Probability Theory and Related Fields
Linear Volterra backward stochastic differential equations
--Manuscript Draft--

Manuscript Number:	
Full Title:	Linear Volterra backward stochastic differential equations
Article Type:	Original Article
Corresponding Author:	Bernt Øksendal Universitetet i Oslo NORWAY
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Universitetet i Oslo
Corresponding Author's Secondary Institution:	
First Author:	Yaozhong Hu
First Author Secondary Information:	
Order of Authors:	Yaozhong Hu
	Bernt Øksendal
Order of Authors Secondary Information:	

| Abstract: | We present an explicit solution triplet (Y, Z, K) of the backward stochastic Volterra integral equation (BSVIE) of linear type, driven by a Brownian motion and a compensated Poisson random measure. The process Y is expressed by an integral whose kernel is explicitly given. The processes Z and K are expressed by Hida-Malliavin derivatives involving Y.

We apply this to show that under some conditions the solution components Z=Z(t,s) and K=K(t,s,\zeta) of a linear BSVIE are C^1 with respect to t. This result has applications to optimal control of Volterra SDEs |

Linear Volterra backward stochastic differential equations

Yaozhong Hu[1,2] and Bernt Øksendal[2]

2 August 2017

Abstract

We present an explicit solution triplet (Y, Z, K) to the backward stochastic Volterra integral equation (BSVIE) of linear type, driven by a Brownian motion and a compensated Poisson random measure. The process Y is expressed by an integral whose kernel is explicitly given. The processes Z and K are expressed by Hida-Malliavin derivatives involving Y.

We apply this to show that under some conditions the solution components $Z = Z(t, s)$ and $K = K(t, s, \zeta)$ of a linear BSVIE are C^1 with respect to t. This result has applications to optimal control of Volterra SDEs.

1 Introduction and main theorem

Backward stochastic Volterra integral equations (BSVIEs) were introduced to solve stochastic optimal control problems for controlled Volterra type systems. Yong (see e.g. [8], [9]) gives a systematic study, including the existence and uniqueness of the solution of general nonlinear equations. In general the classical backward stochastic differential equations are hard to solve, and those of Volterra type are even worse. In this paper we aim to present an explicit solution formula for linear BSVIEs.

[1]Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1. Email: `yaozhong@ualberta.ca`.
[2]Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N–0316 Oslo, Norway. Email: `oksendal@math.uio.no`.

1

To present our main result we start with a basic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual condition. Let $(B(t), 0 \leq t \leq T)$ be a one dimensional Brownian motion and let $\{N(t, A), 0 \leq t \leq T, A \in \mathcal{B}(\mathbb{R})\}$ be an independent Poisson random measure with Lévy measure ν. Both the Brownian motion B and Poisson random measure N are adapted to the filtration $\mathbb{F}$. Let $\{F(t), 0 \leq t \leq T\}$ be a given stochastic process, not necessarily adapted.

Let $(\Phi(t, s), 0 \leq t < s \leq T)$ and $(\xi(s), \beta(s, \zeta); 0 \leq s \leq T, \zeta \in \mathbb{R})$ be given measurable functions of t, s, and ζ, with values in $\mathbb{R}$. For simplicity we assume that these functions are bounded, and we assume that there exists $\varepsilon > 0$ such that $\beta(s, \zeta) \geq -1 + \varepsilon$ for all s, ζ. We consider the following linear backward stochastic Volterra integral equations in the unknown process triplet $(Y(t), Z(t, s), K(t, s, \zeta))$:

$$
\begin{aligned}
Y(t) = F(t) + \int_t^T & \left[\Phi(t, s) Y(s) + \xi(s) Z(t, s) + \int_{\mathbb{R}} \beta(s, \zeta) K(t, s, \zeta) \nu(d\zeta) \right] ds \\
& - \int_t^T Z(t, s) dB(s) - \int_t^T \int_{\mathbb{R}^d} K(t, s, \zeta) \tilde{N}(ds, d\zeta),
\end{aligned}
\tag{1.1}
$$

where $0 \leq t \leq T$ and $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$ is the compensated Poisson random measure. We want to find three processes $(Y(t), Z(t, s), K(t, s, \zeta), 0 \leq t \leq s \leq T, \zeta \in \mathbb{R})$ which are adapted (meaning that $Y(s), Z(t, s), K(t, s, \zeta) \in \mathcal{F}_s$ for any $s \in [t, T]$ and any $\zeta \in \mathbb{R}$) such that the above equation (1.1) is satisfied.

To find such a solution, we first try to get rid of the unknowns $Z(t, s)$ and $K(t, s, \zeta)$ inside the first integral in (1.1). To this end, we define the probability measure $\mathbb{Q}$ by

$$
d\mathbb{Q} = M(T) d\mathbb{P} \text{ on } \mathcal{F}_T,
\tag{1.2}
$$

where

$$
\begin{aligned}
M(t) := \exp \Big(& \int_0^t \xi(s) dB(s) - \frac{1}{2} \int_0^t \xi^2(s) ds + \int_0^t \int_{\mathbb{R}} \ln(1 + \beta(s, \zeta)) \tilde{N}(ds, d\zeta) \\
& + \int_0^t \int_{\mathbb{R}} \{\ln(1 + \beta(s, \zeta)) - \beta(s, \zeta)\} \nu(d\zeta) ds \Big); \quad 0 \leq t \leq T.
\end{aligned}
\tag{1.3}
$$

Then (see e.g. [7] Ch.1) under the probability measure $\mathbb{Q}$ the process

$$
B_{\mathbb{Q}}(t) := B(t) - \int_0^t \xi(s) ds, \quad 0 \leq t \leq T.
\tag{1.4}
$$

is a Brownian motion, and the random measure

$$
\tilde{N}_{\mathbb{Q}}(dt, d\zeta) := \tilde{N}(dt, d\zeta) - \beta(t, \zeta) \nu(d\zeta) dt
\tag{1.5}
$$

is the $\mathbb{Q}$-compensated Poisson random measure of $N(\cdot, \cdot)$, in the sense that the process

$$
\tilde{N}_{\gamma}(t) := \int_0^t \int_{\mathbb{R}} \gamma(s, \zeta) \tilde{N}_{\mathbb{Q}}(ds, d\zeta)
$$

is a local $\mathbb{Q}$-martingale, for all predictable processes $\gamma(t, \zeta)$ such that

$$\int_0^T \int_{\mathbb{R}} \gamma^2(t, \zeta)\beta^2(t, \zeta)\nu(d\zeta)dt < \infty. \tag{1.6}$$

We also introduce, for $0 \le t \le r \le T$,

$$\Phi^{(1)}(t, r) = \Phi(t, r), \quad \Phi^{(2)}(t, r) = \int_t^r \Phi(t, s)\Phi(s, r)ds$$

and inductively

$$\Phi^{(n)}(t, r) = \int_t^r \Phi^{(n-1)}(t, s)\Phi(s, r)ds, \quad n = 3, 4, \cdots. \tag{1.7}$$

Remark 1.1. *Note that if $|\Phi(t, r)| \le C$ (constant) for all t, r, then by induction*

$$|\Phi^{(n)}(t, r)| \le \frac{C^n T^n}{n!} \tag{1.8}$$

for all t, r, n. Hence,

$$\sum_{n=1}^{\infty} |\Phi^{(n)}(t, r)| < \infty \tag{1.9}$$

for all t, r.

With these notations, we can state our main theorem of this paper as follows:

Theorem 1.2. *Put*

$$\Psi(t, r) = \sum_{n=1}^{\infty} \Phi^{(n)}(t, r). \tag{1.10}$$

Then we have the following explicit form of the solution triplet:

(i) The Y component of the solution triplet is given by

$$Y(t) = \mathbb{E}_{\mathbb{Q}}\left[F(t)\Big|\mathcal{F}_t\right] + \int_t^T \Psi(t, r)\mathbb{E}_{\mathbb{Q}}\left[F(r)\Big|\mathcal{F}_t\right] dr$$

$$= \mathbb{E}_{\mathbb{Q}}\left[F(t) + \int_t^T \Psi(t, r)F(r)dr\Big|\mathcal{F}_t\right]. \tag{1.11}$$

(i) The Z and K components of the solution triplet are given by the following:
    Define

$$U(t) = F(t) + \int_t^T \Phi(t, r)Y(r)dr - Y(t); \quad 0 \le t \le T. \tag{1.12}$$

3

Then $Z(t, s)$ and $K(t, s, \zeta)$ can be expressed by the Hida-Malliavin derivatives D_s and $D_{s,\zeta}$ with respect to B and N, respectively, as follows:

$$Z(t, s) = \mathbb{E}_{\mathbb{Q}}[D_s U(t) - U(t) \int_s^T D_s \xi(r) dB_{\mathbb{Q}}(r) | \mathcal{F}_s]; \quad 0 \le t \le s \le T \tag{1.13}$$

and

$$K(t, s, \zeta) = \mathbb{E}_{\mathbb{Q}}[U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_{s,\zeta} U(t) | \mathcal{F}_s]; \quad 0 \le t \le s \le T, \tag{1.14}$$

where

$$\tilde{H}_s = \exp\Big[\int_0^s \int_{\mathbb{R}} [D_{s,x}\beta(r, x) + \log(1 - \frac{D_{s,x}\beta(r, x)}{1 - \beta(r, x)})(1 - \beta(r, x)]\nu(dx)dr$$
$$+ \int_0^s \int_{\mathbb{R}} \log(1 - \frac{D_{s,x}\beta(r, x)}{1 - \beta(r, x)} \tilde{N}_{\mathbb{Q}}(dr, dx)\Big]. \tag{1.15}$$

Proof. With the processes $B_{\mathbb{Q}}$ and $\tilde{N}_{\mathbb{Q}}$ defined in (1.4)-(1.5) we can eliminate the unknowns $Z(t, s)$ and $K(t, s, \zeta)$ inside the first integral in (1.1). More precisely, we can rewrite equation (1.1) as

$$Y(t) = F(t) + \int_t^T \Phi(t, s)Y(s)ds - \int_t^T Z(t, s)dB_{\mathbb{Q}}(s) - \int_t^T \int_{\mathbb{R}} K(t, s, \zeta)\tilde{N}_{\mathbb{Q}}(ds, d\zeta), \tag{1.16}$$

where $0 \le t \le T$. Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_t$, we get

$$Y(t) = \mathbb{E}_{\mathbb{Q}}\left[F(t) + \int_t^T \Phi(t, s)Y(s)ds \big| \mathcal{F}_t \right]$$
$$= \tilde{F}(t, t) + \int_t^T \Phi(t, s)\mathbb{E}_{\mathbb{Q}}\left[Y(s) \big| \mathcal{F}_t \right] ds, \quad 0 \le t \le T. \tag{1.17}$$

Here, and in what follows, we denote

$$\tilde{F}(t, s) = \mathbb{E}_{\mathbb{Q}}\left[F(t) \big| \mathcal{F}_s \right]. \tag{1.18}$$

Fix $r \in [0, t]$. Taking the conditional $\mathbb{Q}$-expectation on $\mathcal{F}_r$ of (1.17), we get

$$\mathbb{E}_Q [Y(t)|\mathcal{F}_r] = \tilde{F}(t, r) + \int_t^T \Phi(t, s)\mathbb{E}_Q \left[Y(s)|\mathcal{F}_r \right] ds, \quad r \le t \le T \tag{1.19}$$

Denote

$$\tilde{Y}(s) = \mathbb{E}_Q [Y(s)|\mathcal{F}_r], \quad r \le s \le T.$$

Then the above equation can be written as

$$\tilde{Y}(t) = \tilde{F}(t, r) + \int_t^T \Phi(t, s)\tilde{Y}(s)ds, \quad r \le t \le T.$$

4

Substituting $\tilde{Y}(s) = \tilde{F}(s,r) + \int_s^T \Phi(s,u)\tilde{Y}(u)du$ in the above equation, we obtain

$$
\begin{aligned}
\tilde{Y}(t) &= \tilde{F}(t,r) + \int_t^T \Phi(t,s)\left\{\tilde{F}(s,r) + \int_s^T \Phi(s,u)\tilde{Y}(u)du\right\}ds \\
&= \tilde{F}(t,r) + \int_t^T \Phi(t,s)\tilde{F}(s,r)ds + \int_t^T \Phi^{(2)}(t,u)\tilde{Y}(u)duds, \quad r \le t \le T,
\end{aligned}
$$

By repeatedly using the above argument, we get

$$
\begin{aligned}
\tilde{Y}(t) &= \tilde{F}(t,r) + \sum_{n=1}^{\infty} \int_t^T \Phi^{(n)}(t,u)\tilde{F}(u,r)du \\
&= \tilde{F}(t,r) + \int_t^T \Psi(t,u)\tilde{F}(u,r)du, \tag{1.20}
\end{aligned}
$$

where Ψ is defined by (1.10). Now substituting $\mathbb{E}_Q(Y(s)|\mathcal{F}_t) = \tilde{Y}(s)$ (with $r = t$) into (1.17) we obtain part (i) of the theorem.

It remains to prove (1.13)-(1.14). By (1.16) we have

$$
U(t) = \int_t^T Z(t,s)dB_Q(s) + \int_t^T \int_{\mathbb{R}} K(t,s,\zeta)\tilde{N}_Q(ds,d\zeta); \quad 0 \le t \le s \le T. \tag{1.21}
$$

Note that by the Clark-Ocone formula under change of measure (see [5], [6]), extended to $L^2(\mathbb{Q}, \mathcal{F}_T)$ as in [1], we get

$$
Z(t,s) = \mathbb{E}_Q[D_s U(t) - U(t)\int_s^T D_s\xi(r)dB_Q(r)|\mathcal{F}_s]; \quad t \le s \le T \tag{1.22}
$$

and

$$
K(t,s,\zeta) = \mathbb{E}_Q[U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_{s,\zeta}U(t)|\mathcal{F}_s]; \quad t \le s \le T \tag{1.23}
$$

where

$$
\begin{aligned}
\tilde{H}_s = \exp\Big[&\int_0^s \int_{\mathbb{R}} [D_{s,x}\beta(r,x) + \log(1 - \frac{D_{s,x}\beta(r,x)}{1 - \beta(r,x)})(1 - \beta(r,x)]\nu(dx)dr \\
&+ \int_0^s \int_{\mathbb{R}} \log(1 - \frac{D_{s,x}\beta(r,x)}{1 - \beta(r,x)}\tilde{N}_Q(dr,dx)\Big],
\end{aligned} \tag{1.24}
$$

as claimed. $\qquad\square$

2 Application to smoothness of the solution triplet

It is of interest to study when the solution components $Z(t,s), K(t,s,\zeta)$ are smooth (C^1) with respect to t. Such smoothness properties are important in the study of optimal control (see e.g. [3]). It is also important in the numerical solutions (see e.g. [4] and references therein). Using the explicit form of the solution triplet (Theorem 1.2) we can give sufficient conditions for such smoothness in the linear case.

Theorem 2.1. *Assume that ξ, β are deterministic and that $F(t)$ and $\Phi(t,s)$ are C^1 with respect to t satisfying*

$$\mathbb{E}_{\mathbb{Q}}\Big[\int_0^T (\int_t^T \Big\{ F^2(t) + \Phi^2(t,s) + (\frac{dF(t)}{dt})^2 + (\frac{\partial\Phi(t,s)}{\partial t})^2 \Big\} ds)dt \Big] < \infty. \qquad (2.1)$$

Then, for $t < s \leq T$,

$$Z(t,s) = \mathbb{E}_{\mathbb{Q}}[D_s F(t) + \int_t^T D_s(\phi(t,r)Y(r)dr)|\mathcal{F}_s], \qquad (2.2)$$

$$K(t,s,\zeta) = \mathbb{E}_{\mathbb{Q}}[D_{s,\zeta} F(t) + \int_t^T D_{s,\zeta}(\phi(t,r)Y(r)dr)|\mathcal{F}_s]. \qquad (2.3)$$

In particular, we have

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (\int_t^T \left(\frac{\partial Z}{\partial t}(t,s)\right)^2 ds)dt + \int_0^T (\int_t^T \int_{\mathbb{R}} \left(\frac{\partial K}{\partial t}(t,s,\zeta)\right)^2 \nu\,(d\zeta)\,ds)dt \right] < \infty. \qquad (2.4)$$

Proof. Since $Y(t)$ is $\mathcal{F}_t$-measurable, we get that $D_s Y(t) = D_{s,\zeta} Y(t) = 0$ for all $s > t$. Hence by (1.12)

$$\mathbb{E}_{\mathbb{Q}}[D_s U(t)|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[D_s F(t) + \int_t^T D_s(\Phi(t,r)Y(r))dr|\mathcal{F}_s] \qquad (2.5)$$

and

$$\mathbb{E}_{\mathbb{Q}}[D_{s,\zeta} U(t)|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[D_{s,\zeta} F(t) + \int_t^T D_{s,\zeta}(\Phi(t,r)Y(r))dr|\mathcal{F}_s]. \qquad (2.6)$$

Then the result follows from (1.13) and (1.14). $\qquad\qquad\square$

References

[1] Aase, K., Øksendal, B., Privault, N. and Ubøe, J. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. Finance Stochast. 4 (2000), 465-496.

[2] Agram, N. and Øksendal, B. Malliavin calculus and optimal control of stochastic Volterra equations. J. Optim. Theory Appl. DOI10.1007/s10957-015-0735-5 (2015).

[3] Agram, N., Øksendal, B. and Yakhlef, S. Optimal control of forward-backward stochastic Volterra equations. In *F. Gesztezy et al (editors): Partial Differential equations, Mathematical Physics, and Stochastic Analysis. A Volume in Honor of Helge Holden's 60th Birthday.* EMS Congress Reports. http://arxiv.org/abs/1669.409v2

[4] Hu, Y.; Nualart, D. and Song, X. Malliavin calculus for backward stochastic differential equations and application to numerical schemes. The Annals of Applied Probability Vol. 21 (2011), no. 6, 2379-2423.

[5] Huehne, F. A Clark-Ocone-Haussmann formula for optimal portfolio under Girsanov transformed pure-jump Lévy processes. Working paper 2005.

[6] Okur, Y. Y. An extension of the Clark-Ocone formula under change of measure for Lévy processes. Preprint Series, Dept. of Mathematics, University of Oslo January 2008.

[7] Øksendal, B. and Sulem, A. *Applied Stochastic Control of Jump Diffusions.* Second Edition, Springer 2007.

[8] Yong, J. Backward stochastic Volterra equations and some related problems, Stochastic Processes and their Applications 116 (2006), 779-795.

[9] Yong, J. Backward stochastic Volterra integral equations - a brief survey. Appl. Math. J. Chinese Univ. 28(4) (2013), 383-394.

Your PDF file "HO_linearvolterra_2_August_2017.pdf" cannot be opened and
processed. Please see the common list of problems, and suggested
resolutions below.

Reason:

Other Common Problems When Creating a PDF from a PDF file

You will need to convert your PDF file to another format or fix the
current PDF file, then re-submit it.