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A Lagrangian study of internal Gerstner- and Stokes-type gravity waves

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HIGHLIGHTS

- A Lagrangian study of two types of internal gravity waves is presented.
- Gerstner-type waves have no mean drift, and no change of mean isopycnals levels.
- Stokes-type waves induce a mean drift as well as a mean isopycnal displacement.

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ABSTRACT

By using a direct Lagrangian formulation, two different solutions for the nonlinear motion in inviscid internal gravity waves between horizontal planes in a non-rotating fluid are discussed. The first solution assumes a constant Brunt–Väisälä frequency N. The main findings are that the vertical mean displacement of the isopycnals and the horizontal mean drift are both zero to second order (Sanderson, 1985). We here point out that this is characteristic for a Gerstner-type wave. Attention is drawn to a second solution for arbitrary stable stratification which yields a non-zero vertical mean isopycnal displacement. Although internal waves are rotational, this solution characterizes a Stokes-type wave. The vertical mean displacement, which can be positive or negative depending on the spatial location and the wave mode in question, is due to the divergence effect in Lagrangian terms. It is shown that the vertical mean displacement at a certain level is proportional to the depth-dependent, or partial Stokes flux. Finally, it is demonstrated that the two wave types have different mean vorticities to second order. For constant N, they are equal in magnitude, but have opposite signs. In the Appendix the existence of a mean drift in internal waves of the Stokes-type is verified by a direct Lagrangian calculation for a slightly viscous fluid with constant N.

1. Introduction

Applying a Lagrangian formulation, a nonlinear solution to second order for inviscid internal gravity waves between horizontal planes was given by Sanderson [1] in the case of constant Brunt-Väisälä frequency. This solution was purely periodic, i.e. there was no horizontal mean drift due to the wave motion. Although not pointed out in [1], the derived solution corresponds to a Gerstner-type internal wave; see Weber [2] for interfacial gravity waves. For short, a Gerstner-type internal wave will hereafter be referred to as a G-wave.

The nonlinear vertical mean displacement of the isopycnals was zero in [1]. This is typical for periodic wave motion with vorticity and no mean drift (Gerstner [3]). However, for irrotational surface waves; see [4–6], it was demonstrated

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that the Lagrangian wave-induced increase in mean surface level is related to the Stokes mass transport [7]. Generalizing to continuous stratification, it is easily seen that the nonlinear continuity equation for internal waves in [1] also allows for a solution with a non-zero vertical mean displacement of the isopycnal levels. This indicates that a Lagrangian solution with non-zero mean drift is possible. In fact, Wunsch [8] demonstrated from Eulerian calculations with constant Brunt-Väisälä frequency N that the Stokes drift in internal Kelvin waves was non-zero, using the formula by Longuet-Higgins [9]. Later, Al-Zanaidi and Dore [10] calculated the Stokes drift in internal waves in the case of a thin thermocline with constant N. The Stokes drift results were generalized to arbitrary stable stratification in [11] with application to equatorial internal Kelvin waves. In the present paper we establish a new connection between the vertical mean displacement of the isopycnals at a certain level, and the integrated Stokes drift in the Stokes-type internal wave from the bottom to that specific level. This result is valid for arbitrary Brunt–Väis älä frequency. In the future we refer to the Stokes-type internal wave as an S-wave.

Finally, we consider the vorticity in Lagrangian coordinates. For irrotational surface waves, it is readily found that by setting the mean vorticity equal to zero, one finds the Stokes drift in these waves. On the other hand, by assuming zero mean drift, the mean vorticity in the surface Gerstner-type wave is obtained. For internal waves, which are inherently rotational, this problem is not so simple, as will be demonstrated here.

The present analysis is based on the Lagrangian description of fluid motion, which historically has not been the main-stream approach to flows in continuous media; see the recent review by Weber [12] for surface waves. However, the number of Lagrangian studies is rapidly increasing, e.g. [13–15].

It should also be mentioned that there is an interesting ongoing research in Lagrangian form on inviscid internal waves in the presence of an underlying uniform current, especially with application to the equatorial β-plane; see e.g. [16–21]. However, the present investigation will focus on internal wave motion in a non-rotating system. Furthermore, the important problem of internal wave stability [22,23] is outside the scope of the present investigation.

This paper is organized as follows: In Section 2 we formulate the problem mathematically, using a Lagrangian description of fluid motion. In Section 3 we state the trivial linear problem, and in Section 4 we point out that the nonlinear solution derived in [1] is a G-wave. In Section 5 we consider the relation between the Stokes drift and the mean vertical displacement in a S-wave (a novel derivation of the Stokes drift from a direct Lagrangian approach for a viscous stratified fluid is given in the Appendix), and in Section 6 we calculate and compare the mean vorticity in G-waves and S-waves. Finally, Section 7 contains a short summary and some concluding remarks.

2. Mathematical formulation

We study wave motion in an inviscid stratified fluid between horizontal planes separated by a distance H. The effect of the earth's rotation is disregarded, and we consider two-dimensional motion in the vertical plane. The motion is described by using a Lagrangian formulation. Let a fluid particle (a, c) initially have coordinates (X_0, Z_0). Its position (X, Z) at later times will then be a function of a, c and time t. Here the X-axis is horizontal, and situated at the upper plane, while the Z-axis is vertical, and positive upwards. Velocity components and accelerations are given by (X_t, Z_t) and (X_{tt}, Z_{tt}), respectively, where subscripts denote partial differentiation. The initial density of a particle is ρ_0, and the density at subsequent times is ρ. The horizontal and vertical momentum equations in the Lagrangian plane become (Lamb [24]):

$$X_{tt}X_a + Z_{tt}Z_a + gZ_a = -P_a/\rho, \tag{1}$$

$$X_{tt}X_c + Z_{tt}Z_c + gZ_c = -P_c/\rho, \tag{2}$$

where P denotes the pressure. The conservation of mass can be written

$$\rho J\,(X, Z) = \rho_0 J(X_0, Z_0), \tag{3}$$

where $J\,(F, G) = F_aG_c - F_cG_a$ is the two-dimensional Jacobian. We assume that the fluid is incompressible, i.e. $\rho\,(a, c, t) = \rho_0(a, c, 0)$, so (3) reduces to

$$J\,(X, Z) = J(X_0, Z_0). \tag{4}$$

Even though (a, c) is not the initial particle position, it is convenient to expand the displacements and the pressure in the following perturbation series (Pierson [25]):

$$X = a + x\,(a, c, t) = a + x_1 + x_2 + .. \tag{5}$$

$$Z = c + z\,(a, c, t) = c + z_1 + z_2 + .. \tag{6}$$

$$P = P_0\,(a, c) + p\,(a, c, t) = P_0 + p_1 + p_2 + .. \tag{7}$$

Here the various terms on the right-hand side contain increasing powers of a small parameter. In principle, the initial density must be expanded in the same way. However, in the absence of the Coriolis force, there is no way to balance a small a-dependence in ρ_0 in (1), so we take that $\rho_0 = \rho_0(c)$; see [1]. To lowest order we obtain from (1)–(2) that $P_0 = P_0(c)$, and $P_{0c} = -\rho_0 g$. Applying the Boussinesq-approximation, the deviations from (a, c, P_0) must satisfy

$$x_{tt} + x_{tt}x_a + z_{tt}z_a + g(\rho_0/\rho_r)z_a = -p_a/\rho_r, \tag{8}$$

$$z_{tt} + x_{tt}x_c + z_{tt}z_c + g(\rho_0/\rho_r)z_c = -p_c/\rho_r, \tag{9}$$

where ρ_r is a constant reference density. The continuity equation (4) becomes

$$x_a + z_c + J(x, z) = x_{0a} + z_{0c} + J(x_0, z_0)\,. \tag{10}$$

3. The linear problem

The linear problem is trivial. However, the linear solutions are needed in order to solve the second order problem in a perturbation procedure. Hence, it eases the reading if they are stated explicitly. To avoid too many repeated subscripts, we introduce a tilde to denote the linearized solutions, i.e. $x_1 = \tilde{x}$ etc. From the linearized versions of (8)–(9) we find for the vorticity:

$$\tilde{x}_{ttc} - \tilde{z}_{tta} - N^2\tilde{z}_a = 0, \tag{11}$$

where N is the Brunt-Väisälä frequency defined by

$$N^2 = -(g/\rho_r)d\rho_0/dc. \tag{12}$$

The continuity equation (10) reduces to

$$\tilde{x}_a + \tilde{z}_c = \tilde{x}_{0a} + \tilde{z}_{0c}. \tag{13}$$

By differentiating in time

$$\tilde{x}_{at} + \tilde{z}_{ct} = 0. \tag{14}$$

For a progressive wave with frequency ω, we have from (14) that $\omega(\tilde{x}_a + \tilde{z}_c) = 0$. Hence, since $\omega \neq 0$, we have to first order

$$\tilde{x}_a + \tilde{z}_c = 0. \tag{15}$$

Accordingly, from (13), we have that $\tilde{x}_{0a} + \tilde{z}_{0c} = 0$ to this order.

From (15) we can introduce the linear stream function, or equivalently (11) yields for the vertical displacement:

$$\tilde{z}_{ttcc} + \tilde{z}_{ttaa} + N^2\tilde{z}_{aa} = 0. \tag{16}$$

We assume that the variables separate, and write the solution in the form of complex Fourier components:

$$\tilde{z} = F(c)\exp i(ka - \omega t), \tag{17}$$

where k is the real wave number. Furthermore, we consider long waves, i.e. $\omega^2 < N^2$. Then (16) reduces to

$$F'' + (N^2/C^2)F = 0, \tag{18}$$

where a prime denotes differentiations with respect to c, and $C = \omega/k$ is the phase speed. The boundary conditions are

$$F(c = 0) = 0, F(c = -H) = 0. \tag{19}$$

The eigenvalue problem (18)–(19) is easily solved when N is constant. However, we need not simplify at this stage. In general we have that $N = N(c)$, and we write F as the sum of eigenfunctions $\varphi_n(c)$, subject to:

$$\left.\begin{array}{l} \varphi_n'' + (N^2/C_n^2)\varphi_n = 0, n = 1, 2, 3, .. \\ \varphi_n = 0,\;\; c = 0, -H, \end{array}\right\} \tag{20}$$

where C_n is the constant eigenvalue. Here, in the non-rotating case, C_n is just the phase speed for mode n. Assuming that the variables are real, we obtain from (15) and (17):

$$\tilde{x} = -\sum_{n=1}^{\infty}(A_n/k_n)\varphi_n'\sin\theta_n, \tag{21}$$

$$\tilde{z} = \sum_{n=1}^{\infty}A_n\varphi_n\cos\theta_n, \tag{22}$$

where the phase function is given by $\theta_n = k_n a - \omega t$. The use of orthogonal eigenfunctions is very convenient for studying the baroclinic response to wind-forcing; see for example [26]. In the present problem, the eigenfunctions become part of the nonlinear wave-forcing of the mean flow. For a specific $N(c)$, the eigenvalue problem (20) is easily solved numerically.

In the case of constant N, the problem is very simple, and the vertical modes are sinusoidal in the Lagrangian coordinates [1]:

$$\tilde{x}_n = -(\gamma_n A_n/k_n)\cos(\gamma_n c)\sin\theta_n, \tag{23}$$

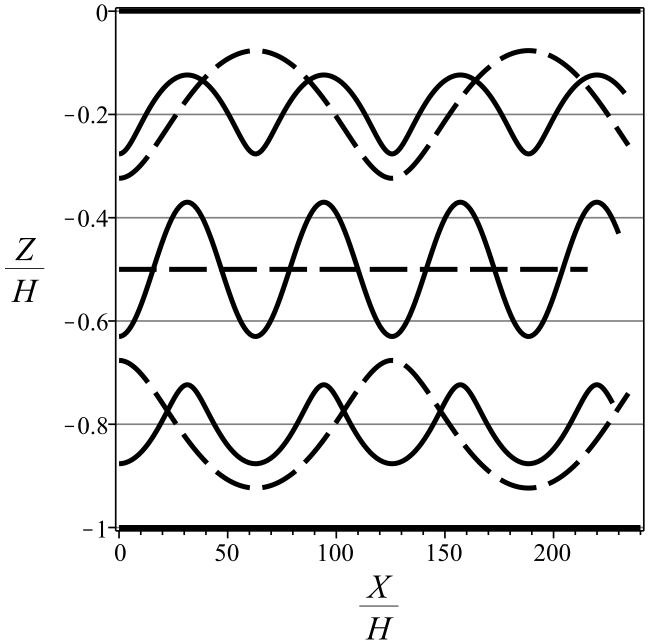

Fig. 1. Wave shape in Lagrangian formulation for linear internal waves at non-dimensional positions $c/H = -0.2, -0.5, -0.8$. Solid lines: first mode. Dashed lines: second mode. Here $A_1/H = A_2/H = 0.13$, and $k_1H = 0.1$, $k_2H = 0.05$.

$$\tilde{z}_n = A_n \sin(\gamma_n c) \cos\theta_n, \tag{24}$$

where $\gamma_n = n\pi/H$. The eigenvalues are $C_n = N/\gamma_n = NH/(n\pi)$.

The full linear solution becomes

$$X = a + \tilde{x}, \tag{25}$$

$$Z = c + \tilde{z}. \tag{26}$$

It is readily shown from (25)–(26) with constant N that the shape of the internal wave from Lagrangian linear theory is trochoidal near the top and bottom, and sinusoidal in the middle [1], while in Eulerian coordinates the linear shape is purely sinusoidal everywhere. Similar non-sinusoidal features are also seen in the Lagrangian solution for interfacial waves [2]. In Fig. 1 we have displayed to first two modes from (25)–(26) for constant N at three different positions.

It should be pointed out that individual particle trajectories are closed to first order. We obtain for each mode from (22) and (23):

$$\frac{(X-a)^2}{A_n^2 \left(\varphi_n'\right)^2 / k_n^2} + \frac{(Z-c)^2}{A_n^2 \varphi_n^2} = 1. \tag{27}$$

Hence, each particle moves in an elliptic path in the Lagrangian plane with centre at (a, c). Since $\left(\varphi_n'\right)^2 / k_n^2 \gg \varphi_n^2$ for long waves, the major axis is always in the X-direction. At the upper and lower boundaries the minor axis tends to zero, and here the particles move in straight horizontal lines.

4. A nonlinear Gerstner type solution

In [1] it is assumed that the initial particle position are $X_0 = a$ and $Z_0 = c$. But this is strictly speaking not true. We have demonstrated from linear theory that generally (a, c) is the centre of the elliptic particle trajectory in the Lagrangian plane; see (27). This was also shown in [1] for constant N. If ε is the small parameter in the power series expansions (5)–(7), we find by utilizing (15) for periodic wave motion that (10) can be written

$$x_a + z_c + J(x, z) = O\left(\varepsilon^2\right) + O\left(\varepsilon^3\right) + .. \tag{28}$$

In [1] the right-hand side is identically zero. Lagrangian variables that fulfil this condition are sometimes referred to as Miche's coordinates after [27]; see Clamond [28]. More correctly, if the right-hand side is of $O\left(\varepsilon^3\right)$, then (28) becomes to second order

$$x_{2a} + z_{2c} = -\tilde{x}_a \tilde{z}_c + \tilde{x}_c \tilde{z}_a. \tag{29}$$

For constant N it is then shown in [1] that a second-order solution for the displacement field is given by

$$x_{2n} = [\gamma_n^2 A_n^2/(2k_n)] \sin 2\theta_n, \tag{30}$$

$$z_{2n} = 0, \tag{31}$$

where we have used the present notation, and introduced real parts. The modification to the linear wave shape from (30)–(31) is given in [1].

We remark that the solution (30)–(31) is purely periodic in the horizontal direction, and does not introduce any mean drift, i.e.

$$\bar{x}_{2nt} = 0, \tag{32}$$

where the over-bar denotes average over the wave cycle. The full solution to second order,

$$X = a + \tilde{x} + x_2, \tag{33}$$

$$Z = c + \tilde{z} + z_2, \tag{34}$$

represents therefore a G-wave; see [2] for interfacial Gerstner-type waves in a two-layer fluid. This fact has apparently been overlooked in [1]. It should also be pointed out that Thorpe [29] in his nonlinear Eulerian solution for the shape of the internal wave assumes that the vertical mean displacement is zero.

5. The Stokes drift and the vertical mean displacement

In [1] we note that $z_2 = 0$, i.e. there are no vertical mean displacement of the isopycnals. But it is easily seen that (29) has an alternative solution. By averaging in the horizontal direction, we find that

$$\bar{x}_{2a} + \bar{z}_{2c} = -\overline{\tilde{x}_a \tilde{z}_c} + \overline{\tilde{x}_c \tilde{z}_a}, \tag{35}$$

where the averaging process again is denoted by an over-bar. Assuming that the mean quantities do not vary in the horizontal, i.e. $\bar{x}_{2a} = 0$, we find from (35) by inserting from the linear solutions (21)–(22), valid for $N = N(c)$:

$$\bar{z}_{2n} = \frac{1}{2} A_n^2 \varphi_n \varphi_n', \tag{36}$$

satisfying $\bar{z}_{2n}\,(c = 0) = \bar{z}_{2n}\,(c = -H) = 0$. We note that $\bar{z}_2$ generally is different from zero. This is the divergence effect for internal waves, as first shown in [6] for surface waves. For constant N, i.e. $\varphi_n = \sin(\gamma_n c)$, the result simply becomes:

$$\bar{z}_{2n}\,(c) = \frac{1}{4} A_n^2 \gamma_n \sin(2\gamma_n c), \tag{37}$$

where $\gamma_n = n\pi/H$.

For surface waves, an increase of the Lagrangian mean fluid level is associated with a positive mean momentum flux, or a Stokes flux; see [4]. For the stratified problem, internal changes of mean particle levels must also be related to the Stokes drift, as will be demonstrated below. Unlike the result in [1], a non-zero Stokes drift has been found for gravity waves in stratified fluids; see e.g. [8,10,11]. In Lagrangian notation the Stokes drift u_S in [9] can be written to second order

$$u_S = \overline{\tilde{x}_{ta}\tilde{x}} + \overline{\tilde{x}_{tc}\tilde{z}}. \tag{38}$$

Alternatively, the Stokes drift can be determined by a direct Lagrangian calculation. This is a novel approach, and a brief account is given in the Appendix.

Inserting from our linear solutions (23)–(24) into (38), we find

$$u_{Sn} = \frac{1}{2} C_n A_n^2 (\varphi_n \varphi_n')', \tag{39}$$

see [11], where also the Stokes drift for the two first modes in the case of a strong pycnocline (Equatorial Pacific) has been calculated.

By combining (36) and (39), we find that

$$d\bar{z}_{2n}/dc = u_{Sn}/C_n. \tag{40}$$

By integrating (40) in the vertical, using that $\bar{z}_{2n}\,(c = -H) = 0$, we obtain

$$\bar{z}_{2n}\,(c) = \frac{1}{C_n} \int_{-H}^{c} u_{Sn}\, dc = U_{Sn}(c)/C_n. \tag{41}$$

This is a new result, and shows that the change of the isopycnal mean level for mode n at a given vertical position is proportional to the Stokes flux $U_{Sn}(c)$ in the layer from the bottom to that specific level. $U_{Sn}(c)$ will be denoted as the partial Stokes flux, as opposed to the total Stokes flux resulting from integration of the Stokes drift across the entire layer.

The result (41) has a nice similarity with the one-layer case of in [12]. Here it is shown that the difference between the Lagrangian mean surface elevation h_L and the Eulerian mean elevation h_E in [5] for surface waves in a layer of finite depth can be written as

$$h_L - h_E = U_S/C. \tag{42}$$

In (42) C is the surface wave speed and U_S is the total Stokes flux. It should be noted that for the internal case, both sides of (41) vanish identically when integrating between the rigid boundaries; see [11].

6. The mean vorticity in internal waves

So what characterizes the two different nonlinear solutions for internal waves? For surface waves in an inviscid homogeneous fluid the vorticity is the key issue. Irrotational surface waves (Stokes waves) possess a mean forward drift, while in rotational waves of the Gerstner-type, the particles move in closed orbits, and there is no mean drift. For small amplitude surface waves, the surface Stokes wave and the surface Gerstner-type wave both have zero vorticity to first order in the Lagrangian formulation; see Clamond [28]. By calculating the vorticity to second order, the assumption of vanishing mean vorticity directly yields an expression for the Stokes drift in Stokes waves [30]. On the other hand, by assuming that there is no mean drift, we obtain the non-zero mean vorticity in the Gerstner-type surface wave.

For internal waves it is more complicated, since vorticity is an inherent part of the wave motion. Already to first order the vorticity is non-zero, as can be seen from (11). In the Eulerian formulation the fluid vorticity ζ perpendicular to the X, Z-plane is linearly dependent on the velocity shear. In Lagrangian notation it becomes nonlinear. It can be written [2,28]:

$$\zeta = [J(X, X_t) - J(Z_t, Z)]/J(X, Z). \tag{43}$$

From (43) and (21)–(22) we find for the linear vorticity in the long wave approximation:

$$\tilde{\zeta}_n = \tilde{x}_{ntc} - \tilde{z}_{nta} = C_n A_n \varphi_n'' \cos\theta_n. \tag{44}$$

There is no difference to this order between G-waves and S-waves. For the second order vorticity, we note from (28) that the Jacobian equivalently can be written

$$J(X, Z) = J(X_0, Z_0) = 1 + O(\varepsilon^2). \tag{45}$$

Hence, to second order we find from (43) by using (5)–(6):

$$\bar{\zeta}_2 = \bar{x}_{2tc} - \bar{z}_{2ta} + \overline{\tilde{x}_{tc}\tilde{x}_a} - \overline{\tilde{x}_{ta}\tilde{x}_c} - \overline{\tilde{z}_{ta}\tilde{z}_c} + \overline{\tilde{z}_{tc}\tilde{z}_a}. \tag{46}$$

Assuming that the mean quantities do not vary in the horizontal, and applying the general solutions (21)–(22), (46) reduces for long waves to

$$\bar{\zeta}_{2n} = \bar{x}_{2ntc} - C_n A_n^2 \varphi_n' \varphi_n''. \tag{47}$$

We note that when $\bar{x}_{2nt} = 0$, which is the case considered in [1], the mean vorticity becomes, when we apply (20):

$$\bar{\zeta}_{Gn} = \frac{1}{C_n} A_n^2 N^2 \varphi_n \varphi_n', \tag{48}$$

where the subscript G refers to the G-wave. On the other hand, if we have a non-zero mean drift, i.e. $\bar{x}_{2nt} = u_{Sn}$ from (39), (47) yields that

$$\bar{\zeta}_{Sn} = -\frac{1}{2C_n} A_n^2 (N^2 \varphi_n^2)'. \tag{49}$$

Here the subscript S alludes to the S-wave. We note that for constant N, (49) yields a direct relation between the Stokes drift and the derivative of the mean vorticity, i.e. $u_{Sn} = -H^2 \bar{\zeta}_{Sn}'/(2n^2\pi^2)$. For a somewhat different problem ([31], internal waves with a free upper surface and the presence of a shear flow), a related dependence of the mean drift on the mean vorticity was found for arbitrary stratification.

In general we obtain that

$$\bar{\zeta}_{Sn} = -\bar{\zeta}_{Gn} + \frac{g}{2\rho_r C_n} A_n^2 \varphi_n^2 \rho_0''. \tag{50}$$

Hence, if N is constant, we have $\bar{\zeta}_{Sn} = -\bar{\zeta}_{Gn}$, i.e. the mean vorticities are equal in magnitude everywhere, but have opposite signs. Generally, in cases with a pronounced pycnocline in the fluid layer, like the Pacific equatorial thermocline (Colin et al. [32]), we have $\rho_0'' = 0$ at the (smoothed) peak of the pycnocline. Accordingly, at this level we always have that $\bar{\zeta}_{Sn} = -\bar{\zeta}_{Gn}$.

7. Summary and concluding remarks

We have here defined and discussed the G- and the S-waves in a Lagrangian formulation. These wave-types are identical to first order: the particle trajectories are closed curves (ellipses), and they possess the same vorticity. It is the nonlinear properties that divide them: the G-wave induces no net forward particle drift to second order, and does not change the mean position of the isopycnals. The S-wave, on the other hand, induces a Stokes drift in the wave propagation direction, and also alters the mean position of the material surfaces in the fluid.

The fact that S-waves induce small second-order Lagrangian changes in the positions of the isopycnals, has a clear parallel to surface waves. McIntyre [6] pointed out that the continuity equation for the mean variables in Lagrangian form was divergent. For periodic deep-water surface waves this leads to a positive Lagrangian mean sea level increase. The Lagrangian increase in sea level was earlier shown by Longuet-Higgins [4] by a different approach, and he was the first to relate this increase to the mean momentum (the total Stokes flux) in the wave motion. We have here done a similar Lagrangian analysis for internal gravity waves in an arbitrary stratified fluid between horizontal planes, and we find non-zero mean vertical displacements of the isopycnal levels, which are related to the partial Stokes flux. Although the wave motion is rotational, it appears that the internal gravity wave we consider is of the Stokes-type. This is verified by an independent Lagrangian calculation of the Stokes drift in internal waves in a slightly viscous fluid with constant stratification; see the Appendix.

When we compare the G-wave with the S-wave, the mean vorticities are different. In particular, for constant stratification, the mean vorticities are equal in magnitude, but have different sign. It will be an interesting future task to investigate (and separate) these internal wave types in the laboratory.

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Appendix. A Lagrangian calculation of the Stokes drift

The mean drift in internal waves of the Stokes-type can also be calculated from a direct Lagrangian approach. However, in this case we need to include a small non-zero viscosity into the problem; see the comprehensive discussion in [12] for surface waves. Introducing the effect of viscosity into (8), we can write

$$(x_{tt} - \nu\nabla^2 x_t)(1 + x_a) + (z_{tt} - \nu\nabla^2 z_t)z_a - cN^2 z_a = -p_a/\rho_r, \tag{A.1}$$

where ν is a constant (eddy) viscosity. We simplify further, and take that N is constant. To second order the Laplacian operator in Lagrangian form becomes; see also [25]:

$$\nabla^2 R = \nabla_L^2 R - R_a \nabla_L^2 x - R_c \nabla_L^2 z - 2\,(x_c + z_a)\,R_{ac} - 2x_a R_{aa} - 2z_c R_{cc}, \tag{A.2}$$

where we have defined $\nabla_L^2 = \partial^2/\partial a^2 + \partial^2/\partial c^2$.

We consider long waves, i.e. $\left|\partial^2/\partial c^2\right| \gg \left|\partial^2/\partial a^2\right|$. Introducing the stream function $\tilde{\psi}$ such that $\tilde{x} = \tilde{\psi}_c$, and $\tilde{z} = -\tilde{\psi}_a$, we find for the linear problem from (A.1):

$$\nu\tilde{\psi}_{tcccc} - \tilde{\psi}_{ttcc} - N^2\tilde{\psi}_{aa} = 0. \tag{A.3}$$

In this problem we assume that the waves are temporally damped. The stream function is written $\tilde{\psi} = \varphi\,(c)\exp(ika + nt)$, where $n = -i\omega - \beta$, and β is the small damping rate ($|\beta/\omega| \ll 1$). Then (A.3) reduces to

$$n\nu\varphi'''' - n^2\varphi'' + k^2N^2\varphi = 0, \tag{A.4}$$

where the prime denote differentiation with respect to c. For simplicity we take that the boundaries are impermeable, i.e. (19), and stress-free. The last condition implies

$$\varphi''\,(c = 0) = 0,\ \ \varphi''\,(c = -H) = 0. \tag{A.5}$$

The solution to (A.4) contains a temporally modified field of the inviscid type (21)–(22), plus viscous solutions in the thin boundary layers close to the upper and lower boundaries. The latter contributions are important for the development of the Eulerian part of the Lagrangian current, as first shown in [9] for surface waves. In the present investigation we are interested in the Stokes drift, which is related to the slowly damped inviscid solution; see for example [33]. Accordingly, our solution, which satisfies (19) and (A.5), becomes

$$\varphi \propto \sin\gamma_n c, \tag{A.6}$$

where $\gamma_n = n\pi/H$. Inserting into (A.4), the real part yields to lowest order that $k_n = n\pi\omega/(NH)$, as before when $N \gg \omega$. From the imaginary part of (A.4) we find for the damping rate

$$\beta_n = \nu\gamma_n^2/2. \tag{A.7}$$

Hence, our viscosity-modified linear solutions can be written

$$\tilde{x}_n = -(\gamma_n A_n/k_n)\exp(-\beta_n t)\cos(\gamma_n c)\sin\theta_n, \tag{A.8}$$

$$\tilde{z}_n = A_n \exp(-\beta_n t)\sin(\gamma_n c)\cos\theta_n. \tag{A.9}$$

By averaging (A.1) in the horizontal, we find to second order, using that $\left|\partial^2/\partial c^2\right| \gg \left|\partial^2/\partial a^2\right|$, and $\tilde{z}_c = -\tilde{x}_a$:

$$\nu u_{cc} - u_t = -cN^2\bar{z}_{2a} + \bar{p}_{2a}/\rho_r + \overline{\tilde{x}_{tt}\tilde{x}_a} + \overline{\tilde{z}_{tt}\tilde{z}_a} - \nu[4\overline{\tilde{x}_{tcc}\tilde{x}_a} + 3\overline{\tilde{z}_{tcc}(\tilde{x}_c + \tilde{z}_a)}], \tag{A.10}$$

where we have defined the Lagrangian mean drift $u = \bar{x}_{2t}$. We assume that the mean variables do not vary in the horizontal, which means that the two first terms on the right-hand side vanishes identically. By inserting from (A.8) and (A.9), (A.10) reduces to

$$\nu u_{ncc} - u_{nt} = -(3/2)\nu C_n \gamma_n^4 A_n^2 \exp(-2\beta_n t)\cos(2\gamma_n c), \tag{A.11}$$

where $C_n = \omega/k_n$ and β_n is given by (A.7). It is readily seen that the solution of (A.11) becomes

$$u_n = \frac{1}{2} C_n \gamma_n^2 A_n^2 \exp(-2\beta_n t)\cos(2\gamma_n c). \tag{A.12}$$

When inserting $\varphi_n = \sin(\gamma_n c)$, we note that (A.12) is just the temporally damped version of the inviscid Stokes drift (39) obtained from Longuet-Higgins' formulae (38). We should point out here that the Lagrangian solution (A.12) for the Stokes drift is valid however small the viscosity ν is, as long as it is nonzero. The importance of a nonzero viscosity in Lagrangian drift calculations has been discussed at some length in [12].

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