RIGIDITY FOR EQUIVARIANT PSEUDO PRETHEORIES

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Abstract. We prove versions of the Suslin and Gabber rigidity theorems in the setting of equivariant pseudo pretheories of smooth schemes over a field with an action of a finite group. Examples include equivariant algebraic K-theory, presheaves with equivariant transfers, equivariant Suslin homology, and Bredon motivic cohomology.

1. Introduction

The classical rigidity theorems for algebraic K-theory are due to Suslin [Sus83] for extensions of algebraically closed fields, Gabber [Gab92] for Hensel local rings, and Gillet-Thomason [GT84] for strictly Hensel local rings. All known proofs rely on $\mathbb{A}^1$-homotopy invariance and existence of transfer maps with certain nice properties. In his work on motives, Voevodsky introduced homotopy invariant pretheories as contravariant functors on smooth schemes over a field enjoying certain transfer maps [Voe00a, Definition 3.1]. While algebraic K-theory admits transfer maps for relative smooth curves, it is not an example of a pretheory [Voe00a, §3.4]. However, it is the motivating example of a pseudo pretheory in the sense of Friedlander-Suslin [FS02, Section 10]. The work of Suslin-Voevodsky [SV96] established rigidity theorems in the context of homotopy invariant pseudo pretheories.

In this paper, we generalize the notion of pseudo pretheories to the equivariant setting of finite group actions (Definition 3.3). Equivariant algebraic K-theory is an example, as well as equivariant Suslin homology, and Bredon motivic cohomology in the sense of [HVØ15, Section 5].

Our main results establish equivariant analogs of the Suslin-Voevodsky rigidity theorems in [SV96, Section 4] (see Theorem 5.1, Theorem 5.4).

Theorem 1.1. *Let k be a field, G be a finite group whose order is invertible in k, and let Sm_k^G denote the category of smooth schemes over k equipped with an action of G. Let F be a homotopy invariant equivariant pseudo pretheory on Sm_k^G. Suppose that F is torsion of exponent coprime to* $\mathrm{char}(k)$*.*

1. *Let $S = \mathrm{Spec}(\mathcal{O}^h_{W,Gw})$ be the Henselization of a smooth affine G-scheme W at the orbit Gw of a closed point. Let $X \to S$ be a smooth affine G-scheme of relative dimension one, admitting an equivariant good compactification. Then for all equivariant sections $i_1, i_2 : S \to X$ which coincide on the closed orbit of S, we have*
$$i_1^* = i_2^* : F(X) \to F(S).$$
2. *Let X be a smooth affine G-scheme and let $x \in X$ be a closed point such that $k \subseteq k(x)$ is separable. If every representation of G over k is a direct*

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sum of one dimensional representations, then there is a naturally induced isomorphism

$$F(Gx) \xrightarrow{\cong} F(\mathrm{Spec}(\mathcal{O}^h_{X,Gx})).$$

The condition in the second part of the theorem is satisfied whenever G is abelian and k contains a primitive dth root of unity, where d is the exponent of the group, by a theorem of Brauer, see e.g., [CR62, Theorem 41.1, Corollary 70.24].

Rigidity theorems have been established for equivariant algebraic K-theory in [YØ09] and [Kri10, Theorem 1.4] at points with trivial stabilizers. The novelty in Theorem 1.1 is that we allow points with *nontrivial* stabilizers. Note, however, that in [YØ09] the groups are more general, and [Kri10] deals with connected split reductive groups. For works on rigidity results in related contexts, see e.g., [AD], [Ayo14], [CD16], [HY07], [Jan], [Nes14], [PY02], [RØ06], [RØ08], [Tab], and [Yag11].

A brief overview of the paper follows. Section 2 recalls notions in G-equivariant algebraic geometry and shows an equivariant proper base change theorem for étale cohomology of Henselian pairs. After recalling equivariant divisors and equivariant correspondences, we define and give examples of equivariant pseudo pretheories in Section 3. Next in Section 4 we discuss the equivariant Nisnevich topology and equivariant good compactification for smooth affine relative curves. Our main results are shown in Section 5. Finally, in Section 6 we show that exactness of the Gersten complex for equivariant algebraic K-theory fails for the group $G = \mathbb{Z}/2\mathbb{Z}$ of order two acting on the affine line $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ by $t \mapsto -t$. This follows by applying rigidity to the G-equivariant Grothendieck group K_0^G of the Henselization $\mathcal{O}^h_{\mathbb{A}^1_k,Gx}$ at the orbit of the closed point $x = (t) \in \mathbb{A}^1_k$.

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2. Preliminaries

Throughout k is a field and G is a finite group whose order is coprime to $\mathrm{char}(k)$ (abusing the terminology we say that n is coprime to $\mathrm{char}(k)$ if n is coprime to the exponential characteristic of k, i.e., n is invertible in k). We view G as a group scheme $\coprod_G \mathrm{Spec}(k)$ over $\mathrm{Spec}(k)$. Let Sch^G_k be the category of separated, finite type schemes over $\mathrm{Spec}(k)$ equipped with a left G-action, and equivariant morphisms. The smooth G-schemes over $\mathrm{Spec}(k)$ form a full subcategory $\mathrm{Sm}^G_k \subseteq \mathrm{Sch}^G_k$. A G-scheme X is equivariantly irreducible if there exists an irreducible component X_0 of X such that $G \cdot X_0 = X$. The fiber product $X \times Y$ of $X, Y \in \mathrm{Sch}^G_k$ is a G-scheme with the diagonal G-action. For a finite dimensional k-vector space V, let $\mathbb{A}(V) := \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ and $\mathbb{P}(V) := \mathrm{Proj}(\mathrm{Sym}(V^\vee))$. If V is a G-representation over k, we view $\mathbb{A}(V)$ and $\mathbb{P}(V)$ as G-schemes via the G-action on V.

For $X \in \mathrm{Sch}_k^G$ we denote the categorical quotient of X by G (in the sense of [MFK94, Definition 0.5]) by X/G, provided it exists. Since G is a finite group, the categorical quotient map $\pi : X \to X/G$ is in fact a uniform geometric quotient ([MFK94, Definitions 0.6, 0.7]). If X is quasi-projective, then a quotient by a finite group $\pi : X \to X/G$ always exists.

Let $H \subseteq G$ be a subgroup and $X \in \mathrm{Sch}_k^H$. Then $G \times X$ is an H-scheme with the action $h(g, x) = (gh^{-1}, hx)$, and we define $G \times^H X := (G \times X)/H$. The scheme $G \times^H X$ has a left G-action through the action of G on itself. Since the H-action on $G \times X$ is free, $\pi : G \times X \to G \times^H X$ is a principle H-bundle. In particular, π is étale and surjective. It follows that if X is smooth, then so is $G \times^H X$. This defines a left adjoint to the restriction functor $\mathrm{Sm}_k^G \to \mathrm{Sm}_k^H$, given by $G \times^H - : \mathrm{Sm}_k^H \to \mathrm{Sm}_k^G$.

For $X \in \mathrm{Sch}_k^G$ and $x \in X$ a point, the *set-theoretic stabilizer* of x is the subgroup $G_x \subseteq G$ defined by $G_x = \{g \in G | gx = x\}$. The *orbit* of x is $Gx := G \times^{G_x} \{x\}$, with underlying set $\{gx | g \in G\}$.

2.1. **G-sheaves.** A G-sheaf on X is basically a sheaf with a G-action which is compatible with the G-action on X. The precise definition goes as follows.

Definition 2.1. Let τ be a Grothendieck topology on X and $\mathcal{F}$ a τ-sheaf of abelian groups. Write $\mathrm{pr}_2 : G \times X \to X$ for the projection and $\mu : G \times X \to X$ for the action map.

(1) A *G-linearization* of $\mathcal{F}$ is an isomorphism $\phi : \mu^*\mathcal{F} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{F}$ of sheaves on $G \times X$ which satisfies the cocycle condition $\mathrm{pr}_{23}^*(\phi) \circ (Id_G \times \mu)^*(\phi) = (m \times Id_X)^*(\phi)$ on $G \times G \times X$. Here $m : G \times G \to G$ is the multiplication and $\mathrm{pr}_{23} : G \times G \times X \to G \times X$ is the projection to second and third factors.
(2) A *G-sheaf* (in the τ-topology) on X is a pair consisting of a τ-sheaf $\mathcal{F}$ together with a G-linearization ϕ of $\mathcal{F}$. We simply write $\mathcal{F}$ for a G-sheaf, leaving the G-linearization understood.
(3) A *G-module* $\mathcal{M}$ on X is a G-sheaf where $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module and the G-linearization $\phi : \mu^*\mathcal{M} \cong pr_2^*\mathcal{M}$ is an $\mathcal{O}_{G\times X}$-module isomorphism. A *G-vector bundle* on X is a G-module $\mathcal{V}$ whose underlying quasi-coherent $\mathcal{O}_X$-module is locally free.

Remark 2.2. Since G is finite, the data of a G-linearization of $\mathcal{F}$ is equivalent to giving a sheaf isomorphism $\phi_g : \mathcal{F} \xrightarrow{\cong} g_*\mathcal{F}$ for each $g \in G$ subject to the conditions $\phi_e = id$ and $\phi_{gh} = h_*(\phi_g) \circ \phi_h$ for all $g, h \in G$.

Remark 2.3. Recall that if G acts on a commutative ring R, the skew group ring $R \wr G$ is the free left R-module with basis $\{[g] \,|\, g \in G\}$ and multiplication is defined by setting $(r[g])(s[h]) = r(g{\cdot}s)[gh]$ and extending linearly. If G acts trivially on R, then $R \wr G$ is simply the usual group ring RG.

If $X = \mathrm{Spec}(R)$, then the category of G-modules on X is equivalent to the category of left $R\wr G$-modules. Further, if the order of G is invertible in R, then the category of G-vector bundles on X is equivalent to the category of left $R\wr G$-modules which are projective as R-modules. See e.g., [LS08, Section 1.1] for details.

A *G-equivariant morphism* $f : (\mathcal{E}, \phi_{\mathcal{E}}) \to (\mathcal{F}, \phi_{\mathcal{F}})$ of G-sheaves is a morphism $f : \mathcal{E} \to \mathcal{F}$ of sheaves compatible with the G-linearizations in the sense that $\phi_{\mathcal{F}} \circ \mu^* f = pr_2^* f \circ \phi_{\mathcal{E}}$, or equivalently $\phi_g \circ f = g_*(f) \circ \phi_g$ for all $g \in G$. Write $Ab_\tau(G, X)$ for the category of G-sheaves on X in the τ-topology. We note that $Ab_\tau(G, X)$ has enough injectives.

Given a G-sheaf $(\mathcal{F}, \phi_g)$, the morphisms ϕ_g induce an action of the group G on the group of global sections $\Gamma(X, \mathcal{F})$. We write $\Gamma_X^G(\mathcal{F}) = \Gamma(X, \mathcal{F})^G$ for the set of G-invariants of $\Gamma(X, \mathcal{F})$. This defines a functor $\Gamma_X^G : \mathrm{Ab}_\tau(G, X) \to \mathrm{Ab}$ from the category of G-sheaves to the category of abelian groups. The τ-G-cohomology groups $H_\tau^p(G; X, \mathcal{M})$ are defined as right derived functors

$$H_\tau^p(G; X, \mathcal{F}) := R^p\Gamma_X^G(\mathcal{F}).$$

Here $\Gamma_X^G = (-)^G \circ \Gamma(X, -)$ is a composite of left exact functors. Since the global sections functor $\Gamma(X, -)$ sends injective G-sheaves to injective $\mathbb{Z}[G]$-modules, the Grothendieck spectral sequence for this composition yields the bounded, convergent spectral sequence

$$E_2^{p,q} = H^p(G, H_\tau^q(X, \mathcal{F})) \Rightarrow H_\tau^{p+q}(G; X, \mathcal{F}), \tag{2.4}$$

where $H^*(G, -)$ denotes the group cohomology of G. Moreover, the spectral sequence induces a finite filtration on each $H_\tau^n(G; X, \mathcal{F})$.

Definition 2.5. The *G-equivariant Picard group* $\mathrm{Pic}^G(X)$ of X is the group of G-line bundles on X modulo equivariant isomorphisms, with group operation given by tensor product. For an invariant closed subscheme $Y \subseteq X$, let $\mathrm{Pic}^G(X, Y)$ denote the group consisting of pairs $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a G-line bundle on X and $\phi : \mathcal{O}_Y \xrightarrow{\cong} \mathcal{L}|_Y$ is an isomorphism of G-line bundles on Y, modulo equivariant isomorphisms respecting the trivializations on Y. The group $\mathrm{Pic}^G(X, Y)$ is called the *relative equivariant Picard group* of X relative to Y.

The following cohomological interpretations of the equivariant and the relative equivariant Picard groups are standard, see [HVØ15, Theorem 2.7, Lemma 6.7].

Theorem 2.6. *Let X be a G-scheme.*
(1) *There is a natural isomorphism* $\mathrm{Pic}^G(X) \xrightarrow{\cong} H^1_{\acute{e}t}(G; X, \mathcal{O}_X^*)$.
(2) *Let $i : Y \hookrightarrow X$ be an invariant closed subscheme. Then there is a natural isomorphism* $\mathrm{Pic}^G(X, Y) \xrightarrow{\cong} H^1_{\acute{e}t}(G; X, \mathbb{G}_{X,Y})$*, where $\mathbb{G}_{X,Y}$ is the étale G-sheaf defined to be the kernel of the equivariant homomorphism $\mathcal{O}_X^* \to i_*\mathcal{O}_Y^*$.*

We end this section by recording an equivariant version of Gabber's proper base change theorem for the cohomology of torsion étale G-sheaves, which will be needed to establish the equivariant version of Suslin's rigidity theorem in Section 5.

Definition 2.7. ([Ray70, Chapter XI, Definition 3]) Let A be a commutative ring and $I \subseteq A$ an ideal which is contained in the Jacobson radical of A. The pair (A, I) is said to be a *Henselian pair* provided $\mathrm{Hom}_A(B, A) \to \mathrm{Hom}_A(B, A/I)$ is surjective for any étale A-algebra B. A G-action on a Henselian pair (A, I) is simply a G-action on A such that the ideal I is invariant.

Theorem 2.8 (Equivariant Proper Base Change)**.** *Let (A, I) be a Henselian pair with G-action. Let $f : Y \to \mathrm{Spec}(A)$ be a proper equivariant map and define Y_0 by the pull-back*

$$\begin{tikzcd} Y_0 \arrow[r, "i"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ \mathrm{Spec}(A/I) \arrow[r, "j"] & \mathrm{Spec}(A). \end{tikzcd}$$

Let $\mathcal{F}$ be a torsion étale G-sheaf on Y and write $\mathcal{F}_0 = i^*\mathcal{F}$. Then the restriction map induces an isomorphism $H^n_{\acute{e}t}(G; Y, \mathcal{F}) \cong H^n_{\acute{e}t}(G; Y_0, \mathcal{F}_0)$ for each n.

Proof. Restriction induces a G-equivariant map $H^p_{\acute{e}t}(Y, \mathcal{F}) \to H^p_{\acute{e}t}(Y_0, \mathcal{F}_0)$. Gabber's base change theorem [Gab94, Corollary 1] shows this is an isomorphism, and therefore it induces an isomorphism in group cohomology. Thus the induced comparison maps of spectral sequences (2.4) for $(Y, \mathcal{F})$ and $(Y_0, \mathcal{F}_0)$ is an isomorphism on the E_2-page. This implies the desired isomorphism.

□

3. Equivariant divisors and pseudo pretheories

We begin by recalling the notion of equivariant Cartier divisors and their properties.

3.1. **Equivariant divisors.** Let X be a G-scheme and $Y \subseteq X$ an invariant closed subscheme.

Definition 3.1.
1. An *equivariant Cartier divisor* on X is an element of $\Gamma^G_X(\mathcal{K}^*_X/\mathcal{O}^*_X)$. The group of equivariant Cartier divisors on X is denoted by $\mathrm{Div}^G(X)$. An effective Cartier divisor D on X such that $D \in \Gamma^G_X(\mathcal{K}^*_X/\mathcal{O}^*_X)$ is called an *equivariant effective Cartier divisor*.
2. A *relative equivariant Cartier divisor* on X relative to Y is an equivariant Cartier divisor D on X such that $\mathrm{Supp}(D) \cap Y = \emptyset$. Write $\mathrm{Div}^G(X, Y)$ for the subgroup of $\mathrm{Div}^G(X)$ consisting of relative equivariant Cartier divisors.
3. A *principal equivariant Cartier divisor* is an invariant rational function on X, i.e., an element in the image of $\Gamma^G_X(\mathcal{K}^*_X)$ in $\Gamma^G_X(\mathcal{K}^*_X/\mathcal{O}^*_X)$. In the relative setting, a *principal equivariant Cartier divisor* f on X is said to be a *principal relative equivariant Cartier divisor* if f is defined and equal to 1 at points of Y.
4. Let $\mathrm{Div}^G_{rat}(X)$ denote the group of equivariant Cartier divisors on X modulo the principal equivariant Cartier divisors, and likewise write $\mathrm{Div}^G_{rat}(X, Y)$ in the relative setting.

Given a Cartier divisor $D = \{(U_i, f_i)\}$ on X, we have an associated line bundle $\mathcal{L}_D$ defined by $\mathcal{L}_D|_{U_i} = \mathcal{O}_{U_i} f_i^{-1}$. When D is an equivariant Cartier divisor it is easy to verify that the line bundle $\mathcal{L}_D$ has a canonical G-linearization; write $\mathcal{L}_D$ for the G-line bundle defined by this choice of linearization. If D is a relative equivariant Cartier divisor relative to Y it is straightforward that $\mathcal{L}_D|_Y$ is trivial.

Let $\mathcal{Z}_d(X)$ (respectively $\mathcal{Z}^d(X)$) denote the free group on dimension d (respectively codimension d) cycles on X. The homomorphism $cyc : Div(X) \to \mathcal{Z}^1(X)$ is defined by $cyc(D) = \sum_{Z \in X^1} ord_Z(D) Z$, where X^1 is the set of closed integral codimension one subschemes. For a G-scheme X, the groups $\mathcal{Z}_d(X)$ and $\mathcal{Z}^d(X)$ have natural G-actions and cyc is an equivariant homomorphism. Therefore we conclude the following.

Lemma 3.2. *([HVØ15, Lemma 2.11]) For a smooth G-scheme X, $cyc : \mathrm{Div}(X) \to \mathcal{Z}^1(X)$ is an equivariant isomorphism.*

3.2. **Equivariant pseudo pretheories.** An equivariant pseudo pretheory is defined as a presheaf on Sm^G_k with transfer maps associated to certain equivariant correspondences subject to some natural axioms.

Definition 3.3. An *equivariant pseudo pretheory* on Sm_k^G is an additive presheaf $F : (\mathrm{Sm}_k^G)^{op} \to \mathrm{Ab}$ (i.e., $F(X \coprod Y) = F(X) \oplus F(Y)$) with transfer maps $\mathrm{Tr}_D : F(X) \to F(S)$ for any equivariant relative smooth affine curve X/S and effective equivariant Cartier divisor D on X which is finite and surjective over a component of S, such that the following holds.

(1) The transfer maps are compatible with pullbacks.
(2) If $D(i)$ is the divisor associated to an equivariant section $i : S \to X$, then
$$\mathrm{Tr}_{D(i)} = F(i).$$
(3) Let $\mathcal{L}_D$ be the G-line bundle associated to D. If the restriction of $\mathcal{L}_D$ to D' is trivial, then
$$\mathrm{Tr}_D + \mathrm{Tr}_{D'} = \mathrm{Tr}_{D+D'}.$$

As usual we extend all functors defined on the category Sm_k^G to limits of smooth G-schemes with G-action (including semilocalizations of all smooth affine G-schemes at closed G-orbits) by taking direct limits. The above properties obviously remain true after such an extension as well.

Definition 3.4. A presheaf F on Sm_k^G (or Sch_k^G) is said to be *homotopy invariant* if for any $X \in \mathrm{Sm}_k^G$ (respectively in Sch_k^G) the projection map $p_1 : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p_1^* : F(X) \xrightarrow{\cong} F(X \times \mathbb{A}^1)$, where the G-action on $X \times \mathbb{A}^1$ is induced by the given G-action on X and the trivial G-action on $\mathbb{A}^1$.

3.3. **Examples of equivariant pseudo pretheories.** In the following we discuss examples of equivariant pseudo pretheories such as equivariant algebraic K-theory, equivariant Suslin homology, K_0^G-presheaves with transfers, presheaves with equivariant transfers, and equivariant motivic representable theories.

Example 3.5. Presheaves with equivariant transfers. For smooth schemes X, Y, the group of correspondences $\mathrm{Cor}_k(X,Y) \subseteq \mathcal{Z}_{\dim(X)}(X \times Y)$ is the subgroup of $\mathcal{Z}_{\dim(X)}(X \times Y)$ of cycles on $X \times Y$ which are finite over X and surjective over some component of X. The category Cor_k has the same objects as Sm/k and $\mathrm{Cor}_k(X,Y)$ are the morphisms between X and Y in this category. The *equivariant correspondences* $\mathrm{Cor}_k^G(X,Y)$ between smooth G-schemes are correspondences $Z : X \to Y$ such that the square

$$\begin{tikzcd} G \times X \arrow[r, "Z \times id"] \arrow[d, "\mu"'] & G \times Y \arrow[d, "\mu"] \\ X \arrow[r, "Z"] & Y \end{tikzcd}$$

commutes in Cor_k [HVØ15, Section 4]. Unravelling definitions we have
$$\mathrm{Cor}_k^G(X,Y) = \mathrm{Cor}_k(X,Y) \cap \mathcal{Z}_{\dim X}(X \times Y)^G.$$
Let Cor_k^G denote the category whose objects are smooth G-schemes and morphisms are equivariant correspondences. There is a canonical inclusion $\mathrm{Sm}_k^G \subseteq \mathrm{Cor}_k^G$ which sends $f : X \to Y$ to its graph $\Gamma_f \subseteq X \times Y$.

Definition 3.6. [HVØ15, Definition 4.1] A *presheaf with equivariant transfers* is a presheaf of abelian groups on the category Cor_k^G.

Given an equivariant relative smooth affine curve X/S and an effective equivariant Cartier divisor D on X which is finite and surjective over S, note that $D \in \mathrm{Cor}_k^G(S, X)$. Moreover, if $D(i)$ is the divisor associated to an equivariant section $i : S \to X$, then $D(i) = \Gamma_i$ in $\mathrm{Cor}_k^G(S, X)$. Therefore if F is a presheaf with equivariant transfers, then F defines an additive presheaf on $\mathrm{Sm}_k^G \subseteq \mathrm{Cor}_k^G$ such that for a divisor D as above, $\mathrm{Tr}_D := F(D) : F(X) \to F(S)$ satisfies conditions (1), (2) and (3) of Definition 3.3.

Example 3.7. Equivariant K-theory. The G-equivariant algebraic K-theory group $K_i^G(X)$ of a scheme X with G-action is the ith homotopy group of the algebraic K-theory spectrum $K^G(X)$ of the exact category of G-vector bundles on X. For $n \geq 2$, the equivariant K-groups with mod-n coefficients are defined as $K_i^G(X; n) := \pi_i(K^G(X) \wedge \mathbb{S}/n)$, for the mod-$n$ Moore spectrum $\mathbb{S}/n$.

The equivariant algebraic K-theory groups K_i^G define functors on Sch_k^G (and Sm_K^G) by considering the category of "big G-vector bundles" ([FS02, Appendix C.4, C.5]). Let $p : X \to S$ be an equivariant relative smooth affine curve in Sm_k^G and let $i_D : D \hookrightarrow X$ be an effective equivariant Cartier divisor on X such that $p_D := p|_D : D \to S$ is finite and surjective. Then $p_D : D \to S$ is also flat. Let $\mathrm{Tr}_D : K_i^G(X) \to K_i^G(S)$ denote the map induced by the functor $F_D : \mathrm{Vect}^G(X) \to \mathrm{Vect}^G(S)$ between the categories of G-vector bundles on X and S defined by $P \mapsto p_{D_*} \circ i_D^*(P)$. By [Tho87, Theorem 4.1, Corollary 5.8(2)], K_i^G is a homotopy invariant functor on Sm_k^G. We show that K_i^G is an equivariant pseudo pretheory on Sm_k^G, so that $K_i^G(-; n)$ is a homotopy invariant equivariant pseudo-pretheory on Sm_k^G with n-torsion values.

Lemma 3.8. *If D and D' are effective equivariant Cartier divisors on X such that the restriction of the G-line bundle $\mathcal{L}_D$ to the G-scheme D' is a trivial G-line bundle, then* $\mathrm{Tr}_{D+D'} = \mathrm{Tr}_D + \mathrm{Tr}_{D'}$.

Proof. We write $i : D \hookrightarrow D + D'$ and $i' : D' \hookrightarrow D + D'$ for the corresponding G-equivariant closed immersions. Let $f \in \Gamma_{D'}^G(\mathcal{L}_D|_{D'})$ define the trivialization of $\mathcal{L}_D$ on D'. Since $\mathcal{L}_D$ defines the ideal sheaf of D, we have an exact sequence of G-equivariant coherent sheaves on $D + D'$:

$$0 \to i'_*(\mathcal{O}_{D'}) \xrightarrow{f} \mathcal{O}_{D+D'} \to i_*(\mathcal{O}_D) \to 0, \tag{3.9}$$

where the maps are G-equivariant. Given $P \in \mathrm{Vect}^G(X)$, the above exact sequence gives the following exact sequence:

$$0 \to i'_* \circ i_{D'}^*(P) \to i_{D+D'}^*(P) \to i_* \circ i_D^*(P) \to 0.$$

Pushforward by the equivariant, finite, and flat map $p_{D+D'}$ gives an exact sequence of G-vector bundles on S:

$$0 \to p_{D'_*} \circ i_{D'}^*(P) \to p_{D+D'_*} \circ i_{D+D'}^*(P) \to p_{D_*} \circ i_D^*(P) \to 0,$$

which by definition of the transfer maps is the exact sequence of functors:

$$0 \to \mathrm{Tr}_{D'}(P) \to \mathrm{Tr}_{D+D'}(P) \to \mathrm{Tr}_D(P) \to 0.$$

Therefore by Waldhausen's additivity theorem, [Wal85, Proposition 1.3.2(4)], we conclude that $\mathrm{Tr}_{D+D'} = \mathrm{Tr}_D + \mathrm{Tr}_{D'}$. ∎

Example 3.10. Equivariant Suslin Homology. For $n \in \mathbb{N}$, the algebraic n-simplex Δ^n is

$$\Delta^n := \operatorname{Spec}\left(\frac{k[t_0, \cdots, t_n]}{(\sum_i t_i - 1)}\right)$$

and $\Delta^\bullet = \{\Delta^n\}_{n\geq 0}$ is a cosimplicial scheme with face and degeneracy maps given by:

$$\partial_r(t_j) = \begin{cases} t_j & \text{if } j < r \\ 0 & \text{if } j = r \\ t_{j-1} & \text{if } j > r \end{cases} \qquad \delta_r(t_j) = \begin{cases} t_j & \text{if } j < r \\ t_j + t_{j+1} & \text{if } j = r \\ t_{j+1} & \text{if } j > r. \end{cases}$$

We view $\Delta^\bullet$ as a cosimplicial G-scheme with trivial G-action.

For a smooth morphism $f : X \to S$, let $C_0(X/S) \subseteq \mathrm{Cor}_k(S, X)$ denote the group of cycles on X which are finite and surjective over a component of S. If $X, S \in \mathrm{Sch}_k^G$ and f is G-equivariant, then $C_0(X/S)$ is a G-invariant subset of $\mathrm{Cor}_k(S, X)$. We let $C_\bullet(X/S)^G$ denote the chain complex associated to the simplicial abelian group $n \mapsto C_n(X/S)^G$, where $C_n(X/S) := C_0(X \times \Delta^n / S \times \Delta^n)$.

Definition 3.11. The nth *equivariant Suslin homology* of X/S is defined as the nth homology group of the complex of abelian groups $C_\bullet(X/S)^G$:

$$\mathrm{H}_n^{\mathrm{Sus}}(G; X/S) := H_n C_\bullet(X/S)^G.$$

For a smooth G-scheme X over k, let $\mathbb{Z}_{tr,G}(X)$ denote the presheaf with equivariant transfers given by the representable functors $\mathbb{Z}_{tr,G}(X)(U) := C_0(X \times U/U)^G = \mathrm{Cor}_k^G(U, X)$ for each $U \in \mathrm{Sm}_k^G$. When G is trivial, this is the same as the presheaf $c_{equi}(X/\operatorname{Spec}(k), 0)$ studied in [Voe00b, Section 5.3]. Similarly for each n, the presheaf $U \mapsto \mathrm{H}_n^{\mathrm{Sus}}(G; X \times U/U)$ is a homotopy invariant presheaf with equivariant transfers. Therefore this defines a family of homotopy invariant equivariant pseudo pretheories.

Lemma 3.12. *Let F be a homotopy invariant equivariant pseudo pretheory on Sm_k^G. Let S be an equivariantly irreducible smooth semilocal G-scheme and X/S be a relative smooth affine curve. Let D and D' be effective equivariant Cartier divisors on X which are finite and surjective over S. If the image of $(D - D')$ in $\mathrm{H}_0^{\mathrm{Sus}}(G; X/S)$ vanishes, then $\mathrm{Tr}_D = \mathrm{Tr}_{D'}$. Here Tr_D and $\mathrm{Tr}_{D'}$ denote the transfer maps associated to D and D', respectively.*

Proof. The proof follows as in [HVØ15, Lemma 6.3]. □

Example 3.13. K_0^G-presheaves. The notion of K_0-presheaves was introduced and studied by Walker in [Wal96] (see also [Sus03, Section 1]). Homotopy invariant K_0-presheaves satisfy many properties enjoyed by presheaves with transfers. An equivariant generalisation of this notion was developed in [HKØ15, Section 6.2]. We briefly recall the definition here.

For $X, Y \in \mathrm{Sch}_k^G$, let $\mathcal{P}^G(X, Y)$ denote the category of coherent G-modules on $X \times Y$ which are flat over X and whose support is finite over X. This is an exact subcategory of the abelian category of coherent G-modules on $X \times Y$. Define $K_0^G(X, Y) := K_0(\mathcal{P}^G(X, Y))$. Given $X, Y, Z \in \mathrm{Sm}_k^G$, we have a natural biexact bifunctor $\mathcal{P}^G(X, Y) \times \mathcal{P}^G(Y, Z) \to \mathcal{P}^G(X, Z)$ given by $(P, Q) \mapsto (p_{XZ})_*(p_{XY}^*(P) \otimes p_{YZ}^*(Q))$, where the tensor product is taken over $\mathcal{O}_{X\times Y\times Z}$. Thus we get a natural composition pairing of exact categories $\circ : K_0^G(X, Y) \times K_0^G(Y, Z) \to K_0^G(X, Z)$ and all these composition laws are associative. This allows us to define an additive category $K_0(\mathrm{Sm}_k^G)$ by taking the objects of Sm_k^G to be the objects and defining $\mathrm{Hom}_{K_0(\mathrm{Sm}_k^G)}(X,Y) = K_0^G(X,Y)$. A K_0^G-*presheaf* is an additive presheaf of abelian groups on the category $K_0(\mathrm{Sm}_k^G)$. Equivariant algebraic K-theory $K_i^G(-)$ is a K_0^G-presheaf for all i; therefore, Example 3.7 is a special case of this one.

There is a functor $\mathrm{Sm}_k^G \to K_0(\mathrm{Sm}_k^G)$ which is the identity on objects and sends a morphism $q : X \to Y$ to the structure sheaf $\mathcal{O}_{\Gamma_q}$ of the graph $\Gamma_q \subseteq X \times Y$. In particular, a K_0^G-presheaf is also a presheaf on Sm_k^G and we discuss below that it is in fact an equivariant pseudo pretheory.

Given an equivariant relative smooth affine curve $p : X \to S$ and an effective equivariant Cartier divisor $i_D : D \hookrightarrow X$ which is finite and surjective over S, the map $p_D := p|_D : D \to S$ is a finite and flat equivariant map. Let $\Gamma^t_{p_D} \subseteq S \times D$ denote the transpose of the graph of p_D and let $\mathcal{O}_{\Gamma^t_{p_D}}$ denote its structure sheaf. Then $\mathcal{F}^t_D := (Id_S \times i_D)_*(\mathcal{O}_{\Gamma^t_{p_D}}) \in \mathcal{P}^G(S,X)$. Define $\mathrm{Tr}_D : F(X) \to F(S)$ to be $F(\mathcal{F}^t_D)$. Then the transfer maps Tr_D are clearly compatible with pullbacks and sections. If D and D' are as in Lemma 3.8, then the exact sequence (3.9) gives an exact sequence of coherent sheaves in $\mathcal{P}^G(S,X)$:

$$0 \to \mathcal{F}^t_{D'} \to \mathcal{F}^t_{D+D'} \to \mathcal{F}^t_D \to 0.$$

Using the additivity in $K_0^G(S,X)$, it follows that $\mathrm{Tr}_{D+D'} = \mathrm{Tr}_D + \mathrm{Tr}_{D'}$.

Example 3.14. Bredon motivic cohomology. Bredon motivic cohomology introduced in [HVØ15, Section 5] and further studied in [HVØ16] (for smooth varieties equipped with $\mathbb{Z}/2\mathbb{Z}$-action) is an equivariant generalization of motivic cohomology for finite group actions.

For a smooth G-scheme X over k, recall that $\mathbb{Z}_{tr,G}(X)$ denotes the presheaf with equivariant transfers given by $\mathbb{Z}_{tr,G}(X)(-) := \mathrm{Cor}_k^G(-,X)$. If F is a presheaf of abelian groups on Sm_k^G, write $C^*F(X)$ for the cochain complex associated to the simplicial abelian group $F(X \times \Delta_\bullet)$. For a finite dimensional representation V of G, let $\mathbb{Z}_G(V)$ denote the complex of presheaves with equivariant transfers given by:

$$\mathbb{Z}_G(V) := C^*(\mathbb{Z}_{tr,G}(\mathbb{P}(V \oplus 1))/\mathbb{Z}_{tr,G}(\mathbb{P}(V)))[-2\dim(V)].$$

The *Bredon motivic cohomology* of a smooth G-variety X is defined to be the equivariant Nisnevich hypercohomology with coefficients in $\mathbb{Z}_G(V)$:

$$H^n_G(X, \mathbb{Z}(V)) := H^n_{GNis}(X, \mathbb{Z}_G(V)).$$

(See Section 4.1 for the definition of the equivariant Nisnevich site.)

The fact that Bredon motivic cohomology define presheaves with equivariant transfers follows from [Voe00c, Proposition 3.1.9] in the case of a trivial group and is proved in [HVØ16, Corollary 3.8] for $\mathbb{Z}/2\mathbb{Z}$. The case of finite groups follows verbatim from the fact that smooth G-schemes have finite equivariant Nisnevich cohomological dimension [HVØ15, Corollary 3.9] and [HVØ15, Theorem 4.15(3)]. Therefore Bredon motivic cohomology define equivariant pseudo pretheories.

4. Equivariant Nisnevich topology and compactifications

In this section we discuss the notions of equivariant Nisnevich topology and equivariant good compactification of equivariant smooth relative curves. We establish some of their properties which are needed in the proofs of our rigidity theorems.

4.1. **Equivariant Nisnevich topology.** We recall briefly the equivariant Nisnevich topology on Sm_k^G for finite groups, first introduced by Voevodsky in [Del09, Section 3.1].

Definition 4.1. A *distinguished square* in Sch_k^G is a cartesian square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, hook, "j"] & X, \end{tikzcd} \tag{4.2}$$

where j is an equivariant open immersion, p an equivariant étale morphism, and the induced map $(Y \smallsetminus B)_{\mathrm{red}} \to (X \smallsetminus A)_{\mathrm{red}}$ is an isomorphism. The collection of distinguished squares forms a *cd*-structure in the sense of [Voe10, Definition 2.1]. The associated Grothendieck topology is called the equivariant *Nisnevich* topology. We write $(\mathrm{Sm}_k^G)_{GNis}$ (resp. $(\mathrm{Sch}_k^G)_{GNis}$) for the respective sites of smooth G-schemes and G-schemes equipped with the *equivariant Nisnevich topology*.

Equivariant Nisnevich covers admit the following equivalent characterizations (see [HKØ15, Propositions 2.15, 2.17]).

Proposition 4.3. *Let $f : Y \to X$ be an equivariant étale map between G-schemes. The following are equivalent.*

(1) *The map f is an equivariant Nisnevich cover.*
(2) *There exists a sequence of invariant closed subschemes*
$$\emptyset = Z_{m+1} \subseteq Z_m \subseteq \cdots \subseteq Z_1 \subseteq Z_0 = X$$
such that $f|_{f^{-1}(Z_i - Z_{i+1})} : f^{-1}(Z_i - Z_{i+1}) \to Z_i - Z_{i+1}$ has an equivariant section.
(3) *For every $x \in X$, there exists a point $y \in Y$ such that f induces isomorphisms of residue fields $k(x) \cong k(y)$ and set-theoretic stabilizers $G_y \cong G_x$.*

Let $X \in \mathrm{Sch}_k^G$ and suppose $x \in X$ has an invariant open affine neighborhood. Then the semilocal ring $\mathcal{O}_{X,Gx}$ has a natural G-action which induces a G-action on the Henselian semilocal ring $\mathcal{O}^h_{X,Gx}$ with a single closed orbit. Any semilocal Henselian affine G-scheme over k with a single orbit is equivariantly isomorphic to $\mathrm{Spec}(\mathcal{O}^h_{Y,Gy})$ for some affine G-scheme Y and $y \in Y$.

For $X \in \mathrm{Sch}_k^G$ and any $x \in X$, let $N_G(Gx)$ denote the filtering category of equivariant étale neighborhoods of Gx. Its objects are pairs $(p : U \to X, s)$, where U is an equivariantly irreducible G-scheme, p is an equivariant étale map, and $s : Gx \to U$ is an equivariant section of p over Gx. A morphism from $(U \to X, s)$ to $(V \to X, s')$ in $N_G(Gx)$ is a map $f : U \to V$ making the evident triangles commute. Although $x \in X$ might not be contained in any G-invariant affine neighborhood, it makes sense to consider $G \times^{G_x} \mathrm{Spec}(\mathcal{O}^h_{X,x})$ and according to [HVØ15, Proposition 3.13] we have:

$$\lim_{U \in N_G(Gx)} U \cong \mathrm{Spec}(\mathcal{O}^h_{G\times^{G_x} X, Gx}) \cong G \times^{G_x} \mathrm{Spec}(\mathcal{O}^h_{X,x}). \tag{4.4}$$

Further if $x \in X$ has an invariant affine neighborhood then there is a canonical G-isomorphism

$$G \times^{G_x} \mathrm{Spec}(\mathcal{O}^h_{X,x}) \xrightarrow{\cong} \mathrm{Spec}(\mathcal{O}^h_{X,Gx}). \tag{4.5}$$

For a Nisnevich sheaf F on Sm_k^G, $X \in \mathrm{Sm}_k^G$, and $x \in X$, we set

$$p_x^* F := F(\mathrm{Spec}(\mathcal{O}^h_{G\times^{G_x}X, Gx})) = \operatorname*{colim}_{U \in N_G(Gx)} F(U).$$

Then p_x^* defines a fiber functor from the category of sheaves to sets, i.e., it commutes with colimits and finite products and so determines a point of the G-equivariant Nisnevich topos. It is known that the set of points $\{p_x^* | x \in X, X \in \mathrm{Sm}_k^G\}$ forms a conservative set of points for $(\mathrm{Sm}_k^G)_{GNis}$ (see [HVØ15, Theorem 3.14]).

4.2. **Suslin homology of equivariant curves.** An equivariant map $p : X \to S$ is an equivariant curve if all of its fibers have dimension one.

Definition 4.6. Say that a smooth equivariant curve $p : X \to S$ admits a *good compactification* if p factors as

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "p"'] & \overline{X} \arrow[d, "\overline{p}"] \\ & S, \end{tikzcd}$$

where $\overline{X}$ is normal, $\overline{p}$ is a proper equivariant curve, j is an equivariant open embedding, and $X_\infty = (\overline{X} \smallsetminus X)_{\mathrm{red}}$ has an invariant open affine neighborhood in $\overline{X}$.

The following lemma about base change is straightforward to verify.

Lemma 4.7. *Let $X \to S$ be an equivariant smooth curve and $S' \to S$ be an equivariant map, where S, S' are affine G-schemes (smooth or a local or semilocal G-scheme which is a limit of smooth G-schemes). If $X \to S$ admits an equivariant good compactification, then the smooth equivariant curve $X' = X \times_S S' \to S'$ also admits an equivariant good compactification.*

If S is affine and $X \to S$ is an equivariant smooth quasi-affine curve with equivariant good compactification $\overline{X}$ and $X_\infty = (\overline{X} \smallsetminus X)_{\mathrm{red}}$, then the equivariant Suslin homology of X/S can be interpreted in terms of relative equivariant Cartier divisors (see [SV96, Theorem 3.1] when G is trivial, and [HVØ15, Theorem 6.12] for an extension to the equivariant case):

$$\mathrm{H}_n^{\mathrm{Sus}}(G; X/S) \cong \begin{cases} \mathrm{Div}^G_{rat}(\overline{X}, X_\infty) & n = 0 \\ 0 & n > 0. \end{cases} \tag{4.8}$$

Lemma 4.9. *Let $S = \lim_{\alpha \in A} S_\alpha$ be a cofiltered limit where the S_α are quasi-projective G-schemes over k and the transition maps are equivariant and affine. If $f : X \to S$ is a finite type equivariant map, then there is λ, a finite type G-scheme X_λ over k, and an equivariant map $f_\lambda : X_\lambda \to S_\lambda$ fitting into a Cartesian square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_\lambda \arrow[d, "f_\lambda"] \\ S \arrow[r] & S_\lambda. \end{tikzcd}$$

Moreover if f is satisfies any of the properties: (i) affine, (ii) open, (iii) smooth, (iv) proper, then f_λ can be chosen to have the same properties.

Proof. Let $T_\alpha = S_\alpha/G$ and $T = \lim_\alpha T_\alpha$. By [Gro66, Théorème 8.8.2] there is β and a map of finite type T_β-schemes $f_\beta : X_\beta \to S_\beta$ such that $X \cong X_\beta \times_{S_\beta} S$ and under this isomorphism f is the pullback of f_β. Moreover if f satisfies some of the properties (i)-(iv), then f_β can be chosen to satisfy the same properties [Gro66, Théorème 8.10.5], [Gro67, Proposition 17.7.8]. For $\alpha \geq \beta$, set $X_\alpha = X_\beta \times_{S_\beta} S_\alpha$. We have that $\mathrm{Aut}_T(X) \cong \mathrm{colim}_\alpha \mathrm{Aut}_{T_\alpha}(X_\alpha)$. Since G is finite, the homomorphism $G \to \mathrm{Aut}_T(X)$ factors through some $\mathrm{Aut}_{T_\lambda}(X_\lambda)$, i.e., we may choose X_λ to have a G-action. Increasing λ we can further assume that f_λ is equivariant. □

Lemma 4.10. *Let $S = \lim_{\alpha \in A} S_\alpha$ be a cofiltered limit where $S_\alpha \in \mathrm{Sm}^G_k$ are affine and the transition maps are equivariant étale. Let $X \to S$ be a smooth equivariant affine curve admitting good compactification.*

1. $\mathrm{H}^{\mathrm{Sus}}_n(G; X/S) \cong \mathrm{colim}_\beta \mathrm{H}^{\mathrm{Sus}}_n(G; X_\beta/S_\beta)$ *where $X_\beta \to S_\beta$ are smooth equivariant curves with good compactification.*
2. $\mathrm{H}^{\mathrm{Sus}}_0(G; X/S) \cong \mathrm{Div}^G_{rat}(\overline{X}, X_\infty)$ *and* $\mathrm{H}^{\mathrm{Sus}}_i(G; X/S) = 0$ *for $i > 0$.*

Proof. Let $X \subseteq \overline{X}$ be an equivariant good compactification. By the previous lemma, there is a smooth, affine, equivariant map $X_\alpha \to S_\alpha$, with equivariant compactification $\overline{X}_\alpha \to S_\alpha$ with $\overline{X}_\alpha \smallsetminus X_\alpha$ has an affine neighborhood, such that $X \cong X_\alpha \times_{S_\alpha} S$ and $\overline{X} \cong \overline{X}_\alpha \times_{S_\alpha} S$. For any generic point $\eta' \in X_\alpha$ lying over a generic point $\eta \in S_\alpha$, we have $\dim(\mathcal{O}_{X_\alpha,\eta'}) = \dim(\mathcal{O}_{S_\alpha,\eta})+1$. Thus there is an open subset of $U \subseteq S_\alpha$ over which the fibers of X_α, $\overline{X}_\alpha$ are one dimensional. Since U contains the image of S in S_α, there is $\lambda \geq \alpha$ such that X_λ and $\overline{X}_\lambda$ are equivariant curves over S_λ, where $X_\beta = X_\alpha \times_{S_\alpha} S_\beta$ for $\beta \geq \alpha$ and similarly for $\overline{X}_\beta$. Replacing $\overline{X}_\lambda$ by its normalization, we see that $X_\lambda \to S_\lambda$ admits good compactification. We thus have that $X \to S$ is isomorphic to the cofiltered limit $\lim_{\beta \geq \lambda}(X_\beta \to S_\beta)$ of smooth affine equivariant curves admitting good compactification. Moreover, we have $\mathrm{colim}_\beta C_n(X_\beta/S_\beta) \cong C_n(X/S)$ and taking fixed points and homology commutes with filtered colimits, yielding (1).

Write $X \to S$ as a filtered limit $\lim_{\beta \in B}(X_\beta \to S_\beta)$ of equivariant curves with good compactification. Moreover we can assume B has a minimal element 0 and $\overline{X}_\beta = \overline{X}_0 \times_{S_0} S_\beta$ is a good compactification of X_β. Write $Y_\beta = \overline{X}_\beta \smallsetminus X_\beta$. Under the isomorphism (4.8), the map $\mathrm{H}^{\mathrm{Sus}}_0(G; X_\beta/S_\beta) \to \mathrm{H}^{\mathrm{Sus}}_0(G; X_\alpha/S_\alpha)$ agrees with the map $\mathrm{Div}^G_{rat}(\overline{X}_\beta, Y_\beta) \to \mathrm{Div}^G_{rat}(\overline{X}_\alpha, Y_\alpha)$ and so $\mathrm{H}^{\mathrm{Sus}}_n(G; X/S) \cong \mathrm{colim}_\beta \mathrm{Div}^G_{rat}(\overline{X}_\beta, Y_\beta)$. Finally, note that $\mathrm{colim}_\beta \mathrm{Div}^G_{rat}(\overline{X}_\beta, Y_\beta) \cong \mathrm{Div}^G_{rat}(\overline{X}, X_\infty)$. □

Corollary 4.11. *Let F be a homotopy invariant equivariant pseudo pretheory on Sm^G_k and $X \to S$ as in the statement of the previous lemma. Then there is a pairing of abelian groups*

$$\mathrm{H}^{\mathrm{Sus}}_0(G; X/S) \otimes F(X) \to F(S).$$

Proposition 4.12. *Let $S = \mathrm{Spec}(\mathcal{O}^h_{W,Gw})$ be the Henselization of a smooth affine G-scheme W at an orbit Gw. Let $p : X \to S$ be a smooth equivariant affine curve with an equivariant good compactification. Let $X_0 \to S_0$ be the fiber over the closed orbit S_0 in S. Then for any n coprime to* $\mathrm{char}(k)$, *restriction induces an injection*

$$\mathrm{H}^{\mathrm{Sus}}_0(G; X/S)/n \hookrightarrow \mathrm{H}^{\mathrm{Sus}}_0(G; X_0/S_0)/n.$$

Proof. Let $\overline{X}$ be the equivariant good compactification of X over S such that $Y = (\overline{X} \smallsetminus X)_{\mathrm{red}}$ has an invariant open neighborhood in $\overline{X}$. By Lemma 4.10(2) and

[HVØ15, Proposition 6.8] it suffices to show that the restriction $\mathrm{Pic}^G(\overline{X}, Y)/n \to \mathrm{Pic}^G(\overline{X}_0, Y_0)/n$ is injective. This follows as in the proof of [SV96, Theorem 4.3], by replacing étale cohomology with $H^*_{et}(G; -)$ and classical proper base change with Theorem 2.8. □

5. Rigidity for equivariant pseudo pretheories

In this section we establish versions of the rigidity theorems of Suslin [Sus83], Gabber [Gab92], and Gillet and Thomason [GT84] in the setting of equivariant pseudo pretheories.

Theorem 5.1 (Equivariant Suslin Rigidity)**.** *Let F be a homotopy invariant equivariant pseudo pretheory on Sm^G_k which takes values in torsion abelian groups of exponent coprime to $\mathrm{char}(k)$. Let $S = \mathrm{Spec}(\mathcal{O}^h_{W,Gw})$ be the Henselization of a smooth affine G-scheme W at a closed orbit, and $X \to S$ a smooth affine equivariant curve admitting good compactification. If $i_1, i_2 : S \to X$ are two equivariant sections which coincide on the closed orbit of S, then $i_1^* = i_2^* : F(X) \to F(S)$.*

Proof. For any n, $F_n = \ker(n : F \to F)$ is again a homotopy invariant equivariant pseudo pretheory and $F = \cup_n F_n$. Thus it suffices to consider the case when $nF = 0$. We may assume that X is equivariantly irreducible. The images of the sections i_j are closed subschemes $W_j \subseteq X$ which are elements of $C_0(X/S)^G$. By definition we have $i_j^* = \mathrm{Tr}_{W_j}$. By Lemma 3.12 it suffices to show that $W_1 - W_2$ becomes zero in $\mathrm{H}^{\mathrm{Sus}}_0(G; X/S)/n$. Proposition 4.12 shows that there is an injection $\mathrm{H}^{\mathrm{Sus}}_0(G; X/S)/n \hookrightarrow \mathrm{H}^{\mathrm{Sus}}_0(G; X_0/S_0)/n$, where X_0 is the fiber over the closed orbit S_0 of S. Since i_1 and i_2 coincide on the closed orbit, we conclude that $W_1 - W_2$ is zero in $\mathrm{H}^{\mathrm{Sus}}_0(G; X/S)/n$.

□

Recall that we write $R \wr G$ for the skew group ring.

Lemma 5.2. *Let $X \to Z$ be a map in Sm^G_k, with X affine, $Z = \mathrm{Spec}(L)$ where L is a field, and $x \in X$ an invariant closed point such that $k(x) \cong L$. Then there is a commutative diagram in Sm^G_k*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & Z, & \end{tikzcd}$$

where $\mathcal{V}$ is an equivariant vector bundle over Z, ϕ is étale at x, and $\phi(x) = 0$.

Proof. Write $X = \mathrm{Spec}(A)$ and $m \subseteq A$ for the maximal ideal corresponding to x. Since $|G|$ is invertible in L, the surjection of $L \wr G$-modules $m \to m/m^2$ has a splitting. The resulting map of $L \wr G$-modules $m/m^2 \to m \subseteq A$ induces the equivariant ring map $\mathrm{Sym}(m/m^2) \to A$. Applying Spec yields the desired map. □

Lemma 5.3. *Let $x \in X$ be an invariant closed point, $X \to \mathrm{Spec}(L)$, and $\mathcal{V}$ be as in the previous lemma. Assume that there is an equivariant vector bundle isomorphism $\mathcal{V} \cong \mathcal{W} \oplus \mathcal{V}'$, where $\mathcal{W}$ has rank $\dim(X) - 1$, and let $p : X \to \mathcal{W}$ be the resulting map. Then there are invariant open affine neighborhoods $U \subseteq X$ and $S \subseteq \mathcal{W}$ of x and 0 respectively, such that p induces a smooth equivariant curve $U \to S$ admitting good compactification.*

Proof. First consider the case where $X \subseteq \mathcal{V}$ is an invariant open subscheme with closure $\overline{X} = \mathcal{W} \times \mathbb{P}(\mathcal{V}' \oplus \mathcal{O}_L)$. For any $a \in X$, the fiber of $X_{p(a)}$ has dimension one and so $(\overline{X} \setminus X)_{p(a)}$ must be finite over $p(a)$ (where $\overline{X} \setminus X$ is considered as a closed subscheme with reduced structure). Since $\overline{X}$ is projective over an affine scheme, there is an invariant affine neighborhood $A \subseteq \overline{X}$ of the finite set of closed points $(\overline{X} \setminus X)_0$. Then $Z = (\overline{X} \setminus X) \setminus ((\overline{X} \setminus X) \cap A)$ is closed in $\overline{X}$ and so has closed image in $\mathcal{W}$. Now let $S \subseteq \mathcal{W}$ be an invariant affine neighborhood of 0 which misses the image of Z and is contained in $p(X)$ (we can find an affine neighborhood with these properties and the intersection over all the translates by $g \in G$ is an invariant neighborhood). Now let $U = X_S$ and $U' = \overline{X}_S$. Then $U' \setminus U$ has an invariant affine neighborhood. Let $\overline{U}$ be the normalization of U'. Then $\overline{U}$ inherits a G-action from that on U' and contains U as an invariant open subscheme. Since $\overline{U} \to U'$ is finite, $\overline{U} \setminus U$ is contained in an invariant affine neighborhood. Now $U \to S$ is a smooth equivariant curve with good compactification $\overline{U}$.

In the general case, since $\phi : X \to \mathcal{V}$ is étale at x, there is an open invariant affine neighborhood on which ϕ is étale, so shrinking X, we may assume ϕ is étale. By the previous paragraph, there are invariant affine neighborhoods $M \subseteq \phi(X)$ of 0 and $S \subseteq \mathcal{W}$ such that $M \to S$ is an equivariant smooth affine curve with good compactification $\overline{M}$. Then $U := \phi^{-1}(M) \to \overline{M}$ is equivariant and quasi-finite and so the equivariant version of Zariski's main theorem (see [LMB00, Theorem 16.5]) yields an equivariant factorization of $U \to \overline{M}$ as the composition of an invariant open immersion $U \hookrightarrow \overline{U}$ and an equivariant finite map $q : \overline{U} \to \overline{M}$. Replacing $\overline{U}$ by its normalization, we may assume $\overline{U}$ is normal. Since $\overline{M}$ is an equivariant good compactification of M over S and q is affine, it follows that $\overline{U}$ is an equivariant good compactification of U over S.

□

Theorem 5.4 (Equivariant Gabber Rigidity)**.** *Assume that every G-representation over k is a direct sum of one dimensional representations. Let F be a homotopy invariant equivariant pseudo pretheory on Sm_k^G with torsion values of exponent coprime to* $\mathrm{char}(k)$*. If X is a smooth affine G-scheme over k of pure dimension d and $x \in X$ is a closed point such that $k \subseteq k(x)$ is separable, then there is an isomorphism:*

$$F(Gx) \xrightarrow{\cong} F(\mathrm{Spec}(\mathcal{O}^h_{X,Gx})).$$

Proof. We proceed by induction on $d = \dim(X)$, the case $d = 0$ being clear. By (4.5), there is an equivariant isomorphism

$$G \times^{G_x} \mathrm{Spec}(\mathcal{O}^h_{X,x}) \xrightarrow{\cong} \mathrm{Spec}(\mathcal{O}^h_{X,Gx}).$$

Thus we are reduced to showing there is an isomorphism

$$\epsilon^* F(\mathrm{Spec}(k(x))) \xrightarrow{\cong} \epsilon^* F(\mathrm{Spec}(\mathcal{O}^h_{X,x})),$$

where $\epsilon(-) = G \times^{G_x} (-)$ and $\epsilon^* F := F \circ \epsilon$. Note that $\epsilon^* F$ is a homotopy invariant equivariant pseudo pretheory on $\mathrm{Sm}_k^{G_x}$ which is torsion of exponent coprime to $\mathrm{char}(k)$. Replacing G by G_x and F by $\epsilon^* F$ it suffices to consider the case where Gx consists of a single point.

The projection $X_x \to X$ sends equivariant étale neighborhoods of $x \in X_x$ to equivariant étale neighborhoods of $x \in X$. If $U \to X$ is an equivariant étale neighborhood of $x \in X$, then $U_x \to X$ is an equivariant étale neighborhood of

$x \in X$ mapping to U. This implies that $\operatorname{Spec}(\mathcal{O}^h_{X_x,x}) \cong \operatorname{Spec}(\mathcal{O}^h_{X,x})$ and so we may replace X with X_x and assume there is an equivariant map $X \to \operatorname{Spec}(L)$, where $L = k(x)$ (equipped with the corresponding G-action). Furthermore, by Lemma 5.2 there is an equivariant vector bundle $\mathcal{V}$ over $\operatorname{Spec}(L)$ such that $\mathcal{O}^h_{X,x} \cong \mathcal{O}^h_{\mathcal{V},0}$ and so it suffices to assume $X = \mathcal{V}$ and $x = 0_L \in \mathcal{V}$.

The assumption on G implies that there is a representation V' over k and an equivariant isomorphism $\mathcal{V} \cong \mathbb{A}(V')_L$, see e.g., the beginning of the proof of [HVØ15, Theorem 8.11]. In particular, $\mathcal{V}$ is a direct sum of equivariant line bundles. Let $i : \mathcal{W} \subseteq \mathcal{V}$ be a rank $d-1$ summand. It now suffices to see that i^* induces an isomorphism $F(\operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0})) \cong F(\operatorname{Spec}(\mathcal{O}^h_{\mathcal{W},0}))$, since $0_L \in \mathcal{W}$ and the induction hypothesis implies that $F(\mathcal{W}) \cong F(0_L)$. The inclusion i is split by the projection $p : \mathcal{V} \to \mathcal{W}$, so it suffices to see that i^* is injective.

Suppose that $[\alpha] \in F(\operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0}))$ is such that $i^*([\alpha]) = 0$. By definition $F(\operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0})) = \operatorname{colim}_{U \to \mathcal{V}} F(U)$, where the colimit is over equivariant étale neighborhoods of $0_L \in \mathcal{V}$. Thus, there is a representative $\alpha \in F(U)$ of $[\alpha]$ where $U \to \mathcal{V}$ is an affine equivariant étale neighborhood of 0_L. There is a canonical equivariant map $\pi : \operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0}) \to U$.

After shrinking U, there is a smooth affine equivariant curve $U \to Y$, admitting a good compactification, by Lemma 5.3, where $Y \subseteq \mathcal{W}$ is an invariant neighborhood of 0. Consider the following commutative diagram of equivariant maps:

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0}) \arrow[dr, "j_i"] \arrow[drrr, bend left, "s_i"] \arrow[ddr, bend right, "\mathrm{id}"'] & & & \\
& \tilde{U} \arrow[rr, "q_2"] \arrow[d, "q_1"] & & U \arrow[d] \\
& \operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0}) \arrow[r, "p"] & \operatorname{Spec}(\mathcal{O}^h_{\mathcal{W},0}) \arrow[r] & Y,
\end{tikzcd}$$

where the rectangle is a pullback. By Lemma 4.7, $\tilde{U} \to \operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0})$ is a smooth affine equivariant curve admitting good compactification. The maps $s_1 := \pi$ and $s_2 := \pi \circ i \circ p$ induce equivariant sections $j_1, j_2 : \operatorname{Spec}(\mathcal{O}^h_{\mathcal{V},0}) \to \tilde{U}$ of q_1. The sections j_1, j_2 agree on the closed orbit by construction and therefore $j_1^* = j_2^*$ by Theorem 5.1. Thus $[\alpha] = \pi^*\alpha = p^*i^*\pi^*\alpha = 0$.

□

6. On the equivariant Gersten resolution

For an affine G-scheme $X \in \mathrm{Sch}^G_k$, let $\mathcal{M}^G(X)$ denote the abelian category of G-equivariant coherent $\mathcal{O}_X$-modules. For $p \geq 0$, let $\mathcal{M}^{G,p}(X) \subset \mathcal{M}^G(X)$ denote the Serre subcategory of coherent sheaves $\mathcal{F}$ whose support is a subscheme of codimension $\geq p$ in X. Since $\mathcal{F}$ is equivariant, the support is an invariant closed subscheme of X. Let $S^{G,p}(X)$ denote the set of all distinct set-theoretic G-orbits $[x]$ in X of codimension p points x of X. Consider the filtration of $\mathcal{M}^G(X)$ by Serre subcategories

$$\mathcal{M}^G(X) = \mathcal{M}^{G,0}(X) \supset \mathcal{M}^{G,1}(X) \supset \mathcal{M}^{G,2}(X) \supset \cdots \supset \mathcal{M}^{G,p}(X) \cdots .$$

Since the natural exact functor $\mathcal{M}^{G,p}(X) \to \coprod_{[x]\in S^{G,p}(X)} \bigcup_n \mathcal{M}^G(\mathrm{Spec}(\mathcal{O}_{X,Gx}/J^n_{Gx}))$ has kernel $\mathcal{M}^{G,p+1}(X)$ and admits a section functor, by [Gab62, Proposition III.2.5] we have an equivalence of categories:

$$\frac{\mathcal{M}^{G,p}(X)}{\mathcal{M}^{G,p+1}(X)} \xrightarrow{\simeq} \coprod_{[x]\in S^{G,p}(X)} \bigcup_n \mathcal{M}^G(\mathrm{Spec}(\mathcal{O}_{X,Gx}/J^n_{Gx})),$$

where J_{Gx} denotes the Jacobson radical of the semilocal ring $\mathcal{O}_{X,Gx}$. The Devissage theorem [Qui73, Theorem 4], the Chinese remainder theorem and the equivalence of equivariant K-theory and G-theory for regular G-schemes [Tho87, Theorem 5.7] imply that

$$K^G_q(\coprod_{y\in[x]} \mathrm{Spec}(k(y))) \simeq G^G_q(\coprod_{y\in[x]} \mathrm{Spec}(k(y))) \simeq K_q(\mathcal{M}^G(\mathrm{Spec}(\mathcal{O}_{X,Gx}/J^n_{Gx}))),$$

for every n. This yields an isomorphism for the union along all n. Further for any $x \in X$, we have the Morita isomorphism [Tho87, Proposition 6.3]

$$K^G_q(\coprod_{y\in[x]} \mathrm{Spec}(k(y))) \simeq K^{G_x}_q(\mathrm{Spec}(k(x))).$$

Combining the above and by [Qui73, Theorem 5], for each $p \geq 0$ there is a localization sequence

$$\cdots \to K_i(\mathcal{M}^{G,p+1}(X)) \to K_i(\mathcal{M}^{G,p}(X)) \to \coprod_{[x]\in S^{G,p}(X)} K^{G_x}_i(\mathrm{Spec}(k(x))) \to K_{i-1}(\mathcal{M}^{G,p+1}(X)) \to \cdots .$$

The above gives rise to a strongly convergent spectral sequence

$$E^{p,q}_1 = \coprod_{[x]\in S^{G,p}(X)} K^{G_x}_{-p-q}(\mathrm{Spec}(k(x))) \Rightarrow G^G_{-p-q}(X).$$

For $X \in Sm^G_k$, the spectral sequence yields a sequence of abelian groups

(6.1)
$$0 \to K^G_n(X) \to \coprod_{[x]\in S^{G,0}(X)} K^{G_x}_n(\mathrm{Spec}(k(x))) \xrightarrow{d_1} \coprod_{[x]\in S^{G,1}(X)} K^{G_x}_{n-1}(\mathrm{Spec}(k(x))) \xrightarrow{d_1} \coprod_{[x]\in S^{G,2}(X)} K^{G_x}_{n-2}(\mathrm{Spec}(k(x))) \xrightarrow{d_1} \cdots ,$$

where d_1 is the differential on the E_1-terms of the spectral sequence.

The Gersten conjecture states that (6.1) is exact if G is trivial and $X = \mathrm{Spec}(R)$, where R is a regular local ring. This is known for regular local rings containing a field, the geometric case was proved by Quillen [Qui73, Theorem 5.11] and the general equicharacteristic case was proved by Sherman [She78] in the 1-dimensional case and Panin [Pan03] for higher dimensions. If X is a regular local ring containing a field with a trivial G-action, where G is a finite diagonalizable group, then the Gersten sequence (6.1) is simply the tensor product of the non-equivariant Gersten sequence with the group ring $\mathbb{Z}[G]$ (by [Ser68, Section 3.4]), and is therefore exact. If the action of G is non-trivial, we discuss in Example 6.2 below that the sequence (6.1) need not be exact even for $n = 0$.

Example 6.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $X = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ via the map $t \mapsto -t$. For the closed point $x = (t) \in \mathbb{A}^1_k$ the Henselization $\mathcal{O}^h_{X,x}$ is the ring of algebraic formal power series in t over k. We compute the G-equivariant K_0 with mod-l

coefficients of $\mathbb{A}^1_{(x)} := \mathrm{Spec}(\mathcal{O}_{X,Gx})$, $\mathrm{Spec}(\mathcal{O}^h_{X,Gx})$, the orbit Gx, and the generic point $\eta \in X$.

By [Tho87, Proposition 6.2] there is an isomorphism

$$K_0^G(Gx) \xrightarrow{\cong} K_0^{G_x}(\mathrm{Spec}(k)),$$

where the set-theoretic stabilizer G_x of x is equal to $G = \mathbb{Z}/2\mathbb{Z}$. We have

$$K_0^G(\mathrm{Spec}(k); l) \cong K_0^G(\mathrm{Spec}(k)) \otimes \mathbb{Z}/l\mathbb{Z} \cong \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}.$$

Thus for a field k of characteristic coprime to 2, l, Theorem 5.4 implies

$$K_0^G(\mathrm{Spec}(\mathcal{O}^h_{X,Gx}); l) \cong K_0^G(Gx; l) \cong \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}.$$

The natural map $\pi : \mathbb{A}^1_{(x)} \to \mathrm{Spec}(k)$ affords a G-equivariant factorization:

$$\begin{tikzcd} \mathbb{A}^1_{(x)} \arrow[rr, "\pi"] \arrow[dr, "j"'] & & \mathrm{Spec}(k) \\ & \mathbb{A}^1_k. \arrow[ur, "\pi_1"'] & \end{tikzcd}$$

Here $j^* : K_0^G(\mathbb{A}^1_k) \to K_0^G(\mathbb{A}^1_{(x)})$ is surjective by the localization exact sequence, and $\pi_1^* : K_0^G(\mathrm{Spec}(k)) \to K_0^G(\mathbb{A}^1_k)$ is an isomorphism [Tho87, Theorems 2.7, 5.7, 4.1]. It follows that $\pi^* : K_0^G(\mathrm{Spec}(k)) \to K_0^G(\mathbb{A}^1_{(x)})$ is surjective. Since $\pi : \mathbb{A}^1_x \to \mathrm{Spec}(k)$ has an equivariant section given by $t \mapsto 0$, $\pi^* : K_0^G(\mathrm{Spec}(k)) \to K_0^G(\mathbb{A}^1_{(x)})$ is also injective. Therefore $K_0^G(\mathbb{A}^1_{(x)}; l) \cong K_0^G(\mathrm{Spec}(k); l) \cong \mathbb{Z}/l\mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}$.

For the generic point $\eta = \mathrm{Spec}(k(t))$, note that the G-action on $k(t)$ is free and $k(t)^G = k(t^2)$. Therefore, $K_0^G(\eta; l) \cong K_0(k(t^2)) \otimes \mathbb{Z}/l\mathbb{Z} \cong \mathbb{Z}/l\mathbb{Z}$ so that $K_0^G(\mathbb{A}^1_{(x)}; l) \ncong K_0^G(\eta; l)$.

Remark 6.3. As pointed out by the referee, the Gersten complex for $\mathbb{A}^1_{(x)}$ with action of the group $G = \mathbb{Z}/2\mathbb{Z}$ given by $t \mapsto -t$ as in the above example can be analyzed using the localization sequence as follows. Under the notations of example 6.2, we get an exact sequence:

$$\cdots \to K_1^G(\mathrm{Spec}(k(t)) \xrightarrow{\partial} K_0^G(\mathrm{Spec}(k)) \xrightarrow{x_*} K_0^G(\mathbb{A}^1_{(x)}) \xrightarrow{\eta^*} K_0^G(\mathrm{Spec}(k(t))).$$

Now the closed point $x \in \mathbb{A}^1_{(x)}$ can be seen as the zero set of the diagonal section of the line bundle $L = \mathbb{A}^1_{(x)} \times \mathbb{A}^1_k \to \mathbb{A}^1_{(x)}$, where $\mathbb{A}^1_k$ has the above non-trivial G-action. By a variant of the excess intersection formula for equivariant K-theory [Köc98, Theorem 3.8], $x_*(1) = 1 - [L]$, and this class is non-zero in $K_0^G(\mathbb{A}^1_{(x)})$. Thus η^* is not injective. The above considerations give the geometric reason for this: as soon as the top Chern class (in equivariant K-theory of the point) of the normal bundle is non-trivial, then x_* is non-zero and η^* is not injective. In the cases considered in other articles, the normal bundle has trivial action, so the top Chern class is zero and the map η^* is injective.

The rigidity property and the exactness of the Gersten sequence (6.1) are two important properties of algebraic K-theory of semilocal rings. In Example 3.7 and Theorem 5.4, we prove the rigidity theorem for equivariant K-theory of schemes with finite group actions. Example 6.2 (see also [Ngu16, Section 5.3]) shows that the Gersten sequence is not exact for equivariant K-theory of semilocal rings with non-trivial $\mathbb{Z}/2\mathbb{Z}$-actions. In this respect the cases of trivial and non-trivial actions are very different.

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