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Topics in toric geometry: Polar degrees, vector bundles and algebraic cycles

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Geometry is unique and eternal, and it shines in the mind of God. The share of it which has been granted to man is one of the reasons why he is the image of God.

–Johannes Kepler

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List of Papers

Paper I

Nødland, Bernt Ivar Utstøl 'Local Euler obstructions of toric varieties'. In: *Journal of Pure and Applied Algebra* **3** (2018), pp. 1508–533, DOI: 10.1016/j.jpaa.2017.04.016.

Paper II

Helmer, Martin and Nødland, Bernt Ivar Utstøl 'Polar degrees and closest points in codimension two'. In: *Journal of Algebra and Its Applications* **3** (2018), DOI: 10.1142/S0219498819500956.

Paper III

Nødland, Bernt Ivar Utstøl 'Chow groups and pseudoffective cones of complexity one T-varieties'. *Submitted for publication.*

Paper IV

Nødland, Bernt Ivar Utstøl 'Murphy's law for toric vector bundles on smooth projective toric varieties'.

Paper V

Nødland, Bernt Ivar Utstøl 'Some positivity results for toric vector bundles'.

Contents

Chapter 1

Introduction

Algebraic geometry is the study of solution sets of systems of polynomial equations. These sets, known as algebraic varieties, can be highly complicated, and are most of the time impossible to describe explicitly. However, in applications it is often sufficient to know more qualitative information about an algebraic variety, such as its dimension, or its degree. For this reason, a major part of algebraic geometry has been devoted to defining and studying geometric and algebraic invariants of algebraic varieties, and to develop algorithms for computing them. These invariants are thus used as tools to classify and distinguish varieties, but they are also interesting objects in their own right. In this thesis, we study geometric and algebraic invariants of varieties that admit special structures (e.g., torus actions), that allow us to study them via combinatorial methods.

The thesis consists of five papers, which can be roughly divided into two parts by theme. Paper I and Paper II study polar degrees and degrees of dual varieties of projectively embedded toric varieties. The main topics of Paper III, Paper IV and Paper V are divisors and algebraic cycles on varieties having a torus action. The unifying concept in all of these papers is that of a toric variety, or, more generally, an algebraic torus acting on a variety.

1.1 Toric varieties

Given a set S of polynomials in variables $x_1, \ldots, x_n$, the associated variety Z is a subset of affine n-space $\mathbb{A}^n$. Here $\mathbb{A}^n$ naturally sits as an open set inside projective space $\mathbb{P}^n$, which is intuitively obtained by adding "points at infinity" to $\mathbb{A}^n$; one for each line through the origin. The closure of Z inside $\mathbb{P}^n$ is thus a subvariety of projective space, or, in other words, a *projective variety*. For several reasons, it will be more convenient to work with projective varieties rather than subvarieties of affine space. For instance, every subvariety of a projective variety is compact, which makes many formulas easier to define and evaluate. Another basic example of this is given by intersections: in the projective plane any two distinct lines intersect in a point, whereas in the affine plane, the same is true, but with the awkward exception when the lines are parallel. Thus adding the 'points at infinity' yields a more harmonious and elegant intersection theory. In this thesis we will be mostly interested in studying projective varieties.

The standard way to define projective n-space $\mathbb{P}^n$ is as the set of lines through the origin of Euclidean $(n+1)$-space $\mathbb{A}^{n+1}$. This is equivalent to saying that $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ modulo the action of the non-zero elements of the ground field, k^* (we assume k is algebraically closed of characteristic 0), by scaling a vector by any number. Subvarieties of projective space correspond to zero-sets of polynomials in the $n+1$ coordinate variables of $\mathbb{A}^{n+1}$ which are invariant under the action of k^*, in other words to homogeneous polynomials. The natural generalization of this construction is that of a normal toric variety, which is a quotient of an open set in affine space $\mathbb{A}^m$, modulo an action of the torus $(k^*)^\rho$. This is Cox's quotient construction [Cox95], which implies that closed subvarieties of X are zero sets of polynomials in m variables which are invariant under the $(k^*)^\rho$-action. For this reason, toric varieties are natural generalizations of projective space. One of the main advantages of working with toric varieties is that they admit rich combinatorial structures, enabling us to use a wide variety of tools from combinatorics to compute and study their invariants. This makes toric geometry into an important subfield of algebraic geometry, since it is often a source of interesting (counter)examples and explicit illustrations of general theory.

The study of toric varieties goes back to Demazure [Dem70], who introduced them to investigate subgroups of the Cremona group of birational automorphisms of $\mathbb{P}^n$; the automorphism group of a smooth toric variety is a subgroup of the Cremona group. Shortly afterwards, toric varieties (then known as *torus embeddings*) appeared in the work of multiple people; Appendix A of [CLS11] gives an extensive historical account of the subject. Around 1990 three textbooks, which are still standard references on toric geometry today, appeared, by Fulton [Ful93], Gelfand, Kapranov and Zelevinsky [GKZ94] and Oda [Oda88]. In an influential 1994 paper [Bat94], Batyrev showed that one can construct interesting examples of Calabi-Yau varieties as hypersurfaces in certain toric varieties as well as construct their mirror partners which are hypersurfaces in other toric varieties. This led to a surge of interest in toric geometry from theoretical physicists, who were interested in mirror symmetry. Over the last 20 years the interest in toric geometry has only increased, and it is now a small subfield of its own within algebraic geometry. The most comprehensive text on the subject is the book by Cox, Little and Schenck [CLS11]; it will be cited extensively throughout this thesis.

By definition a toric variety X is an irreducible algebraic variety which contains the algebraic torus T as an open dense subset, such that the action of T on itself extends to an action of T on X. To X one can associate certain sheaf cohomology groups, which will be representations of T. Hence they will split into sums of irreducible, one-dimensional, representations, given by characters $m \in M \simeq \mathbb{Z}^n$. For this reason lattice polytopes arise naturally from the study of toric varieties: The convex hull of the set of characters m giving the splitting type of the global sections of a line bundle on a complete X form a convex rational polytope. Moreover, if the line bundle satisfy certain positivity properties (see Section 1.6), many properties of this polytope correspond to properties of the toric variety.

All of the varieties we study are in some sense related to toric varieties. Our main tool will be combinatorics, which we use to study lattice polytopes as well as related combinatorial objects that arise from toric varieties.

1.1.1 Equations of toric varieties

Algebraic varieties are zero-sets of polynomials. From this perspective a toric variety correspond to particularly simple equations: They are all (in some coordinate system) binomials, in other words, polynomials of the form

$$x_1^{m_1} \cdots x_s^{m_s} - x_{s+1}^{ms+1} \cdots x_n^{m_n}.$$

It is, in fact, possible to define a toric variety as a variety whose ideal is a prime binomial ideal. However, studying the equations themselves are not usually the most illuminating method to learn about the variety. We will see that for toric varieties the combinatorial structure extracted from the monomials appearing in the equations captures many algebraic and geometric properties of the toric variety.

Example 1.1.1. Let X be the blowup of $\mathbb{P}^2$ at a point. Then X can be embedded in $\mathbb{P}^4$ using the linear system of quadrics passing through the point. The defining equations in $\mathbb{P}^4 = \operatorname{Proj} k[x_0, x_1, x_2, x_3, x_4]$ are

$$x_0x_2 - x_1^2, x_1x_3 - x_0x_4, x_2x_3 - x_1x_4.$$

Because they are all binomials, X is a toric variety.

1.1.2 Non-normal toric varieties

There are different definitions of toric varieties in the literature, depending on whether one requires the toric variety to be normal or not. In the most general setting, one can define an affine toric variety as the spectrum of a semigroup algebra $k[S]$, for a subsemigroup S of M. Thus, generators of the semigroup correspond to generators of the algebra, and relations between the semigroup generators correspond to relations in the algebra. Any affine toric variety is the spectrum of such a semigroup algebra [GKZ94, Ch. 5, Proposition 2.4].

In the preceding paragraph we constructed a toric variety in terms of generators and relations. For projective toric varieties one may alternatively construct a toric variety parametrically. Let $A = \{a_1, ..., a_s\}$ be a finite set of lattice points in M and denote by χ^{a_i} the character $T \to k^*$, corresponding to a_i. Then, we can define an embedded toric variety X_A in $\mathbb{P}^{s-1}$, via mapping the torus using the elements of A and taking the Zariski closure:

$$T \to \mathbb{P}^{s-1}$$

$$t \mapsto (\chi^{a_1}(t) : \chi^{a_2}(t) : \cdots : \chi^{a_s}(t))$$

There is a natural action by the torus T on the variety X_A. To A we can associate the polytope $P = \operatorname{Conv}(A)$, the convex hull of the points in A. Many combinatorial properties of P correspond to properties of X_A. For instance, the T-orbits of X_A are in bijection with the faces of P.

Example 1.1.2. Let $A = \{0, 1, 2, 3\} \subset \mathbb{Z}$. Then P is a line segment of length 3. The corresponding variety is given by the closure of the image of the map

$$t \mapsto (1 : t : t^2 : t^3),$$

which is the twisted cubic curve.

Let $A' = \{0, 2, 3\} \subset \mathbb{Z}$. Then $X_{A'} \subset \mathbb{P}^2$ is the closure of the image of the map

$$t \mapsto (1 : t^2 : t^3).$$

This is the rational cuspidal curve $y^3 = xz^2$, which is not normal.

In Paper II we will study polar and dual degrees of (possibly non-normal) projective toric varieties of codimension two.

1.1.3 Normal toric varieties

In the above definitions we did not require a toric variety to be normal. However, in the more recent literature on the subject one often requires that it is; notably this is done in the books by Oda [Oda88], Fulton [Ful93] and Cox-Little-Schenck [CLS11]. In combinatorial terms, assuming normality means that we require the semigroup of an affine toric variety to be saturated. Being saturated means that the semigroup coincides with the intersection of the lattice points in its positive linear span with the lattice L generated by the semigroup. The positive linear span of the semigroup form a rational polyhedral cone, from which it is possible to recover the saturated semigroup. Thus any normal affine toric variety correspond to a convex rational polyhedral cone in $L \otimes \mathbb{Q}$ [GKZ94, Ch.5, Proposition 2.8].

Example 1.1.3. Affine n-space $\mathbb{A}^n$ is an affine toric variety given by the cone generated by the standard basis vectors $e_1, ..., e_n$ of the lattice M.

A general toric variety is defined by a collection of affine toric varieties together with the data of how they are glued together. To a cone in $M \otimes \mathbb{Q}$ we can associate the dual cone in the dual vector space $N \otimes \mathbb{Q}$, where $N = \mathrm{Hom}(M, \mathbb{Z})$. Passing to the dual space is convenient for recording the gluing information: The toric variety associated with the intersection of two cones in $N \otimes \mathbb{Q}$ is the intersection of the two toric varieties associated to the individual cones. Thus we can record the gluing data in the combinatorial notion of a fan which is a collection of cones in $N \otimes \mathbb{Q}$ closed under intersections and taking faces. If, on the other hand, one starts with a full-dimensional lattice polytope, one can obtain an abstract toric variety by taking the inner normal fan of the polytope. Therefore, both fans and lattice polytopes are closely linked to the study of toric varieties. Many properties of the associated varieties are in fact expressible in terms of the combinatorics of the fan and/or the polytope.

Example 1.1.4. Let P be the polytope in Figure 1.1. A computation shows that the associated $X_{P \cap M}$ is the blowup of $\mathbb{P}^2$ in a point from Example 1.1.1. The fan that defines this toric variety is also shown in Figure 1.1. The direction of its rays

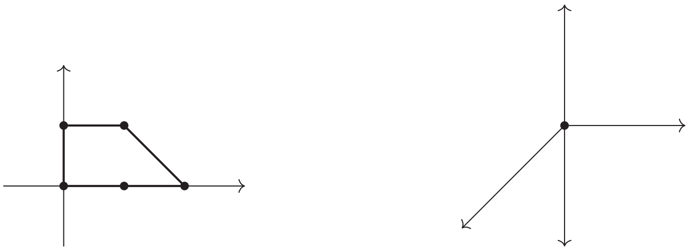

Figure 1.1: Polytope and fan defining blowup of $\mathbb{P}^2$ at a point.

are the normals to the facets of the polytope. Its cones are in dimension-reversing bijection to the faces of P as well as in bijection to torus invariant subvarieties of $X_{P\cap M}$.

The assumption that any toric variety is, by definition, normal will implicitly be made in Paper III, Paper IV and Paper V.

1.2 Projective duality

The first and second paper in the thesis are related to a classical construction in algebraic geometry, namely that of projective duality, which dates back to the early 1800s. We here present a brief introduction to this subject based on [GKZ94, Chapter 1]. Given projective n-space $\mathbb{P}^n$ with coordinates $x_0, ..., x_n$, we can construct the dual projective space $(\mathbb{P}^n)^\vee$ of hyperplanes in $\mathbb{P}^n$. A hyperplane H in $\mathbb{P}^n$ is described by an equation

$$a_0x_0 + a_1x_1 + ... + a_nx_n = 0.$$

The associated point in $(\mathbb{P}^n)^\vee$ is the point with coordinates $(a_0 : a_1 : ... : a_n)$. This gives $(\mathbb{P}^n)^\vee$ the structure of a projective n-space. More generally a linear space L in $\mathbb{P}^n$ corresponds to a linear space in $(\mathbb{P}^n)^\vee$ of dimension $\operatorname{codim} L - 1$. This is a duality in the sense that the dual of $(\mathbb{P}^n)^\vee$ is identified with the projective space we started with. The duality preserves incidences of linear spaces. For instance, when $n = 2$ the intersection of two lines in $\mathbb{P}^2$ is a point p. Inside $(\mathbb{P}^2)^\vee$ the two lines correspond to points x, y and the dual of the point p is exactly the line connecting x and y. One way in which projective duality is useful is that for any statement on incidences of linear spaces in $\mathbb{P}^n$ we get a dual statement in $(\mathbb{P}^n)^\vee$.

We can extend the notion of duality to arbitrary subvarieties X inside projective space by defining $X^\vee$ as the closure of the set of hyperplanes which contain the tangent space to a smooth point $x \in X$. This is a duality in the sense that the dual of $X^\vee$ inside $(\mathbb{P}^n)^\vee$ is equal to X. Thus for a plane curve C the dual curve $C^\vee \subset (\mathbb{P}^2)^\vee$ is the closure of the set of tangent lines to C at smooth points. Invariants and singularities of $C^\vee$ determine interesting invariants of C itself. For instance if the dual curve has a simple node at a point p corresponding

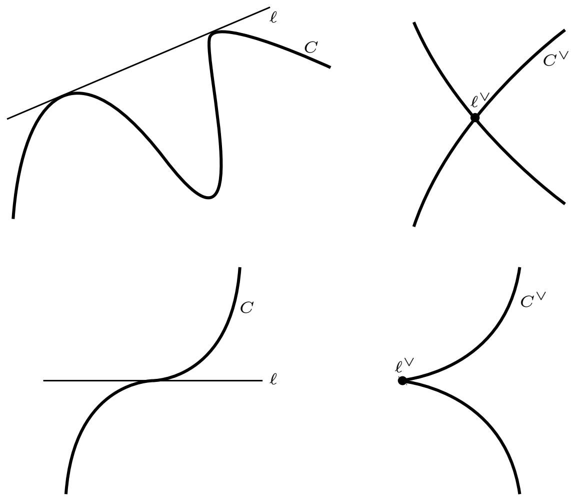

Figure 1.2: A bitangent and an inflectional line corresponding to a node and a cusp of the dual curve.

to the tangent line L_p of C, then the line L_p is actually a double tangent to C. Similarly, if $C^\vee$ has a simple cusp at p, then L_p is an inflectional line of C. In other words the tangent line L_p intersects C to order three. Formulas relating invariants of C and $C^\vee$ are called Plücker formulas, after Julius Plücker. An example is the following:

Theorem 1.2.1. *Assume that C is an irreducible plane curve of degree d with κ cusps and δ simple nodes but no other singularities. Then the degree of the dual curve $C^\vee$ is $d(d-1) - 3\kappa - 2\delta$.*

This is an example of how geometric invariants of a variety correspond to other invariants of the dual variety. By the duality theorem there is also the converse statement: If one knows the degree and singularities of the dual curve $C^\vee$, then one can compute the degree and number of double tangents and inflectional tangents of C.

1.2.1 Polar varieties

A notion which is closely related to dual varieties is that of a *polar variety*. Given an embedded projective variety $X \subset \mathbb{P}^n$ of dimension r, we can associate polar varieties M_k, for any integer k such that $0 \leq k \leq r$. M_k is defined as the subvariety of X which is the closure of the locus consisting of smooth points x whose tangent space intersect a fixed general linear subspace of $\mathbb{P}^n$ of dimension $n - r + k - 2$ non-transversally. The classes of the polar varieties in the Chow ring of X are called *polar classes* (these do not depend on the choice of the general linear subspaces) and their degrees are called *polar degrees*. Polar degrees carry information about invariants of X related to tangency. For example, if C is a plane curve, the first polar degree is the number of tangents to C passing through a general point. Moreover, the top non-zero polar degree is equal to the degree of the dual variety. In her PhD thesis Piene gave several Plücker type formulas relating polar degrees of varieties to geometric invariants [Pie78]. In general, certain signed sums of the polar degrees do not depend on the projective embedding on X, but give fundamental invariants of the underlying abstract variety. This was discovered by Todd and Severi, who used it to define what has later become known as the Chern class of X [Pie15].

The following optimization problem arises in several applications of algebraic geometry and is related to polar degrees: Given an algebraic variety $X \subset \mathbb{A}^n$ and $u \in \mathbb{A}^n$, minimize the Euclidean distance function $d(x, u)$, under the constraint that $x \in X$. For a general choice of u, the number of critical points of $d(x, u)$ is independent of u and this number is called the Euclidean distance degree of X. If X is defined by a homogeneous ideal, then by [Dra+16, Theorem 5.4] the Euclidean distance degree equals the sum of all the polar degrees of the associated projective variety $\widetilde{X} \subset \mathbb{P}^{n-1}$. Therefore polar degrees are also important in applications of algebraic geometry.

In this thesis we will study polar degrees and degrees of dual varieties of toric varieties of small dimension and of small codimension. For toric varieties these degrees can be determined from the combinatorial structure defining the toric variety, as will be described in Paper I and Paper II.

1.3 A-discriminants

The most comprehensive work on duals of toric varieties was done by Gelfand, Kapranov and Zelevinsky [GKZ94], who used them as a tool to study what they call A-discriminants. The classical discriminant of a polynomial $p(x)$ is a polynomial in the coefficients of p which vanishes exactly when p has a double root. The most famous example is the discriminant Δ_p of the quadratic polynomial in one variable,

$$p(x) = ax^2 + bx + c,$$

which is given by

$$\Delta_p = b^2 - 4ac.$$

For a cubic polynomial in one variable,

$$q(x) = ax^3 + bx^2 + cx + d,$$

the discriminant is more complicated [GKZ94, p. 1]:

$$\Delta_q = b^2c^2 - 4b^3d - 4ac^3 - 27a^2d^2 + 18abcd.$$

Generalizing these examples, Gelfand, Kapranov and Zelevinsky define the A-discriminant as follows:

Definition 1.3.1. Let A be a finite set of monomials in d variables, and let k^A be the set of all polynomials whose monomials belong to A. The *A-discriminant* Δ_A is a polynomial in the coefficients of $f \in k^A$ that vanishes whenever f has a multiple root $(x_1, ..., x_d)$, where all $x_i \neq 0$.

The polynomial Δ_A, assuming it exists, may have very large degree and many monomials. Therefore, computing it is usually difficult. However, it is still possible to study geometric and combinatorial properties of the hypersurface: It turns out that Δ_A is the projective dual $X_A^\vee$ of the projective toric variety X_A, defined above, using the same set of monomials A [GKZ94, Ch. 9, Proposition 1.1]. As we have seen before, geometric properties of a variety often correspond to other geometric properties of its projective dual. Hence, we can learn about the A-discriminant by instead studying the toric variety X_A. From this perspective we also expect Δ_A to exist for most sets A; the existence of Δ_A is equivalent to the projective dual of X being a hypersurface and we know from general theory on projective duality that for most varieties the projective dual is a hypersurface.

If X_A is a smooth toric variety, we can compute the degree of the A-discriminant as follows: We denote by P the convex hull of the lattice points A. For each face Q of P there is a lattice $M(Q)$ of dimension equal to $\dim Q$, generated by all lattice points in Q. We denote by Vol_Q the normalized volume function on $M(Q)_\mathbb{R}$, so that the standard simplex has volume 1. This volume function equals $(\dim Q)!$ times the Euclidean volume function. Then by [GKZ94, Ch. 9, Theorem 2.8]

$$\deg \Delta_A = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Vol}_Q(Q).$$

This formula illustrates the usefulness of the combinatorial structure of a toric variety. Indeed, all of the terms in the above sums are invariants of the defining lattice polytope P, and are therefore easily computable. For more general varieties we cannot expect the existence of such simple formulas.

Matsui and Takeuchi generalized the above formula to arbitrary (possibly non-normal) toric varieties X_A [MT11, Corollary 1.6]:

$$\deg \Delta_A = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Eu}(Q) \operatorname{Vol}_Q(Q).$$

Here Eu is the local Euler obstruction function, which for a toric variety is constant on any T–orbit. Eu is an integer-valued function which, in some sense, measures how singular a point is. For smooth points its value is always 1, but it can take any integer value for singular points. Matsui and Takeuchi also gave combinatorial formulas for the local Euler obstruction for toric varieties [MT11, Section 4] as well as for related invariants [MT11, Theorem 1.4], which enabled Helmer and Sturmfels to prove combinatorial expressions for all the degrees of the polar varieties of X_A [HS18, Theorem 1.2]. Many of these formulas are algorithmic and recursive in nature.

1.4 Summary of the first and second paper

In the first half of the thesis we are interested in computing degrees of dual and polar varieties of toric varieties of small dimension and small codimension. In particular, this also implies that we give formulas for degrees of certain corresponding A-discriminants.

We use the recursive formulas by Matsui–Takeuchi, and later those by Helmer–Sturmfels, to compute these invariants in special cases where one can expect to obtain simple formulas.

Paper I The first paper continues the work done in my master thesis [Nød15]. The goal of the paper is to compute the local Euler obstruction and degree of the dual variety of a toric variety. The formulas for these numbers appearing in the literature are algorithmic and recursive in nature, thus primarily interesting from a computational viewpoint. By restricting to the case of varieties of dimension two and three, we are able to give more explicit algorithms and formulas to effectively compute these numbers.

Specializing to examples of weighted projective planes and threefolds, we are in some situations able to find closed-form expressions for these numbers. Moreover, when looking at the case of threefolds with isolated singularities we prove that $\mathrm{Eu}(x) \geq 1$, for any x. We also provide an example of a singular weighted projective threefold for which the local Euler obstruction is constantly equal to 1, thus disproving a conjecture by Matsui and Takeuchi. We use the formulas to give new proofs of some well-known results on which varieties are defective, in other words, when the corresponding dual variety has codimension larger than one.

We also provide several examples where we compute the local Euler obstruction for any point on the variety as well as the degree of the dual variety.

Paper II In the second paper we compute the polar degrees and the Euclidean distance degree of toric varieties of codimension two. This paper is joint work with Martin Helmer, and the idea for the paper was suggested to us by Bernd Sturmfels. While in the first paper I was able to obtain formulas for varieties of low dimension, in this paper we instead study (possibly non-normal) toric varieties of arbitrary dimension, but of codimension two.

To do this we utilize the notion of Gale duality, which is a convenient way of switching between the parametric representation of a toric variety and a description in terms of equations. A codimension two toric variety can be parametrically represented by a $(n-2) \times n$ matrix A. The Gale dual B will be a $2 \times n$ matrix B from which the equations of X_A can be deduced. We translate the formulas for the local Euler obstruction and polar degrees, given in terms of the A-matrix, into formulas in terms of the combinatorial structure of the rows of B. Since they live in $\mathbb{Z}^2$, we end up with significantly simpler formulas. We borrow many ideas from, and are heavily inspired by, the paper of Dickenstein and Sturmfels on elimination theory for codimension two toric varieties [DS02].

The motivation of this paper was partly computational. The fact that we may compute all invariants in $\mathbb{Z}^2$ instead of in $\mathbb{Z}^n$ speeds up all computations significantly. We give several examples of computing invariants, and compare the speed of computation to that of using the A-matrix representation. Additionally, Martin Helmer wrote a Macaulay2 package implementing the formulas, which is available on his website [Hel].

Remark 1.4.1. Both of the papers above are already published in journals. The versions here are almost identical, with the exception that some spelling and typographic errors have been corrected. A few places we have also attempted to made the exposition clearer.

1.5 T-varieties

In the second half of the thesis we study divisors and algebraic cycles on a variety with a torus action as well as positivity properties of these. Toric varieties still play a vital role, since the varieties we study are defined or described in terms of toric geometry.

A T-variety X is a variety with an effective action of an algebraic torus T. Toric varieties are therefore examples of T-varieties, however we now allow X and T to have different dimensions. We define the *complexity* of X to be the difference $\dim X - \dim T$. Altmann and Hausen, and later also Süss, developed a quasi-combinatorial framework for studying T-varieties, in analogy with the case for toric varieties [AH06], [AHS08]. Given a T-variety X, one can define a rational quotient $Y = X/T$ of dimension equal to the complexity of the T-action. By studying the quotient Y and the fibers of the quotient map, which are (possibly non-reduced, non-irreducible) toric varieties, one can deduce many properties of X. As a consequence, studying a T-variety of complexity c and dimension $n + c$ is about as complicated as studying any variety of dimension c, together with a collection of toric varieties of dimension n. This heuristic motivates one to consider T-varieties of low complexity, in particular of complexity one.

The general framework describes an affine T-variety in terms of a *polyhedral divisor* $\mathcal{D}$ on the quotient Y. This is an analogy to a Weil divisor on Y, only the coefficients are now polyhedra instead of integers. From the polyhedral divisor one can define a sheaf of algebras on Y, and the associated T-variety is the spectrum of the global sections of this sheaf. All affine T-varieties arise from this construction [AH06, Theorem 3.4].

As in the case of toric varieties, where a non-affine toric variety corresponds to a fan giving an affine cover, together with the information about how they are glued together, there exists a similar construction describing any T-variety in terms of an affine cover. This is the notion of a divisorial fan, which is a collection of polyhedral divisors satisfying certain compatibility conditions. Any T-variety arises from a divisorial fan [AHS08, Theorem 5.6].

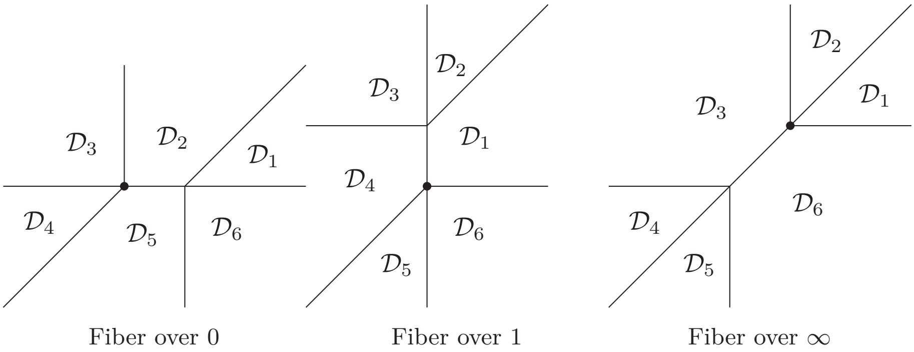

Figure 1.3: Polyhedral divisors defining $\mathbb{P}(T_{\mathbb{P}^2})$.

There is a slightly simpler classification of complexity one T-varieties, due to Ilten and Süss [IS11]. Instead of describing the T-variety in terms of polyhedral divisors, one can record polyhedral subdivisions giving the fibers of the rational quotient map as well as which subvarieties one needs to blow up to resolve the rational quotient map.

Example 1.5.1. The projectivization of the tangent bundle $T_{\mathbb{P}^2}$ on $\mathbb{P}^2$ is a 3-dimensional complexity one T-variety. The rational quotient Y is the projective line. The general fiber of the quotient morphism is the degree six Del Pezzo surface. The structure of the special fibers, of which there are three, are given by the polyhedral subdivisions shown in Figure 1.3. The polyhedral divisors of maximal dimension, $\mathcal{D}_1, \cdots, \mathcal{D}_6$, are labelled in the figure. We see that each special fiber is isomorphic to the union of two Hirzebruch surfaces $\mathcal{H}_1$, intersecting in a projective line. $T_{\mathbb{P}^2}$ is also an example of a toric vector bundle, studied in Paper IV and Paper V.

In Paper III we study algebraic cycles on complexity one T-varieties. The key idea is that, as for toric varieties, any cycle is numerically/rationally equivalent (see the next section for definitions of these notions) to a T-invariant cycle, and these can be described in terms of combinatorics.

1.6 Cycles on varieties

A lot of information about a variety X is encoded in the set of all subvarieties of X. The appropriate place to study these is using the Chow groups of X: For any integer k between 0 and $\dim X$ we can associate the group $A_k(X)$, which is the group $Z_k(X)$ freely generated by subvarieties of X of dimension k, modulo relations $R_k(X)$ coming from principal divisors on subvarieties of dimension $k+1$. Elements of $Z_k(X)$ which are equivalent under this relation are said to be *rationally equivalent*. The structure of the Chow groups $A_k(X)$ are fundamental invariants of the variety X, however they are in general very hard to compute. We remark that all invariants defined from Chow groups are intrinsic to the variety X, in the sense that they only depend on the isomorphism class of X, and not on the particular projective embedding of X. This is in contrast with the cases of polar and dual degrees.

The cycles studied the most are those of codimension one, called divisors. A divisor D is called very ample if it induces a closed embedding $X \to \mathbb{P}(H^0(X, \mathcal{O}(D)))$. D is called ample if some positive multiple kD is very ample. An important reason why divisors provide information about X is that the (very) ample divisors that exist on X tell us how it is possible to embed X in projective space.

Since Chow groups can be very large and complicated, one often considers smaller quotient groups instead, by imposing coarser equivalence relations. We will here consider the numerical groups $N_k(X)$, in which two cycles are regarded as equivalent if they have the same intersection with all subvarieties of complimentary dimension. This is a finitely generated free abelian group. Frequently we wish to take the tensor product with $\mathbb{R}$ to obtain $N_k(X)_{\mathbb{R}}$, which is a finite-dimensional real vector space. The structure of a real vector space enables us to define various cones of cycles inside $N_k(X)_{\mathbb{R}}$. A key example is the cone of effective k-cycles,

$$\mathrm{Eff}_k(X) = \{\sum a_i Z_i \in N_k(X)_{\mathbb{R}} | a_i \geq 0, Z_i \text{ is a subvariety of } X\},$$

as well as the closure $\overline{\mathrm{Eff}}(X)$, the pseudoeffective cone of k-cycles.

An important result in the study of ample line bundles is Kleiman's criterion, which states that D is ample if and only if $D \cdot C > 0$, where C is any element of the pseudoeffective cone of curves $\overline{\mathrm{Eff}}_1(X)$. This implies that ampleness is in fact a property of numerical classes of divisors and it motivates the definition of D being nef if $D \cdot C \geq 0$ for any $C \in \overline{\mathrm{Eff}}_1(X)$. In other words a nef divisor is a limit of ample divisors. One can define other similar properties (known as *positivity* properties) of divisors such as being semi-ample, big, movable and so on; as well as corresponding cones of divisors. These cones are fundamental in the study of the birational geometry of X. Their study and the relation to birational geometry is what is known as Mori theory or the Minimal Model Program.

1.6.1 Cycles on toric varieties

For a normal toric variety X_Σ, associated to the fan Σ, the Chow groups were explicitly described by Fulton and Sturmfels [FS97]. For any cone $\tau \in \Sigma$ we let $M(\tau)$ be defined as $\tau^\perp \cap M$, which is a sublattice of M of dimension $\operatorname{codim} \sigma$. Then there is an exact sequence

$$\bigoplus_{\dim \tau = n-k-1} M(\tau) \to \bigoplus_{\dim \sigma = n-k} \mathbb{Z} \to A_k(X_\Sigma) \to 0.$$

The maps can be explicitly written down in terms of the defining fan Σ. The intuition is that by using the action of the torus T on an arbitrary subvariety Z of X_Σ, we can obtain a sum of subvarieties rationally equivalent to Z, whose components are also invariant under the torus action. Since the only invariant subvarieties are closures of T-orbits, we get that subvarieties corresponding to cones of dimension $n-k$ generate $A_k(X)$. Moreover, exactness of the above sequence implies that all relations between the generators come from relations on invariant subvarieties.

There are characterizations of when a divisor on a toric variety is nef or ample. By the exact sequence describing divisors on a toric variety, we see that any divisor D is equivalent to a sum of T-invariant divisors. Alternatively, it is possible to write the divisor in terms of local defining data, and from this obtain an associated support function $\phi_D : N \otimes \mathbb{Q} \to \mathbb{Q}$ which is a continuous, piecewise linear function, linear on every cone $\sigma \in \Sigma$. A classical result on toric varieties states that a divisor D is nef (ample) if and only if the support function ϕ_D is (strictly) convex [Ful93, p.68].

On a related note, the theory of A-discriminants discussed above is also related to positivity of divisors on the associated toric variety. It turns out that the Newton polytope of the discriminant is what is known as the secondary polytope, which is another polytope associated to A. It describes the set of all coherent subdivsions of the polytope $\text{Conv}(A)$, and it is also intimately related to the birational geometry of X_A, its cones of cycles and to birational maps to other toric varieties, as described in chapter 14 and 15 of [CLS11].

For cycles of higher codimension than one, properties of cones of cycles are not well understood. There are few general theorems, and few examples where one can compute the cones explicitly.

1.7 Summary of the third paper

Paper III In the third paper we study algebraic cycles on T-varieties of complexity one. We prove that the pseudoeffective cones $\overline{\text{Eff}}_k(X)$ of effective cycles on a complexity one T-variety X are always rational polyhedral, generated by classes of invariant subvarieties. This is proved by replacing the class of any effective subvariety by a sum of invariant effective subvarieties, using the T-action as well as proving that there are only finitely many such classes. The generating classes can be found explicitly from the combinatorial data defining X.

We also give a presentation of the Chow groups $A_k(X)$, in terms of generators and relations, generalizing the above sequence for toric varieties. From the theorem on pseudoeffective cones, we obtain a finite set of generators. Moreover, using theory on T-invariant Chow groups, we are able to compute all relations between these generators, obtaining an exact sequence describing $A_k(X)$.

Finally, we provide several examples to illustrate how the exact sequence can be used to compute Chow groups of a complexity one T-variety.

1.8 Toric vector bundles

The final class of varieties we study are toric vector bundles and their projectivizations. A toric vector bundle $\mathcal{E}$ is a vector bundle on a toric variety such that the T-action extends to an action on the geometric vector bundle, and so that the action is linear on each fiber. If $\mathcal{E}$ has rank r, the projectivization $\mathbb{P}(\mathcal{E})$ is a T-variety of complexity at most $r-1$ (if the bundle splits as a direct sum the complexity will be lower). In particular, projectivizations of rank two bundles are examples of complexity one T-varieties.

There is a different take on toric vector bundles which often proves to be fruitful. Through the work of Kaneyama [Kan75] and Klyachko [Kly89] there is a classification of toric vector bundles in terms of combinatorics and linear algebra. This classification has been used to study toric vector bundles from many different perspectives, for example in the work of Payne [Pay08] [Pay09], Hering-Mustata-Payne [HMP10], González-Hering-Payne-Süss [Gon+12] and Di Rocco-Jabbusch-Smith [DJS18].

We will use the classification theorem of toric vector bundles by Klyachko, which goes as follows. Fix a normal toric variety corresponding to a fan Σ and let $\mathcal{E}$ be a toric vector bundle of rank r on X_Σ. We denote by $E \simeq k^r$ the fiber over the identity of the torus. Then $\mathcal{E}$ corresponds to filtrations of E, one for every ray of Σ, satisfying a certain compatibility condition. The structure of these filtrations is equivalent to the fact that any toric vector bundle on an affine toric variety splits a sum of line bundles. Furthermore, Klyachko showed that there is an equivalence of categories: any equivariant map of toric vector bundles $\mathcal{E} \to \mathcal{F}$ corresponds to a linear map of vector spaces $E \to F$, respecting the filtrations.

The starting point for our work on toric vector bundles is the paper by Di Rocco, Jabbusch and Smith [DJS18], describing in greater detail a generating set for the global sections $H^0(X_\Sigma, \mathcal{E})$. For a line bundle $\mathcal{L}$ on a toric variety it is well-known that there is an associated polytope P such that

$$H^0(X_\Sigma, \mathcal{L}) \simeq \bigoplus_{u \in P \cap M} k\chi^u.$$

Similarly, to a toric vector bundle Di Rocco, Jabbusch and Smith associate a collection of polytopes, called the parliament of polytopes, whose lattice points correspond to generators of $H^0(X_\Sigma, \mathcal{E})$. In contrast to the line bundle case these generators are not necessarily linearly independent. The polytopes are indexed by an associated matroid $M(\mathcal{E})$, whose structure corresponds to the relations between the various global sections of $\mathcal{E}$.

Example 1.8.1. In Example 1.5.1 we introduced the projectivization of the tangent bundle on $\mathbb{P}^2$ as an example of a complexity one T-variety. Now we will describe it as a toric vector bundle.

$\mathbb{P}^2$ is described as a toric variety by the complete fan with rays with primitive lattice generators equal to $\rho_0 = (-1,-1), \rho_1 = (1,0), \rho_2 = (0,1)$. The filtrations

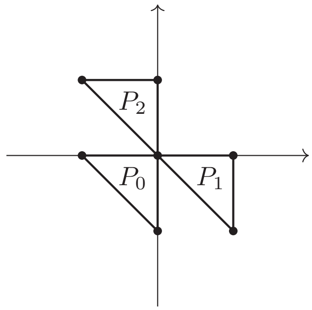

Figure 1.4: The 3 polytopes of the parliament of $T_{\mathbb{P}^2}$.

are given by [Kly89, Example 2.3.5]

$$E^{\rho_i}(j) = \begin{cases} E & \text{if } j \leq 0 \\ \rho_i & \text{if } j = 1 \\ 0 & \text{if } 1 < j \end{cases}.$$

The matroid $M(T_{\mathbb{P}^2})$ will in this case have ground set $\{\rho_0, \rho_1, \rho_2\}$. The polytopes giving the global sections can be seen in Figure 1.4.

We observe that there are 9 lattice points in the parliament, counted with multiplicity. We also see that the lattice point 0 is present in all three polytopes. The span of the vectors indexing the polytopes is only two-dimensional, which means that the three sections corresponding to 0 are not linearly independent. Therefore $H^0(\mathbb{P}^2, T_{\mathbb{P}^2})$ has dimension 8. Figure 1.4 is actually a way of visualizing the well-known Euler-sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2}(1) \to T_{\mathbb{P}^2} \to 0.$$

The goal of our work on toric vector bundles was to use the description of parliaments of polytopes to study positivity of line bundles on the projectivization $\mathbb{P}(\mathcal{E})$. For instance, if $\mathcal{E}$ is an ample vector bundle, which by definition means that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an ample line bundle on $\mathbb{P}(\mathcal{E})$, we may ask: For which k is the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) + K_{\mathbb{P}(\mathcal{E})}$ globally generated or very ample? Fujita conjectured that $k \geq \dim \mathbb{P}(\mathcal{E}) + 1$ is always sufficient for global generation and that $k \geq \dim \mathbb{P}(\mathcal{E}) + 2$ is sufficient for very ampleness. For bundles of rank two Fujita's conjecture on global generation was proved by Altmann and Ilten [AI17] using techniques on T-varieties. Unfortunately we were not able to achieve a proof (or counterexample) to Fujita's conjecture for toric vector bundles. However in the process of working on this problem, we were able to prove several auxiliary results on toric vector bundles, that we think are interesting in themselves.

We believe that fully understanding the positivity of toric vector bundles is a hard problem: It requires us to understand sections of symmetric powers of the bundles. This is as complicated as studying effective divisors on any iterated blow-up of projective space in linear spaces, which is well-known to be difficult.

1.8.1 Moduli of toric vector bundles

A moduli space is a space whose points correspond to geometric objects of a certain type; in other words a space parametrizing the isomorphism classes of certain geometric objects. Payne constructed a moduli space M_c of toric vector bundles with fixed equivariant Chern class c [Pay08]. Fixing the equivariant Chern class is equivalent to fixing the steps at which the Klyachko filtrations change as well as restricting how different subspaces in the filtrations can intersect. Using this, Payne showed that M_c is realized as a locally closed subscheme of a product of partial flag varieties. Moreover, he proved that if you consider the disjoint union of all moduli schemes of rank three toric vector bundles over all quasi-affine toric varieties, then this space will satisfy what Vakil called *Murphy's law* [Vak06]: any singularity type defined over $\mathbb{Z}$ arises on one of these spaces, which implies that these spaces can have all sorts of strange singularities.

1.9 Summary of the fourth paper

Paper IV In the fourth paper we prove that the moduli space of rank three toric vector bundles on smooth projective toric varieties satisfies Murphy's law. This answers a question posed by Payne [Pay08, Remark 4.4]. The proof is achieved by constructing a class of smooth projective toric varieties and rank 3 Chern classes on them, such that any incidence scheme between points and lines in $\mathbb{P}^2$ arises as some M_c. Then the result follows by Mnëv's universality theorem [Mnë88].

An implication of our result is that there exist toric vector bundles on smooth projective toric varieties which are definable in characteristic p, but which cannot be lifted to characteristic 0.

1.10 Cox rings

To a normal projective variety X we may associate the Cox ring $\mathrm{Cox}(X)$, also known as the total coordinate ring of X. This is the ring consisting of all sections of all line bundles on X, thus $\mathrm{Cox}(X)$ is the ring

$$\mathrm{Cox}(X) = \oplus_{\mathcal{L}\in \mathrm{Cl}(X)} H^0(X, \mathcal{L}),$$

where $\mathrm{Cl}(X)$ is the divisor class group. See [Arz+15] for more details. If X is $\mathbb{Q}$-factorial, then finite generation of the Cox ring is equivalent to being what Hu and Keel called a Mori Dream Space [HK00]. Being a Mori Dream Space has many implications for the birational geometry of X, for instance that the pseudoeffective cones of curves and divisors are rational polyhedral. Even simple examples may not be Mori Dream Spaces however, though both toric varieties and rational complexity one T-varieties are.

For a toric variety the Cox ring is as simple as it can be: It is a polynomial ring. This leads to the description of a toric variety alluded to earlier, as a quotient of an affine space modulo some action of a torus. The affine space is the spectrum of the Cox ring, and the torus is the torus $\mathcal{T} = \mathrm{Hom}(A_{n-1}(X), k^*)$. where $\dim X = n$. Thus, subvarieties of X are defined by the vanishing of polynomials in the Cox ring which are invariant under the action of $\mathcal{T}$, and coherent sheaves on X correspond to $\mathrm{Pic}(X)$-graded $\mathrm{Cox}(X)$-modules.

1.11 Summary of the fifth paper

Paper V In the fifth paper we present various results on positivity of toric vector bundles. We study the Cox ring of a projectivized toric vector bundle $\mathcal{E}$. We give a criterion for it to be finitely generated and derive a presentation for its Cox ring, both are related to the structure of the matroids $M(S^k\mathcal{E})$ of symmetric powers $S^k\mathcal{E}$. This is also related to the Cox ring of a certain iterated blow-up of projective space in linear spaces coming from the matroid. Many of the results on Cox rings were already known using techniques from the theory of T-varieties [Gon+12]. However, the upside to our proof is that it uses the Klyachko data directly, thus yielding a better understanding of the interplay between Klyachko-filtrations, the symmetric powers $S^k\mathcal{E}$ and their associated matroids $M(S^k\mathcal{E})$.

We provide a criterion for a toric vector bundle to be big, in terms of the parliament of symmetric powers of the bundle. We also give some other positivity results and examples of toric vector bundles with particular properties. A general theme to our results in this paper is that properties which behave nicely for line bundles, are significantly more complicated for vector bundles. In particular we provide counterexamples to several statements that one might suspect to be natural generalizations of results on toric line bundles.

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Papers

Paper I

Local Euler obstructions of toric varieties

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Abstract

We use Matsui and Takeuchi's formula for toric A-discriminants to give algorithms for computing local Euler obstructions and dual degrees of toric surfaces and 3-folds. In particular, we consider weighted projective spaces. As an application we give counterexamples to a conjecture by Matsui and Takeuchi. As another application we recover the well-known fact that the only defective normal toric surfaces are cones.

Contents

I.1 Introduction

The local Euler obstruction was used by MacPherson [Mac74] in his construction of Chern classes for singular varieties. For a variety X the local Euler obstruction is a constructible function $\mathrm{Eu} : X \to \mathbb{Z}$ which takes the value 1 at smooth points of X. It is related to the Chern–Mather class and to the Chern–Schwartz–MacPherson class of X (see Remark I.2.1).

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Several equivalent definitions of the local Euler obstruction have been given, such as Kashiwara's definition of the local characteristic for a germ of an irreducible analytic space [Kas73]. The first algebraic formula was given by González-Sprinberg and Verdier [Gon81]. Matsui and Takeuchi use a topological definition [MT11], which defines the local Euler obstruction of X inductively using the Whitney stratification of X. They use this definition to prove a formula for the local Euler obstruction on a (not necessarily normal) toric variety X. In this article we will apply this formula to compute the local Euler obstructions of toric varieties of dimension ≤ 3.

For a normal toric surface X, we have that X is smooth if and only if $\mathrm{Eu}(X) = \mathbb{1}_X$ [MT11, Cor 5.7]. [MT11] conjecture that the corresponding statement should also hold for a higher dimensional normal and projective toric variety. As an application we present counterexamples to this conjecture.

A motivation for studying Euler obstructions comes from formulas for the degrees of dual varieties. Given a projective variety variety $X \subset \mathbb{P}^N$, its dual variety $X^\vee \subset \mathbb{P}^{N^\vee}$ is the closure of the set of hyperplanes $H \in \mathbb{P}^{N^\vee}$ such that there exists a smooth point $x \in X$ with $T_x X \subset H$. Generally $X^\vee$ will be a hypersurface in $\mathbb{P}^{N^\vee}$. Finding its equation is usually very difficult, but there are results which give the degree. Gelfand, Kapranov and Zelevinsky [GKZ94] proved a combinatorial formula for the degree of the dual variety of an embedded smooth toric variety. Matsui and Takeuchi generalized this formula to singular toric varieties, by weighting the terms by the local Euler obstruction. We will use this to describe algorithms to compute the degree of the dual variety of some toric varieties, in particular weighted projective spaces of dimension ≤ 3.

There has been recent interest in the local Euler obstruction. Aluffi studied Chern–Mather and Chern–Schwartz–MacPherson classes in [Alu16]. Helmer and Sturmfels studied polar degrees and the local Euler obstruction in [HS18] related to the problem of finding the Euclidean distance degree of a variety. This problem is closely related to the contents of the current paper, since the Euclidean distance degree is expressible in terms of polar degrees, which in turn is expressible in terms of Matsui and Takeuchi's formulas involving the local Euler obstruction. In particular Helmer and Sturmfels study codimension one toric varieties [HS18, Thm 3.7], and also they briefly study surfaces.

In Section 2 we define the local Euler obstruction. We recall some basic facts about toric varieties.

In Section 3 we present Matsui and Takeuchi's method for computing the local Euler obstruction of toric varieties and the degree of dual varieties.

In Section 4 we introduce our main examples of study, the weighted projective spaces. We describe them via toric geometry.

In Section 5 we follow Chapter 5 of [Mor11] and apply the theory to toric surfaces. This relates to Hirzebruch–Jung continued fractions and the minimal resolution of singularities. We then do explicit computations for weighted projective planes.

In Section 6 we consider the local Euler obstruction of toric 3-folds. We prove that for a toric 3-fold $X_{P\cap M}$ with isolated singularities, the local Euler obstruction is always greater than or equal to 1. We find counterexamples to a conjecture by Matsui and Takeuchi [MT11, p.2063].

In Section 7 we apply the above to describe which toric surfaces are dual defective, and to say something about which 3-dimensional weighted projective spaces are dual defective.

In the appendix we collect some computations of the local Euler obstruction and degrees of dual varieties for some weighted projective spaces.

I.2 The local Euler obstruction

Given a complex projective variety X of dimension d, consider the (generalized) Grassmann variety $\mathrm{Grass}_d(\Omega^1_X)$ representing locally free rank d quotients of Ω^1_X. The Nash blowup $\tilde{X}$ of X is the closure of the image of the morphism $X_{\mathrm{sm}} \to \mathrm{Grass}_d(\Omega^1_X)$, sending a smooth point to its tangent space. Let $\pi\colon \tilde{X} \to X$ denote the projection. The Nash sheaf $\tilde{\Omega}$ is the restriction to $\tilde{X}$ of the tautological rank d sheaf on $\mathrm{Grass}_d(\Omega^1_X)$. There is a surjection $\pi^*\Omega^1_X \to \tilde{\Omega}$, and the Nash blowup is universal with respect to birational morphisms $f : Y \to X$ such that there is a locally free sheaf $\mathcal{F}$ of rank d on Y and a surjection $f^*\Omega^1_X \to \mathcal{F}$. Let $\tilde{T}$ denote the dual of $\tilde{\Omega}$.

The local Euler obstruction of a point $x \in X$ is the integer

$$\mathrm{Eu}(x) = \int_{\pi^{-1}(x)} c(\tilde{T}|_{\pi^{-1}(x)}) \cap s(\pi^{-1}(x), \tilde{X}).$$

On the smooth locus of a variety the local Euler obstruction takes the value 1. It is a local invariant, thus we can compute it on an open affine cover.

This is the usual algebraic definition, used by amongst others [Ful98, Ex. 4.2.9]. When the ambient variety is clear we will simply write Eu for the local Euler obstruction, however if there are different ambient varieties we sometimes write Eu_X for the local Euler obstruction on X.

Remark I.2.1. The Chern–Mather class $c^M(X)$ of a variety X is defined by

$$c^M(X) = \pi_*(c(\tilde{T}) \cap [\tilde{X}]).$$

There is an isomorphism T from cycles on X to constructible functions on X given by $\sum n_i[V_i](p) \mapsto \sum n_i \,\mathrm{Eu}_{V_i}(p)$. Letting c_* be $c^M \circ T^{-1}$, we have that c_* is the unique natural transformation from constructible functions on X to the homology of X such that on a non-singular X we have that $c^{SM}(X)$ is the Poincare dual of the total Chern class of X [Mac74, Thm 1]. The Chern-Schwartz-MacPherson class $c^{SM}(X)$ is defined as $c_*(\mathbb{1}_X)$.

I.2.1 Definitions and notation for toric varieties

We shall use the notation and definitions from [CLS11] for toric varieties. Let T be the torus $(\mathbb{C}^*)^n$ and let M denote its character lattice $\mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$. The dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of M we denote by N. Any normal toric variety is of the form X_Σ for a fan $\Sigma \subset N_{\mathbb{R}}$, and has the open affine cover $\{U_\sigma | \sigma \in \Sigma\}$.

We will sometimes be interested in toric varieties which are not normal: For a finite set of lattice points $A \subset M$, we can associate the toric variety $X_A \subset \mathbb{P}^{\#A-1}$ by mapping the torus via the characters corresponding to the lattice points in A and taking the Zariski closure. These varieties are not necessarily normal.

For a subset $S \subset M_{\mathbb{R}}$ we denote by $\mathrm{Conv}(S)$ the convex hull of the points of S. Setting $P = \mathrm{Conv}(A)$, we get a (possibly different) embedding $X_{P\cap M} \subset \mathbb{P}^{\#P\cap M-1}$. The variety X_A is also the image of the projection from $\mathbb{P}^{\#P\cap M-1}$ to $\mathbb{P}^{\#A-1}$ given by forgetting the coordinates corresponding to $P \cap M \setminus A$.

To a lattice polytope P we can also associate the normal toric variety X_P which equals $X_{kP\cap M}$ for any $k \in \mathbb{N}$ such that kP is very ample (this is equivalent to a certain divisor on $X_{kP\cap M}$ being very ample), thus it is independent of any specific embedding and not necessarily isomorphic to $X_{P\cap M}$. We have that X_P is isomorphic to X_{Σ_P}, the toric variety associated to the normal fan $\Sigma_P \subset N_{\mathbb{R}}$ of P.

If the polytope P is itself very ample, we will sometimes, by abuse of notation, identify the abstract variety X_P with the embedded variety $X_{P\cap M}$. For instance all 2-dimensional polytopes are very ample.

I.3 The local Euler obstruction of toric varieties

Consider a toric variety X_A associated to a finite set A in $M \simeq \mathbb{Z}^n$. We will use the formula for the local Euler obstruction of toric varieties proved in [MT11, Ch. 4]. It is proved using an equivalent topological definition of the Euler obstruction, defined by induction on the codimension of the strata of a Whitney stratification of the variety.

Remark I.3.1. One can quite explicitly describe both the Nash blowup of a toric variety, and its normalization as a toric variety, see [Ata+11],[GT14]. It would be interesting to prove Matsui and Takeuchi's formula for the local Euler obstruction directly from the algebraic definition, for instance if one could describe the Nash sheaf as a module over the Cox ring of X_A.

Let P be the convex hull of A. We may assume this has dimension n. Then P is a lattice polytope in M. For a toric variety X_A there is a one-to-one correspondence between faces of P and orbits of X_A by [GKZ94, Prop. 1.9]. The local Euler obstruction is constant on each orbit, hence for a face $\Delta \preceq P$ we can denote by $\mathrm{Eu}(\Delta)$ the common value of the local Euler obstruction on the orbit corresponding to Δ. Matsui and Takeuchi describe the Euler obstruction combinatorially by induction on the codimension of the faces of P.

For a face Δ of P, let $L(\Delta)$ be the smallest affine subspace in $M_{\mathbb{R}}$ containing Δ. The dimension of $L(\Delta)$ is equal to $\dim \Delta$. We can also associate a lattice to Δ: M_Δ is the lattice generated by $A \cap \Delta$ in $L(\Delta)$.

Given faces Δ_α and Δ_β of P such that $\Delta_\beta \preceq \Delta_\alpha$, we can associate a lattice $M_{\alpha,\beta} := M_\alpha \cap L(\Delta_\beta)$. We have that $M_\beta \subseteq M_{\alpha,\beta}$, but they are not necessarily equal (see Examples I.3.6 and I.3.7). They are however both of maximal rank in $L(\Delta_\beta)$ which motivates the following definition:

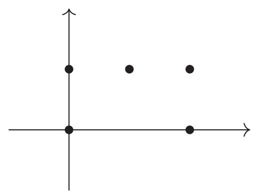

Figure I.1: The set A from Example I.3.6.

Definition I.3.2. Given faces Δ_α and Δ_β of P such that $\Delta_\beta \preceq \Delta_\alpha$, we define the index $i(\Delta_\alpha, \Delta_\beta)$ to be $[M_{\alpha,\beta} : M_\beta]$.

For two faces Δ_α and Δ_β of P such that $\Delta_\beta \preceq \Delta_\alpha$ we may, after a translation, assume that 0 is a vertex of Δ_β. We denote by S_α the semigroup generated by $A \cap \Delta_\alpha$ in M_α. Let $S_{\alpha,\beta}$ denote the image of S_α in the quotient lattice $M_\alpha/M_{\alpha,\beta}$.

Definition I.3.3. Given faces Δ_α and Δ_β of P such that $\Delta_\beta \preceq \Delta_\alpha$, we define the normalized relative subdiagram volume $\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha \Delta_\beta)$ of Δ_α along Δ_β by

$$\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) = \mathrm{Vol}(S_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}),$$

where $\Theta_{\alpha,\beta}$ is the convex hull of $S_{\alpha,\beta} \cap M_\alpha/M_{\alpha,\beta} \setminus \{0\}$ in $(M_\alpha/M_{\alpha,\beta})_{\mathbb{R}}$. The volume is normalized with respect to the $(\dim \Delta_\alpha - \dim \Delta_\beta)$-dimensional lattice $M_\alpha/M_{\alpha,\beta}$. If $\Delta_\alpha = \Delta_\beta$ we set $\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) = 1$.

Corollary I.3.4. *[MT11, Thm 4.7] The local Euler obstruction of X_A is described as follows: The value* $\mathrm{Eu}(\Delta_\beta)$ *for a face Δ_β of P is determined by induction on the codimension of the faces of P by the following:*

$$\mathrm{Eu}(P) = 1,$$

$$\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \precneqq \Delta_\alpha} (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} i(\Delta_\alpha, \Delta_\beta)\, \mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta)\, \mathrm{Eu}(\Delta_\alpha).$$

Remark I.3.5. By [Pie16, Th. 2] the value of the local Euler obstruction at a torus orbit is the coefficient of the orbit in the the Chern–Mather class of X, i.e. $c^M(X_A) = \sum_{\Delta \preceq \mathrm{Conv}(A)} \mathrm{Eu}(\Delta)[\Delta]$.

Example I.3.6. Let A be the following lattice points in $M \simeq \mathbb{Z}^2$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Let e be the edge of $P = \mathrm{Conv}(A)$ generated by the vector $(1,0)$. Then $M_{P,e}$ is the lattice $\mathbb{Z}(1,0)$. However since $(1,0) \notin A$ we have that the lattice $M_e = \mathbb{Z}(2,0)$. Thus the index $i(P,e) = 2$.

In the example above we do not have that A equals $M \cap \mathrm{Conv}(A)$. One might suspect that this is the only way to get a nontrivial index, however the following example show this to be wrong.

Example I.3.7. Let Q be the 3-dimensional polytope in $M \simeq \mathbb{Z}^4$ with lattice points

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\2\\0\end{bmatrix}.$$

Assume Q is the facet of a 4-dimensional polytope P cut out by setting the last coordinate equal to 0. Furthermore assume that P has enough lattice points so that $\mathbb{Z}P = M$. Then $M_{P,Q} = \mathbb{Z}^3$, but $M_Q = \mathbb{Z}^2 \oplus 2\mathbb{Z}$, so $i(P,Q) = 2$.

Next we state Matsui and Takeuchi's formula for the degree of the dual variety of a toric variety.

Proposition I.3.8. *[MT11, Cor 1.6] Assume A is a finite subset of $M \simeq \mathbb{Z}^n$ such that $X_A^\vee$ is a hypersurface in $\mathbb{P}^{N^\vee}$. Setting $P = \mathrm{Conv}(A)$, we have*

$$\deg X_A^\vee = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Eu}(Q) \operatorname{Vol}(Q).$$

where $\operatorname{Eu}(Q)$ *is the (constant) value of the local Euler obstruction on the torus orbit associated to Q, and* $\operatorname{Vol}(Q)$ *is the normalized volume of Q with respect to the sublattice spanned by lattice points of Q.*

Remark I.3.9. By [MT11, Thm 1.4] $X_A^\vee$ is a hypersurface if and only if the formula above yields a non-zero number.

If we take A to be all lattice points of a polytope, we can simplify some calculations:

Lemma I.3.10. *Assume $A = \mathrm{Conv}(A) \cap M$. If $\dim \Delta_\alpha - \dim \Delta_\beta = 1$ then* $\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) = 1$.

Proof. This follows almost by construction: The quotient lattice $M_\alpha / M_{\alpha,\beta}$ will be isomorphic to $\mathbb{Z}$. Then S_α / Δ_β must be generated by either 1 or -1, thus it follows $\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) = 1$. ■

Corollary I.3.11. *Assume $A = \mathrm{Conv}(A) \cap M$. For any $(n-1)$-dimensional face $\Delta \subsetneq P$ we have* $\operatorname{Eu}(\Delta) = i(P, \Delta)$.

We need the following well-known fact: Given set of linearly independent vectors $b_1, ..., b_n \in M$ let

$$T(b_1, ..., b_n) = \{\sum_{i=1}^{n} c_i b_i | 0 \leq c_i < 1\} \subseteq M_{\mathbb{R}} = M \otimes \mathbb{R}.$$

Lemma I.3.12. *The vectors $b_1, ..., b_n$ form a basis for the lattice M if and only if $T(b_1, ..., b_n) \cap M = \{0\}$.*

The following lemma will be useful when we study surfaces and 3-folds:

Lemma I.3.13. *If A is the set of lattice points of a convex lattice polytope of dimension ≤ 3, then for any two faces $\Delta_\beta \preceq \Delta_\alpha \preceq P$ we have $i(\Delta_\alpha, \Delta_\beta) = 1$.*

Proof. Let $d = \dim \Delta_\beta$. We check each value of d separately. We need to check that $M_{\alpha,\beta} \subset M_\beta$. We will do this by showing that $M_{P,\beta} \subset M_\beta$. Again we fix 0 as a common vertex of Δ_β and Δ_α.

If $d = 0$ there is nothing to prove.

If $d = 1$ pick the first lattice point along the ray generated by Δ_β, starting at 0. By construction of A this necessarily generates all lattice points of M which are contained in $L(\Delta_\beta)$.

If $d = 2$ we do something similar: Pick a pair of primitive lattice points $v, w \in \Delta_\beta$ such that the only lattice points of M contained in the set $R_{v,w} = \{av + bw | 0 \leq a, b \leq 1, a + b \leq 1\}$ are $0, v, w$. We claim this can always be done.

Indeed, pick any primitive v', w'. Then $R_{v',w'}$ contains finitely many lattice points. If there exists $u \in R_{v',w'}, u \neq 0, v', w'$, we may without loss of generality assume u is primitive, and consider $R_{u,v'}$ which have fewer lattice points. Iterating this proves the claim.

Now we claim that v, w is a basis for $M \cap L(\Delta_\beta)$. If not, then by Lemma I.3.12 there is a lattice point $p \in M$ such that $p = av + bw$ with $0 \leq a, b < 1$. By assumption $a + b > 1$. But then $v + w - p = v(1-a) + w(1-b)$ is a lattice point in $R_{v,w}$ different from $0, v, w$ which is a contradiction. ∎

Assuming the polytope P is very ample, we have that $X_{P\cap M} \simeq X_{\Sigma_P}$. In this case it will be convenient to be able to compute the local Euler obstruction using the language of fans (for instance when we relate it to the resolution of singularities for surfaces), so we describe how this is done.

We have the identification, for a vertex v of P, of $C_v = \text{Cone}(P \cap M - v)$ with a cone $\sigma^\vee \subset M_\mathbb{R}$ dual to a maximal cone σ in the normal fan Σ_P. This is compatible with face inclusions: If Δ_α is a face of P containing v, there is a corresponding face τ_α of σ. We then have $\text{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) = \text{RSV}_\mathbb{Z}(\tau_\alpha^\vee, \tau_\beta^\vee)$ where the last expression means:

Let $M'_\beta = M_\mathbb{R}/L(\tau_\beta^\vee)$ and let $K_{\alpha,\beta}$ be the image of $\tau_\alpha^\vee$ in M'_β. Then $\text{RSV}(\tau_\alpha^\vee, \tau_\beta^\vee)$ equals $\text{Vol}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta})$ where $\Theta_{\alpha,\beta}$ is $\text{Conv}(K_{\alpha,\beta} \cap M'_\beta \setminus \{0\})$, and the volume is normalized with respect to the lattice $M'_\beta \cap L(K_{\alpha,\beta})$.

I.4 Weighted projective spaces

Our main examples in this paper are the weighted projective spaces(wps), which are defined as follows:

Let $q_0, ..., q_n \in \mathbb{N}$ satisfy $\gcd(q_0, ..., q_n) = 1$. Define $\mathbb{P}(q_0, ..., q_n) = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $\sim$ is the equivalence relation:

$$(a_0, ...a_n) \sim (b_0, ..., b_n) \Leftrightarrow a_i = \lambda^{q_i} b_i \text{ for all i, for some } \lambda \in \mathbb{C}^*.$$

We call $\mathbb{P}(q_0, ..., q_n)$ the wps corresponding to $q_0, ..., q_n$. Observe that $\mathbb{P}(1, ..., 1) \simeq \mathbb{P}^n$. We can construct a wps as a toric variety by the following:

Given natural numbers $q_0, ..., q_n$ with $\gcd(q_0, ..., q_n) = 1$, consider the quotient lattice $\mathbb{Z}^{n+1}$ by the subgroup generated by $(q_0, ..., q_n)$, and write $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, ..., q_n)$. Let u_i for $i = 0, ..., n$ be the images in N of the standard basis vectors of $\mathbb{Z}^{n+1}$. This means that in N we have the relation

$$q_0 u_0 + ... + q_n u_n = 0.$$

Let Σ be the fan consisting of all cones generated by proper subsets of $\{u_0, ..., u_n\}$. Then $X_\Sigma = \mathbb{P}(q_0, ..., q_n)$. By the quotient construction of toric varieties one gets by [CLS11, Example 5.1.14] that X_Σ is a geometric quotient, whose points agree with the set-theoretic definition given above.

From [RT12] we can also describe the wps as embedded in projective space via a polytope P giving $\mathbb{P}(q_0, ..., q_n) \simeq X_P$:

Given $(q_0, ..., q_n)$ and $M \cong \mathbb{Z}^{n+1}$, let $\delta = \operatorname{lcm}(q_0, ..., q_n)$. Consider the $n+1$ points of $M_\mathbb{R} \cong \mathbb{R}^{n+1}$:

$$v_i = (0, ..., \frac{\delta}{q_i}, ...0), i = 0, ..., n.$$

Let Δ be the convex hull of 0 and all v_i. Intersecting Δ with the hyperplane $H = \{(x_0, ..., x_n) | \sum_{i=0}^n x_i q_i = \delta\}$, we get a n-dimensional polytope P. Then $X_P \cong \mathbb{P}(q_0, ..., q_n)$ and the associated divisor D_P will be $\frac{\delta}{q_0} D_0$. This divisor is very ample and its class generates $\operatorname{Pic}(\mathbb{P}(q_0, ..., q_n)) \simeq \mathbb{Z}$. When we speak of the degree of the dual variety of a weighted projective space, we will always mean using the embedding given by D_P.

There are characterizations of when $\mathbb{P}(q_0, ..., q_n) \simeq \mathbb{P}(s_0, ..., s_n)$ in terms of the weights, see for instance [RT12]. The upshot is that we can assume the weights are reduced, i.e., that for all i $\gcd(q_0, ..., \overline{q_i}, ...q_n) = 1$. We will always make this assumption.

Following [Ian00, p. 5.15] we can describe the singular locus of the wps: Recall that the fan Σ is the collection of cones $\operatorname{Cone}(u_j | j \in J)$ for all proper subsets $J \subset \{0, ..., n\}$. Set $\sigma_{j_1,...,j_k} = \operatorname{Cone}(u_{j_1}, ..., u_{j_k})$. Fixing one such cone $\sigma_{j_1,...,j_k}$, let $I = \{i_0, ..., i_{n-k}\} = \{0, ..., n\} \setminus \{j_1, ..., j_k\}$. Then we have:

Proposition I.4.1. *[Nød15, Prop 2.1.7]* $\mathbb{P}(q_0, ..., q_n)$ *is nonsingular in codimension k if for all $\{j_1, ..., j_k\}$, the corresponding* $\gcd(q_{i_0}, ..., q_{i_{n-k}}) = 1$. *In particular:*

$\mathbb{P}(q_0, ..., q_n)$ *is nonsingular in codimension* 1 .

$\mathbb{P}(q_0, ..., q_n)$ *has isolated singularities if and only if* $\gcd(q_i, q_j) = 1$ *for all i, j.*

Thus for surfaces we will always have isolated singularities, but in larger dimensions we might have larger singular locus, for instance $\mathbb{P}(2, 2, 3, 3)$ does not have isolated singularities.

I.5 The surface case

In this section we will let A consist of all lattice points of a 2-dimensional lattice polytope P. Recall that then we have $X_A = X_{P \cap M} \simeq X_P \simeq X_{\Sigma_P}$ and X_A is normal. From Proposition I.3.8 we have

$$\deg X_P^\vee = 3\operatorname{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \operatorname{Eu}(v),$$

where $E(P)$ is the sum of the normalized lengths of the edges of P. Thus we need to compute the Euler obstruction of the singular vertices. By Lemma I.3.13 all indices $i(\Delta_\alpha, \Delta_\beta)$ are equal to 1.

By Corollary I.3.4 we get for a vertex v, letting e_1, e_2 be the edges of P containing v:

$$\operatorname{Eu}(v) = \operatorname{RSV}_{\mathbb{Z}}(e_1, v)\operatorname{Eu}(e_1) + \operatorname{RSV}_{\mathbb{Z}}(e_2, v)\operatorname{Eu}(e_2) - \operatorname{RSV}_{\mathbb{Z}}(P, v),$$

By Lemma I.3.10, $\operatorname{RSV}_{\mathbb{Z}}(P, e_i) = 1$ and $\operatorname{RSV}_{\mathbb{Z}}(e_i, v) = 1$, while by Corollary I.3.11, $\operatorname{Eu}(e_i) = 1$, for $i = 1, 2$. Thus we reduce calculations to:

$$\operatorname{Eu}(v) = 2 - \operatorname{RSV}_{\mathbb{Z}}(P, v).$$

To calculate $\operatorname{RSV}_{\mathbb{Z}}(P, v)$ we get that $M_P/M_{P,v}$ will equal M. Hence $S_{P,v}$ will be the semigroup generated by the lattice points of the polytope P, after translating P such that v is the origin. Then $\operatorname{RSV}_{\mathbb{Z}}(P, v)$ will be the area removed, if we instead of P consider the convex hull of the points of $(P \setminus \{v\}) \cap M$.

Lemma I.5.1. *For a 2-dimensional lattice polytope P and a vertex v we have*

$$\operatorname{Eu}(v) = 1 - c,$$

where c is the number of internal lattice points of P which are boundary points of $\operatorname{Conv}((P \setminus v) \cap M)$.

Proof. By the above discussion

$$\operatorname{Eu}(v) = 2 - \operatorname{Vol}(P) + \operatorname{Vol}(\operatorname{Conv}((P \setminus v) \cap M)).$$

(This formula is also found in [HS18, Corollary 3.2], [Mor11, Proposition 5.2.12],[Nød15, Proposition 1.11.7].) Let i be the number of interior lattice points of P and b be the number of boundary lattice points. By Pick's formula

$$\operatorname{Vol}(P) = 2i + b - 2,$$

$$\operatorname{Vol}(\operatorname{Conv}((P \setminus v) \cap M)) = 2(i - c) + (b + c - 1) - 2,$$

hence

$$\operatorname{Eu}(v) = 2 - (1 + c) = 1 - c.$$

■

One can also describe the Euler obstruction in terms of a resolution of singularities:

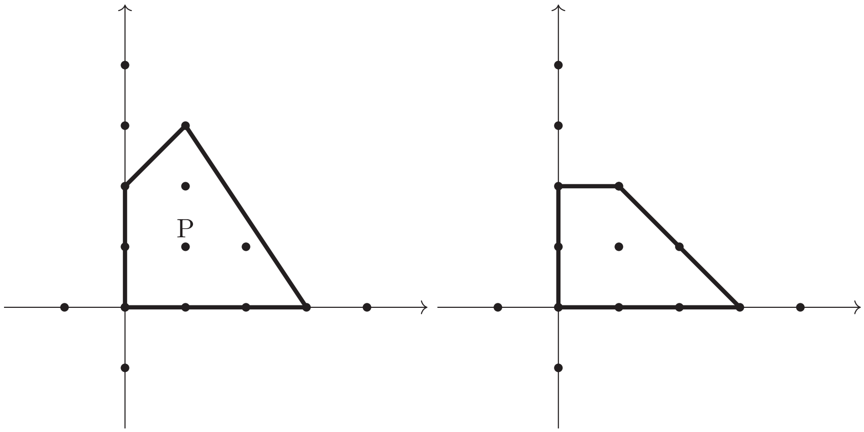

Figure I.2: The polytope $P = \mathrm{Conv}((0,0),(0,2),(1,3),(3,0))$. Removing the vertex $(1,3)$ we get the right figure. $\mathrm{Vol}(P) = 11$ while the volume of the new polytope is 8. Hence $\mathrm{Eu}(1,3) = 2 - 11 + 8 = -1$.

Proposition I.5.2. *[Gon82] Let $p \in S$ be a normal cyclic surface singularity, and $X \to S$ a minimal resolution of the singularity p with exceptional curves E_i. Then*

$$\mathrm{Eu}(p) = \sum_i (2 + E_i \cdot E_i).$$

We will relate these two descriptions of the Euler obstruction.

One can describe resolutions of singularities for toric varieties in general (see for instance [CLS11, Ch. 11.1]), and for surfaces the minimal resolution can be made quite explicit (we follow descriptions in [Pop07] and [Dai06]). Now we switch to the language of fans.

Given a rational number λ, we can consider the Hirzebruch–Jung (HJ) continued fraction

$$\lambda = b_1 - \cfrac{1}{b_2 - \frac{1}{\ldots - \frac{1}{b_r}}},$$

which we will denote by $[b_1, \ldots, b_r]^-$.

Since this is a local computation, we do this cone by cone, so we assume σ is a 2-dimensional cone. We include the proof of the following result, which is well-known, because it shows how to construct the integers k and d:

Lemma I.5.3. *Given any singular 2-dimensional cone σ, one can choose a basis $\{e_1, e_2\}$ for the lattice L such that in this basis $\sigma = Cone(e_1, ke_1 + de_2)$, where $d > k > 0$ and $\gcd(d,k) = 1$.*

Proof. We can always choose a primitive generator u of an edge of σ as the first basis vector of our lattice. Let $(e_1 = u, e_2')$ be a basis for the lattice. The other facet of the cone will in this basis be generated by a vector $w = ae_1 + be_2'$. Now let $d = |b|$ and $k = a \mod d$, where $0 < k < d$.

Then $w = (a-k+k)e_1 + \text{sign}(b)de_2' = ke_1 + d(\text{sign}(b)e_2' + \frac{a-k}{d}e_1)$. Thus we see that in the new basis $\{e_1, e_2 = \text{sign}(b)e_2' + \frac{a-k}{d}e_1\}$, $w = ke_1 + de_2$. ■

Definition I.5.4. We say that a cone σ is of type (d,k) if it can be written as in Proposition I.5.3 with parameters d, k.

Note also that some literature, notably [CLS11] and [Ful93], use a different convention for a (d,k)-cone, so that some results sometimes appear a bit different.

Lemma I.5.5. *[Dai06, Lemma 3.3] Assume $\sigma^\vee$ is a (d,k)-cone in $M_\mathbb{R}$ with respect to $\{e_1, e_2\}$. Then σ is a $(d, d-k)$-cone in $N_\mathbb{R}$ with respect to the basis $\{e_2^*, e_1^* - e_2^*\}$.*

Construction I.5.6. *[Pop07, Section 4] Set $K(\sigma) = \text{Conv}(\sigma \cap (N \setminus \{0\}))$. Let $P(\sigma)$ be the boundary of $K(\sigma)$ and $V(\sigma)$ the set of vertices. $P(\sigma)$ is a connected polygonal line with endpoints coinciding with the generators of σ.*

Let the primitive generators of σ be v_1, v_2. Let $A_0 = v_1$. Define A_i, $i \geq 0$ as the sequence of lattice points as one goes along the compact edges of $P(\sigma)$. This is a finite sequence and the last point v_2 is denoted by A_{r+1}.

By construction each pair (A_i, A_{i+1}) is a basis for N, since the triangle formed by $0, A_i, A_{i+1}$ has no other lattice points. Also the slopes of the set $\{A_i\}$ have to increase with increasing i, since A_i are on the boundary of a convex set. Thus we have relations:

$$rA_{i-1} + sA_i = A_{i+1},$$

$$tA_i + uA_{i+1} = A_{i-1},$$

where $r, s, t, u \in \mathbb{Z}$. This implies

$$(rt + s)A_i + (ru - 1)A_{i+1} = 0,$$

$$rt + s = 0, \; ru = 1.$$

If $r = u = 1$ we get $s = -t$ and

$$sA_i + A_{i-1} = A_{i+1}.$$

But this contradicts the increasing of the slopes. Thus we must have $r = u = -1$ and $s = t$, resulting in the relation

$$A_{i-1} + A_{i+1} = b_i A_i.$$

By convexity we must have $b_i \geq 2$.

Proposition I.5.7. *[Pop07, Prop. 4.3] By Construction I.5.6 for a (d,k)-cone σ, we get that $[b_1 ..., b_r]^- = \frac{d}{d-k}$.*

Example I.5.8. In Figure I.3 we see Construction I.5.6 for $(d,k) = (8,3)$. The lattice points A_i are the following:

$$A_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

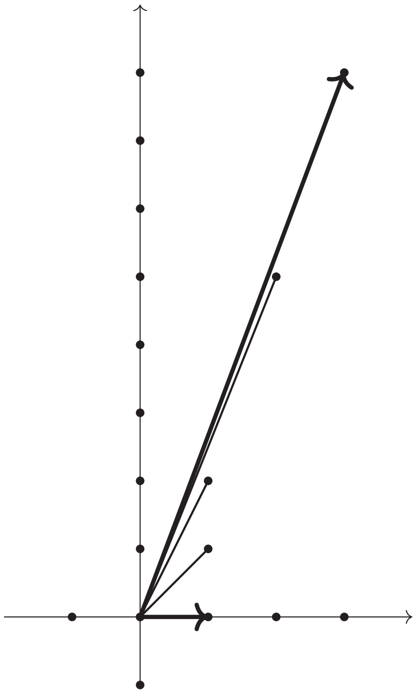

Figure I.3: Construction I.5.6 for (d,k)=(8,3).

The continued fraction $\frac{d}{d-k} = \frac{8}{5}$ equals $[2, 3, 2]^{-1}$. By Proposition I.5.7 this is equivalent to the fact that

$$A_0 + A_2 = 2A_1, \quad A_1 + A_3 = 3A_2, \quad A_2 + A_4 = 2A_3.$$

Given σ, construct the points A_i as in Construction I.5.6. Let $\sigma_i = \mathrm{Cone}(OA_i)$. Let Σ be the fan with 2-dimensional cones $\mathrm{Cone}(\sigma_i, \sigma_{i-1})$ for $i = 0, ..., r$. The identity map on the lattice N induces toric morphisms $U_{\sigma_i} \to U_\sigma$ which glue to a morphism $\phi : X_\Sigma \to U_\sigma$.

Proposition I.5.9. *[Dai06, Thm. 3.20] The morphism ϕ is a minimal resolution of singularities for U_σ with r exceptional components $E_1, ..., E_r$ and $E_i^2 = -b_i$.*

By doing this cone by cone, one obtains a global resolution of singularities by gluing the local constructions. Combining Proposition I.5.9 and Proposition I.5.2 we obtain:

Corollary I.5.10. *Given a (d, k)-cone in $M_\mathbb{R}$ (equivalently a $(d, d-k)$-cone in $N_\mathbb{R}$), let v be the torus fixed point of U_σ. Write*

$$\frac{d}{k} = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_r}}}.$$

Then $\mathrm{Eu}(v) = \sum_{i=1}^r (2 - b_i)$.

We give our own proof of this in the toric case using the formula of Matsui and Takeuchi, without referring to Proposition I.5.2. We need a technical lemma:

Lemma I.5.11. *[Oda88, Lemma 1.22] Let $\frac{d}{k} = [b_1, ..., b_r]^-$ and $\frac{d}{d-k} = [c_1, ..., c_s]^-$. Then*

$$s = 1 + \sum_{i=1}^{r} (b_i - 2).$$

Proof of Corollary I.5.10. Given any normal toric surface, consider a vertex v. We have that $\mathrm{Eu}(v) = 1 - c$ where c is the number of internal lattice points of $\sigma^\vee$ which are boundary points of $\mathrm{Conv}((\sigma^\vee \setminus \{0\}) \cap M)$. Writing $\frac{d}{d-k} = [c_1, ..., c_s]^-$ we have by Construction I.5.6 and Proposition I.5.7 that $c = s$. By Lemma I.5.11 $\mathrm{Eu}(v) = \sum_{i=1}^{r} (2 - b_i)$. ■

Remark I.5.12. If the cone is smooth, it is isomorphic to $\mathrm{Cone}(e_1, e_2)$, if we by convention set the corresponding continued fraction equal to $[1]^-$, then all formulas for the Euler-obstructions are true also for smooth cones.

Combining the above we obtain:

Proposition I.5.13. *Assume P is a 2-dimensional lattice polytope. Construct the minimal resolution of singularities of X_{Σ_P} and let $E_{v,i}$ be the exceptional divisors for the singularities v. Let*

$$\delta = 3\,\mathrm{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \sum_i (2 + E_{v,i}^2).$$

Then $X_P^\vee$ is a hypersurface if and only if δ is non–zero. Assuming $X_P^\vee$ is a hypersurface, it has degree δ.

More explicitly, let $\sigma_1, ..., \sigma_r$ be the maximal cones of Σ_P. Assume σ_i is a $(d_i, d_i - k_i)$-cone and write $\frac{d_i}{k_i} = [b_{i,1}, ..., b_{i,s_i}]^-$. Then

$$\delta = 3\,\mathrm{Vol}(P) - 2E(P) + \sum_{i=1}^{r} \sum_{j=1}^{s_i} (2 - b_{i,j}).$$

We can classify which normal toric surfaces are smooth or Gorenstein using the Euler obstruction.

Corollary I.5.14. *[MT11, Cor. 5.7] For any point v in a normal toric surface we have that v is smooth if and only if* $\mathrm{Eu}(v) = 1$.

Proof. This follows directly from Corollary I.5.10 and the fact that the b_i in Construction I.5.6 are always ≥ 2. ■

Remark I.5.15. Let A be the lattice points from Example I.3.6

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Let v denote the origin and e_1, e_2 the edges of $\mathrm{Conv}(A)$ containing v. Then we have that $\mathrm{Eu}(v) = i(P, e_1) + i(P, e_2) - \mathrm{RSV}_{\mathbb{Z}}(P, v) = 2 + 1 - 2 = 1$, even if v corresponds to a singular point of the non-normal variety X_A.

Corollary I.5.16. *A singular point on a normal toric surface has Euler-obstruction* 0 *if and only if the surface is Gorenstein in a neighbourhood of the point.*

Proof. By [CLS11, Exc. 8.2.13] a singular affine toric surface U_σ is Gorenstein if and only if σ is a $(d, 1)$-cone.

Let the singularity be given as a (d, k)-cone in $N_{\mathbb{R}}$. Let $\frac{d}{d-k} = [b_1, ..., b_r]$. By Corollary I.5.10 the Euler-obstruction is 0 if and only if all $b_i = 2$. Now if the singularity is Gorenstein, then $k = 1$, so $\frac{d}{d-k} = \frac{d}{d-1}$. It is easy to check that the HJ-fraction of $\frac{d}{d-1}$ is a chain of $d - 1$ 2's.

Conversely if the singularity has Euler-obstruction 0, then all b_i's are 2, but by the above this implies that in $M_{\mathbb{R}}$ it is a $(d, d-1)$-cone, so it is a $(d, 1)$-cone in $N_{\mathbb{R}}$. ∎

Remark I.5.17. For a surface X the degree of the dual variety given by an embedding by the very ample line bundle L equals the Severi degree $N^{L,1}$. For a smooth surface one has from [KP99]

$$N^{L,1} = 3L^2 + 2L \cdot K_X + c_2(X),$$

however in the singular case this does not hold. Now fix a toric surface X_P. Using Ehrhart theory and Riemann-Roch [CLS11, Prop. 10.5.6] we obtain that

$$D_P \cdot D_P = \mathrm{Vol}(P),$$

$$-D_P \cdot K_{X_{\Sigma_P}} = E(P).$$

We can combine this with Corollary I.5.13 to obtain

$$N^{D_P,1} = \deg X_P^\vee = 3D_P^2 + 2D_P \cdot K_{X_P} + \sum_v \mathrm{Eu}(v),$$

$$= 3D_P^2 + 2D_P \cdot K_{X_P} + \sum_v \sum_i (2 + E_{v,i}^2),$$

thus $\sum_v \mathrm{Eu}(v)$ acts as a sort of "corrected" version of c_2 for singular surfaces. Indeed, by Remark I.3.5 $\sum_v \mathrm{Eu}(v)$ equals the degree of the second Chern-Mather class $c_2^M(X)$ of the surface. One would have hoped that this correction could work for higher Severi degrees $N^{L,\delta}$, however this seems not to be the case, see for instance [AB13], [LO18], [Nød15, Ch. 4].

I.5.1 Weighted projective planes

We wish to apply the results of the previous section to the weighted projective planes $\mathbb{P}(k, m, n)$ and the 2-dimensional polytope P defined as the convex hull in $\mathbb{R}^3$ of the points $v_1 = (mn, 0, 0), v_2 = (0, kn, 0)$ and $v_3 = (0, 0, km)$. Denote by $\sigma_i^\vee$ the 2-dimensional cone generated by the edges of P emanating from v_i (the dual is chosen to remind us that the polytope is in M).

Proposition I.5.18. *Find minimal natural numbers a, b, c such that*

$$\begin{aligned} m + an &\equiv 0 \pmod{k} \\ n + bk &\equiv 0 \pmod{m} \\ k + cm &\equiv 0 \pmod{n} \end{aligned}$$

Then $\sigma_1^\vee$ is a $(k, k-a)$-cone, $\sigma_2^\vee$ is a $(m, m-b)$-cone and σ_3 is a $(n, n-c)$-cone.

Proof. We prove this for $\sigma_1^\vee$. $\sigma_1^\vee$ is generated as a cone by the vectors $u_1 = (-n, 0, k)$ and $u_2 = (-m, k, 0)$. Picking any a such that $m+an \equiv 0 \pmod{k}$ gives a lattice point in the hyperplane $kx + my + nz = kmn$ of the form $v = (d, 1, a)$. Picking a minimal and positive ensures that v is in P and moreover that $k - a$ is positive, which is needed for our convention of a $(k, k-a)$-cone. Then $w = v - v_1$ and u_1 is a basis for the lattice spanned by P. We have that $u_2 = -au_1 + kw$, thus $\sigma_1^\vee$ is a $(k, k-a)$-cone. ■

Theorem I.5.19. *Given $\mathbb{P}(k, m, n)$, find natural numbers a, b, c as in Proposition I.5.18. Let $\frac{k}{k-a} = [a_1, ..., a_t]^-$, $\frac{m}{m-b} = [b_1, ..., b_s]^-$, $\frac{n}{n-c} = [c_1, ..., c_r]^-$.*

Then $\deg \mathbb{P}(k, m, n)^\vee$ equals

$$3kmn - 2(k + n + m) + \sum_{i=1}^{r} (2 - a_i) + \sum_{i=1}^{s} (2 - b_i) + \sum_{i=1}^{t} (2 - c_i).$$

Using Theorem I.5.19 it is easier to find closed formulas in special cases.

Corollary I.5.20. *For $k \geq 1$, $\deg \mathbb{P}(2k-1, 2k, 2k+1)^\vee = 24k^3 - 20k + 3$.*

Proof. We wish to find minimal a, b, c satisfying

$$\begin{aligned} 2k + a(2k+1) &\equiv 0 \pmod{2k-1}, \\ 2k + 1 + b(2k-1) &\equiv 0 \pmod{2k}, \\ 2k - 1 + c2k &\equiv 0 \pmod{2k+1}. \end{aligned}$$

Some easy algebra shows that a, b, c must satisfy

$$\begin{aligned} 2a &\equiv -1 \pmod{2k-1}, \\ b &\equiv 1 \pmod{2k}, \\ c &\equiv -2 \pmod{2k+1}. \end{aligned}$$

Resulting in $a = k-1, b = 1, c = 2k-1$. Now

$$\begin{aligned} \frac{2k-1}{2k-1-(k-1)} &= \frac{2k-1}{k} = [2, k]^-, \\ \frac{2k}{2k-1} &= [2, ..., 2]^-, \\ \frac{2k+1}{2k+1-(2k-1)} &= \frac{2k+1}{2} = [k+1, 2]^-. \end{aligned}$$

Combining these yields the formula. ■

Corollary I.5.21. $\deg \mathbb{P}(m,n,m+n)^\vee = 3mn(m+n) - 5(m+n) + 4$.

Corollary I.5.22. *For odd* $m > 1$,

$$\deg \mathbb{P}(m-2,m,m+2)^\vee = 3m^3 - 19m + 3.$$

Corollary I.5.23. $\deg \mathbb{P}(m,n,m+2n)^\vee = 6mn^2 + 3m^2n - 7n - \frac{9}{2}m + \frac{5}{2}$.

Proof. Following Theorem I.5.19 we want minimal a, b, c such that

$$\begin{aligned} n + a(m+2n) &\equiv 0 \pmod{m}, \\ mb + m + 2n &\equiv 0 \pmod{n}, \\ m + cn &\equiv 0 \pmod{m+2n}. \end{aligned}$$

One sees that $a = \frac{m-1}{2}, b = n-1, c = 2$ (m has to be odd, if not then $\gcd(m, m+2n) \neq 1$). Now $\frac{m+2n}{m+2n-2} = 2 - \frac{m+2n-4}{m+2n-2} = 2 - \frac{1}{\frac{m+2n-2}{m+2n-4}} = [2, ..., 2, 3]^-$ where the 3 is by induction, since $\frac{3}{1} = [3]^-$. The Hirzebruch–Jung fraction $\frac{n}{n-(n-1)} = \frac{n}{1} = [n]^-$. Also $\frac{m}{m-\frac{m-1}{2}} = \frac{m}{\frac{m+1}{2}} = [2, \frac{m+1}{2}]^-$. Combining these yields the formula. ∎

Example I.5.24. For sufficiently small examples, these calculations can be double-checked using Macaulay2[GS]. According to Corollary I.5.21 $\deg \mathbb{P}(1,2,3)^\vee = 7$. The lattice points of the polytope defining $\mathbb{P}(1,2,3)$ corresponds to monomials $1, s, s^2, s^3, t, st, t^2$. We run the following code:

```
R = ZZ/101[s,t,y1,y2,y3,y4,y5,y6,y7];
f=y1+y2*s+y3*s^2+y4*s^3+y5*t+y6*s*t+y7*t^2;
I=ideal{f,diff(s,f),diff(t,f)};
I =saturate(I,ideal{s*t});
J=eliminate(I,s);
K=eliminate(J,t);
degree K
```

This outputs the correct answer 7.

I.6 3-folds

Here we let A be the lattice points of a 3-dimensional lattice polytope P. We have from Proposition I.3.8:

$$\deg X^\vee_{P\cap M} = 4\operatorname{Vol}(P) - 3\sum_{f \lneq P} \operatorname{Eu}(f)\operatorname{Vol}(f) + 2\sum_{e \lneq P} \operatorname{Eu}(e)\operatorname{Vol}(e) - \sum_{v \in P} \operatorname{Eu}(v),$$

where $\{f\}$ is the collection of all facets of P, $\{e\}$ the is collection of all edges of P, and the last sum is over all vertices v of P.

Again we recall Lemma I.3.13 saying that for any two faces $\Delta_\alpha \preceq \Delta_\beta \preceq P$ we have that $i(\Delta_\alpha, \Delta_\beta) = 1$. Combining this with Corollary I.3.11 we see that $\operatorname{Eu}(f) = 1$ for any facet f of P.

For an edge e of P we have by I.3.4 that

$$\mathrm{Eu}(e) = -\mathrm{RSV}_{\mathbb{Z}}(P,e)\,\mathrm{Eu}(P) + \sum_{e \preceq f, \dim f = 2} \mathrm{RSV}_{\mathbb{Z}}(f,e)\,\mathrm{Eu}(f_i) = -\mathrm{RSV}_{\mathbb{Z}}(P,e) + f_e,$$

where f_e is the number of facets of P containing e.

By unraveling the definition of $\mathrm{RSV}_{\mathbb{Z}}$ we see that the term $\mathrm{RSV}_{\mathbb{Z}}(P,e)$ is nothing but $\mathrm{Vol}(\overline{P} \setminus \mathrm{Conv}((\overline{P} \setminus \overline{e}) \cap \overline{M}))$, where $\overline{M}$ is the quotient $M/\mathbb{Z}e$ and $\overline{P}$, $\overline{e}$ are the images of P and e in $\overline{M}$ (Note that $\overline{e}$ is the origin of $\overline{M}$ and will be a vertex of $\overline{P}$). But this we can calculate: Write the 2-dimensional cone generated by $\overline{P}$ with apex $\overline{e}$ as a (d,k)-cone and write $\frac{d}{k} = [b_1, ..., b_r]^-$. Then $\mathrm{RSV}(P,e) = 2 + \sum_{i=1}^{r}(b_i - 2)$ by the arguments in the surface case. Summing up we get

$$\mathrm{Eu}(e) = f_e - 2 + \sum_{i=1}^{r}(2 - b_i).$$

For a vertex v of P we have

$$\mathrm{Eu}(v) = \mathrm{Eu}(P)\,\mathrm{RSV}_{\mathbb{Z}}(P,v) - \sum_i \mathrm{Eu}(f_i)\,\mathrm{RSV}_{\mathbb{Z}}(f_i,v) + \sum_j \mathrm{Eu}(e_j)\,\mathrm{RSV}_{\mathbb{Z}}(e_j,v)$$

$$= \mathrm{RSV}_{\mathbb{Z}}(P,v) - \sum_{v \preceq f, \dim f = 2} \mathrm{RSV}_{\mathbb{Z}}(f,v) + \sum_{v \preceq e, \dim e = 1} \mathrm{Eu}(e).$$

Calculating most of these terms are easy, $\mathrm{Eu}(e)$ we did above, while similarly to before $\mathrm{RSV}_{\mathbb{Z}}(f,v) = 2 + \sum_{i=1}^{s}(c_i - 2)$, where the cone spanned by f with apex v is a (d,k)-cone with $\frac{d}{k} = [c_1, ..., c_s]^-$. The remaining term $\mathrm{RSV}_{\mathbb{Z}}(P,v)$, however, is problematic, we need to compute the 3-dimensional $\mathrm{Vol}(P \setminus \mathrm{Conv}((P \setminus v) \cap M))$. There seems to be no known general method for doing this. However for sufficiently small polytopes, computer programs capable of calculating convex hulls and volumes can do this, for instance Macaulay2. Collecting the above we get:

Algorithm I.6.1. *To calculate the degree of the dual variety of a toric 3-fold $X_{P\cap M}$, do the following:*

(1) *Calculate the volume V of P.*

(2) *Calculate the sum of the areas of facets of P, denoted A.*

(3) *For each edge e calculate the length of e, denoted $L(e)$.*

(4) *For each edge e, let σ_e be the cone generated by P with apex e in $M/e\mathbb{Z}$. Write σ_e as a (d,k)-cone, and write $\frac{d}{k} = [b_1, ..., b_r]^-$. Then $\mathrm{Eu}(e) = f_e - 2 + \sum_{i=1}^{r}(2 - b_i)$.*

(5) *For each vertex calculate $\mathrm{RSV}_{\mathbb{Z}}(P,v)$.*

(6) *For each pair consisting of a vertex v and a facet f containing it, write the cone generated by edges of f emanating from v as a (d_f, k_f)-cone and write $\frac{d_f}{k_f} = [c_{f,1}, ..., c_{f,s}]^-$. Then $\mathrm{RSV}_{\mathbb{Z}}(f,v) = 2 + \sum_{i=1}^{s}(c_{f,i} - 2)$.*

(7) *For each vertex v calculate*

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P,v) - \sum_f [2 + \sum_{i=1}^{s}(c_{f,i} - 2)] + \sum_e \mathrm{Eu}(e),$$

where the sums are over faces containing v.

Then $\deg X_P^\vee = 4V - 3A + 2\sum_e \mathrm{Eu}(e)L(e) - \sum_v \mathrm{Eu}(v)$.

I.6.1 Weighted projective 3-folds

We will compute the local Euler obstruction and dual degree for weighted projective spaces of the form $\mathbb{P}(1,k,m,n)$. We may assume $\gcd(k,m,n) = 1$.

Set $d = \mathrm{lcm}(k,m,n)$, and let P be the convex hull in $M_{\mathbb{R}}$ of $v_0 = (0,0,0), v_1 = (\frac{d}{k},0,0), v_2 = (0,\frac{d}{m},0), v_3 = (0,0,\frac{d}{n})$. Then $X_P \simeq \mathbb{P}(1,k,m,n)$.

Since every cone containing v_0 is smooth, we only need to calculate for faces containing v_1, v_2, v_3. Thus we will do this for v_1, the rest is obtained by cyclic permutation.

Denoting $\gcd(a,b)$ by (a,b), the primitive vectors emanating from v_1 are

$$e_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} -\frac{n}{(n,k)} \\ 0 \\ \frac{k}{(n,k)} \end{pmatrix}, e_3 = \begin{pmatrix} -\frac{m}{(m,k)} \\ \frac{k}{(m,k)} \\ 0 \end{pmatrix}.$$

Let $f_1 = \mathrm{Cone}(e_2,e_3), f_2 = \mathrm{Cone}(e_1,e_3), f_3 = \mathrm{Cone}(e_1,e_2)$.

Then f_2 is a $(\frac{k}{(n,k)}, n')$-cone where $n' \equiv \frac{n}{(n,k)} \pmod{\frac{k}{(n,k)}}$.

f_3 is a $(\frac{k}{(m,k)}, m')$-cone where $m' \equiv \frac{m}{(m,k)} \pmod{\frac{k}{(m,k)}}$.

For f_1 we first need to choose a basis for the lattice containing f_1:

Lemma I.6.2. *Pick a, c such that $ak + cn = -m(n,k)$. Then the vectors*

$$w = \begin{pmatrix} a \\ (n,k) \\ c \end{pmatrix}, \quad e_2 = \begin{pmatrix} -\frac{n}{(n,k)} \\ 0 \\ \frac{k}{(n,k)} \end{pmatrix},$$

are a basis for the lattice M_{f_1}.

Proof. It is easily verified that M_{f_1} consists of all lattice points (x,y,z) satisfying

$$kx + my + nz = d,$$

hence w is a vector in M_{f_1}. We will apply Lemma I.3.12 to show that $\{w, e_2\}$ is a basis for M_{f_1}.

First we claim that for any (a,b,c) in M_{f_1} we must have

$$b \equiv 0 \pmod{(n,k)}.$$

Indeed, $mb = d - ak - cn$ is congruent to 0 modulo (n,k), and since $\gcd(k,m,n) = 1$ we must have b congruent to 0 modulo (n,k).

Assume now that $sw + te_2$, $0 \le s,t < 1$ is a point in M_{f_1}. By the above claim we must have $s = 0$. But then also $t = 0$, hence we are done. ∎

It will be convenient to choose a particular basis corresponding to the pair (a, c) from Lemma I.6.2, hence we require that c is the minimal non–negative number satisfying $ak+cn = -m(n, k)$, for some a. Dividing by $(k, n)(k, m)(m, n)$ and considering this $\pmod{\frac{k}{(n,k)(m,k)}}$ it is clear that this c satisfies $c(n, k) < k$. Then since

$$\begin{pmatrix} -\frac{m}{(m,k)} \\ \frac{k}{(m,k)} \\ 0 \end{pmatrix} = -\frac{c}{(m,k)} \begin{pmatrix} -\frac{n}{(n,k)} \\ 0 \\ \frac{k}{(n,k)} \end{pmatrix} + \frac{k}{(m,k)(n,k)} \begin{pmatrix} a \\ (n,k) \\ c \end{pmatrix},$$

and $0 < k - c(n, k) < k$, f_1 is a $(\frac{k}{(m,k)(n,k)}, \frac{k-c(n,k)}{(m,k)(n,k)})$-cone. From this we can compute the terms $\mathrm{RSV}_{\mathbb{Z}}(f_i, v_1)$ using HJ-fractions.

For the Euler-obstruction of the edges, we have $\mathrm{Eu}(e_1) = 1$ since the cone generated by the image of the two other vectors in $\mathbb{Z}^3/e_1\mathbb{Z}$ is smooth.

To calculate $Eu(e_2)$, set $a = \frac{n}{(n,k)}, b = \frac{k}{(n,k)}$. Choose integers such that $ea + fb = 1$. Then the following will be a basis for $\mathbb{Z}^3$:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -f \\ e \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -a \\ 0 \\ b \end{pmatrix},$$

Since $e_1 = bev_1 + bv_2 + ev_3$, the image in the quotient lattice $\mathbb{Z}^3/e_2$ is (be, b).

Setting $c = \frac{m}{(m,k)}, d = \frac{k}{(m,k)}$, we have $e_2 = (fbd - bcd)v_1 + (ad + bc)v_2 + (ce - fd)v_3$. In the quotient this is $(fbd - bce, ad + bc)$.

Writing out the details and cancelling common factors (to get primitive vectors) we get that the cone with apex 0 generated by the image of P is $\mathrm{Cone}((fk - em, n + m), (e, 1))$. Now since

$$\begin{pmatrix} fk - em \\ n + m \end{pmatrix} = (n + m) \begin{pmatrix} e \\ 1 \end{pmatrix} + (n, k) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

we get a $((n, k), m \pmod{(n, k)})$–cone.

Similarly for $\mathrm{Eu}(e_3)$ we get a $((m, k), n \pmod{(m, k)})$-cone. Using this and HJ-fractions we can compute the terms $\mathrm{Eu}(e_i)$.

Example I.6.3. We will apply the above to $\mathbb{P}(1, 6, 10, 15)$. Then $v_0 = (0, 0, 0), v_1 = (5, 0, 0), v_2 = (0, 3, 0), v_3 = (0, 0, 2)$. We will do all the steps of Algorithm I.6.1.

We calculate that $V(P) = 30$ and that $A(P) = 1 + 15 + 10 + 6 = 32$.

Denote the edge connecting v_i and v_j by e_{ij}. Denote the facets containing v_i, v_j, v_k by f_{ijk}. Then

$$L(e_{01}) = 5, L(e_{02}) = 3, L(e_{03}) = 2, L(e_{12}) = 1, L(e_{1,3}) = 1, L(e_{2,3}) = 1.$$

Applying the discussion above we can further conclude the following

$$\mathrm{Eu}(e_{0i}) = 1 \text{ for } i = 1, 2, 3, \mathrm{Eu}(e_{12}) = 0, \mathrm{Eu}(e_{13}) = -1, \mathrm{Eu}(e_{23}) = -3.$$

We know that $\mathrm{RSV}(P, v_0) = 1$. Using Macaulay2 we calculate that

$$\mathrm{RSV}_{\mathbb{Z}}(P, v_1) = 4, \mathrm{RSV}_{\mathbb{Z}}(P, v_2) = 6, \mathrm{RSV}(P, v_2) = 7.$$

For a fixed vertex v_i and a facet f containing it we will then write the cone with apex v_i generated by edges of the facet f as a (d, k)-cone.

vertex	facet	corresponding (d, k)
v_1	f_{123}	$(1, 0)$
v_1	f_{012}	$(2, 1)$
v_1	f_{013}	$(3, 2)$
v_2	f_{123}	$(1, 0)$
v_2	f_{012}	$(2, 1)$
v_2	f_{023}	$(5, 3)$
v_3	f_{123}	$(1, 0)$
v_3	f_{013}	$(3, 1)$
v_3	f_{023}	$(5, 2)$

Then we have that

$$\mathrm{Eu}(v_0) = 1, \mathrm{Eu}(v_1) = -1, \mathrm{Eu}(v_2) = -2, \mathrm{Eu}(v_3) = -2.$$

Thus we can in turn conclude that

$$\deg \mathbb{P}(1, 6, 10, 15)^\vee = 4 \cdot 30 - 3 \cdot 32 + 2(5 + 3 + 2 - 4) - (1 - 1 - 2 - 2) = 40.$$

I.6.2 Isolated singularities

If we assume the variety $X_{P\cap M}$ has only isolated singularities we know that $\mathrm{Eu}(e) = 1$ for every edge. Thus we can reduce to

$$\deg X^\vee_{P\cap M} = 4 \operatorname{Vol}(P) - 3A(P) + 2E(P) - \sum_{v\in P} \mathrm{Eu}(v),$$

where $A(P)$ is the sum of areas of facets of P, while $E(P)$ is the sum of lengths of edges of P. For a singular point v associated to a vertex of P

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) - \sum_{v\preceq f, \dim f=2} \mathrm{RSV}_{\mathbb{Z}}(f, v) + e, \qquad \text{(I.1)}$$

where e is the number of edges of P containing v.

We need a generalization of Pick's formula to estimate the volume $\mathrm{RSV}_{\mathbb{Z}}(P, v)$. To do this we make the following definitions:

Definition I.6.4. A piecewise linear lattice polygon (pllp) K is a union $\cup_{i=1}^n K_i$ of some facets of a 3-dimensional convex lattice polytope P which is contractible and connected in codimension one, meaning that for any pair K_i, K_j there is a chain $K_i = K_{l_1}, ..., K_{l_s} = K_j$ such that K_{l_r} and $K_{l_{r+1}}$ has 1-dimensional intersection, for $1 \leq r \leq s - 1$.

A lattice point x in K is a boundary point if it is also contained in some facet F of P which is not contained in K. If x is not a boundary lattice point, then it is an internal lattice point.

Proposition I.6.5 (Generalized Pick's formula)**.** *For a pllp $K = \cup_{i=1}^n K_i$, let K_i be contained in the plane H_i. Let A_i be the area of K_i, normalized with respect to the lattice generated by lattice points in H_i. Then the normalized area of K, defined as $A_K := \sum_{i=1}^n A_i$, equals $2i + b - 2$, where b is the number of boundary lattice points, and i is the number of internal lattice points.*

Proof. We do induction on n. If $n = 1$ this is just the usual Pick's formula in the plane. Assume we have showed the proposition for $n - 1$, and let $K = \cup_{i=1}^n K_i$. We have $A_K = A_{K_n} + A_{K'}$ where $K' = \cup_{i=1}^{n-1} K_i$. Without loss of generality we may assume that we have chosen K_n such that K' is a pllp. Let i', b' be the internal and boundary lattice points of K' respectively. By the inductive hypothesis we have

$$A_{K'} = 2i' + b' - 2,$$

and by Pick's formula in the plane we have

$$A_{K_n} = 2i_n + b_n - 2,$$

where i_n, b_n are internal and boundary lattice points of K_n. Now we have to compute i and b. The boundary points of K' which intersect K_n either are internal in K (call the number of such k) or remain boundary points in K (call the number of such s). If we let l be the number of boundary points of K_n not in any K_i, $i \neq n$, then we have

$$\begin{aligned} b &= b' - k + l, \\ i &= i' + i_n + k. \end{aligned}$$

Then we get

$$2i + b - 2 = 2i' + 2i_n + 2k + b' - k + l - 2 = A_{K'} + 2i_n + k + l.$$

Thus if we can show that $b_n - 2 = k + l$ we are done. By construction $b_n = k + l + s$, hence we need to show that $s = 2$:

Consider the set $S = P \setminus K$ where P is the ambient polytope $\mathrm{Conv}(K)$. If S is nonempty and not connected, then it is clear that K cannot be contractible. Thus we have that S is connected. Then the boundary of K is S intersected with K, which again has to be connected. Now if $s > 2$ we have that the boundary of K intersected with K_n cannot be connected. But this implies that the boundary of K' cannot be connected, which contradicts it being a pllp. ■

When attempting to compute $\mathrm{RSV}_{\mathbb{Z}}(\sigma^\vee, v)$ for the vertex of a 3-dimensional cone σ, there naturally arises a pllp : Let K be the union of the compact faces of the convex hull of the set $(\sigma^\vee \setminus \{v\}) \cap M$. It is a pllp whose ambient polytope is the convex hull of K.

Proposition I.6.6. *For an isolated singular point v on a toric 3-fold $X_{P \cap M}$ we always have* $\mathrm{Eu}(v) \geq 1$.

Proof. Consider the cone $\sigma^\vee$ generated by P with apex v. Let e be he number of rays of $\sigma^\vee$ which is always ≥ 3. Let K be the pllp associated to $\sigma^\vee$. By Construction I.5.6 we see that $\sum_{v \preceq f, \dim f = 2} \mathrm{RSV}_{\mathbb{Z}}(f, v)$ equals the number of boundary points of K. By Pick's formula the area of K is $2i + b - 2$ where i is the number internal lattice points of K. Since $\mathrm{RSV}_Z(P, v) \geq A_K = 2i + b - 2$ we get

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) + e - b \geq 2i + b - 2 + 3 - b = 2i + 1 \geq 1.$$

■

In Example I.6.3 we saw that this is not true for non-isolated singularities. Observe also that by the proof the only way one can have $\mathrm{Eu}(v) = 1$ is if there are just 3 edges emanating from v.

Corollary I.6.7. *For an isolated singular point v on a toric 3-fold $X_{P \cap M}$ one has $\mathrm{Eu}(v) = 1$ if and only if (i) there are exactly 3 edges emanating from v, (ii) the associated pllp $K = \cup_{i=1}^n K_i$ has no internal lattice points, and (iii) for each plane H_i containing K_i, the integer distance from H_i to the origin equals 1.*

The integer distance of a point v and an integer plane H is the index of the lattice generated by vectors joining v and all integer points of H, modulo the lattice M_P generated by lattice points of P. See [Kar13, Rmk. 14.8] for details.

We will again compute the local Euler obstruction for a 3-dimensional wps, now with isolated singularities. This assumption simplifies some of the calculations. By Proposition I.4.1 $\mathbb{P}(1, k, m, n)$ has isolated singularities if and only if $\gcd(m, n) = \gcd(k, n) = \gcd(k, m) = 1$. In this case one can calculate that

$$\mathrm{Vol}(P) = k^2 m^2 n^2,$$

$$A(P) = kmn + k^2 mn + km^2 n + kmn^2,$$
$$E(P) = k + m + n + mn + kn + km.$$

All this is straightforward, except for the first term of $A(P)$, but this is [Nød16, Prop 3.4] for a surface of weights (k, m, n).

The vertex $v_0 = (0, 0, 0)$ is smooth, thus $\mathrm{Eu}(v_0) = 1$. Since every vertex is contained in 3 facets, we get for a vertex v

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) - 3 + \sum_{v \preceq f, \dim f = 2} (2 - c_{f,i}).$$

For the vertex $v_1 = (mn, 0, 0)$, choose $0 < m', n', s < k$ such that

$$\begin{aligned} m' &\equiv m \pmod{k}, \\ n' &\equiv n \pmod{k}, \\ m + sn &\equiv 0 \pmod{k}. \end{aligned}$$

Then the 2-dimensional cones emanating from v_1 are $(k, m'), (k, n'), (k, k - s)$-cones. Using HJ-fractions one can then calculate $\mathrm{Eu}(v_1)$. The rest of the vertices are treated similarly.

Example I.6.8. Consider $\mathbb{P}(1,2,3,5)$. The polytope P has vertices $v_0 = (0,0,0), v_1 = (15,0,0), v_2 = (0,10,0), v_3 = (0,0,6)$. Using Macaulay2 we calculate that

$$RSV_{\mathbb{Z}}(P, v_1) = 4,$$
$$RSV_{\mathbb{Z}}(P, v_2) = 5,$$
$$RSV_{\mathbb{Z}}(P, v_3) = 6.$$

The cones emanating from v_1 are all $(2,1)$-cones, thus all $c_{f,1} = 2$, hence $\mathrm{Eu}(v_1) = 4 - 3 + 0 = 1$.

For v_2 we have $(3,2),(3,2),(3,1)$-cones, giving HJ-fractions $[2,2]^-, [2,2]^-, [3]^-$. Hence $\mathrm{Eu}(v_2) = 5 - 3 - 1 = 1$.

For v_3 we have $(5,2),(5,3),(5,4)$-cones, giving HJ-fractions $[3,2]^-, [2,3]^-, [2,2,2,2]^-$. Hence $\mathrm{Eu}(v_2) = 6 - 3 - 1 - 1 = 1$. We then get:

$$\deg \mathbb{P}(1,2,3,5)^\vee = 4 \cdot 900 - 3 \cdot 330 + 2 \cdot 41 - 4 = 2688.$$

Remark I.6.9. This example is somewhat surprising, as it exhibits a variety with isolated singularities which has Euler-obstruction constantly equal to 1. Matsui and Takeuchi [MT11] shows that for normal and projective toric surfaces, the Euler-obstruction is constantly equal to 1 if and only if the variety is smooth. They conjectured the similar statement in higher dimensions. This is a counterexample to that conjecture. There are also some other examples, see Appendix I.A.

In the appendix we list some computations done in Macaulay2 for the local Euler obstructions of weighted projective 3-folds. It isn't easy to see a clear pattern. This might be analogous to the computations of the Nash blow-up of toric varieties in [Ata+11], which in principle could be be used to compute the local Euler obstruction. The authors write "Almost every straightforward conjecture one might make about the patterns in the Nash resolution seems to be false."

One would have hoped to be able to compute $\mathrm{RSV}_{\mathbb{Z}}(P, v)$ for a 3-dimensional polytope in a way similar to the 2-dimensional case, for instance using some form of generalized theory of multidimensional continued fractions. However little is still known about this. Karpenkov writes "... with the number of compact faces greater than 1 almost nothing is known" [Kar13, p.219]. The number of faces corresponds to the number of compact polytopes in the pllp .

I.7 Dual defective varieties

For a variety $X \subset \mathbb{P}^N$, one defines the dual defect $\operatorname{def} X$ of X to be $\operatorname{def} X = N - 1 - \dim X^\vee$ (i.e., $\operatorname{def} X = 0$ if and only if $X^\vee$ is a hypersurface in $\mathbb{P}^{N^\vee}$). If $\operatorname{def} X > 0$ we say that X is defective. Using the theory from the previous sections we give a new proof of the well-known result:

Proposition I.7.1. *The only normal and projective toric surfaces which are defective are those of the form* $\mathbb{P}(1,1,n)$.

First we prove an easier result:

Lemma I.7.2. *The only normal and projective toric surfaces associated to a triangle P, which are defective, are those of the form $\mathbb{P}(1,1,n)$.*

Proof. We have by [MT11, Thm 1.4] that def $X > 0$ if and only if the expression

$$3\operatorname{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \operatorname{Eu}(v) \tag{I.2}$$

equals 0. We have that $E(P) = b$, where b is the number of boundary points of P, and letting i be the number of internal lattice points, we have by Picks' formula

$$\operatorname{Vol}(P) = 2i + b - 2.$$

We also have that $\operatorname{Eu}(v) = 2 - \operatorname{RSV}_Z(P, v)$. Thus we get

$$3(2i + b - 2) - 2b + 6 - \sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v)$$

$$= 6i + b - \sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v).$$

We now claim that

$$\sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v) \leq 3i + b,$$

which would imply that (I.2) is ≥ 0 with equality only possible if $i = 0$. To see that the claim is true, let b_1, b_2, b_3 be the number of lattice points on the 3 edges of P. By doing Construction I.5.6 for a vertex we construct a sequence of points $A_0, ..., A_{r+1}$. By the construction we see that each of the points $A_1, ..., A_r$ has to be either an inner point of P or an inner point of the edge opposite to the vertex. Then we get $r \leq i + b_j - 2$, thus $\operatorname{RSV}_{\mathbb{Z}}(P, v) = r + 1 \leq i + b_j - 1$, hence

$$\sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v) \leq \sum_{j=1}^{3} i + b_j - 1 = 3i + b,$$

proving the claim.

If $i = 0$, then we need to check when $b = \sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v)$. Assuming there are two different edges with internal lattice points, we see by Construction I.5.6 that $\sum_{v \text{ vertex } \in P} \operatorname{RSV}_Z(P, v) = 3$. Hence the only way in which a triangle can satisfy

$$3\operatorname{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \operatorname{Eu}(v) = 0,$$

is if it has two edges with no internal lattice points. After a change of basis this will always be a polytope of the form $\operatorname{Conv}((0,0),(n,0),(0,1))$ which is isomorphic to $\mathbb{P}(1,1,n)$. That def $\mathbb{P}(1,1,n) > 0$ can be easily calculated from Theorem I.5.19. Alternatively this also follows from the fact that all cones have positive defect and $\mathbb{P}(1,1,n)$ is the cone over the n-th Veronese embedding of $\mathbb{P}^1$, i.e., the rational normal curve of degree n. ∎

Using this we can prove the general case:

Proof. Let the polytope have vertices $v_1, ..., v_n$, indexed such that v_j is connected to v_{j-1} and v_{j+1} via an edge (take indices modulo n when necessary). To estimate $\mathrm{RSV}_{\mathbb{Z}}(P, v_j)$ we will consider the triangle $T_j := v_{j-1}v_jv_{j+1}$. Let i_j be the number of internal lattice points of P contained in T_j. By a similar argument as in the previous lemma, by Construction I.5.6 we have that $RSV_{\mathbb{Z}}(P, v) \leq i_j + 1$. Since an internal vertex of P at most can be contained in two triangles T_j, we get that $\sum_{j=1}^{n} i_j \leq 2i$. Thus

$$\sum_{v \text{ vertex } \in P} \mathrm{RSV}_Z(P, v) \leq \sum_{j=1}^{n} i_j + 1 \leq 2i + n.$$

The expression we wish to consider is

$$3\,\mathrm{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \mathrm{Eu}(v)$$

$$= 3(2i + b - 2) - 2b + 2n - \sum_{v \text{ vertex } \in P} \mathrm{RSV}_Z(P, v)$$

$$= 6i+b-6+2n- \sum_{v \text{ vertex } \in P} \mathrm{RSV}_Z(P, v) \geq 6i+b-6+2n-2i-n = 4i+b+n-6.$$

This last expression is always greater than 0 when $n > 3$. ∎

For 3-folds it is again more difficult to get general results, however for a subclass of wps we can get similar results:

Proposition I.7.3. *The only defective 3-dimensional wps of the form $\mathbb{P}(1, k, m, n)$ with only isolated singularities are those of the form $\mathbb{P}(1, 1, 1, n)$.*

Proof. As before, by [MT11, Thm 1.4] for a toric 3-fold X with isolated singularities, def $X > 0$ if and only if the expression

$$4\,\mathrm{Vol}(P) - 3A(P) + 2E(P) - \sum_{v \in P} \mathrm{Eu}(v) \tag{I.3}$$

equals 0. For $\mathbb{P}(1, k, m, n)$ we have as before

$$\begin{aligned} \mathrm{Vol}(P) &= k^2m^2n^2, \\ A(P) &= knm(1 + k + m + n), \\ E(P) &= k + m + n + mn + kn + km, \end{aligned}$$

and for a vertex v of P

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) + 3 - \sum_{v \preceq f, \dim f = 2} \mathrm{RSV}_{\mathbb{Z}}(f, v).$$

We now claim that for the vertex $v_1 = (mn, 0, 0)$, $\mathrm{Eu}(v_1) \leq k^2$.

Indeed, by using the description of P from Section I.6.1 we have that the volume which equals $\mathrm{RSV}_{\mathbb{Z}}(P, v)$ is enclosed in a polygon with volume

$$\det \begin{bmatrix} -1 & -n & -m \\ 0 & 0 & k \\ 0 & k & 0 \end{bmatrix} = k^2.$$

Thus $\mathrm{RSV}_{\mathbb{Z}}(P, v) \leq k^2$. Also for any face f containing v, $\mathrm{RSV}_{\mathbb{Z}}(f, v) \geq 1$. Combining this we get

$$\mathrm{RSV}_{\mathbb{Z}}(P, v) + 3 - \sum_{v \preceq f, \dim f = 2} \mathrm{RSV}_{\mathbb{Z}}(f, v) \leq k^2 + 3 - 3 = k^2.$$

By symmetry we also have $\mathrm{Eu}(v_2) \leq m^2, \mathrm{Eu}(v_3) \leq n^2$. Thus (I.3) reduces to

$$4k^2m^2n^2 - 3knm(1 + k + m + n) + 2(k + m + n + mn + kn + km) - \sum_{v \in P} \mathrm{Eu}(v)$$

$$\geq 4k^2m^2n^2 - 3knm(1+k+m+n) + 2(k+m+n+mn+kn+km) - 1 - k^2 - m^2 - n^2.$$

If we are not in the case $\mathbb{P}(1, 1, 1, n)$, we may assume without loss of generality that $k \geq 3, m \geq 2$ and $k > m > n$. We have that

$$\begin{aligned} k^2m^2n^2 - 3km^2n &= km^2n(kn - 3) \geq 0, \\ k^2m^2n^2 - 3kmn^2 &= kmn^2(mk - 3) \geq 0, \\ 2k^2m^2n^2 - 3k^2mn - 3kmn - k^2 &= k(k(mn(2mn - 3) - 1) - 3mn). \end{aligned}$$

Now unless $m = 2$ and $n = 1$, we have $2mn - 3 \geq 2$, thus $mn(2mn - 3) - 1 \geq mn$, implying $k(mn(2mn - 3) - 1) \geq 3mn$. Hence

$$2k^2m^2n^2 - 3k^2mn - 3kmn - k^2 \geq 0.$$

Also we have that

$$\begin{aligned} kn - n^2 &\geq 0, \\ km - m^2 &\geq 0. \end{aligned}$$

Combining all these we get

$$\begin{aligned} 4k^2m^2n^2 - 3knm(1+k + m + n) + 2(k + m + n + mn + kn + km) \\ -1 - k^2 - m^2 - n^2 \geq 2(k + n + m + mn) + kn + km - 1 > 0. \end{aligned}$$

One can easily verify that the exception $\mathbb{P}(1, k, 2, 1)$ has defect 0.

That $\operatorname{def} \mathbb{P}(1, 1, 1, n) > 0$ follows from the fact that it is the cone over the n-th Veronese embedding of $\mathbb{P}^2$. ■

Using our algorithms for calculations of degrees of dual varieties, we have checked which wps of the form $\mathbb{P}(1, k, m, n)$ that do not necessarily have isolated singularities, are defective. For $k, m, n \leq 10$ we have computed that the only defective wps of the form $\mathbb{P}(1, k, m, n)$ are $\mathbb{P}(1, 1, 1, l), \mathbb{P}(1, 1, m, lm), \mathbb{P}(1, k, m, km)$ which are cones over $(\mathbb{P}^2, \mathcal{O}(l)), (\mathbb{P}(1, 1, m), \mathcal{O}(l)), (\mathbb{P}(1, k, m), \mathcal{O}(1))$ respectively. Based on the numerical data we conjecture the following.

Conjecture I.7.4. *The only defective wps are those which are cones over a wps (not necessarily with reduced weights) of lower dimension.*

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Appendix I.A Computations

The table below shows weights (W), Euler-obstructions (E1,E2,E3) and $\mathrm{RSV}_{\mathbb{Z}}(P, v)$ (R1,R2,R3) for $\mathbb{P}(1, k, m, n)$, where $k, m, n \leq 10$ and the singularities are isolated. The computations were done using the Macaulay2 package EulerObstruction-WPS which can be found at the author's webpage[Nød]. Note also that the package EDPolytope[HS] by Helmer and Sturmfels can in principle calculate the degree of the dual variety of toric varieties X_A of any dimension. However unless A is quite small, their computation will not terminate.

W			E1	E2	E3	R1	R2	R3
1	1	1	1	1	1	1	1	1
1	1	2	1	1	1	1	1	4
1	1	3	1	1	3	1	1	9
1	1	4	1	1	7	1	1	16
1	1	5	1	1	13	1	1	25
1	1	6	1	1	21	1	1	36
1	1	7	1	1	31	1	1	49
1	1	8	1	1	43	1	1	64
1	1	9	1	1	57	1	1	81
1	1	10	1	1	73	1	1	100
1	2	3	1	1	1	1	4	5
1	2	5	1	1	5	1	4	13
1	2	7	1	1	13	1	4	25
1	2	9	1	1	25	1	4	41
1	3	4	1	3	1	1	9	6
1	3	5	1	1	3	1	5	11
1	3	7	1	3	7	1	9	17
1	3	8	1	1	11	1	5	24
1	3	10	1	3	19	1	9	34
1	4	5	1	7	1	1	16	7
1	4	7	1	1	7	1	6	19
1	4	9	1	7	9	1	16	21
1	5	6	1	13	1	1	25	8
1	5	7	1	5	3	1	13	13
1	5	8	1	3	5	1	11	16
1	5	9	1	1	13	1	7	29
1	6	7	1	21	1	1	36	9
1	7	8	1	31	1	1	49	10
1	7	9	1	13	3	1	25	15
1	7	10	1	7	7	1	17	22
1	8	9	1	43	1	1	64	11

1	9	10	1	57	1	1	81	12
2	3	5	1	1	1	4	5	6
2	3	7	1	1	3	4	5	10
2	5	7	1	3	1	4	11	7
2	5	9	1	1	5	4	6	15
2	7	9	1	7	1	4	19	8
3	4	5	1	1	1	5	6	6
3	4	7	3	1	1	9	6	7
3	5	7	1	1	3	5	6	10
3	5	8	1	5	1	5	13	7
3	7	8	1	7	1	5	17	7
3	7	10	3	3	1	9	13	8
4	5	7	1	1	3	6	6	10
4	5	9	7	1	1	16	7	8
4	7	9	1	3	1	6	12	7
5	6	7	5	1	1	13	8	7
5	7	8	1	3	1	6	13	7
5	7	9	1	1	5	6	7	15
5	8	9	1	5	1	6	16	8
7	8	9	13	1	1	25	10	8
7	9	10	3	3	1	10	15	8

The table below shows weights (W), Euler-obstructions (E1,E2,E3) and $\mathrm{RSV}_{\mathbb{Z}}(P, v)$ (R1,R2,R3) for $\mathbb{P}(1, k, m, n)$, where $k, m, n \leq 6$, where the singularities are not isolated.

W			E1	E2	E3	R1	R2	R3
1	2	2	1	0	0	1	2	2
1	2	4	1	0	2	1	2	8
1	2	6	1	0	8	1	2	18
1	3	3	1	-1	-1	1	3	3
1	3	6	1	-1	3	1	3	12
1	4	4	1	-2	-2	1	4	4
1	4	6	1	2	2	1	8	10
1	5	5	1	-3	-3	1	5	5
1	6	6	1	-4	-4	1	6	6
2	2	3	0	0	1	2	2	5
2	2	5	0	0	3	2	2	11
2	3	3	1	0	0	4	2	2
2	3	4	0	1	0	2	5	4
2	3	6	0	0	1	2	2	6
2	4	5	0	2	1	2	8	6
2	5	5	1	-1	-1	4	3	3
2	5	6	0	5	0	2	13	5
3	3	4	-1	-1	1	3	3	6
3	3	5	0	0	5	2	2	13
3	4	4	3	0	0	9	2	2
3	4	6	-1	0	-1	3	4	4
3	5	5	1	-1	-1	5	3	3
3	5	6	0	3	1	2	11	4
4	4	5	-2	-2	1	4	4	7
4	5	5	7	0	0	16	2	2
4	5	6	2	1	2	8	7	5
5	5	6	-3	-3	1	5	5	8
5	6	6	13	0	0	25	2	2

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Paper II

Polar Degrees and Closest Points in Codimension Two

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Abstract

Suppose that $X_A \subset \mathbb{P}^{n-1}$ is a toric variety of codimension two defined by an $(n-2) \times n$ integer matrix A, and let B be a Gale dual of A. In this paper we compute the Euclidean distance degree and polar degrees of X_A (along with other associated invariants) combinatorially working from the matrix B. Our approach allows for the consideration of examples that would be impractical using algebraic or geometric methods. It also yields considerably simpler computational formulas for these invariants, allowing much larger examples to be computed much more quickly than the analogous combinatorial methods using the matrix A in the codimension two case.

Contents

II.1 Introduction

To a projective variety $X \subset \mathbb{P}^n$ we may associate the *polar varieties* of X; these are subvarieties of X whose points have tangent spaces which intersect non-transversally with a fixed linear subspace. The classes of the polar varieties in the Chow ring are invariants of the projective embedding; in particular their degrees, which are often refereed to as *polar degrees*, are projective invariants of X. As projective invariants, polar varieties and polar degrees have been historically important in the study and classification of projective varieties [Alu18; Ban+10; Ful13; Hol88; Kle+86; Pie15; Pie78; Tev06]. In particular knowing the polar degrees of a smooth variety is equivalent to knowing the Chern classes of the tangent bundle, giving a simple expression for this Chern class. Polar varieties also arise in science and engineering problems where one tests the accuracy of mathematical models against observed data. In this setting it is natural to measure distance using the Euclidean norm and to compute the closest real point to some observed data within the model being studied. In the context of this Euclidean distance optimization problem the polar degrees can be used to compute the *Euclidean distance degree*, a projective invariant which quantifies the difficulty of solving the optimization problem [Dra+16; HS18; OSS14].

In this paper we consider the situation where X is a codimension two projective variety parameterized by monomials, i.e., X is a codimension two projective toric variety. In this case we develop computationally simple formulas for the quantities which determine the polar degrees, Chern-Mather class, and Euclidean distance degree of a projective toric variety.

We introduce the objects to be studied in this paper with an example from classical algebraic geometry which also arises in cell biology when studying pore forming cytotoxins used by numerous pathogenic bacteria [AH17; Los+13]. Let

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

this matrix gives rise to the *twisted cubic curve* in $\mathbb{P}^3$ via the closure of the image of a monomial map defined by A, that is

$$X_A = \overline{\{(t_1^3t_2 : t_1^2t_2 : t_1t_2 : t_2) \mid (t_1, t_2) \in (\mathbb{C}^*)^2\}} \subset \mathbb{P}^3.$$

The toric variety X_A has codimension two. Let $k[x_0, x_1, x_2, x_3]$ be the coordinate ring of $\mathbb{P}^3$, the toric ideal of X_A in this ring is the prime binomial ideal

$$I = (x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2).$$

Consider the matrix

$$B = \begin{bmatrix} -2 & -1 \\ 3 & 1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

The rows of the matrix B generate the kernel of the linear map defined by the matrix A, we refer to A and B as Gale dual matrices and call B the Gale dual of A.

Let $k[y_0, y_1, y_2, y_3]$ be the coordinate ring of $(\mathbb{P}^\vee)^3$. The conormal variety $\mathrm{Con}(X_A)$ of X_A in $\mathbb{P}^3 \times (\mathbb{P}^\vee)^3$ parametrizes pairs of smooth points $x \in X_A$ and planes containing $T_{X_A,x}$. Its bigraded ideal is defined by the sum of the ideal I

and the ideal defined by the 3×3 minors ($3 = \operatorname{codim}(X_A) + 1$) of the matrix

$$\begin{bmatrix} 0 & -x_3 & 2x_2 & -x_1 \\ -x_3 & x_2 & x_1 & -x_0 \\ -x_2 & 2x_1 & -x_0 & 0 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix}.$$

The multidegree of the bigraded ideal defining $\mathrm{Con}(X_A)$ are the coefficients of the polynomial $4H^3h + 3H^2h^2$, which represents the class $[\mathrm{Con}(X_A)]$ in the Chow ring

$$\mathrm{CH}^*\left(\mathbb{P}^3 \times (\mathbb{P}^\vee)^3\right) \cong \mathbb{Z}[h, H]/(h^4, H^4).$$

Here h denotes the class of a hyperplane in $\mathbb{P}^3$ and H denotes the class of a hyperplane in $(\mathbb{P}^\vee)^3$. The polar degrees of X_A are by definition this multidegree

$$(\delta_0(X_A), \delta_1(X_A)) = (4, 3).$$

The first nonzero polar degree is the degree of the projective dual, in this case $\deg(X_A^\vee) = \delta_0(X_A) = 4$.

From the polar degrees we may also determine the Chern-Mather class of X_A, $c_M(X_A)$ (since X_A is smooth the Chern-Mather class agrees with the Chern class of the tangent bundle, i.e. $c_M(X_A) = c(T_{X_A}) \cap [X_A]$). The Chern-Mather class of X_A (pushed forward to $A^*(\mathbb{P}^3)$) is

$$c_M(X_A) = 2h^3 + 3h^2 \in A^*(\mathbb{P}^3) \cong \mathbb{Z}[h]/(h^4).$$

The Euclidean distance (ED) problem associated to X_A seeks to determine the closest point in X_A to a fixed generic point $u \in \mathbb{R}^4$. More specifically we wish to solve the optimization problem

$$\text{Minimize the function } f(t) = (t_1^3t_2 - u_1)^2 + (t_1^2t_2 - u_2)^2 + (t_1t_2 - u_3)^2 + (t_2 - u_4)^2. \tag{II.1}$$

The critical points associated to this optimization problem are the solutions of the system of polynomial equations

$$\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial t_2} = 0.$$

This polynomial systems will have 7 non-zero complex solutions for generic data u, this number is the *ED degree* of X_A. Observe that the ED degree of X_A is equal to the sum of all non-zero polar degrees of X_A. This will be true in general (see [Dra+16]). In the context of systems biology solving this ED problem corresponds to testing if a particular model for pore-forming toxins describes experimentally measured data [AH17].

For a general projective variety the computation of the ED degree and polar degrees can become quite difficult as the degree of the generators and the dimension of the ambient space grows. This is true for all applicable algebraic or geometric methods (i.e. Gröbner basis, homotopy continuation, etc.). For toric varieties, however, we may avoid the potentially time consuming algebraic or geometric methods and compute these invariants combinatorially.

For a projective toric variety X_A methods to compute the polar degrees, Chern-Mather class, and EDdegree based on the polyhedral combinatorics of the polytope $\mathrm{Conv}(A)$ are given in [HS18]. It is also shown in [HS18] that much more computationally efficient formulas may be given in terms of the Gale dual of the matrix A when $\mathrm{codim}(X_A) = 1$, i.e. when X_A is a toric hypersurface.

In this paper we develop analogous formulas for the polar degrees, Chern-Mather classes and ED degree of X_A in terms of the Gale dual matrix, B, of A for the next interesting case, when X_A has codimension two. These formula yield substantially simpler expressions which are much faster to evaluate using a computer.

The methods developed here build on the work of [HS18] and [DS02]. We also note that a method to compute the degree of the A-discriminant, i.e., the projective dual of X_A, in the codimension two case using the Gale dual matrix is given in [DS02]. Since this number must appear as the first non-zero polar degree our results also, in a sense, generalize their result.

Explicitly, by theorems of [MT11] and [HS18], we know that the ED degree and polar degrees of a projective toric variety X_A are determined by the relative subdiagram volumes of the faces of the polytope $\mathrm{Conv}(A)$. Our contribution (detailed in §II.3) is to give explicit formula for these subdiagram volumes in terms of the Gale dual matrix B of A when X_A has codimension two.

Working with the Gale dual is advantageous in low codimensions since in that case we work in a low dimensional integer lattice. This allows us compute the ED degree and polar degrees of large examples with a complicated face structure quickly. For example, in §II.4 we consider a projective toric variety X_{A_6} of degree 581454473 in $\mathbb{P}^9$ (the matrix A_6 is given in Appendix II.A). Using the methods developed in this paper we compute $\mathrm{EDdegree}(X_{A_6}) = 74638158177$ in less than 30 seconds on a laptop, to find this number using algebraic or geometric methods would require computing the degree of a zero dimensional variety with over 74 *billion* isolated points in $\mathbb{P}^9$. Such a computation is unfeasible with current algebraic or geometric methods, even using a super computer. Using the combinatorial methods developed in [HS18] this computation takes over 2600 seconds, hence our new combinatorial method gives a speed up of about 98 times in this case (see Table II.3). A Macaulay2 [GS] package implementing the results developed in this paper can be found at the link (II.2) below:

http://martin-helmer.com/Software/toricED_Codim2.html (II.2)

The paper is organized as follows, in §II.2 we review background on computational tools and formulas that will be need in later sections. The main results are given in §II.3. In §II.4 we test the performance of computer implementations on a variety of examples and analyze the theoretical computational complexity of our new combinatorial methods.

II.2 Background and Preliminaries

In this section we give background on toric varieties and their polar degrees, introduce Gale duality and gather some technical results needed in §II.3.

II.2.1 Toric Varieties, ED Degrees and Polar Degrees

Let A be a $d \times n$ integer matrix with columns $a_1, a_2, \ldots, a_n$, and rank d such that the vector $(1, 1, \ldots, 1)$ lies in the row space of A over $\mathbb{Q}$. Note that we allow A to have negative entries. Each column vector a_i defines a monomial $t^{a_i} = t_1^{a_{1i}} t_2^{a_{2i}} \cdots t_d^{a_{di}}$. The *affine toric variety* defined by A is

$$\tilde{X}_A = \overline{\{(t^{a_1}, \ldots, t^{a_n}) : t \in (\mathbb{C}^*)^d\}} \subset \mathbb{C}^n,$$

that is, $\tilde{X}_A$ is the closure in $\mathbb{C}^n$ of the monomial parametrization specified by A. The affine toric variety $\tilde{X}_A$ is the affine cone over the *projective toric variety* $X_A \subset \mathbb{P}^{n-1}$, that is X_A is the closure in $\mathbb{P}^{n-1}$ of the image of the same monomial map. We have that $\dim(X_A) = d - 1$ and $\dim(\tilde{X}_A) = d$. To the projective toric variety X_A we can associate a polytope $P = \mathrm{Conv}(A)$, which is the convex hull of the lattice points specified by the columns of the matrix A.

We have a particular interest in the case when the toric variety X_A has codimension two in $\mathbb{P}^{n-1}$; in this case A is a $(n-2) \times n$ integer matrix. Let B be a Gale dual matrix of A, i.e. a $(n \times 2)$ matrix such that the image of B equals the kernel of A. We will refer to a finitely generated free abelian group as a *lattice*. Let $\mathbb{Z}B \subset \mathbb{Z}^n$ be the lattice spanned by the columns of the matrix B. We may associate a *lattice ideal*, I_B, to the lattice $\mathbb{Z}B$ as follows:

$$I_B = \left(x^{l^+} - x^{l^-} \mid l \in \mathbb{Z}B \right), \tag{II.3}$$

where l_i^+ is equal to l_i if $l_i > 0$ and 0 otherwise, and where l_i^- is equal to $|l_i|$ if $l_i < 0$ and 0 otherwise.

Example II.2.1. The following example will be used throughout the paper to illustrate definitions and results. Consider the surface $X_A \subset \mathbb{P}^4$ arising from the matrix

$$A = \begin{bmatrix} -2 & -2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \text{ with Gale dual } \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \\ -4 & 0 \\ 1 & -3 \end{bmatrix}.$$

Explicitly the surface X_A is given by

$$\overline{\left\{ \left(\frac{t_2^4 t_3}{t_1^2} : \frac{t_3}{t_1^2} : t_3 t_1 : t_2 t_1 : t_3 \right) \mid t \in (\mathbb{C}^*)^3 \right\}} = V(x_1 x_3^2 x_5 - x_4^4, x_2 x_3^2 - x_5^3) \subset \mathbb{P}^4. \tag{II.4}$$

The polytope $P = \mathrm{Conv}(A)$ associated to X_A is given in Figure II.1. P has 3 vertices v_1, v_2, v_3 and 3 edges e_1, e_2, e_3.

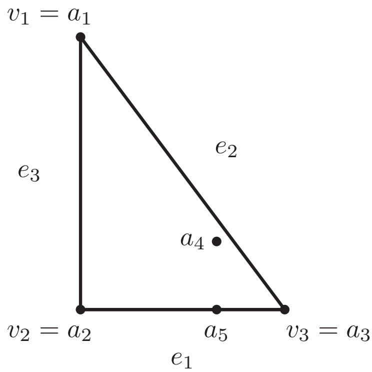

Figure II.1: $P = \text{Conv}(A)$.

The rows of the matrix B are denoted by b_i, $i = 1, ..., n$. We assume as in [DS02, p.13] that all b_i are non-zero. This assumption is equivalent to saying that X_A is not a cone over a coordinate point. We also note that since X_A is irreducible, the rows of B necessarily generate $\mathbb{Z}^2$ [DS02, p.9]: If B has index $i > 1$ in $\mathbb{Z}^2$, then there has to be an f, such that $f^i \in I_B$, but $f \notin I_B$, contradicting that I_B is prime.

We now review the ED problem for toric varieties. For what follows we fix a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive real numbers. We define the λ-weighted Euclidean norm on $\mathbb{R}^n$ to be $||x||_\lambda = (\sum_{i=1}^n \lambda_i x_i^2)^{1/2}$. For a given $u \in \mathbb{R}^n$, we wish to compute a real point $v \in \tilde{X}_A$ which is closest to the given u. In particular the Euclidean distance problem is the constrained optimization problem:

$$\text{Minimize } ||u - v||_\lambda \text{ such that } v \in \tilde{X}_A \cap \mathbb{R}^n. \tag{II.5}$$

Alternatively, using the parametric description of X_A, we can for generic u formulate the ED problem as the unconstrained optimization problem

$$\text{Minimize } \sum_{i=1}^{n} \lambda_i (u_i - t^{a_i})^2 \text{ over all } t = (t_1, \ldots, t_d) \in (\mathbb{R} \setminus \{0\})^d. \tag{II.6}$$

For generic u and λ the number of complex critical points of (II.5) is constant, we refer to this number as the *Euclidean distance degree* of X_A and denote it EDdegree(X_A). This matches the definition of ED degree given in [Dra+16; HS18; OSS14] for the toric variety X_A. The ED degree quantifies the inherent algebraic complexity of finding and representing the exact solutions to the ED problems (II.5) and (II.6). For instance note that EDdegree(X_A) is an upper bound for the number of local minima of the ED problem associated to X_A. Since the degree of the monomial map defining X_A may be greater than one, i.e. we could have $[\mathbb{Z}^d : \mathbb{Z}A] > 1$, the number of complex critical points of (II.6) is given by EDdegree$(X_A) \cdot [\mathbb{Z}^d : \mathbb{Z}A]$.

The relation between the ED degree and polar degrees is the following: The ED degree of a projective variety $X \subset \mathbb{P}^{n-1}$ is equal to the sum of the polar degrees of X, see [Dra+16, Theorem 5.4], that is

$$\text{EDdegree}(X) \;=\; \delta_0(X) + \delta_1(X) + \cdots + \delta_{n-1}(X). \tag{II.7}$$

We now define the polar degrees of X following the conventions of Fulton [Ful13], Holme [Hol88], Piene [Pie78] and others. The j-th *polar degree* of X, written $\delta_j(X)$, is the degree of the j-th *polar variety* of X with respect to a general linear subspace $\ell_j = \mathbb{P}^{j+\text{codim}(X)} \subset \mathbb{P}^{n-1}$:

$$P_j \;=\; \overline{\{x \in X_{\text{smooth}} \mid \dim(T_x X \cap \ell_j) \;\geq\; j+1\}} \;\subset\; \mathbb{P}^{n-1}.$$

Following Kleiman [Kle+86], we can also obtain the polar degrees $\delta_j(X)$ from the rational equivalence class of the conormal variety in the Chow ring $\text{CH}^*(\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee) \cong \mathbb{Z}[H,h]/(H^n, h^n)$; in this convention H denotes the rational equivalence class of a generic hyperplane from the $\mathbb{P}^{n-1}$ factor and h denotes the rational equivalence class of a generic hyperplane from the $(\mathbb{P}^{n-1})^\vee$. The *conormal variety* of X is

$$\text{Con}(X) = \overline{\{(p,L) \mid p \in X_{\text{reg}} \text{ and } L \supseteq T_pX\}} \subset \mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee.$$

The ideal of $\text{Con}(X)$ can be constructed as follows. Let I_X be the ideal defining X in the coordinate ring of $\mathbb{P}^{n-1}$ and let $\mathbb{C}[y] = \mathbb{C}[y_1, \ldots, y_n]$ be the coordinate ring of $(\mathbb{P}^{n-1})^\vee$. Set $c = \text{codim}(X)$, and let $\mathcal{J}$ be the ideal defined by the $(c+1) \times (c+1)$-minors of the matrix $[J(X)\; y]^T$, where $J(X)$ is the Jacobian of X. The ideal of $\text{Con}(X)$ in $\mathbb{C}[x,y]$ is $\mathcal{K} = (I_X + \mathcal{J}) : (I_{\text{Sing}(X)})^\infty$. The Chow class of $\text{Con}(X)$ is

$$[\text{Con}(X)] = \delta_0 H^{n-1}h + \cdots + \delta_{n-2} H h^{n-1} \in A^*(\mathbb{P}^{n-1} \times (\mathbb{P}^{n-1})^\vee),$$

where the integers $\delta_0 = \delta_0(X), \ldots, \delta_{n-2} = \delta_{n-2}(X)$ are the polar degrees of X defined above. From the point of view of commutative algebra (i.e., in the terminology of Miller and Sturmfels [MS04]) the polar degrees are the multidegree of the bigraded ideal $\mathcal{K}$.

II.2.2 Polar Degrees and the Chern-Mather class of X_A via $\text{Conv}(A)$

The Chern-Mather class was first introduced by MacPherson in [Mac74] and is a generalization of the total Chern class of the tangent bundle to singular varieties. In projective space we may express the Chern-Mather class in terms of the polar classes, and conversely may express the polar classes in terms of the Chern-Mather class [Pie78],[Alu18]; in the remainder of this paper we will employ the latter point of view. To this end we now review formulas for the polar degrees and ED degree of a projective toric variety X_A in terms of the Chern-Mather class of X_A, $C_M(X_A)$. In the context of toric varieties, we see that the coefficients of this characteristic class $C_M(X_A)$ take the form of weighted normalized lattice volumes, which we will refer to as the *Chern-Mather* volumes. The Chern-Mather volume will agree with the usual normalized volume when X_A is smooth.

The Chern-Mather volumes are defined in terms of the local Euler obstruction associated to a face of the polytope of a projective toric variety. The local Euler obstruction of a variety is a constructible function $\mathrm{Eu} : X \to \mathbb{Z}$. It was originally used by MacPherson to construct Chern classes for singular varieties [Mac74].

Definition II.2.2. Given faces $\beta \subset \alpha$ of $P = \mathrm{Conv}(A)$ we define $i(\alpha, \beta)$ as the index $[\mathbb{Z}\alpha \cap \mathbb{R}\beta : \mathbb{Z}\beta]$, where $\mathbb{R}\beta$ is the linear subspace of $\mathbb{R}^d$ spanned by β. We also define the relative normalized subdiagram volume $\mu(\alpha, \beta)$ (cf. [GKZ94, Definition 3.8]) as follows: let A_α/β denote the image of the lattice points $A \cap \alpha$ in the quotient lattice $\mathbb{Z}\alpha/\mathbb{Z}\alpha \cap \mathbb{R}\beta$, then

$$\mu(\alpha, \beta) = \mathrm{Vol}(\mathrm{Conv}(A_\alpha/\beta)) - \mathrm{Vol}(\mathrm{Conv}(A_\alpha/\beta \setminus \{0\}), \tag{II.8}$$

where the volume is normalized with respect to the lattice $\mathbb{Z}\alpha/\mathbb{Z}\alpha \cap \mathbb{R}\beta$.

Proposition II.2.3. *[MT11, Thm. 4.7] For a projective toric variety $X_A \subset \mathbb{P}^{n-1}$, the local Euler obstruction is constant on any torus-orbit and can be computed recursively as follows*

(1) $\mathrm{Eu}(P) = 1,$

(2) $\mathrm{Eu}(\beta) = \sum_{\substack{\alpha \text{ s.t. } \beta \text{ is a} \\ \text{proper face of } \alpha}} (-1)^{\dim(\alpha)-\dim(\beta)-1} \mu(\alpha, \beta) i(\alpha, \beta) \mathrm{Eu}(\alpha).$

Example II.2.4. Consider the surface X_A from Example II.2.1. We have that $i(P, e_3) = 4$, while $i(P, e_1) = i(P, e_2) = 1$. To compute the subdiagram volume μ using the matrix A we compute normalized volumes. For instance we see that $\mu(P, v_1) = \mathrm{Vol}(P) - \mathrm{Vol}(\mathrm{Conv}(a_2, a_3, a_4, a_5)) = 12 - 3 = 9$ and that $\mu(P, e_1) = 1, \mu(P, e_2) = 1, \mu(P, e_3) = 2$.

Since this example is a surface all these numbers are easily computable from the definitions above using the A-matrix. However when X_A has large dimension this approach becomes much harder. In §II.3 we develop formulas in terms of the B-matrix, which in this example recovers the above numbers, but has the advantage of working easily for any X_A of codimension two (even when the dimension is very large).

Definition II.2.5. For a face β of P we define $\mathrm{Vol}(\beta)$ as the volume of β, normalized with respect to the lattice $\mathbb{Z}\beta$.

Definition II.2.6. Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety and let $P = \mathrm{Conv}(A)$. The dimension i *Chern-Mather volume*, V_i, of X_A is given by

$$V_i = \sum_{\substack{\beta \text{ a face of } P \\ \text{with } \dim(\beta)=i}} \mathrm{Vol}(\beta)\, \mathrm{Eu}(\beta).$$

When X_A is smooth we have that $\mathrm{Eu}(\beta) = 1$ for all faces of P, and V_i is the sum of the normalized volumes of all dimension i faces of P.

Let $\mathrm{CH}^*(\mathbb{P}^{n-1}) \cong \mathbb{Z}[h]/(h^n)$ denote the Chow ring of $\mathbb{P}^{n-1}$, with h representing the rational equivalence class of a hyperplane in $\mathbb{P}^{n-1}$. We may express the push-forward of the Chern-Mather class of X_A to $\mathbb{P}^{n-1}$ as

$$c_M(X_A) = \sum_{i=0}^{\dim(X_A)} V_i h^{n-i} \in \mathrm{CH}^*(\mathbb{P}^{n-1}), \tag{II.9}$$

where the V_i are the Chern-Mather volumes of Definition II.2.6. From [Pie16] we have that the Chern-Mather class, in the Chow ring of X_A, may be written as

$$C_M(X_A) = \sum_{\alpha} \mathrm{Eu}(\alpha)[X_\alpha] \in \mathrm{CH}^*(X_A). \tag{II.10}$$

Using [HS18, Thm. 1.2] we may also write the polar degrees of a projective toric variety $X_A \subset \mathbb{P}^{n-1}$ in terms of the Chern-Mather volumes of X_A as

$$\delta_i(X_A) = \sum_{j=i+1}^{n-2} (-1)^{n-3-j} \binom{j}{i+1} V_{j-1} \tag{II.11}$$

for $i = 0, ..., n-3$. Using the formula above and the fact that the ED degree is the sum of the polar degrees (see also [HS18, Thm. 1.1]) we obtain:

$$\mathrm{EDdegree}(X_A) = \sum_{j=0}^{n-3} (-1)^{n-3-j}(2^{j+1} - 1)V_j. \tag{II.12}$$

The main task, from a practical point of view, when computing the invariants discussed above is computing the expressions $\mu(\alpha, \beta)$ appearing in Proposition II.2.3; giving formulas for this expression will be the main focus of §II.3. In proving these results we will make use of the method of Helmer and Sturmfels [HS18, Remark 2.2] stated here as Proposition II.2.7.

Proposition II.2.7. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety with associated polytope $P = \mathrm{Conv}(A)$, let α be a face of P and let β be a face of α. Order the columns of A so that those in β comes first, then those from $\alpha \setminus \beta$ and finally those in $A \setminus \alpha$. The row Hermite normal form of this reordered matrix has block structure*

$$\begin{bmatrix} * & * & * \\ 0 & C & * \\ 0 & 0 & * \end{bmatrix}$$

where the integer matrix C has $\dim(\alpha) - \dim(\beta)$ rows and

$$\mu(\alpha, \beta) = \mathrm{Vol}(\mathrm{Conv}(C \cup 0)) - \mathrm{Vol}(\mathrm{Conv}(C)).$$

II.2.3 Working with the Gale Dual

When X_A is a codimension two projective toric variety the Gale dual matrix B of A has only two columns, meaning if we use B we may work over integer lattices in $\mathbb{Z}^2$ rather than in the (often) much larger integer lattices in $\mathbb{Z}^{\dim(X_A)}$. This approach yields significant benefits in computational speed (see §II.4) and also adds theoretical insights. In order to take advantage of this approach to compute polar degrees and other invariants we will need some basic results relating the structure of the Gale dual and the face structure of the polytope $P = \mathrm{Conv}(A)$.

Proposition II.2.8 ([CLS11, Lemma 14.3.3])**.** *Fix $I \subset \{1, ..., n\}$. The following are equivalent:*

- *There is a face β of P such that $a_i \in \beta$ if and only if $i \in I$.*
- *There are positive numbers t_i such that $\displaystyle\sum_{i \in I^c} t_i b_i = 0$.*

Let $P = \mathrm{Conv}(A)$ for a $d \times n$ integer matrix A. Motivated by Proposition II.2.8 we define the following notations, for a face α of P

$$A_\alpha = \{a_i \mid a_i \in A \cap \alpha\}, \quad B_\alpha = \{b_i \mid a_i \notin A_\alpha\}. \tag{II.13}$$

Using a slight abuse of notation we let A_α (resp. B_α) denote both the sets above and also the matrices with columns a_i (resp. with rows b_i). We will also let $\mathfrak{I}_\alpha$ be the set of integer indices of the rows of B_α.

In the case where $X_A \subset \mathbb{P}^{n-1}$ is a codimension two projective toric variety we can give a more specific description of the faces of P. In this case A is an $(n-2) \times n$ integer matrix. For any proper face α of P we have that A_α has either $\dim \alpha + 1$ or $\dim \alpha + 2$ lattice points; otherwise we would contradict the assumption that all b_i are non-zero. In the first of these cases α will be a simplex. Following [DS02] we make the following definition which, in terms of the B-matrix, singles out the faces of P which are not simplices.

Definition II.2.9. A line through 0 in $\mathbb{R}^2$ is said to be a *relevant line* if it contains two vectors b_r, b_s in opposite directions.

Proposition II.2.10. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety and let $P =$ $\mathrm{Conv}(A)$ be the associated polytope. Let β be a face of P. If all rows of B_β are contained in the same relevant line then $\dim \beta = n - |B_\beta| - 2$. If not then $\dim \beta = n - |B_\beta| - 1$, in which case β is a simplex.*

Proof. Assume all b_i are contained in a relevant line. Let $\beta = \beta_0 \subset \beta_1 \subset \cdots \subset \beta_r \subset P$ be a maximal chain of face inclusions. Since all b_i are relevant we see that we can remove one b_i from β_0 to get to the face β_1, remove two b_i to get β_2 and so on. Thus $|B_{\beta_r}| = |B_\beta| - r$. By Proposition II.2.8 a facet with points from a relevant line necessarily has 2 lattice points. Hence $|B_\beta| = r + 2$ and $\dim \beta = (n-4) - r = n - 4 - (|B_\beta| - 2) = n - |B_\beta| - 2$.

Assume that not all b_i are contained in the same relevant line. By a similar argument as above we can consider a maximal inclusion of faces. Either the facet β_r has 3 lattice points, in which case $\dim \beta = n - |B_\beta| - 1$, or it has two, which then has to be contained in a relevant line. However now there has to be an inclusion $\beta_i \subset \beta_{i+1}$ such that all lattice points of β_{i+1} are in the relevant line, but not all in β_i. By Gale duality we must have that $|\beta_i| - |\beta_{i+1}| \geq 2$. From this it follows that $\dim \beta = n - |B_\beta| - 1$. Since β has $n - |B_\beta|$ lattice points it is a simplex. ∎

Example II.2.11. For the matrix A in Example II.2.1 the corresponding Gale dual matrix B is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \\ -4 & 0 \\ 1 & -3 \end{bmatrix},$$

note that $A \cdot B = 0$. We see that the span of $(1,0)$ is a relevant line which corresponds to the fact that the edge e_1 in Figure II.1 has three lattice points, instead of two.

Definition II.2.12. For a Gale dual matrix B of an $(n-2) \times n$ integer matrix A, we define the notation $[i,j] := \det(b_i, b_j)$, where b_ℓ denotes the ℓ^{th} row of B.

Proposition II.2.13. *Let B be a given $2 \times n$ matrix such that the rows of B span $\mathbb{Z}^2$ over $\mathbb{Z}$. Assume without loss of generality that $[1,2] \neq 0$. Then B is the Gale dual of the matrix*

$$A = \begin{bmatrix} [2,3] & -[1,3] & [1,2] & 0 & 0 & \cdots & 0 \\ [2,4] & -[1,4] & 0 & [1,2] & 0 & \cdots & 0 \\ [2,5] & -[1,5] & 0 & 0 & [1,2] & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & 0 \\ [2,n-1] & -[1,n-1] & 0 & 0 & \cdots & [1,2] & 0 \\ 1 & 1 & 1 & 1 & & 1 \cdots & 1 \end{bmatrix}.$$

Proof. Writing out the matrix multiplication we see that $AB = 0$, hence $\mathrm{im}(B) \subseteq \ker(A)$. Letting v, w be generators of $\ker(A)$ we see that the columns c_1, c_2 of B have to be of the form

$$c_1 = pv + qw, \quad c_2 = sv + tw \quad \text{with} \quad D = \begin{vmatrix} p & q \\ s & t \end{vmatrix} \neq 0.$$

A computation shows that every 2×2 minor of B will have D as a factor. Since the rows of B span $\mathbb{Z}^2$ there must exist vectors v, w in the rowspan of B with $\det(v,w) = 1$. Write $v = \sum_{i=1}^n a_i b_i, w = \sum_{i=1}^n c_i b_i$. Then $1 = \det(v,w) = \sum_{i,j=1}^n a_i c_j \det(b_i, b_j)$, hence $|D|$ must be a factor of 1 thus $|D| = 1$ and the columns of B form a basis of $\ker(A)$. ∎

II.2.4 Computing Lattice Indices

Lattice indices appear frequently in the main results presented in §II.3. Our primary motivation in §II.3 is to provide effective formula to compute the invariants discussed in §II.2.2; hence we require explicit methods for lattice index computation. Consider a $d \times n$ integer matrix A of full rank d. Let $\mathbb{Z}A$ denote the integer span of the columns of the matrix A. We wish to compute the lattice index $[\mathbb{Z}^d : \mathbb{Z}A]$.

Proposition II.2.14. *Let A be an $d \times n$ integer matrix with full rank,* $\operatorname{rank}(A) = d$. *Also let M_A denote the $d \times d$ integer matrix specified by the non-zero columns of the Hermite normal form of A (computed by elementary integer column operations on A). We have that*

$$[\mathbb{Z}^d : \mathbb{Z}A] = \det(M_A).$$

Proof. The matrix A has rank d (over $\mathbb{Z}$), this implies that the column space is spanned by d vectors, and hence when we perform the elementary integer column operations to compute the Hermite normal form we will retain only d non-zero columns. The matrix M_A is then a square matrix whose entries are the coefficients of $\mathbb{Z}A$ in the standard basis for $\mathbb{Z}^d$, by [Rot10, Corollary 9.63] the conclusion follows. ■

A second way to compute the lattice index $[\mathbb{Z}^d : \mathbb{Z}A]$ is given by the following proposition.

Proposition II.2.15. *Let A be an $d \times n$ integer matrix with full rank,* $\operatorname{rank}(A) = d$. *Let $v = \binom{n}{d}$ and let $c_1, \dots, c_v$ be the v maximal (that is $d \times d$) minors of A. Then we have that*

$$[\mathbb{Z}^d : \mathbb{Z}A] = \gcd(c_1, \dots, c_v).$$

Proof. The $d \times d$ minors of A generate what is called the d^{th} Fitting ideal of A, $\mathrm{Fit_d(A)}$. By [Kea98, Lemma 11.2.1] we have that the Fitting ideal is preserved by elementary (integer in our case) row or column operations on A, i.e. we have that $\mathrm{Fit_d(A)} = \mathrm{Fit_d(Herm(A))}$ where Herm(A) is the (column-wise) Hermite normal form of A. Since $\mathbb{Z}$ is a principal ideal domain $\mathrm{Fit_d(A)}$ is generated by $\gcd(c_1, \dots, c_v)$, and by Proposition II.2.14 we have that $\mathrm{Fit_d(Herm(A))}$ is generated by $[\mathbb{Z}^d : \mathbb{Z}A]$; this gives the stated result. ■

We note that the second method to compute $[\mathbb{Z}^d : \mathbb{Z}A]$ is less computationally efficient, but could still be convenient in some cases.

Remark II.2.16. Assume that $X_A \subset \mathbb{P}^{n-1}$ is a toric variety of codimension two and let B be a Gale dual of A. Then, since the rows of B span $\mathbb{Z}^2$ [DS02, pg. 4], we have by Proposition II.2.15 that $\gcd(\{[i,j]\}_{i,j}) = 1$.

Remark II.2.17. We say that an integer vector v in $\mathbb{Z}^l$ is a *primitive vector* if v is not a non-trivial integer multiple of another integer vector in $\mathbb{Z}^l$. Let B be a $n \times 2$ integer matrix whose rows span $\mathbb{Z}^2$ and let v be a primitive vector in $\mathbb{Z}^2$. It is well known that we can choose a basis for $\mathbb{Z}^2$ consisting of v and another vector w, such that $\det(v, w) = 1$. Using this basis we can write $b_i = a_i v + c_i w$ for some integers a_i, c_i. Then

$$\det(v, b_i) = c_i, \quad \text{and} \quad \det(b_i, b_j) = a_i c_j - a_j c_i.$$

Note that if $\gcd(\det(v, b_i)) = \gcd(c_i) > 1$ then $\gcd([i, j]) > 1$, which contradicts the assumption that the rows of B span $\mathbb{Z}^2$ by Remark II.2.16. Hence $\gcd(\det(v, b_i)) = 1$.

Proposition II.2.18. *Let A be the $d \times n$ integer matrix from Proposition II.2.13. Then we have that $[\mathbb{Z}^d : \mathbb{Z}A] = [1, 2]^{n-4}$.*

Proof. Let S denote the set of the last $n - 4$ columns of the matrix A from Proposition II.2.13; we will now apply Proposition II.2.15. Each maximal minor of A is the determinant of a $(n-2) \times (n-2)$ matrix $\mathfrak{m}$. This matrix $\mathfrak{m}$ will have at least $(n - 4)$ columns coming from the set S, i.e. $\mathfrak{m}$ will have at least $n - 4$ columns with only two non-zero entries.

If $\mathfrak{m}$ has $(n - 2)$ vectors from S then $\mathfrak{m}$ is a lower triangular matrix and $\det(\mathfrak{m}) = [1, 2]^{n-3}$. If $\mathfrak{m}$ has $(n - 3)$ vectors from S then $\mathfrak{m}$ has one of the vectors a_1 or a_2 as a column. Performing determinant preserving row and column operations on $\mathfrak{m}$ yields a diagonal matrix, from this we obtain one of the following:

$$\det(\mathfrak{m}) = \pm[1, 2]^{n-4}[2, i], \quad \text{or} \quad \det(\mathfrak{m}) = \pm[1, 2]^{n-4}[1, i], \ \text{for } i = 3, ..., n-1, \quad \text{or}$$

$$\det(\mathfrak{m}) = \pm[1, 2]^{n-4} \left([1, 2] - \sum_{i=3}^{n-1} [2, i] \right), \quad \text{or} \quad \det(\mathfrak{m}) = \pm[1, 2]^{n-4} \sum_{i=2}^{n-1} [1, i]$$

with the choice depending on which of a_1 or a_2 appears in $\mathfrak{m}$ and on which column in S does not appear in $\mathfrak{m}$.

Observe that since $\sum_{i=1}^{n} b_i = 0$ then, by elementary properties of determinants, we have that the two last cases can be rewritten as:

$$\det(\mathfrak{m}) = \pm[1, 2]^{n-4} \left([1, 2] - \sum_{i=3}^{n-1} [2, i] \right) = \pm[1, 2]^{n-4} (\sum_{i=1}^{n-1} [i, 2]) = \pm[1, 2]^{n-4}[2, n],$$

and

$$\det(\mathfrak{m}) = \pm[1, 2]^{n-4} \sum_{i=2}^{n-1} [1, i] = \pm[1, 2]^{n-4}[n, 1].$$

In the final case, if $\mathfrak{m}$ has $(n - 4)$ vectors from S, then both a_1, a_2 appear as columns of $\mathfrak{m}$. Let k_1, k_2 be the indices of the columns vectors from S which do not appear in $\mathfrak{m}$; note $k_1, k_2 \in \{3, \ldots, n\}$. The current case has two subcases, first the situation where $k_1 \neq k_2 \neq n$ and second the situation where one of k_1 or k_2 is equal to n. We may again perform elementary row and column operations to obtain a diagonal matrix. In the first subcase, where $k_1 \neq k_2 \neq n$, this computation gives:

$$\det(\mathfrak{m}) = \pm[1, 2]^{n-5}(-[2, k_1][1, k_2] + [1, k_1][2, k_2]), \ k_1 \neq k_2, \ k_1, k_2 \in \{3, 4, \ldots, n-1\}.$$

By the Plücker relation defining $G(2,4) \subset \mathbb{P}^5$ we have that

$$-[2,k_1][1,k_2] + [1,k_1][2,k_2] = [1,2][k_1,k_2].$$

Hence $\det(\mathfrak{m}) = [1,2]^{n-4}[k_1,k_2]$, where $3 \leq k_1 \neq k_2 \leq n-1$. Now consider the second subcase, that is the case where one of k_1 or k_2 is equal to n (i.e where the n^{th} column of A does not appear in $\mathfrak{m}$). Suppose (without loss of generality) that $k_2 = n$, then we have that:

$$\det(\mathfrak{m}) = \pm[1,2]^{n-5}\left([1,k_1]\sum_{i=3,\ i\neq k_1}^{n-1}[2,i] - [2,k_1]\sum_{i=3,\ i\neq k_1}^{n-1}[1,i] - [1,2]([1,k_1]+[2,k_1])\right).$$

Again applying the Plücker relations as above we have that

$$\begin{aligned}\det(\mathfrak{m}) &= \pm\,[1,2]^{n-5}\left([1,2]\sum_{i=3,\ i\neq k_1}^{n-1}[k_1,i] - [1,2]([1,k_1]+[2,k_1])\right)\\ &= \pm\,[1,2]^{n-4}\left(\sum_{i=3,\ i\neq k_1}^{n-1}[k_1,i] - [1,k_1] - [2,k_1]\right)\\ &= \pm\,[1,2]^{n-4}\sum_{i=1}^{n-1}[k_1,i] = \pm[1,2]^{n-4}[k_1,n].\end{aligned}$$

Putting all the cases together we see that the maximal minors of A all have values $\pm[1,2]^{n-4}[i,j]$ for some $i \neq j$. Hence their greatest common divisor equals $[1,2]^{n-4}\gcd([i,j])$. By Remark II.2.16 we have that $\gcd([i,j]) = 1$; the conclusion follows. ■

II.2.5 Other Results Needed for Gale Dual Computations

In this subsection we collect some results on the degrees of lattice ideals, these results will be needed in §II.3.

The degree of a codimension one lattice ideal is the degree of the defining polynomial.

Proposition II.2.19. *The degree of a homogeneous lattice ideal I_B associated to a $n \times 1$ matrix B is given by*

$$\deg(I_B) = \sum_{i|b_i>0} b_i.$$

If now I_B is a codimension two lattice ideal we define the following: For each i, j, if b_i and b_j lie in the interior of opposite quadrants, then define

$$\nu_{ij} := \min\{|b_{i1}b_{j2}|, |b_{i2}b_{j1}|\}.$$

Let β_i be the sum of all positive entries in the i^{th} column of B.

Proposition II.2.20. *[DS02, Corollary 2.2] The degree of a homogeneous lattice ideal I_B associated to a $n \times 2$ integer matrix B is given by*

$$\deg(I_B) = \beta_1\beta_2 - \sum_{i,j} \nu_{ij}.$$

Corollary II.2.21. *If I_B is a prime homogeneous lattice ideal associated to an integer matrix B with Gale dual A then*

$$\mathrm{Vol}(\mathrm{Conv}(A)) = \beta_1\beta_2 - \sum_{i,j} \nu_{ij}.$$

Example II.2.22. For the B-matrix in Example II.2.1, all ν_{ij} equal zero, hence $\deg I_B = \beta_1\beta_2 = 3 \cdot 4 = 12$.

Definition II.2.23. For an inclusion $L \subset M$ of abelian groups, we define $T(M/L)$ to be the torsion subgroup and $|T(M/L)|$ to be its order.

When proving our main results in §II.3 we will sometimes need to compute volumes of convex hulls of lattice points where some lattice points appear more than once. The following proposition shows that this is also expressible as a degree of a lattice ideal, hence the results above can be applied.

Theorem II.2.24. *[OPV14, Theorem 4.6] Given an integer matrix B whose rows generate a r-dimensional lattice $\mathbb{Z}B \subset \mathbb{Z}^n$ and defining a codimension r lattice ideal I_B, there exists a $(n-r) \times n$ matrix $A = [v_1, ..., v_n]$ of rank $n-r$ such that $\mathbb{Z}B \subset \ker(A)$ and*

$$\deg(I_B) = |T(\mathbb{Z}^n/\mathbb{Z}B)| \, \mathrm{Vol}(\mathrm{Conv}(0, v_1, ..., v_n))$$

where the volume is normalized with respect to $\mathbb{Z}A$.

II.3 Computing invariants of codimension two toric varieties

Let X_A be a codimension two projective toric variety and let B be the Gale dual of the matrix A. In this case A is an $(n-2) \times n$ integer matrix and B is an $n \times 2$ integer matrix. From the results in §II.2.2 we see that to compute the Chern-Mather volumes, polar degrees, and the ED degree of X_A, we must compute both the normalized relative subdiagram volumes of all chains of faces and the normalized volumes of all faces of the polytope $P = \mathrm{Conv}(A)$. In this section we present our main results. These results give explicit closed form expressions for the required normalized volume and normalized subdiagram volume computations in terms of the Gale dual matrix B. Both a theoretical analysis and practical testing shows that the methods using the matrix B offer a quite substantial computational performance gain relative to the methods of [HS18] when $\mathrm{codim}(X_A) = 2$, see §II.4 for a discussion of this.

Example II.3.1. As discussed above our goal in this paper is to compute the polar degrees and ED degree (and other associated invariants) using the Gale dual matrix B of A when X_A has codimension two. Continuing Example II.2.1 we now summarize the volumes and subdiagram volumes of the faces of the polytope $P = \text{Conv}(A)$ from Figure II.1 in Table II.1. In this section we will develop the necessary results to fill in this table using only the matrix B. Using the

α	B_α	$\text{Vol}(\alpha)$	$\mu(P,\alpha)$	$i(P,\alpha)$	$\text{Eu}(\alpha)$
e_1	$\{b_1, b_4\}$	3	1	1	1
e_2	$\{b_2, b_4, b_5\}$	1	1	1	1
e_3	$\{b_3, b_4, b_5\}$	1	2	4	8
v_1	$\{b_2, b_3, b_4, b_5\}$	1	9	1	0
v_2	$\{b_1, b_3, b_4, b_5\}$	1	8	1	2
v_3	$\{b_1, b_2, b_4, b_5\}$	1	2	1	0

Table II.1: Invariants of P.

information in Table II.1 along with Definition II.2.6 we obtain the Chern-Mather volumes

$$V_0 = 0 + 2 + 0 = 2,$$

$$V_1 = 3 \cdot 1 + 1 \cdot 1 + 1 \cdot 8 = 12,$$

$$V_2 = 12.$$

Substituting these values into (II.11), and (II.12) we have that the polar degrees and ED degree are:

$$\delta_0(X_A) = 3V_2 - 2V_1 + V_0 = 14,$$

$$\delta_1(X_A) = 3V_2 - V_1 = 24,$$

$$\delta_2(X_A) = V_2 = 12,$$

$$\text{EDdegree}(\mathrm{X_A}) = \delta_0(\mathrm{X_A}) + \delta_1(\mathrm{X_A}) + \delta_2(\mathrm{X_A}) = 50.$$

II.3.1 Gale Dual Formulas for Subdiagram Volumes in Codimension Two

As above we consider a codimension two toric variety X_A in $\mathbb{P}^{n-1}$ and let $P = \text{Conv}(A)$. In this subsection we present several formulas for subdiagram volumes covering all possible expressions which could appear in the computation of the polar degrees and ED degree of X_A. Let α and β be faces of P. These subdiagram volumes can be broadly grouped into two types, those of the form $\mu(P, \beta)$ and those of the form $\mu(\alpha, \beta)$ where $\beta \subset \alpha$.

II.3.1.1 Subdiagram volumes $\mu(P, \beta)$

Let β be a face such that B_β only has vectors from the same relevant line. Let v be a primitive vector in the relevant line and define λ_i by $b_i = \lambda_i v$, for $b_i \in B_\beta$. With these notations we define

$$v_+^\beta = \{i \mid b_i \in B_\beta, \lambda_i > 0\}$$

$$v_-^\beta = \{i \mid b_i \in B_\beta, \lambda_i < 0\}.$$

Theorem II.3.2. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety of codimension 2 and $P = \mathrm{Conv}(A)$. Let β be a face of P having codimension r with only lattice points from a relevant line with primitive vector v. Let the set $\mathfrak{I}_\beta$ index the rows of B_β, then*

$$\mu(P, \beta) = \frac{\min\left(\sum_{i \in v_+^\beta} |\lambda_i|, \sum_{i \in v_-^\beta} |\lambda_i|\right)}{\gcd\left(\lambda_i \mid b_i = \lambda_i v\right)_{i \in \mathfrak{I}_\beta}}, \text{ and } \ i(P, \beta) = \gcd\left(\lambda_i \mid b_i = \lambda_i v\right)_{i \in \mathfrak{I}_\beta}.$$

Proof. We will apply Proposition II.2.7. After reordering we may assume that B_β consists of the rows $b_2, ..., b_r$ of B. Since all lattice points of B_β are contained in the same relevant line we have $[2, i] = 0$ for all $i = 3, ..., r$. This implies that after reordering the columns as in Proposition II.2.7 A has block form

$$A = \begin{bmatrix} D & * \\ 0 & C \end{bmatrix}. \tag{II.14}$$

The submatrix C has $r - 2$ rows and is given by

$$C = \begin{bmatrix} -[1,3] & [1,2] & 0 & 0 & \cdots & 0 \\ -[1,4] & 0 & [1,2] & 0 & \cdots & 0 \\ -[1,5] & 0 & 0 & [1,2] & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ -[1,r] & 0 & 0 & \cdots & \cdots & [1,2] \end{bmatrix}.$$

By considering maximal minors we see that the lattice spanned by C has index $j = [1,2]^{r-3} \gcd([1,i])_{i \in \mathfrak{I}_\beta}$. We know that $b_i = \lambda_i v$ for all $i = 2, \ldots, r$, let $|b_1, v|$ be the value of the determinant of the matrix with rows b_1 and v, then $[1, i] = \lambda_i |b_1, v|$ and we have

$$C = |b_1, v| \cdot \begin{bmatrix} -\lambda_3 & \lambda_2 & 0 & 0 & \cdots & 0 \\ -\lambda_4 & 0 & \lambda_2 & 0 & \cdots & 0 \\ -\lambda_5 & 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ -\lambda_r & 0 & 0 & \cdots & \cdots & \lambda_2 \end{bmatrix}. \tag{II.15}$$

Reformulating we see that the index $j = \lambda_2^{r-3} |v, b_1|^{r-2} \gcd(\lambda_i)_{i \in \mathfrak{I}_\beta}$. Let $P_1 = \mathrm{Conv}\,(C)$, $P_2 = \mathrm{Conv}\,(C \cup \{0\})$. By Proposition II.2.7 we have that

$$\mu(A, \beta) = \mathrm{Vol}(\mathrm{Conv}\,(C \cup \{0\})) - \mathrm{Vol}(\mathrm{Conv}\,(C)) = \mathrm{Vol}(P_2) - \mathrm{Vol}(P_1), \tag{II.16}$$

where Vol is the normalized $(r-2)$-dimensional volume. First we compute

$$\operatorname{Vol}(P_1) = \pm|v,b_1|^{r-2}\begin{vmatrix} -\lambda_3 & \lambda_2 & 0 & 0 & \cdots & 0 \\ -\lambda_4 & 0 & \lambda_2 & 0 & \cdots & 0 \\ -\lambda_5 & 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \lambda_2 \\ -\lambda_r-\lambda_2 & -\lambda_2 & -\lambda_2 & \cdots & \cdots & -\lambda_2 \end{vmatrix}.$$

After doing row and column operations we get that

$$\operatorname{Vol}(P_1) = |v,b_1|^{r-2}\lambda_2^{r-3}\sum_{i=2}^{r} -\lambda_i.$$

Hence after taking the absolute value and normalizing with respect to the index j we get

$$\operatorname{Vol}(P_1) = \frac{|\sum_{i=2}^{r}\lambda_i|}{\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}}.$$

Now consider the polytope $P_2 = \operatorname{Conv}(C\cup\{0\})$. Volume is preserved under taking cones, so we may instead consider the normalized $r-2$ dimensional volume of the convex hull of

$$\tilde{C} = |b_1,v|\cdot\begin{bmatrix} -\lambda_3 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 \\ -\lambda_4 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ -\lambda_5 & 0 & 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ -\lambda_r & 0 & 0 & \cdots & \cdots & \lambda_2 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

$\tilde{C}$ corresponds to a codimension one toric variety $X_{\tilde{C}}$, by Theorem II.2.24 we have that

$$\operatorname{Vol}(P_2) = \frac{\deg(I_{\tilde{B}})}{|T(\mathbb{Z}^r/\mathbb{Z}\tilde{B})|}$$

where $\tilde{B} = \begin{bmatrix}\lambda_2 & \lambda_3 & \cdots & \lambda_r & -\sum_{i=2}^{r}\lambda_i\end{bmatrix}$ is the Gale dual of $\tilde{C}$. Applying Proposition II.2.19, we have that

$$\operatorname{Vol}(P_2) = \frac{\max\left(\sum_{i\in v_+^\beta}|\lambda_i|, \sum_{i\in v_-^\beta}|\lambda_i|\right)}{|T(\mathbb{Z}^r/\mathbb{Z}\tilde{B})|}.$$

Note that $|T(\mathbb{Z}^r/\mathbb{Z}\tilde{B})|$ equals the greatest common divisor of elements of $\tilde{B}$, i.e. $\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}$. Substituting the computed values into (II.16) we have:

$$\begin{aligned}\mu(P,\beta) &= \frac{\max\left(\sum_{i\in v_+^\beta}|\lambda_i|, \sum_{i\in v_-^\beta}|\lambda_i|\right)}{\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}} - \frac{|\sum_{i=2}^{r}\lambda_i|}{\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}} \\ &= \frac{\min\left(\sum_{i\in v_+^\beta}|\lambda_i|, \sum_{i\in v_-^\beta}|\lambda_i|\right)}{\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}}.\end{aligned}$$

Now compute the index $i(P,\beta)$. Consider the submatrix D from (II.14),

$$D = \begin{bmatrix} [2,r+1] & [1,2] & 0 & 0 & \cdots & 0 & 0 \\ [2,r+2] & 0 & [1,2] & 0 & \cdots & 0 & 0 \\ [2,r+3] & 0 & 0 & [1,2] & \cdots & \dots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & \dots & 0 \\ [2,n-1] & \vdots & \cdots & \ddots & \cdots & [1,2] & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

After taking the column-wise Hermite normal form of A, without switching the order of the columns, we will have $(n-2)$ non-zero columns. By Proposition II.2.14 we have that the index $[Z^{n-2} : \mathbb{Z}A]$ is equal to the determinant of these columns, that is $[Z^{n-2} : \mathbb{Z}A] = [1,2]^{n-4}$. Projecting onto the linear space spanned by β, we get that $[\mathbb{Z}^{r-2} : \mathbb{Z}A \cap \mathbb{R}\beta] = \frac{[1,2]^{n-4}}{\det D'}$ where D' is the nonzero submatrix of the Hermite normal form of A corresponding to D. By Lemma II.2.15 $\det(D')$ is equal to the greatest common divisor of the maximal minors of D. Consider the inclusion of lattices $\mathbb{Z}^{r-2} \supset \mathbb{Z}A \cap \mathbb{R}\beta \supset \mathbb{Z}\beta$, we have that

$$[\mathbb{Z}^{r-2} : \mathbb{Z}\beta] = [\mathbb{Z}^{r-2} : \mathbb{Z}A \cap \mathbb{R}\beta] i(A,\beta).$$

Let c be the greatest common divisor of the maximal minors of C and d be greatest common divisor of the maximal minors of D. Then $[\mathbb{Z}^{r-2} : \mathbb{Z}\beta] = c$, thus

$$i(P,\beta) = \frac{cd}{[1,2]^{n-4}}.$$

One computes that $c = [1,2]^{r-3} \gcd([1,i])_{i \in \mathfrak{I}_\beta}$ and that $d = [1,2]^{n-r-2} \gcd([2,j])_{j=1}^n$. Thus

$$\begin{aligned} i(P,\beta) &= \frac{cd}{[1,2]^{n-4}} = \frac{[1,2]^{r-3} \cdot \gcd([1,i])_{i \in \mathfrak{I}_\beta} \cdot [1,2]^{n-r-2} \cdot \gcd([2,j])_{j=1}^n}{[1,2]^{n-4}} \\ &= \frac{\gcd([1,i])_{i \in \mathfrak{I}_\beta} \cdot \gcd([2,j])_{j=1}^n}{[1,2]} \\ &= \frac{\gcd(\lambda_i \det(b_1, v))_{i \in \mathfrak{I}_\beta} \cdot \lambda_2 \cdot \gcd(\det(v, b_j))_{j=1}^n}{\lambda_2 \det(b_1, v)} \\ &= \gcd(\lambda_i)_{i \in \mathfrak{I}_\beta} \cdot \gcd(\det(v, b_j))_{j=1}^n. \end{aligned}$$

By Remark II.2.17 $\gcd(\det(v, b_j))_{j=1}^n = 1$. ∎

Example II.3.3. In Example II.2.1, the edge e_1 in Figure II.1 is not a simplex, hence by Theorem II.3.2 we have that $i(P, e_1) = \gcd(1,4) = 1$, and $\mu(P, e_1) = \min\{1,4\} = 1$.

Theorem II.3.4. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety with $\operatorname{codim}(X_A) = 2$ and set $P = \operatorname{Conv}(A)$. Take a face β of P such that not all $b_i \in B_\beta$ are contained* *in the same relevant line. Let* $w_\beta = \sum_{i \in \mathfrak{I}_\beta} -b_i$ *and let* B'_β *be the matrix obtained by adding* w_β *as an extra row of* B_β*. Then*

$$\mu(P,\beta) = \frac{\deg(I_{B'_\beta})}{|T(\mathbb{Z}^{r+1}/\mathbb{Z}B'_\beta)|} - \frac{\sum_{j\in\mathfrak{I}_\beta \mid \det(w_\beta,b_j)>0} \det(w_\beta,b_j)}{|T(\mathbb{Z}^{r+1}/\mathbb{Z}B'_\beta)|},$$

$$i(P,\beta) = |T(\mathbb{Z}^{r+1}/\mathbb{Z}B'_\beta)| = \gcd([i,j])_{i,j\in\mathfrak{I}_\beta}.$$

Proof. Again we use Proposition II.2.7. After rearranging the columns, the matrix has the following form:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & [2,3] & -[1,3] & [1,2] & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & [2,4] & -[1,4] & 0 & [1,2] & \ddots & 0 \\ 0 & 0 & 0 & 0 & [2,5] & -[1,5] & 0 & 0 & \ddots & [1,2] \\ [1,2] & 0 & 0 & 0 & \vdots & \vdots & 0 & 0 & \ddots & 0 \\ 0 & [1,2] & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & [2,n-1] & -[1,n-1] & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that, by Proposition II.2.10, β has codimension $r-2$. Now consider the matrix A, the only row operations we need to perform to pick out the correct submatrix C in Proposition II.2.7 is exchanging the top and bottom rows. Let $\{c_1,...,c_r\}$ denote the columns of the resulting $(r-2)\times r$ submatrix C. To compute the subdiagram volume $\mu(P,\beta)$ we first compute $\mathrm{Vol}(\mathrm{Conv}(c_1,...,c_r,0))$. Consider the $(r-1)\times(r+1)$ matrix matrix A' with rows $a'_1,...,a'_{r+1}$ of the form

$$A' = \begin{bmatrix} c_1 & c_2 & \cdots & c_r & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Observe that by construction A' has rank $r-1$ and that $B'_\beta \subset \ker A'$. We want to compute $\mathrm{Vol}(\mathrm{Conv}(c_1,...,c_r,0))$ normalized with respect to the lattice spanned by C. We have that

$$\mathrm{Vol}(\mathrm{Conv}(c_1,...,c_r,0)) = \mathrm{Vol}(\mathrm{Conv}(a'_1,...,a'_{r+1})),$$

since the second convex hull is equivalent to taking the cone over $\mathrm{Conv}(c_1,...,c_r,0)$ and normalized volume is preserved under taking cones.

B'_β generates a lattice ideal $I_{B'_\beta}$ of codimension two. Since $B'_\beta \subset \ker(A')$ applying Theorem II.2.24 gives

$$\mathrm{Vol}(\mathrm{Conv}(c_1,...,c_r,0)) = \mathrm{Vol}(\mathrm{Conv}(a'_1,...,a'_{r+1})) = \frac{\deg I_{B'_\beta}}{[\mathbb{Z}^{r+1} : \mathbb{Z}B'_\beta]}.$$

Now consider the volume of the convex hull of $\{c_1, ..., c_r\}$, normalized with respect to the lattice spanned by C. Let A'' be the $(r-1)\times r$ matrix obtained by adding the row $(1, ..., 1)$ to C. There are two cases. If $(1, ..., 1)$ is already contained in the row span of C then all lattice points c_i are contained in an affine hyperplane. Dimension considerations dictate that the resulting normalized volume is zero. Note that in this case we automatically get by Gale duality that $w_\beta = 0$, thus verifying this result. If $(1, ..., 1)$ is not in the row span of C then A'' has rank $r-1$. Consider the $1 \times r$ matrix B'' with rows $\det(w_\beta, b_i)$. Then B'' is contained in $\ker(A'')$. We have that $\mathrm{Vol}(\mathrm{Conv}(c_1, ..., c_r)) = k\,\mathrm{Vol}(\mathrm{Conv}(0, a''_1, ..., a''_r))$ where

$$k = \frac{[\mathbb{Z}^{r-1} : \mathbb{Z}A'']}{[\mathbb{Z}^{r-2} : \mathbb{Z}C]}. \tag{II.17}$$

The equality above follows from the fact that, for a polytope $\mathcal{P}$ of dimension $\mathfrak{n}$, the $\mathfrak{n}$-dimensional Euclidean volume of $\mathcal{P}$ is equal to the $\mathfrak{n}+1$-dimensional Euclidean volume of a pyramid of height one over $\mathcal{P}$. The integer k in (II.17) arises since the lattice indices of the polytopes generated by the points c_i and the points a''_i may differ.

We claim that $[\mathbb{Z}^{r-1} : \mathbb{Z}A''] = [1,2]^{r-3}\gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}$ and $[\mathbb{Z}^{r-2} : \mathbb{Z}C] = [1,2]^{r-3}\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}$ hence

$$k = \frac{\gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}}{\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}}.$$

Indeed $[\mathbb{Z}^{r-2} : \mathbb{Z}C]$ has to equal $[\mathbb{Z}^{r-1} : \mathbb{Z}A']$ which we know, by the proof of Proposition II.2.18, is equal to $[1,2]^{r-3}\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}$. The claim for the last of the indices can be proved using elementary row operations; A'' is a $r-1\times r$ matrix. Denote by m_i the maximal minor obtained by deleting column i. We see that

$$m_1 = [1,2]^{r-3}([1,2]+[1,3]+\cdots+[1,r]) = [1,2]^{r-3}\det(1, \sum_{i=1}^{r} b_i) = [1,2]^{r-3}\det(w_\beta, b_i).$$

Similarly $m_2 = [1,2]^{r-3}\det(b_2, w_\beta)$. For $k > 2$ we see that after column and row operations

$$m_k = \pm[1,2]^{r-4}\left([2,k]\sum_{i=1\ i\neq k}^{r}[1,i] + [1,k]\sum_{i=1\ i\neq k}^{r}[i,2]\right),$$

which after substituting for w_β, rearranging and using the Plucker relation, equals $\pm[1,2]^{r-3}\det(b_k, w_\beta)$. This proves the claim that $[\mathbb{Z}^{r-1} : \mathbb{Z}A''] = [1,2]^{r-3}\gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}$. Using this and Theorem II.2.24 we get that

$$\begin{aligned}\mathrm{Vol}(\mathrm{Conv}(c_1, ..., c_r)) &= \mathrm{Vol}(\mathrm{Conv}(0, a''_1, ..., a''_r))\frac{\gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}}{\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}}\\ &= \frac{\deg I_{B''}}{|T(\mathbb{Z}^{r+1}/\mathbb{Z}B'')|}\frac{\gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}}{\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}}.\end{aligned}$$

Now $|T(\mathbb{Z}^{r+1}/\mathbb{Z}B'')| = \gcd(\det(w_\beta, b_i))_{i\in\mathfrak{I}_\beta}$ and by Proposition II.2.19 we have that

$$\deg I_{B''} = \sum_{j\in\mathfrak{I}_\beta, \det(w_\beta,b_j)>0} \det(w_\beta, b_j),$$

hence

$$\mathrm{Vol}(\mathrm{Conv}(c_1, ..., c_r)) = \frac{\sum_{j\in\mathfrak{I}_\beta,\det(w_\beta,b_j)>0} \det(w_\beta, b_j)_{i\in\mathfrak{I}_\beta}}{\gcd([i,j])_{i,j\in I_\beta}}.$$

Finally it remains to prove that $i(A, \beta) = |T(\mathbb{Z}^r/\mathbb{Z}B'_\beta)|$ (the latter is equal to $\gcd([i,j])_{i,j\in\mathfrak{I}_\beta}$). By considering the block form of A from above we see that the lattice points of β generate a lattice of index $[1, 2]^{n-r-1}$. We also know that $[\mathbb{Z}^n : \mathbb{Z}A] = [1, 2]^{n-4}$. We get that

$$[1, 2]^{n-4} i(A, \beta) = [1, 2]^{n-r-1} [\mathbb{Z}^{r-2} : \mathbb{Z}C].$$

Since $[\mathbb{Z}^{r-2} : \mathbb{Z}C] = [1, 2]^{r-3} |T(\mathbb{Z}^r/\mathbb{Z}B'_\beta)|$ we obtain the desired result.

■

Example II.3.5. In Example II.2.1, the edge e_3 in Figure II.1 is a simplex, by Theorem II.3.4 we have

$$i(P, e_3) = \gcd(8, -8, 12) = 4, \text{ and } \mu(P, e_3) = \frac{12}{4} - \frac{4}{4} = 2$$

where the last number is obtained as follows. The matrix B'_{e_3} from Theorem II.3.4 is

$$B'_{e_3} = \begin{bmatrix} 2 & 2 \\ -4 & 0 \\ 1 & -3 \\ 1 & 1 \end{bmatrix}$$

and the vector w_{e_3} equals $(1, 1)$. The degree of $I_{B'_{e_3}}$ is 12 and is computed using Proposition II.2.20. Also note that in this case we have

$$(\det(w_{e_3}, b_3), \det(w_{e_3}, b_4), \det(w_{e_3}, b_5)) = (0, 4, -4).$$

We may now compute the Euler obstructions $\mathrm{Eu}(e_i) = i(P, e_i)\mu(P, e_i)$ for any edge e_i, the results are summarized in Table II.1.

To find the Euler obstruction of the vertex v_1 we will again apply Theorem II.3.4 to compute $\mu(P, v_1)$. For any vertex $i(P, \alpha)$ is automatically 1. The matrix B'_{v_1} is in this case just B itself, which has degree 12 (see Example II.2.11). We have that w_{v_1} equals $(1, 0)$ hence

$$(\det(w_{v_1}, b_2), \det(w_{v_1}, b_3), \det(w_{v_1}, b_4) \det(w_{v_1}, b_5)) = (1, 2, 0, -3),$$

giving

$$\mu(P, v_1) = 12 - 3 = 9.$$

To complete the computation of the Euler obstructions of the vertices (using Proposition II.2.3) we also need to compute $i(e_i, v_j)$ and $\mu(e_i, v_j)$. We develop the necessary tools to do this using the matrix B in §II.3.1.2 below.

II.3.1.2 Subdiagram volumes $\mu(\alpha, \beta)$

We now consider the subdiagram volumes of two proper faces of the polytope associated to a codimension two projective toric variety.

Theorem II.3.6. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety with* $\operatorname{codim}(X_A) = 2$*, set $P = \operatorname{Conv}(A)$, and let $\beta \subset \alpha$ be faces of P. We have that*

$$\mu(\alpha, \beta) = 1, \quad \text{and} \quad i(\alpha, \beta) = 1$$

if any of the following conditions hold:

(i) *not all rows of B_α are contained in a relevant line,*

(ii) *all rows of B_β are contained in the same relevant line.*

Proof. First consider case (i), where not all rows of B_α are contained in a relevant line. In this case both β and α are simplices; this means that the $\dim(\alpha)$-dimensional volume of $\alpha \backslash \beta$ is zero, and hence $\mu(\alpha, \beta) = \operatorname{Vol}(\operatorname{Conv}(A_\alpha)) = 1$, since α is a simplex. This concludes the proof of (i).

Now consider consider case (ii), where all rows of B_β are contained in the same relevant line. Set $r = |\mathfrak{I}_\beta|$ and $s = |\mathfrak{I}_\alpha|$. To calculate $\mu(\alpha, \beta)$ we are must pick out the correct submatrix C in Proposition II.2.7 and compute the resulting normalized volumes. By our assumption on the face structure of α and β the correct submatrix will be an $r - s \times r - s$ matrix of full rank. But the convex hull of $r - s$ points in $r - s$ dimensional space has volume 0, meaning the second term in the expression for $\mu(\alpha, \beta)$ in Proposition II.2.7 is zero. Thus $\mu(\alpha, \beta)$ is equal to the volume of the convex hull of the columns of C after we add 0. By the argument above this is a simplex. The volume of a simplex (inside the lattice spanned by the submatrix) equals one.

The lattice points of α span a linear subspace of dimension $n - s - 2$. The lattice points of β span a linear subspace of dimension $n - r - 2$. The lattice points of $\alpha \setminus \beta$ are $s - r$ lattice point which span a linear subspace of dimension $s - r$. Hence each $a_i \in \alpha \setminus \beta$ is part of a basis of the lattice L generated by α. Thus any lattice point of L which also lies in L' has to be in the lattice generated by β. It follows that the index $i(\alpha, \beta) = 1$. ∎

Theorem II.3.7. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety with* $\operatorname{codim}(X_A) = 2$ *and set $P = \operatorname{Conv}(A)$. Consider faces $\beta \subset \alpha$ of P where not all rows of B_β are contained in the same relevant line, but all rows B_α are contained in the same relevant line. Let v be primitive vector of the relevant line containing the rows of B_α and let $\gamma_i = \det(v, b_i)$, for $i \in \mathfrak{I}_\beta$. Then*

$$\mu(\alpha, \beta) = \frac{\min\left(\sum_{i\,:\,\gamma_i > 0} |\gamma_i|, \sum_{i\,:\,\gamma_i < 0} |\gamma_i|\right)}{\gcd(\gamma_i)_{i \in \mathfrak{I}_\beta}}, \quad \text{and} \quad i(\alpha, \beta) = \gcd(\gamma_i)_{i \in \mathfrak{I}_\beta}.$$

Proof. This proof of this result is very similar to that of Theorem II.3.2. We may order the rows of the matrix B (whose Gale dual defines P) so that $B_\alpha = \{b_1, b_3, \dots, b_{r+1}\}$ and $B_\beta = \{b_1, b_2, \dots, b_s\}$ where b_2 is not in the relevant line v. Then we have that $[1,3] = [1,4] = ... = [1, r+1] = 0$ since they are all contained in a relevant line. Hence we see that the correct submatrix C in Proposition II.2.7 is given by

$$C = \begin{bmatrix} -[1,r+2] & [1,2] & 0 & 0 & \cdots & 0 \\ -[1,r+3] & 0 & [1,2] & 0 & \cdots & 0 \\ -[1,r+4] & 0 & 0 & [1,2] & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ -[1,s] & 0 & 0 & \cdots & \cdots & [1,2] \end{bmatrix}.$$

This matrix has the same form as the matrix in (II.15), hence the remainder of the proof proceeds similarly to that of Theorem II.3.2. In the resulting formula we obtain sums over i such that $b_i \in B_\beta \setminus B_\alpha$, however since $\gamma_i = 0$ when $i \in B_\alpha$ the same formula is true when looping over all rows of B_β.

■

Example II.3.8. Resuming Example II.2.1 and Example II.3.5 we may now complete the computation of the Euler obstruction $\mathrm{Eu}(v)$ for a vertex v in Figure II.1. To compute the Euler obstructions of the vertices we need to compute the numbers $i(e_i, v_j)$ and $\mu(e_i, v_j)$ using Theorems II.3.6, II.3.6, and II.3.7. Since v_j is a vertex we have that $i(e_i, v_j) = 1$ whenever it is defined. Table II.2 gives the numbers $\mu(e_i, v_j)$. Putting this together we have that

$$\mathrm{Eu}(v_1) = -\mu(P, v) + \mu(e_2, v_1)\,\mathrm{Eu}(e_2) + \mu(e_3, v_1)\,\mathrm{Eu}(e_3) = -9 + 1 \cdot 1 + 1 \cdot 8 = 0.$$

The Euler obstructions of the other vertices are computes similarly and are summarized in Table II.1.

$\mu(e_i, v_j)$	v_1	v_2	v_3
e_1	$*$	2	1
e_2	1	$*$	1
e_3	1	1	$*$

Table II.2: Subdiagram volumes $\mu(e_i, v_j)$. We write $*$ when there is no containment relation between e_i and v_j, i.e. when $\mu(e_i, v_j)$ is undefined.

II.3.2 Volume Calculation Via the Gale Dual in Codimension two

In this subsection we consider the problem of computing the volume of faces of the polytope associated to a codimension two projective toric variety X_A using the Gale dual B of A.

Proposition II.3.9. *Let* $X_A \subset \mathbb{P}^{n-1}$ *be a projective toric variety with* $\operatorname{codim}(X_A) = 2$ *and set* $P = \operatorname{Conv}(A)$*. Let* β *be a face such that all rows of* B_β *are contained in a relevant line. Let* v *be a primitive vector in the relevant line. Then*

$$\operatorname{Vol}(\beta) = \sum_{j \in \mathfrak{I}_\beta^c \,|\, \det(v,b_j)>0} \det(v, b_j).$$

Proof. The lattice points of β defines a toric variety X_β of codimension 1. We have that $\deg X_\beta = \operatorname{Vol}(\beta)$. By Proposition II.2.19 we have that

$$\deg X_\beta = \sum_{j \in \mathfrak{I}_\beta^c \,|\, \det(v,b_j)>0} \det(v, b_j).$$

■

We may now fill in all values in Table II.1 using only the Gale dual matrix B and obtain the ED degree and polar degrees of the variety X_A from Example II.2.1. The results detailed in this section allow us to compute the ED degree and polar degrees of much larger examples much faster, as discussed in §II.4 below.

Finally we remark that we would have hoped to solve the recursion for the Euler obstruction and find compact formulas for the polar degrees and the ED degree of a projective toric variety of codimension two, similar to the codimension one case [HS18, Theorem 3.7]. Unfortunately we have not been able to simplify the expressions sufficiently to find satisfying formulas in the codimension two case. While the lack of such formulas has little effect on the computational performance of the codimension two methods finding them would be mathematically appealing.

II.4 Computational Performance

In this section we briefly compare the computational performance of the specialized codimension two methods developed in §II.3 which use the Gale dual matrix B with the performance of the general purpose (i.e. for any codimension) A-matrix methods described in [HS18]. We will refer to these as the "B-matrix method" and the "A-matrix method", respectively.

When computing the polar degrees or ED degree of a projective toric variety using (II.11), with either the A-matrix or B-matrix method, the primary computational task is to compute the Chern-Mather volumes of X_A. While the number of steps in the recursive loops for both the A and B matrix methods is the same the computational cost of computing the subdiagram volumes $\mu(\alpha, \beta)$ differs quite substantially. We will focus on analyzing the cost of this computation in the case where $\operatorname{codim}(X_A) = 2$ (i.e. where the methods of §II.3 are applicable).

Example	Size A	A-matrix method [HS18]	B-matrix method §II.3	Find faces	Speedup factor
X_{A_1}	4×6	4.6s	0.2s	0.3s	23.0
X_{A_2}	4×6	6.1s	0.2s	0.3s	30.5
X_{A_3}	6×8	106.1s	1.6s	2.3s	66.3
X_{A_4}	6×8	118.4s	1.8s	2.7s	65.8
X_{A_5}	7×9	495.5s	5.3s	8.1s	93.5
X_{A_6}	8×10	2611.5s	26.7s	37.7s	97.8

Table II.3: Average run times to compute the polar degrees of a codimension two projective toric variety X_A. The run time to generate the face lattice of $\mathrm{Conv}(A)$ is listed separately in the fifth column since both methods must perform this step (hence the total run time is the sum the time to find the faces and the time of either the A-matrix or B-matrix method).

II.4.1 Computational Cost of $\mu(\alpha, \beta)$ for General A

First consider the A-matrix method. Let X_A be a projective toric variety with $P = \mathrm{Conv}(A)$ and let $\alpha \supset \beta$ be faces of P. Further suppose that the face lattice has already been computed and that the relevant Hermite normal forms (needed for Proposition II.2.7) have been stored in the process. Let $r = \dim(\alpha) - \dim(\beta)$. To compute the subdiagram volume (using (II.8) or Proposition II.2.7) we must compute the following things:

- the convex hulls of two collections of at least $r + 1$ lattice points in $\mathbb{R}^r$
- the volumes of two dimension r polytopes.

For computing the convex hull of m points in $\mathbb{R}^r$ there exist known (optimal) algorithms of complexity

$$O\left(m \log(m) + m^{\lfloor \frac{r}{2} \rfloor}\right),$$

see [Cha93]. Calculating the volume of a polytope in dimension r is known to be a $\#P$-hard problem [DF88, Theorem 1]. For existing algorithms (to the best of our knowledge) there is not a known compact (i.e. readable/meaningful) run time bound for finding the dimension of an arbitrary polytope in dimension r and different algorithms may vary from being exponential to factorial in r for different polytopes. Using known algorithms, the computational cost of computing the volume of the dimension r hypercube varies from being approximately factorial in r, i.e. $O(r!)$, to being approximately exponential in r, i.e. $O(r^2 4^r)$, depending on the algorithm chosen. See [BEF00] for an in depth discussion of current algorithms. Hence, in particular, the cost of computing the subdiagram volume $\mu(\alpha, \beta)$ will be (at least) exponential, possibly factorial, in the relative dimension $r = \dim(\alpha) - \dim(\beta)$. Further note that r may be as large as $d - 2$ for a $d \times n$ integer matrix A.

II.4.2 Computational Cost of $\mu(\alpha, \beta)$ using §II.3 when $\mathbf{codim}(X_A) = 2$

Let X_A be a codimension two projective toric variety with $P = \mathrm{Conv}(A)$ and let $\alpha \supset \beta$ be faces of P. We again suppose that the face lattice has already been computed, and that faces contained in relevant lines have been identified during this process (this includes the computation of the scaling factors λ of each vector $b = \lambda v$ for v the primitive vector defining the relevant line). By precomputing all lattice indices $[\mathbb{Z}^d : \mathbb{Z}A_\alpha]$ for each face α of P we may compute many of the expressions $\mu(\alpha, \beta)$ in constant time (i.e. only a table lookup is needed). Note that, using Proposition II.2.14, the computation of the lattice index $[\mathbb{Z}^d : \mathbb{Z}A_\alpha]$ for each face requires the computation of one Hermite normal form and one determinant of the resulting square matrix; many efficient algorithms exist for these computations. Assuming the above precomputations, the number of operations required to compute $\mu(\alpha, \beta)$ using the methods of §II.3 is as follows:

- constant if α is a proper face of P and either all lattice points defining β are in a relevant line or if neither α nor β is contained in a relevant line
- linear in ℓ_β, where ℓ_β is the number of rows of B_β, if α is a proper face of P where all rows of B_α are contained in a relevant line but all rows of B_β are not
- proportional to the number of operations to compute a greatest common divisor of ℓ_β integers if $\alpha = P$ and all rows of B_β are contained in a relevant line
- quadratic in ℓ_β if $\alpha = P$ and the rows of B_β are not contained in a relevant line (in this case we are summing 2×2 determinants)

In particular we see that for the vast majority of possible pairs of faces $\alpha \supset \beta$ the cost of computing $\mu(\alpha, \beta)$ will be linear or constant, and at worst will be polynomial in the number of rows of B_β, which will always be less than or equal to the number of rows of B.

II.4.3 Summary

Suppose A is a $(n-2) \times n$ integer matrix defining a codimension 2 projective toric variety X_A. Examining the definitions of the Chern-Mather volume and Euler obstruction we see that to compute all Chern-Mather volumes (and hence to compute the ED degree or polar degrees) we must compute the subdiagram volume $\mu(\alpha, \beta)$ for all possible pairs of faces $\alpha \supset \beta$ of $P = \mathrm{Conv}(A)$. Let F denote the number of faces of P, there are $\frac{F^2-F}{2}$ such pairs. Using the general purpose A-matrix methods we must perform computations which are at least exponential, possibly factorial, in $\dim(\alpha) - \dim(\beta)$; for the majority of pairs this will mean computations that are *at least* exponential in a number larger than $\frac{n}{3}$.

Let ℓ_β denote the number of rows in the matix B_β; this number will always be less than or equal to n. With the specialized B-matrix methods of §II.3 the computation of $\mu(\alpha, \beta)$ will be done in constant or linear time relative to ℓ_β for $\frac{F^2-3F}{2}$ of the pairs with the remaining F pairs being done in at most quadratic, $O(\ell_\beta^2)$, time relative to the number of rows of B_β.

In light of the discussion above the significant runtime gains yielded by the B-matrix methods, as displayed in Table II.3, are not surprising. It should, however, be noted that both combinatorial methods, either A-matrix or B-matrix, will be able to compute ED degrees and other invariants for projective toric varieties X_A which simply would not be feasible using other current methods. For example $\mathrm{EDdegree}(X_{A_5}) = 301137686$ (see Table II.4) represents the degree of the variety defined by the critical equations of the Euclidean distance function for X_{A_5}. Computing this number using algebraic/geometric methods (i.e. Gröbner basis, numerical algebraic geometry, etc.) would require finding the degree of a zero dimensional variety consisting of *greater than* 300 *million* isolated points in $\mathbb{P}^8$. Such a computation would be infeasible with current algebraic/geometric methods even over a span of weeks running on a super computer whereas the A-matrix or B-matrix methods compute this number (on a laptop) in a matter of minutes or seconds, respectively.

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Appendix II.A Computational Examples List

In this appendix we list the integer matrices A defining the codimension two toric varieties listed in Table II.3.

$$A_1 = \begin{bmatrix} 10 & 1 & 0 & -7 & 0 & 0 \\ -7 & 0 & 1 & 5 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 3 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & 0 & 0 & 1 & 1 & 2 \\ 3 & 5 & 0 & 2 & 1 & 3 \\ 0 & 1 & 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & -31 & 0 & -7 & 1 & -31 \\ 0 & 0 & 0 & -12 & 0 & -2 & 0 & -11 \\ 0 & 0 & 1 & -2 & -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 7 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 13 & 0 & 3 & 0 & 13 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \\ 3 & 5 & 0 & 2 & 1 & 3 & 12 & 11 \\ 5 & 1 & 9 & 10 & 12 & 3 & 7 & 9 \\ 3 & 1 & 2 & 19 & 7 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 2 & 0 & 5 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 3 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 7 \\ 3 & 5 & 0 & 2 & 1 & 3 & 12 & 11 & 12 \\ 5 & 1 & 9 & 10 & 12 & 3 & 7 & 9 & 3 \\ 3 & 1 & 2 & 19 & 7 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 5 & 7 & 21 \\ 3 & 1 & 5 & 11 & 22 & 10 & 15 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 2 & 3 & 4 & 0 & -1 & -2 & 5 & 9 & 7 & 0 \\ 13 & 10 & -2 & 21 & -1 & 2 & 5 & 2 & 1 & 4 \\ 1 & 3 & 1 & 0 & -2 & 21 & 31 & 2 & 1 & 2 \\ 7 & 15 & 11 & 3 & 4 & 2 & 6 & 7 & 8 & 1 \\ 14 & 2 & 3 & 1 & 9 & 12 & -1 & -1 & -2 & -1 \\ 1 & -1 & -2 & 0 & 2 & 0 & 4 & 7 & -6 & 15 \\ 31 & 11 & 0 & 5 & 1 & -2 & 4 & 5 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

For reference we include the degree, the degree of the dual variety, and ED degree of the toric varieties defined by the matrices above in Table II.4.

Example	$\deg(X_A)$	$\deg((X_A)^\vee)$	$\mathrm{EDdegree}(X_A)$
X_{A_1}	19	27	170
X_{A_2}	28	45	252
X_{A_3}	70	125	2356
X_{A_4}	16924	30840	641134
X_{A_5}	4570434	8222171	301137686
X_{A_6}	581454473	1056983492	74638158177

Table II.4: The degree, degree of the A-discriminant, and the ED degree of the projective toric varieties appearing in Table II.3.

Appendix II.B Formulas expressed in alternate index convention

In this subsection we restate the results of §II.3.1.1 and §II.3.1.2 (which used the index convention of [MT11; Nød18]) above in the index convention of [HS18]. In [HS18] the lattice index is contained in the normalized volume Vol, while in the convention of [MT11; Nød18] it is contained in Euler obstruction Eu. More precisely we let

$$\begin{aligned}\mathrm{Eu}'(\beta) &= \frac{Eu(\beta)}{i(A,\beta)}\\ \mathrm{Vol}'(\beta) &= \mathrm{Vol}(\beta)i(A,\beta)\\ \mu'(\alpha,\beta) &= \mu(\alpha,\beta)\frac{i(\alpha,\beta)i(A,\alpha)}{i(A,\beta)}.\end{aligned}$$

Let A be a $d \times n$ integer matrix with $(1,\dots,1)$ in its row space, let B be the Gale dual of A and let $P = \mathrm{Conv}(A)$. In this subsection we assume that $[\mathbb{Z}^d : \mathbb{Z}A] = 1$. We can make this assumption without loss of generality since for the purposes of computing the polar degrees we may always find such an A defining a toric variety isomorphic to the original one.

The index convention presented in this subsection is more convenient for volume computations, since we do not have to explicitly compute lattice indexes. This conversion also gives a cleaner expression for the Euler obstruction of a face since the expressions $i(\alpha,\beta)$ do not appear in the formula. More precisely the *Euler obstruction*, of a face β of P, is given by

(1) $\mathrm{Eu}'(P) = 1$,

(2) $\mathrm{Eu}'(\beta) = \sum_{\substack{\alpha \text{ s.t. } \beta \text{ is a} \\ \text{proper face of } \alpha}} (-1)^{\dim(\alpha)-\dim(\beta)-1} \cdot \mu'(\alpha,\beta) \cdot \mathrm{Eu}'(\alpha)$.

As before the dimension i Chern-Mather volume is given by

$$V_i = \sum_{\dim(\alpha)=i} \mathrm{Vol}'(\alpha)\mathrm{Eu}'(\alpha).$$

The formulas for the the polar degrees (II.11), and hence the ED degree (II.7), in terms of the Chern-Mather volumes remain unchanged.

We now restate the expressions for $\mu(\alpha,\beta)$ given in the previous subsections §II.3.1.1 and §II.3.1.2 in terms of this index convention. In the propositions below we let A be a $d \times n$ integer matrix of full rank defining a projective toric variety X_A and let B be the Gale dual of A.

In Proposition II.B.1 the expression $\deg(I_{B'})$ is computed using Proposition II.2.20.

Proposition II.B.1. *Let $X_A \subset \mathbb{P}^{n-1}$ be a projective toric variety with* $\mathrm{codim}(X_A) = 2$ *and set* $P = \mathrm{Conv}(A)$. *Let β be a proper face of P. Then the subdiagram volume $\mu'(P,\beta)$ is as follows:*

(a) *if all rows of* B_β *are contained in a relevant line* v *with* $b_i = \lambda_i v$ *for* b_i *a row of* B_β *then, if* v_+ *indices* $\lambda_i > 0$ *and* v_- *indices* $\lambda_i < 0$*, we have*

$$\mu'(P,\beta) = \frac{\min\left(\sum_{i\in v_+}|\lambda_i|, \sum_{i\in v_-}|\lambda_i|\right)}{\gcd(\lambda_i)_{i\in\mathfrak{I}_\beta}},$$

(b) *if all rows of* B_β *are not contained in a relevant line then*

$$\mu'(P,\beta) = \frac{\deg(I_{B'_\beta})}{|T(\mathbb{Z}^n/\mathbb{Z}B'_\beta)|} - \frac{\sum_{j:\det(w_\beta,b_j)>0}\det(w_\beta,b_j)}{|T(\mathbb{Z}^n/\mathbb{Z}B'_\beta)|}.$$

where B'_β *is the matrix* B_β *with the row* $w_\beta = \sum_{i\in\mathfrak{I}_\beta} -b_i$ *added.*

Proposition II.B.2. *Let* $X_A \subset \mathbb{P}^{n-1}$ *be a projective toric variety with* $\mathrm{codim}(X_A) = 2$ *and set* $P = \mathrm{Conv}(A)$*. Let* $\beta \subset \alpha$ *be proper faces of* P*. Then the subdiagram volume* $\mu'(\alpha,\beta)$ *is as follows:*

(a) *if not all rows of* B_α *are contained in a relevant line or if all rows of* B_β *are contained in a relevant line then*

$$\mu'(\alpha,\beta) = \frac{[\mathbb{Z}^d : \mathbb{Z}A_\alpha]}{[\mathbb{Z}^d : \mathbb{Z}A_\beta]},$$

(b) *if not all rows of* B_β*, but all rows of* B_α*, are contained in a relevant line, then*

$$\mu'(\alpha,\beta) = \frac{[\mathbb{Z}^d : \mathbb{Z}A_\alpha]}{[\mathbb{Z}^d : \mathbb{Z}A_\beta]} \cdot \min\left(\sum_{b_i\,:\,\gamma_i>0}|\gamma_i|, \sum_{b_i\,:\,\gamma_i<0}|\gamma_i|\right),$$

where $\gamma_i = \det(v,b_i)$ *and* i *loops over all* $i \in \mathfrak{I}_\beta$*.*

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Paper IV

Murphy's law for toric vector bundles on smooth projective toric varieties

Bernt Ivar Utstøl Nødland

Abstract

We show that the moduli space of rank three toric vector bundles on smooth toric varieties satisfies Murphy's law.

Contents

IV

IV.1 Introduction

In [Pay08] Payne constructed the moduli space of toric vector bundles of fixed rank and equivariant Chern class. He then proceeded to prove that the moduli space of rank three toric vector bundles on quasi-affine toric varieties satisfies Murphy's law. In other words any singularity type arises on some moduli space of rank three toric vector bundles on some quasi-affine toric variety.

In this paper we use similar techniques to prove that the moduli space of rank three toric vector bundles on smooth projective toric varieties also satisfies Murphy's law, thus answering a question of Payne. This result implies, among other things, that there exist rank three toric vector bundles on smooth projective toric varieties definable in characteristic $p > 0$, which does not lift to characteristic 0.

IV.2 Preliminaries on toric varieties

All of the following preliminary material can be found in any introductory text on toric geometry, for instance [CLS11] and [Ful93]. Let $T = (k^*)^n$ be an algebraic torus and denote by M its character lattice $\mathrm{Hom}(T, k^*)$ and by N its dual lattice of one-parameter subgroups. Both M and N are as groups isomorphic to $\mathbb{Z}^n$ and there is a pairing $M \times N \to \mathbb{Z}$ which, after fixing an isomorphism $M \simeq \mathbb{Z}^n$, is simply the ordinary scalar product. A toric variety X will in this paper denote a normal irreducible variety containing T as an open dense subset, such that the action of T on itself extends to an action on X. It is well-known that toric varieties are classified via combinatorial data.

Definition IV.2.1. By a cone σ we will mean a strongly convex, rational polyhedral cone $\sigma \subset N_{\mathbb{Q}} = N \otimes \mathbb{Q}$.

There is a bijection between cones σ, up to $\mathrm{GL}(n, \mathbb{Z})$, and affine toric varieties obtained by taking the spectrum U_σ of the semigroup algebra $k[\sigma^\vee \cap M]$, where we by $\sigma^\vee$ mean the dual cone of σ, inside $M \otimes \mathbb{Q}$.

A general toric variety is glued from affine pieces corresponding to cones, which is made precise by the following definition.

Definition IV.2.2. By a fan we will mean a collection Σ of finitely many cones σ, which is closed under intersections and taking faces.

There is a bijection between fans, again up to $\mathrm{GL}(n, \mathbb{Z})$, and toric varieties X_Σ obtained by taking the disjoint union of U_σ and gluing U_σ to U_τ along the intersection $U_{\sigma \cap \tau}$.

Fundamental to the theory of toric varieties is the following result, describing the orbit (closures) of the T-action on X_Σ.

Theorem IV.2.3. *There is a bijection between cones $\sigma \subset \Sigma$ and T-orbit (closures) in X_Σ. The correspondence sends a cone of dimension k to a T-orbit (closure) of dimension $n - k$.*

In particular the cones of dimension n corresponds to T-fixed points, while the cones of dimension 1, called rays, corresponds to T-invariant Weil divisors. We often by abuse of notation identify a ray ρ with the unique minimal lattice point on it. We denote by $\Sigma(l)$ the set of l-dimensional cones of Σ and $\sigma(l)$ the set of l-dimensional faces of the cone σ.

There is an exact sequence describing the divisor class group of a toric variety,

$$M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_\Sigma) \to 0,$$

which is left exact if the linear span of the rays of Σ is n-dimensional, for instance if X_Σ is projective [CLS11, Theorem 4.1.3]. In particular this says that any divisor is linearly equivalent to a T-invariant divisor, in other words to a linear combination of divisors D_ρ, where $\rho \in \Sigma(1)$.

On a smooth variety the class group and the Picard group are the same, however if the variety is singular they may be different. The T-invariant Cartier divisors is the subgroup CDiv_T of $\mathbb{Z}^{\Sigma(1)}$ consisting of divisors $D = \sum a_\rho D_\rho$ such that for any cone $\sigma \in \Sigma$ there exists a character $m_\sigma \in M$ with $\langle m_\sigma, \rho \rangle = a_\rho$ for any $\rho \in \sigma(1)$. This last technical condition is simply saying that D has to be trivial on U_σ and that it has to be the divisor of an invariant rational function on T, namely the one corresponding to the character m_σ.

For a T-invariant Cartier divisor D we can also associate a piecewise linear support function

$$\phi_D : N \otimes \mathbb{Q} \to \mathbb{Q}$$

which maps any $x \in N \otimes \mathbb{Q}$ to $\langle m_\sigma, x \rangle \in \mathbb{Q}$ for any σ containing x. This support function uniquely determines D.

IV.3 Toric vector bundles

A toric vector bundle $\mathcal{E}$ is a vector bundle on a toric variety X_Σ together with a T-action on the total space of the vector bundle, making the bundle projection $\mathcal{E} \to X$ into a T-equivariant morphism, such that for any $t \in T, x \in X_\Sigma$ the map $\mathcal{E}_x \to \mathcal{E}_{t \cdot x}$ is linear. The study of toric vector bundles goes back to Kaneyama [Kan75] and Klyachko [Kly89], who both gave classifications of toric vector bundles in terms of combinatorial and linear algebra data. Klyachko applied this to study, among other things, splitting of low rank vector bundles on $\mathbb{P}^n$. We here recollect Klyachko's description:

To a toric vector bundle $\mathcal{E}$ of rank r on X_Σ, we let $E \simeq k^r$ denote the fiber at the identity of the torus. Klyachko shows that there for each ray $\rho \in \Sigma(1)$ is an associated filtration $E^\rho(j)$ of E, indexed over $j \in \mathbb{Z}$. It has the property that for any ray $E^\rho(j) = 0$ for j sufficiently large and $E^\rho(j) = E$ for j sufficiently small. Additionally it satisfies a compatibility condition:

For any maximal cone σ there exists characters $u_1, ..., u_r \in M$ and vectors $L_u \in E$ such that for any ray ρ of σ we have

$$E^\rho(j) = \sum_{j \mid \langle u, \rho \rangle \geq j} L_u.$$

The above decomposition is equivalent to the fact that on the affine U_σ, $\mathcal{E}$ splits into a direct sum of line bundles $\mathcal{O}(u_i)$. Klyachko's classification theorem is the following:

Theorem IV.3.1 ([Kly89, Thm 0.1.1]). *The category of toric vector bundles on X is equivalent to the category of finite dimensional vector spaces E, with filtrations indexed by the rays as described above, satisfying the compatibility condition. A morphism $\mathcal{E} \to \mathcal{F}$ corresponds to a linear map $E \to F$, respecting the filtrations.*

Thus, families of vector bundles give rise to families of filtrations of a fixed vector space.

IV.4 Moduli of toric vector bundles

In this section we recall Payne's construction of the moduli space of toric vector bundles of fixed equivariant Chern class.

The equivariant Chow ring $A_*^T(X_\Sigma)$ of a toric variety X_Σ is isomorphic to the ring of integral piecewise polynomial functions, which are polynomial on each cone σ [Pay06, Theorem 1]. This generalizes the piecewise linear support function associated to a divisor, described above.

For any cone $\tau \in \Sigma$ we let $M_\tau = M/\tau^\perp \cap M$. If σ is a maximal cone containing τ, let $u_1, \dots, u_r$ be the characters such that $\mathcal{E}|_{U_\sigma} = \oplus_{i=1}^r \mathcal{O}(\operatorname{div}(u_i))$. Then the images of $u_1, \dots, u_r$ in $M(\tau)$ define characters $u_1^\tau, \dots, u_r^\tau$ in M_τ which are independent of which maximal cone containing τ one chooses. Denote by $u(\tau)$ the multiset of characters on the cone τ. Payne showed that the equivariant Chern class $c_i^T(\mathcal{E})$ of $\mathcal{E}$ is given on τ by the polynomial $e_i(u(\tau))$, where e_i is the i-th elementary symmetric function [Pay06, Theorem 3].

Definition IV.4.1. A framed rank r toric vector bundle is a toric vector bundle $\mathcal{E}$ together with an isomorphism $\phi : E \to k^r$, where E is the fiber of $\mathcal{E}$ over the identity of the torus.

Payne proved the following:

Theorem IV.4.2 ([Pay08, Theorem 3.9])**.** *Given a toric variety X_Σ and a rank r equivariant Chern class c there exists a fine moduli scheme M_c of framed rank r toric vector bundles on X_Σ with total equivariant Chern class c.*

We note that the scheme M_c is defined over $\operatorname{Spec}\mathbb{Z}$ [Pay08, p. 1207]. The scheme M_c is constructed as a locally closed subscheme of a product of flag varieties, as follows: Let $\mathcal{E}$ be a framed rank r toric vector bundle of equivariant Chern class c. Fixing the Chern class c is equivalent to fixing compatible multisets $u(\sigma)$ of linear functions on each σ. Recall we have fixed the isomorphism $\phi : E \simeq k^r$. For any ray ρ, consider the dimensions of the subspaces appearing in the filtration $E^\rho(j)$. Let $\operatorname{Fl}(\rho)$ be the flag variety of subspaces of k^r having exactly these dimensions. Let $\operatorname{Fl}_c = \prod_\rho \operatorname{Fl}(\rho)$. We see that $\mathcal{E}$ gives an element of Fl_c by taking the the subspaces appearing in the filtrations for any ray.

Any bundle with Chern class c must have the same dimensions of subspaces appearing in the filtrations as $\mathcal{E}$ [Pay08, p. 1205]. Moreover the Chern class c restricts how the subspaces can meet each other in the following way. If $\sigma = \operatorname{Cone}(\rho_1, \dots, \rho_s)$ we require the equality

$$\dim \cap_{i=1}^s E^{\rho_i}(j_i) = \#\{u \in u(\sigma) | \langle u, \rho_i \rangle \leq j_i \text{ for } i = 1, \dots, s\}.$$

These rank conditions correspond to the vanishing of certain polynomials in the Plücker coordinates of the partial flag varieties Fl_ρ as well as the non-vanishing of certain others. Thus the resulting scheme is locally closed in Fl_c and is in fact the scheme M_c.

IV.5 Murphy's law on smooth projective toric varieties

We say that a moduli space M satisfies Murphy's law if any singularity type defined over $\operatorname{Spec}\mathbb{Z}$ arises on M. This notion was introduced by Vakil [Vak06] who proceeded to show that Murphy's law holds for many moduli spaces.

Payne has showed that for moduli of bundles of rank 2, the moduli space M_c is smooth [Pay08, p.1209]. We will now be interested in rank 3 bundles. The starting observation is that if the bundle has rank 3 then all the rank conditions correspond to incidences between points and lines in $\mathbb{P}^2$. If $x_1, \dots, x_d$ are points and $l_1, \dots, l_{d'}$ are lines in $\mathbb{P}^2$, then from a subset

$$I \subset \{1, \dots, d\} \times \{1, \dots, d'\}$$

we can define a set of incidences of points and lines stating that $x_i \in l_j$ if and only if $(i,j) \in I$. There is an associated incidence scheme

$$C_I \subset \prod_{i=1}^{d} \mathbb{P}^2 \times \prod_{i=1}^{d'} \mathbb{P}^{2\vee},$$

parametrizing all such sets of points and lines. Payne uses the scheme C_I to prove Murphy's law for moduli of rank three toric vector bundles on quasi-affine and quasi-projective toric varieties. The proof is via, given a set of incidences I between points and lines in $\mathbb{P}^2$, constructing a quasi-affine toric variety such that M_c is PGL_3-equivariantly isomorphic to C_I and applying Mnëv's universality theorem [Mnë88], which states that any singularity appears on some such incidence scheme. This is essentially the same idea used by Vakil to prove the original formulation of Murphy's law for other moduli spaces [Vak06]. The way Payne proves this for toric vector bundles is by putting all the points and all the lines as non-trivial subspaces of a filtration $E^\rho(l)$ and then choosing the fan such that all pairs of rays are maximal cones. By construction we get the incidence scheme C_I, which has the desired properties.

The material in this section was inspired by the following question:

Question IV.5.1 ([Pay08, Remark 4.4])**.** Does the moduli of toric vector bundles on projective toric varieties satisfy Murphy's law?

Using techniques similar to the above, we can now prove Murphy's law for moduli of rank at least three toric vector bundles on smooth projective toric varieties.

Theorem IV.5.2. *Given an incidence I between points and lines in $\mathbb{P}^2$ there exists a smooth projective toric variety and a rank three Chern class c on it, such that M_c is PGL_3-equivariantly isomorphic to C_I. Thus Murphy's law holds for moduli spaces of rank three framed toric vector bundles on smooth projective toric varieties. Also Murphy's law holds for the coarse moduli scheme of rank three toric vector bundles on smooth projective varieties.*

Proof. Set $n = d + d' - 1$. Let X_n be the toric variety obtained by blowing up $\mathbb{P}^n$ along the following linear spaces of increasing dimension: first all invariant points, then all strict transforms of invariant lines and so on until we have blown up all invariant subvarieties of codimension at least three. We call the new fan Σ. Blowing up a toric variety corresponds to inserting a new ray in the relative interior of the cone corresponding to the subvariety we blow up. The original $n+1 = d + d'$ rays $\rho_1, \ldots, \rho_{n+1}$ of $\mathbb{P}^n$ are thus still rays of Σ. Set

$$y_i = x_i, i = 1, \ldots, d$$

$$y_{d+i} = l_i, i = 1, \ldots, d'.$$

We now specify the equivariant Chern class. In other words we describe the multiset $u(\sigma)$ for each $\sigma \in \Sigma$. For simplicity we first assume that $\sigma = \mathrm{Cone}(\rho_1, \rho_2, \rho_1 + \rho_2 + \rho_3, \rho_1 + \rho_2 + \rho_3 + \rho_4, \ldots, \rho_1 + \ldots + \rho_n)$. Similar to Payne [Pay08, Top of p.1211] we now define $u(\sigma)$ according to the four cases.

$$u(\sigma) = \begin{cases} \{0, e_1^* - e_3^*, e_2^* - e_3^*\} \text{ if } y_1, y_2 \text{ are both points} \\ \{0, e_2^* - e_3^*, e_1^* + e_2^* - 2e_3^*\} \text{ if } y_1 \text{ is a point containing the line } y_2 \\ \{e_1^* - e_3^*, e_2^* - e_3^*, e_2^* - e_3^*\} \text{ if } y_1 \text{ is a point not contained in the line } y_2 \\ \{e_1^* - e_3^*, e_2^* - e_3^*, e_1^* + e_2^* - 2e_3^*\} \text{ if } y_1, y_2 \text{ are both lines} \end{cases}$$

Every maximal cone σ' is of the same form as σ, up to permutation of the ρ_i, thus for other maximal cones the definition of $u(\sigma')$ is done in the analogous way.

The above might seem mysterious, however if we fix a bundle $\mathcal{E}$ with equivariant Chern class as above, the point is that this forces the filtrations on ρ_i to be of the form

$$E^i(j) = \begin{cases} k^3 & \text{if } j \le 0 \\ y_i & \text{if } j = 1 \\ 0 & \text{if } 1 < j \end{cases}$$

and the filtration on any other ray to be trivial, in other words to jump directly from 0 to k^3 at step 0 of the filtration. It is straight-forward to check that Klyachko's compatibility condition is satisfied for these filtrations: the characters on U_σ is exactly the characters $u(\sigma)$. Thus the above filtrations correspond to a toric vector bundle whose Chern classes are the elementary symmetric functions of $u(\sigma)$. Thus $u(\sigma)$ correspond to a well-defined equivariant Chern class. All pairs ρ_i, ρ_j form a two-dimensional cone σ_{ij}, thus from the Chern class on this cone we get all incidences from I. Moreover because we have blown up so much, no three ρ_i, ρ_j, ρ_k form a cone, thus we do not get any extra incidences. Thus $M_c = C_I$ and we are done.

The statement on the coarse moduli scheme follows from the fact that it is the quotient of M_c by GL_3 [Pay08, Corollary 3.11]. By Mnëv's universality theorem [Laf03, Section 1.8] we have that for any affine scheme Y defined over $\operatorname{Spec} \mathbb{Z}$, there exists some incidence scheme C_I on which PGL_3 acts freely such that the quotient C_I/GL_3 is isomorphic to an open subvariety of $Y \times \mathbb{A}^s$ projecting surjectively to Y, for some s. Thus such quotients satisfy Murphy's law which implies that Murphy's law is satisfied for the coarse moduli scheme of rank three toric vector bundles. This argument is the same as Payne's argument in [Pay08, Theorem 4.2]. ■

Remark IV.5.3. The toric varieties X_n from the proof appear in the literature: If one blows up X_n also along the strict transforms of all codimension two linear spaces, the resulting variety is the Losev–Manin space LM_n. By [CT15, Remark 1.5] the blowup at a general point of X_{n+1} is a small modification of a $\mathbb{P}^1$-bundle over $\overline{M}_{0,n}$. Using this, Castravet and Tevelev prove that the blowup at a general point of X_n is not a Mori Dream Space, for large n.

IV.6 Liftability of toric vector bundles

Recall that any toric variety is defined over $\operatorname{Spec}\mathbb{Z}$, since the definition in terms of a fan does not depend on the ground field. Thus, any toric variety in characteristic p can be uniquely lifted to a variety in characteristic 0; the one defined by the same fan. Hence it is natural to ask whether also any toric vector bundle, defined over a field of characteristic p, lifts to characteristic 0.

There are different possible senses in which one could ask for a lifting of a toric vector bundle F from characteristic p to characteristic 0. One could ask if one could lift F, or its projectivization $\mathbb{P}(F)$, as a variety to a variety defined in characteristic 0, as for toric varieties. Alternatively, one could ask if one could lift F as a vector bundle to characteristic 0. In other words whether there exist a flat family $\mathcal{X} = X_\Sigma \times \operatorname{Spec} R$ over a discrete valuation ring R of characteristic 0, satisfying the following: The general fiber is the toric variety defined over the fraction field K of R associated to Σ, while the special fiber is the toric variety associated to Σ, but now defined over the residue field k, which is a field of positive characteristic. There is also a vector bundle $\mathcal{F}$ on $\mathcal{X}$, such that the restriction of $\mathcal{F}$ to the special fiber equals F. Even stronger we may require the vector bundle $\mathcal{F}$ on the total space $\mathcal{X}$ to be equivariant under the torus action. Details on these various notions of liftability, as well as criteria for comparing them are discussed in [Emi18].

The above result implies formally that there exists rank three toric vector bundles on smooth projective toric varieties, definable in characteristic p, which cannot be lifted as an equivariant vector bundle to characteristic 0. This is because there exist singularities of the moduli space M_c, defined over finite fields, having arbitrarily ugly behaviour, and in particular, do not lift to characteristic 0. Given a singular point of M_c whose singularity type is such that it can be defined in characteristic p, but not liftable to characteristic 0, we can pull back the universal family of the fine moduli space along the inclusion of the point into M_c. The resulting scheme parametrizes vector bundles which cannot lift. In other words, there exist vector bundles F on X/k, with the property that there cannot exist a flat family of vector bundles, $\mathcal{F} \to \mathcal{X} \to \operatorname{Spec} R$ over a discrete valuation ring R of characteristic 0, such that the restriction of $\mathcal{F}$ to the special fiber is F. If such a lifting did exist, then the total space of $\mathcal{F}$ would be a lift of the total space of F as a scheme, which would in turn imply that the singularity lifts.

It is tempting to ask whether there are combinatorial explanations for these phenomena. Di Rocco, Jabbusch and Smith defined a representable matroid $M(\mathcal{E})$ associated to a toric vector bundle (recalled in Paper V). It is well known that there exists matroids representable over some fields, but not over others. Motivated by this and the above discussion we pose the following question.

Question IV.6.1. Is non-representability of the matroid $M(\mathcal{E})$ over the field K an obstruction to lifting $\mathcal{E}$ to K (in any of the senses of lifting discussed above)?

From matroid theory it is easy to construct examples of toric vector bundles whose matroid cannot be represented in characteristic 0. An example is the following.

Example IV.6.2. Let X_Σ be any smooth toric surface with 7 rays $\rho_1, \dots, \rho_7$ defined over a field of characteristic 2. Let $\mathcal{E}$ be a rank three toric vector bundle with filtrations given by

$$E^{\rho_i}(j) = \begin{cases} E & \text{if } j \leq 0 \\ H_i & \text{if } j = 1 \\ 0 & \text{if } 1 < j \end{cases},$$

where H_i, $i = 1, ..., 7$ are all seven points of the Fano plane in $\mathbb{P}(E)$, i.e. the seven non-zero points in $\mathbb{P}(E)$ with all coordinates either 0 or 1. Then the matroid $M(\mathcal{E})$ will be the Fano matroid. This matroid is only representable in characteristic 2.

Mnëv's universality theorem, and thus also our proof of Murphy's law, is non-constructive, thus it does not help in answering Question IV.6.1. We do not know how to prove non-liftability for an explicit example such as Example IV.6.2. If a lifting exists over a discrete valuation ring $\operatorname{Spec} R$, then we can lift the filtrations to filtrations of R-modules. However it is not clear to us whether the matroids of the base changes to k or K would be the same or not. We believe techniques used in [AZ17] to prove non-liftability of schemes which are blow-ups of projective space in linear spaces might be useful in answering Question IV.6.1, especially since projectivized toric vector bundles are closely related to blow-ups of projective space in linear spaces, for details see [Gon+12] or the sections on Cox rings in Paper V.

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Paper V

Some positivity results for toric vector bundles

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Abstract

We give a criterion for a projectivized toric vector bundle to be a Mori dream space and describe its Cox ring using generators and relations. Both of these results are in terms of the matroids of all symmetric powers of the bundle. We also give a criterion for a toric vector bundle to be big and describe several interesting examples of toric vector bundles which highlights how positivity properties for toric vector bundles are more complicated than for toric line bundles.

Contents

V.1 Introduction

Positivity is an important notion in the study of the geometry of projective varieties. For toric varieties we have a very good understanding of various positivity properties of line bundles, in terms of the combinatorics defining the variety. In this paper we investigate various positivity properties of toric vector bundles. Equivalently, we study positivity of line bundles on projectivized toric vector bundles.

V

Di Rocco, Jabbusch and Smith [DJS18] associate to a toric vector bundle $\mathcal{E}$ a representable matroid $M(\mathcal{E})$. To each element e in the ground set of $M(\mathcal{E})$, there is an associated divisor D_e, such that $\oplus_e \mathcal{O}(D_e)$ surjects onto $\mathcal{E}$, and the induced map on global sections is surjective.

In this paper we investigate the connection between the matroid and positivity properties of $\mathcal{E}$. Positivity of line bundles is closely related to sections of multiples of the bundle. The analogous notion for vector bundles is symmetric powers. An important property of the matroid $M(\mathcal{E})$ is that it does not necessarily commute with taking symmetric powers. This makes the study of vector bundles significantly harder than the study of line bundles. To be able to study positivity, we thus have to study the matroids of all symmetric powers $S^k\mathcal{E}$ at the same time. To do this we define a set of vectors $\mathfrak{M}(\mathcal{E})$ containing the ground set of the matroid $M(\mathcal{E})$, but also containing all matroid vectors in the ground set of some symmetric power $S^k\mathcal{E}$ which cannot be written as a symmetric product of matroid vectors for lower symmetric powers. The first main result in this paper is the following:

Theorem V.1.1 (Theorem V.4.18)**.** *Let $\mathcal{E}$ be a toric vector bundle on the smooth toric variety X_Σ. Then $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ is finitely generated if and only if $\mathfrak{M}(\mathcal{E})$ is finite.*

The second main result is a presentation of the Cox ring of $\mathbb{P}(\mathcal{E}))$ in terms of generators and relations. The generators correspond bijectively to the set $\mathfrak{M}(\mathcal{E}) \cup \Sigma(1)$. We are also able to describe all relations, in terms of relations between vectors in $\mathfrak{M}(\mathcal{E})$, see Section V.6 and in particular Theorem V.6.4.

Our results on Cox rings reproves many of the results in [Gon+12], using different techniques: We use the Klyachko filtrations directly. Our results are also more general. A heuristic explanation for our results on Cox rings is the following: We have that $M(\mathcal{E} \otimes \mathscr{L}) = M(\mathcal{E})$, for any line bundle $\mathscr{L}$, since tensoring with a 1-dimensional vector space does not change linear algebra relations. Thus the matroid of $\mathcal{E}$ is most naturally considered an invariant of $\mathcal{E} \otimes \mathscr{L}$, where $\mathscr{L}$ is allowed to vary freely. To study the Cox ring we need to study sections of $S^k\mathcal{E} \otimes \mathscr{L}$, where k and $\mathscr{L}$ are allowed to vary. However it is sufficient to restrict to algebra generators of the Cox ring over $\mathrm{Cox}(X_\Sigma)$ and these are exactly sections corresponding to the set $\mathfrak{M}(\mathcal{E})$.

Our third main result is a criterion for a toric vector bundle to be big.

Theorem V.1.2 (Theorem V.7.5)**.** *A toric vector bundle is big if and only if there exists $k > 0$ and $v \in M(S^k\mathcal{E})$, such that the associated polytope is full dimensional.*

Additionally we give some other interesting examples and results which are related to positivity of toric vector bundles:

- A big toric vector bundle with the property that no Minkowski sum of the polytopes in the parliament is full-dimensional (Example V.7.2).
- A way of interpreting the nefness/ampleness of a toric vector bundle in

terms of a notion of concavity for the piecewise linear support function on the associated branched cover of a fan (Proposition V.8.6).

- A toric surface with ample rank two vector bundles $\mathcal{E}_k$ such that $S^k\mathcal{E}_k$ is not globally generated (Section V.9).
- A sequence of toric vector bundles showing that there cannot exist a bound depending on the dimension of the variety and/or the rank of the vector bundle with the following property: If $\mathcal{E}$ is ample and the degree of $\mathcal{E}|_C$ is larger than this bound for any invariant curve C, then $\mathcal{E}$ is globally generated or very ample (Example V.10.4).

V.2 Preliminaries on toric varieties

We here recall some preliminary material on toric varieties and toric vector bundles. Let $T = (\mathbb{C}^*)^n$ be an algebraic torus and denote by M its character lattice $\mathrm{Hom}(T, \mathbb{C}^*)$ and by N its dual lattice of one-parameter subgroups. A toric variety X will in this paper denote a normal irreducible variety containing T as an open dense subset, such that the action of T on itself extends to an action on X. It is well known that any toric variety corresponds to a fan Σ in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. We will denote the toric variety associated to Σ by X_Σ. In this paper we assume that the ground field equals $\mathbb{C}$ and that X_Σ is smooth and complete. This is because the theory of parliaments of polytopes is only developed under these assumptions. We believe that much of the following will remain true with only minor modifications in the case of any $\mathbb{Q}$-factorial toric variety, in other words for any simplicial fan.

Any divisor D on X_Σ is linearly equivalent to a sum $D = \sum_\rho a_\rho D_\rho$ of T-invariant divisors. Alternatively it corresponds to a piecewise linear support function $\phi_D : N_{\mathbb{Q}} \to \mathbb{Q}$ given by

$$x \mapsto \langle m_\sigma, x\rangle,$$

where σ is any cone containing x and m_σ is Cartier data for D on U_σ. In other words m_σ satisfies $a_\rho = \langle m_\sigma, \rho\rangle$ for any ray ρ of σ.

Associated to a divisor D there is a polytope P_D defined by

$$P_D = \{x \in M_{\mathbb{Q}} | \langle x, \rho\rangle \leq a_\rho\}.$$

We have the following well-known formula for the global sections of D [Ful93, p. 66]:

$$H^0(X_\Sigma, \mathcal{O}(D)) \simeq \bigoplus_{m \in P_D \cap M} \mathbb{C}\chi^m.$$

Remark V.2.1. In the above we have used the convention that the Cartier data m_σ of a divisor $D = \sum a_\rho D_\rho$ satisfies $\langle m_\sigma, \rho\rangle = a_\rho$, while many texts on toric geometry use the convention that it satisfies $\langle m_\sigma, \rho\rangle = -a_\rho$. This has the consequence that polytopes will be drawn in the opposite direction of what is usual in toric geometry, for instance in [CLS11] and [Ful93]. This is done because we are largely interested in the parliament of polytopes studied in [DJS18], and they use this convention, thus we have chosen to follow it. This has the consequence that some formulas and results will have an extra minus-sign compared to their usual formulations. For a similar reason we will talk about concave support functions, instead of the usual convex ones.

A toric vector bundle $\mathcal{E}$ is a vector bundle on a toric variety X_Σ together with a T-action on the total space of the vector bundle, making the bundle projection $\mathcal{E} \to X$ into a T-equivariant morphism, such that for any $t \in T, x \in X_\Sigma$ the map $\mathcal{E}_x \to \mathcal{E}_{t \cdot x}$ is linear. The study of toric vector bundles goes back to Kaneyama [Kan75] and Klyachko [Kly89], who both gave classifications of toric vector bundles in terms of combinatorial and linear algebra data. Klyachko applied this to study, among other things, splitting of low rank vector bundles on $\mathbb{P}^n$. We here recollect Klyachko's description:

To a toric vector bundle $\mathcal{E}$ of rank r on X_Σ, we let $E \simeq \mathbb{C}^r$ denote the fiber at the identity of the torus. Klyachko shows that there for each ray $\rho \in \Sigma(1)$ is an associated filtration $E^\rho(j)$ of E, indexed over $j \in \mathbb{Z}$. It has the property that for any ray $E^\rho(j) = 0$ for j sufficiently large and $E^\rho(j) = E$ for j sufficiently small. Additionally it satisfies a compatibility condition:

For any maximal cone σ there exists characters $u_1, ..., u_r \in M$ and vectors $L_u \in E$ such that for any ray ρ of σ we have

$$E^\rho(j) = \sum_{j | \langle u, \rho \rangle \geq j} L_u.$$

The above decomposition is equivalent to the fact that on the affine U_σ, $\mathcal{E}$ splits into a direct sum of line bundles $\mathcal{O}(u_i)$. Klyachko's classification theorem is the following:

Theorem V.2.2 ([Kly89, Thm 0.1.1])**.** *The category of toric vector bundles on X is equivalent to the category of finite dimensional vector spaces E, with filtrations indexed by the rays as described above, satisfying the compatibility condition. A morphism $\mathcal{E} \to \mathcal{F}$ corresponds to a linear map $E \to F$, respecting the filtrations.*

V.3 Parliaments of polytopes

We next briefly recall the notion of a parliament of polytopes introduced in [DJS18], which is a way of describing global sections of $\mathcal{E}$ in terms of lattice points in a collection of polytopes.

For a toric vector bundle the cohomology groups $H^i(X_\Sigma, \mathcal{E})$ decompose as a direct sum $\oplus_{u \in M} H^i(X_\Sigma, \mathcal{E})_u$, over the χ^u-isotypical components. Klyachko showed that

$$H^0(X_\Sigma, \mathcal{E})_u = \cap_{\rho \in \Sigma(1)} E^\rho(\langle u, \rho \rangle).$$

The notion of parliaments of polytopes gives a more detailed way of studying $H^0(X_\Sigma, \mathcal{E})$.

Consider the set of all intersections of the form $\cap_\rho E^\rho(j_\rho)$. There is a unique representable matroid $M(\mathcal{E})$ associated to $\mathcal{E}$, whose ground set is constructed inductively as follows: For any intersection of dimension one, add a vector from this intersection to the ground set. Assume that we have added vectors corresponding to all intersections of dimension i and let V be an intersection of dimension $i+1$. Let G be the set of all such vectors contained in V. Let W be a complementary subspace to $\mathrm{Span}(G)$ inside V. We choose a basis for W and add all the basis vectors to the ground set of the matroid. Performing this process on all possible such intersections, we get a set of vectors whose associated matroid $M(\mathcal{E})$ is uniquely determined by $\mathcal{E}$. We will often, by abuse of notation, identify the matroid with a fixed choice for its ground set.

By construction $M(\mathcal{E})$ has the property that each of the intersections $\cap_\rho^n E^\rho(j_\rho)$ can be written as the span of vectors from a ground set of the matroid. To each vector $e \in M(\mathcal{E})$ we associate a divisor $D_e = \sum_\rho a_\rho D_\rho$, where

$$a_\rho = \max\{j \in \mathbb{Z} : e \in E^\rho(j)\}.$$

We denote by P_e the associated polytope:

$$P_e = \{u \in M_\mathbb{R} | \langle u, \rho\rangle \leq \max(j \in \mathbb{Z} : e \in E^\rho(j)), \text{ for all } \rho\}.$$

Proposition V.3.1 ([DJS18, Proposition 1.1]). *The lattice points in the parliament of polytopes correspond to a torus equivariant generating set of $H^0(X, \mathcal{E})$.*

A crucial difference from the case of line bundles on toric varieties is that the sections associated to all the lattice points aren't necessarily linearly independent. If the polytopes are disjoint then then they actually give a basis, but if there is any overlap there might be relations among them. The matroid structure of the indexing set describes precisely the dimension of a χ^u-isotypical component.

Proposition V.3.2. *Given a toric vector bundle $\mathcal{E}$ and a character $u \in M$ we have*

$$\dim H^0(X_\Sigma, \mathcal{E})_u = \dim \mathrm{Span}\{e \in M(\mathcal{E}) | u \in P_e\}.$$

Proof. This follows by the equivalence from [DJS18, proof of Proposition 1.1]:

$$e \in H^0(X_\Sigma, \mathcal{E})_u = \cap E^\rho(\langle u, \rho\rangle) \Leftrightarrow u \in P_e \cap M.$$

■

An equivalent way of formulating the above statements on global sections is to observe that by construction there is a surjection

$$\mathcal{F} = \bigoplus_{e \in M(\mathcal{E})} \mathcal{O}(D_e) \to \mathcal{E} \to 0,$$

which is surjective on global sections [DJS18, Remark 3.6]. Indeed, due to Klyachko's equivalence of categories such a map corresponds to a surjective map of vector spaces $\psi : F \to E$. The vector space F has a basis consisting of one basis vector w_e for each $e \in M(\mathcal{E})$. The map ψ is simply the map sending w_e to e. By construction it will be surjective on global sections.

We think the following observation will be useful for studying parliaments of polytopes: Observe that the above surjection corresponds to a closed embedding

$$i : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$$

such that $i^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. $\mathbb{P}(\mathcal{F})$ is itself a toric variety $X_{\Sigma'}$ whose fan lives in $N'_{\mathbb{Q}} = N_{\mathbb{Q}} \oplus N''_{Q}$, for a lattice N'' of rank equal to the number of matroid vectors minus one. Thus $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ defines a polytope $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ in $M'_{\mathbb{Q}}$ whose lattice points corresponds to elements of $H^0(X_{\Sigma'}, \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) = H^0(X_{\Sigma}, \mathcal{F})$. Moreover the parliament for $\mathcal{F}$ also corresponds to global sections of $\mathcal{F}$.

We recall the construction of the fan Σ' in $N'_{\mathbb{Q}} = N_{\mathbb{Q}} \oplus N''_{\mathbb{Q}}$. Let $D_0, \ldots, D_s$ be the divisors in the parliament of $\mathcal{E}$ and write

$$D_i = \sum_{\rho} a_{i\rho} D_{\rho}$$

There are rays $w_0, \ldots, w_s$ in $N''_{\mathbb{Q}}$ having the structure of the fan of $\mathbb{P}^s$, exhibiting the $\mathbb{P}^s$-bundle structure of $\mathbb{P}(\mathcal{F})$. For any ray $\rho \in \Sigma(1)$ we have an associated ray ρ' of Σ' with minimal generator

$$\rho' = \rho + \sum_{i} a_{i\rho} w_i.$$

A cone of Σ' is the Minkowski sum of any cone of Σ plus any cone generated by a proper subset of $\{w_0, \ldots, w_s\}$.

Lemma V.3.3. *For fixed i, the rational equivalence class of the divisor*

$$D_{w_i} + \sum_{\rho} a_{i\rho} D_{\rho'},$$

equal that of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. Fixing any one such representation, the polytopes P_e are obtained as the intersections of $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ with the fibers over the $s+1$ lattice points of the standard simplex Δ_s in $M''_{\mathbb{Q}}$, along the map $M_{\mathbb{Q}} \to M''_{\mathbb{Q}}$.

Proof. Set $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}(-D_i)$. Both X_{Σ} and $\mathbb{P}(\mathcal{G})$ are toric varieties, thus their Picard groups are easily computable from the combinatorics of the fans. Since $\mathbb{P}(\mathcal{G})$ is a projective bundle over X_{Σ}, we also know the relationship between these Picard groups. Comparing the two ways of computing these groups, we see that the class of D_{w_i} has to equal the class of $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$, or its negative. But D_{w_i} is effective, thus it has to be the positive $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$. But this implies that on $\mathbb{P}(\mathcal{F})$ the class of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ equals that of $D_{w_i} + D_i$.

We now fix a representation of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$, say $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) = D_{w_0} + \sum_{\rho} a_{0\rho} D_{\rho'}$. The polytope $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ is given by the inequalities, for some $(x, y) \in N_{\mathbb{Q}} \otimes N''_{\mathbb{Q}}$

$$x_1 \leq 0$$

$$\vdots$$

$$x_s \leq 0$$

$$-x_1 - \cdots - x_s \leq 1$$

$$\langle (x, y), \rho' \rangle \leq a_{0\rho}$$

Let F_0 be the subset of $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ where $x_1 = \ldots = x_s = 0$. Then we see that the above inequalities reduce to

$$\langle y, \rho \rangle \leq a_{0\rho}$$

Thus F_0 is exactly $P_{D_0} \times (0, \ldots, 0) \subset N_{\mathbb{Q}} \otimes N''_{\mathbb{Q}}$. Similarly, if F_i is the locus with $x_i = -1$ and $x_j = 0$ for $j \neq i$, then F_j is the polytope given by

$$\langle y, \rho \rangle + \langle (0, \ldots, -1, \ldots, 0), \sum_j a_{j\rho} w_j \rangle \leq a_{0\rho}$$

which after cancelling $a_{0\rho}$ is given by

$$\langle y, \rho \rangle \leq a_{i\rho}$$

Thus F_i is $P_{D_{w_i}}$ times a point. ■

Corollary V.3.4. *The parliament of polytopes of $\mathcal{E}$ is obtained by projecting the $s+1$ polytopes $M \times v_i \cap P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ (which could be empty), where the v_i are the vertices of the standard s-simplex Δ_s in $M''_{\mathbb{Q}}$. Moreover the lattice points in the parliament of polytopes are the images of all lattice points of $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$.*

Proof. Most of this is clear from the above lemma. Note that the lattice points in $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ are all $m' \in M'$ such that $H^0(X'_\Sigma, \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))_{m'}$ is non-zero. By the inequalities defining the polytope we see that for $m' = (m, m'')$ we must have that m'' lies in the standard simplex in $M''_{\mathbb{Q}}$. But this has only $s+1$ lattice points (corresponding exactly to the polytopes F_i defined in the proof of Lemma V.3.3), hence any corresponding m lies in some face F_i. ■

Corollary V.3.5. *The global sections of $\mathcal{E}$ are obtained by projecting $s+1$ polytopes along $p : M'_{\mathbb{Q}} \to M_{\mathbb{Q}}$ and then identifying sections according to the dependence structure of the matroid $M(\mathcal{E})$.*

Remark V.3.6. When all polytopes in the parliament are non-empty, we have that the big polytope $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ is the Cayley polytope of the polytopes in the parliament. In that case the above statements follow from well-known results on Cayley polytopes. In that case we also have that under the projection $q : M'_{\mathbb{Q}} \to M''_{\mathbb{Q}}$ $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ is mapped to the standard simplex. See for instance [BN08, Section 2] for details on Cayley polytopes.

Remark V.3.7. Fix $\mathcal{E}$, and define the polytope Q as the convex hull of all polytopes in the parliament. If $\mathcal{E}$ is ample, then the fiber of p of any point in the interior of Q intersected with $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ is non-empty, although it need not contain lattice points. Studying the size of the fiber, using a similar construction for complexity one T-varieties, under taking multiples of a line bundle $\mathscr{L}$, is the idea used by Altmann and Ilten to prove Fujita's freeness conjecture for complexity one T-varieties [AI17]. In particular it also is true for rank two toric vector bundles. Thus, for any ample line bundle $\mathscr{L}$ on $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ has rank 2, we have that $(\dim X + 1)\mathscr{L} + K_X$ is basepoint free. One could hope that it would be possible to use this construction to study also higher rank bundles, although the fact that the matroid of the symmetric power $S^k\mathcal{E}$ does not equal the symmetric power of the ground set of $M(\mathcal{E})$ makes it significantly harder. An example of a bundle for which the matroid of the symmetric power does not equal the symmetric power of the matroid can be seen in Example V.7.4.

Remark V.3.8. Di Rocco, Jabbusch and Smith ask [DJS18, p.3] whether there is, for a globally generated toric vector bundle $\mathcal{E}$, a relation between regular triangulations of the parliaments of $\mathcal{E}$ and the equations of $\mathbb{P}(\mathcal{E})$ under the embedding by the linear system $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The above makes this plausible: The linear system $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ gives a rational map $\mathbb{P}(\mathcal{F}) \dashrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X_{\Sigma'}, \mathcal{F}))$. The surjection $H^0(X_\Sigma, \mathcal{F}) \to H^0(X_\Sigma, \mathcal{E})$ implies that $\mathbb{P}(H^0(X_\Sigma, \mathcal{E}))$ is a linear subspace of $\mathbb{P}^N$. Then a regular triangulation of $P_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)}$ induces a regular triangulation of the parliament of $\mathcal{E}$, under the projection p.

V.4 Cox rings of projectivized toric vector bundles

Let X_Σ be a smooth projective toric variety. Let $\mathcal{E}$ be a toric vector bundle on X_Σ; we denote the natural map from $\mathbb{P}(\mathcal{E})$ to X_Σ by π. We wish to study the Cox ring of $\mathbb{P}(\mathcal{E})$. Let $\rho_1, \dots, \rho_n$ be the rays of Σ, and denote by D_i the torus-invariant divisor associated to ρ_i. By the description of the Picard group of a projective bundle we have

$$\mathrm{Cox}(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{k, k_1, \dots, k_n \in \mathbb{Z}} H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(k) + \pi^* k_1 D_1 + \ldots + \pi^* k_n D_n)$$

$$\simeq \bigoplus_{k, k_1, \dots, k_n \in \mathbb{Z}} H^0(X_\Sigma, S^k\mathcal{E} \otimes k_1 D_1 \otimes \cdots \otimes k_n D_n).$$

We will study the latter $\mathbb{C}$-algebra.

By construction the matroid $M(\mathcal{E})$ has the property that each of the intersections $\cap_\rho E^\rho(j_\rho)$ can be written as the span of vectors from the ground set of the matroid. We denote by $L(\mathcal{E})$ the set of all such intersections. Recall that we constructed the divisor D_e associated to a matroid vector $e \in M(\mathcal{E})$, given by $D_e = \sum_\rho a_\rho D_\rho$, where

$$a_\rho = \max\{j \in \mathbb{Z} : e \in E^\rho(j)\}.$$

We will now need a slight generalization of these divisors; to each linear space $V \in L(\mathcal{E})$ we associate a divisor $D_V = \sum_\rho a_\rho D_\rho$, where

$$a_\rho = \max\{j \in \mathbb{Z} : V \subset E^\rho(j)\}.$$

We denote by P_V the associated polytope. If V is one-dimensional we often identify it with a non-zero vector e in its span and identify the associated divisor (and polytope) with D_e (and P_e). The parliament of polytopes is the collection of polytopes P_e, for e in the ground set of the matroid $M(\mathcal{E})$.

In the following paragraphs we present several technical results on these divisors. The main motivation behind these is to study the multiplication maps

$$H^0(\mathbb{P}(\mathcal{E}), \mathscr{L}_1) \otimes H^0(\mathbb{P}(\mathcal{E}), \mathscr{L}_2) \to H^0(\mathbb{P}(\mathcal{E}), \mathscr{L}_1 \otimes \mathscr{L}_2).$$

We show that any such multiplication map can be lifted to a multiplication map of sections of line bundles on the toric variety X_Σ. Using this, we give a criterion for $\mathbb{P}(\mathcal{E})$ to be a Mori Dream Space.

Many of the conclusions we obtain on Cox rings were already shown in the paper [Gon+12], using different methods. However we think that our perspective is illuminating for understanding toric vector bundles, since our results are formulated and proved using the Klyachko filtrations directly. Moreover, we are able to isolate precisely what the generators of the Cox ring of $\mathbb{P}(\mathcal{E})$ are: They are pullbacks of sections from the base, together with sections corresponding to matroid vectors of some $S^k\mathcal{E}$ which are not symmetric powers of matroid vectors of symmetric powers of $\mathcal{E}$ smaller than k. A key ingredient for doing this is knowing the Klyachko filtrations of tensor products and symmetric powers of $\mathcal{E}$, which was described in [Gon11]:

Proposition V.4.1 ([Gon11, Corollary 3.2])**.** *Let $\mathcal{E}_1, ...\mathcal{E}_s$ be toric vector bundles on X_Σ. Then the Klyachko filtrations of their tensor product $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_s$ are given by*

$$(E_1 \otimes \cdots \otimes E_s)^\rho(j) = \sum_{j_1+\cdots+j_s=j} E_1^\rho(j_1) \otimes \cdots \otimes E_s^\rho(j_s),$$

for any ray $\rho \in \Sigma$ and $j \in \mathbb{Z}$.

Proposition V.4.2 ([Gon11, Corollary 3.5])**.** *Let $\mathcal{E}$ be toric vector bundles on X_Σ. Then the Klyachko filtrations of its symmetric power $S^k\mathcal{E}$ are given by*

$$(S^k E)^\rho(j) = \sum_{j_1+\cdots+j_k=j} \operatorname{Im}(E^\rho(j_1) \otimes \cdots \otimes E^\rho(j_k) \to S^k E),$$

for any ray $\rho \in \Sigma$ and $j \in \mathbb{Z}$.

We now proceed to use these descriptions to study what happens to the divisors in the parliament of polytopes under taking tensor and symmetric products.

Lemma V.4.3. *Given two vector bundles $\mathcal{E}$ and $\mathcal{F}$ and vector subspaces $V \subset E$ and $W \subset F$, we have that*

$$D_V + D_W = D_{V \otimes W},$$

where $V \otimes W$ is regarded as a vector subspace in $E \otimes F$, the fiber over $\mathcal{E} \otimes \mathcal{F}$ at the identity.

Proof. The filtrations of $\mathcal{E} \otimes \mathcal{F}$ is given by the tensor product of the filtrations of E and F. The claim is local, so in other words we can check the coefficient of each ray separately. Restricted to a fixed ray we can write the filtration of $\mathcal{E} \otimes \mathcal{F}$ in terms of a basis for E containing a basis for V and a basis for F containing a basis for W. Then we see that

$$c_\rho = \max\{l \mid V \otimes W \subset (E \otimes F)^\rho(l)\},$$

is exactly the sum $a_\rho + b_\rho$, where

$$a_\rho = \max\{l \mid V \subset E^\rho(l)\},$$

$$b_\rho = \max\{l \mid W \subset F^\rho(l)\},$$

proving the claim. ■

Corollary V.4.4. *If $\mathcal{E}_1, \ldots, \mathcal{E}_s$ are vector bundles and $V_i \subset E_i$ are subspaces, then*

$$D_{V_1} + \ldots + D_{V_s} = D_{V_1 \otimes \cdots \otimes V_s}.$$

Proof. By induction on s. ■

Lemma V.4.5. *Given a vector bundle $\mathcal{E}$ and subspaces $V_i \subset E$, then*

$$D_{V_1} + \ldots + D_{V_s} = D_{V_1 \cdots V_s},$$

where $V_1 \cdots V_s$ is the subspace of $S^s E$ which by definition is the image of $V_1 \otimes \cdots \otimes V_s$ under the natural map $E^{\otimes s} \to S^s E$.

Proof. The filtrations of $S^s\mathcal{E}$ is given by taking sums of symmetric products of the subspaces appearing in the filtrations Proposition V.4.2. The claim is local, so in other words we can check the coefficient of each ray separately. Writing $D_{V_1 \cdots V_s} = \sum_\rho c_\rho D_\rho$ and $D_{V_i} = \sum_\rho a^i_\rho D_\rho$ we have that

$$c_\rho = \max\{l \mid V_1 \cdots V_s \subset (S^s E)^\rho(l)\}.$$

By Proposition V.4.2 we have that c_ρ has to be greater than or equal to the sum $\sum_i a^i_\rho$. If it is actually greater, then we contradict Corollary V.4.4: there has to be some $v \in \cap(E^{\otimes s})^\rho(c_\rho)$ which is not in the vector space $W_0 = \cap_\rho (E^{\otimes s})^\rho(\sum_i a^i_\rho)$. We can map v to its image $\overline{v}$ in $S^s E$ and on to $w_1 \in W_1 = \cap_\rho (E^s)^\rho(\sum_i a^i_\rho)$. Since $W_0 \to W_1$ is simply induced from the quotient morphism $E^{\otimes s} \to S^s E$ we can lift w_1 to some w_0 in W_0.

By Proposition V.4.1 we observe that the vector spaces appearing in the filtrations of $\mathcal{E}^{\otimes s}$ are invariant under the action of the symmetric group on s letters. Thus if w_0 is in W_0, then v also has to be there, which is a contradiction. ■

Recall that for a character $u \in M$, $H^0(X_\Sigma, \mathcal{E})_u = \cap E^\rho(\langle u, \rho \rangle)$.

Proposition V.4.6. *The natural map $H^0(\mathcal{E})_u \otimes H^0(\mathcal{F})_v \to H^0(\mathcal{E}\otimes\mathcal{F})_{u+v}$ is given by sending $s \in \cap E^\rho(\langle u, \rho\rangle)$ and $t \in \cap F^\rho(\langle v, \rho\rangle)$ to $s \otimes t \in \cap(E \otimes F)^\rho(\langle u+v, \rho\rangle)$.*

Proof. We consider the local situation: On any open affine U_σ, where $\sigma \in \Sigma$, we have that the bundles split as a sum of line bundles, i.e.

$$\mathcal{E}|_{U_\sigma} \simeq \oplus_{i=1}^{s} \mathcal{O}(u_i),$$

$$\mathcal{F}|_{U_\sigma} \simeq \oplus_{j=1}^{t} \mathcal{O}(v_j).$$

Thus the multiplication map corresponds to a map

$$\oplus_{i=1}^{s} H^0(\mathcal{O}(u_i)) \otimes \oplus_{j=1}^{t} H^0(\mathcal{O}(v_j)) \to \oplus_{i,j=1}^{s,t} H^0(\mathcal{O}(u_i + v_j)),$$

which is given simply by sending $\chi^{u_i} \in H^0(\mathcal{O}(u_i), U_\sigma), \chi^{v_j} \in H^0(\mathcal{O}(v_j), U_\sigma)$ to $\chi^{u_i+v_j} \in H^0(\mathcal{O}(u_i + v_j), U_\sigma)$.

Now by construction of the matroids, u_i corresponds to some $e_i \in M(\mathcal{E})$ and v_j corresponds to some $f_j \in M(\mathcal{F})$, thus the above can be written as

$$H^0(U_\sigma, D_{e_i}) \otimes H^0(U_\sigma, D_{f_j}) \to H^0(U_\sigma, D_{e_i} + D_{f_j}) = H^0(U_\sigma, D_{e_i \otimes f_j}).$$

Thus locally the multiplication of sections s, t is given by $s \otimes t$, and the result follows. ■

Proposition V.4.7. *Given a map $\mathcal{E} \to \mathcal{F}$ of vector bundles, corresponding to the linear map $\phi : E \to F$, the induced map $H^0(\mathcal{E}) \to H^0(\mathcal{F})$ is given by sending $s \in H^0(\mathcal{E})_u = \cap E^\rho(\langle u, \rho\rangle)$ to $\phi(s) \in H^0(\mathcal{F})_u$.*

Proof. This follows from Klyachko's classification theorem. ■

Corollary V.4.8. *The natural map $S^a\mathcal{E}\otimes S^b\mathcal{E} \to S^{a+b}\mathcal{E}$ induces the map $H^0(S^a\mathcal{E}\otimes S^b\mathcal{E}) \to H^0(S^{a+b}\mathcal{E})$ given by $s \otimes t \mapsto st$.*

By the above we also get

Proposition V.4.9. *The natural map $H^0(S^a\mathcal{E})\otimes H^0(S^b\mathcal{E}) \to H^0(S^{a+b}\mathcal{E})$ is given by $s \otimes t \mapsto st$.*

Proof. This follows from the above, since it is the composition

$$H^0(S^a\mathcal{E}) \otimes H^0(S^b\mathcal{E}) \to H^0(S^a\mathcal{E} \otimes S^b\mathcal{E}) \to H^0(S^{a+b}\mathcal{E}).$$

■

Lemma V.4.10. *If $\mathcal{E}$ is a vector bundle and $\mathcal{L}$ a line bundle then $M(\mathcal{E} \otimes \mathcal{L}) \simeq M(\mathcal{E})$ under any isomorphism $E \otimes L \simeq E$. If $v \in M(\mathcal{E})$ then the corresponding $v' \in M(\mathcal{E} \otimes \mathcal{L})$ satisfies $D_{v'} = D_v \otimes \mathcal{L}$.*

Proof. We will think of $\mathcal{L}$ as a divisor, thus we can write $\mathcal{L} = \sum a_i D_i$. Then we have that $E^\rho(j) \simeq (E \otimes L)^\rho(j + a_\rho)$. The matroid is constructed by choosing bases (in a particular manner) of all possible subspaces of the form $\cap_\rho E^\rho(j_\rho)$. By the above equality we see that under the isomorphism $E \otimes L \simeq E$ we get exactly the same subspaces for $\mathcal{E} \otimes \mathcal{L}$ thus the matroids are equal.

Given $v \in M(\mathcal{E})$ we can write $D_v = \sum b_\rho D_\rho$. Then by the above we see that $D_{v'} = \sum (a_\rho + b_\rho) D_\rho$, which equals $D_v \otimes \mathcal{L}$. ∎

The above has the following consequences: Constructing the parliament of polytopes for a $\mathcal{E}$ is equivalent to constructing the surjection

$$\bigoplus_{e \in M(\mathcal{E})} \mathcal{O}(D_e) \to \mathcal{E} \to 0.$$

We see by the lemma that constructing the parliament for $\mathcal{E} \otimes \mathcal{L}$ corresponds to the tensored sequence

$$\bigoplus_{e \in M(\mathcal{E})} \mathcal{O}(D_e) \otimes \mathcal{L} \to \mathcal{E} \otimes \mathcal{L} \to 0.$$

In other words the sequence will remain surjective on global sections after tensoring with any line bundle $\mathcal{L}$.

Proposition V.4.11. *Given a vector bundle $\mathcal{E}$ and line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X_Σ and vectors e_1, e_2 in the ground set of $M(\mathcal{E})$, there exists a commutative diagram*

$$\begin{array}{ccc} H^0(\mathcal{O}(D_{e_1}) \otimes \mathcal{L}_1)_u \otimes H^0(\mathcal{O}(D_{e_2}) \otimes \mathcal{L}_2))_v & \xrightarrow{\phi} & H^0(\mathcal{O}(D_{e_1 e_2}) \otimes \mathcal{L}_1 \otimes \mathcal{L}_2))_{u+v} \\ \big\downarrow f & & \big\downarrow g \\ H^0(S^k\mathcal{E} \otimes \mathcal{L}_1)_u \otimes H^0(S^l\mathcal{E} \otimes \mathcal{L}_2)_v & \xrightarrow{\eta} & H^0(S^{k+l}\mathcal{E} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)_{u+v} \end{array}$$

where η and ϕ are multiplication maps and f is the map coming from the parliament of polytopes. Also, g is induced from the map of vector spaces sending $1 \in \mathbb{C}$ to $e_1 e_2$ in $S^{k+l}\mathcal{E}$.

Proof. We have that $\phi(\chi^u \otimes \chi^v) = \chi^{u+v}$. Moreover $g(\chi^{u+v}) = e_1 e_2 \in H^0(S^{k+l}\mathcal{E} \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)_{u+v} = \cap_\rho (E \otimes L_1 \otimes L_2)^\rho(\langle u, \rho \rangle)$.

Going in the other direction $f(\chi^u \otimes \chi^v) = e_1 \otimes e_2$, where we now consider $e_1 \in H^0(S^k E \otimes L_1)_u = \cap (E \otimes L_1)^\rho(\langle u, \rho \rangle)$ and $e_2 \in H^0(S^k E \otimes L_2)_v = \cap (E \otimes L_2)^\rho(\langle u, \rho \rangle)$. We see that $\eta(e_1 \otimes e_2) = e_1 e_2$. ∎

In particular all multiplication maps of global sections of line bundles on $\mathbb{P}(\mathcal{E})$ can be lifted to multiplication of global sections of line bundles on X_Σ.

We now come to a key definition of this section, namely that of a set of vectors $\mathfrak{M}(\mathcal{E})$, whose elements correspond to matroid vectors of some $S^k\mathcal{E}$ which are not symmetric products of matroid vectors for smaller k. We will subsequently see that the elements of $\mathfrak{M}(\mathcal{E})$ (together with generators for $\mathrm{Cox}(X_\Sigma)$) correspond to generators of $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$.

Definition V.4.12. Given a toric vector bundle $\mathcal{E}$ we construct the set $\mathfrak{M}(\mathcal{E})$ by induction on k as follows: The step $k = 1$ corresponds to adding all vectors in the ground set of $M(\mathcal{E})$ to $\mathfrak{M}(\mathcal{E})$. Assume now we have completed step $k - 1$. Step k amounts to adding all vectors of $M(S^k\mathcal{E})$ which cannot be written as a symmetric product of vectors already in $\mathfrak{M}(\mathcal{E})$.

For a vector $e \in \mathfrak{M}(\mathcal{E})$ we denote by $\deg e$ the integer such that $e \in M(S^{\deg e}\mathcal{E})$.

Proposition V.4.13. *Given a toric vector bundle $\mathcal{E}$, there exists a surjective map of $\mathbb{C}$-algebras*

$$\bigoplus_{t_1,\dots,t_n\in\mathbb{Z}}\ \bigoplus_{s_i\in\mathbb{Z}_{\geq 0}|e_i\in\mathfrak{M}(\mathcal{E})} H^0(X_\Sigma, \mathcal{O}(\sum_i s_iD_{e_i} + t_1D_1 + \ldots + t_nD_n))$$

$$\to \bigoplus_{\substack{k\in\mathbb{Z}_{\geq 0},\\ t_1,\dots,t_n\in\mathbb{Z}}} H^0(X_\Sigma, S^k\mathcal{E} \otimes \mathcal{O}(t_1D_1 + \ldots + t_nD_n)),$$

where in each summand in the above sum only finitely many s_i are non-zero.

Proof. The first double sum is simply the ring of all sections of all integral linear combinations of the T-invariant divisors on X_Σ, as well as of all positive sums of divisors D_e, where $e \in \mathfrak{M}(\mathcal{E})$. Thus it is a $\mathbb{C}$-algebra under multiplication of sections of line bundles on X_Σ.

The map is defined as follows: We fix a summand, thus we pick $t_1, \dots, t_n$ and some non-zero s_i, which we denote by $s_1, \dots s_m$. We define $\mathcal{L}_1$ as $\mathcal{O}(\sum_i s_iD_{e_i})$ and $\mathcal{L}_2$ as $\mathcal{O}(\sum_i t_iD_i)$. We set $k = \sum_i s_i \deg e_i$. Then there is, by the assumptions on e_i and Klyachko's equivalence of categories, a map of vector bundles

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \to S^k\mathcal{E} \otimes \mathcal{L}_2,$$

given by sending $1 \in L_1 \otimes L_2 \simeq \mathbb{C}$ to $e_1^{s_1} \cdots e_m^{s_m}$. This in turn induces a map $H^0(X_\Sigma, \mathcal{L}_1 \otimes \mathcal{L}_2) \to H^0(X_\Sigma, S^k\mathcal{E} \otimes \mathcal{L}_2)$ which is the map in the statement above. We note that if $e_1^{s_1} \cdots e_m^{s_m}$ is in $M(S^k\mathcal{E})$ then this map is the same as the corresponding summand in the map induced by the parliaments of polytopes construction.

Fix the numbers $k, t_1, \dots, t_n$. They correspond to the bundle $\mathcal{F} = S^k\mathcal{E} \otimes \mathcal{O}(t_1D_1 + \ldots + t_nD_n)$. By Lemma V.4.10 we have that $M(\mathcal{F}) = M(S^k\mathcal{E})$. All summands in the map

$$\bigoplus_{w\in M(\mathcal{F})} H^0(X_\Sigma, \mathcal{O}(D_w)) \to H^0(X_\Sigma, S^k\mathcal{E} \otimes \mathcal{O}(t_1D_1 + \ldots + t_nD_n))$$

induced by the parliament of polytopes for $S^k\mathcal{E}$ will appear in the big sum in the proposition statement, by the remark at the end of the preceding paragraph and by the construction of $\mathfrak{M}(\mathcal{E})$. Thus the map is surjective for any graded piece $k, t_1, ..., t_n$, hence it is surjective.

By Proposition V.4.11 the map is compatible with the multiplication maps, hence the map is a map of $\mathbb{C}$-algebras. ∎

Motivated by the above we make the following definition.

Definition V.4.14. For a natural number k we define $S^k M(\mathcal{E})$ as all symmetric k-products of vectors in the ground set of the matroid $M(\mathcal{E})$. By $M(S^a\mathcal{E})M(S^b\mathcal{E})$ we mean all symmetric products e_1e_2 of vectors $e_1 \in M(S^a\mathcal{E})$ and $e_2 \in M(S^b\mathcal{E})$.

Before coming to the characterization of being a Mori Dream Space we need the following technical lemmas.

Lemma V.4.15. *Assume that for a natural number c and all integers k_ρ we have the equality*

$$\mathrm{Span}_{a,b,j_\rho,l_\rho}\{\cap_\rho S^a E^\rho(j_\rho) \cap_\rho S^b E^\rho(l_\rho) | j_\rho + l_\rho = k_\rho, a+b=c\} = \cap_\rho S^c E^\rho(k_\rho).$$

Then $M(S^c\mathcal{E}) \subset \cup_{a,b|a+b=c} M(S^a\mathcal{E})M(S^b\mathcal{E})$.

Proof. The matroid $M(S^c\mathcal{E})$ is constructed by [DJS18, Algorithm 3.2] from the intersections $\cap_\rho S^c E^\rho(k_\rho)$. We are supposed to, in a certain order depending on the dimension of the intersection, choose a basis for the intersection, or of a quotient. By our assumption we know that we can pick a basis for each such intersection consisting of vectors of the form v_1v_2 where $v_1 \in M(S^a\mathcal{E})$, $v_2 \in M(S^b\mathcal{E})$. By elementary linear algebra we can also choose such a basis for each quotient. The proposition follows. ∎

Lemma V.4.16. *Assume that for a natural number c and fixed integers k_ρ we have*

$$\mathrm{Span}_{a,b,j_\rho,l_\rho}\{\cap_\rho S^a E^\rho(j_\rho) \cap_\rho S^b E^\rho(l_\rho) | j_\rho + l_\rho = k_\rho, a+b=c\} \subsetneq \cap_\rho S^c E^\rho(k_\rho).$$

Then there exists a line bundle $\mathcal{L}$ and $s \in H^0(S^c\mathcal{E} \otimes \mathcal{L})$ which is not in the image of the map $\bigoplus_{a,b,\mathcal{L}_1,\mathcal{L}_2:\mathcal{L}_1\otimes\mathcal{L}_2=\mathcal{L}} H^0(S^a\mathcal{E}\otimes\mathcal{L}_1) \otimes H^0(S^b\mathcal{E}\otimes\mathcal{L}_2) \to H^0(S^c\mathcal{E}\otimes\mathcal{L})$.

Proof. Let $W = \cap_\rho S^c E^\rho(k_\rho)$. We define $\mathcal{L} = -D_W = \sum_\rho -k_\rho D_\rho$. Then

$$H^0(S^c E \otimes \mathcal{L})_0 = \cap(S^c E \otimes L)^\rho(0) = \cap S^c \mathcal{E}^\rho(k_\rho) = W,$$

thus we obtain W as the space of T-invariant global sections of $\mathcal{E}\otimes\mathcal{L}$. The image of

$$\bigoplus_{\substack{a,b,\mathcal{L}_1,\mathcal{L}_2\\ \mathcal{L}_1\otimes\mathcal{L}_2=\mathcal{L}}} H^0(S^a\mathcal{E}\otimes\mathcal{L}_1)\otimes H^0(S^b\mathcal{E}\otimes\mathcal{L}_2),$$

of degree 0 can be described as follows: First of all to map to the character 0 we can pick any character v in the first factor and $-v$ in the second factor, thus we get this sum equals

$$\bigoplus_{\substack{v,a,b,\mathcal{L}_1,\mathcal{L}_2\\ \mathcal{L}_1\otimes\mathcal{L}_2=\mathcal{L}}} H^0(S^a\mathcal{E}\otimes\mathcal{L}_1)_v\otimes H^0(S^b\mathcal{E}\otimes\mathcal{L}_2)_{-v}.$$

This is equal to the direct sum

$$\bigoplus_{\substack{a,b,j'_\rho,l'_\rho \\ j'_\rho+l'_\rho=k_\rho}} \cap_\rho S^a E^\rho(j'_\rho + \langle v,\rho\rangle) \cap_\rho S^b E^\rho(l'_\rho - \langle v,\rho\rangle),$$

because varying j'_ρ and l'_ρ corresponds exactly to varying the line bundles $\mathcal{L}_1, \mathcal{L}_2$. We set $j_\rho = j'_\rho + \langle v,\rho\rangle$ and $l_\rho = l'_\rho - \langle v,\rho\rangle$. By the assumption the image of this direct sum is contained in a space of smaller dimension than W, thus there must exist some $s \in W$, $s \notin \{\cap_\rho S^a E^\rho(j_\rho) \cap_\rho S^b E^\rho(l_\rho) | j_\rho + l_\rho = k_\rho\}$. Hence we are done. ■

Remark V.4.17. The above proof shows that a linear space $V \subset E$ is in $L(\mathcal{E})$ if and only if there exists a line bundle $\mathcal{L}$ and a character u such that $H^0(\mathcal{E}\otimes\mathcal{L})_u = V$.

Theorem V.4.18. *$\mathbb{P}(\mathcal{E})$ is a Mori Dream Space if and only if the set $\mathfrak{M}(\mathcal{E})$ is finite. Equivalently if and only if there exists an integer c such that for any $k \geq c$ we have that*

$$M(S^k\mathcal{E}) \subset \cup_{a+b=k} M(S^a\mathcal{E})M(S^b\mathcal{E}).$$

Proof. By the way we constructed $\mathfrak{M}(\mathcal{E})$ it is clear that if such a c exists then $\mathfrak{M}(\mathcal{E})$ is finite.

Assume that $\mathfrak{M}(\mathcal{E})$ is finite. Then by Proposition V.4.13 we have that $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ is the image of the section ring of finitely many line bundles on X_Σ, hence it is finitely generated.

Lastly we will prove that if no such c exists, then $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ cannot be finitely generated. Since no such c exist we must have an infinite sequence $c_1, c_2, c_3, \ldots$ satisfying

$$M(S^{c_i}\mathcal{E}) \not\subset \cup_{a+b=c_i} M(S^a\mathcal{E})M(S^b\mathcal{E})$$

For any fixed such c_i we see by Lemma V.4.15 that the assumptions of Lemma V.4.16 has to be satisfied for some k^i_ρ. Set $\mathcal{L}_i = \sum_\rho -k^i_\rho D_\rho$. By Lemma V.4.16 there exists a section $s_i \in H^0(S^{c_i}\mathcal{E}\otimes\mathcal{L}_i)$ which cannot be in the algebra generated by sections of bundles of the form $S^k\mathcal{E}\otimes\mathcal{L}$, for $k < c_i$. Thus to generate the Cox ring there has to be at least one generator which is a section of some $S^{c_i}\mathcal{E}\otimes\mathcal{L}$. Since this is true for any c_i, of which there are infinitely many, there cannot be a finite generating set for $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$. This concludes the proof. ■

V.5 Examples of Cox rings

Proposition V.5.1. *Assume that $M(S^k\mathcal{E}) \subset S^k M(\mathcal{E})$ for all k. Then $\mathbb{P}(\mathcal{E})$ is a Mori Dream Space.*

Proof. This follows from Theorem V.4.18. ■

Proposition V.5.2. *Assume that $\mathcal{E}$ is a toric vector bundle of rank r such that all subspaces appearing in the filtrations are either of dimension $0, 1, r$. Then $M(S^k\mathcal{E}) \subset S^k(M(\mathcal{E}))$, and in particular $\mathbb{P}(\mathcal{E})$ is a Mori Dream Space.*

Proof. It is clearly sufficient to consider only the rays which have a one-dimensional space in the filtration. Let $v_1, \dots, v_s$ be the vectors in these one-dimensional spaces, thus $M(\mathcal{E}) = \{v_1, \dots, v_s\}$.

Let the filtration on ray ρ_i be given by

$$E^i(j) = \begin{cases} E & \text{if } j \le a_i \\ v_i & \text{if } a_i < j \le b_i \\ 0 & \text{if } \ b_i < j \end{cases}.$$

Then the filtration of $S^k\mathcal{E}$ is given by

$$S^kE^i(j) = \begin{cases} S^kE & \text{if } j \le ka_i \\ v_i w & \text{if } ka_i < j \le (k-1)a_i + b_i \\ \vdots & \\ v_i^{k-1}w & \text{if } 2a_i + (k-2)b_i < j \le a_i + (k-1)b_i \\ v_i^k & \text{if } a_i + (k-1)b_i < j \le kb_i \\ 0 & \text{if } kb_i < j \end{cases},$$

where in each step w can be any vector not in the span of v_i. Thus to compute the matroid we need to consider the spaces

$$\cap_i S^kE^i(j_i) = \{f = v_i^{l_i} z_i \in S^kE \text{ where } (k-l_i+1)a_i + (l_i-1)b_i < j_i \le (k-l_i)a_i + l_i b_i\},$$

where, for each i, z_i can be any vector in $S^{k-l_i}E$. If we can show that each such space is spanned by symmetric powers of $\{v_1, \dots, v_s\}$ then we are done. We see that an f in this intersection corresponds to a hypersurface f of degree k in $\mathbb{P}(E)$ vanishing to order l_i at the hyperplane $H_i = V(v_i)$. But if f vanishes to order l_i at H_i then $f = v_i^{l_i} g$ for some hypersurface g of degree $k - l_i$. Iterating we obtain that $f = v_1^{l_1} v_2^{l_2} \cdots v_s^{l_s}$, which implies exactly what we wish to prove. ■

The above give new proofs of Theorem 5.7 and Theorem 5.9 in [HS18]:

Corollary V.5.3. *If $\mathcal{E}$ has rank 2 or if $\mathcal{E}$ is a tangent bundle, then $\mathbb{P}(\mathcal{E})$ is a Mori Dream Space.*

V.6 A presentation of the Cox ring

We here give a presentation of the Cox ring of $\mathbb{P}(\mathcal{E})$, in terms of generators and relations, defined from the matroids $M(S^k\mathcal{E})$. The essential idea is that the relations between matroid vectors in $M(S^k\mathcal{E})$ correspond to all relations between sections of line bundles of the same degree $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k)$. Moreover all relations between sections which are products of sections of different $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$-degree can be described in terms of the difference between $S^kM(\mathcal{E})$ and $M(S^k\mathcal{E})$. Using this we obtain a presentation of the Cox ring of any projectivized toric vector bundle.

To each ray $\rho \in \Sigma(1)$ we associate a variable S_ρ. These correspond to pullbacks of all generators for $\mathrm{Cox}(X_\Sigma)$ and will be some of the generators of $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$. To each vector $v \in \mathfrak{M}(\mathcal{E})$ we associate a variable T_v. Each T_v will also be a generator of $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$. The variable T_v correspond to the following element in $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$: Write $D_v = \sum_\rho a_\rho D_\rho$. Consider the bundle $S^{\deg v}\mathcal{E} \otimes \mathcal{O}(-D_v)$. Then v will be in $M(S^{\deg v}\mathcal{E} \otimes \mathcal{O}(-D_v))$ under the isomorphism $E \otimes \mathbb{C} \simeq E$, by Lemma V.4.10. The associated divisor D'_v is the trivial divisor, hence it has a unique global section. Its image in $H^0(X_\Sigma, S^{\deg v}\mathcal{E} \otimes \mathcal{O}(-D_v))$ under the map given by the parliament of polytopes is the section which we label T_v.

We will show that S_ρ and T_v are all generators of $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$. Moreover we will also describe the ideal of relations between these generators. It is a sum of two ideals I and J, which we now introduce.

Let $\mathrm{Rel}(S^k\mathcal{E})$ denote the space of linear relations between vectors in $M(S^k\mathcal{E})$. For any relation $r = \sum_i \lambda_i v_i$ in $\mathrm{Rel}(S^k\mathcal{E})$ we associate the divisor

$$D_r = \sum_\rho \min_i \{\alpha_{\rho i} | \lambda_i \neq 0\} D_\rho := \sum_\rho \beta^r_\rho D_\rho.$$

Associated to relations between matroid vectors for a fixed symmetric power k we get relations in the Cox ring given by the ideal

$$I = \langle \sum_i \lambda_i S^i_r T_{v_i} | r = \sum \lambda_i v_i \in \mathrm{Rel}(S^k\mathcal{E}) \rangle,$$

where $S^i_r = \prod_\rho S_\rho^{\alpha_{\rho i} - \beta^r_\rho}$.

The reason the polynomials in the ideal I give relations in the Cox ring is the following: For simplicity we consider what happens when $k = 1$, in other words for $M(\mathcal{E})$. Any $v \in M(\mathcal{E})$ defines a divisor D_v and thus a polytope P_v giving global sections. This polytope could be empty or full-dimensional or anything in between. However if we consider $\mathcal{F} = \mathcal{E} \otimes \mathcal{O}(-D_v)$ then by construction $v \in M(\mathcal{F})$ and the associated divisor D'_v is the trivial divisor. In particular it has a unique global section.

If we now fix a non-trivial relation $r = \sum_i \lambda_i v_i$ in $\mathrm{Rel}(\mathcal{E})$, we have for any v_i a unique associated global section. These are the sections corresponding to the variables T_{v_i}. Multiplying T_{v_i} with a monomial in the S_ρ corresponds to tensoring with effective divisors which are pull-backs from the base. In particular, the polytope will become bigger. The relations in the ideal I are relations obtained from multiplying each variable with monomials in S_ρ, such that the polytopes corresponding to v_i overlap. As soon as they overlap, there must be a relation between the corresponding sections by Proposition V.3.2.

Example V.6.1. Consider the projectivization of the tangent bundle $T_{\mathbb{P}^2}$ of $\mathbb{P}^2$. We denote by D_0, D_1, D_2 the T-invariant divisors. The parliament of polytopes is shown in Figure V.1. The matroid has three divisors, equal to D_0, D_1 and D_2. Thus there are three T-variables in $\mathrm{Cox}(\mathbb{P}(T_{\mathbb{P}^2}))$: T_0, T_1, T_2. T_i is the unique section of $\mathcal{O}_{T_{\mathbb{P}^2}}(1) - D_i$ of weight 0.

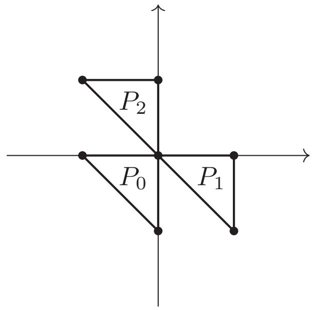

Figure V.1: The 3 polytopes of the parliament of $T_{\mathbb{P}^2}$.

Since the three matroid vectors are linearly dependent, we know that there will be a relation in the Cox ring involving T_0, T_1, T_2. The relation will be given by multiplying up with S_i variables (corresponding to the D_i as pullbacks of divisors from the base) until the three polytopes overlap and are of the same degree in $\mathrm{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$. This is in this case exactly of degree $\mathcal{O}_{T_{\mathbb{P}^2}}(1)$ and is shown in the image. Thus the associated relation is

$$T_0S_0 + T_1S_1 + T_2S_2.$$

Remark V.6.2. Any $v \in M(S^k\mathcal{E})$ is either in $\mathfrak{M}(\mathcal{E})$ or is a symmetric product of elements in $\mathfrak{M}(\mathcal{E})$. If it is the latter then by the letter T_v we denote the monomial $\prod_{i=1}^{l} T_{v_i}$ where v equals the symmetric product $v_1 \cdots v_l$, with $v_i \in \mathfrak{M}(\mathcal{E})$. Thus, for any $v \in M(S^k\mathcal{E})$ the symbol T_v correspond to a distinguished element in $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$.

In addition to the relations in I, we also get relations between generators of different degrees. Fix $v_1 \in M(S^a\mathcal{E})$ and $v_2 \in M(S^b\mathcal{E})$ and consider the associated sections: They are $T_{v_1} \in H^0(S^a\mathcal{E} \otimes -D_{v_1})_0$ and $T_{v_2} \in H^0(S^b\mathcal{E} \otimes -D_{v_2})_0$, respectively. Then the product $T_{v_1}T_{v_2}$ will lie in $H^0(S^{a+b}\mathcal{E} \otimes -D_{v_1} - D_{v_2})_0$. Choosing a basis $\{w_1, \ldots, w_s\} \subset M(S^{a+b}\mathcal{E} \otimes -D_{v_1} - D_{v_2})$ of the latter space, we can write $v_1v_2 = \sum a_iw_i$. Then we get the relation of sections

$$T_{v_1}T_{v_2} = \sum_i a_iT_{w_i}S_{w_i},$$

where $T_{w_i} \in H^0(S^{a+b} \otimes -D_{w_i})_0$ and S_{w_i} is defined as follows: We can write $D_{w_i} - D_{v_1} - D_{v_2} = \sum m_\rho D_\rho$. By assumption this is an effective divisor (since v_1v_2 lies in the intersection spanned by w_i) thus we can define $S_{w_i} = \prod_\rho S_\rho^{m_\rho}$. We let J be the ideal generated by all relations such as this. The ideal J in some sense measures the difference between $M(S^k\mathcal{E})$ and $S^kM(\mathcal{E})$; if $(S^k\mathcal{E}) = S^kM(\mathcal{E})$ for all k, then the ideal J is empty.

Even though we defined the generators of J in terms of symmetric products of two matroid vectors, the following lemma shows that they imply that the analogous relations for symmetric products of arbitrarily many vectors lies in J.

Lemma V.6.3. *Given $v_1, \ldots, v_p$ with $v_i \in M(S^{a_i}\mathcal{E})$ and let $\{w_1, \ldots, w_s\}$ be a basis of $H^0(S^{\sum a_i}\mathcal{E} \otimes (\sum_i -D_{v_i}))_0$ and write $v_1 \cdots v_p = \sum b_i w_i$. Then the relation*

$$\prod_i T_{v_i} - \sum_i b_i T_{w_i} S_{w_i}$$

lies in J, where S_{w_i} is some monomial in S_ρ, defined as above.

Proof. We will prove this for $p = 3$; the general case follows by iterating the process. Thus we assume that $v_1 v_2 v_3 = \sum b_i w_i$, where $w_i \in M(S^3\mathcal{E})$.

Let $v_1 v_2 = \sum c_k z_k$, where $z_k \in M(S^2\mathcal{E})$, thus we have an associated relation in J

$$T_1 T_2 = \sum_k c_k T_{z_k} S_{z_k}.$$

Multiplying the above by T_3 we get

$$T_1 T_2 T_3 = \sum_k c_k T_{z_k} T_3 S_{z_k}.$$

For a fixed k write $z_k v_3 = \sum d_{k,j} w_j$, thus we have an associated relation

$$T_{z_k} T_3 = \sum_j d_{k,j} T_{w_j} S_{k,j}.$$

Substituting this into the above we get

$$T_1 T_2 T_3 = \sum_k c_k S_{z_k} \sum_j d_{k,j} T_{w_j} S_{k,j} = \sum_j T_{w_j} (\sum_k c_k d_{k,j} S_{z_k} S_{k,j}).$$

We have that by construction Sz_k corresponds to the effective divisor $D_{z_k} - D_{v_1} - D_{v_2}$ and $S_{k,i}$ corresponds to $D_{w_i} - D_{z_k} - D_{v_3}$, hence their product $S_{z_k} S_{k,i}$ corresponds to the sum $D_{w_i} - Dv_1 - D_{v_2} - D_{v_3} = S_{w_i}$. Observe also that by combining the expressions for $v_1 v_2 v_3$ we see that $b_i = \sum_k c_k d_{k,i}$, thus the relation above simplifies to

$$T_1 T_2 T_3 = \sum_i b_i T_{w_i} S_{w_i},$$

which is what we wanted to show. ■

Theorem V.6.4. *The Cox ring of $\mathbb{P}(\mathcal{E})$ is*

$$\mathbb{C}[S_\rho, T_v|/(I + J).$$

Proof. From the fact that the Cox ring of the toric variety is $\mathbb{C}[S_\rho]$ [Cox95] and the existence of the surjections

$$\bigoplus_{v_i \in M(S^k\mathcal{E}} \mathcal{O}(D_{v_i}) \otimes \mathcal{L} \to S^k\mathcal{E} \otimes \mathcal{L},$$

which are surjective on global sections, it is clear that S_ρ and T_v generate the Cox ring of $\mathbb{P}(\mathcal{E})$, thus we need to determine the relations. Since $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ is graded by $\mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ we must have that any relation is between elements of the same class in the Picard group, i.e. between elements of some fixed class $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) + \sum_\rho t_\rho D_\rho$. We also have a finer grading given by the character u of the torus. After tensoring with $\mathrm{div}(-u)$ we may assume that $u = 0$. First we show that both I and J is contained in the ideal of the Cox ring.

Fixing a relation $r = \sum_i \lambda_i v_i$ in $\mathrm{Rel}(S^k\mathcal{E})$ we have, for each i, an associated trivial divisor D'_{v_i} of $S^k\mathcal{E} \otimes \mathcal{O}(-D_{v_i})$ and a section in $H^0(S^k\mathcal{E} \otimes \mathcal{O}(-D_{v_i}))_0$ which corresponds to T_{v_i}. By construction S^i_r corresponds to the effective divisor $\sum_\rho (a_{\rho i} - \beta^r_\rho) D_\rho$, thus a section in $H^0(\mathcal{E})_0$. Multiplying T_{v_i} and S^i_r together gives a section of $S^k\mathcal{E} \otimes \mathcal{O}(\sum_\rho -\alpha_{\rho i} D_\rho + \sum_\rho (\alpha_{\rho i} - \beta^r_\rho) D_\rho) = S^k\mathcal{E} \otimes \mathcal{O}(-D_r)$ of weight 0.

We have that $\lambda_i T_{v_i} S^i_r$ corresponds to the section $\lambda_i v_i \in (S^k E \otimes \mathcal{O}(-D_r))^\rho(0) = H^0(S^k\mathcal{E}(-D_r))_0$. Thus $\sum_i \lambda_i T_{v_i} S^i_r$ corresponds to $\sum_i \lambda_i v_i \in H^0(S^k\mathcal{E} \otimes \mathcal{O}(-D_r))_0$, which is zero, thus $\sum_i \lambda_i T_{v_i} S^i_r$ must be a relation, so I is contained in the Cox ideal.

Fixing a generator $T_{v_1} T_{v_2} - \sum_i a_i T_{w_i} S_{w_i}$ of J (as defined above) we see that under the isomorphism (since w_i is a basis for the latter space)

$$\oplus_i H^0(\mathcal{O}(D_{w_i}))_0 \to H^0(S^{a+b}\mathcal{E} \otimes \mathcal{O}(-D_{v_1} - D_{v_2}))_0$$

$v_1 v_2$ is sent to the same as $\sum_i a_i w_i$ thus the generator has to be a relation in the Cox ring. Thus J is contained in the Cox ideal.

Conversely, assume that we are given a relation between sections in $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ of multidegree $(k, l_1, \ldots, l_s)$ and weight 0, in other words in $V = H^0(S^k\mathcal{E} \otimes \mathcal{O}(l_1 D_1 + \cdots + l_s D_s))_0$. We can write the relation as

$$\sum_{i \in K} c_i \prod_j T^{n_{i,j}}_{v_{i,j}} \prod_\rho S^{m_{\rho,i}}_\rho = 0,$$

where K is the index-set of all monomials appearing in the relation. Choose a basis $w_1, \ldots, w_q$ of V, where $w_i \in M(S^k\mathcal{E})$.

Fix i and write $z_j = \prod_j v^{n_{i,j}}_{i,j} = \sum_{i=1}^q a_{i,j} w_i$. Then the relations in I and J imply that we have the relation

$$\prod_j T^{n_{i,j}}_{v_{i,j}} = \sum_{i=1}^q a_{i,j} T_{w_i} S_{i,j}$$

where $S_{i,j}$ is some monomial in S_ρ. Thus we can replace $\prod_j T^{n_{i,j}}_{v_{i,j}}$ in the relation with the sum above. Doing this for all monomials z_j we get a polynomial whose only T-variables appearing are T_{w_i}, in other words a polynomial $\sum_{i=1}^q b_i T_{w_i} S_i$, where S_i is some monomial in S_ρ. This is by assumption equal to 0, however it corresponds to the element $\sum_{i=1}^q b_i w_i$ in V. Since the w_i by construction form a basis for V, this forces $b_i = 0$ for all i. Thus the relation we started with is a sum of relations coming from I and J.

■

Remark V.6.5. Note that in the above we do not assume that $\mathrm{Cox}(\mathbb{P}(\mathcal{E}))$ is finitely generated.

Remark V.6.6. This generalizes the presentations of the Cox rings of rank two bundles and tangent bundles given in [HS18, Theorem 5.7, Theorem 5.9].

Example V.6.7. Consider again the tangent bundle of $\mathbb{P}^2$. By Proposition V.5.2 the ideal J is empty. Thus by Example V.6.1

$$\mathrm{Cox}(\mathbb{P}(T_{\mathbb{P}^2})) = \mathbb{C}[T_0, T_1, T_2, S_0, S_1, S_2]/(T_0S_0 + T_1S_1 + T_2S_2).$$

This is consistent with the well-known description of $\mathbb{P}(T_{\mathbb{P}^2})$ as a $(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

Example V.6.8. From the above we can recover all results on Cox rings of toric vector bundles found in [Gon+12]. For example let $\mathcal{E}$ be a bundle with filtrations given as

$$E^\rho(j) = \begin{cases} E & \text{if } j \leq 0 \\ V_\rho & \text{if } j = 1 \\ 0 & \text{if } 1 < j \end{cases}$$

where $\dim V_\rho > 1$ which is what is mostly studied in [Gon+12]. Let Y be the iterated blowup of $\mathbb{P}(E)$ in all linear spaces in $L(\mathcal{E})$; first blow up points (corresponding to hyperplanes V_ρ), then strict transform of lines and so on. Let X be Y minus all exceptional divisors over linear subspaces not equal to some V_ρ. By [Gon+12, Theorem 3.3] the Cox ring of $\mathbb{P}(\mathcal{E})$ is isomorphic to the Cox ring of X. We can see this as follows from our theorem: Effective divisors of some $S^k\mathcal{E} \otimes \mathcal{O}(D)$ correspond exactly to $f \in \cap(S^kE \otimes D)^\rho(0)$, for some $D = \sum_\rho a_\rho D_\rho$. In other words to degree k polynomials f on $\mathbb{P}(E)$ vanishing to order a_ρ at V_ρ. Such effective divisors which are not symmetric products of divisors of lower $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$-degree, correspond exactly to polynomials f which cannot be written as a product of lower degree such polynomials. This corresponds exactly to generators of the Cox ring of the associated blow-up of $\mathbb{P}(E)$. By well-known results on generators of Cox rings of such blow-ups, these are often not finitely generated, for instance if V_ρ consist of 9 or more general hyperplanes [Muk04].

For example we can pick V_ρ to be 5 general hyperplanes in $E \simeq \mathbb{C}^3$, corresponding to five general points of $\mathbb{P}^2 = \mathbb{P}(E)$. Then the Cox ring is generated by sections pulled back from the base, together with the ten lines between the corresponding points of $\mathbb{P}(E)$, corresponding to the fact that $M(\mathcal{E})$ has ten elements, and by the unique quadric passing through the five points; this corresponds to the fact that there is one unique intersection of the filtrations for $S^2\mathcal{E}$ which is not the span of symmetric products of vectors in $M(\mathcal{E})$.

Remark V.6.9. The above results is satisfying from a theoretical point view, since we are able to completely describe the Cox rings of $\mathbb{P}(\mathcal{E})$ in terms of the matroids of $S^k\mathcal{E}$. However, in practice these results are not necessarily as satisfying, since for a specific bundle $\mathcal{E}$ it is not easy to say what the relationship between $M(S^k\mathcal{E})$ and $S^kM(\mathcal{E})$ is for any k: It is at least as hard as describing generators of Cox rings of general iterated blow-ups of projective space in linear spaces, which is well-known to be a hard problem.

The above results does significantly improve our understanding of sections of line bundles on $\mathbb{P}(\mathcal{E})$. Elements of $\mathfrak{M}(\mathcal{E})$ correspond exactly to sections of some such line bundles, which is not in the algebra generated by sections of lower $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$-degree. We will in the next section that this has the consequence that to check whether $\mathcal{E}$ is big we need also to consider the matroids of all $S^k\mathcal{E}$ at once.

V.7 A bigness criterion

Let $\mathcal{E}$ be a toric vector bundle on the toric variety X_Σ. We say that $\mathcal{E}$ is big if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a big line bundle. This is what Jabbusch calls L-big [Jab09]. In this section we investigate when a toric vector bundle is big.

Example V.7.1. If $\mathcal{E}$ is a direct sum of line bundles then $\mathcal{E}$ is big if and only if some positive linear combination of the line bundles is big [Laz04a, Lemma 2.3.2]. Thus $\mathcal{O}(-1) \oplus \mathcal{O}(2)$ on $\mathbb{P}^d$ is big, but not nef.

Di Rocco, Jabbusch and Smith ask whether a toric vector bundle is big if and only if some Minkowski sum of the polytopes in the parliament is full-dimensional [DJS18, p.3]. The following example shows that this is not the case.

Example V.7.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and denote by H_1 and H_2 the pullbacks of the hyperplane classes of each factor. Let $\mathcal{E} = \mathcal{O}_X(H_1) \oplus \mathcal{O}_X(H_2 - H_1)$. Then since $D = H_1 + H_2 = 2H_1 + (H_2 - H_1)$ is big, we see by the previous example that $\mathcal{E}$ is big. However the parliament of polytopes is the following: P_{H_1} is a line segment of length 1 while $P_{H_2-H_1}$ is empty. Thus no Minkowski sum of the polytopes in the parliament is full-dimensional.

For divisors D and E on a toric variety we have the inclusion $P_D + P_E \subset P_{D+E}$, however in general this is not an equality. This is the reason for the above example giving a big vector bundle even if no Minkowski sum is full-dimensional. To rectify this we might ask the similar question which still might be true:

Question V.7.3. Is $\mathcal{E}$ big if and only if a positive linear combination of the divisors in the parliament is big?

The following example shows that this is also not true.

Example V.7.4. Let be $\mathbb{P}^2$ be given as a toric variety as the complete fan with ray generators $\rho_0 = -e_1 - e_2, \rho_1 = e_1, \rho_2 = e_2$. Consider the rank 3 bundle on $\mathbb{P}^2$ given by Klyachko filtrations for $i = 1, 2$:

$$E^i(j) = \begin{cases} E & \text{if } j \leq -1 \\ W_i & \text{if } j = 0 \\ v_i & \text{if } j = 1 \\ 0 & \text{if } 1 < j \end{cases},$$

and for $i = 0$

$$E^0(j) = \begin{cases} E & \text{if } j \leq 0 \\ W_0 & \text{if } j = 1 \\ v_0 & \text{if } j = 2 \\ 0 & \text{if } 1 < j \end{cases}.$$

Here v_i are general vectors of $E \simeq \mathbb{C}^3$ and W_i is a general 2-dimensional vector space containing v_i. Let $l_{ij} = W_i \cap W_j$. Then $M(\mathcal{E}) = \{v_i, l_{ij}\}$ and each associated divisor is trivial. Hence no positive sum of these can be big. However, considering $S^2\mathcal{E}$ we have that the intersection $S^2E^0(1) \cap S^2E^1(0) \cap S^2E^2(0)$ is one dimensional, so there is a matroid vector w in this intersection. We see that $D_w = \mathcal{O}_{\mathbb{P}^2}(1)$, thus it is big, which implies that $H^0(X_\Sigma, S^{2k}\mathcal{E}) = H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2k))$ grows as $k^{\dim \mathbb{P}(\mathcal{E})}$. This means that $\mathcal{E}$ is big.

In the example above we have that $\mathcal{E}$ is big even if no positive sum of divisors in the parliament is big. However we have that there is a big divisor in the parliament of $S^k\mathcal{E}$ for all sufficiently large k (in the example all $k \geq 2$). The following theorem shows that this will always happen.

Theorem V.7.5. *A toric vector bundle is big if and only if there exists $k > 0$ and a vector $v \in M(S^k\mathcal{E})$ such that P_v is full dimensional.*

Proof. By [Laz04b, Example 6.1.23] we have that $\mathcal{E}$ is big if and only if for some (every) ample A we have that $H^0(S^k\mathcal{E} \otimes A^{-1}) \neq 0$ for some $k > 0$.

Assume $\mathcal{E}$ is not big and let A be an ample line bundle. Then for every $k > 0$ we have that $H^0(S^k\mathcal{E} \otimes A^{-1}) = 0$. That means that for any $v \in M(S^k\mathcal{E})$ we have $H^0(\mathcal{O}(D_v) \otimes A^{-1}) = 0$, since the map $\mathcal{O}(D_v) \otimes A^{-1} \to S^k\mathcal{E} \otimes A^{-1}$ is injective on global sections. There is an induced map $\mathcal{O}(lD_v) \otimes A^{-1} \to S^{kl}\mathcal{E} \otimes A^{-1}$ which is non-zero, since it corresponds to the map of vector spaces sending 1 to v^l. If there exists $l > 0$ such that $H^0(\mathcal{O}(lD_v) \otimes A^{-1}) \neq 0$ then we see that also the induced map on global sections is non-zero, thus $H^0(S^{kl}\mathcal{E} \otimes A^{-1}) \neq 0$, contradicting the fact that $\mathcal{E}$ is not big. Thus $H^0(\mathcal{O}(lD_v) \otimes A^{-1}) = 0$ for all l, thus each D_v is not big and thus each polytope P_v is not full dimensional.

Conversely assume that no such v exists. That means that for each $v \in M(S^k\mathcal{E})$ the polytope P_v is not full dimensional. Since P_v is not full dimensional D_v is not big, thus $H^0(\mathcal{O}(D_v) \otimes A^{-1}) = 0$. Thus $H^0(S^k\mathcal{E} \otimes A^{-1}) = 0$ for every k, thus $\mathcal{E}$ is not big. ■

V.8 Positivity and concave support functions

A fundamental result on positivity of divisors on toric varieties is the following equivalences.

Theorem V.8.1 ([CLS11, Theorem 6.1.7, Lemma 6.1.13, Theorem 6.1.14])**.** *Given a divisor $D = \sum_\rho a_\rho D_\rho$ on a toric variety, the following conditions are equivalent*

- *D is nef*

- *$m_\sigma \in P_D$ for all σ*
- *ϕ_D is concave*
- *$\phi_D(v) \geq \langle m_{\sigma'}, v\rangle$ for all σ' not containing v.*

The following conditions are also equivalent

- *D is ample*
- *ϕ_D is strictly concave*
- *$\phi_D(v) > \langle m_{\sigma'}, v\rangle$ for all σ' not containing v.*

As for line bundles, one can ask for various positivity properties of vector bundles. These properties are often defined in terms of corresponding properties for the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the associated projective bundle $\mathbb{P}(\mathcal{E})$ of rank one quotients of $\mathcal{E}$. One has that positivity notions which coincide for line bundles do not necessarily coincide for vector bundles.

Following Hartshorne [Har66] we say that a vector bundle is nef, ample, very ample respectively if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a nef, ample, very ample line bundle, respectively. On a toric variety, there are well known criteria for checking whether a toric vector bundle is positive:

Theorem V.8.2 ([HMP10, Theorem 2.1])**.** *$\mathcal{E}$ is ample (resp. nef) if and only if $\mathcal{E}|_C$ is ample (resp. nef) for all invariant curves $C \subset X_\Sigma$.*

Remark V.8.3. For checking the ampleness of a line bundle it is sufficient to restrict to invariant curves in X_Σ which are extremal in the Mori cone of curves, since invariant curves generate the Mori cone [CLS11, Theorem 6.3.20] and since the ample cone is the interior of the dual of the Mori cone. In the case of higher rank vector bundles it is easy to construct examples showing that it is not sufficient to restrict to extremal curves in X_Σ.

Theorem V.8.4 ([DJS18, Theorem 1.2])**.** *A toric vector bundle $\mathcal{E}$ is globally generated if and only if for all maximal cones σ, where $\mathcal{E}|_{U_\sigma} = \oplus_{i=1}^r \mathcal{O}(\mathrm{div}(u_i))$, there exists linearly independent $e_1, \ldots, e_r \in M(\mathcal{E})$ such that $u_i \in P_{e_i}$ for all i.*

Theorem V.8.5 ([DJS18, Corollary 6.7])**.** *A toric vector bundle $\mathcal{E}$ is very ample if and only if, for all maximal cones σ, where $\mathcal{E}|_{U_\sigma} = \oplus_{i=1}^r \mathcal{O}(\mathrm{div}(u_i))$, there exist linearly independent $e_1, \ldots, e_r \in M(\mathcal{E})$ such that $u_i \in P_{e_i}$ for all i and such that the following is true: Each u_i is automatically a vertex of P_{e_i}. We require that the edges of P_{e_i} emanating from the vertex u_i generate a cone which is a translate of $\sigma^\vee$ for all i.*

We will now reinterpret the criterion to be nef/ample in terms of concave support functions, which will lead us to a result of similar spirit to Theorem V.8.1 for higher rank toric vector bundles.

V.8.1 Branched covers of fan

We here briefly recall Payne's notion of the branched cover of a fan which is associated to $\mathcal{E}$ [Pay09]. Payne gives a systematic treatment of cone complexes and how to associate a cone complex which is a branched cover of the fan Σ to a toric vector bundle $\mathcal{E}$. We will a bit more naively construct the branched cover as in [Pay09, Example 1.2].

We are given a toric variety X_Σ and a toric vector bundle $\mathcal{E}$ on X_Σ and we will construct a topological space $\mathbf{\Sigma}_\mathcal{E}$ together with projection map $f : \mathbf{\Sigma}_\mathcal{E} \to \Sigma$ with the structure of a rank r branched cover. This means that for any cone σ the inverse image of σ under f is isomorphic to r copies of σ, counted with multiplicity. We will also construct an associated piecewise linear support function $\Psi_\mathcal{E}$ on $\mathbf{\Sigma}_\mathcal{E}$.

For each maximal cone σ_i we can consider the restriction $\mathcal{E}|_{U_{\sigma_i}} = \sum_{k=1}^{r} \mathcal{O}(\operatorname{div} u_{ik})$ where u_{ik} are characters of the torus. For each of the r summands we take a copy of σ_i, denoted by σ_{ik}. We define $\boldsymbol{\sigma_{i\mathcal{E}}}$ to be $\coprod_k \sigma_{ik}/\sim$ where the equivalence relation $\sim$ is given as follows: If τ is a face of σ_i such that $u_{ik} = u_{il}$ on τ we identify the corresponding copies of τ: $\tau_{ik} \sim \tau_{il}$. The piecewise linear function $\Psi_\mathcal{E}$ is linear on each cone σ_{ik}, given by $v \mapsto \langle u_{ik}, v\rangle$.

The topological space $\mathbf{\Sigma}_\mathcal{E}$ is given by $\coprod_i \boldsymbol{\sigma_{i\mathcal{E}}}/\equiv$ where the equivalence relation $\equiv$ is given as follows. Fix a cone τ of codimension one, it is contained in two maximal cones σ_1 and σ_2. Set $\mathcal{E}|_{U_{\sigma_1}} = \sum_k \mathcal{O}(\operatorname{div}(u_{1k}))$ and $\mathcal{E}|_{U_{\sigma_2}} = \sum_k \mathcal{O}(\operatorname{div}(u_{2k}))$. Then τ corresponds to a curve C which is isomorphic to $\mathbb{P}^1$ and any vector bundle on $\mathbb{P}^1$ splits as a sum of line bundles. A result by Hering, Mustata and Payne shows that for an invariant curve C, $\mathcal{E}|_C$ splits equivariantly as a sum of line bundles. Moreover, the numbers a_j such that the splitting type on C is $\oplus_j \mathcal{O}_{\mathbb{P}^1}(a_j)$, correspond to unique pairs of characters (u_{1k}, u_{2l}): $u_{1k} - u_{2l} = a_j \rho_\tau$, where by ρ_τ we mean the primitive generator of $\tau^\perp$ positive on σ. Thus, up to permutation, we may assume that u_{1k} is paired with u_{2k} for all k. We then glue σ_{1k} to σ_{2k} via identifying the face τ_{1k} with the face τ_{2k}.

In other words if one is in a fixed sheet σ_{1k} above a maximal cone σ and moves towards a neighbouring maximal cone σ_2 through a cone of codimension one τ, then the sheet σ_{2l} you end up in is the one corresponding to the pairing of the characters on σ_1, σ_2 when we restrict $\mathcal{E}$ to C. Note that this well-defined: If, for instance, $u_{11} = u_{12}$, then it seems we could reorder and glue different sheets together, however then the corresponding σ_{11} and σ_{12} will already be identified via $\sim$, so we get the same object.

The piecewise linear function $\Psi_\mathcal{E}$ on $\mathbf{\Sigma}_\mathcal{E}$ is obtained from the local versions above. There is also a projection map $\pi : \mathbf{\Sigma}_\mathcal{E} \to |\Sigma|$ given by mapping each σ_{ik} isomorphically onto σ_i. By a line segment in $\mathbf{\Sigma}_\mathcal{E}$, we will mean a connected subset l such that the projection $\pi(l)$ is a line segment in $|\Sigma|$ and so that the fiber over each point of $\pi(l)$ is a single point. We will say that $\Psi_\mathcal{E}$ is concave, if it is concave when restricted to any line segment.

Fix a line segment in $\mathbf{\Sigma}_\mathcal{E}$ with endpoints v and w. The projection $\pi(w)$ has to be contained in at least one maximal cone σ. There are r sheets in $\mathbf{\Sigma}_\mathcal{E}$, each mapping isomorphically to σ. The sheets are by construction in bijection to characters u_i such that $\mathcal{E}|_{U_\sigma} \simeq \oplus_{i=1}^r \mathcal{O}(\mathrm{div}(u_i))$. Thus which sheet w is contained in corresponds to a distinguished character u_i. We say that any such u_i is a character obtained by moving in a straight line from v inside $\mathbf{\Sigma}_\mathcal{E}$.

Proposition V.8.6. *Given a toric vector bundle $\mathcal{E}$ on a smooth complete toric variety X_Σ, the following are equivalent.*

(1) *$\mathcal{E}$ is nef (ample)*

(2) *$\Psi_\mathcal{E}$ is (strictly) concave for all i*

(3) *$\Psi_\mathcal{E}(v) \geq \langle u_i, v\rangle$ ($\Psi_\mathcal{E}(v) > \langle u_i, v\rangle$) where u_i is any character obtained by moving in a straight line from v inside $\mathbf{\Sigma}_\mathcal{E}$.*

Proof. Note that (2) and (3) only say something about $\Psi_\mathcal{E}$ restricted to line segments, thus we fix a line segment l in $\mathbf{\Sigma}_\mathcal{E}$. The projection $\pi(l) \subset |\Sigma|$ is a line segment in $N_\mathbb{Q}$. Let Σ' be the subfan of Σ consisting of all cones in Σ intersecting $\pi(l)$, as well as all their faces. The line segment l determines a character u_σ on each maximal cone $\sigma \in \Sigma'$. The collection of these characters define a divisor D_l on $X_{\Sigma'}$. By construction we see that $\Psi_\mathcal{E}|_l = \phi_{D_l} \circ \pi|_l$. We claim that $\mathcal{E}$ is nef (ample) if and only if $\phi_{D_l}|_l$ is (strictly) concave for all l. Assuming this claim we see that the proposition follows from the corresponding statement for divisors.

The proof of the claim depends on an adaption of standard techniques on toric line bundles. We put the detailed statement in Lemma V.8.7 below. To prove the claim note $\mathcal{E}$ is nef if and only if $\mathcal{E}|_C$ is nef for any invariant curve C. By [HMP10, Corollary 5.5] $\mathcal{E}|_C$ splits as a sum of line bundles $\mathcal{O}_C(D_i)$, thus nefness is equivalent to the inequalities $D_i \cdot C \geq 0$, for all i, C. Let C be given by the cone τ of codimension one, and assume τ is contained in the maximal cones σ and σ'. By [CLS11, Proposition 6.3.8]

$$D_i \cdot C = \langle u_{\sigma'} - u_\sigma, v\rangle = \phi_d(v) - \langle u_\sigma, v\rangle,$$

where v is the generator of $\tau^\perp$ which is positive on σ' and D_i corresponds to the characters $u_\sigma, u_{\sigma'}$.

By the above inequality we see that if $\mathcal{E}$ is nef then clause (4) of Lemma V.8.7 is satisfied, thus by (1) the support function $\phi_D|_l$ is concave for any l.

Conversely if the support function is concave for all lines l, then we can pick l starting in the generator of $\tau^\perp$ which is positive on σ' and ending in the interior of σ. Then the inequality (2) of Lemma V.8.7 is exactly stating that the corresponding divisor satisfies $D_i \cdot C \geq 0$. Since this is true for all divisors and curves, $\mathcal{E}$ is nef.

The case of ampleness is the same argument, only with concavity replaced with strict concavity, as well as that we require inequalities to be strict. ∎

Lemma V.8.7 (cf. [CLS11, Lemma 6.1.5])**.** *Let D be a Cartier divisor on a toric variety corresponding to the fan Σ. Fix a line l which is contained in the support of Σ. Then the following are equivalent:*

(1) *The support function $\phi_D|_l : l \to \mathbb{R}$ is concave.*

(2) $\phi_D(v) \geq \langle u_\sigma, v\rangle$ *for all* $v \in l$ *and maximal cones* σ *intersecting* l.

(3) $\phi_D(v) = \max\langle u_\sigma, v\rangle$, *where* $v \in l$ *and the maximum is over maximal cones* σ *intersecting* l.

(4) *For every cone of codimension one* $\tau = \sigma \cap \sigma'$ *such that* σ *and* σ' *intersects* l *there is* $v_0 \in \sigma' \setminus \sigma$ *with* $\phi_D(v_0) \geq \langle u_\sigma, v_0\rangle$.

Proof. The proof is essentially identical to the proof of [CLS11, Lemma 6.1.5], except that one has to replace all instances of the support of Σ with l and only consider cones which intersects l. ■

Remark V.8.8. In [KM18] Kaveh and Manon prove that a toric vector bundle is nef (ample) if and only if $\mathcal{E}$ is what they call buildingwise (strictly) convex. This statement is similar to the above, although formulated in a different language.

V.9 A counterexample

We here describe a sequence of ample bundles $\mathcal{E}_k$ such that $S^k\mathcal{E}_k$ is not globally generated. This example was worked out jointly with Greg Smith (private correspondence).

Let X_Σ be the smooth complete toric surface with rays

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\end{bmatrix}, v_3 = \begin{bmatrix}1\\2\end{bmatrix}, v_4 = \begin{bmatrix}0\\1\end{bmatrix}, v_5 = \begin{bmatrix}-1\\0\end{bmatrix}, v_6 = \begin{bmatrix}0\\-1\end{bmatrix}.$$

Let $\mathcal{E}$ be a rank 2 toric vector bundle on X_Σ. Assume that for each ray, the filtration has both a one-dimensional and a two-dimensional vector space appearing. Let the integers for which the filtration jumps on ray v_i be given by a_i, b_i, where $b_i > a_i$. Letting p_i be the one-dimensional space appearing in ray v_i, we assume that $q := p_1 = p_2 = p_3 = p_4$ and that q, p_5, p_6 are pairwise distinct.

We have that $\mathcal{E}$ is ample if and only if the restriction to any invariant curve is ample. This is equivalent to the list of inequalities below. These can be derived as follows. An invariant curve C corresponds to a ray of the fan. Restricted to the curve $\mathcal{E}|_C$ splits as a sum of line bundles D_1 and D_2. The numbers below are $C \cdot D_i$, for all i, C. Fix for instance the curve C corresponding to v_2. The splitting type is determined by the restriction of $\mathcal{E}$ to the maximal cones $\sigma_{12} = \mathrm{Cone}(v_1, v_2)$ and $\sigma_{23} = \mathrm{Cone}(v_2, v_3)$ containing v_2. We have that σ_{12} corresponds to the characters $(b_1, b_2 - b_1)$ and $(a_1, a_2 - a_1)$ and σ_{23} corresponds to $(2b_2 - b_3, b_3 - b_2)$ and $(2a_2 - a_3, a_3 - a_2)$. By [HMP10, Corollary 5.10] the restriction to C is determined by a unique pairing of the characters from σ_{12} and σ_{23}, such that the differences of the characters are parallel to $v_2^\perp$. Doing this we obtain the first two inequalities in the list, the ten others correspond to the other five curves.

$$\begin{aligned} b_1 + b_3 - 2b_2 &> 0 & a_1 + a_3 - 2a_2 &> 0 \\ b_2 + b_4 - b_3 &> 0 & a_2 + a_4 - a_3 &> 0 \\ b_3 + a_5 - 2b_4 &> 0 & a_3 + b_5 - 2a_4 &> 0 \\ b_4 + b_6 &> 0 & a_4 + a_6 &> 0 \\ b_5 + b_1 &> 0 & a_5 + a_1 &> 0 \\ b_6 + a_2 - a_1 &> 0 & a_6 + b_2 - b_1 &> 0. \end{aligned}$$

Let the bundle $\mathcal{E}_k, k \geq 1$ be defined by

$$\begin{aligned} a_1 &= 1 & b_1 &= 4 \\ a_2 &= -2 & b_2 &= 6k-1 \\ a_3 &= 3k-6 & b_3 &= 12k-5 \\ a_4 &= 6k-5 & b_4 &= 6k-3 \\ a_5 &= 0 & b_5 &= 9k-3 \\ a_6 &= 6-6k & b_6 &= 4. \end{aligned}$$

We can check that the ampleness inequalities are satisfied for $\mathcal{E}_k$. However we can show that $S^k\mathcal{E}_k$ is not globally generated:

The characters on the cone $\sigma_{2,3}$ are given by $u_1 = (2a_2 - a_3, a_3 - a_2), u_2 = (2b_2 - b_3, b_3 - b_2)$. On σ_{23} we need to choose a matroid vector corresponding to the character $u = u_1 + (k-1)u_2$. This will be of the form $q^{k-1}p_5$ or $q^{k-1}p_6$. In the first case we need to have

$$ka_6 \geq a_2 - a_3 + (k-1)(b_2 - b_3)$$

while in the second case we need to have

$$ka_5 \geq a_3 - 2a_2 + (k-1)(b_3 - 2b_2)$$

Inserting the chosen values and cleaning up we see that these inequalities are

$$0 \geq k$$

$$0 \geq 1$$

Both of these are clearly false, hence $S^k\mathcal{E}_k$ cannot be globally generated.

In conclusion, this example shows that on a toric variety X there cannot exist a number k such that for any ample toric vector bundle $\mathcal{E}$, we have that $S^k\mathcal{E}$ is always globally generated, not even for rank two bundles on a toric surface.

V.10 Pullbacks under multiplication maps

For a toric variety X_Σ there is, for any positive integer k, a toric morphism $f_k : X_\Sigma \to X_\Sigma$ called "multiplication by k" or the "toric Frobenius map". It is given by the map of lattices $N \to N$ which multiplies any element by k. For a toric vector bundle $\mathcal{E}$ on X_Σ we thus have bundles $f_k^*\mathcal{E}$, for any positive integer k.

The interest in the multiplication maps come from the fact that for a line bundle $\mathscr{L}$, the pullback $f_k^*\mathscr{L}$ is "more positive" than $\mathscr{L}$, since $f_k^*\mathscr{L} = \mathscr{L}^{\otimes k}$. This is similar to the Frobenius map for varieties in characteristic p, which was used by Deligne-Illusie-Reynard to prove the Kodaira vanishing theorem [DI87]. The toric Frobenius has been used to great effect in proving vanishing theorems on toric varieties [Fuj07], [CLS11, Chapter 9]. Thus, one would expect that applying f_k^* to a toric vector bundle $\mathcal{E}$ will only "increase the positivity" of $\mathcal{E}$. However, here we show that such analogous statements are not true for several positivity properties of toric vector bundles.

We now fix k and set $\mathcal{F} = f_k^*\mathcal{E}$. By [CLS11, Proposition 6.2.7] the pullback f_k^*D is just kD, multiplication by k. Since $\mathcal{E}$ locally is a sum of divisors this shows that the Klyachko filtrations of $\mathcal{F}$ is given by $E^i(j) = F^i(jk)$. Thus the vector spaces in the filtrations are the same, in particular $M(\mathcal{E}) = M(\mathcal{F})$. Also for $v \in M(\mathcal{F})$ we have that the associated divisor $D_v = kE_v$, where E_v is the divisor associated to $v \in M(\mathcal{E})$.

Lemma V.10.1. *For any k, l we have that $S^l f_k^*\mathcal{E} \simeq f_k^* S^l\mathcal{E}$.*

Proof. By the above we know the Klyachko filtrations of $f_k^*\mathcal{E}$. We also know the Klyachko filtrations of a symmetric power. Writing out the filtrations for any ray, we see that they are identical. ■

Proposition V.10.2. *For any positive integer k we have that $f_k^*\mathcal{E}$ is globally generated, nef, big or ample if and only if $\mathcal{E}$ is globally generated, nef, big or ample, respectively.*

Proof. For nef (resp. ample) the argument is easy: $\mathcal{E}$ is nef (resp. ample) if and only if for each T-invariant curve C, $\mathcal{E}|_C$ is nef (resp. ample). Now $\mathcal{E}|_C$ is a sum of line bundles D_i and $f_k^*\mathcal{E}|_C$ is the sum of kD_i. Since D_i is nef (ample) if and only if kD_i is nef (resp. ample), the result follows.

The statement on bigness follows directly from Theorem V.7.5. Since the polytopes for $f_k^*\mathcal{E}$ are simply k times the polytopes for $\mathcal{E}$ we see that a polytope in the parliament of one of the bundles being full-dimensional is equivalent to the corresponding polytope for the other bundle being full-dimensional.

For global generation we use the criterion in Theorem V.8.4. $\mathcal{E}$ is globally generated if and only if for any maximal cone σ, with $\mathcal{E}|_{U_\sigma} \simeq \oplus_{i=1}^r \mathcal{O}(\mathrm{div}(u_i))$ there exists a basis $v_1, \ldots, v_r \in M(\mathcal{E})$ such that $u_i \in P_{v_i}$. Similarly $f_k^*\mathcal{E}$ is globally generated if and only if there exists a basis $w_1, \ldots, w_r \in M(f_k^*\mathcal{E})$ such that $ku_i \in P_{w_i}$. Now since $D_{w_i} = kD_{v_i}$, $P_{w_i} = kP_{v_i}$ and $M(\mathcal{E}) = M(f_k^*\mathcal{E})$ we see that the two criteria above imply each other.

■

Remark V.10.3. The end of the above proof shows that the corresponding equivalence can fail for some other notions of positivity: in particular [DJS18, Theorem 6.2] implies that if $\mathcal{E}$ separates 1-jets then $f_k^*\mathcal{E}$ separates k-jets, but clearly the converse is not true. This is because separating k-jets correspond to certain edges having lattice length at least k. Thus any toric vector bundle $\mathcal{E}$ where one of the relevant edges has length 1 does not separate k-jets for any $k \geq 2$, however $f_k^*\mathcal{E}$ will separate k-jets.

Example V.10.4. Fix an ample toric vector bundle $\mathcal{E}$ on X_Σ which is not globally generated or very ample, for instance the examples in Section V.9. Then $\mathcal{E}|_C$ is a vector bundle $\sum \mathcal{O}(a_i)$ on $C \simeq \mathbb{P}^1$ for any invariant curve $C \subset X_\Sigma$ with $a_i > 0$. Now $\mathcal{F} = f_k^*\mathcal{E}$ will satisfy $\mathcal{F}|_C = \sum \mathcal{O}(ka_i)$. In particular, the restriction to any invariant curve is greater than or equal to k, which can be picked arbitrarily large. However by Theorem V.10.2, $\mathcal{F}$ is still not globally generated or very ample. Thus there cannot in general exist any bound depending on dimension, or even the fan, guaranteeing that if each summand in $\mathcal{E}|_C$ is more positive than this bound, then $\mathcal{E}$ is globally generated or very ample. For line bundles this exists, depending only on dimension, by the statements implying Fujita's conjecture [Pay06].

The above examples suggests that the multiplication maps might be less useful for studying toric vector bundles, compared to their usefulness for line bundles: Many of the vanishing theorems in toric geometry follow from the fact that for a line bundle one obtains injections $H^i(X_\Sigma, \mathcal{O}(D)) \subset H^i(X_\Sigma, \mathcal{O}(f_k^*D)) = H^i(X_\Sigma, \mathcal{O}(kD))$ [Fuj07]. If D is ample then f_k^*D is very ample for large k. For toric vector bundles, the analogous statement does not hold, thus it is not clear if one can use this technique to get vanishing theorems for positive toric vector bundles.

Example V.10.5. Let $\mathcal{E} = T_{\mathbb{P}^n}(-1)$ and let $\mathcal{F} = f_{n+1}^*\mathcal{E}$. From the Euler sequence one obtains the two exact sequences

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to \mathcal{E} \to 0,$$

$$0 \to \mathcal{O}(-n-1) \to \mathcal{O}^{n+1} \to \mathcal{F} \to 0.$$

The second sequence is obtained by pulling back the Euler exact sequence along f_{n+1}^*; pullback is exact for vector bundles. We see that all higher cohomology of $\mathcal{E}$ vanishes, however the same is not the case for $\mathcal{F} = f_{n+1}^*\mathcal{E}$.

Letting $\mathcal{G} = (f_{n+1}^* T_{\mathbb{P}^n})(-n)$, we have the exact sequence

$$0 \to \mathcal{O}(-n) \to \mathcal{O}(1)^{n+1} \to \mathcal{G} \to 0.$$

In particular $\mathcal{G}$ is very ample and $H^i(\mathbb{P}^n, \mathcal{G}) = 0$ for $i > 0$. However, the higher cohomology of $f_k^*\mathcal{G}$ is nonzero for $k \geq 2$. We note that we also have that $f_k^*\mathcal{G}$, $k \geq 2$ is a very ample toric vector bundle such that all polytopes in the parliament are very ample, but with non-vanishing higher cohomology.

Remark V.10.6. Let $\mathcal{E}$ be a toric vector bundle and let $\mathcal{F} = f_k^*\mathcal{E}$. Then, since restricted to U_σ the bundle splits as a direct sum, we see that $c_i(\mathcal{F}) = k^i c_i(\mathcal{E})$. This implies that for any weighted degree n polynomial P in the Chern classes, where c_i has weight i, we have that $P(c_1(\mathcal{F}), c_2(\mathcal{F}), \ldots, c_r(\mathcal{F})) = k^n P(c_1(\mathcal{E}), c_2(\mathcal{E}), \ldots, c_r(\mathcal{E}))$. Then Example V.10.5 implies that there cannot exist any homogeneous polynomial in the Chern classes of $\mathcal{E}$ such that if this polynomial is larger than some constant, all higher cohomology of very ample bundles vanishes.

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