4–manifolds and intersection forms with local coefficients

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Abstract

We extend Donaldson's diagonalization theorem to intersection forms with certain local coefficients, under some constraints. This provides new examples of non-smoothable topological 4–manifolds.

1 Introduction

A celebrated early theorem of Donaldson [3, 4] says that if the intersection form of a closed, oriented smooth 4–manifold V is negative definite, then it is *standard*, i.e. there is a basis for $H^2(V;\mathbb{Z})$/torsion with respect to which the form is diagonal. The proof involved a careful study of a certain SU(2)–instanton moduli space over V. Later, Fintushel and Stern [7] found a simpler proof using SO(3)–instanton moduli spaces in the case when $H_1(V;\mathbb{Z})$ contains no 2–torsion. (The assumption on the torsion can be removed by using results from [4], see [12].) In either variant of the proof an essential point is the link between the intersection form of V and Abelian reducibles in the moduli spaces, which are represented by connections with stabilizer U(1). In SO(3)–moduli spaces there is also a second type of reducible, namely the *twisted reducibles*, which are represented by connections with stabilizer $\mathbb{Z}/2$ (among all automorphisms of the SO(3)–bundle). In this paper we will show that these are related to the intersection forms of V with certain local coefficients. We use this to partially extend Donaldson's theorem to such forms. We will now explain our result in more detail.

We generalize the setup somewhat and consider a compact, connected, oriented, smooth 4–manifold X with boundary Y. Let $\ell \to X$ be any bundle

*This work was supported by a ProDoc grant at the ETH, Zürich, and by QGM (Centre for Quantum Geometry of Moduli Spaces) funded by the Danish National Research Foundation.

of infinite cyclic groups. Recall that the set of isomorphism classes of such bundles form an Abelian group isomorphic to $H^1(X;\mathbb{Z}/2)$. Let $H^*(X;\ell)$ be the singular cohomology with ℓ as bundle of coefficients. Since $\ell\otimes\ell=\mathbb{Z}$, the cup product defines a homomorphism

$$H^2(X;\ell)\otimes H^2(X,Y;\ell)\to H^4(X,Y;\mathbb{Z})=\mathbb{Z}. \tag{1}$$

Now suppose Y is an integral homology sphere. Then $H^2(X,Y;\ell)=H^2(X;\ell)$, and (1) induces a unimodular quadratic form $Q_{X,\ell}$ on $H^2(X;\ell)/\text{torsion}$, which we refer to as the intersection form of X with coefficients in ℓ. When ℓ is trivial this is of course the usual intersection form of X. The signature of $Q_{X,\ell}$ is independent of ℓ. As observed in [18, p. 587], the same holds for the quantity

$$b_0(X;\ell)-b_1(X;\ell)+b^+(X;\ell),$$

where $b_j(X;\ell):=\operatorname{rank}H^j(X;\ell)$ and $b_2^+(X;\ell)$ denotes the dimension of a maximal positive subspace for $Q_{X,\ell}$. For any non-trivial ℓ one therefore has

$$-b_1(X;\ell)+b^+(X;\ell)=1-b_1(X)+b^+(X), \tag{2}$$

where $b_j(X):=b_j(X;\mathbb{Z})$ and $b^+(X):=b^+(X;\mathbb{Z})$.

For any Abelian group G let $\mathrm{HF}^*(Y;G)$ denote the instanton Floer cohomology group with coefficients in G, see [8, 5]. This is the cohomology of a cochain complex $\mathrm{CF}^*\otimes G$, where CF^q is the free Abelian group generated by gauge equivalence classes of irreducible (perturbed) flat SO(3)–connections over Y of index $q\in\mathbb{Z}/8$, and the differential $d:\mathrm{CF}^q\to\mathrm{CF}^{q+1}$ counts instantons over the cylinder $\mathbb{R}\times Y$ interpolating between two given irreducible flat connections. Counting SO(3)–instantons over $\mathbb{R}\times Y$ with trivial flat limit at $+\infty$ yields a homomorphism $\delta:\mathrm{CF}^4\to\mathbb{Z}$ which satisfies $\delta d=0$ (see [12]) and therefore induces a homomorphism

$$\delta_0:\mathrm{HF}^4(Y;G)\to G.$$

Before stating the main result of this paper we need one more definition:

$$\begin{aligned}\tau(X)&:=\dim_{\mathbb{Z}/2}[\text{torsion}(H_1(X;\mathbb{Z}))\otimes\mathbb{Z}/2]\\&=b_1(X;\mathbb{Z}/2)-b_1(X),\end{aligned}$$

where $b_j(X;\mathbb{Z}/2):=\dim_{\mathbb{Z}/2}H_j(X;\mathbb{Z}/2)$.

Theorem 1.1 *Let X be any compact, connected, oriented, smooth 4–manifold whose boundary Y is an integral homology sphere, and such that*

$$\tau(X)+b^+(X)\le 2. \tag{3}$$

Let $\ell \to X$ be any non-trivial bundle of infinite cyclic groups. If $Q_{X,\ell}$ is non-standard negative definite and $H^2(X;\ell)$ contains no element of order 4 *then*

$$\delta_0 : HF^4(Y;\mathbb{Z}/2) \to \mathbb{Z}/2$$

is non-zero.

Corollary 1.1 *Let V be any closed, connected, oriented, smooth* 4*–manifold such that*

$$\tau(V) + b^+(V) \leq 2.$$

Let $\ell \to V$ be any non-trivial bundle of infinite cyclic groups such that $Q_{V,\ell}$ is negative definite and $H^2(V;\ell)$ contains no element of order 4*. Then $Q_{V,\ell}$ is standard.*

Proof. This follows from the theorem by taking X to be the complement of an open 4–ball in V, and recalling that $\mathrm{HF}^*(S^3;\mathbb{Z}/2) = 0$. □

Remarks. **(i)** Under the hypotheses of the corollary, V cannot be spin. For in that case the usual intersection form Q_V would be even with negative signature, so Q_V could not be definite by Donaldson's theorem. The condition $b^+(V) \leq 2$ would then violate a theorem of Furuta [14].

(ii) If $b_1(X) = 1$ and $\tau(X) = 0$ then $H^2(X;\ell)$ does not even have any element of order 2, see Proposition 2.1.

(iii) The author does not know whether the theorem holds without the assumptions on $\tau(X) + b^+(X)$ and (in general) elements of order 4 in $H^2(X;\ell)$, despite attempts at finding counterexamples.

(iv) The statement of the theorem holds when ℓ is trivial too, and without the assumption $\tau(X) \leq 2$. However, we prefer to take that up in a separate paper.

(v) One reason for the appearance of the term $\tau + b^+$ in the theorem is that this quantity is invariant under surgery on any circle in the interior of X which represents a non-zero class in $H_1(X;\mathbb{Z}/2)$, see Lemma 8.1.

Proposition 1.1 *Let V be any closed, oriented topological* 4*–manifold whose intersection form Q_V is non-standard negative definite. Suppose $H_1(V;\mathbb{Z})$ contains no element of order* 4*. Let either*

(i) *$W = \Sigma \times S^2$, where Σ is any closed, oriented, connected surface of genus at least* 1*, or*

(ii) *$W = Y \times S^1$, where Y is any closed, oriented* 3*–manifold.*

If $\tau(V)+\tau(W)+b^+(W)\leq 2$, then $V\#W$ does not admit any smooth structure.

Of course, if $W=\Sigma\times S^2$ then $\tau(W)=0$ and $b^+(W)=1$, whereas if $W=Y\times S^1$ then $\tau(W)=\tau(Y)$ and $b^+(W)=b_1(Y)$.

Proof. (i) We may assume that V is connected and that Q_V is negative definite. Let $\ell\to W:=\Sigma\times S^2$ be any non-trivial bundle of infinite cyclic groups. The exact sequence (4) below yields

$$\mathrm{torsion}(H_1(W;\ell))=\mathbb{Z}/2,\qquad H_2(W;\ell)=\mathbb{Z}/2.$$

Let $\ell'\to V':=V\#W$ be the bundle which corresponds to the trivial bundle over V and to ℓ over W. Then the group

$$H_1(V';\ell')=H_1(V;\mathbb{Z})\oplus H_1(W;\ell)$$

contains no element of order 4. By the universal coefficient theorem (see (6) below) the same holds for $H^2(V';\ell')$. As for the intersection forms one has

$$Q_{V',\ell'}=Q_V,$$

so it follows from Corollary 1.1 that V' cannot admit any smooth structure.

(ii) Let $\ell\to W:=Y\times S^1$ be the pull-back of the non-trivial $\mathbb{Z}$–bundle over S^1. Using the exact sequence (4) below one finds that $H_k(W;\ell)$ is a finite group for all k, and that

$$H_1(W;\ell)=H_1(Y;\mathbb{Z})\,/\,2H_1(Y;\mathbb{Z})\approx(\mathbb{Z}/2)^r$$

for some r. We can now argue as in (i). $\square$

When combined with Freedman's classification of simply-connected, closed, oriented topological 4–manifolds [10] this yields many examples of non-smoothable indefinite 4–manifolds, also with odd intersection form. In the case of even intersection form such examples can also be found using Rochlin's theorem or Furuta's theorem.

Note that if V is simply connected and negative definite, say, then $V\#\mathbb{CP}^2$ is smoothable, since by Freedman's theorem and the classification of odd indefinite forms it is homeomorphic to $\mathbb{CP}^2\#(-n\mathbb{CP}^2)$ for some n.

In a slightly different direction, Friedl–Hambleton–Melvin–Teichner [11] have proved that a certain negative definite closed, oriented topological 4–manifold V with $\pi_1(V)=\mathbb{Z}$ and $b_2(V)=4$ is not smoothable by applying

Donaldson's diagonalization theorem to the finite coverings of V. (A survey of related material can be found in [15].)

After some preliminaries in Section 2 on (co)homology with local coefficients, Section 3 introduces what is probably the main novelty in the paper as far as gauge theory is concerned: Given any SO(3)–bundle $E \to Z$, where Z is a smooth, compact manifold, and any loop $\gamma : S^1 \to Z$, we define a double covering $\Xi_\gamma \to U_\gamma$, where U_γ is a certain open subset of the orbit space $\mathcal{B}(E)$ of all connections in E (of a given Sobolev type). The subset U_γ contains all irreducible connections as well as some reducibles including all Abelian ones. In Section 4 we classify non-flat twisted reducible instantons over certain 4–manifolds W with a tubular end. The local structure around these reducibles is described in Section 5, whereas Abelian reducibles are discussed in Section 6. Section 7 proves three lemmas on Banach manifolds. Section 8 contains the proof of the theorem. This begins by reducing the problem to the case $b_1(X) = 1 + b^+(X)$ by doing surgery on a suitable collection of disjoint circles in X. We then study the moduli space M_k of instantons with trivial limit in a certain SO(3)–bundle over $W := X \cup_Y (\mathbb{R}_+ \times Y)$. The irreducible part M_k^* is cut down to a 1–manifold using sections of the real line bundles corresponding to suitable double coverings Ξ_γ. The ends of this 1–manifold are associated to twisted reducibles in M_k and factorizations over the end of W. Of course, the number of ends must be zero modulo 2.

The advantage of reducing to the case $b_1 = 1 + b^+$ is that then, generically, all non-flat twisted reducibles in the moduli spaces are isolated. Working directly with the original manifold X would require dealing with positive-dimensional families of twisted reducibles. This technically more difficult situation has been studied by Teleman [24]. However, it is not clear to this author whether one can expect stronger results with such a direct approach.

After this paper was submitted the preprint [20] appeared, which addresses similar issues for closed 4–manifolds, using Seiberg–Witten theory.

Acknowledgement: The author is grateful to the anonymous referee for the careful reading of the manuscript and many suggestions. He would also like to thank Ian Hambleton and Bjørn Jahren for helpful correspondence.

2 Homology and cohomology with local coefficients

This section contains mostly background material.

(I) This part is concerned with singular (co)homology with local coefficients. Let X be any space. For any bundle $E \to X$ of discrete Abelian groups we denote by $C_*(X;E)$ the singular chain complex of X with values in E, as defined in [16]. A short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of morphisms of such bundles induces a short exact sequence of chain complexes

$$0 \to C_*(X;E') \to C_*(X;E) \to C_*(X;E'') \to 0$$

which in turn yields a long exact sequence relating the corresponding homology groups $H_*(X;\cdot)$. Similar statements hold for the singular cochain complexes and cohomology groups $H^*(X;\cdot)$.

Now let $p : \tilde{X} \to X$ be any double covering and $\ell \to X$ the associated bundle of infinite cyclic groups. Consider the $\mathbb{Z}^2$–bundle

$$E := \tilde{X} \underset{\mathbb{Z}/2}{\times} \mathbb{Z}^2$$

over X, where $1 \in \mathbb{Z}/2$ acts on $\tilde{X}$ by flipping the sheets of the covering and on $\mathbb{Z}^2$ by permuting the factors. Then

$$H_*(X; E \otimes G) = H_*(\tilde{X}; G)$$

for any Abelian group G, and similarly for cohomology. There is a canonical short exact sequence of bundles

$$0 \to \ell \to E \to \mathbb{Z} \to 0$$

which induces a long exact sequence

$$\cdots \to H_k(X;\ell) \to H_k(\tilde{X};\mathbb{Z}) \overset{p_*}{\to} H_k(X;\mathbb{Z}) \to H_{k-1}(X;\ell) \to \cdots . \tag{4}$$

We will use the notation λ (resp. $\underline{\lambda}$) for $\ell \otimes \mathbb{R}$ thought of as a real line bundle (resp. a bundle with discrete fibres) over X. By the universal coefficients theorem (see [22, p. 283]) one has

$$H^*(X;\ell) \otimes \mathbb{R} = H^*(X;\underline{\lambda})$$

in each degree in which $H_*(X;\ell)$ is finitely generated. There is a canonical isomorphism of bundles $\mathbb{R} \oplus \underline{\lambda} \overset{\approx}{\to} E \otimes \mathbb{R}$, which induces an isomorphism

$$H^*(\tilde{X};\mathbb{R}) = H^*(X;\mathbb{R}) \oplus H^*(X;\underline{\lambda}).$$

The two summands correspond to the ± 1 eigenspaces of the endomorphism of $H^*(\tilde{X};\mathbb{R})$ induced by the involution of $\tilde{X}$ (i.e. the action of $1 \in \mathbb{Z}/2$). If

X is a smooth manifold then $H^*(X;\underline{\lambda})$ can be computed as the de Rham cohomology associated to the flat bundle λ (see [1]). When working with de Rham cohomology it is natural to write $b_j(X;\lambda)$ instead of $b_j(X;\ell)$, and similarly for b^+.

There is also a relationship with mod 2 (co)homology, for arbitrary X: The short exact sequence

$$0 \to \ell \xrightarrow{\cdot 2} \ell \to \mathbb{Z}/2 \to 0$$

of bundles gives rise to a long exact sequence

$$\cdots \to H_q(X;\ell) \xrightarrow{\cdot 2} H_q(X;\ell) \to H_q(X;\mathbb{Z}/2) \to H_{q-1}(X;\ell) \to \cdots \quad (5)$$

as well as a similar sequence for cohomology. Furthermore, because $\ell^* = \ell$ the universal coefficient theorem yields a split short exact sequence

$$0 \to \operatorname{Ext}(H_{q-1}(X;\ell),\mathbb{Z}) \to H^q(X;\ell) \to \operatorname{Hom}(H_q(X;\ell),\mathbb{Z}) \to 0. \quad (6)$$

Proposition 2.1 *Let X be any compact manifold (with or without boundary) such that $H_1(X;\mathbb{Z}/2) = \mathbb{Z}/2$. Let $\ell \to X$ be any non-trivial bundle of infinite cyclic groups. Then $H_1(X;\ell)$ is a finite group of odd order, hence by (6) the group $H^2(X;\ell)$ contains no 2–torsion.*

Proof. Because X is a manifold and ℓ is non-trivial, $H_0(X;\ell) = \mathbb{Z}/2$. Thus (5) yields an exact sequence

$$H_1(X;\ell) \xrightarrow{\cdot 2} H_1(X;\ell) \to 0.$$

Since X is a *compact* manifold, $H_*(X;\ell)$ is finitely generated, hence $H_1(X;\ell)$ must be a finite group on which multiplication by 2 is an isomorphism. □

We will now state a version of Poincaré duality for local coefficients. Let X be a closed topological n–manifold and $\mathcal{O}_X \to X$ the orientation bundle, whose fibre over $x \in X$ is

$$\mathcal{O}_x = H_n(X, X\backslash\{x\};\mathbb{Z}).$$

Let $[X] \in H_n(X;\mathcal{O}_X)$ be the fundamental class, which is the unique class whose image in

$$H_n(X, X\backslash\{x\};\mathcal{O}_X) = H_n(X, X\backslash\{x\};\mathcal{O}_x) = \mathcal{O}_x \otimes \mathcal{O}_x = \mathbb{Z}$$

is 1 for every $x \in X$. Let R be a commutative ring with identity.

Proposition 2.2 *For any closed topological n–manifold X and any bundle $E \to X$ of R–modules, cap product with $[X]$ defines an isomorphism*

$$H^p(X;E) \stackrel{\approx}{\to} H_{n-p}(X;E \otimes \mathcal{O}_X)$$

for every p.

Proof. The proof in [16] for R–oriented X and $E = R$ carries over with virtually no changes. $\square$

Other duality theorems for (co)homology with local coefficients can be found in [23].

Now suppose X is a closed oriented topological n–manifold and $\lambda \to X$ a bundle of infinite cyclic groups. Then it follows from Proposition 2.2 and the universal coefficient theorem (6) (recalling that $H_*(X;\ell)$ is finitely generated) that the intersection form $Q_{X,\ell}$ is unimodular. The same holds if X has an integral homology sphere as boundary, as one can see by applying the previous result to the double of X and noting that the intersection form of the double is the orthogonal sum of the intersection forms of the two pieces.

(II) In this part we use Čech cohomology. Recall that for any paracompact space X the first Chern class induces an isomorphism between the group of isomorphism classes of complex line bundles over X and the cohomology group $H^2(X;\mathbb{Z})$. We will now give a similar interpretation of $H^2(X;\ell)$. Let $\tilde{X}, \lambda$ be as in (I) and set

$$K := \tilde{X} \times_{\mathbb{Z}/2} \mathbb{C} = \mathbb{R} \oplus \lambda,$$

where $1 \in \mathbb{Z}/2$ acts on $\tilde{X}$ by flipping the sheets and on $\mathbb{C}$ by complex conjugation. Here $\mathbb{C}$ has the Euclidean topology, so that K is a real vector bundle over X. Since conjugation is a field automorphism, K is a bundle of fields isomorphic to $\mathbb{C}$. Let $K^* \subset K$ be the subspace of non-zero vectors thought of as a bundle of multiplicative groups, and let $\mathcal{K}$ and $\mathcal{K}^*$ denote the sheaves of continuous sections of K and K^*, resp. By a *K–line bundle* we mean a bundle $L \to X$ such that each fibre L_x is a 1–dimensional vector space over K_x, and such that these data satisfy the usual axiom of local triviality. A *local trivialization* of L over an open subset $U \subset X$ is an isomorphism $L|_U \stackrel{\approx}{\to} K|_U$ of $K|_U$–modules. An atlas of such local trivializations gives rise to a Čech cocycle with values in $\mathcal{K}^*$. Standard arguments show that L is classified up to isomorphism by the corresponding cohomology class $\tilde{c}_1(L) \in H^1(X;\mathcal{K}^*)$. If X is paracompact then the short exact sequence of sheaves

$$0 \to \ell \to \mathcal{K} \stackrel{\exp}{\to} \mathcal{K}^* \to 1 \qquad (7)$$

yields an isomorphism $H^1(X;\mathcal{K}^*) \xrightarrow{\approx} H^2(X;\ell)$, and we obtain:

Proposition 2.3 *For any paracompact space X the characteristic class $\tilde{c}_1$ induces an isomorphism between the group of isomorphism classes of K–line bundles and the cohomology group $H^2(X;\ell)$.* $\square$

Note that $\Lambda^2 L = \lambda$, so for the first Stiefel–Whitney class one has

$$w_1(L) = w_1(\lambda).$$

Furthermore, $\tilde{c}_1(L)$ maps to $w_2(L)$ under the homomorphism $H^2(X;\ell) \to H^2(X;\mathbb{Z}/2)$.

By a *Hermitian* K–line bundle we mean a K–line bundle equipped with a Euclidean metric such that multiplication with any unit vector in K_x is an orthogonal transformation of L_x, for any $x \in X$.

We now turn to the smooth category. The proof of the following proposition is similar to that of Proposition 2.3.

Proposition 2.4 *For any smooth manifold X the characteristic class $\tilde{c}_1$ induces an isomorphism between the group of isomorphism classes of smooth Hermitian K–line bundles and the cohomology group $H^2(X;\ell)$.* $\square$

Let $L \to X$ be a smooth Hermitian K–line bundle. If A is any (orthogonal) connection in L then its curvature F_A is a 2–form on X with values in the bundle $\mathrm{so}(L)$ of skew-symmetric endomorphisms of L. Under the isomorphism $\lambda \xrightarrow{\approx} \mathrm{so}(L)$ (defined by multiplication with elements from λ) the closed form $F_A \in \Omega^2(X;\lambda)$ represents the image of $-2\pi\tilde{c}_1(L)$ in $H^2(X;\underline{\lambda})$. (One can deduce this last statement from the known case when ℓ is trivial by pulling A back to $\tilde{X}$ and noting that $H^2(X;\underline{\lambda}) \to H^2(\tilde{X};\mathbb{R})$ is injective.)

3 SO(3)–connections and holonomy

Let Z be a connected smooth n–manifold, possibly with boundary, and $E \to Z$ an oriented, Euclidean 3–plane bundle. Fix $p > n$ and let A be an (orthogonal) $L^p_{1,\mathrm{loc}}$ connection in E. Let Γ_A denote the group of $L^p_{2,\mathrm{loc}}$ automorphisms of E which preserve A. Just as for smooth connections, Γ_A is isomorphic to the centralizer of the holonomy group $\mathrm{Hol}_z(A) \subset \mathrm{Aut}(E_z) \approx \mathrm{SO}(3)$ at any point $z \in Z$. Recall that any positive-dimensional proper closed subgroup of SO(3) is conjugate to either U(1) or O(2), and these subgroups have centralizer U(1) and $\mathbb{Z}/2$, resp. We will call the connection A

- *irreducible* if $\Gamma_A = \{1\}$, otherwise *reducible*,
- *Abelian* if $\Gamma_A \approx \mathrm{U}(1)$,
- *twisted reducible* if $\Gamma_A \approx \mathbb{Z}/2$.

Now suppose A is smooth. Then A is reducible if and only if it preserves a rank 1 subbundle $\lambda \subset E$. If in addition A is not flat then λ is unique (because a non-flat connection A has holonomy close to but different from 1 around suitable small loops in Z). In that case A is Abelian if λ is trivial and twisted reducible otherwise.

Now suppose Z is compact. Let $\mathcal{A}$ denote the affine Banach space consisting of all L^p_1 connections in E and let $\mathcal{G}$ be the Banach Lie group of all L^p_2 automorphisms of E. Then $\mathcal{G}$ acts smoothly on $\mathcal{A}$ and we denote the quotient space by $\mathcal{B} = \mathcal{B}(E)$. It follows easily from the local slice theorem (see [6, p. 132 and p. 192] and [13, Section 2.5]) that $\mathcal{B}$ is a regular topological space. Since $\mathcal{B}$ is also second countable, it is metrizable by the Urysohn metrization theorem [17]. Hence $\mathcal{B}$ is paracompact, and the same holds for any subspace of $\mathcal{B}$.

Let $\mathcal{A}^* \subset \mathcal{A}$ be the subset of irreducible connections. Then $\mathcal{B}^* := \mathcal{A}^*/\mathcal{G}$ is a Banach manifold. In the proof of the theorem we will take p to be an even integer, to make sure that $\mathcal{B}^*$ possesses smooth partitions of unity. (In [19] the existence of smooth partitions of unity is established for paracompact Hilbert manifolds. The proof carries over to paracompact Banach manifolds B modelled on a Banach space $(E, \|\cdot\|)$ such that $\|\cdot\|^t$ is a smooth function on E for some $t > 0$. This includes $B = \mathcal{B}^*$ when p is an even integer, with $t = p$.)

Recall that the Lie group $\mathrm{Aut}(E_z) \approx \mathrm{SO}(3)$ has a non-trivial double covering

$$\widetilde{\mathrm{Aut}}(E_z) \to \mathrm{Aut}(E_z), \tag{8}$$

where $\widetilde{\mathrm{Aut}}(E_z)$ is isomorphic to the group $\mathrm{Sp}(1)$ of unit quaternions. Let $\mathcal{G}$ act on $\mathrm{Aut}(E_z)$ by conjugation with $u(z)$ and on $\widetilde{\mathrm{Aut}}(E_z)$ by conjugation with any lift of $u(z)$ to $\widetilde{\mathrm{Aut}}(E_z)$. Then the covering map (8) is $\mathcal{G}$–equivariant. It follows from the local slice theorem that $\mathcal{A}^* \to \mathcal{B}^*$ is a principal $\mathcal{G}$–bundle, hence

$$\mathcal{A}^* \times_{\mathcal{G}} \widetilde{\mathrm{Aut}}(E_z) \to \mathcal{A}^* \times_{\mathcal{G}} \mathrm{Aut}(E_z) \tag{9}$$

is a double covering. Now let $\gamma : S^1 \to Z$ be a loop based at z. Pulling back (9) by the smooth map

$$\mathcal{B}^* \to \mathcal{A}^* \times_{\mathcal{G}} \mathrm{Aut}(E_z), \quad [A] \mapsto [A, \mathrm{Hol}_\gamma(A)]$$

yields a double covering of $\mathcal{B}^*$. We will now show that this extends to a double covering $\Xi_\gamma \to U_\gamma$, where $U_\gamma \subset \mathcal{B}$ contains $\mathcal{B}^*$ as well as certain reducibles.

Definition 3.1 **(i)** *Let $U_\gamma \subset \mathcal{B}$ be the subspace consisting of those $[A]$ such that there are two points in $\mathcal{A} \times_{\mathcal{G}} \widetilde{Aut}(E_z)$ lying above $[A, Hol_\gamma(A)] \in \mathcal{A} \times_{\mathcal{G}} Aut(E_z)$.*

(ii) *Let $\Xi_\gamma \subset \mathcal{A} \times_{\mathcal{G}} \widetilde{Aut}(E_z)$ be the subspace consisting of those $[A, g]$ such that $[A] \in U_\gamma$ and $g \in \widetilde{Aut}(E_z)$ is a lift of $Hol_\gamma(A)$.*

Remark: Note that $[A] \in \mathcal{B}$ lies in the complement of U_γ if and only if there exists a $u \in \Gamma_A$ such that u interchanges the two points in $\widetilde{\mathrm{Aut}}(E_z)$ lying above $\mathrm{Hol}_\gamma(A)$, or equivalently, such that $u(z)$ and $\mathrm{Hol}_\gamma(A)$ are both reflections and have perpendicular axes of rotation.

Proposition 3.1 *Let $[A] \in \mathcal{B}$.*

(i) *If A is Abelian, then $[A] \in U_\gamma$.*

(ii) *Let A be twisted reducible and let $\lambda \subset E$ be the 1–eigenspaces of the non-trivial element of Γ_A. Then $[A] \in U_\gamma$ if and only if $\gamma^*\lambda$ is trivial.*

Note that elements of $\mathcal{G}$ are of class C^1 by the Sobolev embedding theorem, hence the subbundle $\lambda \subset E$ in (ii) is of class C^1.

Proof. (i) If $\Gamma_A \approx \mathrm{U}(1)$ then Γ_A is the centralizer of any non-trivial element $x \in \Gamma_A$ with $x^2 \neq 1$. Hence $\mathrm{Hol}_z(A) \subset \Gamma_A$, so $[A] \in U_\gamma$ by the above remark.

(ii) Since A preserves the subbundle λ, the holonomy $\mathrm{Hol}_\gamma(A)$ acts as $\epsilon = \pm 1$ on the fibre λ_z. Therefore, $[A] \in U_\gamma$ if and only if $\epsilon = 1$, or equivalently, if $\gamma^*\lambda$ is trivial. $\square$

Proposition 3.2 *U_γ is an open subset of $\mathcal{B}$, and the canonical projection $\Xi_\gamma \to U_\gamma$ is a double covering.*

Proof. We give a proof which does not require the local slice theorem. After choosing a framing of E_z we can identify the covering (8) with the adjoint representation $\mathrm{Sp}(1) \to \mathrm{SO}(3)$. Fix $A \in \mathcal{A}$ with $[A] \in U_\gamma$ and a lift $q \in \mathrm{Sp}(1)$ of $\mathrm{Hol}_\gamma(A)$. For $\epsilon > 0$ set

$$P_\epsilon := A + \overset{\circ}{D}_\epsilon,$$

where $\overset{\circ}{D}_\epsilon \subset L^p_1(Z;\mathrm{so}(E))$ is the open ϵ–ball about the origin. Let $\pi : \mathcal{A} \to \mathcal{B}$ be the projection. This is an open map, since $\mathcal{B}$ is the quotient of $\mathcal{A}$ with respect to a group action. Hence $\pi(P_\epsilon) \subset \mathcal{B}$ is an open neighbourhood of $[A]$. If $B \in P_\epsilon$ with ϵ sufficiently small then $\mathrm{Hol}_\gamma(B) \cdot \mathrm{Hol}_\gamma(A)^{-1}$ will not be a reflection and so has a unique lift $g(B) \in \mathrm{Sp}(1)$ with positive real part. Then

$$f(B) := g(B)q$$

is a lift of $\mathrm{Hol}_\gamma(B)$. A simple convergence argument shows that if ϵ is sufficiently small and $B \in P_\epsilon$, $u \in \mathcal{G}$ are such that $u(B) \in P_\epsilon$ then

$$f(u(B)) = u \cdot f(B).$$

For such ϵ we have $\pi(P_\epsilon) \subset U_\gamma$, and the map $[B] \mapsto [B, f(B)]$ is a continuous section of Ξ_γ over $\pi(P_\epsilon)$. Changing the sign of f yields a different section and altogether a trivialization of Ξ_γ over $\pi(P_\epsilon)$. $\square$

4 Moduli spaces and twisted reducibles

Let W be any oriented, connected, Riemannian 4–manifold with one cylindrical end $\mathbb{R}_+ \times Y$, where Y is an integral homology sphere. (Thus, the complement of $\mathbb{R}_+ \times Y$ is compact). Let $E \to W$ be an oriented Euclidean 3–plane bundle. Choose a trivialization of $E|_{\mathbb{R}_+ \times Y}$. For any non-degenerate flat connection ρ in the product SO(3)–bundle $E_0 \to Y$ let $M(E,\rho)$ denote the moduli space of instantons in E that are asymptotic to ρ over the end. We briefly recall the construction of this moduli space, following [5, 13]. Choose a smooth reference connection A_{ref} in E whose restriction to the $\mathbb{R}_+ \times Y$ is the pull-back of ρ. Introduce the space

$$\mathcal{A} = \mathcal{A}(E,\rho) := A_{\mathrm{ref}} + L^{p,w}_1(W;\mathrm{so}(E))$$

of Sobolev connections, where w is a small exponential weight as in [13, Subsection 2.1] (which is actually only needed when ρ is reducible). There is a Banach Lie group $\mathcal{G}$ (consisting of certain $L^p_{2,\mathrm{loc}}$ gauge transformations) acting on $\mathcal{A}$, and $M(E,\rho)$ is the subspace of the quotient space $\mathcal{B} := \mathcal{A}/\mathcal{G}$ consisting of all $[A]$ satisfying $F^+_A = 0$.

If $u : Y \to \mathrm{SO}(3)$ then the moduli spaces with limits ρ and $u(\rho)$, resp., can be identified if u is null-homotopic; otherwise the expected dimensions of these moduli spaces differ by $4\deg(u)$. Let $\mathcal{R}_Y$ denote the space of gauge equivalence classes of flat connections in E_0, and let $\mathcal{R}^*_Y$ be the irreducible part of $\mathcal{R}_Y$. It will be convenient to denote a moduli space $M(E,\rho)$ of expected dimension d by $M_{\alpha,d}$, where $\alpha = [\rho] \in \mathcal{R}_Y$. In the particular case when ρ is trivial, however, we will usually label the moduli space by $k = -p_1(E,\rho) \in H^4_c(W;\mathbb{Z}) = \mathbb{Z}$, where $p_1(E,\rho)$ is the relative Pontryagin class. Note that as ρ varies, k runs through a set of the form $k_0 + 4\mathbb{Z}$, $k_0 \in \mathbb{Z}$. Thus, M_k will denote the moduli space with trivial limit and expected dimension

$$\dim M_k = 2k - 3\delta(W),$$

where

$$\delta(W) := 1 - b_1(W) + b^+(W). \tag{10}$$

If M_k is non-empty then for every $[A] \in M_k$ one has

$$8\pi^2 k = \int_W \operatorname{tr}(F_A \wedge F_A) = \int_W |F_A^-|^2 \geq 0. \tag{11}$$

After perturbing the Riemannian metric on W in a small ball we may assume that there is no $[A] \in M_k$, for any $k > 0$, such that A preserves a real line bundle $\lambda \subset E$ with $b^+(W;\lambda) > 0$. (This can be proved along the same lines as the untwisted case [6, Corollary 4.3.15], cf. [18, Lemma 2.4].)

For the remainder of this section assume

$$k > 0, \qquad \delta(W) = 0. \tag{12}$$

Then $M_k = M_{\theta,2k}$, where $\theta \in \mathcal{R}_Y$ is the class of trivial connections. Let $M_k^*, M_k^{\mathrm{red}}, M_k^{\mathrm{tred}}$ be the subsets of M_k consisting of the irreducible, reducible, and twisted reducible points, resp.

Proposition 4.1 *There is a canonical bijection between M_k^{tred} and the set P of equivalence classes of pairs (ℓ, c), where $\ell \to W$ is a non-trivial bundle of infinite cyclic groups, $c \in H^2(W;\ell)$, and such that for $\lambda := \ell \otimes \mathbb{R}$ one has*

$$b^+(W;\ell) = 0, \quad w_1(\lambda)^2 + [c]_2 = w_2(E), \quad c^2 = -k,$$

where $[c]_2$ denotes the image of c in $H^2(W;\mathbb{Z}/2)$.

Here two such pairs (ℓ, c), (ℓ', c') are deemed equivalent if there is an isomorphism $\ell \overset{\approx}{\to} \ell'$ such that $c \mapsto c'$ under the induced isomorphism $H^2(W;\ell) \overset{\approx}{\to} H^2(W;\ell')$.

Proof. (i) To define this bijection, let $[A] \in M_k^{\mathrm{tred}}$. We may assume A is smooth. Since A is not flat, it preserves a unique non-trival rank 1 subbundle $\lambda \subset E$. Let $K = \mathbb{R} \oplus \lambda$ be the corresponding bundle of fields as in Section 2. The orthogonal complement $L \subset E$ of λ is in a canonical way a K–line bundle. The module structure is given as follows: For $x \in W$, $(a, b) \in \mathbb{R} \oplus \lambda_x$, $v \in L_x$ set

$$(a, b) \cdot v := av + b \times v, \tag{13}$$

where $b \times v$ is the cross product in the 3–dimensional, oriented, Euclidean vector space E_x. Let $\ell \subset \lambda$ denote the lattice of vectors of integer length and set $c := \tilde{c}_1(L) \in H^2(W; \ell)$. It is clear that different representatives A of the same point in M_k^{tred} are mapped to equivalent pairs (ℓ, c).

We now verify that (ℓ, c) has the required properties. By choice of metric on W we must have $b^+(W; \ell) = 0$. Furthermore,

$$w_2(E) = w_2(\lambda \oplus L) = w_1(\lambda) \cup w_1(L) + w_2(L) = w_1(\lambda)^2 + [c]_2.$$

Secondly, let B denote the connection in L induced by A. Then F_B takes values in λ, and one has

$$\mathrm{tr}(F_A \wedge F_A) = -2 F_B \wedge F_B \in \Omega^4(W).$$

Since F_B decays exponentially, we obtain

$$\int_W \mathrm{tr}(F_A \wedge F_A) = -2 \int_W F_B \wedge F_B = -8\pi^2 c^2,$$

hence $c^2 = -k$.

(ii) Now suppose $A, A' \in \mathcal{A}$ are smooth connections representing points in M_k^{tred}, and that the corresponding pairs (ℓ, c), (ℓ', c') are equivalent through an isomorphism $f : \ell \xrightarrow{\approx} \ell'$. Let $E = \lambda \oplus L$ and $E = \lambda' \oplus L'$ be the splittings preserved by A and A', resp., and let K, K' be the bundles of fields corresponding to λ, λ', resp. Let $\phi : K \to K'$ be the isomorphism induced by f. By means of ϕ, we turn L' into an Hermitian K–line bundle which we denote by L'_ϕ. It is easy to check that $f_*(\tilde{c}_1(L'_\phi)) = \tilde{c}_1(L')$, so by Proposition 2.4 there is an isomorphism $\psi : L \to L'_\phi$ of Hermitian K–line bundles. Combining $\phi|_\lambda$ and ψ we obtain an isomorphism of Euclidean vector bundles

$$u : E = \lambda \oplus L \to \lambda' \oplus L' = E.$$

To see that u preserves orientations, let $a \in \lambda_x$ and $b \in L_x$ be of unit length. Then $(a, b, a \times b)$ is a positive orthonormal basis for E_x. Under u this is mapped to $(\phi(a), \psi(b), \phi(a) \times \psi(b))$, which is also a positive orthonormal basis.

We may assume A and A' are in temporal gauge. Then L and L' will be translationary invariant over the end $W^+ := \mathbb{R}_+ \times Y$ with respect to the chosen trivialization of $E|_{W^+}$. Let v be the non-trivial element of Γ_A and let $A|_{W^+} = d + a$, where d denotes the product connection. Then over the end one has

$$0 = d_A v = dv + av - va.$$

Since v is translationary invariant over the end and

$$\int_{[t,t+1]\times Y} |a|^p \to 0 \quad \text{as } t \to \infty,$$

we conclude that $dv = 0$ on W^+. The same holds for the non-trivial element of $\Gamma_{A'}$. Hence

$$L|_{W^+} = W^+ \times C, \qquad L'|_{W^+} = W^+ \times C'$$

for some 2–dimensional subspaces $C, C' \subset \mathbb{R}^3$. Now $\psi|_{W^+}$ is given by a smooth map

$$\tilde{\psi} : W^+ \to \mathrm{SO}(C, C'),$$

where $\mathrm{SO}(C, C') \approx S^1$ is one specific component of the space of linear isometries $C \to C'$, the component being determined by the isomorphism $f : \ell \to \ell'$. But every map $C \to S^1$ is null-homotopic, since $H^1(W^+; \mathbb{Z}) = 0$. We may therefore choose the isomorphism ψ such that $\tilde{\psi}$ is constant on $[1, \infty)\times Y$, say. Then $du = 0$ on $[1, \infty)\times Y$, so $u \in \mathcal{G}$. Set $A'' := u^{-1}(A') \in \mathcal{A}$.

Recall that the cross product on E defines a canonical isomorphism $E \overset{\approx}{\to} \mathrm{so}(E)$. Under this isomorphism, the difference $b := A'' - A$ is a 1–form with values in λ. More precisely, $b \in L^{p,w}_1(W; \Lambda^1 \otimes \lambda)$. Moreover,

$$F_{A''} = F_A + db,$$

so $d^+b = 0$. Since $b_1(W; \ell) = 0$ by (2) there is a section $\xi \in L^{p,w}_2(W; \lambda)$ such that $d\xi = b$. Set $v = \exp(\xi)$. Then $v(A'') = A$, so A and A' represent the same point in M^{tred}_k.

(iii) We will now show that every class $[\ell, c] \in P$ is the image of some point $[A] \in M^{\mathrm{tred}}_k$. Define λ, K in terms of ℓ as in Section 2. Choose a K–line bundle $L \to W$ with $\tilde{c}_1(L) = c$. The hypotheses on ℓ, c imply that $\lambda \oplus L$ and E have the same second Stiefel–Whitney class, hence these bundles are isomorphic (see [2, p. 674] and [9, Theorem E.8]); we will identify them. Since L is trivial over the end of W, there is an orthogonal connection A' in E which respects the given splitting and is flat over the end of W. Since $d^+ : \Omega^1(W; \lambda) \to \Omega^+(W; \lambda)$ induces a surjective map $L^{p,w}_1 \to L^{p,w}$ between Sobolev spaces with a small positive weight (cf. the proof of [13, Prop. 5.1.2]), there is an $a \in L^{p,w}_1$ such that $A := A' + a$ satisfies $F^+_A = 0$. Clearly, $[A] \in M^{\mathrm{tred}}_k$ is mapped to $[\ell, c]$. $\square$

Now fix $\ell \to W$ and let P_ℓ be the set of points in P of the form $[\ell, c]$. Suppose $P_\ell \neq \emptyset$ (which implies $b^+(W;\ell) = 0$) and choose a c with $[\ell, c] \in P_\ell$. Let $\mathcal{T}_\ell$ be the torsion subgroup of $H^2(W;\ell)$ and for any $v \in H^2(W;\ell)$ let $\bar{v}$ denote the image of v in $H^2(W;\ell)/\mathcal{T}_\ell$. Set

$$P_c := \{\{r, s\} \subset H^2(W;\ell)/\mathcal{T}_\ell \,|\, r \cdot s = 0;\; r + s = \bar{c}\},$$

where $\{r, s\}$ means the *unordered* set.

Proposition 4.2 $|P_\ell| = |2\mathcal{T}_\ell| \cdot |P_c|$.

Here $|\cdot|$ denotes the cardinality of the given set. Note that $2\mathcal{T}_\ell$ has even order if and only if $H^2(W;\ell)$ contains an element of order 4.

Proof. Let $\tilde{P}_\ell$ be the set of all $v \in H^2(W;\ell)$ such that $[\ell, v] \in P$. Set

$$\alpha : \tilde{P}_\ell \to P_\ell, \quad v \mapsto [\ell, v].$$

Since the only non-trivial automorphism of ℓ is given by multiplication by -1, we have

$$\alpha(v) = \alpha(v') \iff v = \pm v'.$$

Because $k \neq 0$ it follows that α is two-to-one, hence

$$|\tilde{P}_\ell| = 2|P_\ell|.$$

Now let $\tilde{P}_c$ be the set of all ordered pairs (r, s) such that $\{r, s\} \in P_c$. Because $k \neq 0$ one has $r \neq s$ for all such r, s, hence

$$|\tilde{P}_c| = 2|P_c|.$$

It follows from the long exact sequence

$$\cdots \to H^2(W;\ell) \xrightarrow{\cdot 2} H^2(W;\ell) \to H^2(W;\mathbb{Z}/2) \to \cdots \tag{14}$$

(see Section 2) that the map

$$\tilde{P}_\ell \to \tilde{P}_c, \quad v \mapsto \left(\frac{\bar{c} + \bar{v}}{2}, \frac{\bar{c} - \bar{v}}{2}\right)$$

induces a bijection $\tilde{P}_\ell / 2\mathcal{T}_\ell \to \tilde{P}_c$, where $2\mathcal{T}_\ell$ acts on $\tilde{P}_\ell$ by translation, hence

$$|\tilde{P}_\ell| = |2\mathcal{T}_\ell| \cdot |\tilde{P}_c|$$

and the proposition is proved. $\square$

5 Local structure around twisted reducibles

We continue the discussion of the previous section, under the assumptions (12).

We do not know if the twisted reducibles in M_k are regular points of M_k for a generic tubular end metric on W (although there is a generic metric theorem of this kind for closed 4–manifolds, see [18, Lemma 2.4]). However, regularity of these reducibles can be achieved by a simple local perturbation of the instanton equation which is similar in spirit to that used in [3, p. 292]. To describe this perturbation, let $M_k \subset \mathcal{B} = \mathcal{A}/\mathcal{G}$ as in Section 4, and suppose $B \in \mathcal{A}$ satisfies $F_B^+ = 0$ and preserves a splitting $E = \lambda \oplus L$, where λ is a non-trivial real line bundle. Then the non-trivial element of Γ_B acts on any fibre of $\lambda \oplus L$ by $(a, b) \mapsto (a, -b)$. For any $\epsilon > 0$ set

$$S_{0,\epsilon} = \{a \in L_1^{p,w}(W; \mathrm{so}(E)) \,|\, d_B^* a = 0,\ \|a\|_{L_1^{p,w}} < \epsilon\},$$

where the Sobolev norm is defined in terms of B. This norm is equivalent to the corresponding norm defined by the reference connection A_{ref} because of the Sobolev embedding $L_1^p \subset L^\infty$ in $\mathbb{R}^4$. (Recall that we are assuming $p > 4$.) If ϵ is sufficiently small then $S_\epsilon := B + S_{0,\epsilon}$ is a local slice to the action of $\mathcal{G}$. This means, firstly, that there is an open neighbourhood U of $1 \in \mathcal{G}$ such that

$$U \times S_\epsilon \to \mathcal{A}, \quad (u, A) \mapsto u(A)$$

is a diffeomorphism onto an open subset of $\mathcal{A}$, and secondly, that the projection $S_\epsilon/\Gamma_B \to \mathcal{B}$ is injective. Then S_ϵ/Γ_B maps homeomorphically onto an open neighbourhood of $[B]$ in $\mathcal{B}$, and the irreducible part of S_ϵ/Γ_B maps diffeomorphically onto an open subset of $\mathcal{B}^*$. The operator

$$-d_B^* + d_B^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+,$$

acting on forms on W with values in $\mathrm{so}(E) \approx E$, induces a Fredholm operator $\mathcal{D} : L_1^{p,w} \to L^{p,w}$ whose index is the expected dimension of M_k, i.e. $\mathrm{ind}(\mathcal{D}) = 2k > 0$. Therefore, there is a compact operator P such that $\mathcal{D} + P$ is surjective. We will choose such a P of a particular kind. To describe this, first note that

$$\mathcal{D} = \mathcal{D}_\lambda \oplus \mathcal{D}_L, \tag{15}$$

where $\mathcal{D}_\lambda$ and $\mathcal{D}_L$ act on forms with values in λ and L, resp. Now, $\mathcal{D}_\lambda$ is an isomorphism, because λ is non-trivial and $b^+(W; \lambda) = \delta(W) = 0$. Therefore, $\mathcal{D} + P$ is surjective if P is given by

$$Pa = \sum_{j=1}^{r} \langle a, \phi_j \rangle_{L^2} \cdot \omega_j,$$

where r is the dimension of the cokernel of $\mathcal{D}_L$, and $\phi_j \in \Omega^1_c(W;L)$, $\omega_j \in \Omega^+_c(W;L)$ are suitably chosen. Choose a smooth function $\kappa : [0,\infty) \to [0,\infty)$ such that $\kappa(t) = 1$ for $t \leq \epsilon/3$ and $\kappa(t) = 0$ for $t \geq 2\epsilon/3$. For any $a \in S_{0,\epsilon}$ set

$$\mathfrak{p}(B+a) := \kappa(\|a\|_{L^{p,w}_1}) \cdot Pa.$$

Then $\mathfrak{p}$ is a smooth Γ_B–equivariant map $S_\epsilon \to \Omega^+_c(W;\mathrm{so}(E))$. Moreover, $\mathfrak{p}$ extends uniquely to a smooth $\mathcal{G}$–equivariant map $\mathcal{A} \to L^{p,w}(W;\Lambda^+ \otimes \mathrm{so}(E))$ which vanishes outside $\mathcal{G}S_\epsilon$. This extension will also be denoted $\mathfrak{p}$.

The perturbed instanton equation that we have in mind is then

$$F^+_A + \mathfrak{p}(A) = 0, \tag{16}$$

for $A \in \mathcal{A}$. Clearly, the linearization of this equation at B is surjective, since it restricts to $d^+_B + P$ on $\ker d^*_B$. Note that adding the perturbation $\mathfrak{p}$ does not affect the compactness properties of the corresponding moduli space. If we take $\epsilon > 0$ sufficiently small, then the classification of twisted reducibles in Proposition 4.1 is also not affected.

More generally, we may add one such local perturbation $\mathfrak{p}$ for each of a finite number of twisted reducibles in $\mathcal{B}$. Usually, the perturbations will be suppressed from notation.

Having resolved the regularity issue, we now describe the local structure around a regular twisted reducible in M_k.

In the next lemma Z will denote a compact, connected codimension 0 submanifold of W. Consider the double covering $\Xi_\gamma \to U_\gamma$ associated to the bundle $E|_Z$ and a loop $\gamma : S^1 \to Z$ based at $z \in Z$.

Lemma 5.1 *Suppose $[B]$ is a regular point of M_k such that B preserves a non-trivial real line bundle $\lambda \subset E$. Then under the restriction map $R : M^*_k \to U_\gamma$, the pull-back of the double covering $\Xi_\gamma \to U_\gamma$ is trivial over the link of $[B]$ in M_k if and only if $\gamma^*\lambda$ is trivial.*

The fact that $\Xi_\gamma \to U_\gamma$ is a double covering was proved in Proposition 3.2. By the "link" we mean the boundary $\partial N \approx \mathbb{RP}^{2k-1}$ of a compact neighbourhood N of $[B]$ in M_k to be constructed in the proof.

Proof. If $\gamma^*\lambda$ is trivial then $[B] \in U_\gamma$ by Proposition 3.1, so there is a well-defined restriction map

$$\bar{R} : M^*_k \cup \{[B]\} \to U_\gamma.$$

Since $\Xi_\gamma \to U_\gamma$ is locally trivial, $\bar{R}^*\Xi_\gamma$ is trivial on a neighbourhood of $[B]$.

Now suppose $\gamma^*\lambda$ is non-trivial. We may assume $B \in \mathcal{A}$ is smooth. Recall that the kernel K of the operator (15) consists entirely of forms with values in L. Therefore the non-trivial element of Γ_B acts as -1 on K.

For a small $r > 0$ let $D_r \subset K$ be the closed r–ball around the origin with respect to some Γ_B–invariant inner product on K. By the local slice theorem there is a smooth Γ_B–equivariant embedding

$$Q := B + D_r \to \mathcal{A}$$

whose composition with the projection $\mathcal{A} \to \mathcal{B}$ induces a homeomorphism of Q/Γ_B onto a compact neighbourhood N of $[B]$ in M_k.

Let $\tilde{Q} \to Q$ be the pull-back of the double covering (8) under $\mathrm{Hol}_\gamma : Q \to \mathrm{Aut}(E_z)$. Since Q is contractible, $\tilde{Q} \to Q$ is a trivial double covering. There is now a commutative diagram

$$\begin{array}{ccc} \partial\tilde{Q}/\Gamma_B & \to & R^*\Xi_\gamma \\ \downarrow & & \downarrow \\ \partial Q/\Gamma_B & \to & M_k^* \end{array}$$

where the horizontal maps are the embeddings induced by $Q \to \mathcal{A}$. The image of the bottom map is ∂N, so what we need to show is that the left-most map is a non-trivial covering. Since $\gamma^*\lambda$ is non-trivial, $h := \mathrm{Hol}_\gamma(B)$ acts as -1 on λ_z and hence by a reflection on L_z. In a suitable orthogonal basis for E_z the two lifts of $h \in \mathrm{SO}(3)$ to $\mathrm{Sp}(1)$ are $\pm j$, and σ acts on $\mathrm{Sp}(1)$ by conjugation with i. Since $iji^{-1} = -j$, we see that σ interchanges the two points in $\tilde{Q}$ lying above B. Thus we can identify $\tilde{Q} \to Q$ with

$$D^{2k} \times \{\pm 1\} \to D^{2k},$$

where D^{2k} is the unit disk in $\mathbb{R}^{2k}$, and σ acts on $D^{2k} \times \{\pm 1\}$ by $(x, t) \mapsto (-x, -t)$. Restricting to $\partial D^{2k} = S^{2k-1}$ and dividing out by Γ_B we obtain the usual covering $S^{2k-1} \to \mathbb{RP}^{2k-1}$, which is non-trivial. $\square$

6 Abelian instantons

Let M_k be as in Section 4, assuming (12). In this section we use the unperturbed instanton equation.

We will need to deal with both reducibles in M_k and reducibles appearing in weak limits of sequences in M_k. Such reducibles are contained in the set

$$M^{\mathrm{red}} := \coprod_{s \geq 0} M^{\mathrm{red}}_{k-4s}.$$

Let M^{ared} and M^{tred} be the subsets of M^{red} consisting of Abelian and twisted reducibles, resp. If $b^+(W) > 0$ then M^{ared} is empty, by choice of the metric on W. If $b^+(W) = 0$ (in which case $b_1(W) = 1$) then M^{ared} consists of a finite disjoint union of circles. Let $Z \subset W$ be any compact, connected, codimension 0 submanifold and γ a loop in Z. The restriction map

$$R : M^{\text{ared}} \to U_\gamma \subset \mathcal{B}(E|_Z)$$

maps each circle $S \subset M^{\text{ared}}$ onto either a circle or a point, depending on whether $H^1(W;\mathbb{R}) \to H^1(Z;\mathbb{R})$ is non-zero or not. Moreover, the double covering $\Xi_\gamma \to U_\gamma$ pulls back to a trivial covering of any S if and only if the class in $H_1(W;\mathbb{Z})/\text{torsion}$ represented by γ is divisible by 2; however, we will make no use of this. Note that different circles S have disjoint images in $\mathcal{B}(E|_Z)$, by the unique continuation property of self-dual closed 2–forms (applied to the curvature forms).

7 Three lemmas on Banach manifolds

The results of this section will be used in Section 8.2 below.

Lemma 7.1 *Let B be a smooth (Hausdorff) Banach manifold which admits smooth partitions of unity. Let $L \to B$ be a smooth real 2–plane bundle. If L admits a continuous non-vanishing section, then it admits a smooth non-vanishing section.*

Proof. Of course, this is well known if B is finite-dimensional (and at least in that case it holds for bundles of any finite rank). For general B one can use the following Čech cohomology argument:

Choose a smooth Euclidean metric on L. Set $\lambda := \Lambda^2 L$ and let $K := \mathbb{R} \oplus \lambda$ be the associated bundle of fields as in Section 2. Then L has a canonical structure as a smooth K–line bundle as defined in (13). Clearly, L is trivial as a smooth (resp. continuous) K–line bundle (meaning $L \approx K$) if and only if L admits a non-vanishing smooth (resp. continuous) section.

Let $\mathcal{K}^*$ and $\mathcal{K}^*_\infty$ denote the sheaves of continuous and smooth sections of K^*, resp. Then L is determined up to smooth isomorphism by its class in $H^1(B;\mathcal{K}^*_\infty)$. But the inclusion $\mathcal{K}^*_\infty \to \mathcal{K}^*$ induces an isomorphism

$$H^1(B;\mathcal{K}^*_\infty) \xrightarrow{\approx} H^1(B;\mathcal{K}^*),$$

as is easily seen by considering the morphism between the exponential short exact sequences for $\mathcal{K}^*_\infty$ and $\mathcal{K}^*$ (see (7)). $\square$

In the following two lemmas, B will be a metric space and R a compact subspace. The open subspace $B^* := B \setminus R$ of B will have the structure of a smooth Banach manifold admitting smooth partitions of unity.

Lemma 7.2 *Suppose R is a finite set. Let $\Theta \to B$ be a Euclidean (real) line bundle. Let $\Theta|_{B^*}$ have the obvious smooth structure. Then there exists a smooth section of $\Theta|_{B^*}$ which is nowhere zero in $B^* \cap V$ for some neighbourhood V of R in B.*

Proof. Choose an open neighbourhood N of R and a section σ of $\Theta|_N$ which has (pointwise) unit length. Then σ is smooth in $N \cap B^*$. Since B is a normal space there are disjoint open neighbourhoods V, V' of R and $B \setminus N$, resp. Using a smooth partition of unity of B^* subordinate to the open cover $\{V', B^* \cap N\}$ one can construct a smooth section s of Θ over B^* which agrees with σ on $B^* \cap V$. In particular, s is nowhere zero in $B^* \cap V$. $\square$

Lemma 7.3 *Suppose R is the disjoint union of three sets,*

$$R = R_0 \sqcup R_1 \sqcup R_2,$$

where R_1 and R_2 are finite sets and R_0 is a finite disjoint union of subspaces each of which is homeomorphic to a circle. For $i = 1, 2$ let $\Theta_i \to B^* \cup R_0 \cup R_i$ be a Euclidean line bundle. Let $\hat{\Theta}$ be the direct sum of the restrictions of Θ_1 and Θ_2 to $B^* \cup R_0$. Then there exists a smooth section of $\hat{\Theta}$ over B^* which is nowhere zero in $B^* \cap V$ for some neighbourhood V of R in B.

Proof. Choose pairwise disjoint open sets H_0, H_1, H_2 in B such that $R_i \subset H_i$ for $i = 0, 1, 2$. It is easy to see that $\hat{\Theta}|_{R_0}$ admits a non-vanishing section σ. Since R_0 is compact we can cover R_0 by finitely may open sets U_j in H_0 such that $\hat{\Theta}|_{U_j}$ is trivial for each j. By the Tietze extension theorem there is a section σ_j of $\hat{\Theta}|_{U_j}$ which agrees with σ on $R_0 \cap U_j$. Patching together the sections σ_j by means of a partition of unity yields a section $\tilde{\sigma}$ of Θ over $U := \cup_j U_j$ such that $\tilde{\sigma} = \sigma$ on R_0. Then the locus N_0 where $\tilde{\sigma} \neq 0$ is an open neighbourhood of R_0 in B. By Lemma 7.1 there exists a non-vanishing smooth section s_0 of $\hat{\Theta}$ over $B^* \cap N_0$.

For $i = 1, 2$ choose a unit length section τ_i of Θ_i over some open neighbourhood N_i of R_i in H_i. Combining τ_i with the zero-section of Θ_{3-i} yields a smooth non-vanishing section s_i of $\hat{\Theta}$ over $B^* \cap N_i$.

Set $N := N_0 \cup N_1 \cup N_2$, which is an open neighbourhood of R in B. By means of a smooth partion of unity as in the proof of Lemma 7.2 we can then construct a smooth section s of $\hat{\Theta}|_{B^*}$ which agrees with s_i in $B^* \cap V \cap N_j$ for some neighbourhood V of R in N. In particular, s is nowhere zero in $B^* \cap V$. $\square$

8 Proof of theorem

Assuming the hypotheses of the theorem are satisfied we will show that $\delta_0 \neq 0$.

For any ℓ set $F_\ell := H^2(X;\ell)/\text{torsion}$. If $b^+(X;\ell) = 0$ let $D_\ell \subset F_\ell$ be the subgroup generated by vectors of square -1. Let $\hat{F}_\ell \subset F_\ell$ be the orthogonal complement of D_ℓ, so that

$$F_\ell = D_\ell \oplus \hat{F}_\ell. \tag{17}$$

By assumption there is a non-trivial ℓ such that $\hat{F}_\ell \neq 0$ and $H^2(X;\ell)$ contains no element of order 4. Fix such an ℓ. Note that there is a class $x \in \hat{F}_\ell$ with $x^2 \not\equiv 0 \mod 4$. (Proof: Since $\hat{F}_\ell$ is unimodular we can find elements $a, b \in \hat{F}_\ell$ with $a \cdot b = 1$. If $a^2, b^2 \equiv 0 \mod 4$ then $(a+b)^2 \equiv 2 \mod 4$.) Let k be the smallest (positive) integer $\not\equiv 0 \mod 4$ such that there exists an $x \in \hat{F}_\ell$ with $x^2 = -k$.

8.1 Reduction to $\delta(X) = 0$

By (2) we have $\delta(X) \leq 0$. We will now reduce the remaining part of the proof to the case $\delta(X) = 0$.

Lemma 8.1 *Let N be any compact, connected oriented smooth 4–manifold, and let C be any embedded circle in int(N) which represents a non-zero class in $H_1(N;\mathbb{Z}/2)$. Let N' be obtained from N by surgery on C. Then*

$$\tau(N') + b^+(N') = \tau(N) + b^+(N).$$

Here τ is the invariant defined just before Theorem 1.1.

Proof. Defining $\delta(N)$ as in (10) we have

$$\begin{aligned} b_1(N;\mathbb{Z}/2) + \delta(N) &= (b_1(N) + \tau(N)) + (1 - b_1(N) + b^+(N)) \\ &= \tau(N) + b^+(N) + 1. \end{aligned}$$

Now, $b_1(N;\mathbb{Z}/2)$ drops by one by surgery on C, whereas $\delta(N)$ increases by one by surgery on *any* circle in $\text{int}(N)$. (A highbrow proof of the latter statement applies the excision principle for indices to the elliptic operator $d^* + d^+ : \Omega^1 \to \Omega^0 \oplus \Omega^+$ on some close-up V of N, recalling that the index of that operator is $-\delta(V)$.) □

Every element of $H_1(X;\ell)$ can be represented by an embedded, oriented circle C in the interior of X together with a trivialization of $\ell|_C$. Set

$\mathfrak{d} := -\delta(X)$ and let X' be obtained from X by performing surgery on a collection of disjoint oriented circles $C_1, \dots, C_{\mathfrak{d}}$ in $\mathrm{int}(X)$ which, together with a trivialization of ℓ over $\tilde{C} := \cup_j C_j$, represent a basis for $H_1(X;\ell)$/torsion. Let $\ell' \to X'$ be the bundle obtained by trivially extending $\ell|_{X\setminus\tilde{C}}$. Then $b_1(X';\ell') = 0$, and there is a canonical isomorphism $H^2(X;\ell) \to H^2(X';\ell')$ which induces an isomorphism between the intersection forms. It follows from the long exact sequence

$$\cdots \to H_1(X;\ell) \xrightarrow{\cdot 2} H_1(X;\ell) \to H_1(X;\mathbb{Z}/2) \to \cdots$$

that the circles C_j represent linearly independent classes in $H_1(X;\mathbb{Z}/2)$, so by Lemma 8.1 the invariant $\tau + b^+$ takes the same value on X and X'.

We have shown that X', ℓ' satisfy all the hypotheses of the theorem, and that ℓ', k satisfy the same minimality condition as ℓ, k. We may therefore from now on assume that $b_1(X;\ell) = 0 = \delta(X)$. This implies that

$$b := b_1(X;\mathbb{Z}/2) = \tau(X) + (b^+(X) + 1) \leq 3, \tag{18}$$

where we have used assumption (3) of the theorem.

8.2 Choosing the sections

Let W be the result of attaching a half-infinite cylinder $[0,\infty) \times Y$ to X. We extend the bundle $\ell \to X$ to all of W and, abusing notation, denote the new bundle also by ℓ. Choose a $c \in H^2(W;\ell)$ whose image in F_ℓ lies in $\hat{F}_\ell$ and such that $c^2 = -k$. Define λ, K in terms of ℓ as in Section 2 and let $L \to W$ be a Hermitian K–line bundle with $\tilde{c}_1(L) = c$. Then $E := \lambda \oplus L$ is an oriented, Euclidean 3–plane bundle over W.

We will use the same notation for moduli spaces associated to E as in Section 4. Choose a Riemannian metric on W which is on product form on the end and which is generic as assumed in the beginning of Section 4.

Let M_k^λ be the set of all $[A] \in M_k$ such that A preserves a subbundle of E isomorphic to λ. After perhaps perturbing the instanton equation as in Section 5 we may assume that every element of M_k^λ is a regular point in M_k.

We also add holonomy perturbations over the end of W corresponding to a small generic perturbation of the Chern–Simons functional over Y (which is in general needed to construct the Floer homology of Y), as well as small holonomy perturbations obtained from a finite number of thickened loops in W. (In order not to obscure the main ideas, we usually ignore holonomy perturbations in this paper.)

Let $M_k^{\#}$ be obtained from M_k by deleting the interior of a small compact neighbourhood N_ω of every $\omega \in M_k^\lambda$, where N_ω is as constructed in the proof of Lemma 5.1. Let M_k^- be the irreducible part of $M_k^{\#}$.

We are going to cut down M_k^- to a 1–manifold with boundary in the following way. For $i = 1, \dots, 2k-1$ let $Z_i' \subset W$ be a compact, connected codimension 0 submanifold and $\gamma_i : S^1 \to Z_i'$ a loop. Let $\Theta_{\gamma_i} \to U_{\gamma_i}$ be the real line bundle associated to the double covering $\Xi_{\gamma_i} \to U_{\gamma_i}$. Let s_i' be a smooth section of Θ_{γ_i} over the irreducible part of U_{γ_i} and set

$$\hat{M}_k := \{\omega \in M_k^- \,|\, s_i'(\omega|_{Z_i'}) = 0 \text{ for } i = 1, \dots, 2k-1\}.$$

For generic sections s_i' the space $\hat{M}_k$ will be a smooth 1–manifold with boundary (see [21]). We will show that for a suitable choice of loops and sections the manifold $\hat{M}_k$ will have an odd number of boundary points and no ends coming from reducibles (i.e. points or circles in M^{red}). We briefly outline how this will be achieved.

Consider the set

$$Q := \{w \in H^1(W; \mathbb{Z}/2) \,|\, (\gamma_i)^* w \neq 0 \text{ for } i = 1, \dots, 2k-1\}.$$

If $w_1(\lambda) \in Q$ then, as we will see in Section 8.3, $\hat{M}_k$ will have an odd number of boundary points. If $w_1(\lambda)$ is the *unique* point in Q then the sections s_i' can be chosen so that $\hat{M}_k$ has no ends associated to twisted reducibles in M_k. Note that $|Q| = 1$ is indeed possible, since $b \leq 3 \leq 2k-1$.

To avoid ends of $\hat{M}_k$ associated to circles in M^{ared} we choose $Z_{2j-1}' = Z_{2j}'$ for $j = 1, \dots, k-1$ and exploit the fact that the direct sum of two real line bundles admits a non-vanishing section over any circle (see Lemma 7.3); furthermore, we take the $Z_j := Z_{2j-1}'$, $j = 1, \dots, k$ to be disjoint.

Finally, to arrange, in addition, that there are no ends in $\hat{M}_k$ coming from twisted reducibles in lower strata M_ℓ, $\ell < k$ we rotate the classes represented by the γ_i in a suitable way.

We now make precise the choice of loops and sections. Choose a basis $\{e_1, \dots, e_b\}$ for $H_1(W; \mathbb{Z}/2)$ such that $\langle e_h, w_1(\lambda) \rangle = 1$ for each h. Also choose

- disjoint compact, connected, codimension 0 submanifolds $Z_1, \dots, Z_k$ of W,
- two loops $\gamma_{2j-1}, \gamma_{2j}$ in Z_j for $j = 1, \dots, k-1$,
- a loop γ_{2k-1} in Z_k

such that γ_i represents e_h when $i \equiv h \mod b$. For instance, Z_k may be a closed tubular neighbourhood of an embedded circle in W, whereas for $j = 1, \dots, k-1$ one can take Z_j to be an internal connected sum of two such tubular neighourhoods.

We will write $U_i, \Xi_i, \mathcal{B}_j$ instead of $U_{\gamma_i}, \Xi_{\gamma_i}, \mathcal{B}(E|_{Z_j})$. Let $\mathcal{B}_j^*$ denote the irreducible part of $\mathcal{B}_j$. Let $\Theta_i \to U_i$ be the real line bundle associated to the double covering $\Xi_i \to U_i$.

For $j = 1, \dots, k-1$ let $R_j \subset \mathcal{B}_j$ be the image of M^{red} under the restriction map. We have observed that R_j is the disjoint union of finitely many circles and a finite set. Note that, by Lemma 3.1, all these circles are contained in $U_{2j-1} \cap U_{2j}$. Let $\hat{\Theta}_j$ be the direct sum of the restrictions of Θ_{2j-1} and Θ_{2j} to $\mathcal{B}_j^*$. Let s_j be a generic smooth section of $\hat{\Theta}_j$ which is nowhere zero on $\mathcal{B}_j^* \cap V_j$ for some neighbourhood V_j of the compact set $R_j \cap (U_{2j-1} \cup U_{2j})$ in $\mathcal{B}_j$. The existence of sections of this kind follows from Lemma 7.3. (The fact that $\mathcal{B}_j$ is metrizable was pointed out in Section 3.)

Let $R_k \subset \mathcal{B}_k$ be the image of M^{tred} under the restriction map. Let s_k be a generic smooth section of Θ_{2k-1} over $\mathcal{B}_k^*$ which is nowhere zero on $\mathcal{B}_k^* \cap V_k$ for some neighbourhood V_k of $R_k \cap U_{2k-1}$ in $\mathcal{B}_k$. The existence of sections of this kind follows from Lemma 7.2.

8.3 Ends and boundary points

Set

$$\hat{M}_k := \{\omega \in M_k^- \mid s_j(\omega|_{Z_j}) = 0 \text{ for } j = 1, \dots, k\}.$$

Modulo 2 the number of boundary points of the smooth 1–manifold $\hat{M}_k$ is

$$\#\partial \hat{M}_k \equiv \sum_\omega \langle [\partial N_\omega], e(\Theta_\omega) \rangle \equiv |P_\ell| \equiv 1 \mod 2, \tag{19}$$

where e denotes the Euler class with coefficients in $\mathbb{Z}/2$ and Θ_ω is the direct sum of the pull-backs of the line bundles $\Theta_1, \dots, \Theta_{2k-1}$ to the boundary $\partial N_\omega \approx \mathbb{RP}^{2k-1}$ of N_ω.

To prove the second congruence in (19), note that Θ_i pulls back to a non-trivial bundle over each ∂N_ω by Lemma 5.1. Since the Euler class is multiplicative under finite direct sums, we conclude that each term in the sum in (19) is one. The last congruence in (19) follows from Proposition 4.2 because $|P_c| = 1$ by the minimality property of k, and $|2\mathcal{T}_\ell|$ is odd since by assumption $H^2(W;\ell)$ contains no element of order 4.

It remains to determine the ends of $\hat{M}_k$. For any moduli space $M_{\alpha,d}$ with α irreducible set

$$\hat{M}_{\alpha,d} := \{\omega \in M_{\alpha,d} \mid s_j(\omega|_{Z_j}) = 0 \text{ for } j = 1, \dots, k\}.$$

Proposition 8.1 *Any sequence in $\hat{M}_k$ has a subsequence which either converges in $\hat{M}_k$ or chain-converges to an element of*

$$\hat{M}_{\alpha,2k-1} \times \check{M}(\alpha,\theta)$$

for some $\alpha \in \mathcal{R}_Y^*$, where $M(\alpha,\theta)$ is the one-dimensional moduli space over $\mathbb{R} \times Y$ with limits α at $-\infty$ and θ at ∞, and $\check{M} := M/\mathbb{R}$.

Proof of proposition: Let $\{[A_n]\}$ be a sequence in $\hat{M}_k$. After passing to a subsequence we may assume that $\{[A_n]\}$ chain-converges weakly in the sense of [5]. Let $([A], x_1, \ldots, x_q)$ be the weak limit over W, where $[A] \in M_{\alpha,d}$, $\alpha \in \mathcal{R}_Y$, and $x_1, \ldots, x_q \in W$, $q \geq 0$. We are going to show that A must be irreducible. First we establish the following lemma.

Lemma 8.2 *If A is reducible then there is a $j \in \{1, \ldots, k\}$ with the following two properties:*

(i) *Z_j contains none of the points $x_1, \ldots, x_q$.*

(ii) *$[A|_{Z_j}] \in V_j$.*

Proof of lemma: If A is reducible, then $[A] \in M^{\text{red}}_{k-4s}$ for some non-negative integer s. Observe that

$$q \leq s < \frac{k}{4} < k - 1. \tag{20}$$

The second inequality holds because $k - 4s \geq 0$ by (11) and we have chosen $k \not\equiv 0 \mod 4$. Hence there is certainly a $j < k$ satisfying (i).

Case 1: A Abelian. Then (ii) is satisfied for any $j < k$, so the lemma holds in this case.

Case 2: A twisted reducible. Let $E = \lambda' \oplus L'$ be the splitting preserved by A, where λ' is a non-trivial real line bundle.

Case 2a: $\lambda' \approx \lambda$. We will show that this cannot occur. Let $\ell' \subset \lambda'$ be the lattice of vectors of integer length and set $c' := \tilde{c}_1(L') \in H^2(W;\ell')$. Choose an isomorphism $f : \ell' \xrightarrow{\approx} \ell$ and set $\zeta := f_* c' \in H^2(W;\ell)$. Since $[\zeta]_2 = [c']_2 = [c]_2$ (the last equality by Proposition 4.1) it follows from the exact sequence (14) that there is an $x \in H^2(W;\ell)$ such that $\zeta = c + 2x$. For any $v \in H^2(W;\ell)$ let $\bar{v}$ be the image of v in F_ℓ and let $\hat{v}$ be the component of $\bar{v}$ in $\hat{F}_\ell$ with respect to the splitting (17). Since $\bar{c} \in \hat{F}_\ell$ by assumption, we have $\hat{\zeta} = \bar{c} + 2\hat{x}$, so $(\hat{\zeta})^2 \equiv c^2 = -k \not\equiv 0 \mod 4$. Hence $-(\hat{\zeta})^2 \geq k$ by the minimality of k, so

$$k - 4s = -\zeta^2 \geq -(\hat{\zeta})^2 \geq k.$$

Thus, $s = 0$ and $[A] \in M_k$. It follows that the sequence $\{[A_n]\}$ converges in M_k (see [5]). Since $M_k^{\#}$ is a closed subset of M_k, we must have $[A] \in M_k^{\#}$. This is a contradiction, since $M_k^{\#}$ was obtained from M_k by deleting neighbourhoods of all twisted reducibles preserving a line bundle isomorphic to λ.

Case 2b: $\lambda' \not\approx \lambda$. Then $b \geq 2$.

Case 2b1: $b = 2$. Then for $h = 1$ or 2 we have

$$1 = \langle e_h, w_1(\lambda) + w_1(\lambda') \rangle = 1 + \langle e_h, w_1(\lambda') \rangle,$$

so $\langle e_h, w_1(\lambda') \rangle = 0$. As observed in the beginning of the proof we can find a $j < k$ satisfying (i). For $i = 2j - 1$ or $2j$ the loop γ_i represents e_h, in which case $(\gamma_i)^* \lambda'$ is trivial. This in turn implies $[A|_{Z_j}] \in U_i$ by Proposition 3.1, so $[A|_{Z_j}] \in V_j$.

Case 2b2: $b = 3$. Set

$$m := \left[\frac{k-1}{3} \right].$$

Case 2b2a: $m = 0$. Then $k \leq 3$, so $q = 0$ by (20). The same argument as in the case $b = 2$ shows that $(\gamma_i)^* \lambda'$ is trivial for some $i \in \{1, 2, 3\}$, hence $[A|_{Z_j}] \in V_j$ for $j = 1$ or 2.

Case 2b2b: $m \geq 1$. Choose h with $\langle e_h, w_1(\lambda') \rangle = 0$. Then for at least $2m$ integers $j \in \{1, \ldots, k-1\}$ one of the loops γ_{2j-1} or γ_{2j} will represent e_h, in which case (ii) holds. Because

$$q < \frac{k}{4} < 2m,$$

one can choose j such that (i) holds as well. $\square$

Lemma 8.3 *A is irreducible.*

Proof of lemma: Assume to the contrary that A were reducible, and let j satisfy the two properties of Lemma 8.2. Then

$$[A_n|_{Z_j}] \to [A|_{Z_j}] \quad \text{in } \mathcal{B}_j \text{ as } n \to \infty.$$

But V_j is open, so for sufficently large n we have $[A_n|_{Z_j}] \in \mathcal{B}_j^* \cap V_j$ and therefore $s_j(A_n|_{Z_j}) \neq 0$. This contradicts $[A_n] \in \hat{M}_k$ and the lemma is proved. $\square$

We can now complete the proof of the proposition. First suppose $[A] \in M_k$, which implies $q = 0$. Then $\{[A_n]\}$ converges in M_k, so $[A] \in M_k^{\#}$. But $[A]$ is irreducible, so $[A] \in \hat{M}_k$.

Now suppose $[A] \notin M_k$. Then

$$d \leq \min(2k - 1, 2k - 8q). \tag{21}$$

Set

$$J := \{j \in \{1, \dots, k-1\} \mid Z_j \text{ contains none of the points } x_1, \dots, x_q\}.$$

Then $s_j(A|_{Z_j}) = 0$ for every $j \in J$. Since the sections s_j are generic, we must have $2|J| \leq d$, where $|J|$ denotes the cardinality of the set J. Combining this with (21) yields

$$2|J| \leq 2k - 8q.$$

Setting $t := k - 1 - |J|$ we deduce

$$4q \leq t + 1 \leq q + 1,$$

so $q = 0$. Hence $s_j(A|_{Z_j}) = 0$ for $j = 1, \dots, k$, so $d \geq 2k - 1$. Combining this with (21) we obtain $d = 2k - 1$. This is only possible when α is irreducible, so the proposition is proved. $\square$

We can now complete the proof of the theorem. An argument similar to the proof of Proposition 8.1 shows that $\hat{M}_{\alpha,2k-1}$ is compact, hence a finite set (since it is 0–dimensional). By gluing theory the number of ends of $\hat{M}_k$ is δh, where

$$h := \sum_{\alpha} [\#\hat{M}_{\alpha,2k-1}]\alpha \in \mathrm{CF}^4(Y) \otimes \mathbb{Z}/2.$$

The proof of the proposition applied to moduli spaces $M_{\beta,2k}$ with β irreducible shows that h is a cocycle.

Since the 1–manifold $\hat{M}_k$ has an odd number of boundary points, it must also have an odd number of ends, so $\delta_0([h]) = 1$. $\square$

References

[1] G. E. Bredon. *Sheaf Theory. Second edition.* Springer, 1997.

[2] A. Dold and H. Whitney. Classification of oriented sphere bundles over a 4–complex. *Annals of Math.*, 69(3):667–677, 1959.

[3] S. K. Donaldson. An application of gauge theory to four dimensional topology. *J. Diff. Geom.*, 18:279–315, 1983.

[4] S. K. Donaldson. The orientation of Yang–Mills moduli spaces and 4–manifold topology. *J. Diff. Geom.*, 26:397–428, 1987.

[5] S. K. Donaldson. *Floer Homology Groups in Yang–Mills Theory*. Cambridge University Press, 2002.

[6] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford University Press, 1990.

[7] R. Fintushel and R. J. Stern. Definite 4–manifolds. *J. Diff. Geom.*, 28:133–141, 1988.

[8] A. Floer. An instanton invariant for 3–manifolds. *Comm. Math. Phys.*, 118:215–240, 1988.

[9] D. S. Freed and K. K. Uhlenbeck. *Instantons and Four-Manifolds*. MSRI Publications. Springer-Verlag, second edition, 1991.

[10] M. Freedman and F. Quinn. *Topology of 4–manifolds*, volume 39 of *Princeton Mathematical Series*. Princeton University Press, 1990.

[11] S. Friedl, I. Hambleton, P. Melvin, and P. Teichner. Non-smoothable four-manifolds with infinite cyclic fundamental group. *Int. Math. Res. Not.*, (11), 2007.

[12] K. A. Frøyshov. Equivariant aspects of Yang–Mills Floer theory. *Topology*, 41:525–552, 2002.

[13] K. A. Frøyshov. *Compactness and gluing theory for monopoles*, volume 15 of *Geometry & Topology Monographs*. Geometry & Topology Publications, 2008.

[14] M. Furuta. Monopole equation and the $\frac{11}{8}$–conjecture. *Math. Res. Lett.*, 8:279–291, 2001.

[15] I. Hambleton. Intersection forms, fundamental groups and 4-manifolds. In *Proceedings of Gökova Geometry-Topology Conference 2008*, pages 137–150, 2009.

[16] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[17] J. L. Kelley. *General Topology*. Van Nostrand, 1955.

[18] P. B. Kronheimer and T. S. Mrowka. Embedded surfaces and the structure of Donaldson's polynomial invariants. *J. Diff. Geom.*, 41:573–734, 1995.

[19] S. Lang. *Differential Manifolds. Second edition.* Springer-Verlag, 1985.

[20] N. Nakamura. $\mathrm{Pin}^-(2)$–monopole equations and intersection forms with local coefficients of 4–manifolds. arXiv:1009.3624.

[21] S. Smale. An infinite dimensional version of Sard's theorem. *Am. J. Math.*, 87:861–866, 1965.

[22] E. H. Spanier. *Algebraic Topology. Corrected reprint.* Springer, 1981.

[23] E. H. Spanier. Singular homology and cohomology with local coefficients and duality for manifolds. *Pacific J. Math.*, 160:165–200, 1993.

[24] A. Teleman. Harmonic sections in sphere bundles, normal neighborhoods of reduction loci, and instanton moduli spaces on definite 4–manifolds. *Geometry & Topology*, 11:1681–1730, 2007.

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