A GROTHENDIECK-LEFSCHETZ THEOREM FOR EQUIVARIANT PICARD GROUPS

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Abstract. We prove a Grothendieck-Lefschetz theorem for equivariant Picard groups of non-singular varieties with finite group actions.

1. Introduction

The geometry and K-theory of schemes with group scheme actions have been extensively studied by various authors in recent years (e.g., see [7], [9], [8]). The generalization of some of the fundamental theorems of algebraic geometry to the equivariant setting has played an important role in the development of this subject. The classical Lefschetz-type theorems compare the various algebraic invariants of non-singular projective varieties and their hyperplane sections. Let X be a non-singular projective variety over a field k of characteristic zero and let Y be a non-singular subvariety of X, of dimension $\geqslant 3$, which is a scheme-theoretic complete intersection in X. The Grothendieck-Lefschetz theorem for Picard groups (see [4, Théoremè XI.3.1], [6, Corollary IV.3.3]) states that the Picard groups of X and Y are isomorphic. The purpose of this article is to prove an analogous result for varieties with finite group actions.

For a variety X with G-action, let $\mathrm{Pic}^G(X)$ denote the equivariant Picard group of X (see [10, 1.3, page 32]). Our main result is the following.

Theorem 1.1. *Let k be a field of characteristic zero and let G be a finite group. Let X be a non-singular projective variety over k with G-action and let Y be a non-singular G-invariant subvariety of dimension $\geqslant 3$, which is a scheme-theoretic complete intersection in X. Then the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ is an isomorphism.*

In view of the Kodaira-Akizuki-Nakano vanishing theorem, Theorem 1.1 is a straightforward consequence of the technical result Theorem 3.3, which is proved by closely following the proof of the Grothendieck-Lefschetz theorems given in [6, Chapter IV]. The main idea is to use the formal completion of X along Y and a suitable equivariant generalization of the Lefschetz conditions, which is discussed in (2.2). As a corollary to Theorem 3.3, we also deduce that if G acts on a projective space X over k (a field of arbitrary characteristic) and Y is a G-invariant scheme-theoretic complete intersection in X such that $\dim(Y) \geqslant 3$, then the equivariant Picard groups of X and Y are isomorphic (see Corollary 3.4, [6, Corollary IV.3.2] for the non-equivariant case).

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2. Preliminaries

We will work over a base field k of arbitrary characteristic. All schemes are assumed to be separated and of finite type over k. The term variety will refer to an integral scheme over k. In this note, G will always denote a finite group.

2.1. Group action on formal schemes.

In this section, we recall briefly the notion of G-action on a locally ringed space and equivariant sheaves. In the process we set up notations and terminologies for the rest of the paper.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. A *G-action* on $(X, \mathcal{O}_X)$ is a group homomorphism from G to the group of automorphisms of $(X, \mathcal{O}_X)$. A morphism $\theta : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of locally ringed spaces with G-actions is said to be *G-equivariant* if it is compatible with the G-actions on $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$.

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space with a given G-action.

1. A *G-sheaf* of abelian groups on X is a sheaf of abelian groups $\mathcal{F}$ together with a collection of sheaf isomorphisms $\phi_g : \mathcal{F} \xrightarrow{\simeq} g_*\mathcal{F}$, for each $g \in G$, which are subject to the conditions $\phi_e = id$ and $\phi_{gh} = h_*(\phi_g) \circ \phi_h$, for all $h \in G$. We shall denote a G-sheaf in the sequel by $(\mathcal{F}, \{\phi_g\})$.
2. A *G-module* is a G-sheaf $(\mathcal{F}, \{\phi_g\})$ such that $\mathcal{F}$ is an $\mathcal{O}_X$-module and each ϕ_g is an $\mathcal{O}_X$-module isomorphism. A *locally free* (resp. *invertible*) *G-sheaf* is a G-module $(\mathcal{F}, \{\phi_g\})$, where $\mathcal{F}$ is a locally free (resp. invertible) sheaf of $\mathcal{O}_X$-module.
3. A *G-equivariant morphism* of G-sheaves $f : (\mathcal{F}, \{\phi_g\}) \to (\mathcal{G}, \{\phi'_g\})$ is a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ such that $\phi'_g \circ f = g_*(f) \circ \phi_g$, for all $g \in G$. The set of G-equivariant morphisms from $\mathcal{F}$ to $\mathcal{G}$ is denoted by $\mathrm{Hom}^G(\mathcal{F}, \mathcal{G})$. If $\mathcal{F}$ and $\mathcal{G}$ are G-modules, the set of G-equivariant $\mathcal{O}_X$-module homomorphisms is denoted by $\mathrm{Hom}^G_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Example 2.2. When X is a k-scheme, a G-action on the locally ringed space X defined as above coincides with the usual notion of group scheme action on schemes, where G is viewed as a finite constant group scheme over k. Let $\sigma : G \times X \to X$ denote the action map. It is easy to verify that a G-module structure on a sheaf F of $\mathcal{O}_X$-modules as above is equivalent to giving an isomorphism of $\mathcal{O}_{G\times X}$-modules, $\phi : \sigma^*F \to p_2^*F$, over $G \times X$. Therefore F is a G-module in the sense of [10].

Example 2.3. Let X be a noetherian scheme with G-action and let Y be a G-invariant closed subscheme, defined by a sheaf of ideals $\mathcal{I}$ (which is a G-submodule of $\mathcal{O}_X$). Then $(\widehat{X}, \mathcal{O}_{\widehat{X}})$, the formal completion of X along Y, has an induced G-action, as the direct image functor commutes with inverse limits. The canonical morphism $i : \widehat{X} \to X$ is then G-equivariant. Given a G-equivariant coherent $\mathcal{O}_X$-module F, the completion $\widehat{F}$ of F along Y, has a natural G-equivariant $\mathcal{O}_{\widehat{X}}$-module structure. Furthermore, the functor $F \mapsto \widehat{F}$ from the category of coherent $\mathcal{O}_X$-modules to the category of coherent $\mathcal{O}_{\widehat{X}}$-modules is exact (see [5, Corollary II.9.8]) and therefore it is easy to see that it induces an exact functor on the category of coherent G-modules.

Let $(X, \mathcal{O}_X)$ be a locally ringed space with G-action. Let $Sh^G(X)$ denote the category of G-sheaves, which is an abelian category with enough injectives. Given a G-sheaf $\mathcal{F}$ on X, the group G acts on the global sections $\Gamma(X, \mathcal{F})$. Let $\Gamma(X, \mathcal{F})^G$ denote the G-invariant global sections, and let $H^p(G; X, -)$ denote the right derived functors of the functor $\Gamma(X, -)^G$. The groups $H^p(G; X, \mathcal{F})$ are the G-cohomology groups of $\mathcal{F}$.

Lemma 2.4. *Let $(X, \mathcal{O}_X)$ be a locally ringed space with G-action and let $(\mathcal{F}, \{\phi_g\})$, $(\mathcal{G}, \{\phi'_g\})$ be G-modules. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ has an induced G-module structure such that $\operatorname{Hom}^G_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^G$.*

Proof. For each $g \in G$, let $\rho_g : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(g_*\mathcal{F}, g_*\mathcal{G})$ be the $\mathcal{O}_X$-module homomorphism defined as follows. Given an open subset U of X and $f \in Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$, let $\rho_g|_U(f) := (\phi'_g|_U) \circ f \circ (\phi_g^{-1}|_U)$. Let $\widetilde{\rho_g} = \theta_g \circ \rho_g$, where $\theta_g : \mathcal{H}om_{\mathcal{O}_X}(g_*\mathcal{F}, g_*\mathcal{G}) \xrightarrow{\theta_g} g_*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$ are the canonical isomorphisms. Then $(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}), \{\rho_g\})$ is a G-module. Now,

$$\begin{aligned} f \in \operatorname{Hom}^G_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) &\Leftrightarrow \rho_g(X)(f) = \phi'_g \circ f \circ (\phi_g)^{-1} = g_*(f), \forall g \in G \\ &\Leftrightarrow \widetilde{\rho_g}(X)(f) = f, \forall g \in G \\ &\Leftrightarrow f \in \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^G. \end{aligned}$$

□

Remark 2.5. If $\mathcal{F}$ and $\mathcal{G}$ are G-sheaves then one can show using the same argument as above that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a G-sheaf and $\operatorname{Hom}^G(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{F}, \mathcal{G})^G$.

Corollary 2.6. *Let $(X, \mathcal{O}_X)$ be a locally ringed space with G-action and let $\mathcal{F}$ be an invertible G-sheaf on X. There is a G-equivariant isomorphism $\mathcal{O}_X \xrightarrow{\simeq} \mathcal{F}$, where $\mathcal{O}_X$ has the canonical G-action, if and only if $\Gamma(X, \mathcal{F})^G$ has a nowhere vanishing section.*

Proof. The proof follows from Lemma 2.4, since $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ as G-sets and isomorphisms $\mathcal{O}_X \to \mathcal{F}$ correspond to nowhere vanishing sections in $\Gamma(X, \mathcal{F})$. □

2.2. **The equivariant Lefschetz Conditions.** In [4, Section X.2], Grothendieck introduced the Lefschetz conditions for pairs (X, Y), inspired by Lefschetz, where X is a locally noetherian scheme and Y is a closed subscheme of X. These were essential in the proof of Grothendieck's theorems comparing the Picard groups and the fundamental groups of a projective variety X with a complete intersection subvariety Y. For schemes with action of a finite group

G, we define the equivariant Lefschetz conditions in this section and prove equivariant analogues of some results in the Grothendieck-Lefschetz theory that will be useful in the sequel.

Definition 2.7. Let X be a noetherian scheme with G-action, and let $Y \subseteq X$ be a G-invariant closed subscheme. Let $\widehat{X}$ be the formal completion of X along Y. Then $\widehat{X}$ is a locally ringed space with G-action as discussed in Example 2.3.

(1) The pair (X, Y) satisfies the *equivariant Lefschetz condition*, written $\mathrm{L}^G(X, Y)$, if for every G-invariant open set $U \supseteq Y$, and every G-equivariant locally free sheaf E on U, there exists a G-invariant open set U' with $Y \subseteq U' \subseteq U$ such that the natural map $\Gamma(U', E|_{U'})^G \xrightarrow{\simeq} \Gamma(\widehat{X}, \widehat{E})^G$ is an isomorphism.

(2) The pair (X, Y) satisfies the *equivariant effective Lefschetz condition*, written $\mathrm{eL}^G(X, Y)$, if $\mathrm{L}^G(X, Y)$ holds, and in addition, for every G-equivariant locally free sheaf $\mathcal{E}$ on $\widehat{X}$, there exists a G-invariant open set $U \supseteq Y$ and a G-equivariant locally free sheaf E on U such that $\widehat{E} \simeq \mathcal{E}$ as G-modules.

With (X, Y) as above, let E and F be locally free G-sheaves defined on G-invariant open neighbourhoods U and V of Y, respectively. We write $E \sim F$ if there exists a G-invariant open set W with $Y \subseteq W \subseteq U \cap V$ such that $E|_W \simeq F|_W$ as G-sheaves. We define the category $\mathrm{LF}^0_G(Y)$ of germs of locally free G-sheaves around Y as follows. An object of this category is a class of locally free G-sheaves defined on G-invariant open neighbourhoods of Y under the equivalence relation $\sim$. For any two objects $[E]$ and $[F]$ in $\mathrm{LF}^0_G(Y)$, the set of homomorphisms from $[E]$ to $[F]$ is defined to be the set $\varinjlim_U \mathrm{Hom}^G_{\mathcal{O}_U}(E, F)$, where the colimit is taken over all G-invariant open neighbourhoods U of Y such that both E and F are defined over U. Let $\mathrm{LF}_G(\widehat{X})$ denote the category of locally free G-sheaves on $\widehat{X}$.

Lemma 2.8. *Let $\wedge : \mathrm{LF}^0_G(Y) \to \mathrm{LF}_G(\widehat{X})$ be the functor sending $E \mapsto \widehat{E}$.*

(1) *If $\mathrm{L}^G(X, Y)$ holds, then $\wedge$ is fully faithful.*

(2) *If $\mathrm{eL}^G(X, Y)$ holds, then $\wedge$ is an equivalence of categories.*

Proof. Suppose $\mathrm{L}^G(X, Y)$ holds. Let $E, F \in \mathrm{LF}^0_G(Y)$. Without loss of generality, we may assume that E, F are G-equivariant locally free $\mathcal{O}_V$-modules for some G-invariant open neighbourhood V of Y. Let U be any G-invariant open neighbourhood of Y such that $U \subseteq V$. $\mathcal{H}om_{\mathcal{O}_U}(E, F)$ is a G-equivariant locally free $\mathcal{O}_U$-module, by Lemma 2.4. Since $\mathrm{L}^G(X, Y)$ holds, there exists a G-invariant open set U' with $Y \subseteq U' \subseteq U$ such that the natural map $\Gamma(U', \mathcal{H}om_{\mathcal{O}_U}(E, F))^G \xrightarrow{\simeq} \Gamma(\widehat{X}, \mathcal{H}om_{\mathcal{O}_U}(E, F)^\wedge)^G$ is an isomorphism. Since $\mathcal{H}om_{\mathcal{O}_U}(E, F)^\wedge \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{E}, \widehat{F})$ as G-sheaves, $\Gamma(\widehat{X}, \mathcal{H}om_{\mathcal{O}_U}(E, F)^\wedge)^G \xrightarrow{\simeq} \Gamma(\widehat{X}, \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{E}, \widehat{F}))^G$ is an isomorphism and hence $\Gamma(U', \mathcal{H}om_{\mathcal{O}_U}(E, F))^G \xrightarrow{\simeq} \Gamma(\widehat{X}, \mathcal{H}om_{\mathcal{O}_{\widehat{X}}}(\widehat{E}, \widehat{F}))^G$ is an isomorphism. By Lemma 2.4, the above isomorphism can be rewritten as $\mathrm{Hom}^G_{\mathcal{O}_{U'}}(E, F) \xrightarrow{\simeq} \mathrm{Hom}^G_{\mathcal{O}_{\widehat{X}}}(\widehat{E}, \widehat{F})$. This proves that the functor $\wedge$ is fully faithful. If $\mathrm{eL}^G(X, Y)$ holds, $\wedge$ is further essentially surjective (by definition), and therefore an equivalence of categories. □

Proposition 2.9. *Let X be a non-singular projective variety with G-action. Let $Y \subseteq X$ be a G-invariant closed subscheme, which is a scheme-theoretic complete intersection in X. If $\dim(Y) \geqslant 2$, then $\mathrm{eL}^G(X, Y)$ holds.*

Proof. Let $U \supseteq Y$ be any G-invariant open set, and let E be a locally free G-sheaf on U. Since Y is a complete intersection, by [6, Corollary IV.1.2] and the proof of [6, Proposition IV.1.1], the G-equivariant restriction map $\Gamma(U, E) \xrightarrow{\simeq} \Gamma(\widehat{X}, \widehat{E})$ is an isomorphism. This induces an isomorphism $\Gamma(U, E)^G \xrightarrow{\simeq} \Gamma(\widehat{X}, \widehat{E})^G$. Therefore $\mathrm{L}^G(X, Y)$ holds.

Let $\widehat{X}$ be the formal completion of X along Y, and let $(\mathcal{E}, \{\phi_g\})$ be a locally free G-sheaf on $\widehat{X}$. Since Y is a scheme-theoretic local complete intersection, by [6, Theorem IV.1.5], we can find an open set $U \supseteq Y$ (not necessarily G-invariant) and a locally free sheaf E on U such that $\theta : \widehat{E} \xrightarrow{\simeq} \mathcal{E}$ non-equivariantly. We may assume that U is G-invariant by replacing U by the open set $\bigcap_{g \in G} gU$. For each $g \in G$, g_*E is then a locally free sheaf on U such that we have induced isomorphisms $\widehat{g_*E} \simeq g_*\mathcal{E}$, since direct image functor commutes with inverse limits. Since E, g_*E are locally free sheaves on U, $\mathcal{H}om_{\mathcal{O}_U}(E, g_*E)$ is a locally free $\mathcal{O}_U$-module. Again as above, we have isomorphisms $\mathrm{Hom}_{\mathcal{O}_U}(E, g_*E) \xrightarrow{\simeq} \mathrm{Hom}_{\widehat{X}}(\widehat{E}, g_*\widehat{E}) \xrightarrow{\simeq} \mathrm{Hom}_{\widehat{X}}(\mathcal{E}, g_*\mathcal{E})$ for each $g \in G$. Therefore, $\phi_g \in \mathrm{Hom}_{\widehat{X}}(\mathcal{E}, g_*\mathcal{E})$ can be uniquely lifted to a morphism $\widetilde{\phi}_g \in \mathrm{Hom}_{\mathcal{O}_U}(E, g_*E)$. Since the lifts are unique and $\{\phi_g\}_{g \in G}$ defines a G-module structure on $\mathcal{E}$, $\{\widetilde{\phi}_g\}_{g \in G}$ defines a G-module structure on E. Further $\theta : \widehat{E} \to \mathcal{E}$ is a G-equivariant morphism, by definition of the G-action on E. Therefore, $\mathrm{eL}^G(X, Y)$ holds. ☐

3. Equivariant Grothendieck-Lefschetz theorem

We prove Theorem 1.1 in this section. The following Lemma identifying the equivariant Picard groups of a variety X and its formal completion $\widehat{X}$ will be crucial for proving our main result.

Lemma 3.1. *Let X be a non-singular variety with G-action and let $Y \subseteq X$ be a G-invariant closed subscheme such that Y meets every effective divisor on X. Let $\widehat{X}$ denote the completion of X along Y with the induced G-action. Assume that $\dim(X) \geqslant 2$ and $\mathrm{eL}^G(X, Y)$ holds. Then the canonical map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(\widehat{X})$ is an isomorphism.*

Proof. Since $\mathrm{eL}^G(X, Y)$ holds, every invertible G-sheaf on $\widehat{X}$ extends uniquely to an invertible G-sheaf on some G-invariant open neighbourhood U of Y by Lemma 2.8. Since Y meets every effective divisor on X, we have $\mathrm{codim}(X - U, X) \geqslant 2$. Therefore by [2, Lemma 2(1)], $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(U)$ is an isomorphism. The canonical morphism $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(\widehat{X})$ factors through $\mathrm{Pic}^G(U)$ for every G-invariant open U such that $Y \subseteq U$. Hence we conclude that $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(\widehat{X})$ is an isomorphism. ☐

Lemma 3.2. *Let X be a proper scheme with G-action and let $Y \subseteq X$ be a G-invariant closed subscheme defined by a G-sheaf of ideals $\mathcal{I}$. For $n \geqslant 1$, let*

Y_n denote the G-invariant closed subscheme defined by the sheaf of ideals $\mathcal{I}^n$. Then $\mathrm{Pic}^G(\widehat{X}) \simeq \varprojlim_n \mathrm{Pic}^G(Y_n)$.

Proof. If $\mathcal{F}$ is an invertible G-sheaf on $\widehat{X}$, then $F_n = \mathcal{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{Y_n}$ is an invertible G-sheaf on Y_n. This defines a map $f : \mathrm{Pic}^G(\widehat{X}) \to \varprojlim_n \mathrm{Pic}^G(Y_n)$.

An element of $\varprojlim_n \mathrm{Pic}^G(Y_n)$ is given by a collection of invertible G-sheaves F_n on $\mathrm{Pic}^G(Y_n)$ along with G-equivariant isomorphisms $F_{n+1} \otimes_{\mathcal{O}_{Y_{n+1}}} \mathcal{O}_{Y_n} \xrightarrow{\simeq} F_n$. Composing with the natural G-equivariant map $F_{n+1} \to F_{n+1} \otimes_{\mathcal{O}_{Y_{n+1}}} \mathcal{O}_{Y_n}$, we get a projective system of invertible G-sheaves of $\mathcal{O}_{\widehat{X}}$-modules. Then $\mathcal{F} = \varprojlim_n F_n$ is an invertible G-sheaf on $\widehat{X}$ with $\mathcal{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{Y_n} \simeq F_n$. Therefore f is surjective. To see that f is injective, let $\mathcal{F}$ be an invertible G-sheaf on $\widehat{X}$ such that for each n, there is a G-equivariant isomorphism $\mathcal{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{Y_n} \xrightarrow{\simeq} \mathcal{O}_{Y_n}$, where $\mathcal{O}_{Y_n}$ has the canonical G-action. By [5, Proposition II.9.2] and since $(-)^G$ is an additive left exact functor preserving products, it follows that the functor $\Gamma(Y, -)^G$ preserves inverse limits. Therefore $\Gamma(\widehat{X}, \mathcal{F})^G = \varprojlim_n \Gamma(Y_n, F_n)^G$, where $F_n := \mathcal{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{Y_n}$ and the inverse system $\Gamma(Y_n, F_n)^G$ satisfies the Mittag-Leffler condition [3, Chapter 0, 13.1.2] (since Y_n is proper, $\Gamma(Y_n, F_n)^G$ is a finite-dimensional k-vector space). By Corollary 2.6, each F_n has a nowhere vanishing G-invariant section. Therefore the stable images in the inverse system have nowhere vanishing sections, so we can find a nowhere vanishing section $s \in \Gamma(\widehat{X}, \mathcal{F})^G$. Therefore, again by Corollary 2.6, $\mathcal{F} \simeq \mathcal{O}_{\widehat{X}}$ is trivial. □

Theorem 3.3. *Let k be a field and let G be a finite group. Let X be a proper non-singular variety over k with G-action and let $Y \subseteq X$ be a G-invariant closed subscheme defined by a G-sheaf of ideals $\mathcal{I}$. Suppose that*

1. $\mathrm{eL}^G(X, Y)$ *holds (see Definition 2.7(2));*
2. *Y meets every effective divisor on X; and*
3. *$H^i(G; Y, I^n/I^{n+1}) = 0$ for $i = 1, 2$ for all $n \geqslant 1$.*

Then the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(Y)$ *is an isomorphism.*

Proof. The natural map in question factors as $\mathrm{Pic}^G(X) \xrightarrow{\alpha} \mathrm{Pic}^G(\widehat{X}) \xrightarrow{\beta} \mathrm{Pic}^G(Y)$, where α and β are the natural restriction maps. The map α is an isomorphism by Lemma 3.1. Factorise the map β further as follows. For $n \geqslant 1$, let Y_n denote the G-invariant closed subscheme defined by the sheaf of ideals $\mathcal{I}^n$. We have natural maps:

$$\mathrm{Pic}^G(\widehat{X}) \to \varprojlim_n \mathrm{Pic}^G(Y_n) \to \cdots \to \mathrm{Pic}^G(Y_{n+1}) \to \mathrm{Pic}^G(Y_n) \to \cdots \to \mathrm{Pic}^G(Y).$$

We will show that all the above maps are isomorphisms. The first map is an isomorphism by Lemma 3.2. Let $n \geqslant 1$ and consider the exact sequence of G-sheaves $0 \to I^n/I^{n+1} \to \mathcal{O}^*_{Y_{n+1}} \to \mathcal{O}^*_{Y_n} \to 0$, where $\mathcal{O}^*$ denotes the multiplicative group of units and the first map is given by $x \mapsto 1 + x$. This gives a long exact sequence of G-cohomology groups:

$$\cdots \to H^1(G; Y, I^n/I^{n+1}) \to H^1(G; Y_{n+1}, \mathcal{O}^*_{Y_{n+1}}) \to H^1(G; Y_n, \mathcal{O}^*_{Y_n}) \\ \to H^2(G; Y, I^n/I^{n+1}) \to \cdots .$$

By hypothesis (3), we conclude that $H^1(G; Y_{n+1}, \mathcal{O}^*_{Y_{n+1}}) \xrightarrow{\simeq} H^1(G; Y_n, \mathcal{O}^*_{Y_n})$. By [7, Theorem 2.7], this shows that $\mathrm{Pic}^G(Y_{n+1}) \to \mathrm{Pic}^G(Y_n)$ is an isomorphism. Consequently, $\varprojlim_n \mathrm{Pic}^G(Y_n)$ is isomorphic to $\mathrm{Pic}^G(Y_n)$ for every $n \geqslant 1$. This completes the proof of the theorem. □

Corollary 3.4. *Suppose G acts on $\mathbb{P}^n_k$ and Y is a G-invariant closed subscheme of dimension $\geqslant 3$ which is a scheme-theoretic complete intersection in $\mathbb{P}^n_k$. Then the natural map $\mathrm{Pic}^G(\mathbb{P}^n_k) \to \mathrm{Pic}^G(Y)$ is an isomorphism.*

Proof. Since Y is a G-invariant scheme-theoretic complete intersection and $\dim(Y) \geqslant 3$, $\mathrm{eL}^G(X, Y)$ holds by Proposition 2.9 and Y meets every effective divisor on $\mathbb{P}^n_k$ by [6, Theorem III.5.1, Proposition IV.1.1]. Further if Y is an intersection of hypersurfaces of degree $d_1, \cdots, d_r$ then $I/I^2 \simeq \mathcal{O}_Y(-d_1) \oplus \cdots \oplus \mathcal{O}_Y(-d_r)$. Hence for all $n \geqslant 1$, I^n/I^{n+1} is a direct sum of sheaves of the form $\mathcal{O}_Y(m)$ for suitable integers $m < 0$. By [11, Proposition 5], $H^i(Y, \mathcal{O}_Y(m)) = 0$ for all $0 \leqslant i < \dim(Y)$ for $m < 0$. Since $\dim(Y) \geqslant 3$, $H^i(Y, I^n/I^{n+1}) = 0$ for $0 \leqslant i \leqslant 2$. Therefore by [7, (2.5)], $H^i(G; Y, I^n/I^{n+1}) = 0$ for $i = 1, 2$. This shows that the hypotheses of Theorem 3.3 are satisfied. □

Proof of Theorem 1.1. It is enough to check as in the above corollary that $H^i(Y, \mathcal{O}_Y(m)) = 0$ for $0 \leqslant i \leqslant 2$ and all $m < 0$. This follows from the Kodaira-Akizuki-Nakano vanishing theorem (see [1, Corollary 2.11]) as $\dim(Y) \geqslant 3$. □

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