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Stochastic Control for Mean-Field Stochastic Partial Differential Equations with Jumps

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Abstract We study optimal control for mean-field stochastic partial differential equations (stochastic evolution equations) driven by a Brownian motion and an independent Poisson random measure, in case of *partial information* control. One important novelty of our problem is represented by the introduction of *general mean-field* operators, acting on both the controlled state

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process and the control process. We first formulate a sufficient and a necessary maximum principle for this type of control. We then prove the existence and uniqueness of the solution of such general forward and backward mean-field stochastic partial differential equations. We apply our results to find the explicit optimal control for an optimal harvesting problem.

Keywords Mean-field stochastic partial differential equation · optimal control · mean-field backward stochastic partial differential equation · stochastic maximum principles.

Mathematics Subject Classification (2000) 60H15 · 93E20 · 35R60

1 Introduction

Over the last years, there has been a growing interest in mean-field (forward and backward) stochastic differential equations (SDEs), as well as associated control problems, due to their various appplications in economics, finance or physics. Many studies are devoted to this topic, we refer the interested reader to [1–6]. Compared to the abundant literature on mean-field SDEs, stochastic *partial* differential equations (SPDEs) of mean-field type have received very little attention. To the best of our knowledge, the only paper that deals with the optimal control of mean-field SPDEs is [7]. Our paper extends [7] in several directions: (i) we consider a more *general mean-field operator*; (ii) we introduce an additional *general mean-field operator* which acts on the control process; (iii) we add jumps; (iv) we study the optimal control problem in the case of *partial information*.

More precisely, we provide necessary and sufficient conditions for the optimality of a control in case of *partial information*, as well as results regarding the existence and the uniqueness of the solution for forward and backward mean-field stochastic partial differential equations with a *general mean-field operator*.

The paper is organized as follows: in Section 2, we show the sufficient and necessary maximum principles for optimal control with *partial information* in the case of a process described by a *mean-field* stochastic *partial* differential equation (in short mean-field SPDE) driven by a Brownian motion and a Poisson random measure. The drift and the diffusion coefficients, as well as the performance functional, depend not only on the state and the control, but also on the distribution of the state process and of the control process. We apply these results to solve explicitly an optimal harvesting problem given as a motivating example of our study. In Section 3, we investigate the existence and the uniqueness of the solution to *forward* and *backward* mean-field SPDEs with jumps and a *general mean-field operator*. In Section 4, we present the conclusions of the paper.

2 Maximum Principles for Optimal Control with Partial Information of General Mean-Field SPDEs with Jumps

2.1 A Motivating Example

As a motivation for our study, we consider the following **optimal harvesting** problem: Suppose we model the density $Y(t,x)$ of a fish population in a lake

D at time t and at point $x \in D$ by an equation of the form:

$$dY(t,x) = \mathbf{E}[Y(t,x)]b(t,x)dt + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}Y(t,x)dt + Y(t,x)\sigma(t,x)dW_t$$
$$+ Y(t,x)\int_{\mathbb{R}^*}\theta(t,x,e)\tilde{N}(dt,de).$$
$$Y(0,x) = y_0(x),\ x \in D, \tag{1}$$

where D is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, and $y_0(x), b(t,x), \sigma(t,x), \theta(t,x,e)$ are given bounded deterministic functions. Here W_t is a Brownian motion and $\tilde{N}(dt,de) = N(dt,de) - \nu(de)dt$ is an independent compensated Poisson random measure, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, P)$. We may heuristically regard (1) as a limit as $n \to \infty$ of a large population interacting system of the form

$$dy^{j,n}(t,x) = \left[\frac{1}{n}\sum_{l=1}^{n} y^{l,n}(t,x)\right] b(t,x)dt + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}y^{j,n}(t,x)dt$$
$$+ y^{j,n}(t,x)\sigma(t,x)dW_t + y^{j,n}(t,x)\int_{\mathbb{R}^*}\theta(t,x,e)\tilde{N}(dt,de),\ = 1,2,...,n$$
$$y^{j,n}(t,x)(0,x) = y_0(x), \tag{2}$$

where we have divided the whole lake into a grid of size n and $y^{j,n}(t,x)$ represents the density in box j of the grid. Now suppose we introduce a harvesting-rate process $u(t,x)$. The density of the corresponding population $Y(t,x) = Y^u(t,x)$ is thus modeled by a controlled mean-field stochastic partial differential equation with jumps of the form:

$$dY(t,x) = \mathbf{E}[Y(t,x)]b(t,x)dt + \frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}Y(t,x)dt + Y(t,x)\sigma(t,x)dW_t$$
$$+Y(t,x)\int_{\mathbb{R}^*}\theta(t,x,e)\tilde{N}(dt,de) - Y(t,x)u(t,x)dt. \tag{3}$$

The *performance functional* is assumed to be of the form

$$J(u) = \mathbf{E}\left[\int_0^T \int_D \log(Y(t,x)u(t,x))dxdt + \int_D \alpha(x)Y(T,x)dx\right]. \quad (4)$$

This may be regarded as the expected total logarithmic utility of the harvest up to time T plus the value of the remaining population at time T.

The problem is to find u^* so that

$$J(u^*) = \sup_{u\in\mathcal{A}} J(u), \quad (5)$$

where $\mathcal{A}$ represents the set of *admissible* controls. This process $u^*(t,x)$ is called an optimal harvesting rate. This is an example of an optimal control problem of a mean-field stochastic reaction-diffusion equation.

We will return to this example is Subsection 3.2.

2.2 Framework and Formulation of the Optimal Control Problem

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a filtered probability space. Let W be a one-dimensional Brownian motion. Let $\mathbb{R}^* := \mathbb{R}\setminus\{0\}$ and $\mathcal{B}(\mathbb{R}^*)$ be its Borel σ-field. Suppose that it is equipped with a σ-finite positive measure ν, satisfying $\int_{\mathbb{R}^*} |e|^2\nu(de) < \infty$ and let $N(dt,de)$ be a independent Poisson random measure with compensator $\nu(de)dt$. We denote by $\tilde{N}(dt,de)$ its compensated process, defined as $\tilde{N}(dt,de) = N(dt,de) - \nu(de)dt$. We assume that D is a bounded domain in $\mathbb{R}$. We introduce the following notation:

- $\mathbf{L}^2(\mathbf{P})$:= the set of random variables X so that $\mathbf{E}[|X|^2] < \infty$.
- $\mathbf{L}^2(\mathbb{R})$:= the set of measurable functions $k : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\int_{\mathbb{R}} k^2(x)dx < \infty$.

- $\mathbf{H}^2$:= the set of real-valued predictable processes $Z(t,x)$ with $\mathbf{E}[\int_0^T \int_D Z^2(t,x)dxdt] < \infty$.
- $\mathbf{L}^2_\nu$:= the set of measurable functions $l : (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ so that $\|l\|^2_{\mathbf{L}^2_\nu} := \int_{\mathbb{R}^*} l^2(e)\nu(de) < \infty$. The set $\mathbf{L}^2_\nu$ is a Hilbert space equipped with the scalar product $\langle l, l' \rangle_\nu := \int_{\mathbb{R}^*} l(e)l'(e)\nu(de)$ for all $l, l' \in \mathbf{L}^2_\nu \times \mathbf{L}^2_\nu$.
- $\mathbf{H}^2_\nu$: the set of predictable real-valued processes $k(t,x,\cdot)$ with $\mathbf{E}[\int_0^T \int_D \|k(t,x,\cdot)\|_{\mathbf{L}^2_\nu} dxdt] < \infty$.

Assume that we are given a subfiltration

$$\mathcal{E}_t \subseteq \mathcal{F}_t; \;\; t \in [0,T],$$

representing the information available to the controller at time t. For example, we could have $\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}$ ($\delta > 0$ constant), meaning that the controller gets a *delayed* information flow compared to $\mathcal{F}_t$.

Consider a controlled mean-field stochastic partial differential equation $Y(t,x)$= $Y^u(t,x)$ at (t,x) in $]0,T[\times D$ of the following form

$$\begin{aligned} dY(t,x) &= \big[LY(t,x) + b(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)))\big]dt \\ &+ \sigma(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)))dW_t \\ &+ \int_{\mathbb{R}^*} \theta(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)),e)\tilde{N}(dt,de), \end{aligned} \tag{6}$$

with boundary conditions

$$Y(0,x) = \xi(x); \;\; x \in D \tag{7}$$

$$Y(t,x) = \eta(t,x); \;\; (t,x) \in]0,T[\times \partial D. \tag{8}$$

In the above equation, $\mathbf{F}, \mathbf{G} : \mathbf{L}^2(\mathbf{P}) \mapsto \mathbb{R}$ are Fréchet differentiable operators. One important example is represented by the expectation operator $\mathbf{E}[\cdot]$.

Here, $dY(t,x) = d_tY(t,x)$ is the differential with respect to t and L is a bounded linear integro-differential operator acting on x. We interpret Y as a weak (variational) solution to (6), in the sense that for $\phi \in C_0^\infty(D)$,

$$\langle Y_t, \phi\rangle_{\mathbf{L}^2(D)} = \langle y_0, \phi\rangle_{\mathbf{L}^2(D)} + \int_0^t \langle Y_s, L^*\phi\rangle ds + \int_0^t \langle b(s, Y_s), \phi\rangle_{\mathbf{L}^2(D)} ds + \int_0^t \langle \sigma(s, Y_s), \phi\rangle_{\mathbf{L}^2(D)} dW_s + \int_0^t \int_{\mathbb{R}^*} \langle \theta(s, Y_s, e), \phi\rangle_{\mathbf{L}^2(D)} \tilde{N}(ds, de), \tag{9}$$

where $\langle \cdot, \cdot \rangle$ represents the duality product between $W^{1,2}(D)$ and $W^{1,2}(D)^*$, with $W^{1,2}(D)$ the Sobolev space of order 1. In the above equation, we have not written all the arguments of b, σ, θ, for simplicity. The operator L^* is the *adjoint* operator of L, which satisfies

$$(L^*\phi, \psi) = (\phi, L\psi), \quad \text{for all } \phi, \psi \in C_0^\infty(\mathbb{R}), \tag{10}$$

where $\langle \phi_1, \phi_2\rangle_{\mathbf{L}^2(\mathbb{R})} := (\phi_1, \phi_2) = \int_{\mathbb{R}} \phi_1(x)\phi_2(x)dx$ is the inner product in $\mathbf{L}^2(\mathbb{R})$. The existence and the uniqueness of the solution of (6) are proved in Section 3. Under this framework, the Itô formula can be applied to such SPDEs. See e.g. Pardoux [8], Prévot and Rockner [9].

The process $u(t, x, \omega)$ is our *control* process, taking values in an open set $\mathbf{A} \subset \mathbb{R}$. We denote by $\mathcal{A}_{\mathcal{E}}$ a given family of *admissible* controls, contained in the set of $\mathcal{E}_t$-predictable stochastic processes $u(t, x) \in \mathbf{A}$ satisfying $\mathbf{E}[\int_0^T \int_D u^2(t, x)dxdt] < \infty$ and so that (6)-(7)-(8) has a unique càdlàg solution $Y(t, x)$. We make the following assumption:

Assumption 2.1

(i) The functions $b : [0,T]\times\Omega\times D\times\mathbb{R}^2\times\mathbf{A}\times\mathbb{R}\mapsto\mathbb{R};\ (t,\omega,x,y,\bar{y},u,\bar{u})\mapsto b(t,\omega,x,y,\bar{y},u,\bar{u})$, $\sigma : [0,T]\times\Omega\times D\times\mathbb{R}^2\times\mathbf{A}\times\mathbb{R}\mapsto\mathbb{R};\ (t,\omega,x,y,\bar{y},u,\bar{u})\mapsto\sigma(t,\omega,x,y,\bar{y},u,\bar{u})$, $\theta : [0,T]\times\Omega\times D\times\mathbb{R}^2\times\mathbf{A}\times\mathbb{R}\times\mathbb{R}^*\mapsto\mathbb{R};$ $(t,\omega,x,y,\bar{y},u,\bar{u},e)\mapsto\theta(t,\omega,x,y,\bar{y},u,\bar{u},e)$ are predictable maps. We assume that b,σ,θ are bounded functions of class C_b^1 with respect to $y,\bar{y},u,\bar{u}$.

ii) Let $f : [0,T]\times\Omega\times D\times\mathbb{R}^2\times\mathbf{A}\times\mathbb{R}\mapsto\mathbb{R}$ and $g : \Omega\times D\times\mathbb{R}^2\mapsto\mathbb{R}$ be a given profit rate function and bequest rate function, respectively. We suppose that

$$\mathbf{E}\left[\int_0^T\left(\int_D|f(t,x,Y(t,x),\mathbf{F}\big(Y(t,x)\big),u(t,x),\mathbf{G}\big(u(t,x)\big))|dx\right)dt\right.$$
$$\left.+\int_D|g(x,Y(T,x),\mathbf{F}\big(Y(T,x)\big)|dx\right]<\infty,$$

where $f(t,\omega,x,y,\bar{y},u,\bar{u})$, $g(\omega,x,y,\bar{y})$ are measurable functions of class C_b^1 with respect to $(y,\bar{y},u,\bar{u})$ and continuous with respect to t.

For each $u\in\mathcal{A}_{\mathcal{E}}$, we define the *performance functional* $J(u)$ by

$$J(u)=\mathbf{E}\left[\int_0^T\left(\int_D f(t,x,Y(t,x),\mathbf{F}\big(Y(t,x)\big),u(t,x),\mathbf{G}\big(u(t,x)\big))dx\right)dt\right.$$
$$\left.+\int_D g(x,Y(T,x),\mathbf{F}\big(Y(T,x)\big)dx\right].\tag{11}$$

We aim to maximize $J(u)$ over all $u\in\mathcal{A}_{\mathcal{E}}$ and our problem is the following: Find $u^*\in\mathcal{A}_{\mathcal{E}}$ so that

$$\sup_{u\in\mathcal{A}}J(u)=J(u^*).\tag{12}$$

Such a process u^* is called an optimal control (if it exists), and the number $J=J(u^*)$ is the *value* of this problem.

2.3 Sufficient Maximum Principle for Partial Information Optimal Control of Mean-Field SPDEs with Jumps

We prove here a sufficient maximum principle for our optimal control problem with *partial information*.

Define the *Hamiltonian* $H : [0,T] \times D \times \mathbb{R}^2 \times \mathbf{A} \times \mathbb{R}^3 \times \mathbf{L}^2_\nu \mapsto \mathbb{R}$ as follows:

$$H(t,x,y,\bar{y},u,\bar{u},p,q,\gamma) := f(t,x,y,\bar{y},u,\bar{u}) + b(t,x,y,\bar{y},u,\bar{u})p \\ +\sigma(t,x,y,\bar{y},u,\bar{u})q + \int_{\mathbb{R}^*} \theta(t,x,y,\bar{y},u,\bar{u},e)\gamma(e)\nu(de).$$

Since the state process and the cost functional are of the mean-field type, the adjoint equation is a *mean-field backward SPDE*. For $u \in \mathcal{A}_{\mathcal{E}}$, we thus consider the following *adjoint* equation in the three unknown processes $p(t,x) \in \mathbb{R}, q(t,x) \in \mathbb{R}, \gamma(t,x,\cdot) \in \mathbf{L}^2_\nu$ called the *adjoint processes*:

$$dp(t,x) = -\left[L^* p(t,x) + \frac{\partial H}{\partial y}(\cdot) + \mathbf{E}\left[\frac{\partial H}{\partial \bar{y}}(\cdot)\right] \nabla \mathbf{F}(Y(t,x))\right] dt \\ + q(t,x)dW_t + \int_{\mathbb{R}^*} \gamma(t,x,e)\tilde{N}(dt,de); \quad (t,x) \in]0,T[\times D. \tag{13}$$

$$p(T,x) = \frac{\partial g}{\partial y}(x, Y(T,x), \mathbf{F}(Y(T,x))) \\ + \mathbf{E}\left[\frac{\partial g}{\partial \bar{y}}(x, Y(T,x), \mathbf{F}(Y(T,x)))\right] \nabla \mathbf{F}(Y(T,x)); \quad x \in D \tag{14}$$

$$p(t,x) = 0; \quad (t,x) \in]0,T[\times \partial D. \tag{15}$$

Here $\frac{\partial H}{\partial y}(\cdot)$ stands for
$\frac{\partial H}{\partial y}(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)),p(t,x),q(t,x),\gamma(t,x,\cdot))$,
and similarly for $\frac{\partial H}{\partial \bar{y}}(\cdot)$. We assume that

$$\mathbf{E}[\int_0^T \int_D \{p^2(t,x) + q^2(t,x) + \int_{\mathbb{R}^*} \gamma^2(t,x,e)\nu(de)\} dx dt] < \infty. \tag{16}$$

Note that (13) is equivalent to

$$\begin{aligned} dp(t,x) = -&\Big[L^* p(t,x) + \frac{\partial f}{\partial y}(\cdot) + \frac{\partial b}{\partial y}(\cdot)p(t,x) + \frac{\partial \sigma}{\partial y}(\cdot)q(t,x) \\ &\quad + \int_{\mathbb{R}^*} \frac{\partial \theta}{\partial y}(\cdot,e)\gamma(t,x,e)\nu(de)\Big]dt \\ &- \mathbf{E}\Big[\frac{\partial f}{\partial \bar{y}}(\cdot) + \frac{\partial b}{\partial \bar{y}}(\cdot)p(t,x) + \frac{\partial \sigma}{\partial \bar{y}}(\cdot)q(t,x) \\ &\quad + \int_{\mathbb{R}^*} \frac{\partial \theta}{\partial \bar{y}}(\cdot,e)\gamma(t,x,e)\nu(de)\Big]\nabla \mathbf{F}(Y(t,x))dt \\ &+ q(t,x)dW_t + \int_{\mathbb{R}^*} \gamma(t,x,e)\tilde{N}(dt,de),\ (t,x) \in]0,T[\times D, \end{aligned} \tag{17}$$

where $\frac{\partial f}{\partial y}(\cdot)$ stands for $\frac{\partial f}{\partial y}(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)))$, and similarly for $\frac{\partial b}{\partial y}(\cdot)$, $\frac{\partial \sigma}{\partial y}(\cdot)$, $\frac{\partial f}{\partial \bar{y}}(\cdot)$, $\frac{\partial b}{\partial \bar{y}}(\cdot)$, $\frac{\partial \sigma}{\partial \bar{y}}(\cdot)$, and $\frac{\partial \theta}{\partial y}(\cdot,e)$ stands for $\frac{\partial \theta}{\partial y}(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)),e)$, and similarly for $\frac{\partial \theta}{\partial \bar{y}}(\cdot,e)$.

We now show a sufficient maximum principle.

Theorem 2.1 (Sufficient Maximum Principle for mean-field SPDEs with jumps) *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with the corresponding solution $\hat{Y}(t,x)$ and suppose that $\hat{p}(t,x), \hat{q}(t,x)$ and $\hat{\gamma}(t,x,\cdot)$ is a solution of the associated adjoint mean-field backward SPDE* (13)-(14)-(15). *Assume the following hold:*

(i) (The concavity assumption) The maps $Y \mapsto g(x,Y,\boldsymbol{F}(Y))$ and $(Y,u) \mapsto H(Y,u) := H(t,x,Y,\boldsymbol{F}(Y),u,\boldsymbol{G}(u),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot))$ are concave functions with respect to Y and (Y,u), respectively, for all (t,x) in $[0,T]\times D$.

(ii) (The maximum condition)

$$\begin{aligned} &\mathbf{E}\Big[H(t,x,\hat{Y}(t,x),\boldsymbol{F}(\hat{Y}(t,x)),\hat{u}(t,x),\boldsymbol{G}(\hat{u}(t,x)),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot)) \mid \mathcal{E}_t\Big] \\ &= ess \sup_{v\in\mathcal{A}_{\mathcal{E}}} \mathbf{E}\Big[H(t,x,\hat{Y}(t,x),\boldsymbol{F}(\hat{Y}(t,x)),v(t,x),\boldsymbol{G}(v(t,x)),\hat{p}(t,x), \\ &\qquad \hat{q}(t,x),\hat{\gamma}(t,x,\cdot)) \mid \mathcal{E}_t\Big] \quad a.s \text{ for all } t\in[0,T] \text{ and } x\in D. \end{aligned} \tag{18}$$

Then $\hat{u}(t)$ *is an optimal control for the random field control problem* (12).

Proof Let us fix $u \in \mathcal{A}_{\mathcal{E}}$ and let $Y(t,x) = Y^u(t,x)$ be the associated solution of (6). Define: $\hat{f} := f(t,x,\hat{Y}(t,x),\mathbf{F}(\hat{Y}(t,x)),\hat{u}(t,x),\mathbf{G}(\hat{u}(t,x)))$, and $\hat{b}$ and $\hat{\sigma}$ similarly. Define $f := f(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)))$, and b and σ similarly. Define $\hat{\theta} := \theta(t,x,\hat{Y}(t,x),\mathbf{F}(\hat{Y}(t,x)),\hat{u}(t,x),\mathbf{G}(\hat{u}(t,x)),e)$ and $\theta := \theta(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)),e)$. Set
$\hat{g} := g(x,\hat{Y}(T,x),\mathbf{F}(\hat{Y}(T,x)))$ and $g := g(x,Y(T,x),\mathbf{F}(Y(T,x)))$. Also set

$$\hat{H} := H(t,x,\hat{Y}(t,x),\mathbf{F}(\hat{Y}(t,x)),\hat{u}(t,x),\mathbf{G}(\hat{u}(t,x)),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot)),$$

$$H := H(t,x,Y(t,x),\mathbf{F}(Y(t,x)),u(t,x),\mathbf{G}(u(t,x)),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot)).$$

Using the above definitions and the definition of the performance functional J, we get that:

$$J(u) - J(\hat{u}) = \mathcal{J}_1 + \mathcal{J}_2, \tag{19}$$

where $\mathcal{J}_1 := \mathbf{E}[\int_0^T \int_D (f-\hat{f})dxdt]$ and $\mathcal{J}_2 := \mathbf{E}[\int_D (g-\hat{g})dx]$.

Now, let us notice the following relations:

$$\begin{cases} \hat{f} = \hat{H} - \hat{b}\hat{p}(t,x) - \hat{\sigma}\hat{q}(t,x) - \int_{\mathbb{R}^*} \hat{\theta}\hat{\gamma}(t,x,e)\nu(de); \\ f = H - b\hat{p}(t,x) - \sigma\hat{q}(t,x) - \int_{\mathbb{R}^*} \theta\hat{\gamma}(t,x,e)\nu(de), \end{cases}$$

which imply

$$\mathcal{J}_1 = \mathbf{E}\left[\int_0^T \int_D \left(H - \hat{H} - (b-\hat{b})\cdot\hat{p} - (\sigma-\hat{\sigma})\cdot\hat{q} - \int_{\mathbb{R}^*} (\theta-\hat{\theta})\cdot\hat{\gamma}\nu(de)\right)\right]. \tag{20}$$

Define a sequence of stopping times $\tau_n;\ n = 1,2,....$ as follows:

$$\tau_n := \inf\{t > 0; \max\{\|\hat{p}(t)\|_{\mathbf{L}^2(D)}, \|\hat{q}(t)\|_{\mathbf{L}^2(D)}, \|\hat{\gamma}(t)\|_{\mathbf{L}^2(D\times\mathbb{R}^*)},$$

$\|\sigma(t)-\hat{\sigma}(t)\|_{\mathbf{L}^2(D)}, \|\theta(t)-\hat{\theta}(t)\|_{\mathbf{L}^2(D\times\mathbb{R}^*)}, \|Y(t)-\hat{Y}(t)\|_{\mathbf{L}^2(D)}\} \geq n\} \wedge T.$

Then $\tau_n \to T$ as $n \to \infty$ and for all n, we have

$$\begin{aligned}
&\mathbf{E}\Bigg[\int_0^{\tau_n}\left(\int_D \hat{p}(t,x)(\sigma(t,x)-\hat{\sigma}(t,x))dx\right)dW_t \\
&+\int_0^{\tau_n}\int_{\mathbb{R}^*}\left(\int_D (\theta(t,x,e)-\hat{\theta}(t,x,e))dx\right)\tilde{N}(dt,de)\Bigg] \\
&=\mathbf{E}\Bigg[\int_0^{\tau_n}\left(\int_D (Y(t,x)-\hat{Y}(t,x))\hat{q}(t,x)dx\right)dW_t \\
&\quad+\int_0^{\tau_n}\int_{\mathbb{R}^*}\left(\int_D (Y(t,x)-\hat{Y}(t,x))\hat{\gamma}(t,x,e)dx\right)\tilde{N}(dt,de)\Bigg]=0.
\end{aligned}$$

Hence by (16) and dominated convergence we have

$$\begin{aligned}
&\mathbf{E}\Bigg[\int_0^{T}\left(\int_D \hat{p}(t,x)(\sigma(t,x)-\hat{\sigma}(t,x))dx\right)dW_t \\
&+\int_0^{T}\int_{\mathbb{R}^*}\left(\int_D (\theta(t,x,e)-\hat{\theta}(t,x,e))dx\right)\tilde{N}(dt,de)\Bigg] \\
&=\mathbf{E}\Bigg[\int_0^{T}\left(\int_D (Y(t,x)-\hat{Y}(t,x))\hat{q}(t,x)dx\right)dW_t \\
&\quad+\int_0^{T}\int_{\mathbb{R}^*}\left(\int_D (Y(t,x)-\hat{Y}(t,x))\hat{\gamma}(t,x,e)dx\right)\tilde{N}(dt,de)\Bigg]=0.
\end{aligned}$$

Since the map $Y \mapsto g(x,Y,F(Y))$ is concave for each $x \in D$, we obtain:

$$\begin{aligned}
g-\hat{g} \leq\ &\frac{\partial g}{\partial y}(x,\hat{Y}(T,x),\mathbf{F}(\hat{Y}(T,x)))\tilde{Y}(T,x) \\
&+\frac{\partial g}{\partial \bar{y}}(x,\hat{Y}(T,x),\mathbf{F}(\hat{Y}(T,x)))\langle\nabla\mathbf{F}(\hat{Y}(T,x)),\tilde{Y}(T,x)\rangle_{\mathbf{L}^2(\mathbf{P})},
\end{aligned}$$

where $\tilde{Y}(t,x) := Y(t,x) - \hat{Y}(t,x)$. We thus obtain, by taking the expectation and applying the Itô formula for jump-diffusion processes,

$$
\begin{aligned}
\mathcal{J}_2 \leq\ & \mathbf{E}\left[\int_D \left(\frac{\partial g}{\partial y}(x, \hat{Y}(T,x), \mathbf{F}(\hat{Y}(T,x)))\tilde{Y}(T,x)\right.\right. \\
&\left.\left. + \frac{\partial g}{\partial \bar{y}}(x, \hat{Y}(T,x), \mathbf{F}(\hat{Y}(T,x)))\langle \nabla \mathbf{F}(\hat{Y}(T,x)), \tilde{Y}(T,x)\rangle_{\mathbf{L}^2(\mathbf{P})}\right) dx\right] \\
=\ & \mathbf{E}\left[\int_D \langle \hat{p}(T,x), \tilde{Y}(T,x)\rangle dx\right] \\
=\ & \mathbf{E}\left[\int_D \left(\hat{p}(0,x)\cdot \tilde{Y}(0,x)\right.\right. \\
+\ & \left.\left.\int_0^T \left(\langle \tilde{Y}(t,x), d\hat{p}(t,x)\rangle + \hat{p}(t,x)d\tilde{Y}(t,x) + (\sigma - \hat{\sigma})\hat{q}(t,x)\right) dt\right) dx\right] \\
& + \mathbf{E}\left[\int_D \left(\int_0^T \int_{\mathbb{R}^*} (\theta - \hat{\theta})\hat{\gamma}(t,x,e) N(dt,de)\right) dx\right] \\
=\ & \mathbf{E}\left[\int_D \int_0^T \hat{p}(t,x)\left(L\tilde{Y}(t,x) + (b - \hat{b})\right)\right. \\
& + \tilde{Y}(t,x)\left(-L^*\hat{p}(t,x) - \frac{\partial \hat{H}}{\partial y}\right. \\
-\ & \left.\left.\mathbf{E}\left[\frac{\partial \hat{H}}{\partial \bar{y}}\right]\langle \nabla \mathbf{F}(\hat{Y}(t,x)), \tilde{Y}(t,x)\rangle_{\mathbf{L}^2(P)}\right) dtdx\right] \\
& + \mathbf{E}\left[\int_D \int_0^T \left((\sigma - \hat{\sigma})\hat{q}(t,x) + \int_{\mathbb{R}^*} (\theta - \hat{\theta})\hat{\gamma}(t,x,e)\nu(de)\right) dtdx\right]. \qquad (21)
\end{aligned}
$$

From (19), (20) and (21), we derive

$$
\begin{aligned}
& J(u) - J(\hat{u}) \leq \mathbf{E}\left[\int_0^T \left(\int_D \hat{p}(t,x)L\widetilde{Y}(t,x) - \widetilde{Y}(t,x)L^*\hat{p}(t,x)dx\right) dt\right] \\
& + \mathbf{E}\left[\int_D \left(\int_0^T \left(H - \hat{H} - \frac{\partial \hat{H}}{\partial y}\cdot \tilde{Y}(t,x)\right.\right.\right. \\
& \left.\left.\left. - \mathbf{E}\left[\frac{\partial \hat{H}}{\partial \bar{y}}\right]\langle \nabla \mathbf{F}(\hat{Y}(t,x)), \tilde{Y}(t,x)\rangle_{\mathbf{L}^2(\mathbf{P})}\right) dt\right) dx\right].
\end{aligned}
$$

Since $\tilde{Y}(t,x) = \hat{p}(t,x) = 0$ for all $(t,x) \in [0,T] \times \partial D$, we obtain by an easy extension of (10) that

$$\int_D \tilde{Y}(t,x) L^* \hat{p}(t,x) dx = \int_D \hat{p}(t,x) L \tilde{Y}(t,x),$$

for all $t \in]0,T[$. We therefore get

$$\begin{aligned} J(u) - J(\hat{u}) \leq \mathbf{E} \Bigg[\int_D \Bigg(\int_0^T \Bigg(H - \hat{H} - \frac{\partial \hat{H}}{\partial y} \cdot \tilde{Y}(t,x) \\ + \mathbf{E} \left[\frac{\partial \hat{H}}{\partial \bar{y}} \right] \langle \nabla \mathbf{F}(\hat{Y}(t,x)), \tilde{Y}(t,x) \rangle_{\mathbf{L}^2(P)} \Bigg) dt \Bigg) dx \Bigg]. \end{aligned}$$

By the concavity assumption we have

$$\begin{aligned} H - \hat{H} &\leq \frac{\partial H}{\partial y}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u}))(Y - \hat{Y}) \\ &+ \frac{\partial H}{\partial \bar{y}}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u})) \langle \nabla \mathbf{F}(\hat{Y}), (Y - \hat{Y}) \rangle_{\mathbf{L}^2(\mathbf{P})} \\ &+ \frac{\partial H}{\partial u}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u}))(u - \hat{u}) \\ &+ \frac{\partial H}{\partial \bar{u}}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u})) \langle \nabla \mathbf{G}(\hat{u}), (u - \hat{u}) \rangle_{\mathbf{L}^2(\mathbf{P})}. \end{aligned}$$

Combining the two above relations we get:

$$\begin{aligned} J(u) - J(\hat{u}) \leq \mathbf{E} \Bigg[\int_D \int_0^T \Bigg(\frac{\partial H}{\partial u}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u}))(u - \hat{u}) \\ + \frac{\partial H}{\partial \bar{u}}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u})) \langle \nabla \mathbf{G}(\hat{u}), (u - \hat{u}) \rangle_{\mathbf{L}^2(\mathbf{P})} \Bigg) dt dx \Bigg]. \end{aligned} \tag{22}$$

By the maximum condition (18), we obtain:

$$\begin{aligned} &\mathbf{E} \left[\frac{\partial H}{\partial u}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u})) \mid \mathcal{E}_t \right] (u - \hat{u}) \\ &+ \mathbf{E} \left[\frac{\partial H}{\partial \bar{u}}(\hat{Y}, \mathbf{F}(\hat{Y}), \hat{u}, \mathbf{G}(\hat{u})) \mid \mathcal{E}_t \right] \langle \nabla \mathbf{G}(\hat{u}), u - \hat{u} \rangle_{\mathbf{L}^2(\mathbf{P})} \leq 0 \text{ a.s.}, \end{aligned} \tag{23}$$

for all $(t,x) \in [0,T] \times D$. From (22) and (23) we conclude that

$$J(u) \leq J(\hat{u}).$$

By arbitrariness of u, we conclude that $\hat{u}$ is optimal. □

2.4 A Necessary-type Maximum Principle for Partial Information Control of Mean-Field SPDEs with Jumps

As in many applications the concavity condition may not hold, we prove a version of the maximum principle which does not need this assumption. Instead, we assume the following:

(A1) For all $s \in [0,T[$ and all bounded $\mathcal{E}_s$-measurable random variables $\theta(\omega,x)$ the control β defined by $\beta_t(\omega,x) = \theta(\omega,x)\chi_{(s,T]}(t)$, for all $t \in [0,T], x \in D$, is in $\mathcal{A}_{\mathcal{E}}$.

(A2) For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded, there exists $\delta > 0$ such that the control

$$u(t) + y\beta(t);\ t \in [0,T]$$

belongs to $\mathcal{A}_{\mathcal{E}}$ for all $y \in]-\delta, \delta[$.

Let us give an auxiliary lemma:

Lemma 2.1 *Let $u \in \mathcal{A}_{\mathcal{E}}$ and $v \in \mathcal{A}_{\mathcal{E}}$. The derivative process*

$$\mathcal{Y}(t,x) := \lim_{z \downarrow 0} \frac{Y^{u+z\beta}(t,x) - Y^u(t,x)}{z} \tag{24}$$

exists and belongs to $\boldsymbol{L}^2(dx \times dt \times dP)$. *We then have that* $\mathcal{Y}$ *satisfies the following mean-field SPDE:*

$$
\begin{aligned}
&d\mathcal{Y}(t,x) = L\mathcal{Y}(t,x) + \Big(\frac{\partial b}{\partial y}(\cdot)\mathcal{Y}(t,x) + \frac{\partial b}{\partial \bar{y}}(\cdot)\langle \nabla \boldsymbol{F}(Y^u(t,x)), \mathcal{Y}(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})} \\
&+\frac{\partial b}{\partial u}(\cdot)\beta(t,x) + \frac{\partial b}{\partial \bar{u}}(\cdot)\langle \nabla \boldsymbol{G}(u(t,x)), \beta(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})}\Big)\, dt \\
&+\Big(\frac{\partial \sigma}{\partial y}(\cdot)\mathcal{Y}(t,x) + \frac{\partial \sigma}{\partial \bar{y}}(\cdot)\langle \nabla \boldsymbol{F}(Y^u(t,x)), \mathcal{Y}(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})} \\
&+\frac{\partial \sigma}{\partial u}(\cdot)\beta(t,x) + \frac{\partial \sigma}{\partial \bar{u}}(\cdot)\langle \nabla \boldsymbol{G}(u(t,x)), \beta(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})}\Big)\, dW_t \\
&+\int_{\mathbb{R}^*}\Big(\frac{\partial \theta}{\partial y}(\cdot,e)\mathcal{Y}(t,x) + \frac{\partial \theta}{\partial \bar{y}}(\cdot,e)\langle \nabla \boldsymbol{F}(Y^u(t,x)), \mathcal{Y}(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})} \\
&+\frac{\partial \theta}{\partial u}(\cdot,e)\beta(t,x) + \frac{\partial \theta}{\partial \bar{u}}(\cdot,e)\beta(t,x)\langle \nabla \boldsymbol{G}(u(t,x)), \beta(t,x)\rangle_{\mathbf{L}^2(\boldsymbol{P})}\Big)\, \tilde{N}(dt,de), \\
&\mathcal{Y}(t,x) = 0, \;\; (t,x) \in]0,T[\times \partial D; \\
&\mathcal{Y}(0,x) = 0, \;\; x \in D.
\end{aligned}
$$

where $\frac{\partial b}{\partial y}(\cdot)$ *stands for* $\frac{\partial b}{\partial y}(t,x,Y^u(t,x), \boldsymbol{F}(Y^u(t,x)), \boldsymbol{G}(u(t,x))$ *and similarly for the other coefficients.*

Proof The result follows by applying the mean theorem.

We omit the details. □

We now provide the necessary-type maximum principle for our optimal control problem for mean-field SPDEs.

Theorem 2.2 (Necessary maximum principle for mean-field SPDEs with jumps) *Let* $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ *with corresponding solutions* (6)-(7)-(8) *and* (13)-(14)-(15). *Assume that Assumptions* ***(A1)-(A2)*** *hold. Then the following are equivalent:*

(i) $\frac{d}{dy}J(\hat{u}+y\beta)|_{y=0}=0$ *for all bounded* $\beta\in\mathcal{A}_{\mathcal{E}}$.

(ii) $\mathbf{E}\left[\nabla_u\hat{H}(t,x)|\mathcal{E}_t\right]=0,$ *for all* $(t,x)\in[0,T]\times D$ *a.s.*,

where

$$\begin{aligned}\nabla_u\hat{H}(t,x):=&\frac{\partial H}{\partial u}(t,x,\hat{u}(t,x),\hat{Y}(t,x),\boldsymbol{F}(\hat{Y}(t,x)),\\&\qquad\boldsymbol{G}(\hat{u}(t,x)),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot))\\&+[\frac{\partial H}{\partial\bar{u}}(t,x,\hat{u}(t,x),\hat{Y}(t,x),\boldsymbol{F}(\hat{Y}(t,x)),\\&\qquad\boldsymbol{G}(\hat{u}(t,x)),\hat{p}(t,x),\hat{q}(t,x),\hat{\gamma}(t,x,\cdot))]\nabla\boldsymbol{G}(\hat{u}(t,x)).\end{aligned}$$

Proof By introducing a sequence of stopping times τ_n as in the proof of the previous theorem, we see that we may assume that all the local martingales that appear in the following calculations are martingales with expectation 0. The assumptions on the coefficients together with the mean theorem and relation (24) yield to:

$$\begin{aligned}&\lim_{y\to0}\frac{1}{y}\left(J(\hat{u}+y\beta)-J(\hat{u})\right)\\&=\mathbf{E}\left[\int_0^T\int_D\left(\frac{\partial\hat{f}}{\partial y}(t,x)\mathcal{Y}(t,x)+\frac{\partial\hat{f}}{\partial\overline{y}}(t,x)\langle\nabla\mathbf{F}(\hat{Y}(t,x)),\mathcal{Y}(t,x)\rangle_{\mathbf{L^2(P)}}\right.\right.\\&\left.\left.+\frac{\partial\hat{f}}{\partial u}(t,x)\beta(t,x)+\frac{\partial\hat{f}}{\partial\bar{u}}(t,x)\langle\nabla\mathbf{G}(\hat{u}(t,x)),\beta(t,x)\rangle_{\mathbf{L^2(P)}}\right)dtdx\right]\\&+\mathbf{E}\left[\int_D(\frac{\partial\hat{g}}{\partial y}(x)\mathcal{Y}(T,x)+\frac{\partial\hat{g}}{\partial\overline{y}}(x)\langle\nabla\mathbf{F}(\hat{Y}(T,x)),\mathcal{Y}(T,x)\rangle_{\mathbf{L^2(P)}})dx\right],\qquad(25)\end{aligned}$$

where $\frac{\partial\hat{f}}{\partial y}(t,x)$ stands for $\frac{\partial f}{\partial y}(t,x,\hat{Y}(t,x),\mathbf{F}(\hat{Y}(t,x)),\hat{u}(t,x),\mathbf{G}(\hat{u}(t,x))$ and similarly for the other partial derivatives, and $\frac{\partial\hat{g}}{\partial y}(x)$ stands for $\frac{\partial g}{\partial y}(x,\hat{Y}(x),\mathbf{F}(\hat{Y}(T,x))$ and similarly for $\frac{\partial\hat{g}}{\partial\overline{y}}(x)$.

The definition of the Hamiltonian H implies:

$$\frac{\partial \hat{f}}{\partial y}(t,x) = \frac{\partial \hat{H}}{\partial y}(t,x) - \frac{\partial \hat{b}}{\partial y}(t,x)\hat{p}(t,x) - \frac{\partial \hat{\sigma}}{\partial y}(t,x)\hat{q}(t,x) - \int_{\mathbb{R}^*} \frac{\partial \hat{\theta}}{\partial y}(t,x,e)\hat{\gamma}(t,x,e)\nu(de),$$

and similarly for $\frac{\partial \hat{f}}{\partial \overline{y}}, \frac{\partial \hat{f}}{\partial u}, \frac{\partial \hat{f}}{\partial \overline{u}}$. Using (25) and (13), we derive

$$\begin{aligned}
&\lim_{y\to 0} \frac{1}{y}\left(J(\hat{u}+y\beta) - J(\hat{u})\right) \\
&= \mathbf{E}\left[\int_0^T \int_D \left(\frac{\partial \hat{H}}{\partial y}(t,x) - \frac{\partial \hat{b}}{\partial y}(t,x)\hat{p}(t,x) - \frac{\partial \hat{\sigma}}{\partial y}(t,x)\hat{q}(t,x)\right.\right. \\
&\left.\left. - \int_{\mathbb{R}^*} \frac{\partial \hat{\theta}}{\partial y}(t,x,e)\hat{\gamma}(t,x,e)\nu(de)\right)\mathcal{Y}(t,x)dxdt\right] \\
&+ \mathbf{E}[\int_0^T \int_D (\frac{\partial \hat{H}}{\partial \overline{y}}(t,x) - \frac{\partial \hat{b}}{\partial \overline{y}}(t,x)\hat{p}(t,x) - \frac{\partial \hat{\sigma}}{\partial \overline{y}}(t,x)\hat{q}(t,x) \\
&- \int_{\mathbb{R}^*} \frac{\partial \hat{\theta}}{\partial \overline{y}}(t,x,e)\hat{\gamma}(t,x,e)\nu(de))\langle \nabla \mathbf{F}(\hat{Y}(t,x)), \mathcal{Y}(t,x)\rangle_{\mathbf{L}^2(\mathbf{P})}dxdt] \\
&+ \mathbf{E}\left[\int_0^T \int_D \left(\frac{\partial \hat{H}}{\partial u}(t,x) - \frac{\partial \hat{b}}{\partial u}(t,x)\hat{p}(t,x) - \frac{\partial \hat{\sigma}}{\partial u}(t,x)\hat{q}(t,x)\right.\right. \\
&\left.\left. - \int_{\mathbb{R}^*} \frac{\partial \hat{\theta}}{\partial u}(t,x,e)\hat{\gamma}(t,x,e)\nu(de)\right)\beta(t,x)dxdt\right] \\
&+ \mathbf{E}\left[\int_0^T \int_D (\frac{\partial \hat{H}}{\partial \overline{u}}(t,x) - \frac{\partial \hat{b}}{\partial \overline{u}}(t,x)\hat{p}(t,x) - \frac{\partial \hat{\sigma}}{\partial \overline{u}}(t,x)\hat{q}(t,x)\right. \\
&\left. - \int_{\mathbb{R}^*} \frac{\partial \hat{\theta}}{\partial \overline{u}}(t,x,e)\hat{\gamma}(t,x,e)\nu(de))\langle \nabla \mathbf{G}(\hat{u}(t,x)), \beta(t,x)\rangle dxdt\right] \\
&+ \mathbf{E}[\int_D \langle \hat{p}(T,x), \mathcal{Y}(T,x)\rangle dx].
\end{aligned}$$

Applying Itô formula to $\langle \hat{p}(T,x), \mathcal{Y}(T,x) >$ and using the dynamics of the adjoint equations, we finally get

$$\lim_{y\to 0} \frac{1}{y}\left(J(\hat{u}+y\beta) - J(\hat{u})\right) = \mathbf{E}\left[\int_0^T \int_D \mathbf{E}\left[\langle \nabla_u \hat{H}(t,x), \beta(t,x)\rangle \mid \mathcal{E}_t\right] dxdt\right],$$

where

$$\langle \nabla_u \hat{H}(t,x), \beta(t,x)\rangle = \frac{\partial H}{\partial u}(t,x)\beta(t,x) + \frac{\partial H}{\partial \bar{u}}(t,x)\langle \nabla \mathbf{G}(\hat{u}), \beta(t,x)\rangle_{\mathbf{L}^2(\mathbf{P})}.$$

We conclude that

$$\lim_{y\to 0} \frac{1}{y}\left(J(\hat{u}+y\beta) - J(\hat{u})\right) = 0$$

if and only if

$$\mathbf{E}\left[\int_0^T \int_D \mathbf{E}[\langle \nabla_u \hat{H}(t,x), \beta(t,x)\rangle | \mathcal{E}_t] dx dt\right] = 0.$$

In particular this holds for all $\beta \in \mathcal{A}_{\mathcal{E}}$ which take the form

$$\beta(t,x) = \theta(\omega,x)\chi_{[s,T]}(t);\ t \in [0,T],$$

for a fixed $s \in [0,T[$, where $\theta(\omega,x)$ is a bounded $\mathcal{E}_s$-measurable random variable. We thus get that this is again equivalent to

$$\mathbf{E}\left[\int_s^T \int_D \mathbf{E}\left[\langle \nabla_u \hat{H}(t,x), \theta\rangle \mid \mathcal{E}_t\right] dx dt\right] = 0.$$

We now differentiate with respect to s and derive that

$$\mathbf{E}\left[\int_D \mathbf{E}\left[\langle \nabla_u \hat{H}(s,x), \theta\rangle \mid \mathcal{E}_s\right] dx\right] = 0.$$

Since this holds for all bounded $\mathcal{E}_s$-measurable random variable θ, we can easily conclude that

$$\lim_{y\to 0} \frac{1}{y}\left(J(\hat{u}+y\beta) - J(\hat{u})\right) = 0$$

is equivalent to $\mathbf{E}\left[\nabla_u \hat{H}(t,x) \mid \mathcal{E}_t\right] = 0$, for all $(t,x) \in [0,T] \times D$ a.s. □

2.5 Application to the Optimal Harvesting Example

We now return to the problem of optimal harvesting from a population in a lake D stated in the motivating example. Thus we suppose the density $Y(t,x)$ of the population at time $t \in [0,T]$ and at point $x \in D$ is given by the stochastic reaction-diffusion equation (1), and the performance criterion is assumed to be as in (4). For simplicity, we assume that D is a bounded domain in $\mathbb{R}$ and $\mathcal{E}_t = \mathcal{F}_t$. In this case the Hamiltonian has the following form

$$\begin{aligned} H(t,x,y,\bar{y},u,\bar{u},p,q,\gamma) &= \log(yu) + [b(t,x)\bar{y} - yu]p \\ &\quad + \sigma(t,x)yq + \int_{\mathbb{R}^*} \theta(t,x,e)y\gamma(e)\nu(de), \end{aligned}$$

and the adjoint BSDE becomes

$$\begin{aligned} dp(t,x) &= [-\frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x) + \frac{1}{Y(t,x)} + \sigma(t,x)q(t,x) \\ &\quad + \int_{\mathbb{R}^*} \theta(t,x,e)\gamma(t,x,e)\nu(de) - u(t,x)p(t,x) - \mathbf{E}[b(t,x)p(t,x)]]dt \\ &\quad + q(t,x)dW_t + \int_{\mathbb{R}^*} \gamma(t,x,e)\tilde{N}(dt,de), \quad (t,x) \in]0,T[\times D \\ p(T,x) &= \alpha(x), \;\; x \in D, \\ p(t,x) &= 0, \;\; (t,x) \in]0,T[\times \partial D. \end{aligned}$$

We now apply the necessary maximum principle which implies the fact that if u is an optimal control then it satisfies the first order condition

$$u(t,x) = \frac{1}{Y(t,x)p(t,x)}.$$

We summarize our results as follows:

Theorem 2.3 *Assume that the conditions of Theorem 2.2 hold. Suppose a harvesting rate process $u(t,x)$ is optimal for the optimization problem (5) in Subsection 2.1. Then*

$$u(t,x) = \frac{1}{Y(t,x)p(t,x)},$$

where $p(t,x)$ solves the mean-field backward SPDE

$$\begin{aligned}
&dp(t,x) = [-\frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x) + \frac{1}{Y(t,x)} + \sigma(t,x)q(t,x) \\
&+ \int_{\mathbb{R}^*} \theta(t,x,e)\gamma(t,x,e)\nu(de) - \mathbf{E}[b(t,x)p(t,x)] - u(t,x)p(t,x)]dt \\
&+ q(t,x)dW_t + \int_{\mathbb{R}^*} \gamma(t,x,e)\tilde{N}(dt,de) \\
&p(T,x) = \alpha(x), \ \ x \in D. \\
&p(t,x) = 0, \ \ (t,x) \in]0,T[\times \partial D.
\end{aligned}$$

3 Existence and Uniqueness Results for General Forward and Backward Mean-Field SPDEs with Lévy noise

3.1 Forward Mean-Field SPDEs

We address here the problem of the existence and uniqueness for the solution of forward mean-field SPDE (6) with a general mean-field operator, introduced in Section 2. In order to do this, we first describe the general framework. Let V, H be two separable Hilbert spaces such that V is continously, densely imbedded in H. Identifying H with its dual we have

$$V \subset H \approxeq H^* \subset V^*,$$

where we have denoted by V^* the topological dual of V. Let L be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist the constants $\chi > 0$ and $\zeta \geq 0$ so that

$$2\langle -Lu, u\rangle + \zeta|u|_H^2 \geq \chi\|u\|_V^2 \quad \text{for all } u \in V, \tag{26}$$

where $\langle Lu, u\rangle = Lu(u)$ denotes the action of $Lu \in V^*$ on $u \in V$ and $|\cdot|_H$ (resp. $\|\cdot\|_V$) the norm associated to the Hilbert space H (resp. V). Let us introduce the notation adopted in this section.

- $\mathcal{P}$ is the predictable σ-algebra on $[0,T] \times \Omega$;
- $\mathbf{L}^2_\nu(H)$ is the set of measurable functions $k : (\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*)) \mapsto (H, \mathcal{B}(H))$ such that $\|k\|_{\mathbf{L}^2_\nu(H)} := \left(\int_{\mathbb{R}^*} |k(e)|_H^2 \nu(de)\right)^{\frac{1}{2}} < \infty$;
- $\mathbf{L}^2(\Omega, H)$ is the set of measurable functions $k : (\Omega, \mathcal{F}) \mapsto (H, \mathcal{B}(H))$ such that $\mathbf{E}[|k|_H^2] < \infty$;
- $\mathbf{L}^2(\Omega, \mathbf{L}^2_\nu(H))$ is the set of measurable functions $k : (\Omega, \mathcal{F}) \mapsto (\mathbf{L}^2_\nu(H), \mathcal{B}(\mathbf{L}^2_\nu(H)))$ such that $\mathbf{E}[\|k\|^2_{\mathbf{L}^2_\nu(H)}] < \infty$;
- $\mathbf{L}^2(\Omega \times [0,T], H)$ (resp. $\mathbf{L}_2(\Omega \times [0,T], V)$) is the set of $\mathcal{F}_t$-adapted H-valued (resp. V-valued) processes $\Phi : \Omega \times [0,T] \mapsto H$ (resp.V) such that $\|\Phi\|^2_{\mathbf{L}^2(\Omega\times[0,T],H)} := \mathbf{E}[\int_0^T |\Phi(t)|_H^2 dt] < \infty$ (resp. $\|\Phi\|^2_{\mathbf{L}^2(\Omega\times[0,T],V)} := \mathbf{E}[\int_0^T \|\Phi(t)\|_V^2 dt] < \infty$);
- $\mathbf{L}^2(\Omega \times [0,T] \times \mathbb{R}^*, H)$ is the set of all the $\mathcal{P} \times \mathcal{B}(\mathbb{R}^*)$-measurable H-valued maps $\theta : \Omega \times [0,T] \times \mathbb{R}^* \mapsto H$ satisfying $\|\theta\|_{\mathbf{L}^2(\Omega\times[0,T]\times\mathbb{R}^*,H)} := \mathbf{E}[\int_0^T \int_{\mathbb{R}^*} |\Phi(t,e)|_H^2 \nu(de)dt] < \infty$;
- $\mathbf{S}^2(\Omega \times [0,T], H)$ is the set of $\mathcal{F}_t$-adapted H-valued càdlàg processes $\Phi : \Omega \times [0,T] \mapsto H$ such that $\|\Phi\|^2_{\mathbf{S}^2(\Omega\times[0,T],H)} := \mathbf{E}\left[\sup_{0\leq t\leq T} |\Phi(t)|_H^2\right] < \infty$;

The mean field SPDE under study is:

$$\begin{aligned} dY_t &= [LY_t + b(t, Y_t, \mathbf{F}(Y_t))]dt + \sigma(t, Y_t, \mathbf{F}(Y_t))dW_t \\ &\quad + \int_{\mathbb{R}^*} \theta(t, Y_t, \mathbf{F}(Y_t), e)\tilde{N}(dt, de); \;\; (t, x) \in]0, T[\times D. \end{aligned} \tag{27}$$

We recall that this equation should be understood in the weak sense.

We make the following assumption on b, σ, θ and the operator $\mathbf{F}$.

Assumption 3.1 The maps $b : \Omega\times[0, T]\times H\times H \mapsto H$, $\sigma : \Omega\times[0, T]\times H\times H \mapsto H$ are $\mathcal{P} \times \mathcal{B}(H) \times \mathcal{B}(H)/\mathcal{B}(H)$-measurable. The map $\theta : \Omega \times [0, T] \times H \times H \times \mathbb{R}^* \mapsto H$ is $\mathcal{P} \times \mathcal{B}(H) \times \mathcal{B}(H) \times \mathcal{B}(\mathbb{R}^*)/\mathcal{B}(H)$-measurable. There exists a constant $C < \infty$ so that

$$\begin{aligned} &|b(t, y_1, \bar{y}_1) - b(t, y_2, \bar{y}_2)|_H + |\sigma(t, y_1, \bar{y}_1) - \sigma(t, y_2, \bar{y}_2)|_H \\ &\quad + \int_{\mathbb{R}^*} |\theta(t, y_1, \bar{y}_1, e) - \theta(t, y_2, \bar{y}_2, e)|_H^2 \nu(de) \\ &\le C(|y_1 - y_2|_H + |\bar{y}_1 - \bar{y}_2|_H) \text{ a.s. for all } (\omega, t) \in \Omega \times [0, T]. \end{aligned}$$

There exists $C < \infty$ so that

$$\begin{aligned} |b(t, y, \bar{y})|_H^2 + |\sigma(t, y, \bar{y})|_H^2 + \int_{\mathbb{R}^*} |\theta(t, y, \bar{y}, e)|_H^2 \nu(de) \le C(1 + |y|_H^2 + |\bar{y}|_H^2), \\ \forall(\omega, t) \in \Omega \times [0, T], y, \bar{y} \in \mathbb{R}. \end{aligned}$$

The operator $\mathbf{F} : \mathbf{L}^2(\Omega; H) \mapsto H$ is Fréchet differentiable.

Theorem 3.1 *Let $h \in H$ given. Under Assumption 3.1, there exists a unique H-valued progressively measurable process $(Y_t)_{t\ge0}$ satisfying:*

(i) $Y \in \boldsymbol{L}^2(\Omega \times [0, T], V) \cap \boldsymbol{S}^2(\Omega \times [0, T], H)$;

(ii) $Y_t = h + \int_0^t [LY_s + b(s, Y_s, \boldsymbol{F}(Y_s))]\, ds + \int_0^t \sigma(s, Y_s, \boldsymbol{F}(Y_s))dW_s$
$+ \int_0^t \int_{\mathbb{R}^*} \theta(s_-, Y_{s_-}, \boldsymbol{F}(Y_{s_-}), e)\tilde{N}(ds, de)$.

Proof ***I. Existence of the solution***

Let $Y^0_t := h,\ t \geq 0$. For $n \geq 0$, we define $Y^{n+1} \in \mathbf{L}^2([0,T];V) \cap \mathbf{S}^2([0,T];H)$ to be the unique solution to the following equation:

$$dY^{n+1}_t = [LY^{n+1}_t + b(t, Y^{n+1}_t, \mathbf{F}(Y^n_t))]dt + \sigma(t, Y^{n+1}_t, \mathbf{F}(Y^n_t))dW_t \\ + \int_{\mathbb{R}^*} \theta(t_-, Y^{n+1}_{t_-}, \mathbf{F}(Y^n_{t_-}), e)\tilde{N}(dt, de). \tag{28}$$

The existence of the solution Y^{n+1} of (28) is given in [10, Proposition 3.1]. We now show that the sequence $\{Y^n,\, n \geq 1\}$ is a Cauchy sequence in the spaces $\mathbf{L}^2(\Omega \times [0,T], V)$ and $\mathbf{S}^2(\Omega \times [0,T], H)$. By applying Itô formula, we get

$$|Y^{n+1}_t - Y^n_t|^2_H = 2\int_0^t \langle Y^{n+1}_s - Y^n_s, L(Y^{n+1}_s - Y^n_s)\rangle ds \\ + 2\int_0^t \langle Y^{n+1}_s - Y^n_s, b(s, Y^{n+1}_s, \mathbf{F}(Y^n_s)) - b(s, Y^n_s, \mathbf{F}(Y^{n-1}_s))\rangle_H ds \\ + 2\int_0^t \langle Y^{n+1}_s - Y^n_s, \sigma(s, Y^{n+1}_s, \mathbf{F}(Y^n_s)) - \sigma(s, Y^n_s, \mathbf{F}(Y^{n-1}_s))\rangle_H dW_s \\ + \int_0^t |\sigma(s, Y^{n+1}_s, \mathbf{F}(Y^n_s)) - \sigma(s, Y^n_s, \mathbf{F}(Y^{n-1}_s))|^2_H ds \\ + \int_0^t \int_{\mathbb{R}^*} [|\theta(s, Y^{n+1}_{s-}, \mathbf{F}(Y^n_{s-}), e) - \theta(s, Y^n_{s-}, \mathbf{F}(Y^{n-1}_{s-}), e)|^2_H]\tilde{N}(ds, de) \\ + 2\int_0^t \int_{\mathbb{R}^*} \langle Y^{n+1}_{s^-} - Y^n_{s^-}, \theta(s, Y^{n+1}_{s^-}, \mathbf{F}(Y^n_{s^-}), e) \\ - \theta(s, Y^n_{s^-}, \mathbf{F}(Y^{n-1}_{s^-}), e)\rangle_H \tilde{N}(ds, de) \\ + \int_0^t \int_{\mathbb{R}^*} [|\theta(s, Y^{n+1}_{s-}, \mathbf{F}(Y^n_{s-}), e) - \theta(s, Y^n_{s-}, \mathbf{F}(Y^{n-1}_{s-}), e)|^2_H \nu(de)ds.$$

Below, C denotes a generic constant whose values might change from line to line. Using the Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities and the coercivity assumption (26) on the operator L, we obtain that

$$\begin{aligned}
&\mathbf{E}\left[\sup_{0\leq s\leq t}|Y_s^{n+1}-Y_s^n|_H^2\right] \leq -\chi\mathbf{E}[\int_0^t \|Y_s^{n+1}-Y_s^n\|_V^2 ds] \\
&+C\mathbf{E}[\int_0^t [|Y_s^{n+1}-Y_s^n|_H^2 ds] + \frac{1}{2}\mathbf{E}[\sup_{0\leq s\leq t}|Y_s^{n+1}-Y_s^n|_H^2 ds] \\
&+C\mathbf{E}[\int_0^t [|b(s,Y_s^{n+1},\mathbf{F}(Y_s^n)) - b(s,Y_s^n,\mathbf{F}(Y_s^{n-1}))|_H^2 ds] \\
&+C\mathbf{E}[\int_0^t [|\sigma(s,Y_s^{n+1},\mathbf{F}(Y_s^n)) - \sigma(s,Y_s^n,\mathbf{F}(Y_s^{n-1}))|_H^2 ds] \\
&+C\mathbf{E}[\int_0^t \int_{\mathbb{R}^*} [|\theta(s,Y_s^{n+1},\mathbf{F}(Y_s^n),e) - \theta(s,Y_s^n,\mathbf{F}(Y_s^{n-1}),e)|_H^2 \nu(de)ds]. \quad (29)
\end{aligned}$$

By the Lipschitz properties of b, σ and θ, we deduce

$$\begin{aligned}
\mathbf{E}\left[\sup_{0\leq s\leq t}|Y_s^{n+1}-Y_s^n|_H^2\right] &\leq C\mathbf{E}[\int_0^t [|Y_s^{n+1}-Y_s^n|_H^2 ds] \\
&+C\mathbf{E}[\int_0^t |\mathbf{F}(Y_s^n)-\mathbf{F}(Y_s^{n-1})|_H^2 ds]. \quad (30)
\end{aligned}$$

We use the mean theorem and obtain the existence for each $n \in \mathbb{N}$, $t \in [0,T]$ of a random variable $\tilde{Y}^n(t) \in \mathbf{L}^2(\Omega, H)$ so that

$$|\mathbf{F}(Y_t^n)-\mathbf{F}(Y_t^{n-1})|_H \leq \|\nabla\mathbf{F}(\tilde{Y}^n(t))\|\|Y_t^n - Y_t^{n-1}\|_{\mathbf{L}^2(\Omega;H)}. \quad (31)$$

The two relations (30) and (31) lead to:

$$\begin{aligned}
&\mathbf{E}\left[\sup_{0\leq s\leq t}|Y_s^{n+1}-Y_s^n|_H^2\right] \\
&\leq C\mathbf{E}[\int_0^t [|Y_s^{n+1}-Y_s^n|_H^2 ds] + C\mathbf{E}[\int_0^t |Y_s^n - Y_s^{n-1}|_H ds]. \quad (32)
\end{aligned}$$

Let us now define $a_t^n = \mathbf{E}\left[\sup_{0\leq s\leq t}|Y_s^n - Y_s^{n-1}|_H^2\right]$ and $A_t^n = \int_0^t a_s^n ds$. Using (32), we obtain:

$$a_t^{n+1} \leq CA_t^{n+1} + CA_t^n. \quad (33)$$

We multiply the above inequality by e^{-Ct} and derive

$$\frac{d(A_t^{n+1}e^{-Ct})}{dt} \leq Ce^{-Ct}A_t^n,$$

which allows us to conclude that

$$A_t^{n+1} \leq Ce^{Ct}\int_0^t e^{-Cs}A_s^n ds \leq Ce^{Ct}tA_t^n.$$

This inequality, together with (33), gives

$$a_t^{n+1} \leq C^2e^{Ct}tA_t^n + CA_t^n \leq C_T\int_0^t A_s^n ds,$$

where C_T is a given constant. By iteration for all n, we finally obtain

$$\mathbf{E}[\sup_{0\leq s\leq T}|Y_s^{n+1} - Y_s^n|_H^2] \leq C\frac{(C_TT)^n}{n!}.$$

This implies that we can find $Y \in \mathbf{S}_2(\Omega \times [0,T]; H)$ such that

$$\lim_{n\to\infty}\mathbf{E}[\sup_{0\leq s\leq t}|Y_s^n - Y_s|_H^2] = 0.$$

By (29) we remark that Y^n also converges to Y in $\mathbf{L}^2(\Omega \times [0,T], V)$. Passing to the limit in (28), we obtain that Y satisfies this equation.

II. Uniqueness of the solution.

Let Y_1 and Y_2 be two solutions in $\mathbf{S}^2(\Omega\times[0,T],H)\cap\mathbf{L}^2(\Omega\times[0,T],V)$. Applying

Itô formula, we have

$$
\begin{aligned}
|Y_t^1 - Y_t^2|_H^2 = &-2\int_0^t \langle Y_s^1 - Y_s^2, L(Y_s^1 - Y_s^2)\rangle ds \\
&+ 2\int_0^t \langle Y_s^1 - Y_s^2, b(s, Y_s^1, \mathbf{F}(Y_s^1)) - b(s, Y_s^2, \mathbf{F}(Y_s^2))\rangle_H ds \\
&+ 2\int_0^t \langle Y_s^1 - Y_s^2, \sigma(s, Y_s^1, \mathbf{F}(Y_s^1)) - \sigma(s, Y_s^2, \mathbf{F}(Y_s^2))\rangle_H dW_s \\
&+ \int_0^t |\sigma(s, Y_s^1, \mathbf{F}(Y_s^1)) - \sigma(s, Y_s^2, \mathbf{F}(Y_s^2))|_H^2 ds \\
&+ \int_0^t \int_{\mathbb{R}^*} \big[|\theta(s, Y_{s^-}^1, \mathbf{F}(Y_{s^-}^1), e) - \theta(s, Y_{s^-}^2, \mathbf{F}(Y_{s^-}^2), e)|_H^2 \\
&+2\langle Y_{s^-}^1 - Y_{s^-}^2, \theta(s, Y_{s^-}^1, \mathbf{F}(Y_{s^-}^1), e) - \theta(s, Y_{s^-}^2, \mathbf{F}(Y_{s^-}^2), e)\rangle\big] \tilde{N}(ds, de) \\
&+ \int_0^t \int_{\mathbb{R}^*} |\theta(s, Y_{s^-}^1, \mathbf{F}(Y_{s^-}^1), e) - \theta(s, Y_{s^-}^2, \mathbf{F}(Y_{s^-}^2), e)|_H^2 ds\nu(de).
\end{aligned}
$$

Using the coercivity assumption on the operator L, the Lipschitz property of b, σ, θ and the boundness of the Fréchet derivative of the operator $\mathbf{F}$, we get:

$$
\begin{aligned}
\mathbf{E}[|Y_t^1 - Y_t^2|_H^2] \leq -\chi\mathbf{E}[\int_0^t ||Y_s^1 - Y_s^2||_V^2 ds] + C\mathbf{E}[\int_0^t |Y_s^1 - Y_s^2|_H^2 ds] \\
+\frac{1}{2}\mathbf{E}[\sup_{0\leq s\leq t} |Y_t^1 - Y_t^2|_H^2] + C\mathbf{E}[\int_0^t |b(s, Y_s^1, \mathbf{F}(Y_s^1)) - b(s, Y_s^2, \mathbf{F}(Y_s^2))|_H^2 ds] \\
+C\mathbf{E}[\int_0^t |\sigma(s, Y_s^1, \mathbf{F}(Y_s^1)) - \sigma(s, Y_s^2, \mathbf{F}(Y_s^2))|_H^2 ds] + \\
C\mathbf{E}[\int_0^t \int_{\mathbb{R}^*} |\theta(s, Y_s^1, \mathbf{F}(Y_s^1)) - \theta(s, Y_s^2, \mathbf{F}(Y_s^2))|_H^2 \nu(de) ds] \\
\leq C\mathbf{E}[\int_0^t |Y_s^1 - Y_s^2|_H^2 ds].
\end{aligned}
$$

We thus deduce that $Y_t^1 = Y_t^2$. □

3.2 Backward Mean-Field SPDEs

We prove now an existence and uniqueness result for the solution of mean-field backward SPDEs with jumps. The analysis is carried out in a general case, where there exists a *general mean-field operator* acting on each composant of the solution. We consider the same framework as in the previous section. Let A be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda \geq 0$ so that

$$2\langle Au, u\rangle + \lambda |u|_H^2 \geq \alpha ||u||_V^2 \quad \text{for all } u \in V,$$

where $\langle Au, u\rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

Assumption 3.2 Let $f : [0,T]\times\Omega\times H\times H\times H\times H\times \mathbf{L}_\nu^2(H)\times\mathbf{L}_\nu^2(H) \to H$ be a $\mathcal{P}\times\mathcal{B}(H)\times\mathcal{B}(H)\times\mathcal{B}(H)\times\mathcal{B}(H)\times\mathcal{B}(\mathbf{L}_\nu^2(H))\times\mathcal{B}(\mathbf{L}_\nu^2(H))/\mathcal{B}(H)$ measurable. There exists a constant $C < \infty$ such that

$$\begin{aligned}
&|f(t,\omega,y_1,\tilde{y}_1,z_1,\tilde{z}_1,q_1,\tilde{q}_1) - f(t,\omega,y_2,\tilde{y}_2,z_2,\tilde{z}_2,q_2,\tilde{q}_2)|_H \\
&\leq C(|y_1-y_2|_H + |\tilde{y}_1-\tilde{y}_2|_H + |z_1-z_2|_H + |\tilde{z}_1-\tilde{z}_2|_H \\
&\quad + |q_1-q_2|_{\mathbf{L}_\nu^2(H)} + |\tilde{q}_1-\tilde{q}_2|_{\mathbf{L}_\nu^2(H)})
\end{aligned}$$

for all $t, y_1, \tilde{y}_1, z_1, \tilde{z}_1, q_1, \tilde{q}_1, y_2, \tilde{y}_2, z_2, \tilde{z}_2, q_2, \tilde{q}_2$. We also assume the integrability condition $\mathbf{E}[\int_0^T |f(t,0,0,0,0,0,0)|_H^2 dt] < \infty$.

Theorem 3.2 *Assume Assumption 3.2 holds. Let* $\xi \in \boldsymbol{L}^2(\Omega;H)$. *Let* $\mathcal{H} : \boldsymbol{L}^2(\Omega;H) \mapsto H$, $\mathcal{J} : \boldsymbol{L}^2(\Omega;H) \mapsto H$ *and* $\mathcal{K} : \boldsymbol{L}^2(\Omega,\boldsymbol{L}_\nu^2(H)) \mapsto \boldsymbol{L}_\nu^2(H)$ *be Fréchet differentiable operators. There exists a unique* $H\times H\times\boldsymbol{L}_\nu^2(H)$*-valued progressively measurable process* (Y_t, Z_t, U_t) *so that*

(i) $\mathbf{E}[\int_0^T \|Y_t\|_V^2] < \infty,\ \mathbf{E}[\int_0^T |Z_t|_H^2] < \infty,\ \mathbf{E}[\int_0^T |U_t|^2_{\mathbf{L}_\nu^2(H)} dt] < \infty.$

(ii) $$\begin{aligned}\xi &= Y_t + \int_t^T AY_s ds + \int_t^T f(s,Y_s,\mathcal{H}(Y_s),Z_s,\mathcal{J}(Z_s),U_s,\mathcal{K}(U_s))ds \\ &\quad + \int_t^T Z_s dW_s + \int_t^T\int_{\mathbb{R}^*} U_s(e)\tilde{N}(ds,de), \quad \text{for all} \quad 0\leq t\leq T.\end{aligned}$$

The equation (ii) should be understood in the dual space V^*.

Proof ***I. Existence of the solution.*** Set $Y_t^0 = 0; Z_t^0 = 0; U_t^0 = 0$. Denote by (Y_t^n, Z_t^n, U_t^n) the unique solution of the mean-field backward equation:

$$
\begin{aligned}
dY_t^n &= AY_t^n dt + f(t, Y_t^n, \mathcal{H}(Y_t^{n-1}), Z_t^n, \mathcal{J}(Z_t^{n-1}), U_t^n, \mathcal{K}(U_t^{n-1}))dt \\
&\quad + Z_t^n dW_t + \int_{\mathbb{R}^*} U_t^n(e)\tilde{N}(dt, de); \quad Y_T^n = \xi.
\end{aligned}
$$

The existence and the uniqueness of a solution (Y_t^n, Z_t^n, U_t^n) to such an equation have been proved in [11]. By applying Itô's formula, we get

$$
\begin{aligned}
0 &= |Y_T^{n+1} - Y_T^n|_H^2 \\
&= |Y_t^{n+1} - Y_t^n|_H^2 + 2\int_t^T \langle A(Y_s^{n+1} - Y_s^n), Y_s^{n+1} - Y_s^n\rangle ds \\
&+ 2\int_t^T \langle f(s, Y_s^{n+1}, \mathcal{H}(Y_s^n), Z_s^{n+1}, \mathcal{J}(Z_s^n), U_s^{n+1}, \mathcal{K}(U_s^n)) \\
&\quad - f(s, Y_s^n, \mathcal{H}(Y_s^{n-1}), Z_s^n, \mathcal{J}(Z_s^{n-1}), U_s^n, \mathcal{K}(U_s^{n-1})), Y_s^{n+1} - Y_s^n\rangle_H ds \\
&+ \int_t^T \int_{\mathbb{R}^*} \left[|Y_{s^-}^{n+1} - Y_{s^-}^n + U_s^{n+1} - U_s^n|_H^2 - |Y_{s^-}^{n+1} - Y_{s^-}^n|_H^2\right] \tilde{N}(ds, de) \\
&+ \int_t^T \int_{\mathbb{R}^*} [|U_s^{n+1}(e) - U_s^n(e)|_H^2]\nu(de)ds \\
&+ 2\int_t^T \langle Y_s^{n+1} - Y_s^n, d(\mathcal{Z}_s^{n+1} - \mathcal{Z}_s^n)\rangle_H + \int_t^T |Z_s^{n+1} - Z_s^n|_H^2 ds,
\end{aligned}
$$

where $\mathcal{Z}_t^n := \int_0^t Z_s^n dW_s$. We thus get, by taking the expectation and using the coercivity assumption on the operator A,

$$
\begin{aligned}
&\mathbf{E}[|Y_t^{n+1}-Y_t^n|_H^2]+\mathbf{E}[\int_t^T|Z_s^{n+1}-Z_s^n|_H^2ds]\\
&\qquad+\mathbf{E}[\int_t^T\int_{\mathbb{R}^*}|U_s^{n+1}-U_s^n|_H^2\nu(de)ds]\\
&=-2\mathbf{E}[\langle A(Y_s^{n+1}-Y_s^n),Y_s^{n+1}-Y_s^n\rangle ds]\\
&-2\mathbf{E}[\int_t^T\langle f(s,Y_s^{n+1},\mathcal{H}(Y_s^n),Z_s^{n+1},\mathcal{J}(Z_s^n),U_s^{n+1},\mathcal{K}(U_s^n))\\
&\qquad-f(s,Y_s^n,\mathcal{H}(Y_s^{n-1}),Z_s^n,\mathcal{J}(Z_s^{n-1}),U_s^n,\mathcal{K}(U_s^{n-1})),Y_s^{n+1}-Y_s^n\rangle ds]\\
&\le\lambda\mathbf{E}[\int_t^T|Y_s^{n+1}-Y_s^n|_H^2ds]-\alpha\mathbf{E}[\int_t^T\|Y_s^{n+1}-Y_s^n\|_V^2ds]\\
&-2\mathbf{E}[\int_t^T\langle f(s,Y_s^{n+1},\mathcal{H}(Y_s^n),Z_s^{n+1},\mathcal{J}(Z_s^n),U_s^{n+1},\mathcal{K}(U_s^n))\\
&\qquad-f(s,Y_s^n,\mathcal{H}(Y_s^{n-1}),Z_s^n,\mathcal{J}(Z_s^{n-1}),U_s^n,\mathcal{K}(U_s^{n-1})),Y_s^{n+1}-Y_s^n\rangle_H ds].
\end{aligned}
\tag{34}
$$

By using the Cauchy Schwarz inequality and the Lipschitz property of the generator f, for each $(t,\omega)\in[0,T]\times\Omega$ we obtain:

$$
\begin{aligned}
&\langle f(s,Y_s^{n+1},\mathcal{H}(Y_s^n),Z_s^{n+1},\mathcal{J}(Z_s^n),U_s^{n+1},\mathcal{K}(U_s^n))\\
&-f(s,Y_s^n,\mathcal{H}(Y_s^{n-1}),Z_s^n,\mathcal{J}(Z_s^{n-1}),U_s^n,\mathcal{K}(U_s^{n-1})),Y_s^{n+1}-Y_s^n\rangle_H\\
&\le|f(s,Y_s^{n+1},\mathcal{H}(Y_s^n),Z_s^{n+1},\mathcal{J}(Z_s^n),U_s^{n+1},\mathcal{K}(U_s^n))\\
&-f(s,Y_s^n,\mathcal{H}(Y_s^{n-1}),Z_s^n,\mathcal{J}(Z_s^{n-1}),U_s^n,\mathcal{K}(U_s^{n-1}))|_H\cdot|Y_s^{n+1}-Y_s^n|_H\\
&\le C\left(|\mathcal{H}(Y_s^n)-\mathcal{H}(Y_s^{n-1})|_H+|\mathcal{J}(Z_s^n)-\mathcal{J}(Z_s^{n-1})|_H\right.\\
&\qquad\left.+|\mathcal{K}(U_s^n)-\mathcal{K}(U_s^{n-1})|_{\mathbf{L}^2_\nu(H)}\right)\cdot|Y_s^{n+1}-Y_s^n|_H\\
&+C\big(|Y_s^{n+1}-Y_s^n|_H+|Z_s^{n+1}-Z_s^n|_H+|U_s^{n+1}-U_s^n|_{\mathbf{L}^2_\nu(H)}\big)|Y_s^{n+1}-Y_s^n|_H.
\end{aligned}
\tag{35}
$$

We now use the mean theorem in Hilbert spaces and obtain the existence for each $t\in[0,T]$ of some random variables $\tilde{Y}^n(t)\in\mathbf{L}^2(\Omega,H)$, $\tilde{Z}^n(t)\in\mathbf{L}^2(\Omega,H)$,

$\tilde{U}^n(t) \in \mathbf{L}^2(\Omega, \mathbf{L}^2_\nu(H))$ so that

$$\begin{aligned}
|\mathcal{H}(Y_t^n) - \mathcal{H}(Y_t^{n-1})|_H &\le \|\nabla \mathcal{H}(\tilde{Y}^n(t))\| \|Y_t^n - Y_t^{n-1}\|_{\mathbf{L}^2(\Omega,H)} \\
|\mathcal{J}(Z_t^n) - \mathcal{J}(Z_t^{n-1})|_H &\le \|\nabla \mathcal{J}(\tilde{Z}^n(t))\| \|Z_t^n - Z_t^{n-1}\|_{\mathbf{L}^2(\Omega,H)} \\
|\mathcal{K}(U_t^n) - \mathcal{K}(U_t^{n-1})|_H &\le \|\nabla \mathcal{K}(\tilde{U}^n(t))\| \|U_t^n - U_t^{n-1}\|_{\mathbf{L}^2(\Omega,\mathbf{L}^2_\nu(H))}.
\end{aligned} \tag{36}$$

Using (34), (35), (36) together with the boundness of the Fréchet derivatives of the operators $\mathcal{H}, \mathcal{J}, \mathcal{K}$ and the inequality $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, we obtain:

$$\begin{aligned}
&\mathbf{E}[|Y_t^{n+1} - Y_t^n|_H^2] + \mathbf{E}[\int_t^T |Z_s^{n+1} - Z_s^n|_H^2 ds] \\
&+ \mathbf{E}[\int_t^T \int_{\mathbb{R}^*} |U_s^{n+1}(e) - U_s^n(e)|_H^2 \nu(de) ds] \\
&\le \lambda \mathbf{E}[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] - \alpha \mathbf{E}[\int_t^T \|Y_s^{n+1} - Y_s^n\|_V^2 ds] - \\
&+ C\varepsilon \mathbf{E}[\int_t^T \left(|Y_s^n - Y_s^{n-1}|_H^2 + |Z_s^n - Z_s^{n-1}|_H^2 + |U_s^n - U_s^{n-1}|_{\mathbf{L}^2_\nu(H)}^2 \right) ds] \\
&+ \frac{1}{\varepsilon} \mathbf{E}[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] \\
&+ C\beta \mathbf{E}[\int_t^T \left(|Y_s^{n+1} - Y_s^n|_H^2 + |Z_s^{n+1} - Z_s^n|_H^2 + |U_s^{n+1} - U_s^n|_{\mathbf{L}^2_\nu(H)}^2 \right) ds] \\
&+ \frac{1}{\beta} \mathbf{E}[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds],
\end{aligned}$$

where C is a constant depending on the Lipschitz constant of f and the bounding constants of the Fréchet derivative operators of $\mathcal{H}, \mathcal{J}, \mathcal{K}$.

Let us choose $\varepsilon \le \frac{1}{4C}$ and $\beta \le \frac{1}{2C}$. We set $\gamma := \lambda + C\beta + \frac{1}{\varepsilon} + \frac{1}{\beta} + \frac{1}{2}$ and then multiply the previous inequality by $e^{\gamma t}$. We thus get

$$
\begin{aligned}
&-\frac{d}{dt}\left(e^{\gamma t}\mathbf{E}[\int_t^T |Y_s^{n+1}-Y_s^n|_H^2 ds]\right)+\frac{1}{2}e^{\gamma t}\mathbf{E}[\int_t^T |Z_s^{n+1}-Z_s^n|_H^2 ds]\\
&+\frac{1}{2}\mathbf{E}[\int_t^T |Y_s^{n+1}-Y_s^n|_H^2 ds]e^{\gamma t}+\frac{1}{2}e^{\gamma t}\mathbf{E}[\int_t^T |U_s^{n+1}-U_s^n|^2_{\mathbf{L}^2_\nu(H)}ds]\\
&+\alpha e^{\gamma t}\mathbf{E}[\int_t^T \|Y_s^{n+1}-Y_s^n\|_V^2 ds]\\
&\leq \frac{1}{4}\mathbf{E}[\int_t^T |Y_s^n-Y_s^{n-1}|_H^2]e^{\gamma t}+\frac{1}{4}\mathbf{E}[\int_t^T |Z_s^n-Z_s^{n-1}|_H^2]e^{\gamma t}\\
&+\frac{1}{4}\mathbf{E}[\int_t^T |U_s^n-U_s^{n-1}|^2_{\mathbf{L}^2_\nu(H)}ds]e^{\gamma t}. \qquad (37)
\end{aligned}
$$

We now integrate between 0 and T and obtain:

$$
\begin{aligned}
&\mathbf{E}[\int_0^T |Y_s^{n+1}-Y_s^n|_H^2 ds]+\frac{1}{2}\int_0^T \mathbf{E}[\int_t^T |Y_s^{n+1}-Y_s^n|_H^2 ds]e^{\gamma t}dt\\
&+\frac{1}{2}\int_0^T \mathbf{E}[\int_t^T |Z_s^{n+1}-Z_s^n|_H^2 ds]e^{\gamma t}dt\\
&+\frac{1}{2}\int_0^T e^{\gamma t}\mathbf{E}[\int_t^T |U_s^{n+1}-U_s^n|^2_{\mathbf{L}^2_\nu(H)}ds]+\int_0^T \alpha\mathbf{E}[\int_t^T \|Y_s^{n+1}-Y_s^n\|_V^2 ds]e^{\gamma t}dt\\
&\leq \frac{1}{4}\int_0^T \mathbf{E}[\int_t^T |Y_s^n-Y_s^{n-1}|_H^2 ds]e^{\gamma t}dt+\frac{1}{4}\int_0^T \mathbf{E}[\int_t^T |Z_s^n-Z_s^{n-1}|_H^2 ds]e^{\gamma t}dt\\
&+\frac{1}{4}\int_0^T \mathbf{E}[\int_t^T |U_s^n-U_s^{n-1}|^2_{\mathbf{L}^2_\nu(H)}ds]e^{\gamma t}dt. \qquad (38)
\end{aligned}
$$

From the above inequality it follows that

$$
\begin{aligned}
&\int_0^T \mathbf{E}[\int_t^T |Y_s^n-Y_s^{n-1}|_H^2]e^{\gamma t}dt+\int_0^T \mathbf{E}[\int_t^T |Z_s^n-Z_s^{n-1}|_H^2]e^{\gamma t}dt\\
&+\int_0^T \mathbf{E}[\int_t^T |U_s^n-U_s^{n-1}|^2_{\mathbf{L}^2_\nu(H)}ds]e^{\gamma t}dt \leq \frac{1}{2^n}C.
\end{aligned}
$$

From (38) one can deduce $\mathbf{E}[\int_0^T |Y_s^{n+1}-Y_s^n|_H^2 ds] \leq \frac{1}{2^n}C$. Using (37), we derive

$$
\begin{aligned}
&\frac{1}{2}\mathbf{E}[\int_0^T |Y_s^{n+1}-Y_s^n|_H^2 ds]+\frac{1}{2}\mathbf{E}[\int_0^T |Z_s^{n+1}-Z_s^n|_H^2 ds]\\
&+\frac{1}{2}\mathbf{E}[\int_0^T |U_s^{n+1}-U_s^n|^2_{\mathbf{L}^2_\nu(H)}ds] \leq \gamma\frac{1}{2^n}C+\frac{1}{4}\mathbf{E}[\int_0^T |Y_s^n-Y_s^{n-1}|_H^2 ds]+\\
&\frac{1}{4}\mathbf{E}[\int_0^T |Z_s^n-Z_s^{n-1}|_H^2 ds]+\frac{1}{4}\mathbf{E}[\int_0^T |U_s^{n+1}-U_s^n|^2_{\mathbf{L}^2_\nu(H)}ds],
\end{aligned}
$$

which implies that

$$\mathbf{E}[\int_0^T |Y_s^{n+1} - Y_s^n|_H^2 ds] + \mathbf{E}[\int_0^T |Z_s^{n+1} - Z_s^n|_H^2 ds]$$
$$+ \mathbf{E}[\int_0^T |U_s^{n+1} - U_s^n|_{\mathbf{L}_\nu^2(H)}^2 ds] \leq \frac{1}{2^n} C\gamma n.$$

The above inequality together with (37) leads to

$$\mathbf{E}[\int_0^T \|Y_s^{n+1} - Y_s^n\|_V^2 ds] \leq (\frac{1}{2})^n (n+1) C\gamma.$$

We can conclude that the sequence (Y^n, Z^n, U^n), $n \geq 1$ is a Cauchy sequence in the Banach space $\mathbf{L}^2(\Omega\times[0,T],V)\times\mathbf{L}^2(\Omega\times[0,T],H)\times\mathbf{L}^2(\Omega\times[0,T],\mathbf{L}_\nu^2(H)$, and thus converges in the corresponding spaces to (Y,Z,U). The limit (Y,Z,U) satisfies:

$$Y_t + \int_t^T AY_s ds + \int_t^T f(s, Y_s, \mathcal{H}(Y_s), Z_s, \mathcal{J}(Z_s), U_s, \mathcal{K}(U_s)) ds + \int_t^T Z_s dW_s$$
$$+ \int_t^T \int_{\mathbb{R}^*} U_s \tilde{N}(ds, de) = \xi \quad \text{a.s.}$$

II. Uniqueness of the solution

The proof of the uniqueness of the solution is classical, but we give it for reader's convenience. Suppose (Y_t, Z_t, U_t) and $(\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t)$ are two solutions. By applying Itô's formula, we obtain

$$\mathbf{E}[|Y_t - \tilde{Y}_t|_H^2] + \mathbf{E}[\int_t^T |Z_s - \tilde{Z}_s|_H^2 ds] + \mathbf{E}[\int_t^T |U_s - \tilde{U}_s|_{\mathbf{L}_\nu^2(H)}^2] ds]$$
$$= -\mathbf{E}[\langle A(Y_s - \tilde{Y}_s), Y_s - \tilde{Y}_s\rangle ds]$$
$$- 2\mathbf{E}[\int_t^T \langle f(s, Y_s, \mathcal{H}(Y_s), Z_s, \mathcal{J}(Z_s), U_s, \mathcal{K}(U_s))$$
$$- f(s, \tilde{Y}_s, \mathcal{H}(\tilde{Y}_s), \tilde{Z}_s, \mathcal{J}(\tilde{Z}_s), \tilde{U}_s, \mathcal{K}(\tilde{U}_s)), \tilde{Y}_s - \tilde{Y}_s\rangle_H ds]$$
$$\leq \lambda\mathbf{E}[\int_t^T |Y_s - \tilde{Y}_s|_H^2 ds] - \alpha\mathbf{E}[\int_t^T |Y_s - \tilde{Y}_s|_V^2 ds] + K\mathbf{E}[\int_t^T |Y_s - \tilde{Y}_s|_H^2 ds]$$
$$+ \frac{1}{2}\mathbf{E}[\int_t^T |Z_s - \tilde{Z}_s|_H^2 ds] + \frac{1}{2}\mathbf{E}[\int_t^T |U_s - \tilde{U}_s|_{\mathbf{L}_\nu^2(H)}^2 ds].$$

We thus derive that

$$\mathbf{E}[|Y_t - \tilde{Y}_t|_H^2] \leq (\lambda + K)\mathbf{E}[\int_t^T |Y_s - \tilde{Y}_s|_H^2].$$

Hence, by Gronwall lemma, we get $Y_t = \tilde{Y}_t$. This also implies that $Z_t = \tilde{Z}_t$ and $U_t = \tilde{U}_t$. □

4 Conclusions

The paper aims at providing an extensive analysis of mean-field stochastic partial differential equations and their associated control problem, in a general framework. More precisely, we have studied the optimal control problem for mean-field stochastic *partial* differential equations with jumps and a *general* mean-field operator, in case of *partial information control*. We have first established necessary and sufficient maximum principles in a general setting, where different mean-field operators are acting on both the controlled state process and the control. We then applied these results in order to solve explicitly an optimal harvesting problem. Finally, we have shown the existence and the uniqueness of the solution for both forward and backward mean-field stochastic partial differential equations with a *general mean-field operator*.

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