

THE NATURE OF GENERALIZED SCALES

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ABSTRACT. The notion of a generalized scale emerged in recent joint work with Afsar–Brownlowe–Larsen on equilibrium states on C^* -algebras of right LCM monoids, where it features as the key datum for the dynamics under investigation. This work provides the structure theory for such monoidal homomorphisms. We establish uniqueness of the generalized scale and characterize its existence in terms of a simplicial graph arising from a new notion of irreducibility inside right LCM monoids. In addition, the method yields an explicit construction of the generalized scale if existent. We discuss applications for graph products as well as algebraic dynamical systems, and reveal a striking connection to Saito’s degree map.

1. INTRODUCTION

The notion of a generalized scale first appeared in [ABLS17] as an ad-hoc definition that allowed for an abstract and unified theory of equilibrium states for a number of quantum statistical mechanical systems [LR10, BaHLR12, LRR11, LRRW14, CaHR16]. More precisely, all these case studies can be viewed as considerations for C^* -algebras of *right LCM monoids*, that is, left cancellative monoids in which the intersection of two principal right ideals is either empty or another principal right ideal again. Here, right LCM refers to the existence of a *right Least Common Multiple* given the presence of a right common multiple. We note that right LCM monoids are commonly referred to as right LCM semigroups in the operator algebraic literature as the presence of a unit is taken for granted in this context. Since this work is part of semigroup theory rather than operator algebras, we decided to correct this potentially misleading terminology.

In order to give the definition of a generalized scale, it is convenient to first recall some notation: Let S be a right LCM monoid. For $s, t \in S$, we follow the notation of [Spi12] and write $s \pitchfork t$ if $sS \cap tS$ is nonempty, and $s \perp t$ otherwise, in which case we call s and t orthogonal. An important submonoid of a right LCM monoid S is its *core* $S_c := \{a \in S \mid a \pitchfork s \text{ for all } s \in S\}$ introduced in [Star15], inspired by [CL07]. As in [ABLS17, Subsection 2.1], $s, t \in S$ are *core equivalent*, denoted $s \sim t$, if there are $a, b \in S_c$ with $sa = tb$. It will be convenient to work with the following definition of a generalized scale that avoids the notion of accurate foundation sets:

Definition 1.1. A *generalized scale* on a right LCM monoid S is a nontrivial homomorphism $N: S \rightarrow \mathbb{N}^\times$, $s \mapsto N_s$ satisfying

- (i) $|N^{-1}(n)/\sim| = n$ for all $n \in N(S)$;

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- (ii) $N_s = N_t$ implies $s \sim t$ or $s \perp t$; and
- (iii) for all $s \in S, n \in N(S)$, there exists $t \in N^{-1}(n)$ with $s \pitchfork t$.

Proposition 2.5 will establish that this definition is equivalent to the original one given in [ABLS17, Definition 3.1(A3)]. The assumption that N has to be nontrivial excludes precisely the left reversible right LCM monoids and is needed to obtain a nontrivial time evolution of the quantum statistical mechanical system based on $C^*(S)$, see [ABLS17, Section 4].

Arguably, the definition of a generalized scale was useful, but rather elusive. It is thus natural to ask basic questions such as:

1. How many generalized scales may a right LCM monoid have?
2. When does a right LCM monoid admit a generalized scale?
3. Which structural aspects are captured by a generalized scale?

Already in [Sta18, Corollary 4.9] it had been observed that right-angled Artin monoids have at most one generalized scale. These monoids were first considered in [CF69] (also known as *free partially commutative monoids* or *trace monoids*). This result can then be further exploited with the help of [ABLS17, Proposition 5.4] to show uniqueness of the generalized scale for self-similar group actions [ABLS17, Subsection 5.3] and Baumslag–Solitar monoids with positive parameters [ABLS17, Subsection 5.5]. Combining [Sta18, Corollary 4.5] with [ABLS17, Proposition 3.6(v)], it follows that there is an abundance of right LCM monoids without generalized scales.

In the present work, we start by answering the first question by establishing a uniqueness theorem for generalized scales, see Theorem 3.6. Its proof relies on the preservation of arithmetic aspects of \mathbb{N}^\times under generalized scales. On the way to this result, we show that condition (A4) from [ABLS17, Definition 3.1] is redundant, see Proposition 3.1. This is of particular relevance to [BLRSt], where, using this work, we obtain a classification of KMS-states for arbitrary right LCM monoids that admit a (necessarily unique) generalized scale. That is to say, the only requirement left for obtaining the conclusions of [ABLS17, Theorem 4.3] is the existence of a generalized scale. The guiding theme for this improvement is to work with the quotient space S/\sim and the action $\alpha: S_c \curvearrowright S/\sim$ induced by left multiplication instead of using arithmetic structures inside the semigroup. In return, [BLRSt] indicates that a very natural direction for further progress on KMS-states in the context of semigroup C^* -algebras is a sophisticated understanding of generalized scales on right LCM monoids, which is the aim of the present work.

The answers to the second and third questions are given in Theorem 4.13 and require more work: A central object in the proof of Theorem 3.6 is the set of irreducible elements $\text{Irr } N(S)$ of the submonoid $N(S) \subset \mathbb{N}^\times$. In Proposition 4.1, we note that their preimages under N are precisely the elements $s \in S$ with the following property: whenever $s = tr$ for $t, r \in S$, then either $t \in S_c$ or $r \in S_c$. We shall call these elements *noncore irreducible*, see Definition 4.2, and denote their collection by $\mathcal{I}(S)$. For a basic understanding of their behavior, we keep track of whether two noncore irreducible elements s and t satisfy $s \perp t$ or $s \pitchfork t$. Since this is a property of $[s], [t] \in S/\sim$ rather than their representatives, we reduce to equivalence classes. By drawing an edge between $[s]$ and $[t]$ whenever $s \pitchfork t$, we obtain a simplicial graph $\Gamma(S)$ in Definition 4.4, the *core graph* of S . It turns out that the restriction of α to $\mathcal{I}(S)/\sim$ induces an action $\beta: S_c \curvearrowright \Gamma(S)$ by graph automorphisms, see Proposition 4.7, which may be of independent interest. The existence of a generalized

scale implies that this action is simply a composition of permutations of the vertex sets of the coconnected components of $\Gamma(S)$, see Corollary 4.8 and Theorem 4.13 (i).

With the core graph at our disposal, we set out to characterize the existence of the generalized scale and to provide an explicit construction in Theorem 4.13. We find that, in addition to quite restrictive features of the core graph, see Theorem 4.13(i),(ii), the existence of a generalized scale also requires that

- (iii) S is *noncore factorable*, that is, every element in $S \setminus S_c$ be core equivalent to a finite product of noncore irreducibles; and that
- (iv) S has *balanced factorization*: for $s, t \in \mathcal{I}(S)$ with $s \pitchfork t$, there are $s', t' \in \mathcal{I}(S)$ such that $sS \cap tS = s't'S$, $st' = ts'$ with $[s']$ and $[t']$ belonging to the same coconnected component of $\Gamma(S)$ as $[s]$ and $[t]$, respectively.

As to be expected, showing that the conditions (i)–(iv) are not only necessary, but in fact sufficient for the existence of a generalized scale is the hard part of the proof. One of the key tools in the proof is an algorithm for computing a right LCM of products of noncore irreducibles, see Remark 4.11.

In the final section, we first describe the core graph and its features for the two elementary examples of self-similar group actions in 5.1 and the $ax+b$ -semigroup over the natural numbers in 5.2. For algebraic dynamical systems, that is, suitable actions of right LCM monoids on discrete groups by injective group endomorphisms, the application of Theorem 4.13 is less straightforward. We examine the sufficient criterion for the existence of a generalized scale from [ABLS17, Proposition 5.11(i)], and find that it is not necessary in general, see Example 5.2. On the other hand, Ledrappier’s shift is an example that appears to lack a generalized scale precisely for the violation of condition (b) of Remark 5.1, see Example 5.3. Algebraic dynamical systems also provide the flexibility to build examples where β switches coconnected components of $\Gamma(S)$, see Example 5.4.

Motivated by the findings in [Sta18], we investigate applications to graph products of right LCM monoids, and obtain a result which reduces the considerations to the coconnected case, see Theorem 5.5. However, already Example 5.7 shows that graph products of coconnected graphs may well behave counterintuitively, depending on the combination of edge set and family of vertex semigroups. Roughly speaking, very few graph products admit generalized scales. This even remains true for the most elementary examples of this type, namely right-angled Artin monoids, see [Sta18, Corollary 4.9].

In summary, the present work shows that the generalized scale represents an intriguing set of characteristic features of the right LCM monoid in question, see Theorem 4.13. These features are common for many types of right LCM monoids, see [ABLS17]. However, there are also many interesting examples where different parts of the requirements for the existence of the generalized scale are not met. This motivates the search for a more flexible notion of such maps, the outcome of which we present in Subsection 5.5: Generalized scales correspond to special kinds of Saito’s degree maps, see [Sai13] for details.

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and for spotting details in the proofs of Lemma 4.12 (i) and Theorem 4.13 that needed extra care.

2. PRELIMINARIES

In an attempt to make this work essentially self-contained, we shall briefly recall the auxiliary results from [ABLS17] that we will need and make a few additional observations to help the reader familiarize with the concepts of this work. Throughout, let S be a right LCM monoid. Let us start with the first part of [ABLS17, Lemma 3.5]:

Lemma 2.1. *Let $a \in S_c$ and $s \in S$. If $b \in S$ satisfies $aS \cap sS = sbS$, then $b \in S_c$.*

Proof. Let $a \in S_c$ and $s, b \in S$ satisfy $sS \cap aS = sbS$ and fix $t \in S$. Then

$$s(tS \cap bS) = stS \cap sS \cap aS = stS \cap aS \neq \emptyset$$

shows $t \mathfrak{m} b$, and hence $b \in S_c$ since t was arbitrary. \square

Recall that a subsemigroup $T \subset S$ is said to be *hereditary* if it has the following property: If $s \in S$ satisfies $t \in sS$ ($t \geq s$) for some $t \in T$, then $s \in T$.

Lemma 2.2. *The core S_c is a hereditary submonoid of S . In particular, if $s, t \in S$ satisfy $s \sim t$, then $sS \cap tS = saS$ for some $a \in S_c$.*

Proof. For the first claim, we note that for $a \in S, b \in S_c$ with $b \in aS$, we have $aS \cap sS \supset bS \cap sS \neq \emptyset$ for all $s \in S$, so that $a \in S_c$. For the second part, $s \sim t$ implies that there are $b, c \in S_c$ with $sb = tc$. So we have $sS \cap tS = saS$ for some $a \in S$ with $b \in aS$. Thus the first part yields $a \in S_c$. \square

Lemma 2.3. *Let $s, t \in S$. Then $s \sim t$ holds if and only if $s \mathfrak{m} r \Leftrightarrow t \mathfrak{m} r$ for all $r \in S$.*

Proof. Let $s, t \in S$. Suppose first that $s \sim t$. Lemma 2.2 shows that $sS \cap tS = saS$ with $sa = tb$ for some $a, b \in S_c$. For $r \in S$, we thus get $r \mathfrak{m} s \Leftrightarrow r \mathfrak{m} sa = tb \Leftrightarrow r \mathfrak{m} t$.

Conversely, assume that $s \mathfrak{m} r \Leftrightarrow t \mathfrak{m} r$ for all $r \in S$. In particular, picking $r := s$ yields $s \mathfrak{m} t$, say $sS \cap tS = saS$ with $sa = tb$ for some $a, b \in S$. We claim that a and b belong to S_c . For $p \in S$, let us consider sp . As $sp \mathfrak{m} s$, we have $sp \mathfrak{m} t$, which allows us to conclude

$$s(spS \cap aS) = spS \cap sS \cap tS = spS \cap tS \neq \emptyset.$$

Hence we obtain $a \mathfrak{m} p$ for all $p \in S$, which is $a \in S_c$. Arguing in the same way for b , we also get $b \in S_c$. \square

Following the terminology of [SY10], the notion of a *foundation set* was introduced in [BRRW14] for right LCM monoids: A foundation set is a finite subset $F \subset S$ (we write $F \subset\subset S$ to indicate this) such that for every $s \in S$ there is $t \in F$ with $s \mathfrak{m} t$. This was refined in [BS16] to the notion of an *accurate foundation set*, which is a foundation set F satisfying $s \perp t$ for all $s, t \in F, s \neq t$.

According to [ABLS17, Definition 3.1(A3)], a nontrivial homomorphism $N: S \rightarrow \mathbb{N}^\times$ is called a *generalized scale* if

- (a) $|N^{-1}(n)/\sim| = n$ for all $n \in N(S)$; and
- (b) for each $n \in N(S)$, every transversal for $N^{-1}(n)/\sim$ is an accurate foundation set for S .

Let us recall some key properties for generalized scales, namely [ABLS17, Proposition 3.6(i)-(iv)] without proof.

Proposition 2.4. *Let $N: S \rightarrow \mathbb{N}^\times$ be a generalized scale in the sense of [ABLS17, Definition 3.6(A3)].*

- (i) $N^{-1}(1) = S_c$, and this is a proper subsemigroup of S .
- (ii) For $s, t \in S$ with $N_s = N_t$, either $s \sim t$ or $s \perp t$ holds.
- (iii) Let F be a foundation set for S with $F \subset N^{-1}(n)$ for some $n \in N(S)$. If $|F| = n$ or F is accurate, then F is a transversal for $N^{-1}(n)/\sim$.
- (iv) Whenever $s, t \in S$ satisfy $sS \cap tS = rS$, then $N_s N(S) \cap N_t N(S) = N_r N(S)$.

In particular, part (i) implies that core equivalent elements have the same image under N . Let us now show that the two competing notions for a generalized scale coincide:

Proposition 2.5. *A nontrivial homomorphism $N: S \rightarrow \mathbb{N}^\times$ is a generalized scale in the sense of [ABLS17, Definition 3.6(A3)] if and only if it satisfies (i)–(iii) of Definition 1.1.*

Proof. As (a) and (i) are identical, we need to show that (b) is equivalent to (ii) and (iii), given (i). We observe that (b) implies (iii) by the definition of a foundation set. Proposition 2.4(ii) establishes (ii).

Conversely, suppose (i)–(iii) of Definition 1.1 hold. We need to prove (b), so let $n \in N(S)$ and fix a transversal F for $N^{-1}(n)/\sim$. The set F is finite by (i) and its elements are mutually orthogonal by (ii). Thus the claim reduces to showing that F is a foundation set. For every $s \in S$, (iii) states that there is $t \in N^{-1}(n)$ with $s \pitchfork t$. If $t' \in F$ is the (unique) element satisfying $t' \sim t$, then Lemma 2.3 shows $s \pitchfork t'$. Thus F is an accurate foundation set, that is, (b) holds. \square

Corollary 2.6. *For every generalized scale N , its image $N(S)$ is a (right) LCM subsemigroup of \mathbb{N}^\times .*

Proof. Let $m, n \in N(S)$. So there is $s \in N^{-1}(m)$, and Definition 1.1 (iii) asserts that there is $t \in N^{-1}(n)$ with $s \pitchfork t$, that is, $sS \cap tS = rS$ for some $r \in S$. Using Proposition 2.4 (iv), we thus get $mN(S) \cap nN(S) = N_r N(S)$, so that N_r is the (unique right) LCM of m and n . \square

Finally, let us recall the part of [ABLS17, Lemma 3.9] that holds without assumptions on core irreducible elements in S , along with its proof.

Lemma 2.7. *Let S be a right LCM monoid. Then left multiplication defines an action $\alpha: S_c \curvearrowright S/\sim$ by bijections $\alpha_a([s]) = [as]$. Every generalized scale N on S is invariant under this action.*

Proof. For every $a \in S_c$, the map α_a is well-defined as left multiplication preserves the core equivalence relation. If $as \sim at$ for some $s, t \in S$, then $s \sim t$ by left cancellation. Hence α_a is injective. On the other hand, Lemma 2.1 states that for $s \in S$ we have $aS \cap sS = atS, at = sb$ for some $b \in S_c$ and $t \in S$. Thus $\alpha_a([t]) = [at] = [s]$, and we conclude that α_a is a bijection. Generalized scales are invariant under this action by Proposition 2.4 (i). \square

3. UNIQUENESS OF THE GENERALIZED SCALE

Let us start with a few observations that will streamline the proof of the uniqueness theorem. For an LCM subsemigroup \mathcal{N} of \mathbb{N}^\times , let us denote its irreducible elements by

$$\text{Irr } \mathcal{N} := \{n \in \mathcal{N} \setminus \{1\} \mid n = k\ell \text{ for } k, \ell \in \mathcal{N} \Rightarrow k = 1 \text{ or } \ell = 1\}.$$

Proposition 3.1. *Every LCM subsemigroup \mathcal{N} of \mathbb{N}^\times is freely generated by its irreducible elements. In particular, $N(S)$ is freely generated by $\text{Irr } N(S)$ for every generalized scale $N: S \rightarrow \mathbb{N}^\times$, that is, condition (A3) of [ABLS17, Definition 3.1] implies (A4).*

Proof. First, we claim that $m\mathcal{N} \cap n\mathcal{N} = mn\mathcal{N}$ whenever $m, n \in \text{Irr } \mathcal{N}$ are distinct. There are unique $k, \ell \in \mathcal{N}$ with $m\mathcal{N} \cap n\mathcal{N} = mk\mathcal{N}$ and $n = k\ell$ because $mn \in mk\mathcal{N}$. As $m \neq n$ forces $k > 1$ and \mathcal{N} is cancellative, irreducibility of n forces $\ell = 1$, that is, $k = n$.

For the general case suppose to the contrary that there were $k, \ell \geq 1$ and

$$m_1, \dots, m_k, n_1, \dots, n_\ell \in \text{Irr } \mathcal{N} \text{ with } m_1 \cdots m_k = n_1 \cdots n_\ell,$$

but there did not exist a bijection $f: \{1, \dots, k\} \rightarrow \{1, \dots, \ell\}$ such that $m_{f(j)} = n_j$ for all $1 \leq j \leq k$. By counting multiplicities of the elements in $\{m_j \mid 1 \leq j \leq k\} \cap \{n_j \mid 1 \leq j \leq \ell\}$, we can choose subsets $K \subset \{1, \dots, k\}$ and $L \subset \{1, \dots, \ell\}$ and build a bijection $f': K \rightarrow L$ with $m_{f'(j)} = n_j$ for all $j \in K$. By choosing K maximal with the above property and removing it from $\{1, \dots, k\}$ (and the corresponding L from $\{1, \dots, \ell\}$), the problem reduces to the following situation: There exist $k', \ell' \geq 1$ ($k = \ell$ and $K = \{1, \dots, k\}$ cannot occur) and

$$m_1, \dots, m_{k'}, n_1, \dots, n_{\ell'} \in \text{Irr } \mathcal{N} \text{ with } m_1 \cdots m_{k'} = n_1 \cdots n_{\ell'} \text{ and } m_i \neq n_j \text{ for all } i, j.$$

In view of the first part of the proof, $n_1 \neq m_j$ for all j gives

$$\begin{aligned} n_1\mathcal{N} \cap m_1 \cdots m_{k'}\mathcal{N} &= n_1\mathcal{N} \cap m_1\mathcal{N} \cap m_1 \cdots m_{k'}\mathcal{N} \\ &= m_1 n_1\mathcal{N} \cap m_1 m_2\mathcal{N} \cap m_1 \cdots m_{k'}\mathcal{N} \\ &= m_1(n_1\mathcal{N} \cap m_2\mathcal{N}) \cap m_1 \cdots m_{k'}\mathcal{N} \\ &\quad \vdots \\ &= m_1 \cdots m_{k'} n_1\mathcal{N}. \end{aligned}$$

As $n_1 > 1$, this contradicts

$$n_1\mathcal{N} \cap m_1 \cdots m_{k'}\mathcal{N} \supset n_1 \cdots n_{\ell'}\mathcal{N} \cap m_1 \cdots m_{k'}\mathcal{N} = m_1 \cdots m_{k'}\mathcal{N}.$$

Hence we conclude that $\text{Irr } \mathcal{N}$ freely generates a submonoid M of \mathcal{N} . To see that $M = \mathcal{N}$, we show by induction on $n \in \mathcal{N}$ that every element in \mathbb{N} factors into a finite product of elements from $\text{Irr } \mathcal{N}$: Suppose $\mathcal{N} \neq \{1\}$ and pick the smallest $n \in \mathcal{N} \setminus \{1\}$ with respect to the total order $\leq_{\mathbb{N}}$ coming from the canonical inclusion $\mathcal{N} \subset \mathbb{N}$. Then $n \in \text{Irr } \mathcal{N}$ as $n = mm'$ forces $m, m' \in \{1, n\}$. Next, assume that a factorization with respect to $\text{Irr } \mathcal{N}$ exists for all $m \in \mathcal{N}, m \leq_{\mathbb{N}} n$ for an arbitrary but fixed $n \in \mathcal{N}$. Let $n' \in \mathcal{N}$ be the smallest number with $n' >_{\mathbb{N}} n$. If $n' \in \text{Irr } \mathcal{N}$, then there is nothing to prove. On the other hand, if $n' = mm'$ for some $m, m' \in \mathcal{N} \setminus \{1\}$, then $m, m' \leq_{\mathbb{N}} n$ since n' was chosen to be minimal with $n' >_{\mathbb{N}} n$. Therefore, both m and m' admit a factorization with respect to $\text{Irr } \mathcal{N}$, and the product of these is a factorization for n' . This completes the induction and we conclude that \mathcal{N} is freely generated by $\text{Irr } \mathcal{N}$.

The claim concerning a generalized scale N now follows from Corollary 2.6. \square

Remark 3.2. The following elegant alternative for the proof of Proposition 3.1 was suggested by the anonymous referee: It is known that an abelian cancellative monoid is free if and only if it

- (a) has trivial group of units;
- (b) satisfies the ascending chain condition on principal ideals; and
- (c) each irreducible element is prime.

In the present case of an LCM subsemigroup $\mathcal{N} \subset \mathbb{N}^\times$, (a) and (b) are inherited from \mathbb{N}^\times . These two conditions imply that every element admits a factorization into a finite product of irreducibles.

In order to establish (c), it suffices to show that if $p \in \text{Irr } \mathcal{N}$ and $q_1^{m_1} \cdots q_k^{m_k} \in p\mathcal{N}$ for some mutually distinct $q_i \in \text{Irr } \mathcal{N}$ and $m_i \geq 1$ for all i , then $q_i \in p\mathcal{N}$ for some i , that is, $p = q_i$ as (a) holds and q_i is irreducible. Suppose that $p \neq q_i$ for all i and that k and $\sum_{1 \leq i \leq k} m_i$ are minimal, that is, $q_1^{m'_1} \cdots q_\ell^{m'_\ell} \in p\mathcal{N}$ implies $\ell \geq k$ and $\sum_{1 \leq i \leq \ell} m'_i \geq \sum_{1 \leq i \leq k} m_i$. Combining the first claim in the proof of Proposition 3.1 with left cancellation, we obtain

$$\begin{aligned} p\mathcal{N} \cap q_1\mathcal{N} \cap \dots \cap q_k\mathcal{N} &= pq_1\mathcal{N} \cap pq_2\mathcal{N} \cap \dots \cap pq_k\mathcal{N} \\ &= p(q_1\mathcal{N} \cap q_2\mathcal{N} \cap \dots \cap q_k\mathcal{N}) \\ &\vdots \\ &= pq_1q_2 \cdots q_k\mathcal{N}. \end{aligned}$$

Thus we get

$$q_1^{m_1} \cdots q_k^{m_k} \in p\mathcal{N} \cap q_1\mathcal{N} \cap \dots \cap q_k\mathcal{N} = pq_1q_2 \cdots q_k\mathcal{N},$$

which implies $q_1^{m_1-1} \cdots q_k^{m_k-1} \in p\mathcal{N}$ by left cancellation. This contradicts minimality of (m_1, \dots, m_k) .

Lemma 3.3. *Let S be a right LCM monoid, $M, N: S \rightarrow \mathbb{N}^\times$ be two generalized scales on S , and $s \in S \setminus S_c$. If M_s is reducible in $M(S)$, then there are $a \in S_c$ and $t, r \in S \setminus S_c$ such that $sa = tr$. In particular, N_s is irreducible if and only if M_s is irreducible.*

Proof. Suppose there are $k, \ell \in M(S)$, $k, \ell > 1$ such that $M_s = k\ell$. By Definition 1.1 (iii), there is $t \in M^{-1}(k)$ such that $t \pitchfork s$, say $tS \cap sS = trS$ with $tr = sa$ for some $r, a \in S$. Due to Proposition 2.4 (iv) and $M_s = M_t\ell$, we know that $M_r = \ell > 1$ and $M_a = 1$. This implies $r \in S \setminus S_c$ and $a \in S_c$, see Proposition 2.4 (i). We conclude from this that $N_s = N_{sa} = N_tN_r$ with $N_t, N_r > 1$, so that N_s is reducible in $N(S)$. \square

Recall from [BS16, Definition 1.2 and Definition 2.1] that a finite subset $\mathcal{T} \subset S$ is an *accurate foundation set* if its elements are mutually orthogonal and for every $s \in S$ there is $t \in \mathcal{T}$ with $s \pitchfork t$.

Lemma 3.4. *Let N be a generalized scale on S . If $s, t \in S$ with $N_s, N_t \in \text{Irr } N(S)$ satisfy $N_s \neq N_t$, then $s \pitchfork t$.*

Proof. According to Proposition 3.1, the right LCM of N_s and N_t in $N(S)$ is N_sN_t . So if $s' \in N^{-1}(N_s)$ and $t' \in N^{-1}(N_t)$ satisfy $s'S \cap t'S = rS$ for some $r \in S$, then $N_r = N_sN_t$ by Proposition 2.4 (iv).

Next, take transversals F_r for $N^{-1}(N_r)/\sim$ with $r \in F_r$ for $r = s, t$. By [ABLS17, Definition 3.1(A3)(b)], F_s and F_t are accurate foundation sets with $|F_r| = N_r$ for $r = s, t$. Hence, every complete minimal set of representatives F for

$$\{s'S \cap t'S \mid s' \in F_s, t' \in F_t\} \setminus \{\emptyset\}$$

is also an accurate foundation set. By the first paragraph, we have $F \subset N^{-1}(N_s N_t)$. According to Proposition 2.4 (iii), F is a transversal for $N^{-1}(N_s N_t)/\sim$, and thus $|F| = N_s N_t$. In view of $|F_s| = N_s$ and $|F_t| = N_t$, this forces $s \pitchfork t$. \square

Lemma 3.5. *Let N be a generalized scale on the right LCM monoid S and $s \in S \setminus S_c$. Then there are $k \geq 1$ and $s_1, \dots, s_k \in S$ with $N_{s_j} \in \text{Irr } N(S)$ for all j satisfying $s \sim s_1 \cdots s_k$.*

Proof. Let $N_s = n_1 \cdots n_k$ with $n_i \in \text{Irr } N(S)$ for all i be the (up to permutation unique) factorization into irreducibles in $N(S) \subset \mathbb{N}^\times$. By Definition 1.1(iii), there is $s_1 \in N^{-1}(n_1)$ with $s_1 \pitchfork s$, say $s_1 S \cap sS = s_1 r_1 S$, $s_1 r_1 = sa_1$ for some $a_1, r_1 \in S$. Since N_{s_1} is a divisor of N_s , Proposition 2.4 (iv) implies that $N_{r_1} = n_2 \cdots n_k$ and $N_{a_1} = 1$. The latter is equivalent to $a_1 \in S_c$, see Proposition 2.4 (i). Thus we obtained $s_1 \in N^{-1}(n_1)$, $a_1 \in S_c$, and $r_1 \in N^{-1}(n_2 \cdots n_k)$ such that

$$sS \cap s_1 S = s_1 r_1 S, s_1 r_1 = sa_1.$$

Applying the same process to (r_1, n_2) in place of (s, n_1) yields $s_2 \in N^{-1}(n_2)$, $a_2 \in S_c$, and $r_2 \in N^{-1}(n_3 \cdots n_k)$ such that $r_1 S \cap s_2 S = s_2 r_2 S$, $s_2 r_2 = r_1 a_2$. After $k - 1$ steps, we have obtained $s_i \in N^{-1}(n_i)$, $a_i \in S_c$ and $r_i \in N^{-1}(\prod_{i+1 \leq j \leq k} n_j)$ for $1 \leq i \leq k - 1$ with

$$sa_1 \cdots a_{k-1} = s_1 \cdots s_{k-1} r_{k-1} \quad \text{with } r_{k-1} \in N^{-1}(n_k).$$

Thus, setting $s_k := r_{k-1}$ gives $s \sim s_1 \cdots s_k$. \square

Theorem 3.6. *A right LCM monoid admits at most one generalized scale.*

Proof. Let S be a right LCM monoid and $M, N: S \rightarrow \mathbb{N}^\times$ be two generalized scales. Fix $s \in S$ with $N_s \in \text{Irr } N(S)$. By Lemma 3.3, we know that $M_s \in \text{Irr } M(S)$. We claim that $N_s = M_s$. According to Definition 1.1(i), this amounts to $|N^{-1}(N_s)/\sim| = |M^{-1}(M_s)/\sim|$. By symmetry, it thus suffices to show $N^{-1}(N_s)/\sim \subset M^{-1}(M_s)/\sim$. Suppose to the contrary that there exists $t \in N^{-1}(N_s)$ with $t \not\sim s$ and $M_t \neq M_s$. By Definition 1.1(ii), $N_t = N_s$ and $t \not\sim s$ force $s \perp t$. However, Lemma 3.3 entails $M_s, M_t \in \text{Irr } M(S)$ as $N_s = N_t \in \text{Irr } N(S)$. Therefore, Lemma 3.4 applied to M gives $s \pitchfork t$, contradicting $s \perp t$. Thus we get $N^{-1}(N_s)/\sim \subset M^{-1}(M_s)/\sim$, and hence $N_s = M_s$ for every $s \in S$ with $N_s \in \text{Irr } N(S)$.

In addition, we know that M and N are homomorphisms with $M|_{S_c} = 1 = N|_{S_c}$. To conclude that $N = M$, let $s \in S \setminus S_c$. By Lemma 3.5, there are $k \geq 1$ and $s_1, \dots, s_k \in S$ with $N_{s_i} \in \text{Irr } N(S)$ such that $s \sim s_1 \cdots s_k$. This allows us to deduce

$$N_s = N_{s_1} \cdots N_{s_k} = M_{s_1} \cdots M_{s_k} = M_s,$$

that is, $M = N$. \square

4. EXISTENCE AND CONSTRUCTION OF GENERALIZED SCALES

The idea behind the final step in the proof of the uniqueness result Theorem 3.6 will serve as our starting point: If a right LCM monoid S admits a generalized scale $N: S \rightarrow \mathbb{N}^\times$, then every element in $S \setminus S_c$ is a nonempty, finite product of elements in $N^{-1}(\text{Irr } N(S))$. We can characterize these minimal elements in the following way:

Proposition 4.1. *Let $N: S \rightarrow \mathbb{N}^\times$ be a generalized scale. Then $s \in S \setminus S_c$ satisfies $N_s \in \text{Irr } N(S)$ if and only if $sa = tr$ for $a \in S_c$ and $t, r \in S$ implies either $t \in S_c$ or $r \in S_c$.*

Proof. First we note that $N_s \in \text{Irr } N(S)$ requires $N_s > 1$, which amounts to $s \in S \setminus S_c$, see Proposition 2.4(i). Likewise, $sa = tr$ with $a \in S_c$ and $t \notin S_c$ or $r \notin S_c$ forces $s \notin S_c$. Thus we can restrict our attention to $s \in S \setminus S_c$. In case there are $a \in S_c$ and $t, r \in S \setminus S_c$ such that $sa = tr$, Proposition 2.4(i) entails that $N_s = N_{sa} = N_t N_r$ with $N_t, N_r > 1$, which means $N_s \notin \text{Irr } N(S)$. Conversely, if $N_s \notin \text{Irr } N(S)$, then there are $t, r \in S$ with $t, r \in S \setminus S_c$ and $a \in S_c$ such that $sa = tr$, see Lemma 3.3. \square

Definition 4.2. Let S be a right LCM monoid. An element $s \in S$ is *noncore irreducible* if $s \notin S_c$ and whenever $sa = tr$ for some $a \in S_c$ and $t, r \in S$, then $t \in S_c$ or $r \in S_c$. The collection of all noncore irreducible elements of S is denoted by $\mathcal{I}(S)$.

Remark 4.3. The notion introduced in Definition 4.2 is not to be confused with *core irreducibility* from [ABLS17]: $s \in S$ is core irreducible if $s = ta$ with $t \in S, a \in S_c$ implies $a \in S^*$, the subgroup of invertible elements in S . We note that noncore irreducibility is preserved under core equivalence.

Definition 4.4. The *core graph* of a right LCM monoid S is the undirected graph $\Gamma(S) := (V, E)$ with vertex set $V := \mathcal{I}(S)/\sim$ and edge set $E := \{([s], [t]) \mid s \bowtie t\}$.

We will simply write Γ whenever this is unambiguous. For convenience, we recall that Γ is characterized by its unique decomposition into coconnected components $(\Gamma_i)_{i \in I}$ with $\Gamma_i = (V_i, E_i)$. Such coconnected components are determined by the vertex set V_i as $E_i = V_i \times V_i \cap E$. A natural characterization of coconnectedness is that the V_i forms a maximal, connected subset of the complimentary graph $(V, V \times V \setminus E)$.

Notation 4.5. For $s \in \mathcal{I}(S)$, we denote by $i(s) \in I$ the index with $[s] \in V_{i(s)}$.

Recall from Lemma 2.7 that $a.[s] := [as]$ defines an action $\alpha: S_c \curvearrowright S/\sim$ by bijections.

Lemma 4.6. *For $a \in S_c$ and $s \in S$, $as \in \mathcal{I}(S)$ holds if and only if $s \in \mathcal{I}(S)$, that is, the action α restricts to an action on $\mathcal{I}(S)/\sim$.*

Proof. Let $a \in S_c$ and $s \in S$. It is clear that $as \in S_c$ if and only if $s \in S_c$, so we may restrict to the case where $s \in S \setminus S_c$. We will prove that $\alpha_a^{-1}([s]) \in \mathcal{I}(S)/\sim$ if and only if $[s] \in \mathcal{I}(S)/\sim$, which is equivalent to the original claim since $\mathcal{I}(S)$ is closed under core equivalence (essentially by definition) and α_a is bijective, see Lemma 2.7. Using Lemma 2.1, there are $b \in S_c$ and $s' \in S \setminus S_c$ such that $sS \cap aS = sbS, sb = as'$. We claim that $s \in \mathcal{I}(S)$ if and only if $s' \in \mathcal{I}(S)$, that is, $[s] \in \mathcal{I}(S)/\sim$ if and only if $[s'] = \alpha_a^{-1}([s]) \in \mathcal{I}(S)$: If there are $c \in S_c, t, r \in S \setminus S_c$ with $s'c = tr$, then $sbc = as'c = (at)r$ implies $s \notin \mathcal{I}(S)$. Thus $s \in \mathcal{I}(S)$ implies $s' \in \mathcal{I}(S)$. Conversely, if there

are $c \in S_c, t, r \in S$ with $sc = tr$, then $bS_c \cap cS_c = bc'S_c, bc' = cb'$ for some $b', c' \in S_c$ by Lemma 2.1. With $trb' = scb' = as'c'$ we get a diagram

$$\begin{array}{ccc}
 & \leftarrow \text{---} b' \text{---} & \\
 r \} & & \swarrow r' \\
 & \leftarrow \text{---} a' \text{---} & c' \\
 t \} & & \swarrow t' \\
 & \leftarrow \text{---} a \text{---} & \leftarrow \text{---} s' \text{---} \\
 & & \downarrow
 \end{array}$$

where $tS \cap aS = ta'S, ta' = at'$ and $r' \in S$ with $rb' = a'r'$. Here r' results from the fact that ta' is the right LCM of t and a , while $trb' = as'c'$ is another common right multiple. The dotted arrows represent elements from S_c , while solid arrows refer to $\mathcal{I}(S)$ (and \rightsquigarrow bear no constraints). We get $s'c' = t'r'$ (upon using left cancellation). Since $a \in S_c$, we have $a' \in S_c$ by Lemma 2.1. Thus $t \in S_c$ holds iff $t' \in S_c$, and $r \in S_c$ iff $r' \in S_c$. Therefore $s' \in \mathcal{I}(S)$ forces $s \in \mathcal{I}(S)$. \square

Proposition 4.7. *The action $\alpha: S_c \curvearrowright S/\sim$ determines an action $\beta: S_c \curvearrowright \Gamma$ by graph automorphisms. In particular, $\beta_a(\Gamma_i)$ is a coconnected component of Γ isomorphic with Γ_i for all $i \in I$ and $a \in S_c$.*

Proof. By Lemma 4.6, α restricts to an action on the vertex set $\mathcal{I}(S)/\sim$. Recall that for $r, s, t \in S$ we have $s \perp t$ if and only if $rs \perp rt$ (using left cancellation). In particular, $([s], [t]) \in E$ is equivalent to $([as], [at]) \in E$ for all $s, t \in \mathcal{I}(S), a \in S_c$. Therefore α_a induces an automorphism β_a of the graph Γ . For every such map, $\beta_a(\Gamma_i)$ is again a coconnected component of Γ as these are determined by the connectivity inside the graph Γ . \square

As an immediate consequence of Proposition 4.7, we note:

Corollary 4.8. *If the coconnected components of Γ are mutually nonisomorphic, then α restricts to an action $\alpha_a: S_c \curvearrowright V_i$ on the vertex sets of the coconnected components Γ_i for all $i \in I$.*

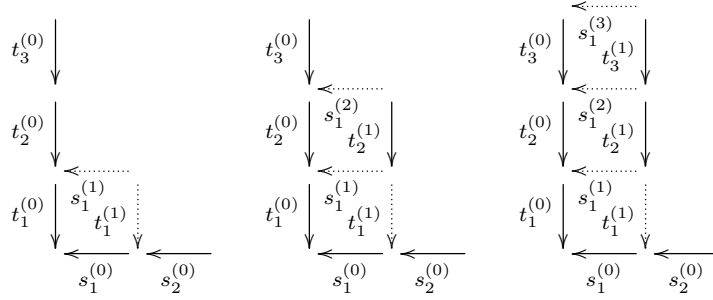
Definition 4.9. Let S be a right LCM monoid.

- (i) The semigroup S is *noncore factorable* if every element in $S \setminus S_c$ is core equivalent to a finite product of noncore irreducibles.
- (ii) The semigroup S has *balanced factorization* if $i(s) \neq i(t)$ implies $sS \cap tS = st'S, st' = ts'$ for some $s', t' \in \mathcal{I}(S)$ with $i(s') = i(s), i(t') = i(t)$ for all $s, t \in \mathcal{I}(S)$, where $i(s) \in I$ is the index of the coconnected component of Γ containing $[s]$.

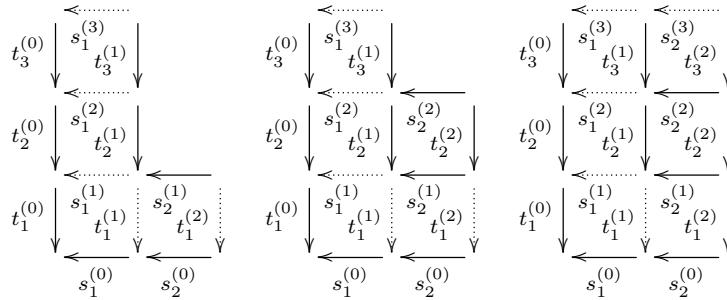
Our next target is to detect the existence of and compute a right LCM for two finite products of noncore irreducibles.

Example 4.10. Assume that balanced factorization holds for S and that the coconnected components of Γ are edge-free. Let $s_1, s_2, t_1, t_2, t_3 \in \mathcal{I}(S)$ and consider $s := s_1s_2, t := t_1t_2t_3$. Suppose for convenience that we already know that $s \pitchfork t$ holds (otherwise the process would terminate at one of the steps to come, and thus signalling $s \perp t$). Then we must have $s_1 \pitchfork t_1$ because $sS \subset s_1S$ and $tS \subset t_1S$. Suppose $[s_1]$ and $[t_1]$ belong to the same coconnected component of Γ . As the component is edge-free, this forces

$s_1 \sim t_1$, so there are $s_1^{(1)}, t_1^{(1)} \in S_c$ with $s_1 S \cap t_1 S = s_1 t_1^{(1)} S, s_1 t_1^{(1)} = t_1 s_1^{(1)}$. Next, we will determine a right LCM for $s_1^{(1)}$ and t_2 : Since $s_1^{(1)} \in S_c$, Lemma 2.1 implies that there are $s_1^{(2)} \in S_c$ and $t_2^{(1)} \in S$ with $s_1^{(1)} S \cap t_2 S = s_1^{(1)} t_2^{(1)} S, s_1^{(1)} t_2^{(1)} = t_2 s_1^{(2)}$. Due to Lemma 4.6, we also know that $t_2^{(1)} \in \mathcal{I}(S)$. A repetition of this argument yields a right LCM for $s_1^{(2)}$ and t_3 : $s_1^{(3)} \in S_c$ and $t_3^{(1)} \in \mathcal{I}(S)$ with $s_1^{(2)} S \cap t_3 S = s_1^{(2)} t_3^{(1)} S, s_1^{(2)} t_3^{(1)} = t_3 s_1^{(3)}$. This completes the first column in our diagram, where solid arrows refer to $\mathcal{I}(S)$ and dotted ones to S_c , as in the proof of Lemma 4.6:



For the bottom square in the second column, the same argument yields a right LCM for s_2 and $t_1^{(1)}$: $s_2^{(1)} \in \mathcal{I}(S)$ and $t_1^{(2)} \in S_c$ with $s_2 S \cap t_1^{(1)} S = s_2 t_1^{(2)} S, s_2 t_1^{(2)} = t_1^{(1)} s_2^{(1)}$. Thus we see that the encounter of a square with $s_{k+1}^{(\ell)} \sim t_{\ell+1}^{(k)}$ simplifies the task tremendously because we can then easily complete the corresponding row and column. Next, we focus on a right LCM for $s_2^{(1)}$ and $t_2^{(1)}$: The assumption $s \pitchfork t$ implies $s_2^{(1)} \pitchfork t_2^{(1)}$. Suppose that $i(s_2^{(1)}) \neq i(t_2^{(1)})$, that is, they belong to distinct coconnected components of Γ . Then balanced factorization grants $s_2^{(2)}, t_2^{(2)} \in \mathcal{I}(S)$ (with $i(s_2^{(1)}) = i(s_2^{(2)})$ and $i(t_2^{(1)}) = i(t_2^{(2)})$) such that $s_2^{(1)} S \cap t_2^{(1)} S = s_2^{(1)} t_2^{(2)} S, s_2^{(1)} t_2^{(2)} = t_2^{(1)} s_2^{(2)}$. For the right LCM for $s_2^{(2)}$ and $t_3^{(1)}$ suppose again that they have the same coconnected component, and thus $s_2^{(2)} \sim t_3^{(1)}$. This leads to the existence of $s_2^{(3)}, t_3^{(2)} \in S_c$ with $s_2^{(2)} S \cap t_3^{(1)} S = s_2^{(2)} t_3^{(2)} S, s_2^{(2)} t_3^{(2)} = t_3^{(1)} s_2^{(3)}$, and our diagram is complete:



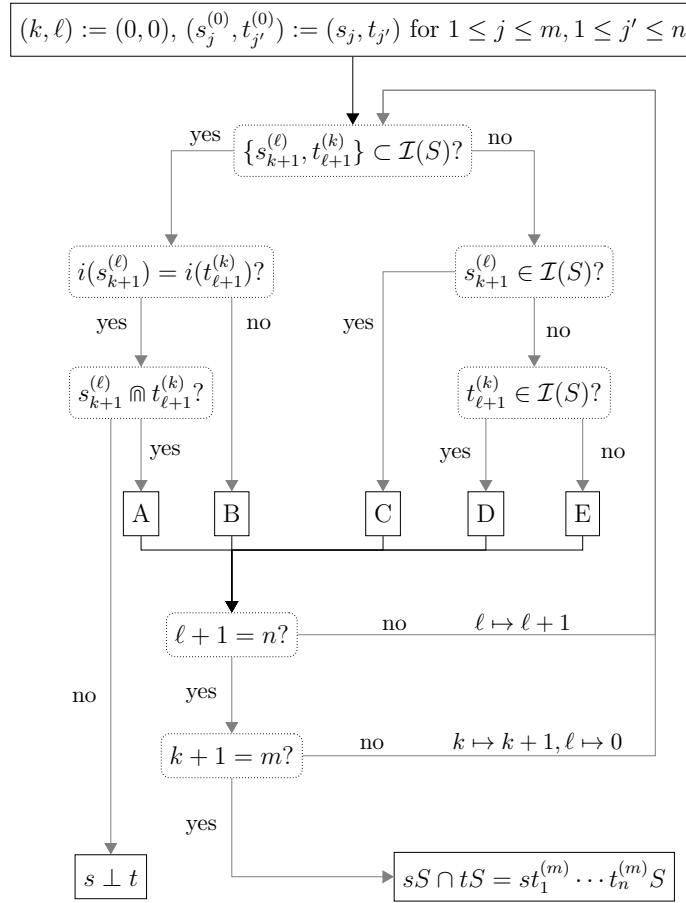
A right LCM for s and t is now given by $st_1^{(2)} t_2^{(2)} t_3^{(2)}$ (or taking the product along any path from the upper right to the lower left corner in the completed diagram).

Remark 4.11. Let $s_1, \dots, s_m, t_1, \dots, t_n \in \mathcal{I}(S)$ and $s := s_1 \cdots s_m, t := t_1 \cdots t_n$. Assume that balanced factorization holds for S and that the coconnected components of Γ are edge-free. We want to describe an algorithm that decides whether $s \pitchfork t$ or not, and produces a right LCM for s and t in case $s \pitchfork t$. Let us first fix the notation: We introduce two indices k, ℓ with initial value $(k, \ell) := (0, 0)$ and range $0 \leq k \leq m-1, 0 \leq$

$\ell \leq n - 1$, and set $s_j^{(0)} := s_j$ for $1 \leq j \leq m$, $t_j^{(0)} := t_j$ for $1 \leq j \leq n$. In order to find a right LCM for s and t , we need to find elements $s_{k+1}^{(\ell)}, t_{\ell+1}^{(k)} \in \mathcal{I}(S) \cup S_c$, where $(0, 0) \leq (k, \ell) \leq (m - 1, n - 1)$, such that

$$(4.1) \quad s_{k+1}^{(\ell)} S \cap t_{\ell+1}^{(k)} S = s_{k+1}^{(\ell)} t_{\ell+1}^{(k+1)} S, \quad s_{k+1}^{(\ell)} t_{\ell+1}^{(k+1)} = t_{\ell+1}^{(k)} s_{k+1}^{(\ell+1)}$$

holds for all $(0, 0) \leq (k, \ell) \leq (m - 1, n - 1)$. In other words, we successively compute right LCMs for the (altered) factors of s and t , and compose these to a right LCM of the product. We find it convenient to employ a flowchart for describing this repetitive process:



Before explaining the output, let us first describe the processes \boxed{A} – \boxed{E} from the flowchart above, in which we compute a right LCM (or rather the missing right factors) for $s_{k+1}^{(\ell)}$ and $t_{\ell+1}^{(k)}$. Apart from \boxed{E} , these did already make an appearance in Example 4.10. We point out that $s_{k+1}^{(\ell)} \cap t_{\ell+1}^{(k)}$ holds whenever we enter any of these processes within the algorithm.

\boxed{A} We have $s_{k+1}^{(\ell)}, t_{\ell+1}^{(k)} \in \mathcal{I}(S)$ such that $[s_{k+1}^{(\ell)}]$ and $[t_{\ell+1}^{(k)}]$ belong to the same connected component of Γ , that is, $i(s_{k+1}^{(\ell)}) = i(t_{\ell+1}^{(k)})$. Then $s_{k+1}^{(\ell)} \cap t_{\ell+1}^{(k)}$ and the assumption of edge-freeness force $s_{k+1}^{(\ell)} \sim t_{\ell+1}^{(k)}$, so Lemma 2.2 implies:

\rightsquigarrow There exist $s_{k+1}^{(\ell+1)}, t_{\ell+1}^{(k+1)} \in S_c$ with (4.1).

- [B]** We have $s_{k+1}^{(\ell)}, t_{\ell+1}^{(k)} \in \mathcal{I}(S)$ and the two vertices $[s_{k+1}^{(\ell)}], [t_{\ell+1}^{(k)}]$ belong to distinct coconnected components of Γ , that is, $i(s_{k+1}^{(\ell)}) \neq i(t_{\ell+1}^{(k)})$. Here, balanced factorization implies:
- \rightsquigarrow There exist $s_{k+1}^{(\ell+1)}, t_{\ell+1}^{(k+1)} \in \mathcal{I}(S)$ with (4.1) satisfying $i(s_{k+1}^{(\ell)}) = i(s_{k+1}^{(\ell+1)})$ and $i(t_{\ell+1}^{(k)}) = i(t_{\ell+1}^{(k+1)})$.
- [C]** We have $s_{k+1}^{(\ell)} \in \mathcal{I}(S)$ and $t_{\ell+1}^{(k)} \in S_c$. Thus Lemma 2.1 and Lemma 4.6 imply:
- \rightsquigarrow There are $s_{k+1}^{(\ell+1)} \in \mathcal{I}(S), t_{\ell+1}^{(k+1)} \in S_c$ with (4.1).
- [D]** We have $s_{k+1}^{(\ell)} \in S_c$ and $t_{\ell+1}^{(k)} \in \mathcal{I}(S)$. Thus Lemma 2.1 and Lemma 4.6 imply:
- \rightsquigarrow There are $s_{k+1}^{(\ell+1)} \in S_c, t_{\ell+1}^{(k+1)} \in \mathcal{I}(S)$ with (4.1).
- [E]** We have $s_{k+1}^{(\ell)}, t_{\ell+1}^{(k)} \in S_c$. So Lemma 2.1 implies:
- \rightsquigarrow There are $s_{k+1}^{(\ell+1)}, t_{\ell+1}^{(k+1)} \in S_c$ with (4.1).

The algorithm starts with loop index $(k, \ell) = (0, 0)$ and ends if it arrives at some level (k', ℓ') with $i(s_{k'+1}^{(\ell')}) = i(t_{\ell'+1}^{(k')})$ and $s_{k'+1}^{(\ell')} \perp t_{\ell'+1}^{(k')}$, or if a right LCM has been obtained for $s_m^{(n-1)} \perp t_n^{(m-1)}$. In the first case, the algorithm shows

$$sS \cap tS \subset s_1 \cdots s_{k'+1} S \cap t_1 \cdots t_{\ell'+1} S = s_1 \cdots s_{k'} t_1^{(k')} \cdots t_{\ell'}^{(k')} (s_{k'+1}^{(\ell')} S \cap t_{\ell'+1}^{(k')} S) = \emptyset,$$

which forces $s \perp t$. Similarly, the second case leads to

$$sS \cap tS = s_1 \cdots s_m t_1^{(m)} \cdots t_n^{(m)} S = t_1 \cdots t_n s_1^{(n)} \cdots s_m^{(n)} S,$$

in which case we obtain the right LCM for s and t by combining the right LCMs for the pairs appearing in the algorithm along a path on the grid from $(0, 0)$ to (m, n) , compare Example 4.10.

Lemma 4.12. *Suppose that all coconnected components of Γ are edge-free and mutually nonisomorphic, and that S has balanced factorization.*

- (i) *Assume in addition that all coconnected components of Γ are finite. If $s_1, \dots, s_n \in \mathcal{I}(S)$ and σ is a permutation of $\{1, \dots, n\}$, then there exist $t_1, \dots, t_n \in \mathcal{I}(S)$ with $s_1 s_2 \cdots s_n \sim t_1 \cdots t_n$ and $i(t_k) = i(s_{\sigma(k)})$ for all k .*
- (ii) *Let $m, n \geq 1$. Whenever $s_1, \dots, s_m, t_1, \dots, t_n \in \mathcal{I}(S)$ satisfy $s_1 \cdots s_m \sim t_1 \cdots t_n$, then $m = n$ and there is a permutation σ of $\{1, \dots, n\}$ such that $i(t_\ell) = i(s_{\sigma(\ell)})$ for every $1 \leq \ell \leq n$.*

Proof. For (i), let $s_1, \dots, s_n \in \mathcal{I}(S)$, $s := s_1 \cdots s_n$. Suppose that σ_1, σ_2 are permutations of $\{1, \dots, n\}$ and $t_1^{(1)}, \dots, t_n^{(1)}, t_1^{(2)}, \dots, t_n^{(2)} \in \mathcal{I}(S)$ satisfy

- (1) $s \sim t_1^{(1)} \cdots t_n^{(1)}$ with $i(t_k^{(1)}) = i(s_{\sigma_1(k)})$ for all k ; and
- (2) $t_1^{(1)} \cdots t_n^{(1)} \sim t_1^{(2)} \cdots t_n^{(2)}$ with $i(t_k^{(2)}) = i(t_{\sigma_2(k)}^{(1)})$ for all k .

Then we deduce that $t_1^{(2)}, \dots, t_n^{(2)} \in \mathcal{I}(S)$ satisfies $s \sim t_1^{(2)} \cdots t_n^{(2)}$ with $i(t_k^{(2)}) = i(t_{\sigma_2(k)}^{(1)}) = i(s_{\sigma_1 \sigma_2(k)})$ for all k . It therefore suffices to prove (i) for the Coxeter-Moore generators $\tau_k := (k, k+1)$, where $1 \leq k \leq n-1$, because every permutation is a finite product of these. So let $1 \leq k \leq n-1$ and consider τ_k for $s := s_1 \cdots s_n$ with $s_1, \dots, s_n \in \mathcal{I}(S)$. If $i(s_k) = i(s_{k+1})$ holds, then $t_\ell := s_\ell$ for all ℓ is a solution. In the case of $i(s_k) \neq i(s_{k+1})$, we start by choosing $t_j := s_j$ for $1 \leq j \leq k-1$. At step k , we need to invoke our assumptions:

For every $t \in \mathcal{I}(S)$ with $i(t) = i(s_{k+1}) \neq i(s_k)$, there is an edge $([t], [s_k])$ in Γ , that is, $t \pitchfork s_k$. Thus balanced factorization yields elements $s_t, t' \in \mathcal{I}(S)$ with $s_k S \cap tS = s_k t' S, s_k t' = t s_t$ and $i(s_t) = i(s_k), i(t') = i(t) = i(s_{k+1})$. For all $r, t \in \mathcal{I}(S)$ with $i(t) = i(r) = i(s_{k+1})$ and $[t] \neq [r]$, edge-freeness of the coconnected components implies $t \perp r$. Therefore we get $s_k t' = t s_t \perp r s_r = s_k r'$, which is equivalent to $t' \perp r'$ by left cancellation. Hence $[t']$ and $[r']$ are distinct vertices from the coconnected component $\Gamma_{i(s_{k+1})}$. Since the vertex set of $\Gamma_{i(s_{k+1})}$ is finite, the map $t \mapsto t'$ is a bijection, so that there is $t \in \mathcal{I}(S)$ with $i(t) = i(s_{k+1})$ and $[t'] = [s_{k+1}]$. Every such t satisfies $s_k s_{k+1} \sim s_k t' = t s_t$, say $s_k s_{k+1} S \cap t s_t S = s_k s_{k+1} a S, s_k s_{k+1} a = t s_t b$ for some $a, b \in S_c$. By setting $(t_k, t_{k+1}) := (t, s_t b)$ for such a t which is fixed from now on (the class $[t]$ is uniquely determined), we have $s_k s_{k+1} a = t_k t_{k+1}$ for some $a \in S_c$ and $i(t_k) = i(s_{k+1})$. Since $t_{k+1} = s_t b \sim s_t$ and $i(s_t) = i(s_k)$, we also get $i(t_{k+1}) = i(s_k)$.

Using left cancellation, $t_j = s_j$ for $1 \leq j \leq k-1$, and $t_k t_{k+1} = t s_t b = s_k s_{k+1} a$, we arrive at

$$t_1 \cdots t_{k+1} S \cap s_1 \cdots s_n S = s_1 \cdots s_{k+1} (a S \cap s_{k+2} \cdots s_n S).$$

Note that this also entails $t_1 \cdots t_{k+1} \pitchfork s_1 \cdots s_n$ due to $a \in S_c$. Since the coconnected components are mutually nonisomorphic, it follows from Corollary 4.8 that $a S \cap s_{k+2} S = a t_{k+2} S, a t_{k+2} = s_{k+2} a_{k+2}$ for some $a_{k+2} \in S_c$ and $t_{k+2} \in \mathcal{I}(S)$ with $i(t_{k+2}) = i(s_{k+2})$. Repeating this procedure for (a_j, s_j) with $k+3 \leq j \leq n$ allows us to arrive at a set of elements $t_{k+3}, \dots, t_n \in \mathcal{I}(S)$ with $i(t_j) = i(s_j)$ for all $j \geq k+2$ and

$$\begin{aligned} t_1 \cdots t_n &= s_1 \cdots s_{k+1} a t_{k+2} t_{k+3} \cdots t_n \\ &= s_1 \cdots s_{k+1} s_{k+2} a_{k+2} t_{k+3} \cdots t_n \\ &\quad \vdots \\ &= s_1 \cdots s_n a_n \\ &\sim s_1 \cdots s_n. \end{aligned}$$

This completes the proof of (i) as $i(t_j) = i(s_{\tau_k(j)})$ holds for all $1 \leq j \leq n$ with this choice of $t_1, \dots, t_n \in \mathcal{I}(S)$.

For (ii), let $s := s_1 \cdots s_m$ and $t := t_1 \cdots t_n$ and suppose $s \sim t$. Then $s S \cap t S = s a S, s a = t b$ for some $a, b \in S_c$, see Lemma 2.2. Thus Remark 4.11 states that $s S \cap t S = s t_1^{(m)} \cdots t_n^{(m)} S = s a S$, which forces $t_1^{(m)} \cdots t_n^{(m)} \in S_c$. As S_c is hereditary, see Lemma 2.2, we get $t_\ell^{(m)} \in S_c$ for every $1 \leq \ell \leq n$. Recall from Remark 4.11 that $t_\ell^{(0)} = t_\ell \in \mathcal{I}(S) \subset S \setminus S_c$, and that $t_\ell^{(k)} \in S_c$ implies $t_\ell^{(k')} \in S_c$ for all $k \leq k' \leq m$ since the flowchart will always end up in process \boxed{C} or \boxed{E} for these cases. It follows that, for each $1 \leq \ell \leq n$, there is a unique minimal $k_\ell \geq 1$ such that $t_\ell^{(k_\ell)} \in S_c$, but $t_\ell^{(k)} \notin S_c$ for $0 \leq k < k_\ell$.

We will now argue that this entails a bijection $\{1, \dots, m\} \rightarrow \{1, \dots, n\}$: According to Remark 4.11, the only process that leads to an output $t_\ell^{(k_\ell)}$ in the core starting from $t_\ell^{(k_\ell-1)} \in \mathcal{I}(S)$ is \boxed{A} . Thus we must have $t_\ell^{(k_\ell-1)} \sim s_{k_\ell}^{(\ell-1)}$ for all ℓ . In particular, this implies that

$$s_{k_\ell}^{(\ell-1)} \in \mathcal{I}(S) \quad \text{with} \quad i(t_\ell^{(k_\ell-1)}) = i(s_{k_\ell}^{(\ell-1)}).$$

Moreover, it follows that the map $\sigma: \{1, \dots, m\} \rightarrow \{1, \dots, n\}, \ell \mapsto k_\ell$ is injective: For every $\ell' > \ell$, we have $s_{k_\ell}^{(\ell'-1)} \in S_c$ because of $s_{k_\ell}^{(\ell)} \in S_c$. Therefore, the right LCM of

$s_{k_\ell}^{(\ell'-1)}$ and $t_{\ell'}^{(k_\ell-1)}$ will be obtained either through process \boxed{D} or \boxed{E} . In the first case we get $t_{\ell'}^{(k_\ell)} \notin S_c$, while the latter case requires $t_{\ell'}^{(k_\ell-1)} \in S_c$, which forces $k_{\ell'} < k_\ell$ due to minimality $k_{\ell'}$.

Switching the role of the s_i and the t_j , we get an injection $\sigma': \{1, \dots, n\} \rightarrow \{1, \dots, m\}$, $k \mapsto \ell_k$. Together with the injective map σ , this shows $m = n$. Moreover, $s_{k_\ell}^{(\ell-1)} \in \mathcal{I}(S)$ satisfies $i(t_\ell^{(k_\ell-1)}) = i(s_{k_\ell}^{(\ell-1)})$, allowing us to deduce $\ell_{k_\ell} = \ell$, that is, $\sigma' \circ \sigma = \text{id}$.

Next, let us note that

- (a) process \boxed{B} preserves the coconnected components due to balanced factorization, that is, $i(s_{k+1}^{(\ell)}) = i(s_{k+1}^{(\ell+1)})$ and $i(t_{\ell+1}^{(k)}) = i(t_{\ell+1}^{(k+1)})$; and that
- (b) processes \boxed{C} and \boxed{D} preserve the coconnected component of the respective noncore irreducible element due to mutually nonisomorphic coconnected components, see Corollary 4.8.

Since only the processes \boxed{B} , \boxed{C} , or \boxed{D} have occurred for computing the right LCM of $s_k^{(\ell-1)}$ and $t_\ell^{(k-1)}$ with $k < k_\ell$, and then \boxed{A} takes place for $k = k_\ell - 1$, we deduce from (a) and (b) that

$$i(t_\ell) = i(t_\ell^{(k_\ell-1)}) = i(s_{k_\ell}^{(\ell-1)}) = i(s_{k_\ell}) = i(s_{\sigma(\ell)}) \quad \text{holds for all } 1 \leq \ell \leq n.$$

□

Theorem 4.13. *A right LCM monoid S admits a generalized scale if and only if the following conditions hold:*

- (i) *The family $(|V_i|)_{i \in I} \subset \mathbb{N}^\times \cup \{\infty\}$ freely generates a nontrivial submonoid of \mathbb{N}^\times .*
- (ii) *The coconnected components Γ_i are edge-free.*
- (iii) *S is noncore factorable.*
- (iv) *S has balanced factorization.*

If S satisfies (i)–(iv), then the generalized scale $N: S \rightarrow \mathbb{N}^\times$ is determined by $N_s := |V_i|$ for $[s] \in V_i$.

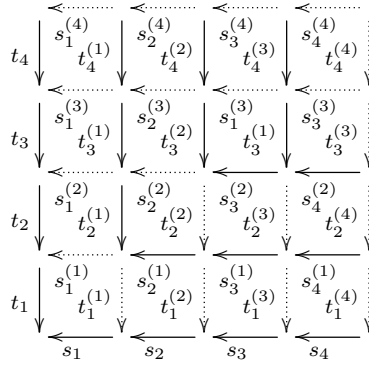
Proof. Suppose first that there is a generalized scale N on S . For $s, t \in \mathcal{I}(S)$ with $s \not\sim t$, Proposition 2.4(ii) shows $([s], [t]) \notin E$ whenever $N_s = N_t$. Thus the coconnected component $\Gamma_i = (V_i, E_i)$ with $[s] \in V_i$ satisfies $N^{-1}(N_s)/\sim \subset V_i$. On the other hand, if $N_s \neq N_t$, then Lemma 3.4 shows $s \pitchfork t$, that is, $([s], [t]) \in E$, because $N_s, N_t \in \text{Irr } N(S)$ by Proposition 4.1. Thus we see that the coconnected components of $\Gamma(S)$ are given by $((N^{-1}(n)/\sim, \emptyset))_{n \in \text{Irr } N(S)}$. This shows (ii) and also (i), as $\text{Irr } N(S)$ freely generates $N(S)$ by Proposition 3.1.

Property (iii) is a direct consequence of Lemma 3.5. Concerning (iv), if $s, t \in \mathcal{I}(S)$ satisfy $N_s \neq N_t$, then $s \pitchfork t$ by Proposition 4.1 and Lemma 3.4. Due to Proposition 2.4(iv) and Proposition 3.1, we have $sS \cap tS = st'S, st' = ts'$ for some $s', t' \in S$ satisfying $N_{st'} = N_s N_t$. This yields $N_{s'} = N_s$ and $N_{t'} = N_t$, and hence (iv).

Conversely, suppose that S satisfies properties (i)–(iv). We first note that (i) forces $|V_i| \geq 2$ for all $i \in I$ and that these are all mutually distinct. Using (i) and (iii), we define $N: S \rightarrow \mathbb{N}^\times$ by $N|_{S_c} = 1$ and $s \sim s_1 \cdots s_m \mapsto \prod_{k=1}^m |V_{i(s_k)}|$, where $s_k \in \mathcal{I}(S)$ for all k . If $s \sim t_1 \cdots t_n$ with $t_\ell \in \mathcal{I}(S)$ for all ℓ , then Lemma 4.12(ii) implies that $m = n$ and that there is a permutation σ of $\{1, \dots, m\}$ such that $i(t_\ell) = i(s_{\sigma(\ell)})$ for all $1 \leq \ell \leq m$.

In particular, we get $\prod_{k=1}^m |V_{i(s_k)}| = \prod_{\ell=1}^n |V_{i(t_\ell)}|$, and hence N is a well-defined map. It is then easy to see that it is in fact a homomorphism of monoids.

Suppose $s, t' \in S$ satisfy $N_s = N_{t'}$, and let $N_s = n_1 \cdots n_m, n_k = |V_{i_k}|$, which is the unique factorization by (i). Due to (iii), $s \sim s_1 \cdots s_m$ and $t' \sim t'_1 \cdots t'_m$ for some $s_k, t'_k \in \mathcal{I}(S)$. In addition, there exists a permutation σ of $\{1, \dots, m\}$ such that $i(s_k) = i(t'_{\sigma(k)})$ for all k . Invoking Lemma 4.12(i), we find $t_1, \dots, t_m \in \mathcal{I}(S)$ with $i(t_k) = i(t'_{\sigma(k)}) = i(s_k)$ for all k and $t' \sim t'_1 \cdots t'_m \sim t := t_1 \cdots t_m$. We can now apply the algorithm of Remark 4.11 to s and t . For $m = 4$ with $s \sim t$, the output diagram from Remark 4.11 would be



where dashed arrows refer to S_c while solid arrows refer to $\mathcal{I}(S)$. More precisely, we get $s \perp t$, unless $s_1 \sim t_1$ due to $i(s_1) = i(t_1)$ and (ii). This leads to $s_1^{(k)}, t_1^{(k)} \in S_c$ for all $k \geq 1$. By iteration, we see that we only need to check that

$$(4.2) \quad s_k^{(k-1)} \sim t_k^{(k-1)}$$

holds for all $1 \leq k \leq m$ to decide upon $s \pitchfork t$, as all other squares will involve (precisely) one element from S_c in its lower half. This stems from the fact that if (4.2) holds for $k = 1, \dots, k_0$, then we get $s_k^{(\ell)}, t_k^{(\ell)} \in S_c$ for all $1 \leq k \leq \ell \leq m, k \leq k_0$. Now suppose there is $\ell \leq m$ such that (4.2) holds for all $k < \ell$. According to the diagram obtained from Remark 4.11 up to $\ell - 1$, (4.2) will hold for ℓ if and only if

$$\alpha_{t_{\ell-1}^{(\ell-1)} \cdots t_1^{(\ell-1)}}^{-1}([s_\ell]) = [s_\ell^{(\ell-1)}] = [t_\ell^{(\ell-1)}] = \alpha_{s_{\ell-1}^{(\ell-1)} \cdots s_1^{(\ell-1)}}^{-1}([t_\ell]).$$

For convenience, let $a := t_{\ell-1}^{(\ell-1)} \cdots t_1^{(\ell-1)}, b := s_{\ell-1}^{(\ell-1)} \cdots s_1^{(\ell-1)} (\in S_c)$. By (i), the connected components have vertex sets of mutually distinct cardinalities, so they are mutually nonisomorphic. Thus Corollary 4.8 applies, showing that α_a and α_b restrict to bijections on $V_{i(s_\ell)}$ and $V_{i(t_\ell)}$, respectively. In particular, we have $i(s_\ell^{(\ell-1)}) = i(s_\ell) = i(t_\ell) = i(t_\ell^{(\ell-1)})$, so that $s_\ell^{(\ell-1)} \pitchfork t_\ell^{(\ell-1)}$ is in fact equivalent to $s_\ell^{(\ell-1)} \sim t_\ell^{(\ell-1)}$.

If (4.2) holds for all k , that is, $s \pitchfork t$, then the top row and the rightmost column of the diagram jointly mediate

$$s \sim s_1 \cdots s_m \sim s_1 \cdots s_m t_1^{(m)} \cdots t_m^{(m)} = t s_1^{(m)} \cdots s_m^{(m)} \sim t \sim t'.$$

Thus N satisfies (ii) of Definition 1.1, that is, $N_s = N_{t'}$ implies $s \sim t'$ or $s \perp t'$.

A second conclusion we draw from the previous consideration is that $|N^{-1}(n)/\sim| = n$ for all $n \in N(S)$: For $n = n_1 \cdots n_m, n_k = |V_{i_k}|$, we can pick

$$\{s_{k,j} \mid 1 \leq k \leq m, 1 \leq j \leq n_k\} \subset \mathcal{I}(S)$$

such that $i(s_{k,j}) = i_k$ and $s_{k,j} \perp s_{k,j'}$ for all $j \neq j'$. This already shows $|N^{-1}(n)/\sim| \geq n$ as $(s_{1,j_1} \cdots s_{m,j_m})_{1 \leq j_k \leq n_k} \subset N^{-1}(n)$ consists of n mutually orthogonal elements. Now for $[t] \in N^{-1}(n)/\sim$, we can assume that $t \sim t_1 \cdots t_m$ with $i(t_k) = i_k$ for all k (using the replacement technique for $t' \mapsto t \sim t'$ from before). We are going to find (j_1, \dots, j_m) such that $t \sim s_{1,j_1} \cdots s_{m,j_m}$: As $n_1 = |V_{i_1}|$, there is a unique $1 \leq j_1 \leq n_1$ such that $t_1 \sim s_{1,j_1}$. Therefore (4.2) holds for $k = 1$ and we get $t_1^{(k)}, s_{1,j_1}^{(k)} \in S_c$ for all $1 \leq k \leq m$. As $\alpha_{t_1^{(k)}} \alpha_{s_{1,j_1}^{(k)}}^{-1}$ restricts to a bijection of V_{i_2} , see Corollary 4.8, there is $1 \leq j_2 \leq n_2$ such that $\alpha_{t_1^{(1)}} \alpha_{s_{1,j_1}^{(1)}}^{-1}([t_2]) = [s_{2,j_2}]$, which is (4.2) for $k = 2$. Iterating this process, where at stage k we find $1 \leq j_k \leq n_k$ with

$$\alpha_{t_{k-1,j_{k-1}}^{(k-1)}} \cdots \alpha_{t_{1,j_1}^{(1)}} \alpha_{s_{k-1,j_{k-1}}^{(k-1)}}^{-1} \cdots \alpha_{s_{1,j_1}^{(1)}}^{-1}([t_k]) = [s_{k,j_k}]$$

that gives (4.2). This shows $|N^{-1}(n)/\sim| = n$.

The above procedure is also useful in proving (iii) of Definition 1.1: Let $s \in S, n \in N(S)$. Without loss of generality, we may assume $s \in S \setminus S_c, n > 1$ as the remaining cases are trivial due to the presence of core elements. Then (iii) asserts that $s \sim s_1 \cdots s_{m_1}$ for some $s_k \in \mathcal{I}(S)$ with $|V_{i(s_k)}| =: n_{1,k}$, while (i) gives $n = n_{2,1} \cdots n_{2,m_2}$ for uniquely determined $n_{2,k} \in \{|V_i| \mid i \in I\}$. By rearrangement of the factors $n_{2,k}$ and possibly switching $s \mapsto s' \sim s$ via Lemma 4.12(i), we can assume that there is $0 \leq \ell \leq m_1 \wedge m_2$ such that $n_{1,k} = n_{2,k}$ for $1 \leq k \leq \ell$, while $n_{1,k} \neq n_{2,k'}$ for all $\ell < k, k'$. In plain words, we extract the greatest common divisor of $n_{1,1} \cdots n_{1,m_1}$ and n . We then simply set $t_k := s_k$ for $1 \leq k \leq \ell$ and fix an arbitrary choice of $t_{\ell+1}, \dots, t_{m_2}$ with $n_{2,k} = |V_{i(t_k)}|$ for $\ell + 1 \leq k \leq m_2$. Looking at Remark 4.11 for $s_{\ell+1} \cdots s_{m_1}$ and $t_{\ell+1}, \dots, t_{m_2}$, we see that due to $n_{1,k} \neq n_{2,k'}$ for all $\ell < k, k'$, we can always complete the diagram as we only apply process \boxed{B} in every step. The cause of this outcome is nothing but balanced factorization. Thus we find $t \in N^{-1}(n)$ with $s \pitchfork t$ as required, and we have shown that N is indeed a generalized scale. \square

Remark 4.14. Due to the explicit description of the generalized scale in terms of $\Gamma(S)$, Theorem 4.13 not only characterizes the existence of the generalized scale, but also implies uniqueness, which we already showed abstractly in Theorem 3.6.

5. EXAMPLES, CHALLENGES, AND SAITO'S DEGREE MAP

5.1. Self-similar group actions. Let (G, X) be a self-similar group action, that is, X is a finite alphabet in at least two letters and G is a group acting on the free monoid X^* generated by X such that for each $x \in X, g \in G$, there are unique elements $g(x) \in X, g|_x \in G$ such that $g(xw) = g(x)g|_x(w)$ for all $w \in X^*$. The associated Zappa-Szép product $S := X^* \bowtie G$ is the monoid with elements $X^* \times G$ and product $(v, g)(w, h) := (vg(w), g|_w h)$, where the expression $g|_w$ is defined recursively through $g|_{xu} := (g|_x)|_u$. It was observed that S is a right LCM monoid in [BRRW14, Theorem 3.8], and that S

admits a generalized scale, see [ABLS17, Subsection 5.3]. We can recover the second part easily from Theorem 4.13:

- (a) $S_c = S^* = \{(\emptyset, g) \mid g \in G\} \cong G$.
- (b) An element (w, g) belongs to $\mathcal{I}(S)$ if and only if $w \in X$.
- (c) For every $(w, g) \in S \setminus S_c$, w is a nontrivial word $w = x_1x_2 \cdots x_n$ in X of length $n \geq 1$. Thus $(w, g) = (x_1, 1)(x_2, 1) \cdots (x_{n-1}, 1)(x_n, g)$ shows that S is noncore factorable.
- (d) For $(x, g), (y, h) \in \mathcal{I}(S)$, we get $(x, g) \mathfrak{m} (y, h)$ if and only if $x = y$ if and only if $(x, g) \sim (y, h)$. Thus, $\Gamma(S)$ is the empty graph on $|X|$ vertices $[(x, 1)], x \in X$, which is coconnected.

Hence conditions (i)–(iv) in Theorem 4.13 are satisfied and $(w, g) \mapsto |X|^{\ell(w)}$ defines a generalized scale, where $\ell: X^* \rightarrow \mathbb{N}$ denotes the word length with respect to X .

5.2. The $\mathbf{ax+b}$ -semigroup over the natural numbers. A famous example of a right LCM monoid with a generalized scale is the $\mathbf{ax+b}$ -semigroup over the natural numbers $S := \mathbb{N} \rtimes \mathbb{N}^\times$, see [ABLS17, Subsection 5.4]. This example features:

- (a) $S_c = \{(n, 1) \mid n \in \mathbb{N}\} \cong \mathbb{N}$.
- (b) An element (m, p) belongs to $\mathcal{I}(S)$ if and only if p is a prime.
- (c) As every positive integer is a finite product of primes, S is noncore factorable.
- (d) Let $(m, p), (n, q) \in \mathcal{I}(S)$. If $p = q$, then $(m, p) \mathfrak{m} (n, q)$ holds if and only if $m - n \in p\mathbb{Z}$, that is, $(m, p) \sim (n, q)$. If $p \neq q$, then $p\mathbb{Z} + q\mathbb{Z} = \gcd(p, q)\mathbb{Z} = \mathbb{Z}$ and $(m, p)S \cap (n, q)S = (m + pn', pq)S, m + pn' = n + qm'$, where $m', n' \in \mathbb{N}$ represent the smallest nonnegative solution for $m + pn' = n + qm'$ (the solution in \mathbb{Z} is unique up to $pq\mathbb{Z}$). We thus get the coconnected components of $\Gamma(S)$ to be $(V_p)_{p \in P}$, where P denotes the primes in \mathbb{N}^\times , with $V_p = \{[(k, p)] \mid 0 \leq k < p\}$ and $E_p = \emptyset$.
- (e) In this example, balanced factorization mirrors the fact that the prime factorization mentioned in (c) is unique (up to permutation of factors).

So Theorem 4.13 implies that $(m, p) \mapsto p$ is the unique generalized scale on S . In passing, we note that Theorem 4.13 and the notion of a generalized scale thereby provide a profound justification of “the obvious choice” made in [LR10] as described in [BLRS19, Remark 2.4].

5.3. Algebraic dynamical systems. Let (G, P, θ) be an algebraic dynamical system as considered in [BLS18], that is, P is a right LCM monoid acting upon a discrete group G by injective group endomorphisms θ_p such that $pP \cap qP = rP$ implies $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$. The right LCM monoid of interest here is $S := G \rtimes_\theta P$, and we assume that $p \in P^*$ whenever $\theta_p \in \text{Aut}(G)$ ^a.

Remark 5.1. The set $\mathcal{I}(G \rtimes_\theta P)$ is given by $G \times \mathcal{I}(P)$. It is shown in [ABLS17, Proposition 5.11(i)] that S admits a generalized scale (given by $N_{(g,p)} := [G : \theta_p(G)]$) provided that:

- (a) The index $N_p := [G : \theta_p(G)]$ is finite for all $p \in P$, and $N_p > 1$ for some $p \in P$.
- (b) If $N_p = N_q$ for some $p, q \in P$, then $p \in qP^*$.

^aOne may always achieve this by replacing P by the acting semigroup of endomorphisms.

It is easy to see from Theorem 4.13(i) and (iii) that part (a) is also necessary: Suppose first that there was $p \in P$ with infinite index. For the semigroup $S = G \rtimes_{\theta} P$ to be noncore factorable, P has to be noncore factorable, so p would have to be expressible as a finite product $p = p_1 q_2 \cdots p_n$ for some $p_k \in \mathcal{I}(P)$. Since the index is multiplicative, there would exist $q \in \mathcal{I}(P)$ with infinite index. We observe that $(g, q) \in \mathcal{I}(S)$ for all $g \in G$ with $(g, q) \sim (h, q)$ if and only if $g^{-1}h \in \theta_q(G)$. Thus the coconnected component Γ_i of $[(g, q)]$ will contain infinitely many vertices, thereby violating condition (i) of Theorem 4.13. So P must act by finite index endomorphisms. If all of them are automorphisms, then S is a group and hence $\Gamma = (\{v\}, \emptyset)$ will be an obstruction to nontriviality in Theorem 4.13 (i). The next example shows that part (b) is too strong in general:

Example 5.2 (Freely doubled $\times 2$). Let P be the free monoid generated by two elements p, q , $G = \mathbb{Z}$, and $\theta_p = \theta_q = \times 2: \mathbb{Z} \rightarrow \mathbb{Z}$. Then $S = G \rtimes_{\theta} P$ is a right LCM monoid with $\mathcal{I}(S) = \{(n, r) \mid n \in \mathbb{Z}, r \in \{p, q\}\}$, see Remark 5.1. We note that $(m, p) \perp (n, q)$ for all $m, n \in \mathbb{Z}$, while $(m, r) \cap (n, r)$ if and only if $m - n \in \theta_r(G) = 2\mathbb{Z}$, that is, $(m, r) \sim (n, r)$, for $r \in \{p, q\}$. Hence $\Gamma(S)$ is the empty graph on four vertices $[(0, p)], [(1, p)], [(0, q)], [(1, q)]$. Therefore, $\Gamma(S)$ is coconnected and conditions (i), (ii), and (iv) of Theorem 4.13 follow immediately, while (iii) boils down to Remark 5.1, noting that p and q generate P by construction. Thus S admits a generalized scale N with $N_{(g,w)} = 4^{\ell(w)}$, where ℓ denotes the word length on P with respect to $\{p, q\}$.

On the other hand, $G \rtimes_{\theta} P$ may fail to have a generalized scale for $P \cong \mathbb{N}^2$ as soon as condition (b) from Remark 5.1 is violated:

Example 5.3 (Ledrappier's shift). Consider the right LCM monoid $S := G \rtimes_{\sigma, \text{id} + \sigma} \mathbb{N}^2$, where $G := \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ and σ is the right shift $(g_0, g_1 \dots) \mapsto (0, g_0, g_1, \dots)$. In other words, S is the semidirect product of the standard restricted wreath product $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{N}$ by \mathbb{N} , where the latter action is given by $\text{id} + \sigma$. This monoid has close ties to the well-known Ledrappier shift from [Led78], see [Sta, Example 2.9] and [ER07, Section 11]. It is also an example of a right LCM monoid arising from an algebraic dynamical system in the sense of [BLS18]. The two generators for the \mathbb{N}^2 -action σ and $\text{id} + \sigma$ are independent^b injective group endomorphisms whose images have index two. More precisely, we have $G = \sigma(G) \sqcup e_0 + \sigma(G) = (\text{id} + \sigma)(G) \sqcup e_0 + (\text{id} + \sigma)(G)$, where $e_k := (\delta_{k,n})_{n \in \mathbb{N}}$. This leads us to the following features:

- (a) $S_c = S^* = G \times \{0\}$.
- (b) An element (g, m) belongs to $\mathcal{I}(S)$ if and only if $m \in \{\sigma, \text{id} + \sigma\}$.
- (c) The semigroup S is noncore factorable because $\langle \sigma, \text{id} + \sigma \rangle \cong \mathbb{N}^2$, see Remark 5.1.
- (d) Let $(g, m), (h, n) \in \mathcal{I}(S)$. If $m = n$, then $(g, m) \cap (h, n)$ if and only if $g_0 = h_0$ if and only if $(g, m) \sim (h, n)$. For $m \neq n$, we always get $(g, m) \cap (h, n)$ with $(g, m)S \cap (h, n)S = (gm(h'), mn)S$, $gm(h') = hn(g')$ for suitable $g', h' \in G$ due to strong independence of σ and $\text{id} + \sigma$, that is, $\sigma(G) + (\text{id} + \sigma)(G) = G$, see [Sta15, Definition 1.3]. This means, $\Gamma(S)$ has two coconnected components

^bThis strong form of commutativity for injective group endomorphisms of discrete groups surfaced in [CV13], and was analysed more closely in [Sta15, Section 1] and [Sta, Section 2], exhibiting a close connection to $*$ -commutativity from [AR97].

$\Gamma_i, i = 1, 2$ given by $V_1 := \{[(g, \sigma)] \mid g \in \{0, e_0\}\}$ and $V_2 := \{[(g, \text{id} + \sigma)] \mid g \in \{0, e_0\}\}$ with $E_i = \emptyset$.

(e) Balanced factorization holds because of independence, see (d).

We note that unlike for the previous examples from 5.1, 5.2, and Example 5.4, Corollary 4.8 does not apply, but its conclusion holds nonetheless. The natural alternative to a generalized scale for S is the homomorphism given by $g \mapsto 1$ for $g \in G$ and $\sigma \mapsto 2, \text{id} + \sigma \mapsto 2$. We expect the corresponding dynamics on $C^*(S)$ to have a KMS-state structure of the same kind as we have established for right LCM monoids admitting a generalized scale, see [ABLS17, Theorem 4.3] and [BLRSt].

While all the previous examples satisfy the conclusion of Corollary 4.8 that $\alpha: S_c \curvearrowright S/\sim$ restricts to permutations on the vertex sets of the coconnected components of $\Gamma(S)$, the following is an easy example, where this is no longer the case. As this requires the existence of two distinct isomorphic coconnected components in $\Gamma(S)$, any such example will fail condition (i) of Theorem 4.13. It will thus not have a generalized scale, which indicates that despite the usefulness of the notion of generalized scales, we ought to consider more flexible analogues eventually.

Example 5.4 (Multiplication with flip). Let $p \in \mathbb{Z}, |p| \geq 2$ and

$$p_0 := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p_1 := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}).$$

Consider $S := \mathbb{Z}^2 \rtimes P$, where $P \subset M_2(\mathbb{Z})$ is the monoid generated by p_0 and x . Then S is a right LCM monoid and $P = \langle p_0, p_1 \rangle \rtimes \langle x \rangle \cong \mathbb{N}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ with $xp_0x = p_1$. We obtain:

- (a) $S_c = S^* = \mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$.
- (b) An element (m, q) belongs to $\mathcal{I}(S)$ if and only if $q = p_i x^\ell$ for $i, \ell \in \{0, 1\}$ as every element in P admits a unique normal form $q = p_{k_1} p_{k_2} \cdots p_{k_n} x^\ell$ with $k_j, \ell \in \{0, 1\}$.
- (c) The normal form described in (b) implies that S is noncore factorable.
- (d) Let $(m, q_0), (n, q_1) \in \mathcal{I}(S)$. If $q_i = p_k x^{\ell_i}$ for some $k, \ell_i \in \{0, 1\}$, then $(m, q_0) \pitchfork (n, q_1)$ holds if and only if $m - n \in p_k \mathbb{Z}^2$, that is, $(m, q_0) \sim (n, q_1)$. On the other hand, if $q_i = p_i x^{\ell_i}$ for some $\ell_i \in \{0, 1\}, i = 1, 2$, then $(m, q_0) \pitchfork (n, q_1)$ always holds as $p_1 \mathbb{Z}^2 + p_2 \mathbb{Z}^2 = \mathbb{Z}^2$. Thus $\Gamma(S)$ is given by two coconnected components Γ_0 and Γ_1 with $V_i := \{[(m, p_0)] \mid 0 \leq m < p e_i\}$ for $i = 0, 1$, where $e_0 = (1, 0), e_1 = (0, 1)$, and $E_i = \emptyset$.

(e) Balanced factorization is due to $p_0 \mathbb{Z}^2 \cap p_1 \mathbb{Z}^2 = p_0 p_1 \mathbb{Z}^2, p_0 p_1 = p_1 p_0$.

For $[(m_0, m_1), p_0] \in V_0$, we get $\alpha_x([(m_0, m_1), p_0]) = [(m_1, m_0), p_1] \in V_1$, so the flip switches the two coconnected components Γ_0 and Γ_1 .

5.4. Graph products. Suppose $\Lambda = (W, F)$ is an undirected graph, and $(S_w)_{w \in W}$ is a family of right LCM monoids. The graph product $S(\Lambda, (S_w)_{w \in W})$ is then defined as the quotient of the free monoid $*_{w \in W} S_w$ by the congruence generated by the relations $st = ts$ for $s \in S_v, t \in S_w$ with $(v, w) \in F$. Whenever there is no ambiguity concerning the vertex semigroups $(S_w)_{w \in W}$, we shall simply write $S(\Lambda)$. This construction goes back to the case of groups first considered in [Gre90], and was studied for monoids in [VdC01, FK09]. Prominent examples of this kind are right-angled Artin monoids, which have already been of interest to operator algebraists, see for instance [CL02, CL07, ELR16, Sta18]. We would like to mention that extensive research has been conducted on such monoids

under the name of *trace monoids* in connection theoretical computer science, see [DR95]. A standard means of studying the structure of graph products is the decomposition of the graph Λ into its *coconnected components* $\Lambda_i, i \in I$. For convenience, let us recall that graph (V, E) is called coconnected if the graph $(V, V \times V \setminus E)$ is connected, see also [Sta18, Subsection 2.2]. Let $(\Lambda_j)_{j \in J}$ be the decomposition into coconnected components for Λ , so that the graph product satisfies $S(\Lambda) = \bigoplus_{j \in J} S(\Lambda_j)$. Now for each $j \in J$, we let $(\Gamma_i)_{i \in I_j}$ with $\Gamma_i = (V_i, E_i)$ denote the decomposition of $\Gamma(S(\Lambda_j))$ into its coconnected components.

Theorem 5.5. *Let $\Lambda = (W, F)$ be an undirected graph and $(S_w)_{w \in W}$ a family of right LCM monoids. Then:*

- (i) $\mathcal{I}(S(\Lambda)) = \bigsqcup_{j \in J} \mathcal{I}(S(\Lambda_j)) \oplus \bigoplus_{j' \in J \setminus \{j\}} S(\Lambda_{j'})_c$.
- (ii) *The coconnected components of $\Gamma(S(\Lambda))$ are edge-free if and only if the coconnected components of $\Gamma(S(\Lambda_j))$ are edge-free for all $j \in J$.*
- (iii) *$S(\Lambda)$ is noncore factorable if and only if $S(\Lambda_j)$ is noncore factorable for all $j \in J$.*
- (iv) *$S(\Lambda)$ has balanced factorization if and only if $S(\Lambda_j)$ has balanced factorization and $\alpha: S(\Lambda_j)_c \curvearrowright S(\Lambda_j)/\sim$ restricts to an action on the vertex sets of the individual coconnected components for all $j \in J$.*

In particular, the graph product $S(\Lambda)$ has a generalized scale if and only if the properties (ii)–(iv) from Theorem 4.13 hold for every $S(\Lambda_j), j \in J$, and $(|V_i|)_{i \in \bigsqcup_{j \in J} I_j} \subset \mathbb{N}^\times \cup \{\infty\}$ freely generates a nontrivial submonoid of \mathbb{N}^\times .

Proof. Part (i) is a straightforward observation using the definition of $S(\Lambda)$, and (iii) then follows easily. For $s = \sum_{j \in J} s_j \in \mathcal{I}(S(\Lambda))$, we let $j(s) \in J$ be the unique index with $s_{j(s)} \in \mathcal{I}(S(\Lambda_{j(s)}))$. Then $s, t \in \mathcal{I}(S(\Lambda))$ satisfy $s \mathfrak{m} t$ if and only if $j(s) \neq j(t)$ or $j(s) = j(t)$ and $s_{j(s)} \mathfrak{m} t_{j(s)}$. Thus the coconnected components of $\Gamma(S(\Lambda))$ are given by $(\Gamma_i)_{i \in \bigsqcup_{j \in J} I_j}$. With this insight, (ii) is an immediate consequence. By (i), balanced factorization for every $j \in J$ is necessary, corresponding to the case of $s, t \in \mathcal{I}(S(\Lambda))$ with $j(s) = j(t)$. For $j(s) \neq j(t)$, the description in (i) yields $sS(\Lambda) \cap tS(\Lambda) = st'S(\Lambda), st' = ts'$ with $t'_j \in S(\Lambda_j)_c$ if and only if $j \neq j(t)$ and $s'_j \in S(\Lambda_j)_c$ if and only if $j \neq j(s)$. At $j(t)$ we have $[t'_{j(t)}] = \alpha_{s_{j(t)}}^{-1}([t_{j(t)}]) \in \mathcal{I}(S(\Lambda_{j(t)}))$, and similarly for s' at $j(s)$. Thus the necessary and sufficient condition for balanced factorization is obtained by adding the assumption that $\alpha: S(\Lambda_j)_c \curvearrowright S(\Lambda_j)/\sim$ restricts to an action on the vertex set V_i of each coconnected component $\Gamma_i, i \in I_j$, for all $j \in J$, compare Corollary 4.8. This proves (iv), and the claim about the existence of generalized scales is now a consequence of Theorem 4.13. \square

Remark 5.6. The condition that $(|V_i|)_{i \in \bigsqcup_{j \in J} I_j} \subset \mathbb{N}^\times \cup \{\infty\}$ freely generates a nontrivial submonoid of \mathbb{N}^\times can also be expressed by saying that

- (a) $(|V_i|)_{i \in I_j} \subset \mathbb{N}^\times \cup \{\infty\}$ freely generates a submonoid M_j of \mathbb{N}^\times for all $j \in J$;
- (b) $M_j \cap M_{j'} = \{1\}$ for all $j, j' \in J, j \neq j'$; and
- (c) there is $j \in J$ such that M_j is nontrivial.

In particular, we see that a necessary condition for $S(\Lambda)$ to admit a generalized scale N is that $S(\Lambda_j)$ admits a generalized scale for all $j \in J$ with nontrivial M_j . In this case, N is determined by this family of generalized scales.

According to Theorem 5.5, the study of generalized scales (or analogues thereof) on graph products can be reduced to the coconnected case. But the situation seems to be quite intricate, as the following example shows:

Example 5.7 (Graph products behaving badly). Let us consider the graph product for $\Lambda = (\{v, w\}, \emptyset)$ with

$$S_v = \langle b \rangle \rtimes \langle q \rangle \cong \mathbb{N} \rtimes \mathbb{N} \cong \langle b \rangle \rtimes \langle q \rangle = S_w$$

for $p, q \in \mathbb{N}_{\geq 2}$, that is, the free product of two one-dimensional subdynamics from 5.2. As Λ is edge-free, $S := S(\Lambda)$ has

- (a) $S_c = S^* = \{1_S\}$; and
- (b) $\mathcal{I}(S) = \{(a, 1), (0, p), (b, 1), (0, q)\}$ with $\Gamma(S)$ having two edges $([(a, 1)], [(0, p)])$ and $([(b, 1)], [(0, q)])$.

So $\Gamma(S)$ is coconnected but has edges, so S does not satisfy (ii) of Theorem 4.13. Therefore S does not admit a generalized scale even though both S_v and S_w do.

It appears that a natural choice for the substitute of the generalized scale in Example 5.7 would be obtained by combining the generalized scales on S_v and S_w suitably with regard to Λ . More precisely, we could consider the homomorphism determined by $(a, 1), (b, 1) \mapsto 2, (0, p) \mapsto 2p$, and $(0, q) \mapsto 2q$. In this specific example we might, for each coconnected component,

- (a) count the number of connected components, and
- (b) pick the generalized scale on each of the connected components

instead.

Such problems cannot occur if we restrict our attention to the special case where all the vertex semigroups are isomorphic to \mathbb{N} , that is, right-angled Artin monoids $S(\Lambda) =: A_\Lambda^+$. Here, the essential obstruction to the existence of the generalized scale in the coconnected case is the presence of edges, as $\Gamma(A_\Lambda^+) \cong \Lambda$, see Theorem 4.13. This already follows from [Sta18, Corollary 4.9], though the approach taken there does not provide for a workaround for graphs with edges. In this respect, Theorem 4.13 is a clear improvement: If Λ is a coconnected graph with at least one edge (hence at least two vertices), then A_Λ^+ satisfies all but condition (ii) of Theorem 4.13.

5.5. Saito's degree map. If S admits a generalized scale $N: S \rightarrow \mathbb{N}^\times$, applying the logarithm results in a homomorphism $\deg: S \rightarrow \mathbb{R}_{\geq 0}$. Under the assumption that the core S_c is nothing more than the invertible elements S^* of S , the map \deg happens to be a particular case of a *degree map* in the sense of [Sai13, Section 4]. Degree maps are employed in [Sai13] to define a growth function and the skew-growth function for the monoid, which are then shown to be inverse to each other as Dirichlet series. In the notation of [Sai13] with a change of variable $t \mapsto e^{-\beta}$, we get

$$P_{\deg}(\beta) = \sum_{[s] \in S/\sim} N_s^{-\beta} = \sum_{n \in N(S)} n^{1-\beta} = \zeta_S(\beta),$$

the ζ -function appearing in [ABLS17, Definition 4.2]. The skew-growth function appears in [ABLS17, Remark 7.4] and is crucial for the reconstruction formula for KMS_β -states on $C^*(S)$ in [ABLS17, Lemma 7.5]. With this perspective and the inversion formula established in [Sai13], degree maps (with extra properties related to the intersection of

right ideals) may yield interesting dynamics on C^* -algebras associated to cancellative monoids, for which the structure of KMS-states can be studied to a satisfactory degree. The assumption $S_c = S^*$ from above is needed to harmonize the two approaches as Saito identifies elements in the monoid up to S^* , while an identification up to S_c is more natural in the approach pursued in [ABLS17]. It may thus be very interesting to see how much of Saito's theory can be transferred from S^* to S_c .

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