THE MOTIVIC HOPF MAP SOLVES THE
HOMOTOPY LIMIT PROBLEM FOR K-THEORY

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ABSTRACT. We solve affirmatively the homotopy limit problem for (effective) K-theory over fields of finite virtual cohomological dimension. Our solution uses the motivic slice filtration and the first motivic Hopf map.

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1 INTRODUCTION

A homotopy limit problem asks for an equivalence between fixed points and homotopy fixed points for a group action [36]. In some contexts, the fixed points are easily described, and one then obtains a description of the otherwise intractable homotopy fixed points. Many distinguished results in algebraic topology take the form of a homotopy limit problem, e.g., the Atiyah-Segal completion theorem linking equivariant K-theory to representation theory, Segal's Burnside ring conjecture on stable cohomotopy, Sullivan's conjecture on the homotopy type of real points of algebraic varieties, and the Quillen-Lichtenbaum conjecture on Galois descent for algebraic K-theory under field extensions.
In this paper we give a surprising solution of the homotopy limit problem for effective K-theory in the stable motivic homotopy category. This is achieved by analyzing the slice filtration for algebraic and hermitian K-theory [29], [30], [38], and completing with respect to the Hopf element η in the Milnor-Witt K-theory ring [19].
To provide context for our approach, recall that complex conjugation of vector bundles gives rise to the Adams operation Ψ^{-1} and an action of the group C_2

of order two on the complex K-theory spectrum $\mathbf{KU}$. Atiyah [1] shows there is an isomorphism

$$\mathbf{KO} \xrightarrow{\simeq} \mathbf{KU}^{hC_2} \tag{1.1}$$

between the real K-theory spectrum $\mathbf{KO}$ and the C_2-homotopy fixed points of $\mathbf{KU}$. For the corresponding connective K-theory spectra, the homotopy cofiber of

$$\mathbf{ko} \longrightarrow \mathbf{ku}^{hC_2} \tag{1.2}$$

is an infinite sum $\bigvee_{i<0} \Sigma^{4i} \mathbf{H}\mathbb{Z}/2$ of suspensions of the mod-2 Eilenberg-MacLane spectrum.

The Grothendieck-Witt group of symmetric bilinear forms, and more generally hermitian K-groups [16], is the algebro-geometric analogue of the Grothendieck group of real vector bundles, represented by $\mathbf{KO}$. In the context of motivic homotopy theory, algebraic and hermitian K-theory are prominent examples of representable cohomology theories, see [9], [39]. For a geometric model for the motivic spectrum representing hermitian K-theory, closely related to the classical construction of the topological spaces appearing in $\mathbf{KO}$, see [32]. Over any field of characteristic not equal to two, the Grothendieck-Witt group is isomorphic to the zeroth stable motivic homotopy group of the motivic sphere spectrum [19]. The close connection between hermitian K-theory and stable motivic homotopy is further exploited in the recent solution of Morel's motivic 1-line conjecture in [30].

Motivated by Atiyah's work and the connection with motivic stable stems we investigate the analogues of (1.1) and (1.2) for algebraic K-theory $\mathbf{KGL}$ with C_2-action given by the Adams operation Ψ^{-1} and hermitian K-theory $\mathbf{KQ}$, see for example [29, §3,4]. Throughout we work in the stable motivic homotopy category $\mathbf{SH}$ over a fixed field F of characteristic $\mathsf{char}(F) \neq 2$. Our starting point is, somewhat unexpectedly in view of (1.2), the naturally induced map between fixed points and homotopy fixed points

$$\gamma \colon \mathbf{kq} \longrightarrow \mathbf{kgl}^{hC_2} \tag{1.3}$$

for the projections of $\mathbf{KGL}$ and $\mathbf{KQ}$ to the effective stable motivic homotopy category $\mathbf{SH}^{\mathsf{eff}}$. The latter is the localizing subcategory of $\mathbf{SH}$ generated by suspension spectra of smooth schemes [38]. Let η be the first motivic Hopf map induced by the natural map of algebraic varieties $\mathbf{A}^2 \smallsetminus \{0\} \longrightarrow \mathbf{P}^1$. Recall that η defines a non-nilpotent element in the homotopy group $\pi_{1,1}\mathbf{1}$ of the motivic sphere spectrum. We let $\pi_\star\mathbf{E}$ denote the bigraded coefficients of a generic motivic spectrum $\mathbf{E}$. The stable cone of η acquires a Bousfield localization functor $\mathbf{L}_\eta$ defined on all motivic spectra [28, Appendix A]. Let $\mathsf{vcd}_2(F)$ denote the mod-2 cohomological dimension of the absolute Galois group of $F(\sqrt{-1})$ [33, Chapter 1,§3].

The main result of this paper solves affirmatively the homotopy limit problem for effective K-theory (1.3) when $\mathsf{vcd}_2(F) < \infty$, by completing with respect to η.

THEOREM 1.1. *Suppose F is a field of $\mathsf{char}(F) \neq 2$ and virtual cohomological dimension $\mathsf{vcd}_2(F) < \infty$. Then (1.3) induces an isomorphism*

$$\mathbf{L}_\eta(\gamma) \colon \mathbf{kq}_\eta^\wedge \xrightarrow{\simeq} \mathbf{kgl}^{hC_2}. \tag{1.4}$$

We show that the homotopy fixed point spectrum $\mathbf{kgl}^{hC_2}$ in (1.4) is η-complete. The proof of Theorem 1.1 invokes the slice filtration

$$\cdots \subset \Sigma_T^{q+1}\mathbf{SH}^{\mathsf{eff}} \subset \Sigma_T^q\mathbf{SH}^{\mathsf{eff}} \subset \Sigma_T^{q-1}\mathbf{SH}^{\mathsf{eff}} \subset \cdots \tag{1.5}$$

introduced by Voevodsky [38, §2]. Using (1.5) one associates to $\mathbf{E}$ an integrally graded family of slices $\mathsf{s}_*(\mathbf{E})$ and a trigraded slice spectral sequence

$$\pi_\star \mathsf{s}_*(\mathbf{E}) \Longrightarrow \pi_\star \mathbf{E}. \tag{1.6}$$

We show that the spectral sequence (1.6) converges conditionally for the η-completion of $\mathbf{kq}$ and also for the homotopy fixed point spectrum $\mathbf{kgl}^{hC_2}$. Many of the techniques entering in the proofs are of more general interest in the setting of the slice filtration. In contrast to the topological situation (1.1), the coefficient rings $\pi_\star \mathbf{KQ}$ and $\pi_\star \mathbf{KGL}$ are largely unknown over general fields. Nonetheless we obtain a proof of Theorem 1.1 by using the computations of $\mathsf{s}_*(\mathbf{KQ})$ and $\mathsf{s}_*(\mathbf{KGL}^{hC_2})$ accomplished in [29].
Next we turn to solving the homotopy limit problem

$$\Upsilon \colon \mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}. \tag{1.7}$$

Here $\mathbf{KGL}^{hC_2}$ is η-complete essentially due to motivic orientability of algebraic K-theory. We proceed by comparing with the effective cocovers of $\mathbf{KQ}$ and $\mathbf{KGL}^{hC_2}$. Remarkably, the first motivic Hopf map η turns Υ into an isomorphism without altering the target in (1.7).

THEOREM 1.2. *Suppose F is a field of $\mathsf{char}(F) \neq 2$ and virtual cohomological dimension $\mathsf{vcd}_2(F) < \infty$. Then (1.7) induces an isomorphism*

$$\mathbf{L}_\eta(\Upsilon) \colon \mathbf{KQ}_\eta^\wedge \xrightarrow{\simeq} \mathbf{KGL}^{hC_2}. \tag{1.8}$$

Supplementing our main results we note that the η-arithmetic square

$$\begin{array}{ccc} \mathbf{KQ} & \longrightarrow & \mathbf{KW} \\ \downarrow & & \downarrow \\ \mathbf{KQ}_\eta^\wedge & \longrightarrow & \mathbf{KQ}_\eta^\wedge[\eta^{-1}] \end{array} \tag{1.9}$$

for $\mathbf{KQ}$ [30, §3.1] coincides up to isomorphism with the Tate diagram [11, (20)]

$$\begin{array}{ccc} \mathbf{KQ} & \longrightarrow & \mathbf{KW} \\ \downarrow & & \downarrow \\ \mathbf{KGL}^{hC_2} & \longrightarrow & \mathbf{KGL}^{tC_2} \end{array} \tag{1.10}$$

for the C_2-action on $\mathbf{KGL}$. Here $\mathbf{KW}$ denotes the higher Witt-theory and $\mathbf{KGL}^{tC_2}$ denotes the Tate K-theory spectrum. Moreover, by representability, (1.8) implies that for every $X \in \mathbf{Sm}_F$ — smooth F-schemes of finite type — there is a naturally induced isomorphism

$$\mathbf{L}_\eta(\Upsilon)_\star \colon \pi_\star \mathbf{KQ}(X)^\wedge_\eta \stackrel{\cong}{\longrightarrow} \pi_\star \mathbf{KGL}(X)^{hC_2}.$$

REMARK 1.3. *The earlier works [2] and [11] identified the 2-adic completion of the homotopy fixed points by showing an isomorphism $\pi_\star \mathbf{KQ}/2 \stackrel{\cong}{\longrightarrow} \pi_\star \mathbf{KGL}^{hC_2}/2$. Explicit calculations are carried out over the complex numbers $\mathbb{C}$ in [12]. The commonplace assumption $\mathsf{vcd}_2(F) < \infty$ is also used in [2], [11], and in the context of the Quillen-Lichtenbaum conjecture for étale K-theory [14, §4].*

REMARK 1.4. *Drew Heard [4] has recently used ∞-categories to prove Theorem 1.2 over a general base scheme with no assumption on vcd_2. An alternate proof will appear in work with Jeremiah Heller and Vesna Stojanoska using C_2-equivariant motivic homotopy theory. Marco Schlichting has indicated another proof based on the homotopy cartesian Tate diagram of topological spectra [31, Theorem 7.6]. It implies the Grothendieck-Witt spectrum of a dg category $\mathcal{A}$ with weak equivalences and duality, assuming $\frac{1}{2} \in \mathcal{A}$, maps by an η-equivalence to the C_2-homotopy fixed points of the algebraic K-theory $K(\mathcal{A})$, for η as defined in [31, (6.1)]. A crucial part is to show the η-completion of the Tate spectrum $K(\mathcal{A})^{tC_2}$ is contractible. By first extending this result to presheaves of S^1-spectra and next to motivic spectra, one should recover the motivic homotopy cartesian Tate diagram (1.10), and the motivic η-contractibility of $\mathbf{KGL}^{tC_2}$. Thus Theorem 1.2 holds for function fields in infinitely many variables. Our main result, Theorem 1.1, does not seem to follow from any of these approaches. It is unclear whether the assumption $\mathsf{vcd}_2(F) < \infty$ can be relaxed for the effective covers, which are natural objects to consider in the context of the slice filtration.*
The results and techniques in this paper produce calculations of hermitian K-groups of rings of integers in number fields [17].

2   The first motivic Hopf map η

We view $\mathbf{A}^2 \smallsetminus \{0\}$ and $\mathbf{P}^1$ as motivic spaces pointed at $(1,1)$ and $[1:1]$, respectively. The canonical projection map $\mathbf{A}^2 \smallsetminus \{0\} \longrightarrow \mathbf{P}^1$ induces the stable motivic Hopf map $\eta \colon \mathbf{G}_{\mathfrak{m}} \longrightarrow \mathbf{1}$ for the motivic sphere spectrum $\mathbf{1}$. Iteration of η yields the cofiber sequence

$$\mathbf{G}_{\mathfrak{m}}^{\wedge n} \stackrel{\eta^n}{\longrightarrow} \mathbf{1} \longrightarrow \mathbf{1}/\eta^n. \tag{2.1}$$

The η-completion $\mathbf{E}^\wedge_\eta$ of a motivic spectrum $\mathbf{E}$ is defined as the homotopy limit $\underset{n \to \infty}{\mathrm{holim}} \, \mathbf{E}/\eta^n$ of the canonically induced diagram

$$\cdots \longrightarrow \mathbf{E}/\eta^{n+1} \longrightarrow \mathbf{E}/\eta^n \longrightarrow \cdots \longrightarrow \mathbf{E}/\eta. \tag{2.2}$$

By (2.1) and (2.2) there is a naturally induced map

$$\mathbf{E} \longrightarrow \mathbf{E}_\eta^\wedge. \tag{2.3}$$

We say that $\mathbf{E}$ is η-*complete* if the map in (2.3) is an isomorphism. The Bousfield localization $\mathbf{L}_\eta \mathbf{E}$ of $\mathbf{E}$ for the cone of η coincides with $\mathbf{E}_\eta^\wedge$. Recall that the algebraic cobordism spectrum $\mathbf{MGL}$ is the universal oriented motivic spectrum, see [20], [22].

LEMMA 2.1. *Every module over an oriented motivic ring spectrum is η-complete.*

Proof. The unit map for algebraic cobordism $\mathbf{1} \longrightarrow \mathbf{MGL}$ factors through the cone $\mathbf{1}/\eta$, see [30, Lemma 3.24] for an explicit factorization, which implies $\mathbf{MGL} \wedge \eta = 0$. The statement for modules follows readily. $\square$

3 The slice filtration

In this section we discuss results for the slice filtration [38] which will be applied in the proofs of our main results in Section 5. Throughout we work over a base field F.

To (1.5) one associates distinguished triangles

$$\mathsf{f}_{q+1}(\mathbf{E}) \longrightarrow \mathsf{f}_q(\mathbf{E}) \longrightarrow \mathsf{s}_q(\mathbf{E}), \tag{3.1}$$

for every motivic spectrum $\mathbf{E}$, see [38, Theorem 2.2]. Here the qth effective cover of $\mathbf{E}$ is the universal map $\mathsf{f}_q(\mathbf{E}) \longrightarrow \mathbf{E}$ from $\Sigma_T^q \mathbf{SH}^{\mathsf{eff}}$ to $\mathbf{E}$. The qth slice $\mathsf{s}_q(\mathbf{E}) \in \Sigma_T^q \mathbf{SH}^{\mathsf{eff}}$ is uniquely determined up to isomorphism by (3.1). Every object of $\Sigma_T^{q+1} \mathbf{SH}^{\mathsf{eff}}$ maps trivially to $\mathsf{s}_q(\mathbf{E})$. It is technically important for many constructions to have a "strict model" for the slice filtration, e.g., by means of model categories as in [7, §3.1], [24, §3.2]. We note that $\mathsf{f}_q \mathsf{s}_{q'} \simeq \mathsf{s}_{q'} \mathsf{f}_q$ follows from [7, (2.2), §6] for all $q, q' \in \mathbb{Z}$.

LEMMA 3.1. *The slice filtration is exhaustive in the sense that there is an isomorphism*

$$\operatorname*{hocolim}_{q \to -\infty} \mathsf{f}_q(\mathbf{E}) \xrightarrow{\simeq} \mathbf{E}. \tag{3.2}$$

Proof. Each generator $\Sigma^{s,t} X_+$ of $\mathbf{SH}$ is contained in $\Sigma_T^{q'} \mathbf{SH}^{\mathsf{eff}}$ for some $q' \in \mathbb{Z}$. Here $s, t \in \mathbb{Z}$ and $X \in \mathbf{Sm}_F$, see for example [5, Theorem 9.1]. Recall that $\mathsf{f}_{q'}$ preserves homotopy colimits [34, Corollary 4.5], [38, Lemma 4.2]. By the universal property of the q'th effective cover it suffices to show there is an isomorphism

$$\mathbf{SH}(\Sigma^{s,t} X_+, \operatorname*{hocolim}_{q < q'} \mathsf{f}_{q'} \mathsf{f}_q(\mathbf{E})) \xrightarrow{\simeq} \mathbf{SH}(\Sigma^{s,t} X_+, \mathsf{f}_{q'}(\mathbf{E})).$$

This follows since $\mathsf{f}_{q'} \mathsf{f}_q \simeq \mathsf{f}_{q'}$ for $q < q'$. $\square$

Lemma 3.2. *The slices of a motivic spectrum are η-complete.*

Proof. Every slice $\mathsf{s}_q(\mathbf{E})$ is a module over the motivic ring spectrum $\mathsf{s}_0(\mathbf{1})$, cf. [7, §6 (iv),(v)] and [24, Theorem 3.6.13(6)]. If F is a perfect field, then $\mathsf{s}_0(\mathbf{1})$ is the motivic cohomology spectrum $\mathbf{MZ}$ by [18, Theorem 10.5.1] and [40, Theorem 6.6]. This extends by base change; every field is essentially smooth over a perfect field [10, Lemma 2.9], and [10, Lemma 2.7(1)] verifies the hypothesis of [25, Theorem 2.12] for an essentially smooth map. To conclude the proof we use Lemma 2.1 and the canonical orientation on $\mathbf{MZ}$ [21, §10]. □

Corollary 3.3. *Algebraic K-theory $\mathbf{KGL}$ and its effective cover $\mathbf{kgl}$ are η-complete.*

Proof. This follows from Lemma 2.1 by using the orientation map $\mathbf{MGL} \longrightarrow \mathbf{KGL}$, see for example [23, Example 2.4] and [35, Examples 2.1, 2.2], and the geometric fact that the algebraic cobordism spectrum is effective [34, Corollary 3.2], [38, §8]. □

Corollary 3.4. *The homotopy fixed points spectra $\mathbf{KGL}^{hC_2}$ and $\mathbf{kgl}^{hC_2}$ are η-complete.*

Proof. Let $\mathbf{E}$ be short for $\mathbf{KGL}$ or $\mathbf{kgl}$. We use homotopy limits to model the homotopy fixed points $\mathbf{E}^{hC_2}$ for the C_2-action given by the Adams operation Ψ^{-1} [8, §18], [29, §3,4]. Corollary 3.3 implies there is an isomorphism

$$\operatorname*{holim}_{C_2} \mathbf{E} \xrightarrow{\simeq} \operatorname*{holim}_{C_2} \operatorname*{holim}_{n\to\infty} \mathbf{E}/\eta^n.$$

Thus the corollary follows by commuting homotopy limits over small categories, i.e.,

$$\operatorname*{holim}_{C_2} \operatorname*{holim}_{n\to\infty} \mathbf{E}/\eta^n \xrightarrow{\simeq} \operatorname*{holim}_{n\to\infty} \operatorname*{holim}_{C_2} \mathbf{E}/\eta^n.$$

□

The qth effective cocover $\mathsf{f}^{q-1}(\mathbf{E})$ of $\mathbf{E}$ is uniquely determined up to isomorphism by the distinguished triangle

$$\mathsf{f}_q(\mathbf{E}) \longrightarrow \mathbf{E} \longrightarrow \mathsf{f}^{q-1}(\mathbf{E}). \tag{3.3}$$

Note that $\mathsf{f}^{q-1}(\mathbf{E})$ is a $(q-1)$-coeffective motivic spectrum, i.e., it is an object of the right orthogonal subcategory of $\Sigma_T^q \mathbf{SH}^{\mathsf{eff}}$. If $q \leq q'$ the isomorphism

$$\mathsf{s}_{q'}\mathsf{f}_q(\mathbf{E}) \xrightarrow{\simeq} \mathsf{s}_{q'}(\mathbf{E}) \tag{3.4}$$

implies $\mathsf{s}_{q'}\mathsf{f}^{q-1}(\mathbf{E}) \simeq *$. When $q = 0$, (3.3) yields a distinguished triangle for the effective cover $\mathbf{e} \in \mathbf{SH}^{\mathsf{eff}}$ of $\mathbf{E}$, i.e.,

$$\mathbf{e} \longrightarrow \mathbf{E} \longrightarrow \mathsf{f}^{-1}(\mathbf{E}). \tag{3.5}$$

We note that all the nonnegative slices of the coeffective motivic spectrum $\mathsf{f}^{-1}(\mathbf{E})$ are trivial.

LEMMA 3.5. *If* $\mathbf{E}$ *has no nontrivial negative slices then* $\mathbf{E} \in \mathbf{SH}^{\mathrm{eff}}$.

Proof. Using (3.1), (3.2) and (3.5) it follows that $\mathsf{f}^{-1}(\mathbf{E}) \simeq *$. $\square$

LEMMA 3.6. *Suppose* $\mathbf{E} \to \mathbf{F}$ *induces an isomorphism* $\mathsf{s}_q \mathsf{f}^{q-1}(\mathbf{E}) \overset{\simeq}{\to} \mathsf{s}_q \mathsf{f}^{q-1}(\mathbf{F})$ *for all* $q \in \mathbb{Z}$. *Then there is a naturally induced isomorphism* $\mathsf{f}^{q-1}(\mathbf{E}) \overset{\simeq}{\Rightarrow} \mathsf{f}^{q-1}(\mathbf{F})$.

Proof. This follows by applying (3.1) and (3.2) to the effective cocovers. $\square$

LEMMA 3.7. *For* $n > 0$ *there is a distinguished triangle*

$$\mathsf{f}_{-n+1}\mathsf{f}^{-1}(\mathbf{E}) \longrightarrow \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}) \longrightarrow \mathsf{s}_{-n}(\mathbf{E}). \tag{3.6}$$

It follows that $\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})$ *is a finite extension of the negative slices of* $\mathbf{E}$.

Proof. This follows from the distinguished triangles:

$$\begin{array}{ccccc}
\mathsf{f}_{-n+1}(\mathbf{e}) & \longrightarrow & \mathsf{f}_{-n+1}(\mathbf{E}) & \longrightarrow & \mathsf{f}_{-n+1}\mathsf{f}^{-1}(\mathbf{E}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{f}_{-n}(\mathbf{e}) & \longrightarrow & \mathsf{f}_{-n}(\mathbf{E}) & \longrightarrow & \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{s}_{-n}(\mathbf{e}) & \longrightarrow & \mathsf{s}_{-n}(\mathbf{E}) & \longrightarrow & \mathsf{s}_{-n}\mathsf{f}^{-1}(\mathbf{E})
\end{array} \tag{3.7}$$

In (3.7) the slice $\mathsf{s}_{-n}(\mathbf{e}) \simeq *$ by the assumption $n > 0$. This implies (3.6). We note the effective cover $\mathsf{f}_0\mathsf{f}^{-1}(\mathbf{E}) \simeq *$ since the map $\mathsf{f}_0(\mathbf{e}) \longrightarrow \mathsf{f}_0(\mathbf{E})$ is an isomorphism. It follows that $\mathsf{f}_{-1}\mathsf{f}^{-1}(\mathbf{E})$ is isomorphic to $\mathsf{s}_{-1}(\mathbf{E})$. The conclusion follows from (3.6) by induction on n. $\square$

REMARK 3.8. *For* $n > 0$, $\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})$ *is* η-*complete by Lemmas 3.2 and 3.7.*

LEMMA 3.9. *For* $n \geq -q > 0$ *there are isomorphisms*

$$\mathsf{s}_q(\mathbf{E}) \overset{\simeq}{\longrightarrow} \mathsf{s}_q\mathsf{f}^{-1}(\mathbf{E}) \overset{\simeq}{\longleftarrow} \mathsf{s}_q\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}). \tag{3.8}$$

Proof. Here we use that $\mathsf{s}_q(\mathbf{e}) \simeq *$ for $q < 0$. The isomorphism for f_{-n} is a special case of (3.4). $\square$

For every $\mathbf{E} \in \mathbf{SH}$ the distinguished triangle (3.3) yields a commutative diagram:

$$\begin{array}{ccccc}
\mathsf{f}_{q+1}(\mathbf{E}) & \longrightarrow & \mathbf{E} & \longrightarrow & \mathsf{f}^q(\mathbf{E}) \\
\downarrow & & \| & & \downarrow \\
\mathsf{f}_q(\mathbf{E}) & \longrightarrow & \mathbf{E} & \longrightarrow & \mathsf{f}^{q-1}(\mathbf{E})
\end{array} \tag{3.9}$$

The slice completion of $\mathbf{E}$ is defined as the homotopy limit

$$\mathsf{sc}(\mathbf{E}) \equiv \operatornamewithlimits{holim}_{q \to \infty} \mathsf{f}^{q-1}(\mathbf{E}). \qquad (3.10)$$

Using (3.9) and (3.10) we conclude there is a distinguished triangle

$$\operatornamewithlimits{holim}_{q \to \infty} \mathsf{f}_q(\mathbf{E}) \longrightarrow \mathbf{E} \longrightarrow \mathsf{sc}(\mathbf{E}). \qquad (3.11)$$

We say that $\mathbf{E}$ is *slice complete* if the homotopy limit $\operatornamewithlimits{holim}_{q \to \infty} \mathsf{f}_q(\mathbf{E})$ in (3.11) is contractible.

LEMMA 3.10. *For every $\mathbf{E} \in \mathbf{SH}$, both $\mathsf{s}_q(\mathbf{E})$ and $\mathsf{f}^q(\mathbf{E})$ are slice complete for all $q \in \mathbb{Z}$.*

Proof. If $q < q'$ there are distinguished triangles

$$\mathsf{f}_{q+1}\mathsf{f}_{q'}(\mathbf{E}) \xrightarrow{\simeq} \mathsf{f}_q\mathsf{f}_{q'}(\mathbf{E}) \longrightarrow \mathsf{s}_q\mathsf{f}_{q'}(\mathbf{E}), \mathsf{f}_{q'}\mathsf{f}_{q+1}(\mathbf{E}) \xrightarrow{\simeq} \mathsf{f}_{q'}(\mathbf{E}) \longrightarrow \mathsf{f}_{q'}\mathsf{f}^q(\mathbf{E}). \qquad (3.12)$$

It follows that $\mathsf{f}_{q'}\mathsf{s}_q(\mathbf{E}) \simeq \mathsf{s}_q\mathsf{f}_{q'}(\mathbf{E}) \simeq *$ and $\mathsf{f}_{q'}\mathsf{f}^q(\mathbf{E}) \simeq *$ for $q < q'$ by (3.12). $\square$

LEMMA 3.11. *Algebraic K-theory $\mathbf{KGL}$ and its effective cover $\mathbf{kgl}$ are slice complete.*

Proof. It suffices to consider $\mathbf{KGL}$. The associated Nisnevich sheaf of homotopy groups $\underline{\pi}_{p,q}\mathbf{KGL}$ is trivial when $p < 2q$. Hence $\mathsf{f}_q(\mathbf{KGL})$ is q-connected by [30, Lemma 3.17], i.e., for every triple (s,t,d) of integers with $s - t + d < q$ and every $X \in \mathbf{Sm}_F$ of dimension $\leq d$, the group $[\Sigma^{s,t}X_+, \mathsf{f}_q(\mathbf{KGL})]$ is trivial. We conclude by letting $q \to \infty$. $\blacksquare$

From Lemma 3.11 we deduce isomorphisms

$$\mathbf{KGL}^{hC_2} \xrightarrow{\simeq} \mathsf{sc}(\mathbf{KGL})^{hC_2}, \mathbf{kgl}^{hC_2} \xrightarrow{\simeq} \mathsf{sc}(\mathbf{kgl})^{hC_2}.$$

However, it is unclear whether $\mathbf{KGL}^{hC_2}$ and $\mathbf{kgl}^{hC_2}$ are slice complete because homotopy fixed points need not commute with effective cocovers or equivalently with effective covers. To emphasize this issue we construct an example in §6, see also Proposition 3.15.

For $n > 0$ there is a naturally induced distinguished triangle

$$\mathbf{e} \longrightarrow \mathsf{f}_{-n}(\mathbf{E}) \longrightarrow \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}). \qquad (3.13)$$

Lemma 3.1, (3.5), and (3.13) imply there is a naturally induced isomorphism

$$\operatornamewithlimits{hocolim}_{n>0} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}) \xrightarrow{\simeq} \mathsf{f}^{-1}(\mathbf{E}). \qquad (3.14)$$

Next we make precise the vagary of identifying the homotopy fixed points $\mathsf{f}^{-1}(\mathbf{KGL})^{hC_2}$ with a homotopy colimit. That is, we identify a homotopy limit with a homotopy colimit. Throughout we let $\mathbf{E}$ be a motivic spectrum equipped with a G-action for a finite group G.

Lemma 3.12. *There is a natural isomorphism*

$$\operatorname*{hocolim}_{n>0} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG} \overset{\simeq}{\longrightarrow} \mathsf{f}^{-1}(\mathbf{E})^{hG}. \qquad (3.15)$$

Proof. For every generator $\Sigma^{s,t}X_+$ of $\mathbf{SH}$ there is a canonically induced map

$$\mathbf{SH}(\Sigma^{s,t}X_+, \operatorname*{hocolim}_{n>0} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}) \longrightarrow \mathbf{SH}(\Sigma^{s,t}X_+, \mathsf{f}^{-1}(\mathbf{E})^{hG}). \qquad (3.16)$$

If $t \geq 0$ the source and target of (3.16) are trivial. If $t < 0$ we show (3.16) is an isomorphism by using the distinguished triangle

$$\mathsf{f}_t\mathsf{f}^{-1}(\mathbf{E})^{hG} \longrightarrow \mathsf{f}^{-1}(\mathbf{E})^{hG} \longrightarrow \mathsf{f}^{t-1}(\mathbf{E})^{hG}, \qquad (3.17)$$

obtained by applying homotopy fixed points to (3.3) for $\mathsf{f}^{-1}(\mathbf{E})$ and identifying $\mathsf{f}^{t-1}\mathsf{f}^{-1}(\mathbf{E})$ with $\mathsf{f}^{t-1}(\mathbf{E})$ by means of the distinguished triangles

$$\mathsf{f}_t(\mathbf{e}) \overset{\simeq}{\longrightarrow} \mathbf{e} \longrightarrow \mathsf{f}^{t-1}(\mathbf{e}), \ \ \mathsf{f}^{t-1}(\mathbf{e}) \longrightarrow \mathsf{f}^{t-1}(\mathbf{E}) \overset{\simeq}{\longrightarrow} \mathsf{f}^{t-1}\mathsf{f}^{-1}(\mathbf{E}).$$

Since $\mathsf{f}^{t-1}(\mathbf{E})^{hG}$ in (3.17) is $(t-1)$-coeffective there is a canonically induced isomorphism

$$\mathbf{SH}(\Sigma^{s,t}X_+, \mathsf{f}_t\mathsf{f}^{-1}(\mathbf{E})^{hG}) \overset{\cong}{\longrightarrow} \mathbf{SH}(\Sigma^{s,t}X_+, \mathsf{f}^{-1}(\mathbf{E})^{hG}).$$

On the other hand there are canonical identifications

$$\mathbf{SH}(\Sigma^{s,t}X_+, \operatorname*{hocolim}_{n>0} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}) \cong \operatorname*{colim}_{n>0} \mathbf{SH}(\Sigma^{s,t}X_+, \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG})$$
$$\cong \mathbf{SH}(\Sigma^{s,t}X_+, \mathsf{f}_t\mathsf{f}^{-1}(\mathbf{E})^{hG}).$$

$\square$

In the following we make the standing assumption that for all $q \in \mathbb{Z}$ there is a naturally induced isomorphism

$$\mathsf{s}_q(\mathbf{E}^{hG}) \overset{\simeq}{\longrightarrow} \mathsf{s}_q(\mathbf{E})^{hG}. \qquad (3.18)$$

The map in (3.18) arises from the standard adjunction between motivic spectra and "naive" G-motivic spectra. That is, with the trivial G-action on the homotopy fixed points there is a naturally induced G-map $\mathsf{s}_q(\mathbf{E}^{hG}) \to \mathsf{s}_q(\mathbf{E})$. Its adjoint is the map in (3.18).

Corollary 3.13. *Assuming* (3.18) *and* $n > 0$ *there is a naturally induced isomorphism*

$$\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}^{hG}) \overset{\simeq}{\longrightarrow} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}. \qquad (3.19)$$

Proof. Follows from Lemma 3.7 under the stated assumptions. $\square$

COROLLARY 3.14. *Assuming* (3.18) *and* $n \geq -q > 0$ *there is a naturally induced isomorphism*

$$\mathsf{s}_q(\mathbf{E}^{hG}) \xrightarrow{\simeq} \mathsf{s}_q(\operatorname*{hocolim}_{n>0} \mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}). \tag{3.20}$$

Proof. From the isomorphisms (3.8) in Lemma 3.9 we obtain

$$\mathsf{s}_q(\mathbf{E}^{hG}) \xrightarrow{\simeq} \mathsf{s}_q\mathsf{f}^{-1}(\mathbf{E}^{hG}) \xleftarrow{\simeq} \mathsf{s}_q\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E}^{hG}). \tag{3.21}$$

Recall that slices commute with homotopy colimits [34, Corollary 4.5], [38, Lemma 4.2]. Thus the target in (3.20) identifies with the homotopy colimit

$$\operatorname*{hocolim}_{n>0} \mathsf{s}_q\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}. \tag{3.22}$$

With the assumption $n \geq -q > 0$ the qth slice $\mathsf{s}_q\mathsf{f}_{-n}\mathsf{f}^{-1}(\mathbf{E})^{hG}$ maps isomorphically to (3.22). It remains to apply the isomorphism (3.19) in Corollary 3.13 and (3.21). □

PROPOSITION 3.15. *Assuming* (3.18) *the slices of* $\mathbf{e}$ *commute with homotopy fixed points in the sense that there is a naturally induced isomorphism*

$$\mathsf{s}_q(\mathbf{e}^{hG}) \xrightarrow{\simeq} \mathsf{s}_q(\mathbf{e})^{hG} \tag{3.23}$$

for every $q \in \mathbb{Z}$. *Moreover,* $\mathbf{e}^{hG}$ *is an effective motivic spectrum.*

Proof. Applying homotopy fixed points to (3.3) yields the distinguished triangle

$$\mathbf{e}^{hG} \longrightarrow \mathbf{E}^{hG} \longrightarrow \mathsf{f}^{-1}(\mathbf{E})^{hG}. \tag{3.24}$$

From (3.24) we deduce the commutative diagram of distinguished triangles:

$$\begin{array}{ccc}
\mathsf{s}_q(\mathbf{e}^{hG}) \longrightarrow \mathsf{s}_q(\mathbf{E}^{hG}) \longrightarrow \mathsf{s}_q(\mathsf{f}^{-1}(\mathbf{E})^{hG}) \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \\
\mathsf{s}_q(\mathbf{e})^{hG} \longrightarrow \mathsf{s}_q(\mathbf{E})^{hG} \longrightarrow \mathsf{s}_q(\mathsf{f}^{-1}(\mathbf{E}))^{hG}
\end{array} \tag{3.25}$$

When $q \geq 0$ it follows that $\mathsf{s}_q(\mathsf{f}^{-1}(\mathbf{E})) \simeq \mathsf{s}_q(\mathsf{f}^{-1}(\mathbf{E})^{hG}) \simeq *$ since homotopy limits preserve coeffective motivic spectra. Since the middle vertical map in (3.25) is an isomorphism, see the assumption (3.18), so is (3.23).
When $q < 0$, (3.15) and (3.20) imply that $\mathsf{s}_q(\mathbf{E}^{hG}) \longrightarrow \mathsf{s}_q(\mathsf{f}^{-1}(\mathbf{E})^{hG})$ is an isomorphism. Lemma 3.5 implies $\mathbf{e}^{hG} \in \mathbf{SH}^{\mathsf{eff}}$ and thus $\mathsf{s}_q(\mathbf{e}^{hG}) \longrightarrow \mathsf{s}_q(\mathbf{e})^{hG}$ is an isomorphism. □

LEMMA 3.16. *Assuming* (3.18) *there are naturally induced isomorphisms*

$$\mathsf{f}^q(\mathbf{e}^{hG}) \xrightarrow{\simeq} \mathsf{f}^q(\mathbf{e})^{hG}, \ \mathsf{f}_q(\mathbf{e}^{hG}) \xrightarrow{\simeq} \mathsf{f}_q(\mathbf{e})^{hG}. \tag{3.26}$$

Proof. We show that all the nonnegative effective cocovers of $\mathbf{e}$ commute with homotopy fixed points. With this in hand the assertion for the effective covers of $\mathbf{e}$ follows from (3.3).

We claim there is a commutative diagram:

$$
\begin{array}{ccc}
\mathsf{s}_0(\mathbf{e}^{hG}) & \xrightarrow{\ \simeq\ } & \mathsf{f}^0(\mathbf{e}^{hG}) \\
{\scriptstyle \simeq} \downarrow & & \downarrow \\
\mathsf{s}_0(\mathbf{e})^{hG} & \xrightarrow{\ \simeq\ } & \mathsf{f}^0(\mathbf{e})^{hG}
\end{array}
\tag{3.27}
$$

Proposition 3.15 shows the left vertical map in (3.27) is an isomorphism and that $\mathbf{e}^{hG}$ is an effective motivic spectrum. Hence $\mathsf{s}_0(\mathbf{e}) \simeq \mathsf{f}^0(\mathbf{e})$ and $\mathsf{s}_0(\mathbf{e}^{hG}) \simeq \mathsf{f}^0(\mathbf{e}^{hG})$ by comparing (3.1) and (3.3). It follows that the natural map $\mathsf{f}^0(\mathbf{e}^{hG}) \longrightarrow \mathsf{f}^0(\mathbf{e})^{hG}$ is also an isomorphism.

The cone of the left vertical map in (3.9) is the qth slice. Hence there is a homotopy cofiber sequence $\mathsf{s}_q(\mathbf{e}) \longrightarrow \mathsf{f}^q(\mathbf{e}) \longrightarrow \mathsf{f}^{q-1}(\mathbf{e})$, and likewise for $\mathbf{e}^{hG}$. Proposition 3.15 and induction on q implies that $\mathsf{f}^q(\mathbf{e})$ commutes with homotopy fixed points. $\square$

COROLLARY 3.17. *Assuming* (3.18) *there is a naturally induced isomorphism*

$$
\mathsf{sc}(\mathbf{e}^{hG}) \xrightarrow{\ \simeq\ } \mathsf{sc}(\mathbf{e})^{hG}.
\tag{3.28}
$$

If $\mathbf{e}$ is slice complete then so is $\mathbf{e}^{hG}$.

Proof. Lemma 3.16 and the fact that homotopy limits commute imply there are canonical isomorphisms

$$
\operatorname*{holim}_{q\to\infty} \mathsf{f}^{q-1}(\operatorname*{holim}_{G} \mathbf{e}) \simeq \operatorname*{holim}_{q\to\infty} \operatorname*{holim}_{G} \mathsf{f}^{q-1}(\mathbf{e})
$$

$$
\simeq \operatorname*{holim}_{q\to\infty} \operatorname*{holim}_{G} \mathsf{f}^{q-1}(\mathbf{e})
$$

$$
\simeq \operatorname*{holim}_{G} \operatorname*{holim}_{q\to\infty} \mathsf{f}^{q-1}(\mathbf{e}).
$$

For slice completeness of $\mathbf{e}^{hG}$ we use the factorization

$$
\mathbf{e}^{hG} \longrightarrow \mathsf{sc}(\mathbf{e}^{hG}) \longrightarrow \mathsf{sc}(\mathbf{e})^{hG}.
$$

$\square$

COROLLARY 3.18. *Assuming* (3.18) *there are naturally induced isomorphisms*

$$
\mathsf{f}^q(\mathbf{E}^{hG}) \xrightarrow{\ \simeq\ } \mathsf{f}^q(\mathbf{E})^{hG}, \ \mathsf{f}_q(\mathbf{E}^{hG}) \xrightarrow{\ \simeq\ } \mathsf{f}_q(\mathbf{E})^{hG}.
\tag{3.29}
$$

Proof. There is a naturally induced commutative diagram of distinguished triangles:

$$
\begin{array}{ccccc}
\mathsf{f}_0(\mathbf{E}^{hG}) & \longrightarrow & \mathbf{E}^{hG} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{E}^{hG}) \\
\downarrow & & \| & & \downarrow \\
\mathsf{f}_0(\mathbf{E})^{hG} & \longrightarrow & \mathbf{E}^{hG} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{E})^{hG}
\end{array}
\tag{3.30}
$$

Proposition 3.15 shows that $f_0(\mathbf{E}^{hG}) \longrightarrow \mathbf{e}^{hG}$ is a map between effective motivic spectra. It follows that $f^{-1}(\mathbf{E}^{hG}) \longrightarrow f^{-1}(\mathbf{E})^{hG}$ induces an isomorphism on all negative slices. Since it is a map between coeffective spectra, it is in fact an isomorphism according to Lemma 3.6.

The general cases follow by using induction on (3.1) and (3.3).   □

We end this section by discussing G-fixed points in more detail. Let $\mathcal{O}_G$ denote the orbit category of G with objects $\{G/H\}$ and morphisms the G-maps $\mathrm{map}_G(G/H, G/K) \cong (G/K)^H$ [37, §1.8]. Let $\mathbf{MSS}$ be a highly structured model for the stable motivic homotopy category, e.g., motivic functors [6], or motivic symmetric spectra [13]. Let $\mathbf{MSS}^{\mathsf{eff}}$ be the Bousfield colocalization of $\mathbf{MSS}$ with respect to the set of objects $\Sigma^{p,0}\Sigma^\infty_{\mathbf{P}^1} X_+$, where $X \in \mathbf{Sm}_F$ and $p \in \mathbb{Z}$ (it suffices to consider $p \le 0$). Its homotopy category is $\mathbf{SH}^{\mathsf{eff}}$. As a model for naive G-motivic spectra we use the functor category $[\mathcal{O}_G^{\mathsf{op}}, \mathbf{MSS}]$ with the projective model structure [8, Theorem 11.6.1]. There is a naturally induced Quillen adjunction:

$$\mathsf{i}_0^G \colon [\mathcal{O}_G^{\mathsf{op}}, \mathbf{MSS}^{\mathsf{eff}}] \rightleftarrows [\mathcal{O}_G^{\mathsf{op}}, \mathbf{MSS}] \colon \mathsf{r}_0^G \tag{3.31}$$

Evaluating a naive G-motivic spectrum $\underline{\mathbf{E}}$ at the orbits corresponding to G and the identity element yields the underlying motivic spectrum $\mathbf{E}$ and the G-fixed points $\mathbf{E}^G$, respectively. Let $\underline{\mathbf{e}}$ be the naive G-motivic spectrum $\mathsf{f}_0^G(\underline{\mathbf{E}})$, where $\mathsf{f}_0^G = \mathbf{L}\mathsf{i}_0^G \circ \mathsf{r}_0^G$. (Forgetting the G-action, $\underline{\mathbf{e}}$ coincides with $\mathbf{e}$.) Since evaluating at the identity orbit commutes with (3.31), we obtain an isomorphism

$$\mathsf{f}_0^G(\underline{\mathbf{E}})^G \cong \mathsf{f}_0(\underline{\mathbf{E}}^G). \tag{3.32}$$

In particular, $\mathbf{kgl}^{C_2}$ coincides with the effective hermitian K-theory spectrum $\mathbf{kq}$, cf. (1.3).

4   The slice spectral sequence

The trigraded slice spectral sequence for $\mathbf{E}$ arising from (1.5) takes the form

$$\pi_\star \mathsf{s}_*(\mathbf{E}) \Longrightarrow \pi_\star \mathbf{E}. \tag{4.1}$$

This is an upper half-plane spectral sequence with entering differentials [3, §7] because $\pi_{p,w}\mathsf{s}_q(\mathbf{E}) = 0$ for $q < w$, cf. [38, §7]. A standard argument shows that (4.1) converges conditionally to the motivic homotopy groups of $\mathsf{sc}(\mathbf{E})$ in the sense of [3, Definition 5.10]. For the following result we refer to [30, Lemma 3.14].

Lemma 4.1. *Suppose $\mathbf{e} \in \mathbf{SH}^{\mathsf{eff}}$ and $\mathbf{e}/\eta$ is slice complete. Then there is a naturally induced isomorphism between $\mathbf{e}_\eta^\wedge$ to $\mathsf{sc}(\mathbf{e})$.*

Proposition 4.2. *Suppose $\mathbf{e} \in \mathbf{SH}^{\mathsf{eff}}$ and $\mathbf{e}/\eta$ is slice complete. There is a conditionally convergent slice spectral sequence*

$$\pi_\star \mathsf{s}_*(\mathbf{e}) \Longrightarrow \pi_\star \mathbf{e}_\eta^\wedge. \tag{4.2}$$

Proof. This follows from Lemma 4.1 and (4.1). ◻

5 PROOFS OF THEOREMS 1.1 AND 1.2

COROLLARY 5.1. *For effective hermitian K-theory there is a conditionally convergent slice spectral sequence*

$$\pi_\star \mathsf{s}_*(\mathbf{kq}) \Longrightarrow \pi_\star \mathbf{kq}_\eta^\wedge. \tag{5.1}$$

Proof. The only issue is to identify the quotient of $\mathbf{kq}$ by η with a slice complete spectrum. By [29, Theorem 3.4] there is a homotopy cofiber sequence

$$\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \mathbf{KQ} \longrightarrow \mathbf{KGL} \tag{5.2}$$

relating algebraic and hermitian K-theory via η. Passing to effective covers in (5.2) identifies the cofiber of $\mathsf{f}_0(\eta\colon \Sigma^{1,1}\mathbf{KQ} \longrightarrow \mathbf{KQ})$ with $\mathbf{kgl}$. Hence the cofiber of $\eta\colon \Sigma^{1,1}\mathbf{kq} \longrightarrow \mathbf{kq}$ is an extension of $\mathbf{kgl}$ by $\Sigma^{1,1}\mathsf{s}_{-1}(\mathbf{KQ}) \simeq \mathsf{s}_0(\Sigma^{1,1}\mathbf{KQ})$, cf. [29, Lemma 2.1], so it is slice complete by Lemmas 3.10 and 3.11. This verifies the assumptions in Proposition 4.2 for $\mathbf{kq}$. ◻

Assuming $\mathsf{vcd}_2(F) < \infty$, (3.18) holds for $\mathbf{KGL}$ by [29, Proposition 4.24]. We summarize some useful consequences of the results in §3.

PROPOSITION 5.2. *The following holds when $\mathsf{vcd}_2(F) < \infty$.*

(1) The homotopy fixed points spectrum of effective K-theory $\mathbf{kgl}^{hC_2}$ is slice complete. There is a conditionally convergent slice spectral sequence

$$\pi_\star \mathsf{s}_*(\mathbf{kgl}^{hC_2}) \Longrightarrow \pi_\star \mathbf{kgl}^{hC_2}. \tag{5.3}$$

(2) There is a naturally induced isomorphism $\mathsf{f}_q(\mathbf{KGL}^{hC_2}) \xrightarrow{\simeq} \mathsf{f}_q(\mathbf{KGL})^{hC_2}$.

Proof. Here (1) follows from Lemma 3.11, Corollary 3.17 and the discussion of (4.1) in §4, while (2) is a special case of Corollary 3.18. ◻

Proof of Theorem 1.1. By [29, Theorems 4.18, 4.25, 4.27, Lemma 4.26] the natural map $\Upsilon\colon \mathbf{KQ} \longrightarrow \mathbf{KGL}^{hC_2}$ in (1.7) induces an isomorphism of slices

$$\mathsf{s}_q(\Upsilon)\colon \mathsf{s}_q(\mathbf{KQ}) \xrightarrow{\simeq} \mathsf{s}_q(\mathbf{KGL}^{hC_2}) \simeq \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \bigvee_{i<0} \Sigma^{2q+2i,q}\mathsf{M}\mathbb{Z}/2 & 2\,|\,q \\ \bigvee_{i<0} \Sigma^{2q+2i+1,q}\mathsf{M}\mathbb{Z}/2 & 2 \nmid q. \end{cases} \tag{5.4}$$

From (5.4) we conclude the natural map $\mathbf{kq} \longrightarrow \mathsf{f}_0(\mathbf{KGL}^{hC_2})$ induces an isomorphism on slices. By composing with $\mathsf{f}_0(\mathbf{KGL}^{hC_2}) \xrightarrow{\simeq} \mathbf{kgl}^{hC_2}$, see Proposition 5.2(2), we conclude there is an isomorphism

$$\mathsf{s}_q(\gamma)\colon \mathsf{s}_q(\mathbf{kq}) \xrightarrow{\simeq} \mathsf{s}_q(\mathbf{kgl}^{hC_2}). \tag{5.5}$$

Thus $\gamma\colon \mathbf{kq} \longrightarrow \mathbf{kgl}^{hC_2}$ in (1.3) induces an isomorphism between the conditionally convergent upper half-plane slice spectral sequences (5.1) and (5.3). The induced map between the filtered target groups is thus an isomorphism [3, Theorem 7.2]. This finishes the proof by passing to Nisnevich sheaves of homotopy groups. $\square$

Proof of Theorem 1.2. Using (3.5) we obtain the naturally induced commutative diagram of distinguished triangles:

$$
\begin{array}{ccccc}
\mathbf{kq} & \longrightarrow & \mathbf{KQ} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{KQ}) \\
{\scriptstyle \mathsf{f}_0(\Upsilon)}\downarrow & & {\scriptstyle \Upsilon}\downarrow & & {\scriptstyle \mathsf{f}^{-1}(\Upsilon)}\downarrow \\
\mathsf{f}_0(\mathbf{KGL}^{hC_2}) & \longrightarrow & \mathbf{KGL}^{hC_2} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{KGL}^{hC_2})
\end{array}
\tag{5.6}
$$

Lemma 3.6 and (5.4) imply that $\mathsf{f}^{-1}(\Upsilon)\colon \mathsf{f}^{-1}(\mathbf{KQ}) \longrightarrow \mathsf{f}^{-1}(\mathbf{KGL}^{hC_2})$ is an isomorphism because it is a map between coeffective spectra and it induces on isomorphism on slices. By composing with the isomorphism $\mathsf{f}_0(\mathbf{KGL}^{hC_2}) \xrightarrow{\simeq} \mathbf{kgl}^{hC_2}$ of Proposition 5.2(2) we obtain a commutative diagram of distinguished triangles:

$$
\begin{array}{ccccc}
\mathbf{kq} & \longrightarrow & \mathbf{KQ} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{KQ}) \\
{\scriptstyle \gamma}\downarrow & & {\scriptstyle \Upsilon}\downarrow & & {\scriptstyle \mathsf{f}^{-1}(\Upsilon)}\downarrow{\scriptstyle \simeq} \\
\mathbf{kgl}^{hC_2} & \longrightarrow & \mathbf{KGL}^{hC_2} & \longrightarrow & \mathsf{f}^{-1}(\mathbf{KGL}^{hC_2})
\end{array}
\tag{5.7}
$$

This shows that $\mathbf{L}_\eta(\gamma)$ is an isomorphism if and only if $\mathbf{L}_\eta(\Upsilon)$ is an isomorphism. It follows that Theorem 1.1 implies Theorem 1.2. $\square$

COROLLARY 5.3. *If* $\mathsf{vcd}_2(F) < \infty$ *then the η-completion of the Tate K-theory spectrum* $\mathbf{KGL}^{tC_2}$ *is contractible.*

Proof. Recall that $\mathbf{KGL}^{tC_2}$ is the cone of the norm map from the homotopy orbits $\mathbf{KGL}_{hC_2}$ to the homotopy fixed points $\mathbf{KGL}^{hC_2}$ in the Tate diagram [11, (20)]

$$
\begin{array}{ccccc}
\mathbf{KGL}_{hC_2} & \longrightarrow & \mathbf{KQ} & \longrightarrow & \mathbf{KW} \\
\| & & {\scriptstyle \Upsilon}\downarrow & & \downarrow \\
\mathbf{KGL}_{hC_2} & \longrightarrow & \mathbf{KGL}^{hC_2} & \longrightarrow & \mathbf{KGL}^{tC_2}
\end{array}
$$

for the C_2-action on $\mathbf{KGL}$. Thus the assertion follows from Theorem 1.2 since the higher Witt-theory spectrum $\mathbf{KW}$ can be identified with $\mathbf{KQ}[\eta^{-1}]$, see e.g., [29, (7)]. $\square$

6 Appendix

By way of example we show that $\mathbf{SH}^{\mathrm{eff}}$ is not closed under homotopy fixed points. In effect, consider the homotopy fixed points for the trivial C_2-action on the effective motivic spectrum

$$\bigvee_{i \geq 0} \Sigma^{i,0} \mathbf{MZ}/2 \simeq \prod_{i \geq 0} \Sigma^{i,0} \mathbf{MZ}/2. \tag{6.1}$$

Here the sum and product are isomorphic by [29, Proposition A.5]. Since C_2-homotopy fixed points commute with products, [29, Lemma 4.22] yields a naturally induced isomorphism

$$\left(\prod_{i \geq 0} \Sigma^{i,0} \mathbf{MZ}/2\right)^{hC_2} \xrightarrow{\simeq} \prod_{i \geq 0} \prod_{j \geq 0} \Sigma^{i-j,0} \mathbf{MZ}/2. \tag{6.2}$$

Assuming (6.2) is an isomorphism in $\mathbf{SH}^{\mathrm{eff}}$, the countably infinite product $\prod_{n \in \mathbb{N}} \mathbf{MZ}/2$ — corresponding to indices $i = j$ — is effective. However, we show that $\prod_{n \in \mathbb{N}} \mathbf{MZ}/2 \notin \mathbf{SH}^{\mathrm{eff}}$.
Recall from [39, §2] the adjunction between $\mathbf{SH}$ and the stable motivic homotopy category of S^1-spectra $\mathbf{SH}_s$:

$$\Sigma_t^\infty \colon \mathbf{SH}_s \rightleftarrows \mathbf{SH} \colon \Omega_t^\infty.$$

Now $\Omega_t^\infty \mathbf{MZ}/2$ is the Eilenberg-MacLane S^1-spectrum $\mathbf{HZ}/2$ associated with the constant presheaf $\mathbf{Z}/2$ by [39, Lemma 5.2]. It follows that $\Omega_t^\infty \prod_{n \in \mathbb{N}} \mathbf{MZ}/2$ is the Eilenberg-MacLane S^1-spectrum $\mathbf{HV}$ associated with the constant presheaf $\mathbf{V}$, where $\mathbf{V}$ is a $\mathbf{Z}/2$-vector space of (uncountable) infinite dimension. If $\prod_{n \in \mathbb{N}} \mathbf{MZ}/2 \in \mathbf{SH}^{\mathrm{eff}}$ we would obtain

$$\Omega_t^\infty \Sigma^{0,1} \prod \mathbf{MZ}/2 \simeq \prod \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2 \in \Sigma_t^1 \mathbf{SH}_s \tag{6.3}$$

since $\Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2 \in \Sigma_t^1 \mathbf{SH}_s$ by [18, Theorem 7.4.1] (as conjectured in [39, Conjecture 4]). In (6.3) we use that $\Sigma^{0,1}$, being an equivalence, commutes with products. In $\mathbf{SH}_s$ there is a canonically induced map

$$\alpha \colon \left(\prod_{n \in \mathbb{N}} \Omega_t^\infty \mathbf{MZ}/2\right) \otimes_{\Omega_t^\infty \mathbf{MZ}/2} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2 \longrightarrow \prod_{n \in \mathbb{N}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2. \tag{6.4}$$

The tensor product in (6.4) is formed in the module category of the Eilenberg-MacLane S^1-spectrum $\Omega_t^\infty \mathbf{MZ}/2$ [15], cf. [26], [27], [39, Lemma 5.2]. In particular, the source of α in (6.4) is 1-effective, because $\Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2 \in \Sigma_t^1 \mathbf{SH}_s$ by [18, Theorem 7.4.1] and $\prod_{n \in \mathbb{N}} \Omega_t^\infty \mathbf{MZ}/2$ is effective, since it is the Eilenberg-MacLane spectrum associated with $\mathbf{V}$. We will prove that $\mathbf{SH}_s(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \alpha)$ is an isomorphism for every $X \in \mathbf{Sm}_F$ and $n \in \mathbb{Z}$; here $\mathbf{G_m}$ denotes the multiplicative group scheme. In effect, choose an uncountable

basis $\mathcal{B}$ of $\mathbf{V}$ and express $\mathbf{V}$ as the filtered colimit of finite dimensional sub-$\mathbf{Z}/2$-vector spaces $\mathbf{V}' \subset \mathbf{V}$ spanned by finite subsets $\mathcal{F} \subset \mathcal{B}$. Since $\Sigma^{n,0} X_+ \wedge \mathbf{G_m}$ is compact in $\mathbf{SH}_s$ there are isomorphisms

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \big(\prod_{n\in\mathbb{N}} \Omega_t^\infty \mathbf{MZ}/2\big) \otimes_{\Omega_t^\infty \mathbf{MZ}/2} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, (\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \prod_{f\in\mathcal{F}} \Omega_t^\infty \mathbf{MZ}/2) \otimes_{\Omega_t^\infty \mathbf{MZ}/2} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \big(\prod_{f\in\mathcal{F}} \Omega_t^\infty \mathbf{MZ}/2 \otimes_{\Omega_t^\infty \mathbf{MZ}/2} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\big)\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \Omega_t^\infty \mathbf{MZ}/2 \otimes_{\Omega_t^\infty \mathbf{MZ}/2} \prod_{f\in\mathcal{F}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \prod_{f\in\mathcal{F}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \Omega_t\big(\prod_{f\in\mathcal{F}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\big)\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \prod_{f\in\mathcal{F}} \Omega_t^\infty \Omega_t \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \prod_{f\in\mathcal{F}} \Omega_t^\infty \Sigma^{-1,0} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \mathrm{colim}_{\mathcal{F}\subset\mathcal{B}} \prod_{f\in\mathcal{F}} \Omega_t^\infty \Sigma^{-1,0} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big((\Sigma^{n,0} X_+, \prod_{n\in\mathbb{N}} \Omega_t^\infty \Sigma^{-1,0} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \prod_{n\in\mathbb{N}} \Omega_t^\infty \Omega_t \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+, \Omega_t \prod_{n\in\mathbb{N}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big) \cong$$

$$\mathbf{SH}_s\Big(\Sigma^{n,0} X_+ \wedge \mathbf{G_m}, \prod_{n\in\mathbb{N}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2\Big),$$

which by canonicity coincides with the map induced by α. If the target in (6.4) is 1-effective, as implied by (6.3), it would follow that α is an isomorphism. One checks that α is not an isomorphism by choosing a field F such that $F^* \otimes \mathbb{Z}/2$ is an infinitely generated $\mathbb{Z}/2$-module, e.g., $F = \mathbb{Q}$. The map $\mathbf{SH}_s(\Sigma^{-1,0}\mathrm{Spec}(F)_+, \alpha)$ coincides with the canonical map

$$\Big(\prod \mathbb{Z}/2\Big) \otimes_{\mathbb{Z}/2} (F^* \otimes \mathbb{Z}/2) \longrightarrow \prod F^* \otimes \mathbb{Z}/2, \tag{6.5}$$

which is not surjective. Hence $\prod_{n\in\mathbb{N}} \Omega_t^\infty \Sigma^{0,1} \mathbf{MZ}/2$ cannot be 1-effective. As

explained above it follows that $(\prod_{i \geq 0} \Sigma^{0,i}\mathbf{MZ}/2)^{hC_2}$ is noneffective.

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