}} c_{i-1}, & \text{if } m-p < i \leq m, \\ c_{i-1}, & \text{if } i > m. \end{cases} \tag{57}$$

The values $\alpha_{i,p,\boldsymbol{\xi}}$ and $\beta_{i,p,\boldsymbol{\xi}}$ in (57) *are nonnegative, and*

$$\alpha_{i,p,\boldsymbol{\xi}} + \beta_{i,p,\boldsymbol{\xi}} = 1, \quad m-p < i \leq m. \tag{58}$$

Proof. From Lemma 5 we immediately deduce that

$$\sum_{j=1}^{n} c_j B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}} = \alpha_{1,p,\boldsymbol{\xi}} c_1 B^{\boldsymbol{w}}_{1,p,\tilde{\boldsymbol{\xi}}} + \beta_{n+1,p,\boldsymbol{\xi}} c_n B^{\boldsymbol{w}}_{n+1,p,\tilde{\boldsymbol{\xi}}} + \sum_{i=2}^{n} (\alpha_{i,p,\boldsymbol{\xi}} c_i + \beta_{i,p,\boldsymbol{\xi}} c_{i-1}) B^{\boldsymbol{w}}_{i,p,\tilde{\boldsymbol{\xi}}},$$

where the α's and β's are nonnegative. This gives (56) with

$$\tilde{c}_i = \alpha_{i,p,\boldsymbol{\xi}} c_i + \beta_{i,p,\boldsymbol{\xi}} c_{i-1}, \quad i = 2, \ldots, n. \tag{59}$$

First, recall from (42) that both sets of B-splines in (56) sum to w_p. Hence, in the case $s = w_p$, (59) implies

$$1 = \alpha_{i,p,\boldsymbol{\xi}} + \beta_{i,p,\boldsymbol{\xi}}, \quad i = 2, \ldots, n.$$

Since $p+1 \le m \le n$, it follows that $\{m-p+1,\ldots,m\} \subseteq \{2,\ldots,n\}$ and we obtain (58). Furthermore, from case (i) in Lemma 5 we have $\alpha_{i-1,p,\boldsymbol{\xi}} = 1$ and $\beta_{i,p,\boldsymbol{\xi}} = 0$ for $2 \le i \le m-p$. We also observe from (54) that $\alpha_{m-p,p,\boldsymbol{\xi}} = 1$. Indeed, if $\mu_{m-p} = p+1$ it is obvious, and otherwise we have $\gamma^{\boldsymbol{w}}_{m-p,k-1,\tilde{\boldsymbol{\xi}}} = \gamma^{\boldsymbol{w}}_{m-p,k-1,\boldsymbol{\xi}}$ for $k = \mu_{m-p},\ldots,p$. Similarly, from case (iii) in the same lemma we have $\alpha_{i,p,\boldsymbol{\xi}} = 0$ and $\beta_{i+1,p,\boldsymbol{\xi}} = 1$ for $m < i \le n$. If $\xi_m = \xi$ then this case also implies $\beta_{m+1,p,\boldsymbol{\xi}} = 1$. If $\xi_m < \xi$ then we can conclude from (55) that $\beta_{m+1,p,\boldsymbol{\xi}} = 1$. This completes the proof. □

From the proof we observe that the α's and β's in (57) are specified in (54) and (55), respectively.

4 Generalized B-Splines

In this section we consider a special subclass of normalized Tchebycheffian B-splines, the so-called generalized B-splines.[4] They can be seen as the minimal extension of classical polynomial splines still offering a wide variety of additional flexibility.

Definition 10. *Given the partition $\Delta := \{\eta_0 < \eta_1 < \cdots < \eta_{\ell+1}\}$ and a nonnegative integer $p \ge 2$, a* **generalized polynomial space** *of degree p is defined as a space of the form*

$$\mathbb{P}_p^{U,V}(\Delta) := \langle 1, x, \ldots, x^{p-2}, U(x), V(x) \rangle, \quad x \in [\eta_0, \eta_{\ell+1}], \tag{60}$$

where $U, V \in C^p([\eta_i^+, \eta_{i+1}^-])$ and $\langle D^{p-1}U, D^{p-1}V \rangle$ is an ET-space on $[\eta_i^+, \eta_{i+1}^-]$ for all $i = 0,\ldots,\ell$.

From Example 12 we conclude that the restriction of the generalized polynomial space $\mathbb{P}_p^{U,V}(\Delta)$ on the interval $[\eta_i^+, \eta_{i+1}^-]$ is an ECT-space of dimension $p+1$ generated by weight functions of the form

$$w_{0,i}(x), \quad w_{1,i}(x), \quad w_{2,i}(x) = \cdots = w_{p,i}(x) = 1, \tag{61}$$

where

$$w_{1,i}(\eta_i) = w_{1,i}(\eta_{i+1}) = 1. \tag{62}$$

Note that the space $\langle D^{p-1}U, D^{p-1}V \rangle$ is also an ECT-space on $[\eta_i^+, \eta_{i+1}^-]$, generated by the weights $w_{0,i}$ and $w_{1,i}$ (see Example 11). The local weights in (61)–(62) allow us to define a global weight vector $\boldsymbol{w} := (w_0,\ldots,w_p)$ such that

$$w_j(x) := w_{j,i}(x), \quad x \in [\eta_i, \eta_{i+1}), \quad i = 0,\ldots,\ell, \quad j = 0,\ldots,p. \tag{63}$$

[4] The term "generalized splines" has several different meanings in the literature. For example, the splines considered here are much less general than those described in [27, Chapter 11]. We follow the definition given in [16]. This definition was already used before for special choices of U and V; see, for example, [14, 15].

From the construction it is easy to check that each weight $w_j \in C^j([\eta_i^+, \eta_{i+1}^-])$, $i = 0, \dots, \ell$ and that $w_j \in C^{j-1}(\eta_i)$, $i = 1 \dots, \ell$.

We now define generalized B-splines of degree p associated with a knot sequence $\boldsymbol{\xi}$ and a generalized polynomial space $\mathbb{P}_p^{U,V}(\Delta)$. The knot sequence $\boldsymbol{\xi} := \{\xi_i\}_{i=1}^{n+p+1}$ is connected to the partition Δ as follows

$$\xi_1, \dots, \xi_{n+p+1} = \overbrace{\eta_0, \dots, \eta_0}^{\mu_0}, \dots, \overbrace{\eta_{\ell+1}, \dots, \eta_{\ell+1}}^{\mu_{\ell+1}}, \tag{64}$$

for some integers $\mu_0, \dots, \mu_{\ell+1}$.

Definition 11. *For a given partition Δ, let $\mathbb{P}_p^{U,V}(\Delta)$ be a generalized polynomial space of degree $p \geq 2$, and let $\boldsymbol{\xi}$ be a knot sequence connected to Δ as in* (64). *For any $\xi_i < \xi_{i+1}$, let u_i, v_i be the unique functions in $\langle D^{p-1}U, D^{p-1}V \rangle$ on $[\xi_i^+, \xi_{i+1}^-]$ satisfying*

$$u_i(\xi_i) = 1, \quad u_i(\xi_{i+1}) = 0, \quad v_i(\xi_i) = 0, \quad v_i(\xi_{i+1}) = 1.$$

Suppose for some integer j that $\xi_j \leq \xi_{j+1} \leq \dots \leq \xi_{j+p+1}$ are $p+2$ real numbers taken from $\boldsymbol{\xi}$. The j-th **generalized B-spline** *$B_{j,p,\boldsymbol{\xi}}^{U,V} : \mathbb{R} \to \mathbb{R}$ of degree p is identically zero if $\xi_{j+p+1} = \xi_j$ and otherwise defined recursively by*

$$B_{j,p,\boldsymbol{\xi}}^{U,V}(x) := \int_{\xi_j}^x \frac{B_{j,p-1,\boldsymbol{\xi}}^{U,V}(y)}{\gamma_{j,p-1,\boldsymbol{\xi}}^{U,V}} \,\mathrm{d}y - \int_{\xi_{j+1}}^x \frac{B_{j+1,p-1,\boldsymbol{\xi}}^{U,V}(y)}{\gamma_{j+1,p-1,\boldsymbol{\xi}}^{U,V}} \,\mathrm{d}y, \tag{65}$$

starting with

$$B_{i,1,\boldsymbol{\xi}}^{U,V}(x) := \begin{cases} v_i(x), & \text{if } x \in [\xi_i, \xi_{i+1}), \\ u_{i+1}(x), & \text{if } x \in [\xi_{i+1}, \xi_{i+2}), \\ 0, & \text{otherwise}. \end{cases} \tag{66}$$

Here, $\gamma_{i,k,\boldsymbol{\xi}}^{U,V}$ is defined as the integral of $B_{i,k,\boldsymbol{\xi}}^{U,V}$,

$$\gamma_{i,k,\boldsymbol{\xi}}^{U,V} := \int_{\xi_i}^{\xi_{i+k+1}} B_{i,k,\boldsymbol{\xi}}^{U,V}(y) \,\mathrm{d}y,$$

and we used the convention that if $\gamma_{i,k,\boldsymbol{\xi}}^{U,V} = 0$ then

$$\int_{\xi_i}^x \frac{B_{i,k,\boldsymbol{\xi}}^{U,V}(y)}{\gamma_{i,k,\boldsymbol{\xi}}^{U,V}} \,\mathrm{d}y := \begin{cases} 1, & \text{if } x \geq \xi_{i+k+1}, \\ 0, & \text{otherwise}. \end{cases}$$

We now show that generalized B-splines are a special instance of normalized Tchebycheffian B-splines, and therefore they enjoy all their properties.

Theorem 11. *Generalized B-splines are normalized Tchebycheffian B-splines generated by the Tchebycheffian B-spline weights $w_0, \dots, w_p$ given in* (63).

Proof. We first note that the global weights $w_0, \dots, w_p$ in (63) satisfy the smoothness conditions in Definition 6, so they are actually Tchebycheffian B-spline weights with respect to $\boldsymbol{\xi}$. Let $\boldsymbol{w} := (w_0, \ldots, w_p)$. A direct computation shows that $B^{\boldsymbol{w}}_{i,1,\boldsymbol{\xi}}(x) = B^{U,V}_{i,1,\boldsymbol{\xi}}(x)$ for all i. Indeed, if $\xi_i < \xi_{i+1}$ then from Definition 7 and the weight property (62) we know that

$$B^{\boldsymbol{w}}_{i,1,\boldsymbol{\xi}}(\xi_i) = 0, \qquad \lim_{\substack{x\to\xi_{i+1}\\ x<\xi_{i+1}}} B^{\boldsymbol{w}}_{i,1,\boldsymbol{\xi}}(x) = \lim_{\substack{x\to\xi_{i+1}\\ x<\xi_{i+1}}} w_1(x) = 1,$$

and from the piecewise Tchebycheff structure that $B^{\boldsymbol{w}}_{i,1,\boldsymbol{\xi}}$ belongs to the ET-space $\langle D^{p-1}U, D^{p-1}V\rangle$ on $[\xi_i^+, \xi_{i+1}^-]$. Since the same properties also hold for $B^{U,V}_{i,1,\boldsymbol{\xi}}(x)$, they must be identical on $[\xi_i, \xi_{i+1})$. A similar argument holds for the interval $[\xi_{i+1}, \xi_{i+2})$. As a consequence, taking into account that $w_2 = \cdots = w_p = 1$, it follows clearly from their definitions that $B^{\boldsymbol{w}}_{j,p,\boldsymbol{\xi}}(x) = B^{U,V}_{j,p,\boldsymbol{\xi}}(x)$ for $p \geq 2$ and they are normalized. In other words, generalized B-splines are a special instance of normalized Tchebycheffian B-splines. □

Example 19. Any linear Tchebycheffian B-spline $B^{\boldsymbol{w}}_{j,1,\boldsymbol{\xi}}(x)$ can be written as $B^{U,V}_{j,1,\boldsymbol{\xi}}(x)$ in the form (66), up to the positive scaling factor $w_1(\xi_{j+1})$. In particular, when $U(x) = \cos(x)$, $V(x) = \sin(x)$, we have

$$B^{U,V}_{j,1,\boldsymbol{\xi}}(x) = \begin{cases} \dfrac{\sin(x-\xi_j)}{\sin(\xi_{j+1}-\xi_j)}, & \text{if } x \in [\xi_j, \xi_{j+1}),\\[2mm] \dfrac{\sin(\xi_{j+2}-x)}{\sin(\xi_{j+2}-\xi_{j+1})}, & \text{if } x \in [\xi_{j+1}, \xi_{j+2}),\\[2mm] 0, & \text{otherwise.} \end{cases}$$

This is the scaled version of the spline in Example 15, with scaling factor $w_1(\xi_{j+1}) = \cos(\xi_{j+1})$.

Example 20. The generalized B-spline of degree $p = 2$ on a knot sequence $\boldsymbol{\xi}$ consisting of simple knots is given by

$$B^{U,V}_{j,2,\boldsymbol{\xi}}(x) = \begin{cases} \delta_j \displaystyle\int_{\xi_j}^{x} v_j(y)\,\mathrm{d}y, & \text{if } x \in [\xi_j, \xi_{j+1}),\\[2mm] 1 - \delta_{j+1} \displaystyle\int_{\xi_{j+1}}^{x} v_{j+1}(y) - \delta_i \int_{x}^{\xi_{j+2}} u_{j+1}(y), & \text{if } x \in [\xi_{j+1}, \xi_{j+2}),\\[2mm] \delta_{j+1} \displaystyle\int_{x}^{\xi_{j+3}} u_{j+2}(y)\,\mathrm{d}y, & \text{if } x \in [\xi_{j+2}, \xi_{j+3}),\\[2mm] 0, & \text{otherwise,} \end{cases}$$

where

$$\delta_i := \big(\gamma^{U,V}_{i,1,\boldsymbol{\xi}}\big)^{-1} = \Big(\int_{\xi_i}^{\xi_{i+1}} v_i(y)\,\mathrm{d}y + \int_{\xi_{i+1}}^{\xi_{i+2}} u_{i+1}(y)\,\mathrm{d}y\Big)^{-1}.$$

The normalized Tchebycheffian B-spline defined in Example 16 is a special case, considering the functions $U(x) = \cos(x)$, $V(x) = \sin(x)$, and the uniform knot sequence $\{i\omega\}_{i=0}^{3}$.

Example 21. Consider the partition $\Delta = \{0, 1, 2, 3\}$, and

$$U(x) = \begin{cases} x, & \text{if } x \in [0,1),\\ \mathrm{e}^{\alpha x}, & \text{if } x \in [1,2),\\ x, & \text{if } x \in [2,3), \end{cases} \qquad V(x) = \begin{cases} x^2, & \text{if } x \in [0,1),\\ \mathrm{e}^{-\alpha x}, & \text{if } x \in [1,2),\\ x^2, & \text{if } x \in [2,3). \end{cases}$$

When taking $\boldsymbol{\xi} = \Delta$, we get for $p = 1$,

$$B^{U,V}_{1,1,\boldsymbol{\xi}}(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ \dfrac{\sinh((2-x)\alpha)}{\sinh(\alpha)}, & \text{if } x \in [1,2), \\ 0, & \text{otherwise}, \end{cases} \quad B^{U,V}_{2,1,\boldsymbol{\xi}}(x) = \begin{cases} \dfrac{\sinh((x-1)\alpha)}{\sinh(\alpha)}, & \text{if } x \in [1,2), \\ 3-x, & \text{if } x \in [2,3), \\ 0, & \text{otherwise}, \end{cases}$$

and for $p = 2$,

$$B^{U,V}_{1,2,\boldsymbol{\xi}}(x) = \frac{1}{1 + \dfrac{\sinh(\beta)}{\beta\cosh(\beta)}} \begin{cases} x^2, & \text{if } x \in [0,1), \\ 1 + \dfrac{\cosh(\beta) - \cosh((3-2x)\beta)}{\beta\sinh(\beta)}, & \text{if } x \in [1,2), \\ (x-3)^2, & \text{if } x \in [2,3), \\ 0, & \text{otherwise}, \end{cases} \tag{67}$$

where $\beta := \alpha/2$. The three non-trivial pieces of $B^{U,V}_{1,2,\boldsymbol{\xi}}$ belong to $\mathbb{P}_2$, $\langle 1, \mathrm{e}^{\alpha x}, \mathrm{e}^{-\alpha x}\rangle$, and $\mathbb{P}_2$, respectively. When α tends to zero, the quadratic GB-spline in (67) tends to the quadratic polynomial cardinal B-spline.

Example 22. If $U(x) = x^{p-1}$ and $V(x) = x^p$, then the space in (60) is nothing else than the polynomial space $\mathbb{P}_p$. In this case,

$$u_i(x) = \frac{\xi_{i+1} - x}{\xi_{i+1} - \xi_i}, \quad v_i(x) = \frac{x - \xi_i}{\xi_{i+1} - \xi_i}, \quad \xi_i < \xi_{i+1},$$

and Definition 11 results in the standard **polynomial B-splines** of degree p. This is in agreement with Example 17.

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