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Stochastically forced cardiac bidomain model[☆]

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Abstract

The bidomain system of degenerate reaction–diffusion equations is a well-established spatial model of electrical activity in cardiac tissue, with "reaction" linked to the cellular action potential and "diffusion" representing current flow between cells. The purpose of this paper is to introduce a "stochastically forced" version of the bidomain model that accounts for various random effects. We establish the existence of martingale (probabilistic weak) solutions to the stochastic bidomain model. The result is proved by means of an auxiliary nondegenerate system and the Faedo–Galerkin method. To prove convergence of the approximate solutions, we use the stochastic compactness method and Skorokhod–Jakubowski a.s. representations. Finally, via a pathwise uniqueness result, we conclude that the martingale solutions are pathwise (i.e., probabilistic strong) solutions.

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1. Introduction

1.1. Background

Hodgkin and Huxley [29] introduced the first mathematical model for the propagation of electrical signals along nerve fibers. This model was later tweaked to describe assorted

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phenomena in biology. Similar to nerve cells, conduction of electrical signals in cardiac tissue rely on the flow of ions through so-called ion channels in the cell membrane. This similarity has led to a number of cardiac models based on the Hodgkin–Huxley formalism [11,13,32,42,45,52]. Among these is the *bidomain model* [54], which is regarded as an apt spatial model of the electrical properties of cardiac tissue [13,52].

The bidomain equations result from the principle of conservation of current between the intra- and extracellular domains, followed by a homogenization process of the cellular model defined on a periodic structure of cardiac tissue (see, e.g., [13]). The bidomain model can be viewed as a PDE system, consisting of a degenerate parabolic (reaction–diffusion) PDE for the transmembrane potential and an elliptic PDE for the extracellular potential. These PDEs are supplemented by a nonlinear ODE system for the conduction dynamics of the ion channels. There are many membrane models of cardiac cells, differing in their complexity and in the level of detail with which they represent the biology (see [11] for a review). Herein we will utilize a simple model for voltage-gated ion channels [37].

The idiom "bidomain" reflects that the intra- and extracellular tissues are viewed as two superimposed anisotropic continuous media, with different longitudinal and transversal conductivities. If these conductivities are equal, then we have the so-called monodomain model (elliptic PDE reduces to an algebraic equation). The degenerate structure of the bidomain PDE system is due to the anisotropy of cardiac tissue [2,15]. Solutions exhibit discontinuous-like propagating excitation fronts. This, together with strongly varying time scales, makes the system difficult to solve by numerical methods.

The bidomain model is a deterministic system. This means that at each moment in time, the solution can be inferred from the prescribed data. This is at variance with several phenomena happening at the microscopic (cellular) and macroscopic (heart/torso) scales, where respectively channel noise and external random perturbations acting in the torso can play important roles. At the macroscopic level, the ECG signal, a coarse-grained representation of the electrical activity in the heart, is often contaminated by noise. One source for this noise is the fluctuating environment of the heart. In [36], the authors argue that such randomness cannot always be suppressed. Occasionally deterministic equations give qualitatively incorrect results, and it is important to quantify the nature of the noise and choose an appropriate model incorporating randomness.

At the cellular level, the membrane potential is due to disparities in ion concentrations (e.g., sodium, calcium, potassium) across the cell membrane. The ions move through the cell membrane due to random transitions between open and close states of the ion channels. The dynamics of the voltage potential reflect the aggregated behavior of the individual ion channels, whose conformational changes control the conductance of each ionic current. The profound role of channel noise in excitable cells is summarized and discussed in [26]. Faithful modeling of channel noise gives rise to continuous-time Markov chains with voltage-dependent transition probabilities. In the limit of infinitely many ion channels, these models lead to deterministic Hodgkin–Huxley type equations. To capture channel noise, an alternative (and computationally much simpler) approach is to add well-placed stochastic terms to equations of the Hodgkin–Huxley type [26,35]. Indeed, recent studies (see [26] for a synthesis) indicate that this approach can give an accurate reproduction of channel fluctuations. For work specifically devoted to cardiac cells, see [19,36,42].

1.2. Deterministic bidomain equations

Fix a final time $T > 0$ and a bounded open subset $\Omega \subset \mathbb{R}^3$ representing the heart (cf. Section 2). Roughly speaking, the bidomain equations result from applying Ohm's electrical conduction law and the continuity equation (conservation of electrical charge) to the intracellular and extracellular domains. Let J_i and J_e denote, respectively, the current densities in the intracellular and extracellular domains. Moreover, denote by I_m the membrane current per unit volume and by I_i, I_e the injected stimulating currents. The continuity equations are

$$\nabla \cdot J_i = -I_m + I_i, \quad \nabla \cdot J_e = I_m + I_e. \tag{1.1}$$

The negative sign in the first equation reflects that the current leaving the intracellular domain is positive. We assume that the intracellular and extracellular current densities can be written in terms of potentials u_i, u_e as follows: $J_i = -M_i \nabla u_i$, $J_e = -M_e \nabla u_e$, where M_i, M_e are the intracellular and extracellular conductivity tensors. The transmembrane potential v is defined as $v := u_i - u_e$. Hence, the continuity equations (1.1) become

$$-\nabla \cdot (M_i \nabla u_i) = -I_m + I_i, \quad -\nabla \cdot (M_e \nabla u_e) = I_m + I_e. \tag{1.2}$$

By adding the equations in (1.2), we obtain

$$-\nabla \cdot ((M_i + M_e)\nabla u_e) - \nabla \cdot (M_i \nabla v) = I_i + I_e \quad \text{in } \Omega \times (0, T). \tag{1.3}$$

The membrane current I_m splits into a capacitive current I_c, since the cell membrane acts as a capacitor, and an ionic current, due to the flowing of ions through different ion channels (and also pumps/exchangers):

$$I_m = \chi_m \left(I_c + I_{\text{ion}}\right), \qquad I_c = c_m \frac{\partial v}{\partial t}, \qquad I_{\text{ion}} = I_{\text{ion}}(v, w), \tag{1.4}$$

where χ_m is the ratio of membrane surface area to tissue volume and $c_m > 0$ is the (surface) capacitance of the membrane per unit area. The (nonlinear) function $I_{\text{ion}}(v, w)$ represents the ionic current per unit surface area, which depends on the transmembrane potential v and a vector w of ionic (recovery, gating, concentrations, etc.) variables. A simplified model, frequently used for analysis, assumes that the functional form of I_{ion} is a cubic polynomial in v. The ionic variables w are governed by an ODE system,

$$\frac{\partial w}{\partial t} = H(v, w) \quad \text{in } \Omega \times (0, T), \tag{1.5}$$

where, as alluded to earlier, various membrane models exist for cardiac cells, giving rise to different choices of H (and I_{ion}). Inserting (1.4) into (1.2), we arrive at

$$\chi_m c_m \frac{\partial v}{\partial t} - \nabla \cdot (M_i \nabla (v + u_e)) + \chi_m I_{\text{ion}}(v, w) = I_i \quad \text{in } \Omega \times (0, T). \tag{1.6}$$

The system (1.3), (1.5), (1.6) is sometimes referred to as the *parabolic–elliptic form* of the bidomain model, as it contains a parabolic PDE (1.6) for the transmembrane potential v and an elliptic PDE (1.3) for the extracellular potential u_e. The bidomain equations are closed by specifying initial conditions for v, w and boundary conditions for u_i, u_e. Electrically isolated heart tissue, for example, leads to zero flux boundary conditions.

Herein we will rely on a slightly different form of the bidomain model, obtained by inserting (1.4) into both equations in (1.2):

$$\begin{aligned}
&\chi_m c_m \frac{\partial v}{\partial t} - \nabla \cdot (M_i \nabla u_i) + \chi_m I_{\text{ion}}(v, w) = I_i \quad \text{in } \Omega \times (0, T), \\
&\chi_m c_m \frac{\partial v}{\partial t} + \nabla \cdot (M_e \nabla u_e) + \chi_m I_{\text{ion}}(v, w) = -I_e \quad \text{in } \Omega \times (0, T).
\end{aligned} \tag{1.7}$$

Consisting of two (degenerate) parabolic PDEs, the system (1.5), (1.7) is occasionally referred to as the *parabolic–parabolic form* of the bidomain model. On the subject of well-posedness, i.e., existence, uniqueness, and stability of properly defined solutions, we remark that standard theory for parabolic–elliptic systems does not apply naturally. The main reason is that the anisotropies of the intra- and extracellular domains differ, entailing the degenerate structure of the system. Moreover, a maximum principle is not available. That being the case, a number of works [1,2,5,6,13,15,23,34,55] have recently provided well-posedness results for the bidomain model, applying differing solution concepts and technical frameworks.

1.3. Stochastic model & main results

The purpose of the present paper is to introduce and analyze a bidomain model that accounts for random effects (noise), by way of a few well-placed stochastic terms. The simplest way to insert randomness is to add Gaussian white noise to one or more of the ionic ODEs (1.5), leading to a system of (Itô) stochastic differential equations (SDEs):

$$dw = H(v, w)\,dt + \alpha\,dW^w, \tag{1.8}$$

where W^w is a cylindrical Wiener process, with noise amplitude α. Formally, we can think of $\alpha\,dW^w$ as $\sum_{k\geq 1} \alpha_k\,dW_k^w(t)$, where $\{W_k^w\}_{k\geq 1}$ is a sequence of independent 1D Brownian motions and $\{\alpha_k\}_{k\geq 1}$ is a sequence of noise coefficients. Interpreting w as gating variables representing the fraction of open channel subunits of varying types, in [26] this type of noise is referred to as *subunit noise*. We will allow for subunit noise in our model, assuming for simplicity that the ionic variable w is a scalar and that the noise amplitude depends on the transmembrane potential v, $\alpha = \alpha(v)$ (multiplicative noise). We will also introduce fluctuations into the bidomain system by replacing the PDEs (1.7) with the (Itô) stochastic partial differential equations (SPDEs)

$$\begin{aligned} \chi_m c_m dv - \nabla\cdot(M_i \nabla u_i)\,dt + \chi_m I_{\mathrm{ion}}(v, w)\,dt &= I_i\,dt + \beta\,dW^v \\ \chi_m c_m dv + \nabla\cdot(M_e \nabla u_e)\,dt + \chi_m I_{\mathrm{ion}}(v, w)\,dt &= -I_e\,dt + \beta\,dW^v, \end{aligned} \tag{1.9}$$

where W^v is a cylindrical Wiener process (independent of W^w), with noise amplitude β. Adding a stochastic term to the equation for the membrane potential v is labeled *current noise* in [26]. Current noise represents the aggregated effect of the random activity of ion channels on the voltage dynamics. Allowing the noise amplitude in (1.9) to depend on the membrane voltage v, we arrive at equations with so-called *conductance noise* [26]. The nonlinear term $I_{\mathrm{ion}}(v, w)$ accounts for the total conductances of various ionic currents, and conductance noise pertains to adding "white noise" to the deterministic values of the conductances, i.e., replacing I_{ion} by $I_{\mathrm{ion}} + \hat{\beta}(v)\,\frac{dW_v}{dt}$, for some function $\hat{\beta}$. Herein we include this case by permitting β in (1.9) to depend on the voltage variable v, $\beta = \beta(v)$.

Our main contribution is to establish the existence of properly defined solutions to the SDE–SPDE system (1.8), (1.9). From the PDE perspective, we are searching for weak solutions in a certain Sobolev space (H^1). From the probabilistic point of view, we are considering martingale solutions, sometimes also referred to as weak solutions. The notions of weak & strong probabilistic solutions have different meaning from weak & strong solutions in the PDE literature. If the stochastic elements are fixed in advance, we speak of a strong (or pathwise) solution. The stochastic elements are collected in a stochastic basis $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, W\right)$, where $W = (W_w, W_v)$ are cylindrical Wiener processes adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Whenever these elements constitute a part of the unknown solution, the relevant notion is that of a martingale solution. The connection between weak and strong solutions to Itô equations is exposed in the famous Yamada–Watanabe theorem, see, e.g., [41]. We reserve the name *weak martingale solution* for solutions that are weak in the PDE sense as well as being probabilistic weak.

We will prove that there exists a weak martingale solution to the stochastic bidomain system. Motivated by the approach in [2] (see also [5]) for the deterministic system, we use the Faedo–Galerkin method to construct approximate solutions, based on an auxiliary nondegenerate system obtained by adding εdu_i and $-\varepsilon du_e$ respectively to the first and second equations in (1.9) (ε is a small positive parameter). The stochastic compactness method is put to use to conclude subsequential convergence of the approximate solutions.

Indeed, we first apply the Itô chain rule to derive some basic a priori estimates. The combination of multiplicative noise and the specific structure of the system makes these estimates notably harder to obtain than in the deterministic case. The a priori estimates lead to strong compactness of the approximations in the t, x variables (in the deterministic context [2]). In the stochastic setting, there is an additional (probability) variable $\omega \in D$ in which strong compactness is not expected. Traditionally, one handles this issue by arguing for weak compactness of the probability laws of the approximate solutions, via tightness and Prokhorov's theorem. The ensuing step is to construct a.s. convergent versions of the approximations using the Skorokhod representation theorem. This theorem supplies new random variables on a new probability space, with the same laws as the original variables, converging almost surely. Equipped with a.s. convergence, we are able to show that the limit variables constitute a weak martingale solution. Finally, thanks to a uniqueness result and the Gyöngy–Krylov characterization of convergence in probability [27], we pass *à la* Yamada–Watanabe from martingale to pathwise (probabilistic strong) solutions.

Martingale solutions and the stochastic compactness method have been harnessed by many authors for different classes of SPDEs, see e.g. [3,4,16,17,21,22,24,28,30,38,43,46,47] for problems related to fluid mechanics. An important step in the compactness method is the construction of almost surely convergent versions of processes that converge weakly. This construction dates back to the work of Skorokhod, for processes taking values in a Polish (complete separable metric) space [16]. The classical Skorokhod theorem is befitting for the transmembrane variable v, but not the intracellular and extracellular variables u_i, u_e. This fact is a manifestation of the degenerate structure of the bidomain system, necessitating the use of a Bochner–Sobolev space equipped with the weak topology. We refer to Jakubowski [31] for a recent variant of the representation theorem that applies to so-called quasi-Polish spaces, specifically allowing for separable Banach spaces equipped with the weak topology, as well as spaces of weakly continuous functions with values in a separable Banach space. We refer to [7–10,40,51] for works making use of Skorokhod–Jakubowski a.s. representations.

The remaining part of this paper is organized as follows: The stochastic bidomain model is presented in Section 2. Section 3 outlines the underlying stochastic framework and list the conditions imposed on the "stochastic" data of the model. Solution concepts and the accompanying main results are collected in Section 4. The approximate (Faedo–Galerkin) solutions are constructed in Section 5. In Section 6 we establish several a priori estimates and prove convergence of the approximate solutions, thereby providing an existence result for weak martingale solutions. A pathwise uniqueness result is established in Section 7, which is then used in Section 8 to upgrade martingale solutions to pathwise solutions.

2. Stochastic bidomain model

The spatial domain of the heart is given by a bounded open set $\Omega \subset \mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$. This three-dimensional slice of the cardiac muscle is viewed as two superimposed (anisotropic) continuous media, representing the intracellular (i) and extracellular (e) tissues. The tissues are connected at each point via the cell membrane. In our earlier outline of the (deterministic) bidomain model, we saw that the relevant quantities are the *intracellular* and *extracellular* potentials

$$u_i = u_i(x,t) \quad \text{and} \quad u_e = u_e(x,t), \qquad (x,t) \in \Omega_T := \Omega \times (0,T),$$

as well as the *transmembrane potential* $v := u_i - u_e$ (defined in Ω_T).

The conductivities of the intracellular and extracellular tissues are encoded in anisotropic matrices $M_i = M_i(x)$, $M_e = M_e(x)$. Herein we do not exploit structural properties of cardiac tissue, and assume that $M_i, M_e > 0$ are general matrices, cf. (2.5) below. For the modeling of electrical conductivities of cardiac tissue, see for example [12,13,52].

The stochastic bidomain model contains two nonlinearly coupled SPDEs involving the potentials u_i, u_e, v. These stochastic reaction–diffusion equations are further coupled to a nonlinear SDE for the gating (recovery) variable w. The dynamics of (u_i, u_e, v, w) is governed by the equations

$$\begin{aligned}
&\chi_m c_m dv - \nabla\cdot\big(M_i \nabla u_i\big)\,dt + \chi_m I_{\text{ion}}(v,w)\,dt = I_i\,dt + \beta(v)\,dW^v \quad \text{in } \Omega_T,\\
&\chi_m c_m dv + \nabla\cdot\big(M_e \nabla u_e\big)\,dt + \chi_m I_{\text{ion}}(v,w)\,dt = -I_e\,dt + \beta(v)\,dW^v \quad \text{in } \Omega_T,\\
&dw = H(v,w)\,dt + \alpha(v)\,dW^w \quad \text{in } \Omega_T,
\end{aligned} \tag{2.1}$$

where $c_m > 0$ is the surface capacitance of the membrane, χ_m is the surface-to-volume ratio, and I_i, I_e are stimulation currents. In (2.1), randomness is represented by cylindrical Wiener processes W^v, W^w with nonlinear noise amplitudes β, α (cf. Section 3 for details).

We impose initial conditions on the transmembrane potential and the gating variable:

$$v(0,x) = v_0(x), \qquad w(0,x) = w_0(x), \qquad x \in \Omega. \tag{2.2}$$

The intra- and extracellular domains are often assumed to be electrically isolated, giving rise to zero flux (Neumann type) boundary conditions on the potentials u_i, u_e [13,52]. From a mathematical point of view, Dirichlet and mixed Dirichlet–Neumann type boundary conditions are utilized in [1] and [2], respectively. Herein we partition the boundary $\partial\Omega$ into regular parts Σ_N and Σ_D and impose the mixed boundary conditions ($j = i, e$)

$$\begin{aligned}
&\big(M_j(x)\nabla u_j\big)\cdot\nu = 0 \quad \text{on } \Sigma_{N,T} := \Sigma_N \times (0,T),\\
&u_j = 0 \qquad\qquad\qquad\; \text{on } \Sigma_{D,T} := \Sigma_D \times (0,T),
\end{aligned} \tag{2.3}$$

where ν denotes the exterior unit normal to the "Neumann part" Σ_N of the boundary, which is defined a.e. with respect to the two-dimensional Hausdorff measure $\mathcal{H}^2$ on $\partial\Omega$.

Observe that the equations in (2.1) are invariant under the change of u_i and u_e into $u_i + k, u_e + k$, for any $k \in \mathbb{R}$. Hence, unless Dirichlet conditions are imposed somewhere ($\Sigma_D \neq \emptyset$), the bidomain system determines the electrical potentials only up to an additive constant. To ensure a unique solution in the case $\Sigma_D := \emptyset$ ($\partial\Omega = \Sigma_N$), we may impose the normalization condition $\int_\Omega u_e(x,t)\,dx = 0$. To avoid making this paper too long, we assume that $\Sigma_D \neq \emptyset$. Moreover, we stick to homogeneous boundary conditions, although we could have replaced the right-hand sides of (2.3) by sufficiently regular functions.

Regarding the "membrane" functions I_{ion} and H, we have in mind the fairly uncluttered FitzHugh–Nagumo model [20,39]. This is a simple choice for the membrane kinetics that is often used to avoid difficulties arising from a large number of coupling variables. The model is specified by

$$I_{\text{ion}}(v,w) = -v\,(v-a)\,(1-v) + w, \qquad H(v,w) = \epsilon(\kappa v - \gamma w),$$

where the parameter a represents the threshold for excitation, ϵ represents excitability, and κ, γ, δ are parameters that influence the overall dynamics of the system. For background material on cardiac membrane models and their general mathematical structure, we refer to the books [13,32,52].

In an attempt to simplify the notation, we redefine M_i, M_e as $\frac{1}{\chi_m c_m} M_i$, $\frac{1}{\chi_m c_m} M_e$, and set $I := \frac{1}{c_m} I_{\text{ion}}$, $\eta := \frac{1}{\chi_m c_m}\beta$. We also assume $I_i, I_e \equiv 0$, as these source terms do not add new difficulties. The resulting stochastic bidomain system becomes

$$\begin{cases} dv - \nabla\cdot\left(M_i \nabla u_i\right) dt + I(v,w)\,dt = \eta(v)\,dW^v & \text{in } \Omega_T, \\ dv + \nabla\cdot\left(M_e \nabla u_e\right) dt + I(v,w)\,dt = \eta(v)\,dW^v & \text{in } \Omega_T, \\ dw = H(v,w)\,dt + \sigma(v)\,dW^w & \text{in } \Omega_T, \end{cases} \tag{2.4}$$

along with the initial and boundary conditions (2.2) and (2.3). The cylindrical Wiener processes W^v, W^w in (2.4) are defined in Section 3.

With regard to the conductivity matrices in (2.4), we assume the existence of positive constants m, M such that for $j = i, e$,

$$M_j \in L^\infty, \quad m\,|\xi|^2 \le \xi^\top M_j(x)\,\xi \le M\,|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \text{ for a.e. } x. \tag{2.5}$$

Motivated by the discussion above on membrane models, we impose the following set of assumptions on the functions I, H in (2.4):

- Generalized FitzHugh–Nagumo model (**GFHN**):

  $$I(v,w) = I_1(v) + I_2(v)w, \qquad H(v,w) = h(v) + c_{H,1} w,$$

  where $I_1, I_2, h \in C^1(\mathbb{R})$ and for all $v \in \mathbb{R}$,

  $$\begin{aligned} &|I_1(v)| \le c_{I,1}\left(1 + |v|^3\right), \quad I_1(v)v \ge \underline{c}_I\,|v|^4 - c_{I,2}\,|v|^2, \\ &I_2(v) = c_{I,3} + c_{I,4} v, \quad |h(v)| \le c_{H,2}\left(1 + |v|^2\right), \end{aligned}$$

  for some positive constants $c_{I,1}, c_{I,2}, c_{I,3}, c_{I,4}, c_{H,1}, c_{H,2}$ and $\underline{c}_I > 0$.
  There exist $\mu, \lambda > 0$ such that

  $$\begin{aligned} &\mu\left(I(v_2,w_2) - I(v_1,w_2)\right)(v_2 - v_1) - \left(H(v_2,w_2) - H(v_1,w_1)\right)(w_2 - w_1) \\ &\quad \ge -\lambda\left(|v_2 - v_1|^2 + |w_2 - w_1|^2\right), \qquad \forall v_1, v_2, w_1, w_2 \in \mathbb{R}. \end{aligned} \tag{2.6}$$

The "dissipative" condition (2.6), involving an appropriate linear combination of I and H, is linked to stability and uniqueness results. It will be used in Lemma 5.2 (existence of Faedo–Galerkin solutions), cf. (5.24), and Theorem 7.2 (L^2 stability and uniqueness). It can be verified for the FitzHugh–Nagumo model. We refer to [6, pages 478–479] for additional details and a more general condition.

The (generalized) FitzHugh–Nagumo model is a simplification of the Hodgkin–Huxley model of voltage-gated ion channels. It is possible to treat other membrane models by blending the arguments used herein with those found in [5,6,55].

We end this section with a remark about the so-called *monodomain model*.

Remark 2.1. The stochastic bidomain model simplifies if $M_i = \lambda M_e$ for some constant $\lambda > 0$. In this case the first two equations in (2.4) can be combined into a single equation; thereby arriving at the stochastic monodomain system

$$\begin{aligned} &dv - \nabla\cdot(M\nabla v)\,dt + I(v,w)\,dt = \eta(v)\,dW^v \quad \text{in } \Omega_T,\\ &dw = H(v,w)\,dt + \sigma(v)dW^w \quad \text{in } \Omega_T, \end{aligned} \tag{2.7}$$

where $M := \frac{\lambda}{1+\lambda}M_i$. The system (2.7) is a significant simplification of the bidomain model (2.4), and even though the assumption of equal anisotropy ratios is very strong, the monodomain model is adequate in certain situations [14].

3. Stochastic framework

We refer to the books [16,41] for relevant notation, basic concepts, and results from stochastic analysis, including the theory of cylindrical Wiener processes and stochastic integration. We consider a complete probability space $(D, \mathcal{F}, P)$, along with a complete right-continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Without loss of generality, we assume that the σ-algebra $\mathcal{F}$ is countably generated. Let $\{W_k\}_{k=1}^{\infty}$ be a sequence of independent one-dimensional Brownian motions adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We refer to

$$\mathcal{S} = \left(D, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, \{W_k\}_{k=1}^{\infty}\right) \tag{3.1}$$

as a (Brownian) *stochastic basis*.

Fix a separable Hilbert space $\mathbb{U}$, equipped with a complete orthonormal basis $\{\psi_k\}_{k\ge1}$. We use cylindrical Wiener processes W evolving over $\mathbb{U}$, namely

$$W(\omega, t, \cdot) := \sum_{k\ge1} W_k(\omega, t)\psi_k(\cdot) \tag{3.2}$$

where the right-hand side of (3.2) converges on a larger Hilbert space $\mathbb{U}_0$, such that the embedding $\mathbb{U} \subset \mathbb{U}_0$ is Hilbert–Schmidt. Via standard martingale arguments, W is almost surely continuous with values in $\mathbb{U}_0$, that is, $W(\omega,\cdot,\cdot) \in C([0,T];\mathbb{U}_0)$ for P-a.e. $\omega \in D$. We also have $W \in L^2(D, \mathcal{F}, P; C([0,T];\mathbb{U}_0))$. Without loss of generality, we assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by W and the initial data. See [16,41] for details.

Let $\mathbb{X}$ be a separable Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{X}}$ and norm $\|\cdot\|_{\mathbb{X}}$. For the bidomain model (2.4), a natural choice is $\mathbb{X} = L^2(\Omega)$. The vector space of all bounded linear operators from $\mathbb{U}$ to $\mathbb{X}$ is denoted $L(\mathbb{U},\mathbb{X})$. We denote by $L_2(\mathbb{U},\mathbb{X})$ the collection of Hilbert–Schmidt operators from $\mathbb{U}$ to $\mathbb{X}$, that is, $R \in L_2(\mathbb{U},\mathbb{X})$ if and only if $R \in L(\mathbb{U},\mathbb{X})$ and $\|R\|^2_{L_2(\mathbb{U},\mathbb{X})} := \sum_{k\ge1}\|R\psi_k\|^2_{\mathbb{X}} < \infty$.

Given a cylindrical Wiener process W, we define the Itô stochastic integral $\int G\,dW$ as follows [16,41]:

$$\int_0^t G\,dW = \sum_{k=1}^{\infty}\int_0^t G_k\,dW_k, \qquad G_k := G\psi_k, \tag{3.3}$$

provided the integrand G is a predictable $\mathbb{X}$-valued process satisfying

$$G \in L^2\left(D, \mathcal{F}, P; L^2((0,T); L_2(\mathbb{U},\mathbb{X}))\right).$$

The stochastic integral (3.3) is an $\mathbb{X}$-valued square integrable martingale, satisfying the Burkholder–Davis–Gundy inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t G\,dW\right\|_{\mathbb{X}}^p\right] \le C\,\mathbb{E}\left[\left(\int_0^T \|G\|_{L_2(\mathbb{U},\mathbb{X})}^2\,dt\right)^{\frac{p}{2}}\right], \tag{3.4}$$

where C is a constant depending on $p \ge 1$.

For the bidomain model (2.4), we take $\mathbb{X} = L^2(\Omega)$. With this choice, we can give meaning to the stochastic terms

$$\int_\Omega \left(\int_0^t \beta(v)\,dW\right)\varphi\,dx, \qquad (\beta, W) = (\eta, W^v) \text{ or } (\sigma, W^w),$$

appearing in the weak formulation of (2.4), with $\varphi \in L^2(\Omega)$. Since $W = \sum_{k\ge1} W_k\psi_k$ is a cylindrical Brownian motion, we can write

$$\begin{aligned}\int_\Omega \left(\int_0^t \beta(v)\,dW\right)\varphi\,dx &= \int_\Omega \left(\sum_{k\ge1}\int_0^t \beta_k(v)\,dW\right)\varphi\,dx\\ &= \sum_{k\ge1}\int_0^t\int_\Omega \beta_k(v)\varphi\,dx\,dW_k,\end{aligned} \tag{3.5}$$

knowing that the series converges in $L^2(D,\mathcal{F},P;C([0,T]))$, where $\beta_k(v) := \beta(v)\psi_k$ are real-valued functions. Sometimes we denote the right-hand side by $\int_0^t\int_\Omega \beta(v)\varphi\,dx\,dW^v$.

We need to impose conditions on the noise amplitudes $\beta = \eta, \sigma$. For each $v \in L^2(\Omega)$, we assume that $\beta(v) : \mathbb{U} \to L^2(\Omega)$ is defined by

$$\beta(v)\psi_k = \beta_k(v(\cdot)), \quad k \ge 1,$$

for some real-valued functions $\beta_k(\cdot) : \mathbb{R} \to \mathbb{R}$ that satisfy

$$\begin{aligned}&\sum_{k\ge1} |\beta_k(v)|^2 \le C_\beta\left(1 + |v|^2\right), \qquad \forall v \in \mathbb{R},\\ &\sum_{k\ge1} |\beta_k(v_1) - \beta_k(v_2)|^2 \le C_\beta\,|v_1 - v_2|^2, \qquad \forall v_1, v_2 \in \mathbb{R},\end{aligned} \tag{3.6}$$

for a constant $C_\beta > 0$. As a result, β becomes a mapping from $L^2(\Omega)$ to $L_2(\mathbb{U}, L^2(\Omega))$. More precisely, we have

$$\begin{aligned}&\|\beta(v)\|_{L_2(\mathbb{U},L^2(\Omega))}^2 \le C_\beta\left(1 + \|v\|_{L^2(\Omega)}^2\right), \quad v \in L^2(\Omega),\\ &\|\beta(v_1) - \beta(v_2)\|_{L_2(\mathbb{U},L^2(\Omega))}^2 \le C_\beta\,\|v_1 - v_2\|_{L^2(\Omega)}^2, \quad v_1, v_2 \in L^2(\Omega).\end{aligned} \tag{3.7}$$

Let $(\beta, W) = (\eta, W^v)$ or (σ, W^w). Given a predictable process

$$v \in L^2\left(D, \mathcal{F}, P; L^2((0,T); L^2(\Omega))\right),$$

the stochastic integral $\int_0^t \beta(v)\,dW^w$ is well-defined, taking values in $L^2(\Omega)$. Indeed,

$$\begin{aligned}\mathbb{E}\left[\left|\int_\Omega\left(\int_0^t \beta(v)\,dW\right)\varphi\,dx\right|^2\right] &\le \mathbb{E}\left[\left\|\int_0^t \beta(v)\,dW\right\|_{L^2(\Omega)}^2\right]\|\varphi\|_{L^2(\Omega)}^2\\ &\overset{(3.4)}{\le} C_\varphi\,\mathbb{E}\left[\int_0^T \|\beta(v)\|_{L_2(\mathbb{U},L^2(\Omega))}^2\,dt\right] \overset{(3.7)}{<} \infty,\end{aligned}$$

for any $\varphi \in L^2(\Omega)$. Hence, (3.5) makes sense.

Remark 3.1. The condition (3.6) on the noise amplitude allows for various additive and multiplicative noises, see e.g. [25, Example 3.2] for a list of representative examples.

It is possible to allow $\beta = \eta, \sigma$ to be time and space dependent, $\beta = \beta(t, x, v)$. Then β must satisfy (3.6) for a.e. $t \in [0, T]$, with a constant C_β that is independent of t. This does not entail additional effort in the proofs, but for simplicity of presentation we suppress the t, x dependency throughout the paper.

We will construct weak martingale solutions by applying the stochastic compactness method to a sequence of approximate solutions. In one step of the argument, we show tightness of the probability laws of the approximations. By the Prokhorov theorem, this is equivalent to exhibiting weak compactness of the laws. Relating to convergence of the approximate solutions, it is essential that we secure strong compactness (a.s. convergence) in the ω variable. To that end, we need of a Skorokhod a.s. representation theorem, delivering a new probability space and new random variables, with the same laws as the original ones, converging almost surely. As alluded to before, our path space is not a Polish space since weak topologies in Hilbert and Banach spaces are not metrizable. Thus the original Skorokhod theorem is not applicable; instead we will use the recent Jakubowski version [31] that applies to so-called quasi-Polish spaces. "Quasi-Polish" refers to spaces $\mathbb{S}$ for which there exists a countable family

$$\{f_\ell : \mathbb{S} \to [-1, 1]\}_{\ell \in L} \tag{3.8}$$

of continuous functionals that separate points (of $\mathbb{S}$) [31]. Quasi-Polish spaces include separable Banach spaces equipped with the weak topology, and also spaces of weakly continuous functions taking values in some separable Banach space. The basic assumption (3.8) gives rise to a mapping between $\mathbb{S}$ and the Polish space $[-1, 1]^L$,

$$\mathbb{S} \ni u \mapsto \tilde{f}(u) = \{f_\ell(u)\}_{\ell \in L} \in [-1, 1]^L, \tag{3.9}$$

which is one-to-one and continuous, but in general $\tilde{f}$ is not a homeomorphism of $\mathbb{S}$ onto a subspace of $\mathbb{S}$. However, if we restrict to a σ-compact subspace of $\mathbb{S}$, then $\tilde{f}$ becomes a measurable isomorphism [31]. In this paper we use the following form of the Skorokhod–Jakubowski theorem [31], taken from [8,40] (see also [9,10]).

Theorem 3.2. *[Skorokhod–Jakubowski a.s. Representations for Subsequences] Let $\mathbb{S}$ be a topological space for which there exists a sequence $\{f_\ell\}_{\ell \geq 1}$ of continuous functionals $f_\ell : \mathbb{S} \to \mathbb{R}$ that separate points of $\mathbb{S}$. Denote by Σ the σ-algebra generated by the maps $\{f_\ell\}_{\ell \geq 1}$. Then*

(1) every compact subset of $\mathbb{S}$ is metrizable;

(2) every Borel subset of a σ-compact set in $\mathbb{S}$ belongs to Σ;

(3) every probability measure supported by a σ-compact set in $\mathbb{S}$ has a unique Radon extension to the Borel σ-algebra $\mathcal{B}(\mathbb{S})$;

(4) if $\{\mu_n\}_{n \geq 1}$ is a tight sequence of probability measures on $(\mathbb{S}, \Sigma)$, then there exist a subsequence $\{n_k\}_{k \geq 1}$, a probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$, and Borel measurable $\mathbb{S}$-valued random variables $\tilde{X}_k$, $\tilde{X}$, such that μ_{n_k} is the law of $\tilde{X}_k$ and $X_k \to X$ $\tilde{P}$-a.s. (in $\mathbb{S}$). Moreover, the law μ of $\tilde{X}$ is a Radon measure.

We will need the Gyöngy–Krylov characterization of convergence in probability [27]. It will be used to upgrade weak martingale solutions to strong (pathwise) solutions, via a pathwise uniqueness result.

Lemma 3.3 (*Gyöngy–Krylov Characterization*)**.** *Let $\mathbb{S}$ be a Polish space, and let $\{X_n\}_{n\geq 1}$ be a sequence of $\mathbb{S}$-valued random variables on a probability space $(D, \mathcal{F}, P)$. For each $n, m \geq 1$, denote by $\mu_{n,m}$ the joint law of (X_n, X_m), that is,*

$$\mu_{n,m}(A) := P\left(\{\omega \in D : (X_n(\omega), X_m(\omega)) \in A\}\right), \quad A \in \mathcal{B}(\mathbb{S} \times \mathbb{S}).$$

Then $\{X_n\}_{n\geq 1}$ converges in probability (and P-a.s. along a subsequence) $\Longleftrightarrow$ for any subsequence $\{\mu_{m_k,n_k}\}_{k\geq 1}$ there exists a further subsequence that converges weakly to some $\mu \in \mathcal{P}(\mathbb{S})$ that is supported on the diagonal: $\mu\left(\{(X, Y) \in \mathbb{S} \times \mathbb{S} : X = Y\}\right) = 1$.

Remark 3.4. As a matter of fact, we need access to the "$\Longleftarrow$" part of the Gyöngy–Krylov lemma for quasi-Polish spaces $\mathbb{S}$. Suppose for any subsequence $\left\{\left(X_{n_k}, X_{m_k}\right)\right\}_{k\geq 1}$ there exists a further subsequence $\left\{\left(X_{n_{k_j}}, X_{m_{k_j}}\right)\right\}_{j\geq 1}$ that converges in distribution to (X, X) as $j \to \infty$, for some $X \in \mathbb{S}$, that is, the joint probability laws $\mu_{m_{k_j},n_{k_j}}$ converge weakly to some $\mu \in \mathcal{P}(\mathbb{S}\times\mathbb{S})$ that is supported on the diagonal. Recalling the mapping $\tilde{f}$ between $\mathbb{S}$ and the Polish space $[-1, 1]^L$, cf. (3.9), and the continuous mapping theorem, it follows that the sequence $\left\{\left(\tilde{f}(X_{n_{k_j}}), \tilde{f}(X_{m_{k_j}})\right)\right\}_{j\geq 1}$ converges in distribution to $(f(X), f(X))$ as $j \to \infty$. In view of the Gyöngy–Krylov lemma, this implies that the sequence $\left\{\tilde{f}(X_n)\right\}_{n\geq 1}$ converges in probability and thus, along a subsequence $\left\{\tilde{f}(X_{n_j})\right\}_{j\geq 1}$, P-almost surely. Since $\{f_\ell\}_{\ell\geq 1}$ separate points of $\mathbb{S}$, it is not difficult to see that this implies that $\left\{X_{n_j}\right\}_{j\geq 1}$ converges P-a.s. as well.

4. Notion of solution and main results

Depending on the (probabilistic) notion of solution, the initial data (2.2) are imposed differently. For pathwise (probabilistic strong) solutions, we prescribe the initial data as random variables $v_0, w_0 \in L^2(D, \mathcal{F}, P; L^2(\Omega))$. For martingale (or probabilistic weak) solutions, of which the stochastic basis is an unknown component, we prescribe the initial data in terms of probability measures μ_{v_0}, μ_{w_0} on $L^2(\Omega)$. The measures μ_{v_0} and μ_{w_0} should be viewed as "initial laws" in the sense that the laws of $v(0)$, $w(0)$ are required to coincide with μ_{v_0}, μ_{w_0}, respectively.

Sometimes we need to assume the existence of a number $q_0 > \frac{9}{2}$ such that

$$\int_{L^2(\Omega)} \|v\|_{L^2(\Omega)}^{q_0} \, d\mu_{v_0}(v) < \infty, \qquad \int_{L^2(\Omega)} \|w\|_{L^2(\Omega)}^{q_0} \, d\mu_{w_0}(w) < \infty. \tag{4.1}$$

As a matter of fact, we mostly need (4.1) with $q_0 > 2$. One exception occurs in Section 6.5, where we use $q_0 > \frac{9}{2}$ to conclude that the transmembrane potential v is a.s. weakly time continuous, cf. part (5) in the definition below (for w this holds with just $q_0 > 2$).

Let us define precisely what is meant by a solution to the stochastic bidomain model. For this, we use the space

$$H_D^1(\Omega) := \text{closure of the set } \left\{v \in C^\infty(\mathbb{R}^3),\ v\big|_{\Sigma_D} = 0\right\} \text{ in the } H^1(\Omega) \text{ norm.}$$

We denote by $(H_D^1(\Omega))^*$ the dual of $H_D^1(\Omega)$, which is equipped with the norm

$$\left\|u^*\right\|_{(H_D^1(\Omega))^*} = \sup_{\substack{\phi\in H_D^1(\Omega) \\ \|\phi\|_{H_D^1(\Omega)}\leq 1}} \left\langle u^*, \phi\right\rangle_{(H_D^1(\Omega))^*, H_D^1(\Omega)}. \tag{4.2}$$

Definition 4.1 (*Weak Martingale Solution*). Let μ_{v_0} and μ_{w_0} be probability measures on $L^2(\Omega)$. A weak martingale solution of the stochastic bidomain system (2.4), with initial–boundary data (2.2)–(2.3), is a collection $\left(\mathcal{S}, u_i, u_e, v, w\right)$ satisfying

(1) $\mathcal{S} = \left(D, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, \left\{W_k^v\right\}_{k=1}^\infty, \left\{W_k^w\right\}_{k=1}^\infty\right)$ is a stochastic basis;
(2) $W^v := \sum_{k\geq 1} W_k^v e_k$ and $W^w := \sum_{k\geq 1} W_k^w e_k$ are two independent cylindrical Brownian motions, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$;
(3) For P-a.e. $\omega \in D$, $u_i(\omega), u_e(\omega) \in L^2((0,T); H_D^1(\Omega))$;
(4) For P-a.e. $\omega \in D$, $v(\omega) \in L^2((0,T); H_D^1(\Omega)) \cap L^4(\Omega_T)$. Moreover, $v = u_i - u_e$;
(5) $v, w : D \times [0,T] \to L^2(\Omega)$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted processes, $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable in $(H_D^1(\Omega))^*$, such that for P-a.e. $\omega \in D$,

$$v(\omega), w(\omega) \in L^\infty((0,T); L^2(\Omega)) \cap C([0,T]; (H_D^1(\Omega))^*);$$

(6) The laws of $v_0 := v(0)$ and $w_0 := w(0)$ are respectively μ_{v_0} and μ_{w_0}.
(7) The following identities hold P-almost surely, for any $t \in [0,T]$:

$$\begin{aligned}
&\int_\Omega v(t)\varphi_i\,dx + \int_0^t \int_\Omega \Big(M_i \nabla u_i \cdot \nabla \varphi_i + I(v,w)\varphi_i\Big)\,dx\,ds \\
&\qquad = \int_\Omega v_0\,\varphi_i\,dx + \int_0^t \int_\Omega \eta(v)\varphi_i\,dx\,dW^v(s), \\
&\int_\Omega v(t)\varphi_e\,dx + \int_0^t \int_\Omega \Big(-M_e \nabla u_e \cdot \nabla \varphi_e + I(v,w)\varphi_e\Big)\,dx\,ds \\
&\qquad = \int_\Omega v_0\varphi_e\,dx + \int_0^t \int_\Omega \eta(v)\varphi_e\,dx\,dW^v(s), \\
&\int_\Omega w(t)\varphi\,dx = \int_\Omega w_0\varphi\,dx + \int_0^t \int_\Omega H(v,w)\varphi\,dx\,ds \\
&\qquad + \int_0^t \int_\Omega \sigma(v)\varphi\,dx\,dW^w(s),
\end{aligned} \tag{4.3}$$

for all $\varphi_i, \varphi_e \in H_D^1(\Omega)$ and $\varphi \in L^2(\Omega)$.

Remark 4.2. In view of the regularity conditions imposed in Definition 4.1, it is easily verified that the deterministic integrals in (4.3) are well-defined. The stochastic integrals are well-defined as well; they have been given special attention in Section 3, see (3.5).

Remark 4.3. We denote by $C\left([0,T]; L^2(\Omega) - \text{weak}\right)$ the space of weakly continuous $L^2(\Omega)$ functions. According to [53, Lemma 1.4], part (5) of Definition 4.1 implies that

$$v(\omega,\cdot,\cdot), w(\omega,\cdot,\cdot) \in C\left([0,T]; L^2(\Omega) - \text{weak}\right), \quad \text{for } P\text{-a.e. } \omega \in D.$$

Our main existence result is contained in

Theorem 4.4 (*Existence of Weak Martingale Solution*). *Suppose conditions* (**GFHN**), *(2.5) and (3.6) hold. Let μ_{v_0}, μ_{w_0} be probability measures satisfying the moment estimates (4.1) (with*

$v_0 \sim \mu_{v_0}$, $w_0 \sim \mu_{w_0}$). Then the stochastic bidomain model (2.4), (2.2), (2.3) possesses a weak martingale solution in the sense of Definition 4.1.

The proof of Theorem 4.4 is divided into a series of steps. We construct approximate solutions in Section 5, which are shown to converge in Section 6. The convergence proof relies on several uniform a priori estimates that are established in Sections 6.1 and 6.2. We use these estimates in Section 6.3 to conclude that the laws of the approximate solutions are tight and that the approximations (along a subsequence) converge to a limit. The limit is shown to be a weak martingale solution in Sections 6.4 and 6.5.

If the stochastic basis $\mathcal{S}$ in Definition 4.1 is fixed in advance (not part of the solution), we speak of a weak solution or weak pathwise solution. A weak solution is thus weak in the PDE sense and strong in the probabilistic sense. In this case, we prescribe the initial data v_0, w_0 as random variables relative to $\mathcal{S}$.

Definition 4.5 (*Weak Solution*)**.** Fix a stochastic basis $\mathcal{S}$ and assume that the initial data v_0, w_0 are $\mathcal{F}_0$-measurable and belong to $L^2(D, \mathcal{F}, P; L^2(\Omega))$. A weak solution of the stochastic bidomain system (2.4), with initial–boundary data (2.2)–(2.3), is a collection $U = \left(u_i, u_e, v, w\right)$ satisfying conditions (3), (4), (5), (7) in Definition 4.1 (relative to $\mathcal{S}$).

Weak solutions are said to be unique if, given any pair of such solutions $\hat{U}, \tilde{U}$ for which $\hat{U}$ and $\tilde{U}$ coincide a.s. at $t = 0$,

$$P\left(\left\{\hat{U}(t) = \tilde{U}(t)\,\forall t \in [0, T]\right\}\right) = 1. \tag{4.4}$$

We establish pathwise uniqueness by demonstrating that $v(t), w(t)$ depend continuously on the initial data v_0, w_0 in $L^2(D, \mathcal{F}, P; L^2(\Omega))$. Moreover, using the Poincaré inequality, we conclude as well the pathwise uniqueness of u_i, u_e.

As alluded to earlier, we use this to "upgrade" martingale solutions to weak (pathwise) solutions, thereby delivering

Theorem 4.6 (*Existence and Uniqueness of Weak Solution*)**.** *Suppose conditions* (**GFHN**)*, (2.5), and (3.6) hold. Then the stochastic bidomain model (2.4), (2.2), (2.3) possesses a unique weak solution in the sense of Definition 4.5, provided the initial data satisfy $v_0, w_0 \in L^{q_0}(D, \mathcal{F}, P; L^2(\Omega))$, $q_0 > 9/2$.*

Regarding the proof of Theorem 4.6, we divide it into two steps. A pathwise uniqueness result is established in Section 7 by exhibiting an L^2 stability estimate for the difference between two solutions. We use this result in Section 8 to upgrade martingale solutions to pathwise solutions.

5. Construction of approximate solutions

In this section we define the Faedo–Galerkin approximations. They are based on a *non-degenerate* system introduced below. In upcoming sections we use these approximations to construct weak martingale solutions to the stochastic bidomain model.

We begin by fixing a stochastic basis

$$\mathcal{S} = \left(D, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, \left\{W_k^v\right\}_{k=1}^{\infty}, \left\{W_k^w\right\}_{k=1}^{\infty}\right), \tag{5.1}$$

and $\mathcal{F}_0$-measurable initial data $v_0, w_0 \in L^2(D; L^2(\Omega))$ with respective laws μ_{v_0}, μ_{w_0} on $L^2(\Omega)$. For each fixed $\varepsilon > 0$, the nondegenerate system reads

$$\begin{aligned} &dv + \varepsilon du_i - \nabla \cdot \left(M_i \nabla u_i\right) dt + I(v, w)\, dt = \eta(v)\, dW^v \quad \text{in } \Omega_T, \\ &dv - \varepsilon du_e + \nabla \cdot \left(M_e \nabla u_e\right) dt + I(v, w)\, dt = \eta(v)\, dW^v \quad \text{in } \Omega_T, \\ &dw = H(v, w)\, dt + \sigma(v) dW^w \quad \text{in } \Omega_T, \end{aligned} \tag{5.2}$$

with boundary conditions (2.3). Regarding (5.2), we must provide initial data for u_i, u_e (not $v = u_i - u_e$ as in the original problem). For that reason, we decompose (arbitrarily) the initial condition v_0 in (2.2) as $v_0 = u_{i,0} - u_{e,0}$, for some $\mathcal{F}_0$-measurable random variables $u_{i,0}$ and $u_{e,0}$,

$$u_{i,0}, u_{e,0} \in L^2\left(D, \mathcal{F}, P; L^2(\Omega)\right), \tag{5.3}$$

such that the law of $u_{i,0} - u_{e,0}$ coincides with μ_{v_0}. We replace (2.2) by

$$u_j(0, x) = u_{j,0}(x) \quad (j = i, e), \qquad w(0, x) = w_0(x), \qquad x \in \Omega. \tag{5.4}$$

In some situations, we make use of the strengthened assumption

$$u_{i,0}, u_{e,0}, w_0 \in L^{q_0}\left(D, \mathcal{F}, P; L^2(\Omega)\right), \quad \text{with } q_0 \text{ defined in (4.1)}. \tag{5.5}$$

Remark 5.1. Modulo some obvious changes, the definitions of weak martingale and weak (pathwise) solutions to the nondegenerate system (5.2)–(5.4)–(2.3) are basically the same as those for the original system.

To construct and justify the validity of the Faedo–Galerkin approximations, we employ a classical Hilbert basis, which is orthonormal in L^2 and orthogonal in H_D^1. We refer for example to [48, Thm. 7.7, p. 87] (see also [44]) for the standard construction of such bases. We operate with the same basis $\{e_l\}_{l=1}^n$ for all the unknowns u_i, u_e, v, w.

We look for a solution to the problem arising as the projection of (5.2), (2.2), (2.3) onto the finite dimensional subspace $\mathbb{X}_n := \text{Span}\{e_l\}_{l=1}^n$. The (finite dimensional) approximate solutions take the form

$$\begin{aligned} &u_j^n : [0, T] \to \mathbb{X}_n, \quad u_j^n(t) = \sum_{l=1}^n c_{j,l}^n(t) e_l \quad (j = i, e), \\ &v^n : [0, T] \to \mathbb{X}_n, \quad v^n(t) = \sum_{l=1}^n c_l^n(t) e_l, \quad c_l^n(t) = c_{i,l}^n(t) - c_{e,l}^n(t), \\ &w^n : [0, T] \to \mathbb{X}_n, \quad w^n(t) = \sum_{l=1}^n a_l^n(t) e_l. \end{aligned} \tag{5.6}$$

We pick the coefficients

$$c_j^n = \left\{c_{j,l}^n\right\}_{l=1}^n \ (j = i, e), \quad a^n = \left\{a_l^n\right\}_{l=1}^n, \tag{5.7}$$

which are finite dimensional stochastic processes relative to (5.1), such that ($\ell = 1, \dots, n$)

$$
\begin{aligned}
&\left(dv^n, e_\ell\right)_{L^2(\Omega)} + \varepsilon_n \left(du_i^n, e_\ell\right)_{L^2(\Omega)} \\
&\quad + \left(M_i \nabla u_i^n, \nabla e_\ell\right)_{L^2(\Omega)} dt + \left(I(v^n, w^n), e_\ell\right)_{L^2(\Omega)} dt \\
&\qquad = \sum_{k=1}^{n} \left(\eta_k^n(v^n), e_\ell\right)_{L^2(\Omega)} dW_k^v(t), \\
&\left(dv^n, e_\ell\right)_{L^2(\Omega)} - \varepsilon_n \left(du_e^n, e_\ell\right)_{L^2(\Omega)} \\
&\quad - \left(M_e \nabla u_e^n, \nabla e_\ell\right)_{L^2(\Omega)} dt + \left(I(v^n, w^n), e_\ell\right)_{L^2(\Omega)} dt \\
&\qquad = \sum_{k=1}^{n} \left(\eta_k^n(v^n), e_\ell\right)_{L^2(\Omega)} dW_k^v(t), \\
&\left(dw^n, e_\ell\right)_{L^2(\Omega)} = \left(H(v^n, w^n), e_\ell\right)_{L^2(\Omega)} dt + \sum_{k=1}^{n} \left(\sigma_k^n(v^n), e_\ell\right)_{L^2(\Omega)} dW_k^w(t),
\end{aligned}
\tag{5.8}
$$

where ε in (5.2) is taken as

$$
\varepsilon = \varepsilon_n := \frac{1}{n}, \qquad n \geq 1. \tag{5.9}
$$

We need to comment on the finite dimensional approximations of the stochastic terms utilized in (5.8). With (β, W) denoting (η, W^v) or (σ, W^w), recall that β maps from $L^2\big((0,T); L^2(\Omega)\big)$ to $L^2\left((0,T); L_2\left(\mathbb{U}, L^2(\Omega)\right)\right)$, where $\mathbb{U}$ is equipped with the orthonormal basis $\{\psi_k\}_{k\geq 1}$ (cf. Section 3). Employing the decomposition $\beta_k(v) = \beta(v)\psi_k$, $\beta_k(v) = \sum_{l\geq 1} \left(\beta_k(v), e_l\right)_{L^2(\Omega)} e_l$, we can write

$$
\beta(v)\, dW = \sum_{k\geq 1} \beta_k(v)\, dW_k = \sum_{k,l\geq 1} \beta_{k,l}(v) e_l\, dW_k, \quad \beta_{k,l}(v) = \left(\beta_k(v), e_l\right)_{L^2(\Omega)}.
$$

In (5.8), we utilize the finite dimensional approximation

$$
\beta^n(v)\, dW^n := \sum_{k,l=1}^{n} \beta_{k,l}(v) e_l\, dW_k = \sum_{k=1}^{n} \beta_k^n(v)\, dW_k, \tag{5.10}
$$

with β^n and W^n then defined by

$$
\beta_k^n(v) = \beta^n(v)\psi_k, \quad \beta_k^n(v) = \sum_{l=1}^{n} \beta_{k,l}(v) e_l, \quad W^n = \sum_{k=1}^{n} W_k \psi_k,
$$

where (β^n, W^n) denotes $(\eta^n, W^{v,n})$ or $(\sigma^n, W^{w,n})$; W^n converges in $C([0,T]; \mathbb{U}_0)$ for P-a.e. $\omega \in D$ and (by a martingale inequality) in $L^2\left(D, \mathcal{F}, P; C([0,T]; \mathbb{U}_0)\right)$.

The initial conditions are

$$
\begin{aligned}
&u_j^n(0) = u_{j,0}^n := \sum_{l=1}^{n} c_{j,l}^n(0) e_l, \quad c_{j,l}^n(0) := \left(u_{j,0}^n, e_l\right)_{L^2(\Omega)}, \quad j = i, e, \\
&v^n(0) = v_0^n := u_{i,0}^n - u_{e,0}^n, \\
&w^n(0) = w_0^n := \sum_{l=1}^{n} a_l^n(0) e_l, \quad a_l^n(0) := \left(w_0, e_l\right)_{L^2(\Omega)}.
\end{aligned}
\tag{5.11}
$$

In (5.11), consider for example $u^n_{j,0}$. Since $u_{j,0} \in L^2\left(D, \mathcal{F}, P; L^2(\Omega)\right)$, we have (by standard properties of finite-dimensional projections, cf. (5.14), (5.16) below) $u^n_{j,0} \to u_{j,0}$ in $L^2(\Omega)$, P-a.s., as $n \to \infty$, and $\left\| u^n_{j,0} \right\|^2_{L^2(\Omega)} \le C \left\| u_{j,0} \right\|^2_{L^2(\Omega)}$. On this account, the dominated convergence theorem implies

$$u^n_{j,0} \to u_{j,0} \text{ in } L^2\left(D, \mathcal{F}, P; L^2(\Omega)\right), \text{ as } n \to \infty. \tag{5.12}$$

Similarly, $w^n_0 \to w_0$, $v^n_0 \to v_0$ in $L^2(\Omega)$, P-a.s., and thus in $L^2_\omega(L^2_x)$.

For the basis $\{e_l\}_{l=1}^\infty$, we introduce the projection operators (see e.g. [8, page 1636])

$$\Pi_n : (H^1_D(\Omega))^* \to \operatorname{Span}\{e_l\}_{j=1}^\infty, \quad \Pi_n u^* := \sum_{l=1}^n \left\langle u^*, e_l \right\rangle_{(H^1_D(\Omega))^*, H^1_D(\Omega)} e_l. \tag{5.13}$$

The restriction of Π_n to $L^2(\Omega)$ is also denoted by Π_n:

$$\Pi_n : L^2(\Omega) \to \operatorname{Span}\{e_l\}_{j=1}^\infty, \quad \Pi_n u := \sum_{l=1}^n (u, e_l)_{L^2(\Omega)} e_l,$$

i.e., Π_n is the orthogonal projection from $L^2(\Omega)$ to $\operatorname{Span}\{e_l\}_{j=1}^\infty$. We have

$$\|\Pi_n u\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}, \qquad u \in L^2(\Omega). \tag{5.14}$$

Note that we have the following equality for any $u^* \in (H^1_D(\Omega))^*$ and $u \in H^1_D(\Omega)$:

$$\left(\Pi_n u^*, u\right)_{L^2(\Omega)} = \left\langle u^*, \Pi_n u \right\rangle_{(H^1_D(\Omega))^*, H^1_D(\Omega)}. \tag{5.15}$$

Furthermore, as $n \to \infty$,

$$\|\Pi_n u - u\|_{H^1_D(\Omega)} \to 0, \quad u \in H^1_D(\Omega). \tag{5.16}$$

Using the projection operator (5.13), we may write (5.8) in integrated form equivalently as equalities between $(H^1_D(\Omega))^*$ valued random variables:

$$\begin{aligned}
v^n(t) + \varepsilon_n u^n_i(t) &= v^n_0 + \varepsilon_n u^n_{i,0} + \int_0^t \Pi_n \left[\nabla \cdot \left(M_i \nabla u^n_i\right) - I(v^n, w^n)\right] ds \\
&\qquad + \int_0^t \eta^n(v^n)\, dW^{v,n}(s) \quad \text{in } (H^1_D(\Omega))^*, \\
v^n(t) - \varepsilon_n u^n_e(t) &= v^n_0 - \varepsilon_n u^n_{e,0} + \int_0^t \Pi_n \left[-\nabla \cdot \left(M_e \nabla u^n_e\right) - I(v^n, w^n)\right] ds \\
&\qquad + \int_0^t \eta^n(v^n)\, dW^{v,n}(s) \quad \text{in } (H^1_D(\Omega))^*, \\
w^n(t) &= w^n_0 + \int_0^t \Pi_n \left(H(v^n, w^n)\right) ds \\
&\qquad + \int_0^t \sigma^n(v^n)\, dW^{w,n}(s) \quad \text{in } (H^1_D(\Omega))^*,
\end{aligned} \tag{5.17}$$

where $v^n_0 = u^n_{i,0} - u^n_{e,0}$ and $u^n_{i,0} = \Pi_n u_{i,0}$, $u^n_{e,0} = \Pi_n u_{e,0}$, $w^n_0 = \Pi_n w_0$.

In coming sections we investigate the convergence properties of the sequences $\left\{u^n_j\right\}_{n\ge1}$ $(j = i, e)$, $\{v^n\}_{n\ge1}$, $\{w^n\}_{n\ge1}$ defined by (5.17). Meanwhile, we must verify the existence of a (pathwise) solution to the finite dimensional system (5.8).

Lemma 5.2. *For each fixed $n \geq 1$, the Faedo–Galerkin equations (5.6), (5.8), and (5.11) possess a unique global adapted solution $(u_i^n(t), u_e^n(t), v^n(t), w^n(t))$ on $[0, T]$. Besides, u_i^n, u_e^n, v^n, w^n belong to $C([0,T]; \mathbb{X}_n)$, and $v^n = u_i^n - u_e^n$.*

Proof. Using the orthonormality of the basis, (5.8) becomes the SDE system ($\ell = 1, \dots, n$)

$$\begin{aligned} d\left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right) &= A_{i,\ell}\, dt + \Gamma_\ell\, dW^{v,n}, \\ d\left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right) &= A_{e,\ell}\, dt + \Gamma_\ell\, dW^{v,n}, \\ da_\ell^n &= A_{H,\ell}\, dt + \zeta_\ell\, dW^{w,n}, \end{aligned} \tag{5.18}$$

for the coefficients $c_j^n = c_j^n(t)$ ($j = i, e$) and $a^n = a^n(t)$, cf. (5.7), where

$$A_{i,\ell} = -\int_\Omega M_i \nabla u_i^n \cdot \nabla e_\ell\, dx - \int_\Omega I(v^n, w^n) e_\ell\, dx,$$

$$A_{e,\ell} = \int_\Omega M_e \nabla u_e^n \cdot \nabla e_\ell\, dx - \int_\Omega I(v^n, w^n) e_\ell\, dx,$$

$$A_{H,\ell} = \int_\Omega H(v^n, w^n) e_\ell\, dx,$$

$$\Gamma_\ell = \left\{\Gamma_{\ell,k}\right\}_{k=1}^n, \ \Gamma_{\ell,k} = \int_\Omega \eta_k^n(v^n)\, e_\ell\, dx, \ \Gamma_\ell\, dW^{v,n} = \sum_{k=1}^n \Gamma_{\ell,k}\, dW_k^v,$$

$$\zeta_\ell = \left\{\zeta_{\ell,k}\right\}_{k=1}^n, \ \zeta_{\ell,k} = \int_\Omega \sigma_k^n(v^n)\, e_\ell\, dx, \ \zeta_\ell\, dW^{v,n} = \sum_{k=1}^n \zeta_{\ell,k}\, dW_k^v.$$

Adding the first and second equations in (5.18) yields ($\ell = 1, \dots, n$)

$$dc_\ell^n = \frac{1}{2+\varepsilon_n}\left[A_{i,\ell} + A_{e,\ell}\right] dt + \frac{2}{2+\varepsilon_n}\Gamma_\ell\, dW^{v,n} =: F_{ie,\ell}\, dt + 2G_\ell\, dW^{v,n}, \tag{5.19}$$

and plugging (5.19) into (5.18) we arrive at ($\ell = 1, \dots, n$)

$$\begin{aligned} d\left(\sqrt{\varepsilon_n} c_{i,\ell}^n\right) &= \left[\frac{1+\varepsilon_n}{\sqrt{\varepsilon_n}(2+\varepsilon_n)} A_{i,\ell} - \frac{1}{\sqrt{\varepsilon_n}(2+\varepsilon_n)} A_{e,\ell}\right] dt \\ &\qquad + \frac{\sqrt{\varepsilon_n}}{2+\varepsilon_n}\Gamma_\ell\, dW^{v,n} =: F_{i,\ell}\, dt + \sqrt{\varepsilon_n} G_\ell\, dW^{v,n}, \\ d\left(\sqrt{\varepsilon_n} c_{e,\ell}^n\right) &= \left[\frac{1}{\sqrt{\varepsilon_n}(2+\varepsilon_n)} A_{i,\ell} - \frac{1+\varepsilon_n}{\sqrt{\varepsilon_n}(2+\varepsilon_n)} A_{e,\ell}\right] dt \\ &\qquad - \frac{\sqrt{\varepsilon_n}}{2+\varepsilon_n}\Gamma_\ell\, dW^{v,n} =: F_{e,\ell}\, dt - \sqrt{\varepsilon_n} G_\ell\, dW^{v,n}, \\ da_\ell^n &= A_{H,\ell}\, dt + \zeta_\ell\, dW^{w,n}. \end{aligned} \tag{5.20}$$

Recalling (2.6), we let

$$C^n = C^n(t) = \left\{c^n(t), \sqrt{\varepsilon_n} c_i^n(t), \sqrt{\varepsilon_n} c_e^n(t), a^n(t)/\mu\right\}$$

be the vector containing all the unknowns in (5.19) and (5.20). For technical reasons, related to (5.22) and (5.23), we write the left-hand sides of the first two equations in (5.20) in terms of the ε_n scaled quantities $\sqrt{\varepsilon_n} c_i^n$, $\sqrt{\varepsilon_n} c_e^n$. Moreover, we view the right-hand sides of all the equations as functions of C^n (involving the ε_n scaled quantities), which can always be done since $\varepsilon_n > 0$ is a fixed number. As a result, the constants below may depend on $1/\varepsilon_n$. Let

$$F(C^n) = \left\{\left\{F_{ie,\ell}(C^n)\right\}_{\ell=1}^n, \left\{F_{i,\ell}(C^n)\right\}_{\ell=1}^n, \left\{F_{e,\ell}(C^n)\right\}_{\ell=1}^n, \left\{A_{H,\ell}(C^n)/\mu\right\}_{\ell=1}^n\right\}$$

be the vector containing all the drift terms, and

$$G(C^n) = \left\{\{2G_\ell\}_{\ell=1}^n, \left\{\sqrt{\varepsilon_n}G_\ell\right\}_{\ell=1}^n, \left\{-\sqrt{\varepsilon_n}G_\ell\right\}_{\ell=1}^n, \{\zeta_\ell/\mu\}_{\ell=1}^n\right\},$$

be the collection of noise coefficients. The vector $\{W^{v,n}, W^{v,n}, W^{v,n}, W^{w,n}\}$ is denoted by W^n. Then (5.19) and (5.20) take the compact form

$$dC^n(t) = F(C^n(t))\,dt + G(C^n(t))\,dW^n(t), \quad C^n(0) = C_0^n, \tag{5.21}$$

where $C_0^n = \left\{c^n(0), \sqrt{\varepsilon_n}c_i^n(0), \sqrt{\varepsilon_n}c_e^n(0), a^n(0)/\mu\right\}$, cf. (5.11).

If F, G are globally Lipschitz continuous, classical SDE theory [41,50] provides the existence and uniqueness of a pathwise solution. However, due to the nonlinear nature of the ionic models, cf. (**GFHN**), the global Lipschitz condition does not hold for the SDE system (5.21). As a replacement, we consider the following two conditions:

- (local weak monotonicity) $\forall C_1, C_2 \in \mathbb{R}^{4n}$, $|C_1|, |C_2| \le r$, for any $r > 0$,
$$2\left(F(C_1) - F(C_2)\right)\cdot(C_1 - C_2) + |G(C_1) - G(C_2)|^2 \le K_r\,|C_1 - C_2|^2, \tag{5.22}$$
for some r-dependent positive constant K_r.
- (weak coercivity) $\forall C \in \mathbb{R}^{4n}$, there exists a constant $K > 0$ such that
$$2F(C)\cdot C + |G(C)|^2 \le K\left(1 + |C|^2\right). \tag{5.23}$$

Below we verify that the coefficients F and G in (5.21) satisfy both these conditions globally (i.e., (5.22) holds independent of r). Then, in view of Theorem 3.1.1 in [41], there exists a unique global adapted solution to (5.21).

Let us verify the weak monotonicity condition. To this end, set

$$u_j^n := u_{j,1}^n - u_{j,2}^n \ (j = i, e), \quad v_k^n := u_{i,k}^n - u_{e,k}^n \ (k = 1, 2),$$
$$v^n := v_1^n - v_2^n, \quad w^n := w_1^n - w_2^n,$$

where $\left(u_{i,1}^n, u_{e,1}^n, w_1^n\right)$ and $\left(u_{i,2}^n, u_{e,2}^n, w_2^n\right)$ are arbitrary functions of the form of (5.6), with corresponding time coefficients $\left(c_{i,1}^n, c_{e,1}^n, a_1^n\right)$ and $\left(c_{i,2}^n, c_{e,2}^n, a_2^n\right)$, respectively. Moreover, set $c_1^n := c_{i,1}^n - c_{e,1}^n$, $c_2^n := c_{i,1}^n - c_{e,1}^n$, $C_k^n := \left\{c_k^n, \sqrt{\varepsilon_n}c_{i,k}^n, \sqrt{\varepsilon_n}c_{e,k}^n, a_k^n/\mu\right\}$ for $k = 1, 2$.

We wish to show that

$$\mathcal{I}_F := \left(F(C_1^n) - F(C_2^n)\right)\cdot\left(C_1^n - C_2^n\right) \le K_F\left|C_1^n - C_2^n\right|^2,$$

i.e., that F is globally one-sided Lipschitz. This requires comparing the "dt-terms" in (5.19) and (5.20) corresponding to the vectors C_1^n and C_2^n, resulting in three different types of terms, linked to the M_j (diffusion) part, the I (ionic) part, and the H (gating) part of the equations, that is, $\mathcal{I}_F = \mathcal{I}_F^M + \mathcal{I}_F^I + \mathcal{I}_F^H$. First,

$$\begin{aligned}\mathcal{I}_F^I = \frac{-2}{2+\varepsilon_n}\sum_{l=1}^n \int_\Omega \left(I(v_1^n, w_1^n) - I(v_2^n, w_2^n)\right) e_l\,dx\left(c_{1,l}^n - c_{2,l}^n\right) \\ + \frac{-(1+\varepsilon_n)+1}{\sqrt{\varepsilon_n}(2+\varepsilon_n)}\sum_{l=1}^n \int_\Omega \left(I(v_1^n, w_1^n) - I(v_2^n, w_2^n)\right) e_l\,dx \\ \times\left(\sqrt{\varepsilon_n}c_{i,1,l}^n - \sqrt{\varepsilon_n}c_{i,2,l}^n\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{-1+(1+\varepsilon_n)}{\sqrt{\varepsilon_n}(2+\varepsilon_n)} \sum_{l=1}^{n} \int_{\Omega} \left(I(v_1^n, w_1^n) - I(v_2^n, w_2^n) \right) e_l \, dx \\
&\qquad\qquad\qquad \times \left(\sqrt{\varepsilon_n} c_{e,1,l}(t) - \sqrt{\varepsilon_n} c_{e,2,l}(t) \right) \\
&= - \int_{\Omega} \left(I(v_1^n, w_1^n) - I(v_2^n, w_2^n) \right) v^n \, dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{I}_F^H &= \sum_{l=1}^{n} \int_{\Omega} \left(H(v_1^n, w_1^n) - H(v_2^n, w_2^n) \right) e_l \, dx \left(a_{1,l}/\mu - a_{2,l}/\mu \right) \\
&= \frac{1}{\mu} \int_{\Omega} \left(H(v_1^n, w_1^n) - H(v_2^n, w_2^n) \right) w^n \, dx,
\end{aligned}
$$

and therefore $\mathcal{I}_F^I + \mathcal{I}_F^H$ becomes

$$
\begin{aligned}
&\frac{1}{\mu} \int_{\Omega} \Big(\left(H(v_1^n, w_1^n) - H(v_2^n, w_2^n) \right) w^n - \mu \left(I(v_1^n, w_1^n) - I(v_2^n, w_2^n) \right) v^n \Big) \, dx \\
&\overset{(2.6)}{\leq} \tilde{K}_{H,I} \left(\left\| v_1^n - v_2^n \right\|_{L^2(\Omega)}^2 + \left\| w_1^n - w_2^n \right\|_{L^2(\Omega)}^2 \right) \leq K_{H,I} \left| C_1^n - C_2^n \right|^2,
\end{aligned}
\tag{5.24}
$$

for some constants $\tilde{K}_{H,I}$, $K_{H,I}$. Finally,

$$
\begin{aligned}
\mathcal{I}_F^M &= \frac{1}{2+\varepsilon_n} \int_{\Omega} \left(-M_i \nabla U_i^n + M_e \nabla U_e^n \right) \cdot \nabla V^n \, dx \\
&\quad + \frac{1}{2+\varepsilon_n} \int_{\Omega} \left(-(1+\varepsilon_n) M_i \nabla U_i^n - M_e \nabla U_e^n \right) \cdot \nabla U_i^n \, dx \\
&\quad + \frac{1}{2+\varepsilon_n} \int_{\Omega} \left(-M_i \nabla U_i^n - (1+\varepsilon_n) M_e \nabla U_e^n \right) \cdot \nabla U_e^n \, dx.
\end{aligned}
$$

Adding the integrands gives

$$
\begin{aligned}
&\left(-M_i \nabla U_i^n + M_e \nabla U_e^n \right) \cdot \nabla V^n + \left(-(1+\varepsilon_n) M_i \nabla U_i^n - M_e \nabla U_e^n \right) \cdot \nabla U_i^n \\
&\qquad + \left(-M_i \nabla U_i^n - (1+\varepsilon_n) M_e \nabla U_e^n \right) \cdot \nabla U_e^n \\
&\quad = -(2+\varepsilon_n) M_i \nabla U_i^n \cdot \nabla U_i^n - (2+\varepsilon_n) M_e \nabla U_e^n \cdot \nabla U_e^n,
\end{aligned}
$$

and thus, cf. (2.5), $\mathcal{I}_F^M = -\sum_{j=i,e} M_j \nabla U_j^n \cdot \nabla U_j^n \leq 0$. Hence, F is globally one-sided Lipschitz. In view of (3.6), it follows easily that G is globally Lipschitz:

$$
\left| G(C_1^n) - G(C_2^n) \right| \leq K_G \left| C_1^n - C_2^n \right|,
$$

for some constant K_G (depending on n). Summarizing, condition (5.22) holds.

In much the same way, again using assumptions (**GFHN**) and (3.6), we deduce that

$$
F(C_1^n) \cdot C_1^n \leq K_F \left(1 + \left| C_1^n \right|^2 \right), \qquad \left| G(C_1^n) \right|^2 \leq K_G \left(1 + \left| C_1^n \right|^2 \right),
$$

for some constants K_F, K_G; that is to say, condition (5.23) holds. □

6. Convergence of approximate solutions

6.1. Basic apriori estimates

To establish convergence of the Faedo–Galerkin approximations, we must supply a series of apriori estimates that are independent of the parameter n (cf. Lemma 6.1). At an informal level, assuming that the relevant functions are sufficiently regular, these estimates are obtained by considering

$$d\,(v+\varepsilon_n u_i) = \left[\nabla\cdot\left(M_i\nabla u_i\right) - I(v,w)\right]dt + \eta(v)\,dW^v$$
$$d\,(v-\varepsilon_n u_e) = \left[-\nabla\cdot\left(M_e\nabla u_e\right) - I(v,w)\right]dt + \eta(v)\,dW^v,$$

where ε_n is defined in (5.9), multiplying the first equation by u_i, the second equation by $-u_e$, and summing the resulting equations. For the moment, let us assume that the noise W^v is one-dimensional and $\eta(v)$ is a scalar function. To proceed we use the stochastic (Itô) product rule. Hence, we need access to the equation for du_i, which turns out to be

$$du_i = \left[\frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)}\nabla\cdot\left(M_i\nabla u_i\right) + \frac{1}{\varepsilon_n(2+\varepsilon_n)}\nabla\cdot\left(M_e\nabla u_e\right) - \frac{1}{2+\varepsilon_n}I(v,w)\right]dt + \frac{1}{2+\varepsilon_n}\eta(v)\,dW^v.$$

Note that this equation "blows up" as $\varepsilon_n \to 0$ (the same is true for the du_e equation below). The stochastic product rule gives

$$\begin{aligned} d\,(u_i\,(v+\varepsilon_n u_i)) &= u_i\,d\,(v+\varepsilon_n u_i) + du_i\,(v+\varepsilon_n u_i) + \frac{1}{2+\varepsilon_n}\eta(v)^2\,dt \\ &= \frac{1}{2+\varepsilon_n}\eta(v)^2\,dt + \left[u_i\,\nabla\cdot\left(M_i\nabla u_i\right) - u_i\,I(v,w)\right]dt \\ &\quad + u_i\,\eta(v)\,dW^v + \Big[\cdots\Big]_i\,dt + \frac{1}{2+\varepsilon_n}\,(v+\varepsilon_n u_i)\,\eta(v)\,dW^v, \end{aligned} \tag{6.1}$$

where

$$\Big[\cdots\Big]_i\,dt = \left[\frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)}\,(v+\varepsilon_n u_i)\,\nabla\cdot\left(M_i\nabla u_i\right) + \frac{1}{\varepsilon_n(2+\varepsilon_n)}\,(v+\varepsilon_n u_i)\,\nabla\cdot\left(M_e\nabla u_e\right) - \frac{1}{2+\varepsilon_n}\,(v+\varepsilon_n u_i)\,I(v,w)\right]dt.$$

Similar computations, this time involving the equation

$$du_e = \left[\frac{1}{\varepsilon_n(2+\varepsilon_n)}\nabla\cdot\left(M_i\nabla u_i\right) + \frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)}\nabla\cdot\left(M_e\nabla u_e\right) + \frac{1}{2+\varepsilon_n}I(v,w)\right]dt - \frac{1}{2+\varepsilon_n}\eta(v)\,dW^v,$$

yield

$$\begin{aligned} d\,(-u_e\,(v-\varepsilon_n u_e)) &= -u_e\,d\,(v-\varepsilon_n u_e) - du_e\,(v-\varepsilon_n u_e) + \frac{1}{2+\varepsilon_n}\eta(v)^2\,dt \\ &= \frac{1}{2+\varepsilon_n}\eta(v)^2\,dt + \left[u_e\,\nabla\cdot\left(M_e\nabla u_e\right) + u_e I(v,w)\right]dt \\ &\quad - u_e\,\eta(v)\,dW^v + \Big[\cdots\Big]_e\,dt + \frac{1}{2+\varepsilon_n}\,(v-\varepsilon_n u_e)\,\eta(v)\,dW^v, \end{aligned} \tag{6.2}$$

where

$$\Big[\cdots\Big]_e\,dt = \left[-\frac{1}{\varepsilon_n(2+\varepsilon_n)}\,(v-\varepsilon_n u_e)\,\nabla\cdot\left(M_i\nabla u_i\right) - \frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)}\,(v-\varepsilon_n u_e)\,\nabla\cdot\left(M_e\nabla u_e\right) - \frac{1}{2+\varepsilon_n}\,(v-\varepsilon_n u_e)\,I(v,w)\right]dt.$$

After some computations we find that

$$\Big[\cdots\Big]_i\,dt + \Big[\cdots\Big]_e\,dt = \Big[2u_i\,\nabla\cdot\big(M_i\nabla u_i\big) + 2u_e\,\nabla\cdot\big(M_e\nabla u_e\big) - 2v\,I(v,w)\Big]\,dt$$

and

$$\begin{aligned} &u_i\,\eta(v)\,dW^v + \frac{1}{2+\varepsilon_n}\,(v+\varepsilon_n u_i)\,\eta(v)\,dW^v \\ &\quad - u_e\,\eta(v)\,dW^v + \frac{1}{2+\varepsilon_n}\,(v-\varepsilon_n u_e)\,\eta(v)\,dW^v = 2v\,\eta(v)\,dW^v. \end{aligned}$$

Whence, adding (6.1) and (6.2),

$$\begin{aligned} d\left(v^2+\varepsilon_n u_i^2+\varepsilon_n u_e^2\right) &= d\left(u_i\,(v+\varepsilon_n u_i)\right) + d\left(-u_e\,(v-\varepsilon_n u_e)\right) \\ &= \Bigg[\frac{2}{2+\varepsilon_n}\eta(v)^2 + 2u_i\,\nabla\cdot\big(M_i\nabla u_i\big) \\ &\qquad + 2u_e\,\nabla\cdot\big(M_e\nabla u_e\big) - 2v\,I(v,w)\Bigg]\,dt + 2v\,\eta(v)\,dW^v. \end{aligned}$$

Adding to this the equation for dw^2, resulting from (5.2) and Itô's formula, the estimates in Lemma 6.1 appear once we integrate in x and t, make use of spatial integration by parts, the boundary conditions (2.3), and properties of the nonlinear functions I, H implying (6.13). Arguing at the level of finite dimensional approximations, we now convert the computations outlined above into a rigorous proof.

Lemma 6.1. *Suppose conditions* (**GFHN**)*,* (2.5)*,* (3.6)*, and* (5.3) *hold. Let*

$$u_i^n(t),\, u_e^n(t),\, v^n(t),\, w^n(t), \qquad t\in[0,T],$$

satisfy (5.8)*,* (5.9)*,* (5.10)*,* (5.11)*. There is a constant $C>0$, independent of n, such that*

$$\begin{aligned} &\mathbb{E}\left[\left\|v^n(t)\right\|_{L^2(\Omega)}^2\right] + \mathbb{E}\left[\left\|w^n(t)\right\|_{L^2(\Omega)}^2\right] \\ &\qquad + \sum_{j=i,e}\mathbb{E}\left[\left\|\sqrt{\varepsilon_n}u_j^n(t)\right\|_{L^2(\Omega)}^2\right] \le C, \qquad \forall t\in[0,T]; \end{aligned} \tag{6.3}$$

$$\sum_{j=i,e}\mathbb{E}\left[\int_0^T\int_\Omega\left|\nabla u_j^n\right|^2\,dx\,dt\right] + \mathbb{E}\left[\int_0^T\int_\Omega\left|v^n\right|^4\,dx\,dt\right] \le C; \tag{6.4}$$

$$\sum_{j=i,e}\mathbb{E}\left[\int_0^T\int_\Omega\left|u_j^n\right|^2\,dx\,dt\right] \le C; \tag{6.5}$$

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]}\left\|v^n(t)\right\|_{L^2(\Omega)}^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left\|w^n(t)\right\|_{L^2(\Omega)}^2\right] \\ &\qquad + \sum_{j=i,e}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sqrt{\varepsilon_n}u_j^n(t)\right\|_{L^2(\Omega)}^2\right] \le C. \end{aligned} \tag{6.6}$$

Proof. We wish to compute $dJ(t)$, $J(t) = \int_\Omega (v^n)^2 + \varepsilon_n \left(u_i^n\right)^2 + \varepsilon_n \left(u_e^n\right)^2 \, dx$:

$$
\begin{aligned}
dJ(t) &= d\int_\Omega u_i^n \left(v^n + \varepsilon_n u_i^n\right) \, dx + d\int_\Omega -u_e^n \left(v^n - \varepsilon_n u_e^n\right) \, dx \\
&= \sum_{\ell=1}^n d\left(c_{i,\ell}^n \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right)\right) + \sum_{\ell=1}^n d\left(-c_{e,\ell}^n \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right)\right),
\end{aligned}
\tag{6.7}
$$

where we have used (5.6) and the orthonormality of the basis.

First, in view of (5.18) and (5.20), the stochastic product rule implies ($\ell = 1, \dots, n$)

$$
\begin{aligned}
d\left(c_{i,\ell}^n(c_\ell^n + \varepsilon_n c_{i,\ell}^n)\right) &= \left(c_{i,\ell}^n d\left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right)\right) + \left(dc_{i,\ell}^n \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right)\right) \\
&\quad + \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 dt \\
&= \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 dt \\
&\quad + \int_\Omega \left(M_i \nabla u_i^n \cdot \nabla e_\ell - I(v^n, w^n) e_\ell\right) \, dx \, c_{i,\ell}^n \, dt \\
&\quad + \sum_{k=1}^n \int_\Omega \eta_k^n(v^n) e_\ell \, dx \, c_{i,\ell}^n \, dW^{v,n} + \Big[\cdots\Big]_i \, dt \\
&\quad + \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \int_\Omega \eta_k^n(v^n) e_\ell \, dx \, \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right) \, dW^{v,n},
\end{aligned}
\tag{6.8}
$$

where

$$
\begin{aligned}
\Big[\cdots\Big]_i \, dt = \Bigg[&\frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)} \int_\Omega M_i \nabla u_i^n \cdot \nabla e_\ell \, dx \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right) \\
&+ \frac{1}{\varepsilon_n(2+\varepsilon_n)} \int_\Omega M_e \nabla u_e^n \cdot \nabla e_\ell \, dx \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right) \\
&- \frac{1}{2+\varepsilon_n} \int_\Omega I(v^n, w^n) e_\ell \, dx \left(c_\ell^n + \varepsilon_n c_{i,\ell}^n\right)\Bigg] dt.
\end{aligned}
$$

Similar computations give ($\ell = 1, \dots, n$)

$$
\begin{aligned}
d\left(-c_{e,\ell}^n, (c_\ell^n + \varepsilon_n c_{e,\ell}^n)\right) &= \left(-c_{e,\ell}^n d\left(c_\ell^n - \varepsilon_n c_\ell^n\right)\right) - \left(dc_{e,\ell}^n \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right)\right) \\
&\quad + \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 dt \\
&= \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 dt \\
&\quad + \int_\Omega (M_e \nabla u_e^n \cdot \nabla e_\ell + I(v^n, w^n) e_\ell) \, dx \, c_{e,\ell}^n \, dt \\
&\quad - \sum_{k=1}^n \int_\Omega \eta_k^n(v^n) e_l \, dx \, c_{e,\ell} \, dW^{v,n} + \Big[\cdots\Big]_e \, dt \\
&\quad + \frac{1}{2+\varepsilon_n} \sum_{k=1}^n \int_\Omega \eta_k^n(v^n) e_\ell \, dx \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right) dW^{v,n},
\end{aligned}
\tag{6.9}
$$

where

$$\Big[\cdots\Big]_e dt = \Big[-\frac{1}{\varepsilon_n(2+\varepsilon_n)}\int_\Omega M_i \nabla u_i^n \cdot \nabla e_\ell \, dx \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right) - \frac{1+\varepsilon_n}{\varepsilon_n(2+\varepsilon_n)}\int_\Omega M_e \nabla u_e^n \cdot \nabla e_\ell \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right) dx - \frac{1}{2+\varepsilon_n}\int_\Omega I(v^n, w^n) e_\ell \, dx \left(c_\ell^n - \varepsilon_n c_{e,\ell}^n\right)\Big] dt.$$

Combining (6.7), (6.8), (6.9) we arrive eventually at

$$\begin{aligned} d\int_\Omega \left|v^n\right|^2 + \varepsilon_n \left|u_i^n\right|^2 + \varepsilon_n \left|u_e^n\right|^2 dx &= \Big[-2\int_\Omega M_i \nabla u_i^n \cdot \nabla u_i^n \, dx - 2\int_\Omega M_e \nabla u_e^n \cdot \nabla u_e^n \, dx - 2\int_\Omega v^n I(v^n, w^n)\, dx \\ &\quad + \frac{2}{2+\varepsilon_n}\sum_{k,l=1}^n \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 dt \Big] dt + 2\int_\Omega v^n \eta^n(v^n)\, dx \, dW^{v,n}. \end{aligned} \tag{6.10}$$

Similarly, in view of (5.6) and (5.20), Itô's lemma gives

$$\begin{aligned} d\int_\Omega \left|w^n\right|^2 dx &= \left[2\int_\Omega w^n H(v^n, w^n)\, dx + \sum_{k,l=1}^n \left(\int_\Omega \sigma_k^n(v^n) e_l \, dx\right)^2\right] dt \\ &\quad + 2\int_\Omega w^n \sigma^n(v^n)\, dW^{w,n}. \end{aligned} \tag{6.11}$$

After integration in time, adding (6.10) and (6.11) delivers

$$\begin{aligned} &\frac{1}{2}\left\|v^n(t)\right\|_{L^2(\Omega)}^2 + \sum_{j=i,e}\frac{1}{2}\left\|\sqrt{\varepsilon_n} u_j^n(t)\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|w^n(t)\right\|_{L^2(\Omega)}^2 . \\ &\quad + \sum_{j=i,e}\int_0^t \int_\Omega M_j \nabla u_j^n \cdot \nabla u_j^n \, dx \, ds \\ &= \frac{1}{2}\left\|v^n(0)\right\|_{L^2(\Omega)}^2 + \sum_{j=i,e}\frac{1}{2}\left\|\sqrt{\varepsilon_n} u_j^n(0)\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|w^n(0)\right\|_{L^2(\Omega)}^2 \\ &\quad + \int_0^t \int_\Omega \left(w^n H(v^n, w^n) - v^n I(v^n, w^n)\right) dx \, ds \\ &\quad + \frac{1}{2+\varepsilon_n}\sum_{k,l=1}^n \int_0^t \left(\int_\Omega \eta_k^n(v^n) e_l \, dx\right)^2 ds + \frac{1}{2}\sum_{k,l=1}^n \int_0^t \left(\int_\Omega \sigma_k^n(v^n) e_l \, dx\right)^2 ds \\ &\quad + \int_0^t \int_\Omega v^n \eta^n(v^n)\, dx \, dW^{v,n}(s) + \int_0^t \int_\Omega w^n \sigma^n(v^n)\, dx \, dW^{w,n}(s), \end{aligned} \tag{6.12}$$

for any $t \in [0, T]$. By (**GFHN**) and repeated applications of Cauchy's inequality,

$$wH(v, w) - vI(v, w) \le -C_1 |v|^4 + C_2\left(|v|^2 + |w|^2\right) + C_3, \tag{6.13}$$

for some constants $C_1 > 0$ and $C_2, C_3 \geq 0$. Recalling that $\{e_l\}_{l\geq 1}$ is a basis for $L^2(\Omega)$,

$$\begin{aligned}
&\sum_{k,l=1}^{n} \int_0^t \left(\int_\Omega \eta_k^n(v^n) e_l \, dx \right)^2 ds + \sum_{k,l=1}^{n} \int_0^t \left(\int_\Omega \sigma_k^n(v^n) e_l \, dx \right)^2 ds \\
&\quad \leq \int_0^t \int_\Omega \sum_{k=1}^{n} \left|\eta_k(v^n)\right|^2 dx\, ds + \int_0^t \int_\Omega \sum_{k=1}^{n} \left|\sigma_k(v^n)\right|^2 dx\, ds \\
&\quad \overset{(3.6)}{\leq} C_4 \left(\int_0^t \int_\Omega \left|v^n\right|^2 dx\, ds + t\,|\Omega| \right),
\end{aligned} \tag{6.14}$$

for some constant $C_4 > 0$. Using (6.13), (6.14), and (2.5) in (6.12), we obtain

$$\begin{aligned}
&\frac{1}{2} \left\| v^n(t) \right\|_{L^2(\Omega)}^2 + \sum_{j=i,e} \frac{1}{2} \left\| \sqrt{\varepsilon_n} u_j^n(t) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| w^n(t) \right\|_{L^2(\Omega)}^2 . \\
&\quad + m \sum_{j=i,e} \int_0^t \int_\Omega \left|\nabla u_j^n\right|^2 dx\, ds + C_1 \int_0^t \int_\Omega |v|^4 \, dx\, ds \\
&\leq \frac{1}{2} \left\| v^n(0) \right\|_{L^2(\Omega)}^2 + \sum_{j=i,e} \frac{1}{2} \left\| \sqrt{\varepsilon_n} u_j^n(0) \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| w^n(0) \right\|_{L^2(\Omega)}^2 \\
&\quad + (C_3 + C_4) t\, |\Omega| \\
&\quad + (C_2 + C_4) \int_0^t \left\| v^n(s) \right\|_{L^2(\Omega)}^2 ds + C_2 \int_0^t \left\| w^n(s) \right\|_{L^2(\Omega)}^2 ds \\
&\quad + \int_0^t \int_\Omega v^n \eta^n(v^n) \, dx\, dW^{v,n}(s) + \int_0^t \int_\Omega w^n \sigma^n(v^n) \, dx\, dW^{w,n}(s).
\end{aligned} \tag{6.15}$$

Since $\mathbb{E}\left[\int_0^T |f(t)|^2 \, dt \right] < \infty$ for $f = \int_\Omega v^n \eta^n(v^n)\, dx$ and $f = \int_\Omega w^n \sigma^n(v^n)\, dx$, the martingale property of stochastic integrals ensures that the expected value of each of the last two terms in (6.15) is zero. Hence, taking the expectation in (6.15), keeping in mind (5.3) and using Grönwall's inequality, we conclude that (6.3) and (6.4) hold.

The refinement of (6.3) into (6.6) comes from a martingale inequality. Indeed, taking the sup over $[0, T]$ and subsequently applying $\mathbb{E}[\cdot]$ in (6.15), it follows that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\| v^n(t) \right\|_{L^2(\Omega)}^2 \right] + \sum_{j=i,e} \mathbb{E}\left[\sup_{t\in[0,T]} \left\| \sqrt{\varepsilon_n} u_j^n(t) \right\|_{L^2(\Omega)}^2 \right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[0,T]} \left\| w^n(t) \right\|_{L^2(\Omega)}^2 \right] \leq C_5 \left(1 + \Gamma_\eta + \Gamma_\sigma \right),
\end{aligned} \tag{6.16}$$

where C_5 is a constant independent of n and

$$\begin{aligned}
\Gamma_\eta &:= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \int_\Omega v^n \eta^n(v^n) \, dx\, dW^{v,n}(s) \right| \right], \\
\Gamma_\sigma &:= \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \int_\Omega w^n \sigma^n(v^n) \, dx\, dW^{w,n}(s) \right| \right].
\end{aligned}$$

To arrive at (6.16) we have used (5.3), (6.3).

We use the Burkholder–Davis–Gundy inequality to handle the last two terms. To be more precise, using (3.4), the Cauchy–Schwarz inequality, the assumption (3.6) on η, Cauchy's inequality "with δ", and (6.3), we obtain

$$\begin{aligned}\Gamma_\eta &\le C_6 \mathbb{E}\left[\left(\int_0^T \sum_{k=1}^n \left|\int_\Omega v^n \eta_k^n(v^n)\,dx\right|^2 dt\right)^{\frac{1}{2}}\right]\\ &\le C_6 \mathbb{E}\left[\left(\int_0^T \left(\int_\Omega |v^n|^2\,dx\right)\left(\sum_{k=1}^n \int_\Omega |\eta_k^n(v^n)|^2\,dx\right) dt\right)^{\frac{1}{2}}\right]\\ &\le \delta \mathbb{E}\left[\sup_{t\in[0,T]} \|v^n(t)\|_{L^2(\Omega)}^2\right] + C_7,\end{aligned} \tag{6.17}$$

for any $\delta > 0$. Similarly, using (3.6) and (6.3),

$$\Gamma_\sigma \le \delta \mathbb{E}\left[\sup_{t\in[0,T]} \|w^n(t)\|_{L^2(\Omega)}^2\right] + C_8. \tag{6.18}$$

Combining (6.16), (6.17) and (6.18), with $\delta > 0$ small, the desired estimate (6.6) follows.

Finally, let us prove (6.5). By the Poincaré inequality, there is a constant $C_9 > 0$, depending on Ω but not n, ω and t, such that for each fixed $(\omega, t) \in D \times [0, T]$,

$$\|u_e^n(\omega, t, \cdot)\|_{L^2(\Omega)}^2 \le C_9 \|\nabla u_e^n(\omega, t, \cdot)\|_{L^2(\Omega)}^2.$$

Hence, by (6.4),

$$\mathbb{E}\left[\int_0^T \|u_e^n(\omega, t, \cdot)\|_{L^2(\Omega)}^2\,dt\right] \le C_{10}. \tag{6.19}$$

Since v^n $(= u_i^n - u_e^n)$ complies with (6.3), it follows that also u_i^n satisfies (6.19). □

In view of the n-independent estimates in Lemma 6.1, passing if necessary to a proper subsequence, we can assume that the following (weak) convergences hold as $n \to \infty$:

$$\begin{cases} u_j^n \rightharpoonup u_j & \text{in} \quad L^2\left(D, \mathcal{F}, P; L^2((0,T); H_D^1(\Omega))\right), \ j = i, e,\\ \varepsilon_n u_j^n \to 0 & \text{in} \quad L^2\left(D, \mathcal{F}, P; L^2((0,T); L^2(\Omega))\right), \ j = i, e,\\ v^n \rightharpoonup v & \text{in} \quad L^2\left(D, \mathcal{F}, P; L^2((0,T); H_D^1(\Omega))\right),\\ v^n \overset{\star}{\rightharpoonup} v & \text{in} \quad L^2\left(D, \mathcal{F}, P; L^\infty((0,T); L^2(\Omega))\right),\\ v^n \rightharpoonup v & \text{in} \quad L^4\left(D, \mathcal{F}, P; L^4(\Omega_T)\right),\\ w^n \overset{\star}{\rightharpoonup} w & \text{in} \quad L^2\left(D, \mathcal{F}, P; L^\infty((0,T); L^2(\Omega))\right). \end{cases} \tag{6.20}$$

The next result, a consequence of Lemma 6.1 and a martingale inequality, supplies high-order moment estimates, useful when converting a.s. convergence into L^2 convergence.

Corollary 6.2. *In addition to the assumptions in Lemma 6.1, suppose (5.5) holds with q_0 defined in (4.1). There exists a constant $C > 0$, independent of n, such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left\|v^n(t)\right\|_{L^2(\Omega)}^{q_0}\right]+\sum_{j=i,e}\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sqrt{\varepsilon_n}u_i^n(t)\right\|_{L^2(\Omega)}^{q_0}\right] + \mathbb{E}\left[\sup_{0\le t\le T}\left\|w^n(t)\right\|_{L^2(\Omega)}^{q_0}\right]\le C. \tag{6.21}$$

Moreover,

$$\sum_{j=i,e}\mathbb{E}\left[\left\|\nabla u_j^n\right\|_{L^2((0,T)\times\Omega)}^{q_0}\right]+\mathbb{E}\left[\left\|v^n\right\|_{L^4((0,T)\times\Omega)}^{2q_0}\right]\le C.$$

Proof. In view of (6.15), we have the following estimate for any $(\omega, t) \in D \times [0, T]$:

$$\begin{aligned}
&\sup_{0\le\tau\le t}\left\|v^n(\tau)\right\|_{L^2(\Omega)}^2+\sum_{j=i,e}\sup_{0\le\tau\le t}\left\|\sqrt{\varepsilon_n}u_j^n(\tau)\right\|_{L^2(\Omega)}^2\\
&\qquad+\sup_{0\le\tau\le t}\left\|w^n(\tau)\right\|_{L^2(\Omega)}^2\\
&\quad\le\left\|v^n(0)\right\|_{L^2(\Omega)}^2+\sum_{j=i,e}\left\|\sqrt{\varepsilon_n}u_j^n(0)\right\|_{L^2(\Omega)}^2+\left\|w^n(0)\right\|_{L^2(\Omega)}^2\\
&\qquad+C_1(1+t)+C_1\int_0^t\left\|v^n(s)\right\|_{L^2(\Omega)}^2\,ds+C_1\int_0^t\left\|w^n(s)\right\|_{L^2(\Omega)}^2\,ds\\
&\qquad+C_1\sup_{0\le\tau\le t}\left|\int_0^\tau\int_\Omega v^n\eta^n(v^n)\,dx\,dW^{v,n}(s)\right|\\
&\qquad+C_1\sup_{0\le\tau\le t}\left|\int_0^\tau\int_\Omega w^n\sigma^n(v^n)\,dx\,dW^{w,n}(s)\right|,
\end{aligned}$$

for some constant C_1 independent of n.

We raise both sides of this inequality to the power $q_0/2$, take the expectation, and apply several elementary inequalities, eventually arriving at

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|v^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]+\sum_{j=i,e}\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|\sqrt{\varepsilon_n}u_i^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]\\
&\qquad+\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|w^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]\\
&\quad\le C_2\mathbb{E}\left[\left\|v^n(0)\right\|_{L^2(\Omega)}^{q_0}\right]+C_2\sum_{j=i,e}\mathbb{E}\left[\left\|\sqrt{\varepsilon_n}u_i^n(0)\right\|_{L^2(\Omega)}^{q_0}\right]\\
&\qquad+C_2\mathbb{E}\left[\left\|w^n(0)\right\|_{L^2(\Omega)}^{q_0}\right]+C_2\left(1+t\right)^{\frac{q_0}{2}}\\
&\qquad+C_2\int_0^t\left\|v^n(s)\right\|_{L^2(\Omega)}^{q_0}\,ds+C_2\int_0^t\left\|w^n(s)\right\|_{L^2(\Omega)}^{q_0}\,ds+\Gamma_\eta+\Gamma_\sigma,
\end{aligned} \tag{6.22}$$

where

$$\Gamma_\eta:=\mathbb{E}\left[\sup_{0\le\tau\le t}\left|\int_0^\tau\int_\Omega v^n\eta^n(v^n)\,dx\,dW^{v,n}(s)\right|^{\frac{q_0}{2}}\right],$$

$$\Gamma_\sigma := \mathbb{E}\left[\sup_{0\le\tau\le t}\left|\int_0^\tau \int_\Omega w^n\sigma^n(v^n)\,dx\,dW^{w,n}(s)\right|^{\frac{q_0}{2}}\right].$$

Arguing as in (6.17), using a martingale inequality and (3.6),

$$\begin{aligned}\Gamma_\eta &\le C_3\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^n \left|\int_\Omega v^n\eta_k^n(v^n)\,dx\right|^2 ds\right)^{\frac{q_0}{4}}\right]\\ &\le C_3\mathbb{E}\left[\left(\int_0^t \left(\int_\Omega |v^n|^2\,dx\right)\left(\sum_{k=1}^n\int_\Omega \left|\eta_k^n(v^n)\right|^2 dx\right) ds\right)^{\frac{q_0}{4}}\right]\\ &\le \delta\mathbb{E}\left[\sup_{\tau\in[0,t]}\left\|v^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right] + C_4\mathbb{E}\left[\int_0^t \left\|v^n(s)\right\|_{L^2(\Omega)}^{q_0} ds\right] + C_5,\end{aligned} \tag{6.23}$$

for any $\delta > 0$. Similarly, relying again on (3.6),

$$\Gamma_\sigma \le \delta\mathbb{E}\left[\sup_{\tau\in[0,t]}\left\|w^n(\tau)\right\|_{L^2(\Omega)}^{q_0} dx\right] + C_6\mathbb{E}\left[\int_0^t \left\|v^n\right\|_{L^2(\Omega)}^{q_0} ds\right] + C_7. \tag{6.24}$$

With δ chosen small, combining (6.23) and (6.24) in (6.22) gives

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|v^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right] + \sum_{j=i,e}\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|\sqrt{\varepsilon_n}u_i^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]\\ &\qquad + \mathbb{E}\left[\sup_{0\le\tau\le t}\left\|w^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]\\ &\quad\le C_8\mathbb{E}\left[\left\|v^n(0)\right\|_{L^2(\Omega)}^{q_0}\right] + C_8\sum_{j=i,e}\mathbb{E}\left[\left\|\sqrt{\varepsilon_n}u_i^n(0)\right\|_{L^2(\Omega)}^{q_0}\right]\\ &\qquad + C_8\mathbb{E}\left[\left\|w^n(0)\right\|_{L^2(\Omega)}^{q_0}\right] + C_8 + C_8\int_0^t \mathbb{E}\left[\left\|v^n(s)\right\|_{L^2(\Omega)}^{q_0} ds\right],\end{aligned} \tag{6.25}$$

for some constant $C_8 > 0$ independent of n. Set

$$\begin{aligned}\Gamma(t) := &\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|v^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right] + \sum_{j=i,e}\mathbb{E}\left[\sup_{0\le\tau\le t}\left\|\sqrt{\varepsilon_n}u_i^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right]\\ &+ \mathbb{E}\left[\sup_{0\le\tau\le t}\left\|w^n(\tau)\right\|_{L^2(\Omega)}^{q_0}\right],\end{aligned}$$

and note that (6.25) reads $\Gamma(t) \le C_8\Gamma(0) + C_8 + C_8\int_0^t \Gamma(s)\,ds$ for $t \in [0,T]$. Now an application of Grönwall's inequality yields the desired result (6.21).

Finally, we can use (6.15), (6.23), (6.24), and (6.21) to conclude that

$$\sum_{j=i,e}\mathbb{E}\left[\left|\int_0^t\int_\Omega \left|\nabla u_i^n\right|^2 dx\,ds\right|^{\frac{q_0}{2}}\right] + \mathbb{E}\left[\left|\int_0^t\int_\Omega \left|v^n\right|^4 dx\,ds\right|^{\frac{q_0}{2}}\right] \le C_9,$$

and (6.21) follows. □

6.2. Temporal translation estimates

To secure strong $L^2_{t,x}$ compactness of the Faedo–Galerkin solutions, via a standard Aubin–Lions–Simon compactness lemma, we need to come up with n-independent temporal translation estimates.

Lemma 6.3. *Suppose conditions* (**GFHN**), (2.5), (3.6), *and* (5.3) *hold. Let*

$$u_i^n(t), u_e^n(t), v^n(t), w^n(t), \quad t \in [0, T],$$

satisfy (5.8), (5.9), (5.10), (5.11). *With* $u^n = v^n$ *or* w^n, *there is a constant* $C > 0$, *independent of* n, *such that for any sufficiently small* $\delta > 0$,

$$\mathbb{E}\left[\sup_{0\le\tau\le\delta}\int_0^{T-\tau}\int_\Omega \left|u^n(t+\tau,x)-u^n(t,x)\right|^2\,dx\,dt\right] \le C\delta^{\frac{1}{4}}. \tag{6.26}$$

Proof. We assume that v^n, u_i^n, u_e^n, w^n and η^n, σ^n have been extended by zero outside the time interval $[0, T]$. Recalling (5.6) (i.e., $v^n = u_i^n - u_e^n$), it follows that

$$\begin{aligned}
\Gamma_{ie}(t) &:= \int_\Omega \left|v^n(t+\tau,x)-v^n(t,x)\right|^2 dx + \varepsilon_n \sum_{j=i,e}\int_\Omega \left|u_j^n(t+\tau,x)-u_j^n(t,x)\right|^2 dx \\
&= \int_\Omega \left(u_i^n(t+\tau,x)-u_i^n(t,x)\right)\left(\int_t^{t+\tau} d\left(v^n(s,x)+\varepsilon_n u_i(s,x)\right)\right) dx \\
&\quad - \int_\Omega \left(u_e^n(t+\tau,x)-u_e^n(t,x)\right)\left(\int_t^{t+\tau} d\left(v^n(s,x)-\varepsilon_n u_e(s,x)\right)\right) dx.
\end{aligned}$$

In view of (5.18), see also (5.17),

$$\begin{aligned}
\Gamma_{ie}(t) &= -\sum_{j=i,e}\int_\Omega \left(\int_t^{t+\tau} M_j(x)\nabla u_j^n(s,x)\,ds\right)\cdot\nabla\left(u_j^n(t+\tau,x)-u_j^n(t,x)\right) dx \\
&\quad - \int_\Omega \left(\int_t^{t+\tau} I\left(v^n(s,x),w^n(s,x)\right) ds\right)\left(v^n(t+\tau,x)-v^n(t,x)\right) dx \\
&\quad + \int_\Omega \left(\int_t^{t+\tau} \eta^n(v^n(s,x))\,dW^{v,n}(s)\right)\left(v^n(t+\tau,x)-v^n(t,x)\right) dx.
\end{aligned}$$

Similarly, using the equation for w^n, cf. (5.18) and also (5.2),

$$\begin{aligned}
\Gamma_w(t) &:= \int_\Omega \left|w^n(t+\tau,x)-w^n(t,x)\right|^2 dx \\
&= \int_\Omega \left(\int_t^{t+\tau} H\left(v^n(s,x),w^n(s,x)\right) ds\right)\left(w^n(t+\tau,x)-w^n(t,x)\right) dx \\
&\quad + \int_\Omega \left(\int_t^{t+\tau} \sigma^n(v^n(s,x))\,dW^{v,n}(s)\right)\left(w^n(t+\tau,x)-w^n(t,x)\right) dx.
\end{aligned}$$

Integrating over $t \in (0, T-\tau)$ and summing the resulting equations gives

$$\int_0^{T-\tau}\Gamma_{ie}(t)\,dt + \int_0^{T-\tau}\Gamma_w(t)\,dt = \Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5, \tag{6.27}$$

where

$$\begin{aligned}
\Gamma_1 &:= -\sum_{j=i,e}\int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau} M_j(x)\nabla u_j^n(s,x)\,ds\right)\\
&\qquad\qquad\cdot\nabla\left(u_j^n(t+\tau,x)-u_j^n(t,x)\right)\,dx\,dt\\
\Gamma_2 &:= -\int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau} I\left(v^n(s,x),w^n(s,x)\right)\,ds\right)\\
&\qquad\qquad\times\left(v^n(t+\tau,x)-v^n(t,x)\right)\,dx\,dt\\
\Gamma_3 &:= \int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau} H\left(v^n(s,x),w^n(s,x)\right)\,ds\right)\\
&\qquad\qquad\times\left(w^n(t+\tau,x)-w^n(t,x)\right)\,dx\,dt\\
\Gamma_4 &:= \int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau} \eta^n(v^n(s,x))\,dW^{v,n}(s)\right)\\
&\qquad\qquad\times\left(v^n(t+\tau,x)-v^n(t,x)\right)\,dx\,dt\\
\Gamma_5 &:= \int_0^{T-\tau}\int_\Omega\left(\int_t^{t+\tau} \sigma^n(v^n(s,x))\,dW^{v,n}(s)\right)\\
&\qquad\qquad\times\left(w^n(t+\tau,x)-w^n(t,x)\right)\,dx\,dt.
\end{aligned}$$

We examine these six terms separately. For the Γ_1 term, noting that

$$\left|\int_t^{t+\tau} M_j(x)\nabla u_j^n(s,x)\,ds\right|^2 \le M\tau\int_t^{t+\tau}\left|\nabla u_j^n(s,x)\right|^2\,ds,$$

thanks to (2.5), we obtain

$$\begin{aligned}
|\Gamma_1| \le \sqrt{M\tau}\sum_{j=i,e}&\left(\int_0^{T-\tau}\int_t^{t+\tau}\int_\Omega\left|\nabla u_j^n(s,x)\right|^2\,dx\,ds\,dt\right)^{\frac12}\\
&\times\left(\int_0^{T-\tau}\int_\Omega\left|\nabla\left(u_j^n(t+\tau,x)-u_j^n(t,x)\right)\right|^2\,dx\,dt\right)^{\frac12},
\end{aligned}$$

using Cauchy–Schwarz's inequality. Hence, by Young's inequality and (6.4),

$$\mathbb{E}\left[\sup_{0\le\tau\le\delta}|\Gamma_1|\right]\le C_1\sqrt{\delta}, \tag{6.28}$$

for some constant $C_1>0$ independent of n.

Next, take notice of the bound

$$\begin{aligned}
\left|\int_t^{t+\tau} I\left(v^n(s,x),w^n(s,x)\right)\,ds\right|^{\frac43} &\le \tau^{\frac13}\int_t^{t+\tau}\left|I\left(v^n(s,x),w^n(s,x)\right)\right|^{\frac43}\,ds\\
&\le C_2\tau^{\frac13}\int_t^{t+\tau}\left(1+|v(s,x)|^4+|w(s,x)|^2\right)\,ds,
\end{aligned} \tag{6.29}$$

where we have used the inequality

$$|I(v,w)|^{\frac43}\le C_2\left(1+|v|^4+|w|^2\right), \tag{6.30}$$

resulting from (**GFHN**) and Young's inequality. Due to (6.29), (6.3) and (6.4),

$$|\Gamma_2| \le C_3 \tau^{\frac{1}{4}} \left(\int_0^{T-\tau} \int_t^{t+\tau} \int_\Omega \left(1 + |v(s,x)|^4 + |w(s,x)|^2\right) \,dx\,ds\,dt \right)^{\frac{3}{4}} \times \left(\int_0^{T-\tau} \int_\Omega \left| v^n(t+\tau,x) - v^n(t,x) \right|^4 \,dx\,dt \right)^{\frac{1}{4}},$$

and for this reason, in view of Young's inequality and (6.4),

$$\mathbb{E}\left[\sup_{0\le\tau\le\delta} |\Gamma_2| \right] \le C_4 \delta^{\frac{1}{4}}. \tag{6.31}$$

Similarly, since $|H(v,w)|^2 \le C_5 \left(1 + |v|^4 + |w|^2\right)$, cf. (**GFHN**), we obtain

$$\mathbb{E}\left[\sup_{0\le\tau\le\delta} |\Gamma_3| \right] \le C_6 \delta^{\frac{1}{2}}. \tag{6.32}$$

Finally, we treat the stochastic terms. By the Cauchy–Schwarz inequality,

$$|\Gamma_4| \le \left(\int_0^T \int_\Omega \sup_{0\le\tau\le\delta} \left| \int_t^{t+\tau} \eta^n(v^n(s,x))\,dW^{v,n}(s) \right|^2 \,dx\,dt \right)^{\frac{1}{2}} \times \left(\int_0^T \sup_{0\le\tau\le\delta} \int_\Omega \left| v^n(t+\tau,x) - v^n(t,x) \right|^2 \,dx\,dt \right)^{\frac{1}{2}}.$$

Applying $\mathbb{E}[\cdot]$ along with the Cauchy–Schwarz inequality, we gather the estimate

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le\tau\le\delta} |\Gamma_4| \right] &\le \left(\mathbb{E}\left[\int_0^T \int_\Omega \sup_{0\le\tau\le\delta} \left| \int_t^{t+\tau} \eta^n(v^n(s,x))\,dW^{v,n}(s) \right|^2 \,dx\,dt \right] \right)^{\frac{1}{2}} \\
&\quad \times \left(\mathbb{E}\left[\sup_{0\le\tau\le\delta} \int_0^{T-\tau} \int_\Omega \left| v^n(t+\tau,x) - v^n(t,x) \right|^2 \,dx\,dt \right] \right)^{\frac{1}{2}} \\
&\le C_7 \left(\mathbb{E}\left[\int_0^T \int_t^{t+\delta} \sum_{k=1}^n \int_\Omega \left| \eta_k^n(v^n(s,x)) \right|^2 \,dx\,ds\,dt \right] \right)^{\frac{1}{2}} \\
&\le C_8 \left(\mathbb{E}\left[\int_0^T \int_t^{t+\delta} \int_\Omega \left(1 + \left| v^n(s,x) \right|^2\right) \,dx\,ds\,dt \right] \right)^{\frac{1}{2}} \le C_9 \delta^{\frac{1}{2}},
\end{aligned} \tag{6.33}$$

where we have also used the Burkholder–Davis–Gundy inequality (3.4) and (3.6), (6.6).

Similarly,

$$\mathbb{E}\left[\sup_{0\le\tau\le\delta} |\Gamma_5| \right] \le C_{10} \delta^{\frac{1}{2}}. \tag{6.34}$$

Collecting the previous estimates (6.28), (6.31), (6.32), (6.33), and (6.34) we readily conclude from (6.27) that the time translation estimate (6.26) holds. □

6.3. Tightness and a.s. representations

To justify passing to the limit in the nonlinear terms in (5.2), we must show that $\{v^n\}_{n\geq 1}$ converges strongly, thereby upgrading the weak L^2 convergence in (6.20). Strong (t,x) convergence is a result of the spatial H^1_D bound (6.4) and the time translation estimate (6.26).

On the other hand, to secure strong (a.s.) convergence in the probability variable $\omega \in D$ we must invoke some nontrivial results of Skorokhod, linked to tightness of probability measures and a.s. representations of random variables. Actually, there is a complicating factor at play here, namely that the sequences $\{u_i^n\}_{n\geq 1}$, $\{u_e^n\}_{n\geq 1}$ only converge weakly in (t,x) because of the degenerate structure of the bidomain model. As a result, we must turn to the Skorokhod–Jakubowski representation theorem [31], which applies to separable Banach spaces equipped with the weak topology and other so-called quasi-Polish spaces. At variance with the original Skorokhod representations on Polish spaces, the flexibility of the Jakubowski version comes at the expense of having to pass to a subsequence (which may be satisfactory in many situations). We refer to [7–10,40,51] for works making use of Skorokhod–Jakubowski a.s. representations.

Following [3,38] (for example), the aim is to establish tightness of the probability measures (laws) generated by the Faedo–Galerkin solutions $\{(U^n, W^n, U_0^n)\}_{n\geq 1}$, where

$$U^n = u_i^n, u_e^n, v^n, w^n, \qquad W^n = W^{v,n}, W^{w,n}, \qquad U_0^n = u_{i,0}^n, u_{e,0}^n, v_0^n, w_0^n. \tag{6.35}$$

Accordingly, we choose the following path space for these measures:

$$\begin{aligned}\mathcal{X} := &\Big[\big(L^2((0,T); H^1_D(\Omega_T))\text{–weak}\big)^2 \times L^2(\Omega_T) \times L^2((0,T); (H^1_D(\Omega_T))^*)\Big] \\ &\times \Big[(C([0,T]; \mathbb{U}_0))^2\Big] \times \Big[\big(L^2(\Omega)\big)^4\Big] =: \mathcal{X}_U \times \mathcal{X}_W \times \mathcal{X}_{U_0},\end{aligned}$$

where the tag "–weak" signifies that the space is equipped with the weak topology. The σ-algebra of Borel subsets of $\mathcal{X}$ is denoted by $\mathcal{B}(\mathcal{X})$. We introduce the $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued measurable mapping Φ_n defined on $(D, \mathcal{F}, P)$ by $\Phi_n(\omega) = \big(U^n(\omega), W^n, U_0^n(\omega)\big)$. On $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, we define the probability measure (law of Φ_n)

$$\mathcal{L}_n(\mathcal{A}) = P\left(\Phi_n^{-1}(\mathcal{A})\right), \qquad \mathcal{A} \in \mathcal{B}(\mathcal{X}). \tag{6.36}$$

We denote by $\mathcal{L}_{u_i^n}, \mathcal{L}_{u_e^n}$ the respective laws of u_i^n, u_e^n on $L^2((0,T); H^1_D(\Omega_T))$-weak, with similar notations for the laws of v^n on $L^2(\Omega_T)$, w^n on $L^2((0,T); (H^1_D(\Omega))^*)$, $W^{v,n}, W^{w,n}$ on $C([0,T]; \mathbb{U}_0)$, and $u_{i,0}^n, u_{e,0}^n, v_0^n, w_0^n$ on $L^2(\Omega)$. Hence,

$$\mathcal{L}_n = \mathcal{L}_{u_i^n} \times \mathcal{L}_{u_e^n} \times \mathcal{L}_{v^n} \times \mathcal{L}_{w^n} \times \mathcal{L}_{u_{i,0}^n} \times \mathcal{L}_{u_{e,0}^n} \times \mathcal{L}_{v_0^n} \times \mathcal{L}_{w_0^n}.$$

Inspired by [3], for any two sequences of positive numbers r_m, ν_m tending to zero as $m \to \infty$, we introduce the set

$$\begin{aligned}\mathcal{Z}^v_{r_m,\nu_m} := \Big\{ &u \in L^\infty\left((0,T); L^2(\Omega)\right) \cap L^2((0,T); H^1_D(\Omega)) : \\ &\sup_{m\geq 1} \frac{1}{\nu_m} \sup_{0\leq\tau\leq r_m} \|u(\cdot+\tau) - u\|_{L^2((0,T-\tau); L^2(\Omega))} < \infty \Big\}.\end{aligned}$$

Then $\mathcal{Z}^v_{r_m,\nu_m}$ is a Banach space under the natural norm

$$\begin{aligned}\|u\|_{\mathcal{Z}^v_{r_m,\nu_m}} := &\|u\|_{L^\infty((0,T);L^2(\Omega))} + \|u\|_{L^2((0,T);H^1_D(\Omega))} \\ &+ \sup_{m\geq 1} \frac{1}{\nu_m} \sup_{0\leq\tau\leq r_m} \|u(\cdot+\tau) - u\|_{L^2((0,T-\tau);L^2(\Omega))}.\end{aligned}$$

Moreover, $\mathcal{Z}^v_{r_m,\nu_m}$ is compactly embedded in $L^2(\Omega_T)$, which is a consequence of an Aubin–Lions–Simon lemma. Suppose $X_1 \subset X_0$ are two Banach spaces, where X_1 is compactly embedded in X_0. Let $\mathcal{Z} \subset L^p((0,T);X_0)$, where $1 \le p \le \infty$. Simon [49] provides several results ensuring the compactness of $\mathcal{Z}$ in $L^p((0,T);X_0)$ (and in $C([0,T];X_0)$ if $p=\infty$). For example, by assuming that $\mathcal{Z}$ is bounded in $L^1_{\mathrm{loc}}((0,T);X_1)$ and $\|u(\cdot+\tau)-u\|_{L^p((0,T-\tau);X_0)} \to 0$ as $\tau \to 0$, uniformly for $u \in \mathcal{Z}$ [49, Theorem 3].

The space $\mathcal{Z}^v_{r_m,\nu_m}$ is relevant for v^n, while for w^n we utilize

$$\mathcal{Z}^w_{r_m,\nu_m} := \Big\{ u \in L^\infty\big((0,T);L^2(\Omega)\big) : \\ \sup_{m\ge1} \frac{1}{\nu_m} \sup_{0\le\tau\le r_m} \|u(\cdot+\tau)-u\|_{L^2((0,T-\tau);(H^1_D(\Omega))^*)} < \infty \Big\},$$

with a corresponding natural norm $\|u\|_{\mathcal{Z}^w_{r_m,\nu_m}}$. Besides, $\mathcal{Z}^v_{r_m,\nu_m}$ is compactly embedded in $L^2((0,T);(H^1_D(\Omega))^*)$.

Lemma 6.4 (*Tightness of Laws (6.36) for Faedo–Galerkin approximations*)**.** *Equipped with the estimates in Lemmas 6.1 and 6.3, the sequence of laws $\{\mathcal{L}_n\}_{\ge1}$ is tight on $(\mathcal{X},\mathcal{B}(\mathcal{X}))$.*

Proof. Given any $\delta > 0$, we need to produce compact sets

$$\mathcal{K}_{0,\delta} \subset L^2((0,T);H^1_D(\Omega))\text{–weak},$$
$$\mathcal{K}_{1,\delta} \subset L^2(\Omega_T), \quad \mathcal{K}_{2,\delta} \subset L^2((0,T);(H^1_D(\Omega))^*),$$
$$\mathcal{K}_{3,\delta} \subset C([0,T];\mathbb{U}_0), \quad \mathcal{K}_{4,\delta} \subset L^2(\Omega),$$

such that, with $\mathcal{K}_\delta = \big(\mathcal{K}_{0,\delta}\big)^2 \times \mathcal{K}_{1,\delta} \times \mathcal{K}_{2,\delta} \times \big(\mathcal{K}_{3,\delta}\big)^2 \times \big(\mathcal{K}_{4,\delta}\big)^4$,

$$\mathcal{L}_n(\mathcal{K}_\delta) = P(\{\omega \in D : \Phi_n(\omega) \in \mathcal{K}_\delta\}) > 1-\delta.$$

This inequality follows if we can show that

$$\mathcal{L}_{u^n}\big(\mathcal{K}^c_{0,\delta}\big) = P\big(\{\omega \in D : u^n(\omega) \notin \mathcal{K}_{0,\delta}\}\big) \le \frac{\delta}{10}, \quad u^n = u^n_i, u^n_e, \tag{6.37}$$

$$\mathcal{L}_{v^n}\big(\mathcal{K}^c_{1,\delta}\big) = P\big(\{\omega \in D : v^n(\omega) \notin \mathcal{K}_{1,\delta}\}\big) \le \frac{\delta}{10}, \tag{6.38}$$

$$\mathcal{L}_{w^n}\big(\mathcal{K}^c_{2,\delta}\big) = P\big(\{\omega \in D : w^n(\omega) \notin \mathcal{K}_{2,\delta}\}\big) \le \frac{\delta}{10}, \tag{6.39}$$

$$\mathcal{L}_{W^n}\big(\mathcal{K}^c_{3,\delta}\big) = P\big(\{\omega \in D : W^n(\omega) \notin \mathcal{K}_{3,\delta}\}\big) \le \frac{\delta}{10}, \quad W^n = W^{v,n}, W^{w,n}, \tag{6.40}$$

$$\mathcal{L}_{u^n_0}\big(\mathcal{K}^c_{4,\delta}\big) = P\big(\{\omega \in D : U^n_0(\omega) \notin \mathcal{K}_{4,\delta}\}\big) \le \frac{\delta}{10}, \quad U^n_0 = u^n_{i,0}, u^n_{e,0}, v^n_0, w^n_0. \tag{6.41}$$

By weak compactness of bounded sets in $L^2((0,T);H^1_D(\Omega))$, the set

$$\mathcal{K}^{0,\delta} := \Big\{ u : \|u\|_{L^2((0,T);H^1_D(\Omega))} \le R_{0,\delta} \Big\},$$

is a compact subset of $L^2((0,T);H^1_D(\Omega))$-weak, where $R_{0,\delta} > 0$ is to be determined later. Recalling the Chebyshev inequality for a nonnegative random variable ξ,

$$P(\{\omega \in D : \xi(\omega) \ge R\}) \le \frac{E\big[\xi^k\big]}{R^k}, \qquad R,k > 0, \tag{6.42}$$

it follows that

$$P\left(\left\{\omega\in D: u^n(\omega)\notin \mathcal{K}^{0,\delta}\right\}\right) = P\left(\left\{\omega\in D: \left\|u^n(\omega)\right\|_{L^2((0,T);H^1_D(\Omega))} > R_{0,\delta}\right\}\right) \le \frac{1}{R_{0,\delta}}\mathbb{E}\left[\left\|u^n(\omega)\right\|_{L^2((0,T);H^1_D(\Omega))}\right] \le \frac{C}{R_{0,\delta}}.$$

To derive the last inequality we used the Cauchy–Schwarz inequality and then (6.4). Clearly, we can choose $R_{0,\delta} > 0$ such that (6.37) holds.

We fix two sequences $\{r_m\}_{m=1}^\infty$, $\{\nu_m\}_{m=1}^\infty$ of positive numbers tending to zero as $m\to\infty$ (independently of n), such that

$$\sum_{m=1}^\infty r_m^{\frac{1}{8}}/\nu_m < \infty, \tag{6.43}$$

and define $\mathcal{K}^{1,\delta} := \left\{u : \|u\|_{\mathcal{Z}^v_{r_m,\nu_m}} \le R_{1,\delta}\right\}$, for a number $R_{1,\delta} > 0$ to be determined later. Evidently, in view of an Aubin–Lions–Simon lemma, $\mathcal{K}^{1,\delta}$ is a compact subset of $L^2(\Omega_T)$. We have

$$\begin{aligned}
&P\left(\left\{\omega\in D: v^n(\omega)\notin \mathcal{K}^{1,\delta}\right\}\right)\\
&\quad\le P\left(\left\{\omega\in D: \left\|v^n(\omega)\right\|_{L^\infty((0,T);L^2(\Omega))} > R_{1,\delta}\right\}\right)\\
&\qquad + P\left(\left\{\omega\in D: \left\|v^n(\omega)\right\|_{L^2((0,T);H^1_D(\Omega))} > R_{1,\delta}\right\}\right)\\
&\qquad + P\left(\left\{\omega\in D: \sup_{0\le\tau\le r_m}\left\|v^n(\cdot+\tau)-v^n\right\|_{L^2((0,T-\tau);L^2(\Omega))} > R_{1,\delta}\,\nu_m\right\}\right)\\
&\quad =: P_{1,1}+P_{1,2}+P_{1,3} \quad (\text{for any } m\ge 1).
\end{aligned}$$

Again by the Chebyshev inequality (6.42), we infer that

$$\begin{aligned}
P_{1,1} &\le \frac{1}{R_{1,\delta}}\mathbb{E}\left[\left\|v^n(\omega)\right\|_{L^\infty((0,T);L^2(\Omega))}\right] \le \frac{C}{R_{1,\delta}},\\
P_{1,2} &\le \frac{1}{R_{1,\delta}}\mathbb{E}\left[\left\|v^n(\omega)\right\|_{L^2((0,T);H^1_D(\Omega))}\right] \le \frac{C}{R_{1,\delta}},\\
P_{1,3} &\le \sum_{m=1}^\infty \frac{1}{R_{1,\delta}\,\nu_m}\mathbb{E}\left[\sup_{0\le\tau\le r_m}\left\|v^n(\cdot+\tau)-v^n\right\|_{L^2((0,T-\tau);L^2(\Omega))}\right]\\
&\le \frac{C}{R_{1,\delta}}\sum_{m=1}^\infty \frac{r_m^{\frac{1}{8}}}{\nu_m},
\end{aligned}$$

where we have used (6.4), (6.6), and (6.26). On the grounds of this and (6.43), we can choose R_δ such that (6.38) holds.

Similarly, with sequences $\{r_m\}_{m=1}^\infty$, $\{\nu_m\}_{m=1}^\infty$ as above, define

$$\mathcal{K}^{2,\delta} := \left\{u : \|u\|_{\mathcal{Z}^w_{r_m,\nu_m}} \le R_{2,\delta}\right\},$$

for a number $R_{2,\delta} > 0$ to be determined later. By an Aubin–Lions–Simon lemma, $\mathcal{K}^{2,\delta}$ is a compact subset of $L^2((0,T);(H_D^1(\Omega))^*)$. We have

$$\begin{aligned} &P\left(\left\{\omega \in D : w^n(\omega) \notin \mathcal{K}^{2,\delta}\right\}\right) \\ &\quad \le P\left(\left\{\omega \in D : \left\|w^n(\omega)\right\|_{L^\infty((0,T);L^2(\Omega))} > R_{2,\delta}\right\}\right) \\ &\qquad + P\left(\left\{\omega \in D : \sup_{0\le\tau\le r_m} \left\|w^n(\cdot+\tau) - w^n\right\|_{L^2((0,T-\tau);(H_D^1(\Omega))^*)} > R_{2,\delta}\,\nu_m\right\}\right) \\ &\quad =: P_{2,1} + P_{2,2} \quad (\text{for any } m \ge 1), \end{aligned}$$

where, using (6.42) and (6.6) as before,

$$P_{2,1} \le \frac{1}{R_\delta}\mathbb{E}\left[\left\|w^n(\omega)\right\|_{L^\infty((0,T);L^2(\Omega))}\right] \le \frac{C}{R_{2,\delta}},$$

and, via (6.26) and (6.43),

$$P_{2,2} \le \sum_{m=1}^{\infty} \frac{1}{R_{2,\delta}\,\nu_m}\mathbb{E}\left[\sup_{0\le\tau\le r_m} \left\|w^n(\cdot+\tau) - w^n\right\|_{L^2((0,T-\tau);(H_D^1(\Omega))^*)}\right] \le \frac{C}{R_{2,\delta}}.$$

Consequently, we can choose $R_{2,\delta}$ such that (6.39) holds.

Recall that the finite dimensional approximations $W^n = W^{v,n}, W^{w,n}$, cf. (5.10), are P-a.s. convergent in $C([0,T];\mathbb{U}_0)$ as $n \to \infty$, and hence the laws $\mathcal{L}_{W^n}$ converge weakly. This entails the tightness of $\{\mathcal{L}_{W^n}\}_{n\ge1}$, i.e., for any $\delta > 0$, there exists a compact set $\mathcal{K}_{3,\delta}$ in $C([0,T];\mathbb{U}_0)$ such that (6.40) holds. Similarly, as the finite dimensional approximations $u_{i,0}^n, u_{e,0}^n, v_0^n, w_0^n$, cf. (5.11), are P-a.s. convergent in $L^2(\Omega)$, the laws $\mathcal{L}_{U_0^n}$ converge weakly ($\mathcal{L}_{v_0^n} \rightharpoonup \mu_{v_0}, \mathcal{L}_{w_0^n} \rightharpoonup \mu_{w_0}$). Hence, (6.41) follows. □

Lemma 6.5. *[Skorokhod–Jakubowski a.s. Representations] By passing to a subsequence (that we do not relabel), there exist a new probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$ and new random variables $\left(\tilde{U}^n, \tilde{W}^n, \tilde{U}_0^n\right)$, $\left(\tilde{U}, \tilde{W}, \tilde{U}_0\right)$, where*

$$\begin{aligned} &\tilde{U}^n = \tilde{u}_i^n, \tilde{u}_e^n, \tilde{v}^n, \tilde{w}^n, \quad \tilde{W}^n = \tilde{W}^{v,n}, \tilde{W}^{w,n}, \quad \tilde{U}_0^n = \tilde{u}_{i,0}^n, \tilde{u}_{e,0}^n, \tilde{v}_0^n, \tilde{w}_0^n, \\ &\tilde{U} = \tilde{u}_i, \tilde{u}_e, \tilde{v}, \tilde{w}, \quad \tilde{W} = \tilde{W}^v, \tilde{W}^w, \quad \tilde{U}_0 = \tilde{u}_{i,0}, \tilde{u}_{e,0}, \tilde{v}_0, \tilde{w}_0, \end{aligned} \tag{6.44}$$

with respective (joint) laws $\mathcal{L}_n$ and $\mathcal{L}$, such that the following strong convergences hold $\tilde{P}$-almost surely as $n \to \infty$:

$$\begin{aligned} &\tilde{v}^n \to \tilde{v} \quad \text{in } L^2((0,T);L^2(\Omega)), \quad \tilde{w}^n \to \tilde{w} \quad \text{in } L^2((0,T);(H_D^1(\Omega))^*), \\ &\tilde{W}^{v,n} \to \tilde{W}^v,\ \tilde{W}^{w,n} \to \tilde{W}^w \quad \text{in } C([0,T];\mathbb{U}_0), \\ &\tilde{u}_{i,0}^n \to \tilde{u}_{i,0},\ \tilde{u}_{e,0}^n \to \tilde{u}_{e,0},\ \tilde{v}_0^n \to \tilde{v}_0,\ \tilde{w}_0^n \to \tilde{w}_0 \quad \text{in } L^2(\Omega). \end{aligned} \tag{6.45}$$

Moreover, the following weak convergences hold $\tilde{P}$-almost surely as $n \to \infty$:

$$\tilde{u}_i^n \rightharpoonup \tilde{u}_i,\ \tilde{u}_e^n \rightharpoonup \tilde{u}_e \quad \text{in } L^2((0,T);H_D^1(\Omega)). \tag{6.46}$$

Proof. Thanks to the Skorokhod–Jakubowski representation theorem (Theorem 3.2), there exist a new probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$ and new $\mathcal{X}$-valued random variables

$$\begin{aligned} \tilde{\Phi}_n &= \left(\tilde{u}_i^n, \tilde{u}_e^n, \tilde{v}^n, \tilde{w}^n, \tilde{W}^{v,n}, \tilde{W}^{w,n}, \tilde{u}_{i,0}^n, \tilde{u}_{e,0}^n, \tilde{v}_0^n, \tilde{w}_0^n\right), \\ \tilde{\Phi} &= \left(\tilde{u}_i, \tilde{u}_e, \tilde{v}, \tilde{w}, \tilde{W}^v, \tilde{W}^w, \tilde{u}_{i,0}, \tilde{u}_{e,0}, \tilde{v}_0, \tilde{w}_0\right) \end{aligned} \tag{6.47}$$

on $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$, such that the law of $\tilde{\Phi}_n$ is $\mathcal{L}_n$ and as $n \to \infty$,

$$\tilde{\Phi}_n \to \tilde{\Phi} \quad \tilde{P}\text{-almost surely (in } \mathcal{X}). \tag{6.48}$$

To be more accurate, the Skorokhod–Jakubowski theorem implies (6.47), (6.48) along a subsequence, but (as usual) we do not relabel the involved variables. Inasmuch as (6.48) is a repackaging of (6.45), (6.46), this concludes the proof. □

Remark 6.6. As mentioned before, since our path space $\mathcal{X}$ is not a Polish space, we use Skorokhod–Jakubowski a.s. representations [31] instead of the classical Skorokhod theorem [16]. For a proof that $L^2((0,T); H_D^1(\Omega_T))$-weak (and thus $\mathcal{X}$) is covered by the Skorokhod–Jakubowski theorem, see for example [8, page 1645].

Lemma 6.7 (*A Priori Estimates*)**.** *The a priori estimates in Lemma 6.1 continue to hold for the new random variables $\tilde{u}_i^n, \tilde{u}_e^n, \tilde{v}^n, \tilde{w}^n$ on $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$, that is,*

$$\begin{cases} \left\| \tilde{u}_j^n \right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^2((0,T);H_D^1(\Omega))\right)} \le C, \quad j = i, e, \\ \left\| \sqrt{\varepsilon_n}\tilde{u}_j^n \right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^\infty((0,T);L^2(\Omega))\right)} \le C, \quad j = i, e, \\ \left\| \tilde{v}^n \right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^2((0,T);H_D^1(\Omega))\right)} \le C, \\ \left\| \tilde{v}^n \right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^\infty((0,T);L^2(\Omega))\right)} \le C, \\ \left\| \tilde{v}^n \right\|_{L^4\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^4(\Omega_T)\right)} \le C, \\ \left\| \tilde{w}^n \right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^\infty((0,T);L^2(\Omega))\right)} \le C, \end{cases} \tag{6.49}$$

for some n-independent constant $C > 0$. The same applies to the estimates in Corollary 6.2, provided (5.5) holds. Namely,

$$\left\| \left(\sqrt{\varepsilon_n}\tilde{u}_i^n, \sqrt{\varepsilon_n}\tilde{u}_e^n, \tilde{v}^n, \tilde{w}^n\right) \right\|_{L^{q_0}\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^\infty((0,T);L^2(\Omega))\right)} \le C, \tag{6.50}$$

$$\left\| \left(\nabla\tilde{u}_i^n, \nabla\tilde{u}_e^n\right) \right\|_{L^{q_0}\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^2((0,T)\times\Omega)\right)}, \left\| \tilde{v}^n \right\|_{L^{2q_0}\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^4((0,T)\times\Omega)\right)} \le C. \tag{6.51}$$

Proof. Since the laws of v^n and $\tilde{v}^n$ coincide and $|\cdot|^2 := \|\cdot\|^2_{L^\infty((0,T);L^2(\Omega))}$ is bounded continuous on $B := L^\infty((0,T); L^2(\Omega))$ (so $|\cdot|^2$ is measurable and B is a Borel set in $\mathcal{X}$), $\int_B |v|^2 \, d\mathcal{L}_{\tilde{v}^n}(v) = \int_B |v|^2 \, d\mathcal{L}_{v^n}(v)$ and thus

$$\tilde{\mathbb{E}}\left[\left\|\tilde{v}^n(t)\right\|^2_{L^\infty((0,T);L^2(\Omega))}\right] = \mathbb{E}\left[\left\|v^n(t)\right\|^2_{L^\infty((0,T);L^2(\Omega))}\right] \overset{(6.6)}{\le} C,$$

where $\tilde{\mathbb{E}}[\cdot]$ is the expectation operator with respect to $(\tilde{P}, \tilde{D})$; hence the fourth estimate in (6.49) holds. As a matter of fact, by equality of the laws, all the estimates in Lemma 6.1 and Corollary 6.2 hold for the corresponding "tilde" functions defined on $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$. □

Let us introduce the following stochastic basis linked to $\tilde{\Phi}_n$, cf. (6.47):

$$\begin{aligned}&\tilde{\mathcal{S}}_n = \left(\tilde{D}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t^n\right\}_{t\in[0,T]}, \tilde{P}, \tilde{W}^{v,n}, \tilde{W}^{w,n}\right),\\ &\tilde{\mathcal{F}}_t^n = \sigma\left(\sigma\left(\tilde{\Phi}_n\big|_{[0,t]}\right)\bigcup\left\{N \in \tilde{\mathcal{F}} : \tilde{P}(N) = 0\right\}\right), \qquad t \in [0, T];\end{aligned} \tag{6.52}$$

thus $\left\{\tilde{\mathcal{F}}_t^n\right\}_{n\geq 1}$ is the smallest filtration making all the relevant processes (6.47) adapted. By equality of the laws and [16], $\tilde{W}^{v,n}$ and $\tilde{W}^{w,n}$ are cylindrical Brownian motions, i.e., there exist sequences $\left\{\tilde{W}_k^{v,n}\right\}_{k\geq 1}$ and $\left\{\tilde{W}_k^{w,n}\right\}_{k\geq 1}$ of mutually independent real-valued Brownian motions adapted to $\left\{\tilde{\mathcal{F}}_t^n\right\}_{t\in[0,T]}$ such that $\tilde{W}^{v,n} = \sum_{k\geq 1} \tilde{W}_k^{v,n}\psi_k$ and $\tilde{W}^{w,n} = \sum_{k\geq 1} \tilde{W}_k^{w,n}\psi_k$, where $\{\psi_k\}_{k\geq 1}$ is the basis of $\mathbb{U}$ and each series converges in $\mathbb{U}_0 \supset \mathbb{U}$ (cf. Section 3). Below we need the n-truncated sums

$$\tilde{W}^{v,(n)} = \sum_{k=1}^{n} \tilde{W}_k^{v,n}\psi_k, \qquad \tilde{W}^{w,(n)} = \sum_{k=1}^{n} \tilde{W}_k^{w,n}\psi_k, \tag{6.53}$$

which converge respectively to $\tilde{W}^v$, $\tilde{W}^w$ in $C([0, T]; \mathbb{U}_0)$, $\tilde{P}$-a.s., cf. (6.45).

We must show that the Faedo–Galerkin equations hold on the new probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$. To do that, we use an argument of Bensoussan [3], developed originally for the stochastic Navier–Stokes equations. For other possible methods leading to the construction of martingale solutions, see for example [16, Chap. 8] and [40].

Lemma 6.8 (*Faedo–Galerkin Equations*)**.** *Relative to the stochastic basis $\tilde{\mathcal{S}}_n$ in (6.52), the functions $\tilde{U}^n$, $\tilde{W}^n$, $\tilde{U}_0^n$ defined in (6.44) satisfy the following equations $\tilde{P}$-a.s.:*

$$\begin{aligned}\tilde{v}^n(t) + \varepsilon_n\tilde{u}_i^n(t) &= \tilde{v}_0^n + \varepsilon_n\tilde{u}_{i,0}^n + \int_0^t \Pi_n\left[\nabla\cdot\left(M_i\nabla\tilde{u}_i^n\right) - I(\tilde{v}^n, \tilde{w}^n)\right] ds\\ &\quad + \int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,(n)}(s) \quad \text{in } (H_D^1(\Omega))^*,\\ \tilde{v}^n(t) - \varepsilon_n\tilde{u}_e^n(t) &= \tilde{v}_0^n - \varepsilon_n\tilde{u}_{e,0}^n + \int_0^t \Pi_n\left[-\nabla\cdot\left(M_e\nabla\tilde{u}_e^n\right) - I(\tilde{v}^n, \tilde{w}^n)\right] ds\\ &\quad + \int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,(n)}(s) \quad \text{in } (H_D^1(\Omega))^*,\\ \tilde{w}^n(t) &= \tilde{w}_0^n + \int_0^t H(\Pi_n\tilde{v}^n, \Pi_n^w\tilde{w}^n)\, ds + \int_0^t \sigma^n(\tilde{v}^n)\, d\tilde{W}^{w,(n)}(s) \quad \text{in } (H_D^1(\Omega))^*,\end{aligned} \tag{6.54}$$

for each $t \in [0, T]$, where ε_n is specified in (5.9) and $\tilde{W}^{v,(n)}$, $\tilde{W}^{w,(n)}$ are defined in (6.53). Moreover,

$$\tilde{v}^n = \tilde{u}_i^n - \tilde{u}_e^n, \quad d\tilde{P}\times dt\times dx \text{ a.e. in } \tilde{D}\times(0, T)\times\Omega, \tag{6.55}$$

and (by construction) $\tilde{U}^n$, $\tilde{W}^n$ are continuous, adapted (and thus predictable) processes. Finally, the laws of $\tilde{v}_0^n$ and $\tilde{w}_0^n$ coincide with the laws of $\Pi_n v_0$ and $\Pi_n w_0$, respectively, where $v_0 \sim \mu_{v_0}$, $w_0 \sim \mu_{w_0}$ (see Definition 4.1).

Proof. We establish the first equation in (6.54), with the remaining ones following along the same lines. In accordance with Lemma 5.2 and (6.35), recall that (U^n, W^n, U_0^n) is the continuous adapted solution to the Faedo–Galerkin equations (5.17) relative to $\mathcal{S}$, cf. (5.1).

Let us introduce the $(H_D^1(\Omega))^*$ valued stochastic processes

$$\begin{aligned}
\mathcal{I}_n(\omega, t) &:= \left(v^n(t) - v_0^n\right) + \varepsilon_n \left(u_i^n(t) - u_{i,0}^n\right) \\
&\quad - \int_0^t \Pi_n \left[\nabla \cdot \left(M_i \nabla u_i^n\right) - I(v^n, w^n)\right] ds - \int_0^t \eta^n(v^n)\, dW^{v,n}(s), \\
\tilde{\mathcal{I}}_n(\omega, t) &:= \left(\tilde{v}^n(t) - \tilde{v}_0^n\right) + \varepsilon_n \left(\tilde{u}_i^n(t) - \tilde{u}_{i,0}^n\right), \\
&\quad - \int_0^t \Pi_n \left[\nabla \cdot \left(M_i \nabla \tilde{u}_i^n\right) - I(\tilde{v}^n, \tilde{w}^n)\right] ds - \int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,(n)}(s),
\end{aligned}$$

and the real-valued random variables, cf. (4.2), $I_n(\omega) := \|\mathcal{I}_n\|^2_{L^2((0,T);(H_D^1(\Omega))^*)}$ and $\tilde{I}_n(\omega) := \left\|\tilde{\mathcal{I}}_n\right\|^2_{L^2((0,T);(H_D^1(\Omega))^*)}$. Note that $I_n = 0$ P-a.s. and so $\mathbb{E}[I_n] = 0$.

If we could write $I_n = L_n(\Phi_n)$ for a (deterministic) bounded continuous functional $L_n(\cdot)$ on $\mathcal{X}$, cf. (6.47), then by equality of the laws, also $\tilde{\mathbb{E}}[\tilde{I}_n] = 0$ and the result would follow. However, this is not immediately achievable since the stochastic integral is not a deterministic function of $W^{v,n}$. Hence, certain modifications are needed to produce a workable proof [3]. First of all, we do not consider I_n but rather the bounded map $I_n/(1 + I_n)$. Noting that $\mathbb{E}[I_n] = 0$ implies

$$\mathbb{E}\left[\frac{I_n}{1 + I_n}\right] = 0, \tag{6.56}$$

the goal is to show that

$$\tilde{\mathbb{E}}\left[\frac{\tilde{I}_n}{1 + \tilde{I}_n}\right] = 0, \tag{6.57}$$

from which the first equation in (6.54) follows.

Recall that, cf. (5.10), $\int_0^t \eta^n(v^n)\, dW^{v,n}(s) = \sum_{k=1}^n \int_0^t \eta_k^n(v^n)\, dW_k^v(s)$. Let $\varrho_\nu(t)$ be a standard mollifier and define (for $k = 1, \dots, n$)

$$\eta_k^{n,\nu} := (\eta_k^n(v^n)) \underset{(t)}{\star} \varrho_\nu, \qquad \nu > 0.$$

By properties of mollifiers,

$$\left\|\eta_k^{n,\nu}\right\|_{L^2\left(D,\mathcal{F},P;L^2((0,T);L^2(\Omega))\right)} \le \left\|\eta_k^n(v^n)\right\|_{L^2\left(D,\mathcal{F},P;L^2((0,T);L^2(\Omega))\right)}$$

and

$$\eta_k^{n,\nu} \to \eta_k^n \quad \text{in } L^2\left(D, \mathcal{F}, P; L^2((0,T); L^2(\Omega))\right) \text{ as } \nu \to 0. \tag{6.58}$$

We define $\tilde{\eta}_k^{n,\nu}$ similarly (with v^n replaced by $\tilde{v}^n$).

An "integration by parts" reveals that

$$\int_0^t \eta_k^{n,\nu}\, dW_k^\nu(s) = \left(\eta_k^{n,\nu}\right)(t)\, W_k^\nu(t) - \int_0^t W_k^\nu(s) \frac{\partial}{\partial s}\left(\eta_k^{n,\nu}\right) ds,$$

i.e., thanks to the regularization of $\eta_k^n(v^n)$ in the t variable, $\int_0^t \eta_k^{n,\nu}\, dW_k^\nu(s)$ can be viewed as a (deterministic) functional of W_k^ν.

Denote by I_n^ν, $\tilde{I}_n^\nu$ the random variables corresponding to I_n, $\tilde{I}_n$ with $\eta_k^n(v^n)$, $\eta_k^n(\tilde{v}^n)$ replaced by $\eta_k^{n,\nu}$, $\tilde{\eta}_k^{n,\nu}$, respectively, and note $\frac{I_n^\nu}{1+I_n^\nu} = L_{n,\nu}(\Phi_n)$, $\frac{\tilde{I}_n^\nu}{1+\tilde{I}_n^\nu} = L_{n,\nu}(\tilde{\Phi}_n)$, for some bounded continuous functional $L_{n,\nu}(\cdot)$ on $\mathcal{X}$. By equality of the laws,

$$\tilde{\mathbb{E}}\left[\frac{\tilde{I}_n^\nu}{1+\tilde{I}_n^\nu}\right] = \int_{\mathcal{X}} L_{n,\nu}(\Phi)\, d\tilde{\mathcal{L}}_n(\Phi) = \int_{\mathcal{X}} L_{n,\nu}(\Phi)\, d\mathcal{L}_n(\Phi) = \mathbb{E}\left[\frac{I_n^\nu}{1+I_n^\nu}\right]. \tag{6.59}$$

One can check that

$$\begin{aligned} \mathbb{E}\left[\left|\frac{I_n}{1+I_n} - \frac{I_n^\nu}{1+I_n^\nu}\right|\right] &\le \mathbb{E}\left[\left|I_n - I_n^\nu\right|\right] \\ &\le C\left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \left\|\eta_k^n(v^n) - \eta_k^{n,\nu}\right\|_{L^2(\Omega)}^2 dt\right]\right)^{\frac{1}{2}} \overset{(6.58)}{\longrightarrow} 0 \quad \text{as } \nu \to 0, \end{aligned} \tag{6.60}$$

and similarly

$$\tilde{\mathbb{E}}\left[\left|\frac{\tilde{I}_n}{1+\tilde{I}_n} - \frac{\tilde{I}_n^\nu}{1+\tilde{I}_n^\nu}\right|\right] \le C\left(\mathbb{E}\left[\int_0^T \sum_{k=1}^n \left\|\eta_k^n(\tilde{v}^n) - \tilde{\eta}_k^{n,\nu}\right\|_{L^2(\Omega)}^2 dt\right]\right)^{\frac{1}{2}} \overset{\nu\downarrow 0}{\to} 0. \tag{6.61}$$

Combining (6.59), (6.60), (6.61), (6.56) we arrive at (6.57).

Finally, let us prove (6.55). By construction, $v^n = u_i^n - u_e^n$ and so

$$\left\|v^n - (u_i^n - u_e^n)\right\|_{L^2\left(D,\mathcal{F},P;L^2((0,T);L^2(\Omega))\right)} = 0.$$

For $\Phi \in \mathcal{X}$, define $L(\Phi) = \frac{\|v-(u_i-u_e)\|^2_{L^2((0,T);L^2(\Omega))}}{1+\|v-(u_i-u_e)\|^2_{L^2((0,T);L^2(\Omega))}}$. Since $L(\cdot)$ is a bounded continuous functional on $\mathcal{X}$ and the laws $\mathcal{L}_n$, $\tilde{\mathcal{L}}_n$ are equal,

$$\tilde{\mathbb{E}}\left[L(\tilde{\Phi}_n)\right] = \mathbb{E}\left[L(\Phi_n)\right] \le \left\|v^n - (u_i^n - u_e^n)\right\|^2_{L^2\left(D,\mathcal{F},P;L^2((0,T);L^2(\Omega))\right)} = 0,$$

i.e., $L(\tilde{\Phi}_n) = 0$ $\tilde{P}$-a.s. and thus, via (6.49),

$$\left\|\tilde{v}^n - (\tilde{u}_i^n - \tilde{u}_e^n)\right\|_{L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P};L^2((0,T);L^2(\Omega))\right)} = 0.$$

This concludes the proof of the lemma. □

6.4. Passing to the limit

We begin by turning the probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$, cf. (6.47) and (6.48), into a stochastic basis,

$$\tilde{\mathcal{S}} = \left(\tilde{D}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t\right\}_{t\in[0,T]}, \tilde{P}, \tilde{W}^v, \tilde{W}^w\right), \tag{6.62}$$

by supplying the natural filtration $\left\{\tilde{\mathcal{F}}_t\right\}_{t\in[0,T]}$, i.e., the smallest filtration with respect to which all the relevant processes are adapted, viz.

$$\tilde{\mathcal{F}}_t = \sigma\left(\sigma\left(\tilde{\Phi}\big|_{[0,t]}\right)\bigcup\left\{N\in\tilde{\mathcal{F}}:\tilde{P}(N)=0\right\}\right),\quad t\in[0,T]. \tag{6.63}$$

Lemma 6.8 shows that $\tilde{U}^n, \tilde{W}^n, \tilde{U}_0^n$ satisfy the Faedo–Galerkin equations (5.17); hence, they are worthy of being referred to as “approximations”. The next two lemmas summarize the relevant convergence properties satisfied by these approximations.

Lemma 6.9 (*Weak Convergence*)**.** *There exist functions $\tilde{u}_i, \tilde{u}_e, \tilde{v}, \tilde{w}$, with*

$$\tilde{u}_i, \tilde{u}_e, \tilde{v} \in L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); H_D^1(\Omega))\right),\ \tilde{v} = \tilde{u}_i - \tilde{u}_e,$$
$$\tilde{v},\ \tilde{w} \in L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^\infty((0,T); L^2(\Omega))\right),\ \tilde{v} \in L^4\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^4(\Omega_T)\right),$$

such that as $n\to\infty$, passing to a subsequence if necessary,

$$\begin{cases} \tilde{u}_i^n \rightharpoonup \tilde{u}_i,\ \tilde{u}_e^n \rightharpoonup \tilde{u}_e \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); H_D^1(\Omega))\right), \\ \varepsilon_n \tilde{u}_i^n \to 0,\ \varepsilon_n \tilde{u}_e^n \to 0 \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); L^2(\Omega))\right), \\ \tilde{v}^n \rightharpoonup \tilde{v} \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); H_D^1(\Omega))\right), \\ \tilde{v}^n \overset{\star}{\rightharpoonup} \tilde{v} \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^\infty((0,T); L^2(\Omega))\right), \\ \tilde{v}^n \rightharpoonup \tilde{v} \quad \text{in } L^4\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^4(\Omega_T)\right), \\ \tilde{w}^n \overset{\star}{\rightharpoonup} \tilde{w} \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^\infty((0,T); L^2(\Omega))\right). \end{cases} \tag{6.64}$$

Proof. The claims in (6.64) follow from the estimates in (6.20) and the sequential Banach–Alaoglu theorem. The relation $\tilde{v}_i = \tilde{u}_i - \tilde{u}_e$, $d\tilde{P}\times dt\times dx$ a.e. in $\tilde{D}\times(0,T)\times\Omega$, is a consequence of (6.55) and the weak convergences in $L^2_{\omega,t,x}$ of $\tilde{v}^n, \tilde{u}_i^n, \tilde{u}_e^n$. The limit functions $\tilde{u}_i, \tilde{u}_e, \tilde{v}, \tilde{w}$ are easily identified with the a.s. representations in Lemma 6.5. □

As a result of (6.50), we can upgrade a.s. to L^2 convergence.

Lemma 6.10 (*Strong Convergence*)**.** *As $n\to\infty$, passing to a subsequence if necessary, the following strong convergences hold:*

$$\begin{aligned} &\tilde{v}^n \to \tilde{v} \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); L^2(\Omega))\right), \\ &\tilde{w}^n \to \tilde{w} \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); (H_D^1(\Omega))^*)\right), \\ &\tilde{W}^{v,n} \to \tilde{W}^v,\ \tilde{W}^{w,n} \to \tilde{W}^w \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; C([0,T]; \mathbb{U}_0)\right). \\ &\tilde{u}_{i,0}^n \to \tilde{u}_{i,0},\ \tilde{u}_{e,0}^n \to \tilde{u}_{e,0},\ \tilde{v}_0^n \to \tilde{v}_0,\ \tilde{w}_0^n \to \tilde{w}_0 \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2(\Omega)\right). \end{aligned} \tag{6.65}$$

Proof. The proof merges the a.s. convergences in (6.45), the high-order moment estimates in (6.50), and Vitali’s convergence theorem. To justify the first claim in (6.65), for example, we consider the estimate $\tilde{\mathbb{E}}\left[\|\tilde{v}^n(t)\|^{q_0}_{L^\infty((0,T);L^2(\Omega))}\right] \le C$ with $q_0 > 2$, see (6.50). From this we infer the equi-integrability (w.r.t. $\tilde{P}$) of $\left\{\|\tilde{v}^n(t)\|^2_{L^2((0,T);L^2(\Omega))}\right\}_{n\ge 1}$. Accordingly, the first claim in (6.65) follows from the $\tilde{P}$-a.s. convergence in (6.45) and Vitali's convergence theorem, with the remaining claims following along similar lines. Regarding the third claim, note also that for $\tilde{W}^n = \tilde{W}^{v,n}$ or $\tilde{W}^{w,n}$,

$$\tilde{\mathbb{E}}\left[\left\|\tilde{W}^n\right\|^q_{C([0,T];\mathbb{U}_0)}\right] = \mathbb{E}\left[\|W^n\|^q_{C([0,T];\mathbb{U}_0)}\right] \le C_T, \qquad \forall q \in [1,\infty), \tag{6.66}$$

which follows from equality of the laws and a martingale inequality. □

For each $n \ge 1$, $\tilde{W}^{v,n}$ and $\tilde{W}^{w,n}$ are (independent) cylindrical Wiener processes with respect to the stochastic basis $\tilde{\mathcal{S}}_n$, see (6.52). Since $\tilde{W}^{v,n} \to \tilde{W}^v$, $\tilde{W}^{w,n} \to \tilde{W}^w$ in the sense of (6.45) or (6.65), it is more or less obvious that also the limit processes $\tilde{W}^v$, $\tilde{W}^w$ are cylindrical Wiener processes. Indeed, we have

Lemma 6.11. *The a.s. representations $\tilde{W} = \tilde{W}^v, \tilde{W}^w$ from Lemma 6.5 are (independent) cylindrical Wiener processes with respect to sequences $\left\{\tilde{W}^v_k\right\}_{k\ge1}$, $\left\{\tilde{W}^w_k\right\}_{k\ge1}$ of mutually independent real-valued Wiener processes adapted to the natural filtration $\left\{\tilde{\mathcal{F}}_t\right\}_{t\in[0,T]}$, cf. (6.62) and (6.63), such that $\tilde{W}^v = \sum_{k\ge1}\tilde{W}^v_k\psi_k$, $\tilde{W}^v = \sum_{k\ge1}\tilde{W}^w_k\psi_k$.*

Proof. The proof is standard, see e.g. [40, Lemma 9.9] or [18, Proposition 4.8]. To be more precise, by the martingale characterization theorem [16, Theorem 4.6], we must show that $\tilde{W}^v, W^w$ are $\{\tilde{\mathcal{F}}_t\}$-martingales. With $\tilde{\Phi}$ defined in (6.47), it is sufficient to show that

$$\tilde{\mathbb{E}}\left[L_s(\tilde{\Phi})\left(\tilde{W}(t) - \tilde{W}(s)\right)\right] = 0, \qquad \tilde{W} = \tilde{W}^v, \tilde{W}^w,$$

for all bounded continuous functionals $L_s(\Phi)$ on $\mathcal{X}$ depending only on the values of Φ restricted to $[0,s]$. Since the laws of Φ_n and $\tilde{\Phi}_n$ coincide, cf. (6.47),

$$\tilde{\mathbb{E}}\left[L_s(\tilde{\Phi}_n)\left(\tilde{W}^n(t) - \tilde{W}^n(s)\right)\right] = \mathbb{E}\left[L_s(\Phi_n)\left(W^n(t) - W^n(s)\right)\right] = 0, \tag{6.67}$$

where the last equality is a result of the $\{\mathcal{F}^n_t\}$-martingale property of $W^n = W^{v,n}, W^{w,n}$. By (6.45), (6.66), and Vitali's convergence theorem, we can pass to the limit in (6.67) as $n \to \infty$. This concludes the proof of the lemma. □

Given the above convergences, the final step is to pass to the limit in the Faedo–Galerkin equations. The next lemma shows that the Skorokhod–Jakubowski representations satisfy the weak form (4.3) of the stochastic bidomain system.

Lemma 6.12 (*Limit Equations*)**.** *Let $\tilde{U}, \tilde{W}, \tilde{v}_0, \tilde{w}_0$ be the a.s. representations constructed in Lemma 6.5, and $\tilde{\mathcal{S}}$ the accompanying stochastic basis defined in (6.62), (6.63), so that*

$\tilde v, \tilde w, \tilde W^v, \tilde W^w$ become $\{\tilde{\mathcal F}_t\}$-adapted processes. Then the following equations hold $\tilde P$-a.s., for a.e. $t \in [0,T]$:

$$\begin{aligned}
&\int_\Omega \tilde v(t)\varphi_i\,dx + \int_0^t\int_\Omega \Big(M_i\nabla\tilde u_i\cdot\nabla\varphi_i + I(\tilde v,\tilde w)\varphi_i\Big)\,dx\,ds \\
&\qquad = \int_\Omega \tilde v_0\,\varphi_i\,dx + \int_0^t\int_\Omega \eta(\tilde v)\varphi_i\,dx\,d\tilde W^v(s), \\
&\int_\Omega \tilde v(t)\varphi_e\,dx + \int_0^t\int_\Omega \Big(-M_e\nabla\tilde u_e\cdot\nabla\varphi_e + I(\tilde v,\tilde w)\varphi_e\Big)\,dx\,ds \\
&\qquad = \int_\Omega \tilde v_0\varphi_e\,dx + \int_0^t\int_\Omega \eta(\tilde v)\varphi_e\,dx\,d\tilde W^v(s), \\
&\int_\Omega \tilde w(t)\varphi\,dx = \int_\Omega \tilde w_0\varphi\,dx + \int_0^t\int_\Omega H(\tilde v,\tilde w)\varphi\,dx\,ds \\
&\qquad\qquad + \int_0^t\int_\Omega \sigma(\tilde v)\varphi\,dx\,d\tilde W^w(s),
\end{aligned} \tag{6.68}$$

for all $\varphi_i,\varphi_e \in H^1_D(\Omega)$ and $\varphi \in L^2(\Omega)$. The laws of $\tilde v(0) = \tilde v_0$ and $\tilde w(0) = \tilde w_0$ are μ_{v_0} and μ_{w_0}, respectively.

Proof. We establish the first equation in (6.68). The remaining equations are treated in the same way. Let $Z \subset \tilde D \times [0,T]$ be a measurable set, and denote by

$$\mathbf 1_Z(\omega,t) \in L^\infty\Big(\tilde D\times[0,T];\tilde d P\times dt\Big) \tag{6.69}$$

the characteristic function of Z. Our aim is to show

$$\begin{aligned}
&\mathbb E\left[\int_0^T \mathbf 1_Z(\omega,t)\left(\int_\Omega \tilde v(t)\varphi_i\,dx\right)dt\right] \\
&\quad + \mathbb E\left[\int_0^T \mathbf 1_Z(\omega,t)\left(\int_0^t\int_\Omega M_i\nabla\tilde u_i\cdot\nabla\varphi_i\,dx\,ds\right)dt\right] \\
&\quad + \mathbb E\left[\int_0^T \mathbf 1_Z(\omega,t)\left(\int_0^t\int_\Omega I(\tilde v,\tilde w)\varphi_i\,dx\,ds\right)dt\right] \\
&\quad = \mathbb E\left[\int_0^T \mathbf 1_Z(\omega,t)\left(\int_\Omega \tilde v_0\varphi_i\,dx\right)dt\right] \\
&\qquad + \mathbb E\left[\int_0^T \mathbf 1_Z(\omega,t)\left(\int_0^t\int_\Omega \eta(\tilde v)\varphi_i\,dx\,d\tilde W^v(s)\right)dt\right].
\end{aligned} \tag{6.70}$$

Then, since Z is an arbitrary measurable set and the simple functions are dense in L^2, we conclude that the first equation in (6.68) holds for $d\tilde P\times dt$ almost every $(\omega,t)\in\tilde D\times[0,T]$ and any $\varphi_i \in H^1_D(\Omega)$.

Fix $\varphi_i \in H^1_D(\Omega)$, and note that (6.54) implies

$$\begin{aligned}&\int_\Omega \tilde{v}^n(t)\varphi_i\,dx + \int_\Omega \varepsilon_n \tilde{u}_i^n(t)\varphi_i\,dx \\ &\quad + \int_0^t\int_\Omega M_i\nabla\tilde{u}_i^n\cdot\nabla\Pi_n\varphi_i\,dx\,ds + \int_0^t\int_\Omega I(\tilde{v}^n,\tilde{w}^n)\Pi_n\varphi_i\,dx\,ds \\ &\qquad = \int_\Omega \tilde{v}_0^n\,\varphi_i\,dx + \int_\Omega \varepsilon_n\tilde{u}_{i,0}^n\,\varphi_i\,dx + \int_0^t\int_\Omega \eta^n(\tilde{v}^n)\varphi_i\,dx\,d\tilde{W}^{v,(n)}(s),\end{aligned} \tag{6.71}$$

using (5.15). We multiply (6.71) by $\mathbf{1}_Z(\omega,t)$, cf. (6.69), integrate over (ω,t), and then attempt to pass to the limit $n\to\infty$ in each term separately.

We will make repeated use of the following simple fact: If $X_n \rightharpoonup X$ in $L^p(\tilde{D}\times(0,T))$, for $p\in[1,\infty)$, then $\int_0^t X_n\,ds \rightharpoonup \int_0^t X\,ds$ in $L^p(\tilde{D}\times(0,T))$ as well.

First, since

$$\mathbf{1}_Z(\omega,t)\varphi_i(x) \in L^2\left(\tilde{D},\tilde{\mathcal{F}},\tilde{P}; L^2((0,T);L^2(\Omega))\right), \tag{6.72}$$

the weak convergence in $L^2_{\omega,t,x}$ of $\tilde{v}^n$, cf. (6.64), implies

$$\tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_\Omega \tilde{v}^n(t)\varphi_i\,dx\right)dt\right] \to \tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_\Omega \tilde{v}(t)\varphi_i\,dx\right)dt\right],$$

as $n\to\infty$. Similarly,

$$\tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_\Omega \varepsilon_n u_i^n\varphi_i\,dx\right)dt\right] \to 0, \quad \text{as } n\to\infty.$$

The initial data terms on the right-hand side of (6.71) can be treated in the same way, using (6.65). Recall also that the laws of $\tilde{v}_0^n$, $\tilde{w}_0^n$ coincide with the laws of $\Pi_n v_0$, $\Pi_n w_0$, respectively, and that $v_0 \sim \mu_{v_0}$, $w_0\sim\mu_{w_0}$. Since $\Pi_n v_0 \to v_0$, $\Pi_n w_0\to w_0$ in $L^2_{\omega,x}$ as $n\to\infty$, cf. (5.12) or (5.16), we conclude that $\tilde{v}(0)=\tilde{v}_0\sim\mu_{v_0}$, $\tilde{w}(0)=\tilde{w}_0\sim\mu_{w_0}$.

Next, note that $\nabla\Pi_n\varphi_i \to \nabla\varphi_i$ in $L^2(\Omega)$ as $n\to\infty$, cf. (5.16). By weak convergence in $L^2_{\omega,t,x}$ of $\tilde{\nabla} u_i^n$, cf. (6.64), and (6.72), it follows that

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t\int_\Omega M_i\nabla\tilde{u}_i^n\cdot\nabla\Pi_n\varphi_i\,dx\,ds\right)dt\right] \\ &\quad \to \tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t\int_\Omega M_i\nabla\tilde{u}_i\cdot\nabla\varphi_i\,dx\,ds\right)dt\right] \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

To demonstrate convergence of the stochastic integral

$$\int_0^t\int_\Omega \eta^n(\tilde{v}^n)\varphi_i\,dx\,d\tilde{W}^{v,(n)}(s) = \int_\Omega\left(\int_0^t \eta^n(\tilde{v}^n)\,d\tilde{W}^{v,(n)}(s)\right)\varphi_i\,dx,$$

we will use [17, Lemma 2.1] to infer that

$$\int_0^t \eta^n(\tilde{v}^n)\,d\tilde{W}^{v,(n)}(s) \to \int_0^t \eta(\tilde{v})\,d\tilde{W}^v(s) \quad \text{in } L^2((0,T);L^2(\Omega)), \tag{6.73}$$

in probability, as $n\to\infty$. Since $\tilde{W}^{v,(n)} \to \tilde{W}^v$ in $C([0,T];\mathbb{U}_0)$, $\tilde{P}$-a.s. (and thus in probability), cf. (6.45), it remains to prove that

$$\eta^n(\tilde{v}^n)\to\eta(\tilde{v}) \quad \text{in } L^2\left((0,T);L_2(\mathbb{U};L^2(\Omega))\right),\ \tilde{P}\text{-almost surely.} \tag{6.74}$$

Before we continue, recall that $\int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,(n)}$ equals $\sum_{k=1}^n \int_0^t \eta_k^n(\tilde{v}^n)\, d\tilde{W}_k^{v,n}$, where $\eta_k^n(\tilde{v}^n) = \eta^n(\tilde{v}^n)\psi_k \in L^2(\Omega)$, $\{\psi_k\}_{k\geq 1}$ is an orthonormal basis of $\mathbb{U}$, $\eta_k^n(\tilde{v}^n)$ equals $\sum_{l=1}^n \eta_{k,l}(\tilde{v}^n) e_l$ with $\eta_{k,l}(\tilde{v}^n) = (\eta_k(\tilde{v}^n), e_l)_{L^2(\Omega)}$ and $\{e_l\}_{l=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$. We have a similar decomposition of $\eta(\tilde{v})$. Note that

$$\begin{aligned}
&\int_0^t \left\| \eta(\tilde{v}) - \eta^n(\tilde{v}^n) \right\|_{L_2(\mathbb{U};L^2(\Omega))}^2 ds \\
&\quad \leq \int_0^t \left\| \eta(\tilde{v}) - \eta(\tilde{v}^n) \right\|_{L_2(\mathbb{U};L^2(\Omega))}^2 ds + \int_0^t \left\| \eta(\tilde{v}) - \eta^n(\tilde{v}) \right\|_{L_2(\mathbb{U};L^2(\Omega))}^2 ds \\
&\quad =: J_1 + J_2.
\end{aligned} \tag{6.75}$$

Exploiting (3.7) and (6.45), we conclude easily that

$$J_1 \to 0, \quad \tilde{P}\text{-almost surely,} \tag{6.76}$$

as $n \to \infty$. We handle the J_2-term as follows:

$$\begin{aligned}
J_2 &= \int_0^t \sum_{k\geq 1} \left\| \eta_k(\tilde{v}) - \eta_k^n(\tilde{v}) \right\|_{L^2(\Omega)}^2 ds \\
&= \int_0^t \sum_{k\geq 1} \left\| \sum_{l\geq 1} \eta_{k,l}(\tilde{v}) e_l - \sum_{l=1}^n \eta_{k,l}(\tilde{v}) e_l \right\|_{L^2(\Omega)}^2 ds \\
&= \int_0^t \sum_{k\geq 1} \left\| \eta_k(\tilde{v}) - \Pi_n\left(\eta_k(\tilde{v})\right) \right\|_{L^2(\Omega)}^2 ds.
\end{aligned}$$

Observe that the integrand can be dominated by an $L^1(0,T)$ function, $\tilde{P}$-a.s.:

$$\begin{aligned}
&\sum_{k\geq 1} \left\| \eta_k(\tilde{v}(t)) - \Pi_n\left(\eta_k(\tilde{v}(t))\right) \right\|_{L^2(\Omega)}^2 \\
&\quad \overset{(5.14)}{\leq} 4 \sum_{k\geq 1} \left\| \eta_k(\tilde{v}(t)) \right\|_{L^2(\Omega)}^2 = 4 \left\| \eta(\tilde{v}(t)) \right\|_{L_2(\mathbb{U};L^2(\Omega))}^2 \overset{(3.7)}{\leq} C\left(1 + \|\tilde{v}(t)\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

where we recall that $\tilde{v} \in L^2_\omega\left(L^\infty_t\left(L^2_x\right)\right)$ and thus $\tilde{v} \in L^2_t\left(L^2_x\right)$ $\tilde{P}$-a.s. (cf. Lemma 6.9). Clearly, by (5.16), $\Pi_n\left(\eta_k(\tilde{v})\right)$ converges as $n \to \infty$ to $\eta_k(\tilde{v})$ in $L^2(\Omega)$, for a.e. t, $\tilde{P}$-almost surely. Therefore, after an application of Lebesgue's dominated convergence theorem,

$$J_2 \overset{n\uparrow\infty}{\to} 0, \quad \tilde{P}\text{-almost surely.} \tag{6.77}$$

Combining (6.76), (6.75), (6.77) we arrive at (6.74) ($\Longrightarrow$ (6.73) via [17, Lemma 2.1]).

Passing to a subsequence (not relabeled), we can replace "in probability" by "$\tilde{P}$-almost surely" in (6.73). Next, fixing any $q > 2$, we verify that

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left\| \int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,(n)} \right\|_{L^2((0,T);L^2(\Omega))}^q \right] \\
&\quad \leq C_T\, \tilde{\mathbb{E}}\left[\left(\int_0^T \sum_{k=1}^n \left\| \eta_k(\tilde{v}^n) \right\|_{L^2(\Omega)}^2 dt \right)^{\frac{q}{2}} \right] \leq C_{\eta,T},
\end{aligned}$$

using the Burkholder–Davis–Gundy inequality (3.4) and (3.6), (6.49). Accordingly, in light of Vitali's theorem, (6.73) implies

$$\int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,n}(s) \overset{n\uparrow\infty}{\to} \int_0^t \eta(\tilde{v})\, d\tilde{W}^v(s) \quad \text{in } L^2\left(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P}; L^2((0,T); L^2(\Omega))\right),$$

and hence

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t \int_\Omega \eta^n(\tilde{v}^n)\varphi_i\, dx\, d\tilde{W}^{v,n}(s)\right) dt\right] \\
&\quad = \tilde{\mathbb{E}}\left[\int_0^T \int_\Omega \left(\int_0^t \eta^n(\tilde{v}^n)\, d\tilde{W}^{v,n}(s)\right) (\mathbf{1}_Z(\omega,t)\Pi_n\varphi_i(x))\, dx\, dt\right] \\
&\quad \to \tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t \int_\Omega \eta(\tilde{v})\varphi_i\, dx\, d\tilde{W}^v(s)\right) dt\right] \quad \text{as } n \to \infty.
\end{aligned}$$

With reference to the nonlinear term in (6.71), according to condition (**GFHN**), we have $I(v,w) = I_1(v) + I_2(v)w$ with $|I_1(v)| \le c_{I,1}\left(1 + |v|^3\right)$ and $I_2(v) = c_{I,3} + c_{I,4}v$. By the first part of (6.65), passing to a subsequence if necessary, we may assume that as $n \to \infty$,

$$\tilde{v}^n \to \tilde{v} \quad \text{for } d\tilde{P} \times dt \times dx \text{ almost every } (\omega,t,x) \in \tilde{D} \times [0,T] \times \Omega.$$

As a result of this, the boundedness of $\tilde{v}^n$ in $L^4_{\omega,t,x}$, cf. (6.49), and Vitali's convergence theorem, we conclude that as $n \to \infty$,

$$\begin{aligned}
&\tilde{v}^n \to \tilde{v} \quad \text{in } L^q(d\tilde{P} \times dt \times dx), \quad \text{for any } q \in [1,4), \\
&I_1(\tilde{v}^n) \to I_1(\tilde{v}) \quad \text{in } L^q(d\tilde{P} \times dt \times dx), \quad \text{for any } q \in [1,4/3).
\end{aligned} \tag{6.78}$$

Fix two numbers q, q' such that $\frac{3}{2} \le q < 2$, $2 < q' \le 3$, $\frac{1}{q} + \frac{1}{q'} = 1$, for example $q = 3/2$ and $q' = 3$. Then, by Hölder's inequality,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\int_0^T \int_\Omega \left|I_2(\tilde{v}^n)\Pi_n\varphi_i - I_2(\tilde{v})\varphi_i\right|^2 dx\, dt\right] \\
&\qquad \le \tilde{\mathbb{E}}\left[\int_0^T \int_\Omega \left|I_2(\tilde{v}^n)\right|^2 |\Pi_n\varphi_i - \varphi_i|^2\, dx\, dt\right] \\
&\qquad\qquad + \tilde{\mathbb{E}}\left[\int_0^T \int_\Omega \left|I_2(\tilde{v}^n) - I_2(\tilde{v})\right|^2 |\varphi_i|^2\, dx\, dt\right] \\
&\qquad \le \left\|I_2(\tilde{v}^n)\right\|^2_{L^{2q}_{\omega,t,x}} \|\Pi_n\varphi_i - \varphi_i\|^2_{L^{2q'}_{\omega,t,x}} + \left\|I_2(\tilde{v}^n) - I_2(\tilde{v})\right\|^2_{L^{2q}_{\omega,t,x}} \|\varphi_i\|^2_{L^{2q'}_{\omega,t,x}} \overset{n\uparrow\infty}{\longrightarrow} 0,
\end{aligned}$$

since $I_2(\tilde{v}^n)$ is bounded and converges strongly in $L^{2q}_{\omega,t,x}$ (with $2q < 4$), consult (6.78). Consequently, $I_2(\tilde{v}^n)\Pi_n\varphi_i \to I_2(\tilde{v})\varphi_i$ in $L^2(\tilde{d}P \times dt \times dx)$. Besides, (6.64) implies $\tilde{w}^n \rightharpoonup \tilde{w}$ in $L^2(\tilde{d}P \times dt \times dx)$. Hence,

$$I_2(\tilde{v}^n)\, \tilde{w}^n\, \Pi_n\varphi_i \overset{n\uparrow\infty}{\rightharpoonup} I_2(\tilde{v})\, \varphi_i\, \tilde{w} \quad \text{in } L^1(\tilde{d}P \times dt \times dx). \tag{6.79}$$

Regarding the I_1 term, fix two numbers q, q' such that $\frac{6}{5} \le q < \frac{4}{3}$, $3 < q' \le 6$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then, similar to the treatment of the I_1 term,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\int_0^T \int_\Omega \left|I_1(\tilde{v}^n)\Pi_n\varphi_i - I_1(\tilde{v})\varphi_i\right| dx\, dt\right] \\
&\quad \le \left\|I_1(\tilde{v}^n)\right\|_{L^q_{\omega,t,x}} \|\Pi_n\varphi_i - \varphi_i\|_{L^{q'}_{\omega,t,x}} + \left\|I_1(\tilde{v}^n) - I_1(\tilde{v})\right\|_{L^q_{\omega,t,x}} \|\varphi_i\|_{L^{q'}_{\omega,t,x}} \overset{n\uparrow\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have used that $I_2(\tilde{v}^n)$ is bounded and converges strongly in $L^q_{\omega,t,x}$ ($q < 4/3$), see (6.78), and the Sobolev embedding theorem to control the $L^{q'}$ norm of φ_i, $\Pi_n\varphi_i - \varphi_i$ in terms of the H^1_D norm ($q' \le 6$). In other words,

$$I_1(\tilde{v}^n)\Pi_n\varphi_i \to I_1(\tilde{v})\varphi_i \quad \text{in } L^1\left(d\tilde{P} \times dt \times dx\right) \text{ as } n \to \infty.$$

Combining this and (6.79), recalling $I(\tilde{v}^n, \tilde{w}^n) = I_1(\tilde{v}^n) + I_2(\tilde{v}^n)\tilde{w}^n$, we arrive finally at

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t\int_\Omega I(\tilde{v}^n,\tilde{w}^n)\Pi_n\varphi_i\,ds\right)dt\right]\\&\quad\longrightarrow \tilde{\mathbb{E}}\left[\int_0^T \mathbf{1}_Z(\omega,t)\left(\int_0^t\int_\Omega I(\tilde{v},\tilde{w})\varphi_i\,ds\right)dt\right] \quad \text{as } n \to \infty.\end{aligned}$$

This concludes the proof of (6.70) and thus the lemma. □

6.5. *Concluding the proof of Theorem 4.4*

As stated in Lemma 6.12, the Skorokhod–Jakubowski representations $\tilde{U}, \tilde{W}, \tilde{v}_0, \tilde{w}_0$ satisfy the weak form (6.68) for a.e. $t \in [0,T]$. Regarding the stochastic integrals in (6.68), the $(H^1_D(\Omega))^*$ valued processes $\tilde{v}(t), \tilde{w}(t)$ are (by construction) $\tilde{\mathcal{F}}_t$-measurable for each t. To upgrade (6.68) to hold for "every t", we will now prove that (cf. also Remark 4.3)

$$\tilde{v}(\omega), \tilde{w}(\omega) \in C([0,T];(H^1_D(\Omega))^*), \quad \text{for } \tilde{P}\text{-a.e. } \omega \in \tilde{D}. \tag{6.80}$$

This weak continuity property also ensures that $\tilde{v}$, $\tilde{w}$ are predictable in $(H^1_D(\Omega))^*$. Hence, conditions (5) and (7) in Definition 4.1 hold. Conditions (1) and (2) are covered by Lemma 6.11, while Lemma 6.9 validates conditions (3) and (4). Lemma 6.12 implies (6).

To conclude the proof of Theorem 4.4, it remains to verify (6.80), which we do for $\tilde{v}$ (the case of $\tilde{w}$ is easier). Fix $\varphi \in H^1_D(\Omega) \subset L^6(\Omega)$, and consider the stochastic process

$$\Psi_\varphi : \tilde{D} \times [0,T] \to \mathbb{R}, \qquad \Psi_\varphi(\omega,t) := \int_\Omega \tilde{v}(\omega,t)\varphi\,dx,$$

relative to $\tilde{\mathcal{S}}$, cf. (6.62) and (6.63). To arrive at (6.80) it will be sufficient to prove that $\Psi_\varphi \in C([0,T])$ $\tilde{P}$-a.s., for any φ in a countable dense subset $\{\varphi_\ell\}_{\ell=1}^\infty \subset H^1_D(\Omega)$. In what follows, let φ denote an arbitrary function from $\{\varphi_\ell\}_{\ell=1}^\infty$.

We are going to use the $L^{q_0}_\omega$ estimates in Corollary 6.2, with $q_0 > \frac{9}{2}$. Fix $t \in [0,T]$, $\vartheta > 0$ (the case $\vartheta < 0$ is treated similarly), and $q \in \left(3, \frac{2}{3}q_0\right]$. Then, using e.g. the first equation in (6.68),

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\left|\Psi_\varphi(t+\vartheta) - \Psi_\varphi(t)\right|^q\right]\\&\le \tilde{\mathbb{E}}\left[\left|\int_t^{t+\vartheta}\int_\Omega M_i\nabla\tilde{u}_i\cdot\nabla\varphi\,dx\,ds\right|^q\right] + \tilde{\mathbb{E}}\left[\left|\int_t^{t+\vartheta}\int_\Omega I(\tilde{v},\tilde{w})\varphi\,dx\,ds\right|^q\right]\\&\quad + \tilde{\mathbb{E}}\left[\left|\int_t^{t+\vartheta}\int_\Omega \eta(\tilde{v})\varphi\,dx\,d\tilde{W}^v(s)\right|^q\right] =: \Gamma_1 + \Gamma_2 + \Gamma_3.\end{aligned}$$

The Γ_1 term is estimated using the Cauchy–Schwarz inequality, the fact that $\nabla \tilde{u}_i \in L^{q_0}_\omega(L^2_{t,x})$, cf. (6.51), and $q \le q_0$:

$$\begin{aligned}\Gamma_1 &\le \tilde{\mathbb{E}}\left[\left(\int_t^{t+\vartheta}\int_\Omega |\nabla \tilde{u}_i|^2\,dx\,ds\right)^{\frac{q}{2}}\left(\int_t^{t+\vartheta}\int_\Omega |\nabla \varphi|^2\,dx\,ds\right)^{\frac{q}{2}}\right]\\ &\le C_1\,|\vartheta|^{\frac{q}{2}}\,\|\nabla\varphi\|^q_{L^2(\Omega)}.\end{aligned}$$

Thanks to Hölder's inequality,

$$\begin{aligned}\Gamma_2 &\le \tilde{\mathbb{E}}\left[\left(\int_t^{t+\vartheta}\int_\Omega |I(\tilde{v},\tilde{w})|^{\frac{4}{3}}\,dx\,ds\right)^{\frac{3q}{4}}\left(\int_t^{t+\vartheta}\int_\Omega |\varphi|^3\,dx\,ds\right)^{\frac{q}{3}}\right]\\ &\le \tilde{C}_2\,|\vartheta|^{\frac{q}{3}}\,\tilde{\mathbb{E}}\left[\left(\int_t^{t+\vartheta}\int_\Omega \left(|\tilde{v}|^4+|\tilde{w}|^2\right)dx\,ds\right)^{\frac{3q}{4}}\right]\|\varphi\|^{\frac{q}{3}}_{L^3(\Omega)}\\ &\le C_2\,|\vartheta|^{\frac{q}{3}}\,\|\varphi\|^q_{L^3(\Omega)},\end{aligned}$$

using (6.30), $\tilde{v} \in L^{2q_0}_\omega(L^4_{t,x})$, cf. (6.51), $\tilde{w} \in L^{q_0}_\omega(L^\infty_t(L^2_x))$, cf. (6.21), and that the relevant exponents satisfy $3q \le 2q_0$, $3q/2 \le q_0$.

Finally, we have

$$\begin{aligned}\Gamma_3 &\le \tilde{\mathbb{E}}\left[\left\|\sup_{\tau\in[0,\vartheta]}\int_t^{t+\tau}\eta(\tilde{v})\,d\tilde{W}^v\right\|^q_{L^2(\Omega)}\right]\|\varphi\|^q_{L^2(\Omega)}\\ &\overset{(3.4)}{\le} \tilde{C}_3\tilde{\mathbb{E}}\left[\left(\int_t^{t+\vartheta}\|\eta(\tilde{v})\|^2_{L_2(\mathbb{U},L^2(\Omega))}\,dt\right)^{\frac{q}{2}}\right]\|\varphi\|^q_{L^2(\Omega)}\\ &\overset{(3.7)}{\le} \hat{C}_3\,|\vartheta|^{\frac{q}{2}}\left(1+\tilde{\mathbb{E}}\left[\|\tilde{v}\|^q_{L^\infty((0,T);L^2(\Omega))}\right]\right)\|\varphi\|^q_{L^2(\Omega)} \le C_3\,|\vartheta|^{\frac{q}{2}}\,\|\varphi\|^q_{L^2(\Omega)},\end{aligned}$$

since $\tilde{v} \in L^{q_0}_\omega(L^\infty_t(L^2_x))$ and $q \le q_0$.

Summarizing, with $t, t+\vartheta \in [0,T]$ and $|\vartheta| < 1$, there exists a constant $C > 0$ such that

$$\tilde{\mathbb{E}}\left[\left|\Psi_\varphi(t+\vartheta)-\Psi_\varphi(t)\right|^q\right] \le C\,|\vartheta|^{\frac{q}{3}}\,\|\varphi\|^q_{H^1_D(\Omega)} = C_\varphi\,|\vartheta|^{1+\frac{q-3}{3}},$$

where $C_\varphi := C\,\|\varphi\|^q_{H^1_D(\Omega)}$. Noting that $\gamma := \frac{1}{3}-\frac{1}{q} > 0$, Kolmogorov's continuity result ensures the existence of a γ-Hölder continuous modification of Ψ_φ.

7. Uniqueness of weak (pathwise) solutions

In this section we prove an L^2 stability estimate and consequently a pathwise uniqueness result. This result is used in the next section to conclude the existence of a unique weak solution to the stochastic bidomain model.

Let $(\mathcal{S}, u_i, u_e, v, w)$ be a weak solution according to Definition 4.1. We need a special case of the infinite dimensional version of Itô's formula [16,33,41]:

$$d\,\|v(t)\|^2_{L^2(\Omega)} = 2\,(dv, v)_{(H^1_D(\Omega))^*, H^1_D(\Omega)} + 2\sum_{k\ge 1}\|\eta_k(v)\|^2_{L^2(\Omega)}\,dt.$$

To compute the first term on the right-hand side, multiply the first equation in (2.4) by u_i, the second equation by $-u_e$, and sum the resulting equations. The outcome is

$$v\,dv - \sum_{j=i,e} \nabla\cdot\left(M_j \nabla u_j\right) u_j\,dt + v I(v,w)\,dt = v\eta(v)\,dW^v.$$

Hence,

$$\begin{aligned}(dv, v)_{(H^1_D(\Omega))^*, H^1_D(\Omega)} = &- \sum_{j=i,e} \left(M_j\nabla u_j, \nabla u_j\right)_{L^2(\Omega)}\,dt - (v, I(v,w))_{L^2(\Omega)}\,dt \\ &+ \sum_{k\ge 1} (v, \eta_k(v))_{L^2(\Omega)}\,dW^v_k.\end{aligned}$$

Therefore, weak solutions of the stochastic bidomain model satisfy the following Itô formula for the squared L^2 norm:

$$\begin{aligned}\|v(t)\|^2_{L^2(\Omega)} &= \|v(0)\|^2_{L^2(\Omega)} - 2\sum_{j=i,e}\int_0^t\int_\Omega M_j\nabla u_j\cdot\nabla u_j\,dx\,ds \\ &- 2\int_0^t\int_\Omega vI(v,w)\,dx\,ds + 2\sum_{k\ge1}\int_0^t\int_\Omega |\eta_k(v)|^2\,dx\,ds + 2\sum_{k\ge1}\int_\Omega v\,\eta_k(v)\,dx\,dW^v_k.\end{aligned} \tag{7.1}$$

Additionally, from the (simpler) w-equation in (2.4) we obtain

$$\begin{aligned}\|w(t)\|^2_{L^2(\Omega)} &= \|w(0)\|^2_{L^2(\Omega)} + 2\int_0^t\int_\Omega wH(v,w)\,dx\,ds \\ &+ 2\sum_{k\ge1}\int_0^t\int_\Omega |\sigma_k(v)|^2\,dx\,ds + 2\sum_{k\ge1}\int_\Omega w\,\sigma_k(v)\,dx\,dW^w_k.\end{aligned} \tag{7.2}$$

Remark 7.1. Definition 4.1 asks that the paths of $v(t)$ are weakly time continuous but not that they belong to $C([0,T];L^2(\Omega))$. Define X, Y by $X(t) := v(t) - \int_0^t \eta(v)\,dW^v$ and $Y(t) := w(t) - \int_0^t \sigma(w)\,dW^w$, and note that P-a.s., $X, Y \in L^2((0,T);H^1_D(\Omega))$ and $\partial_t X, \partial_t Y \in L^2((0,T);(H^1_D(\Omega))^*)$. Consequently, X, Y belong to $C([0,T];L^2(\Omega))$ [53]. According to standard arguments [16], $t \mapsto \int_0^t \beta(v)\,dW \in C([0,T];L^2(\Omega))$, P-almost surely, for $(\beta, W) = (\eta, W^v), (\sigma, W^w)$. We conclude that P-a.s. $v, w \in C([0,T];L^2(\Omega))$.

We are now in a position to prove the stability result.

Theorem 7.2. *Suppose conditions* (**GFHN**), (2.5), *and* (3.6) *hold. Let* $\bar U = \left(\mathcal{S}, \bar u_i, \bar u_e, \bar v, \bar w\right)$ *and* $\hat U = \left(\mathcal{S}, \hat u_i, \hat u_e, \hat v, \hat w\right)$ *be two weak solutions (according to Definition 4.1), relative to the same stochastic basis* $\mathcal{S}$*, cf.* (3.1)*, with initial data* $\bar v(0) = \bar v_0$, $\hat v(0) = \hat v_0$, $\bar w(0) = w_0$, *and* $\hat w(0) = \hat w(0)$, *where* $\bar v_0, \hat v_0, \bar w_0, \hat w_0 \in L^2\left(D, \mathcal{F}, P; L^2(\Omega)\right)$. *There exists a positive constant* $C \ge 1$ *such that*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\bar v(t) - \hat v(t)\right\|^2_{L^2(\Omega)}\right] + \sum_{j=i,e}\mathbb{E}\left[\left\|\bar u_j - \hat u_j\right\|^2_{L^2(\Omega_T)}\right] \\ &\quad + \mathbb{E}\left[\sup_{t\in[0,T]}\left\|\bar w(t) - \hat w(t)\right\|^2_{L^2(\Omega)}\right] \\ &\qquad \le C\left(\mathbb{E}\left[\left\|\bar v_0 - \hat v_0\right\|^2_{L^2(\Omega)}\right] + \mathbb{E}\left[\left\|\bar w_0 - \hat w_0\right\|^2_{L^2(\Omega)}\right]\right).\end{aligned} \tag{7.3}$$

With $\bar v_0 = \hat v_0$, $\bar w_0 = \hat w_0$, *it follows that weak (pathwise) solutions are unique, cf.* (4.4).

Proof of Theorem 7.2. Set $v := \bar{v} - \hat{v}$, $u_i := \bar{u}_i - \hat{u}_i$, $u_e := \bar{u}_e - \hat{u}_e$, and $w := \bar{w} - \hat{w}$. Note that $v = u_i - u_e$. We have P-a.s.,

$$u_i, \bar{u}_i, \hat{u}_i, u_e, \bar{u}_e, \hat{u}_e, v, \bar{v}, \hat{v} \in L^2((0,T); H_D^1(\Omega)),$$
$$v, \bar{v}, \hat{v}, w, \bar{w}, \hat{w} \in L^\infty((0,T); L^2(\Omega)) \cap C([0,T]; (H_D^1(\Omega))^*).$$

Actually, we can replace $C([0,T]; (H_D^1(\Omega))^*)$ by $C([0,T]; L^2(\Omega))$, see Remark 7.1.

Subtracting the $(H_D^1(\Omega))^*$ valued equations for $\bar{U}, \hat{U}$, cf. (2.4), we obtain

$$\begin{aligned} &dv - \nabla \cdot \left(M_i \nabla u_i\right) dt + \left(I(\bar{v}, \bar{w}) - I(\hat{v}, \hat{w})\right) dt = \left(\eta(\bar{v}) - \eta(\hat{v})\right) dW^v, \\ &dv + \nabla \cdot \left(M_e \nabla u_e\right) dt + \left(I(\bar{v}, \bar{w}) - I(\hat{v}, \hat{w})\right) dt = \left(\eta(\bar{v}) - \eta(\hat{v})\right) dW^v, \\ &dw = \left(H(\bar{v}, \bar{w}) - H(\hat{v}, \hat{w})\right) dt + \left(\sigma(\bar{v}) - \sigma(\hat{v})\right) dW^w. \end{aligned} \tag{7.4}$$

We apply the Itô formula to the w-equation, cf. (7.2), and multiply by $1/\mu$, cf. (2.6). We then apply the Itô formula to the v-equations, cf. (7.1). Adding the results and using (2.5), we obtain in the end the following inequality:

$$\begin{aligned} &\frac{1}{2} \|v(t)\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|w(t)\|_{L^2(\Omega)}^2 + m \sum_{j=i,e} \int_0^t \int_\Omega \left|\nabla u_j\right|^2 dx\, ds \\ &\le \frac{1}{2} \|v(0)\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|w(0)\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{\mu} \int_0^t \int_\Omega \Big(w \left(H(\bar{v}, \bar{w}) - H(\hat{v}, \hat{w})\right) - \mu v \left(I(\bar{v}, \bar{w}) - I(\hat{v}, \hat{w})\right)\Big) dx\, ds \\ &\quad + \sum_{k \ge 1} \int_0^t \int_\Omega \left|\eta_k(\bar{v}) - \eta_k(\hat{v})\right|^2 dx\, ds + \frac{1}{\mu} \sum_{k \ge 1} \int_0^t \int_\Omega \left|\sigma_k(\bar{v}) - \sigma_k(\hat{v})\right|^2 dx\, ds \\ &\quad + \sum_{k \ge 1} \int_\Omega v \left(\eta_k(\bar{v}) - \eta_k(\hat{v})\right) dx\, dW_k^v + \frac{1}{\mu} \sum_{k \ge 1} \int_\Omega w \left(\sigma_k(\bar{v}) - \sigma_k(\hat{v})\right) dx\, dW_k^w. \end{aligned} \tag{7.5}$$

We use assumption (2.6) to bound the third term on the right-hand side by a constant times $\int_0^t \left(\|v(s)\|_{L^2(\Omega)}^2 + \|w(s)\|_{L^2(\Omega)}^2 \right) ds$. We use (3.6) to bound the fourth term by a constant times $\int_0^t \|v(s)\|_{L^2(\Omega)}^2\, ds$. The stochastic integrals in (7.5) are square-integrable, zero-mean martingales. Moreover, by the Poincaré inequality, we have

$$\int_0^t \int_\Omega |u_e|^2\, dx\, ds \le \tilde{C} \int_0^t \int_\Omega |\nabla u_e|^2\, dx\, ds,$$

for some constant $\tilde{C} > 0$. Since $u_i = v + u_e$, we control u_i as well. As a result of all this, there is a constant $C > 0$ such that

$$\begin{aligned} &\mathbb{E}\left[\|v(t)\|_{L^2(\Omega)}^2\right] + \sum_{j=i,e} \mathbb{E}\left[\|u_j\|_{L^2(\Omega_T)}^2\right] + \mathbb{E}\left[\|w(t)\|_{L^2(\Omega)}^2\right] \\ &\quad \le C\mathbb{E}\left[\|v(0)\|_{L^2(\Omega)}^2\right] + C\mathbb{E}\left[\|w(0)\|_{L^2(\Omega)}^2\right] \\ &\qquad + C \int_0^t \left(\mathbb{E}\left[\|v(s)\|_{L^2(\Omega)}^2\right] + \mathbb{E}\left[\|w(s)\|_{L^2(\Omega)}^2\right]\right) ds, \qquad t \in [0,T]. \end{aligned}$$

The Grönwall inequality delivers (7.3) "without sup". The refinement (7.3) ("with sup") comes from a martingale inequality (3.4), see (6.16) for a similar situation. □

8. Existence of weak (pathwise) solutions

In this section we prove the existence of a unique weak (pathwise) solution in the sense of Definition 4.5, thereby proving Theorem 4.6. The proof follows the traditional Yamada–Watanabe approach [17,24,41], combining the existence of at least one weak martingale solution (Theorem 4.4) with a pathwise uniqueness result (Theorem 7.2), relying on the Gyöngy–Krylov characterization of convergence in probability (Lemma 3.3).

Fix a stochastic basis $\mathcal{S} = \left(D, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P, W\right)$, where $W = \left(W^v, W^w\right)$ and $W^v = \left\{W_k^v\right\}_{k\geq 1}$, $W^w = \left\{W_k^w\right\}_{k\geq 1}$ are cylindrical Wiener processes. We denote by $U^n = \left(u_i^n, u_e^n, v^n, w^n\right)$, $W^n = \left(W^{v,n}, W^{w,n}\right)$, $U_0^n = \left(u_{i,0}^n, u_{e,0}^n, v_0^n, w_0^n\right)$ the Faedo–Galerkin solution defined on $\mathcal{S}$, cf. Section 5. Let $\mathcal{L}_n$ be the probability law of

$$\Phi_n : D \to \mathcal{X} = \mathcal{X}_U \times \mathcal{X}_W \times \mathcal{X}_{U_0}, \quad \Phi_n(\omega) = \left(U^n(\omega), W^n(\omega), U_0^n(\omega)\right).$$

We intend to show that the approximate solutions U^n converge in probability (in $\mathcal{X}_U$) to a random variable $U = (u_i, u_e, v, w)$ (defined on $\mathcal{S}$). Passing to a subsequence if necessary, we may as well replace convergence in probability by a.s. convergence. We then argue as in Section 6.4 to arrive at the conclusion that the limit U of $\{U^n\}_{n\geq 1}$ is a weak (pathwise) solution of the stochastic bidomain model.

It remains to prove that $\{U^n\}_{n\geq 1}$ converges in probability. To this end, we will use the Gyöngy–Krylov lemma along with pathwise uniqueness. By Lemma 6.4, the sequence $\{\mathcal{L}_n\}_{n\geq 1}$ is tight on $\mathcal{X}$. For $n, m \geq 1$, denote by $\mathcal{L}_{n,m}$ the law of the random variable

$$\Phi_{n,m}(\omega) = \left(U^n(\omega), U^m(\omega), W^n(\omega), U_0^n(\omega), U_0^m(\omega)\right),$$

defined on the extended path space $\mathcal{X}^E := \mathcal{X}_U \times \mathcal{X}_U \times \mathcal{X}_W \times \mathcal{X}_{U_0} \times \mathcal{X}_{U_0}$. Clearly, we have $\mathcal{L}_{n,m} = \mathcal{L}_{U^n} \times \mathcal{L}_{U^m} \times \mathcal{L}_{W^n} \times \mathcal{L}_{U_0^n} \times \mathcal{L}_{U_0^m}$ (see Section 6.3 for the notation). With obvious modifications of the proof of Lemma 6.4, we conclude that $\left\{\mathcal{L}_{n,m}\right\}_{n,m\geq 1}$ is tight on $\mathcal{X}^E$. Let us now fix an arbitrary subsequence $\left\{\mathcal{L}_{n_k,m_k}\right\}_{k\geq 1}$ of $\left\{\mathcal{L}_{n,m}\right\}_{n,m\geq 1}$, which obviously is also tight on $\mathcal{X}^E$.

Passing to a further subsequence if needed (without relabeling as usual), the Skorokhod–Jakubowski representation theorem provides a new probability space $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$ and new $\mathcal{X}^E$-valued random variables

$$\left(\bar{U}^{n_k}, \hat{U}^{m_k}, \tilde{W}^{n_k}, \bar{U}_0^{n_k}, \hat{U}_0^{m_k}\right), \quad \left(\bar{U}, \hat{U}, \tilde{W}, \bar{U}_0, \hat{U}_0\right) \tag{8.1}$$

on $(\tilde{D}, \tilde{\mathcal{F}}, \tilde{P})$, such that the law of $\left(\bar{U}^{n_k}, \hat{U}^{m_k}, \tilde{W}^{n_k}, \bar{U}_0^{n_k}, \hat{U}_0^{m_k}\right)$ is $\mathcal{L}_{n_k,m_k}$ and

$$\left(\bar{U}^{n_k}, \hat{U}^{m_k}, \tilde{W}^{n_k}, \bar{U}_0^{n_k}, \hat{U}_0^{m_k}\right) \xrightarrow{k\uparrow\infty} \left(\bar{U}, \hat{U}, \tilde{W}, \bar{U}_0, \hat{U}_0\right) \quad \tilde{P}\text{-almost surely (in } \mathcal{X}^E).$$

Observe that $\tilde{P}\left(\left\{\omega \in \tilde{D} : \bar{U}_0(\omega) = \hat{U}_0(\omega)\right\}\right) = 1$. Indeed, we have $U_0^{n_k} = \Pi_{n_k} U_0$ and $U_0^{m_k} = \Pi_{m_k} U_0$, and so for any $\ell \leq \min(n_k, m_k)$,

$$\tilde{P}\left(\left\{\omega \in \tilde{D} : \Pi_\ell \bar{U}_0^{n_k} = \Pi_\ell \hat{U}_0^{m_k}\right\}\right) = P\left(\left\{\omega \in D : \Pi_\ell U_0^{n_k} = \Pi_\ell U_0^{m_k}\right\}\right) = 1,$$

by equality of the laws. This proves the claim.

Applying the arguments in Section 6.4 separately to

$$\bar{U}^{n_k}, \tilde{W}^{n_k}, \bar{U}_0^{n_k}, \bar{U}, \tilde{W}, \bar{U}_0 \quad \text{and} \quad \hat{U}^{m_k}, \tilde{W}^{n_k}, \hat{U}_0^{m_k}, \hat{U}, \tilde{W}, \hat{U}_0,$$

it follows that $\left(\bar{U}, \tilde{W}, \bar{U}_0\right)$ and $\left(\hat{U}, \tilde{W}, \hat{U}_0\right)$ are both weak martingale solutions, relative to the same stochastic basis $\tilde{\mathcal{S}} = \left(\tilde{D}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t\right\}_{t\in[0,T]}, \tilde{P}, \tilde{W}\right)$, $\tilde{W} = \tilde{W}^v, \tilde{W}^w$, where

$$\tilde{\mathcal{F}}_t = \sigma\left(\sigma\left(\bar{U}\big|_{[0,t]}, \hat{U}\big|_{[0,t]}, \tilde{W}\big|_{[0,t]}, \bar{U}_0\right) \bigcup \left\{N \in \tilde{\mathcal{F}} : \tilde{P}(N) = 0\right\}\right), \quad t \in [0, T].$$

Denote by μ_{n_k,m_k} and μ the joint laws of $\left(\bar{U}^{n_k}, \hat{U}^{m_k}\right)$ and $\left(\bar{U}, \hat{U}\right)$, respectively. Then, in view of (8.1), $\mu_{n_k,m_k} \rightharpoonup \mu$ as $k \to \infty$. Since $\bar{U}_0 = \hat{U}_0$ $\tilde{P}$-a.s., Theorem 7.2 ensures that $\bar{U} = \hat{U}$ $\tilde{P}$-a.s. (in $\mathcal{X}_U$). Hence, since this implies

$$\mu\left(\{(X, Y) \in \mathcal{X}_U \times \mathcal{X}_U : X = Y\}\right) = \tilde{P}\left(\left\{\omega \in \tilde{D} : \bar{U}(\omega) = \hat{U}(\omega)\right\}\right) = 1,$$

we can appeal to Lemma 3.3, cf. Remark 3.4, to complete the proof.

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