APPROXIMATION OF FORWARD CURVE MODELS IN COMMODITY MARKETS WITH ARBITRAGE-FREE FINITE DIMENSIONAL MODELS

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ABSTRACT. In this paper we show how to approximate a Heath-Jarrow-Morton dynamics for the forward prices in commodity markets with arbitrage-free models which have a finite dimensional state space. Moreover, we recover a closed form representation of the forward price dynamics in the approximation models and derive the rate of convergence uniformly over an interval of time to maturity to the true dynamics under certain additional smoothness conditions. In the Markovian case we can strengthen the convergence to be uniform over time as well. Our results are based on the construction of a convenient Riesz basis on the state space of the term structure dynamics.

1. INTRODUCTION

We develop arbitrage-free approximations to the forward term structure dynamics in commodity markets. The approximating term structure models have finite dimensional state space, and therefore tractable for further analysis and numerical simulation. We provide results on the convergence of the approximating term structures and characterize the speed under reasonable smoothness properties of the true term structure. Our results are based on the construction of a convenient Riesz basis on the state space of the term structure dynamics.

In the context of fixed-income markets, Heath, Jarrow and Morton [22] propose to model the entire term structure of interest rates. Filipović [19] reinterprets this approach in the so-called Musiela parametrisation, i.e., studying the so-called forward rates as solutions of first-order stochastic partial differential equations. This class of stochastic partial differential equations is often referred to as the Heath-Jarrow-Morton-Musiela (HJMM) dynamics. This highly successful method has been transferred to other markets, including commodity and energy futures markets (see Clewlow and Strickland [14] and Benth, Šaltytė Benth and Koekebakker [7]), where the term structure of forward and futures prices are modelled by similar stochastic partial differential equations.

An important stream of research in interest rate modelling has been so-called finite dimensional realizations of the solutions of the HJMM dynamics (see e.g., Björk and Svensson [11], Björk and Landen [10], Filipovic and Teichmann [21] and Tappe [33]). Starting out with an equation for the forward rates driven by a d-dimensional Wiener process, the

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question has been under what conditions on the volatility and drift do we get solutions which belong to a finite dimensional space, that is, when can the dynamics of the whole curve be decomposed into a finite number of factors. This property has a close connection with principal component analysis (see Carmona and Tehranchi [12, Ch. 1]), but is also convenient when it comes to further analysis like estimation, simulation, pricing and portfolio management (see Benth and Lempa [6] for the latter).

In energy markets like power and gas, there is empirical and economical evidence for high-dimensional noise. Moreover, the noise shows clear leptokurtic signs (see Benth, Šaltytė Benth and Koekebakker [7, Ch. 8] and references therein). These empirical insights motivate the use of infinite dimensional Lévy processes driving the noise in the HJMM-dynamics modelling the forward term structure. We refer to Carmona and Tehranchi [12] for a thorough analysis of HJMM-models with infinite dimensional Gaussian noise in interest rate markets. Benth and Krühner [4] introduced a convenient class of infinite dimensional Lévy processes via subordination of Gaussian processes in infinite dimensions. These models were used in analysing stochastic partial differential equations with infinite dimensional Lévy noise in Benth and Krühner [3]. Further, pricing and hedging of derivatives in energy markets based on such models were studied in Benth and Krühner [5].

The present paper is motivated by the need of an arbitrage-free approximation of Heath, Jarrow, Morton style models – using the Musiela parametrisation – in electricity finance. Related research has been carried out by Henseler, Peters and Seydel [24] who construct a finite-dimensional affine model where a refined principle component analysis (PCA) method yields an arbitrage-free approximation of the term structure model. Arbitrage-free approximating models are desirable since they allow for the use of the arbitrage theory to price and hedge options, say, by applying the approximating model instead of the original model. This would come at the cost of a (hopefully) small approximation error, without incurring arbitrage in the analysis.

For the approximation procedure proposed by us, we ask for the following:

(i) A given (arbitrage-free) model f with values in a suitable curve space H is approximated by a sequence $\{f_n\}_{n\in\mathbb{N}}$ of stochastic models, i.e. $f_n \to f$ in a suitable way.
(ii) f_n should have a finite dimensional state space, i.e. there is finite dimensional space H_n such that $f_n(t) \in H_n$.
(iii) f_n itself is asked to be an arbitrage-free HJM-type model.
(iv) Finally, the dynamics of f_n should have a structure which is as simple as possible.

If we think of models $\{f_n\}_{n\in\mathbb{N}}$ satisfying (ii) and (iii) and being a solution to a stochastic partial differential equation (SPDE)

$$df_n(t) = (\mu_P(t))dt + \sigma(t)dW_n(t)$$

where W_n is an H_n-valued Brownian motion and μ_P, σ are suitable coefficients under some probability measure P, then, the no-arbitrage condition yields that there is an equivalent measure $Q \sim P$ such that

$$df_n(t) = \partial_x f_n(t)dt + \sigma(t)dW_n^Q(t)$$

for some Q_n-Brownian motion W_n^Q. Thus, f_n is a finite dimensional realisation (FDR) which have been discussed in Filipovic [18], Björk [9] and Filipovic and Teichmann [21]. For those, the possible state spaces are rather limited imposing strong conditions on the volatility σ. This restricts the possibilities of approximations in (i) (a more detailed discussion is provided in Section 3). To overcome this problem we adapt a specific Galerkin method which is tailored to the specific Hilbert space in our setup as well as being an FDR, cf. Section 4.

Our main result Theorem 5.1 states that the arbitrage-free models for the underlying forward curve process $f(t,x)$, $x \geq 0$ being time to maturity and $t \geq 0$ is current time, can be approximated with processes of the form

$$f_k(t,x) = S_k(t) + \sum_{n=-k}^{k} U_n(t) g_n(x)\,,$$

where S_k denotes the spot prices in the approximation model, $g_{-k},\dots,g_k$ are deterministic functions and $U_{-k},\dots,U_k$ are one-dimensional Ornstein-Uhlenbeck type processes. Obviously, models of this type are much easier to handle in applications than general solutions for the HJMM equation. The approximation f_k is again a solution of an HJMM equation, and as such being an arbitrage-free model for the forward term structure. We prove a uniform convergence in space of f_k to the "real" forward price curve f, pointwise in time. The convergence rate is of order $k^{-1/2}$ when the forward curve $x \mapsto f(t,x)$ is twice continuously differentiable. Our approach is an alternative to numerical approximations of the HJMM dynamics based on finite difference schemes or finite element methods, where arbitrage-freeness of the approximating dynamics is not automatically ensured. We refer to Barth [1] for an analysis of finite element methods applied to stochastic partial differential equations of the type we study.

We refine our results to the Markovian case, where the convergence is slightly strengthened to be uniform over time as well. Our approach goes via the explicit construction of a Riesz basis for a subspace of the so-called Filipović space (see Filipović [19]), a separable Hilbert space of absolutely continuous functions on the positive real line with (weak) derivative disappearing at a certain speed at infinity. The basis will be the functions g_n in the approximation f_k, and the subspace is defined by concentrating the functions in the Filipović space to a finite time horizon $x \leq T$. This space was defined in Benth and Krühner [3], and we extend the analysis here to accomodate the arbitrage-free finite dimensional approximation of the HJMM-dynamics. We rest on properties of C_0-semigroups and stochastic integration with respect to infinite dimensional Lévy processes (see Peszat and Zabczyk [28]) in the analysis.

This paper is organised as follows. In Section 2 we start with the mathematical formulation of the HJMM dynamics for forward rates set in the Filipović space. The following section provides a motivation for our paper by discussing in more detail the problem of arbitrage-free approximations. The Riesz basis that will make the foundation for our proposed approximation scheme is defined and analysed in detail in Section 4. The arbitrage-free finite dimensional approximation to term structure modelling is constructed in Section 5, where we study convergence properties. The Markovian case is analysed in the last Section 6.

2. THE MODEL OF THE FORWARD PRICE DYNAMICS

By $\mathbb{N}$ we denote the set of non-negative integers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. Throughout this paper we use the Hilbert space

$$H_\alpha := \left\{ f \in AC(\mathbb{R}_+, \mathbb{C}) : \int_0^\infty |f'(x)|^2 e^{\alpha x} dx < \infty \right\},$$

where $AC(\mathbb{R}_+, \mathbb{C})$ denotes the space of complex-valued absolutely continuous functions on $\mathbb{R}_+$ and $\alpha > 0$ is some fixed parameter. We endow H_α with the scalar product $\langle f, g\rangle_\alpha := f(0)\overline{g}(0) + \int_0^\infty f'(x)\overline{g}'(x)e^{\alpha x}dx$, and denote the associated norm by $\|\cdot\|_\alpha$. Filipović [19, Section 5] shows that $(H_\alpha, \|\cdot\|_\alpha)$ is a separable Hilbert space. For $x \geq 0$, the evaluation map $\delta_x : H_\alpha \to \mathbb{C}$ defined by $\delta_x(f) = f(x)$ for $f \in H_\alpha$ is a continuous linear functional, cf. [19, Theorem 5.1]. The space H_α has been used in Filipović [19] for term structure modelling of bonds and many mathematical properties have been derived therein. We will frequently refer to H_α as the *Filipović space*. Note that Filipović [19] considers real-valued functions instead. In our context, this minor extension is convenient, as will become clear later.

We next introduce our dynamics for the term structure of forward prices in a commodity market. Denote by $f(t, x)$ the price at time t of a forward contract where time to delivery of the underlying commodity is $x \geq 0$. We treat f as a stochastic process in time with values in the Filipović space H_α. More specifically, we assume that the process $\{f(t)\}_{t\geq 0}$ follows the HJM-Musiela model which we formalize next.

On a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$, where the filtration is assumed to be complete and right continuous, we work with an H_α-valued Lévy process $\{L(t)\}_{t\geq 0}$ (cf. Peszat and Zabczyk [28, Theorem 4.27(i)] for the construction of H_α-valued Lévy processes). We assume that L has finite variance and mean equal to zero and we denote its covariance operator by $\mathcal{Q}$, cf. [28, Definition 4.45]. For $t \geq 0$, denote by $\mathcal{U}_t$ the shift semigroup on H_α defined by $\mathcal{U}_t f = f(t + \cdot)$ for $f \in H_\alpha$. It is shown in Filipović [19, Thm. 5.1.1] that $\{\mathcal{U}_t\}_{t\geq 0}$ is a C_0-semigroup on H_α, with generator ∂_x given by $\partial_x f = f'$ for any $f \in H_\alpha$ which is continuously differentiable with derivative f' in H_α. Let $f_0 \in H_\alpha$ and f be the solution of the SPDE

$$df(t) = \partial_x f(t)dt + \beta(t)dt + \Psi(t)dL(t), \quad t \geq 0, f(0) = f_0 \tag{1}$$

where $\beta \in L^1((\Omega \times \mathbb{R}_+, \mathcal{P}, P \otimes \lambda), H_\alpha)$, $\mathcal{P}$ denotes the predictable σ-field and we have $\Psi \in \mathcal{L}^2_L(H_\alpha) := \bigcup_{T>0} \mathcal{L}^2_{L,T}(H_\alpha)$ where the spaces $\mathcal{L}^2_{L,T}(H_\alpha)$ are defined in Peszat and Zabczyk [28, page 113]. Recall, that any C_0-semigroup admits the bound $\|\mathcal{U}_t\|_{\text{op}} \leq Me^{wt}$ for some $w, M > 0$ and any $t \geq 0$. Here, $\|\cdot\|_{\text{op}}$ denotes the operator norm. In fact, in Filipović [19, Equation (5.10)] and Benth and Krühner [5, Lemma 3.4] it is shown that $\|\mathcal{U}_t\|_{\text{op}} \leq C_{\mathcal{U}}$ for any $t \geq 0$ and the constant $C_{\mathcal{U}} := \sqrt{2(1 \wedge \alpha^{-1})}$. Thus $s \mapsto \mathcal{U}_{t-s}\beta(s)$ is Bochner-integrable and $s \mapsto \mathcal{U}_{t-s}\Psi(s)$ is integrable with respect to L. The unique mild solution of (1) is

$$f(t) = \mathcal{U}_t f_0 + \int_0^t \mathcal{U}_{t-s}\beta(s)\, ds + \int_0^t \mathcal{U}_{t-s}\Psi(s)\, dL(s), \tag{2}$$

which has a càdlàg version according to Tappe [34, Theorem 4.5, Remark 4.6]. We will always refer to a càdlàg version of f.

If we model the forward price dynamics f directly in a risk-neutral setting, the drift coefficient $\beta(t)$ must be equal to zero in order to ensure the (local) martingale property of the process $t \mapsto f(t, \tau - t)$, where $\tau \geq t$ is the time of delivery of the forward. In this case, the probability P is to be interpreted as the equivalent martingale measure (also called the pricing measure). However, with a non-zero drift, the forward model is stated under the market probability and β can be related to the risk premium in the market.

We remark in passing that in energy markets like power and gas, the forward contracts deliver over a period, and forward prices can be expressed by integral operators on the Filipović space applied on f (see Benth and Krühner [3, 5] for more details).

The dynamics of f can be considered as a model for the forward rate in fixed-income theory, see Filipović [19]. This is indeed the traditional application area and point of analysis of the SPDE in (1). Note, however, that the no-arbitrage condition in the HJM approach for interest rate markets is different from and more complex than the condition we use here in the commodity market context. If f is understood as the forward rate modelled in the risk-neutral setting, there is a nonlinear relationship between the drift β, the volatility σ and the covariance of the driving noise L. We refer to Carmona and Tehranchi [12] for a detailed analysis.

3. The problem of arbitrage-free approximation.

In this section we provide some motivation and background for the problem we are going to address in this article. To make the problem we study more precise, we start out with a model for the futures curve dynamics set in H_α under the Musiela parametrisation. Considering a sequence of approximation models restricted to have a *finite dimensional* state space, we identify certain conditions that must be fulfilled and discuss these in view of existing numerical methods and the approach proposed in this paper.

To this end, let f be given as in (2) and assume for simplicity that $L = W$ is a Wiener process and β, Ψ are bounded càdlàg processes. Furthermore, we assume that $\{f_n\}_{n\in\mathbb{N}}$ are H_α-valued processes such that $f_n(t) \in H_{n,\alpha}$ P-a.s. for all $t \geq 0$, for finite dimensional complex subspaces $H_{n,\alpha} \subseteq H_\alpha$. Also, we assume that $H_{n,\alpha}$ are chosen minimal in the sense that for any proper subspace $G \subseteq H_{n,\alpha}$ there is $t \geq 0$ such that $P(f_n(t) \notin G) > 0$. Note that the traded assets in the n-th approximation are forward contracts with forward prices $F_n(t,\tau) := f_n(t, \tau - t), 0 \leq t \leq \tau$, which we suppose to be arbitrage-free in the sense of "NAFLVR" as defined by Cuchiero *et al.* [13] and we assume to be of the form $F_n(t,\tau) = F_n(0,\tau) + B_{n,\tau}(t) + \int_0^t \Sigma_n(s)dW_n(s)$ for some Wiener process W_n, bounded càdlàg integrand Σ_n and some adapted finite variation process $B_{n,\tau}$. Then, Cuchiero *et al.* [13, Theorem 1.1] yields the existence of a probability measure $Q_n \sim P$ such that the price processes

$$F_n(t,\tau) = \mathcal{U}_{\tau-t} f_n(t), \quad 0 \leq t \leq \tau,$$

are local Q_n-martingales. In particular we have

$$df_n(t) = \partial_x f_n(t)dt + \Sigma_n(t)dW_n^Q(t), \quad t \geq 0,$$

for some and a Q_n-Wiener process W_n^Q.

Remark 3.1. Galerkin methods generate dynamics f_n such that $f_n \to f$ in a suitable way and such that the spaces $H_{n,\alpha}$ are finite dimensional. For the use of Galerkin methods to

SPDEs, we refer to Greksch and Kloeden [23] and the books by da Prato and Röckner [30] and Kruse [26] (as well as references therein). The finite element method also satisfies the finite dimensional state space requirement (we refer to Barth [1] for the finite element method applied to SPDEs). However, methods based on finite difference approximations directly discretise in space and time, and the approximation is not an H_α-valued process anymore unless one applies a suitable interpolation in space.

We recall the following important result of Filipović [18, Theorem 4].

Proposition 3.2. *The vector space $H_{n,\alpha}$ is invariant under ∂_x.*

This key insight leads immediately to a restrictive structural condition on the space $H_{n,\alpha}$.

Corollary 3.3. *For given $n \in \mathbb{N}$, denote by $d \in \mathbb{N}$ the dimension of $H_{n,\alpha}$. Then there are constants $a_1, \dots, a_d \in \mathbb{C}$ and polynomials $p_1, \dots, p_d : \mathbb{C} \to \mathbb{C}$ such that*

$$\{x \mapsto p_j(x)e^{a_j x}\}_{j=1,\dots,d}$$

is a vector space basis of $H_{n,\alpha}$.

Proof. Let $g_1, \dots, g_d$ be a vector space basis for $H_{n,\alpha}$. Proposition 3.2 implies that we have $g_1', \dots, g_d' \in H_{n,\alpha}$ and hence there is $C \in \mathbb{C}^{d\times d}$ such that

$$g' = Cg.$$

Choose $D \in \mathbb{C}^{d\times d}$ such that $\widetilde{C} := DCD^{-1}$ is in Jordan normal form. Then

$$(Dg)' = \widetilde{C}(Dg).$$

The claim follows trivially for the basis $h_j := (Dg)_j$, $j = 1, \dots, d$. □

The following example illustrates Corollary 3.3 in view of the Galerkin approximation method:

Example 3.4. Let $e_*(x) := 1$ and

$$e_{n,k}(x) := \begin{cases} 0 & x < n, \\ \frac{e^{(2\pi ik-\alpha/2)x} - e^{-n\alpha/2}}{2\pi ik-\alpha/2} & x \in [n, n+1], \\ \frac{e^{-(n+1)\alpha/2} - e^{-n\alpha/2}}{2\pi ik-\alpha/2} & x > n+1, \end{cases}$$

for any $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Clearly, we have

$$e_{n,k}'(x) = 1_{x\in[n,n+1]}e^{(2\pi k-\alpha/2)x}, \quad x \geq 0$$

for any $n \in \mathbb{N}$, $k \in \mathbb{Z}$, and $\{e_*, \{e_{n,k}\}_{n\in\mathbb{N},k\in\mathbb{Z}}\}$ is an orthonormal basis on H_α which is local in the following sense: if $h_1, h_2 \in H_\alpha$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $h_1(x) = h_2(x)$ for $x \in [n, n+1]$, then $\langle h_1, e_{n,k}\rangle_\alpha = \langle h_2, e_{n,k}\rangle_\alpha$. One could use as an approximation for f the orthonormal expansion relative to any finite enumeration of $\{e_*, \{e_{n,k}\}_{n\in\mathbb{N},k\in\mathbb{Z}}\}$, which is a local Galerkin method. However, the only finite dimensional spaces generated by a finite selection of the functions $e_{n,k}$ as described in Corollary 3.3 are $\mathrm{span}\{e_*\}$ and $\{0\}$. Thus the approximating models cannot be arbitrage-free (unless $x \mapsto f(t,x)$ is constant for any $t \geq 0$.).

We understand from Corollary 3.3 that the subspaces $H_{n,\alpha}$ have to be spanned by curves which can be expressed as polynomial times exponential functions. Two special cases are immediately apparent: either to select subspaces $H_{n,\alpha}$ spanned by polynomial functions, or select subspaces $H_{n,\alpha}$ spanned by exponential functions. Since all polynomials $p \in H_\alpha$ are constants, it is obvious that H_α is unsuitable for approximation with polynomial functions.

Therefore, we will focus on approximations based on exponential basis functions. We believe that the case where the noise term W has a positive definite covariance matrix and where one uses a Galerkin method projecting to finite dimensional subspaces generated by exponential functions does lead to arbitrage-free approximations in most situations. Indeed, in the next section we will identify a Riesz basis consisting of simple and explicit exponential functions for a 'rich' subspace of H_α, cf. Theorem 4.4 below. This Riesz basis is then used for a basis expansion for the coefficients which appear in the SPDE (1). However, unlike the Galerkin approach, we will not discretise the differential operator ∂_x. We emphasise that if the differential operator is discretised, then option prices in the approximation models have to be calculated under an equivalent local martingale measure Q_n depending on n, and the convergence rate of option prices becomes non-obvious (see e.g. Mishura and Munchak [27] and references therein). Therefore, it is additionally desirable that we can use the same pricing measure Q for the initial model f and all the approximation models f_n.

Finally, we like to highlight that our approximations are in fact FDRs of the SPDE with the projected coefficients, and as such our method combines a Galerkin type approximation with FDR. Moreover, if the martingale part $\Psi(t)dL(t)$ is a Lévy process, then our approximating models are affine in the sense of Duffie *et al.* [17], cf. Theorem 5.1.

4. A Riesz basis for the Filipović space

In Section 5 we want to employ the spectral method to an approximation of the SPDE in (1) involving the differential operator on the Filipović space H_α. Thus, it would be convenient to have available the eigenvector basis for the differential operator. However, its eigenvectors do not seem to have nice basis properties, and instead we propose to use a system of vectors which forms a Riesz basis. It turns out that this basis has neat analytical properties and is close to form an eigenvector system for the differential operator.

In this section we introduce such a Riesz basis for a suitable subspace of H_α defined in Benth and Krühner [3, Appendix A] and recall some of its properties. Moreover, we give refined statements for this basis and also identify new results. In particular, we make precise the connection between our suggested Riesz basis and the differential operator, as well as quantifying the convergence speed of the basis expansion. We recall from Young [35, Sect. 1, Theorem 9] that any Riesz basis $\{g_n\}_{n\in\mathbb{N}}$ on a separable Hilbert space can be expressed by $g_n = \mathcal{T}e_n$ where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis and $\mathcal{T}$ is a bounded invertible linear operator. For further properties and definitions of Riesz bases, see Young [35].

Fix $\lambda > 0$, $T > 0$, and introduce

$$\mathrm{cut} : \mathbb{R}_+ \to [0, T)\,, \qquad x \mapsto x - \max\{Tz : z \in \mathbb{Z} : Tz \leq x\}\,, \tag{3}$$

and

$$\mathcal{A} : L^2([0,T),\mathbb{C}) \to L^2(\mathbb{R}_+,\mathbb{C})\,, \qquad f \mapsto \left(x \mapsto e^{-\lambda x} f(\mathrm{cut}(x))\right)\,. \tag{4}$$

Here, $L^2(A,\mathbb{C})$ is the space of complex-valued square integrable functions on the Borel set $A \subset \mathbb{R}_+$ equipped with the Lebesgue measure. The inner product of $L^2(A,\mathbb{C})$ will be denoted $(\cdot,\cdot)_2$ and the corresponding norm $|\cdot|_2$. We remark that the set A will be clear from the context and thus not indicated in the notation for norm and inner product.

We define

$$g_*(x) := 1, \tag{5}$$

$$g_n(x) := \frac{1}{\lambda_n \sqrt{T}} \left(\exp\left(\lambda_n x\right) - 1\right)\,, \tag{6}$$

where

$$\lambda_n := \frac{2\pi i}{T} n - \lambda - \frac{\alpha}{2}\,, \tag{7}$$

for any $n \in \mathbb{Z}$, $x \geq 0$. It is simple to verify that $g_n \in H_\alpha$ for any $n \in \mathbb{Z}$ and $g_* \in H_\alpha$. As we will see, the system of vectors $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ forms a Riesz basis and we will use this basis expansion to obtain arbitrage-free finite-dimensional approximations of the forward price dynamics (1). The remainder of this Section is devoted to the study of the system of vectors $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$.

We start our analysis with some elementary properties of the operator $\mathcal{A}$ defined in (4) which have been proven in Benth and Krühner [3].

Lemma 4.1. *$\mathcal{A}$ is a bounded linear operator and its range is closed in $L^2(\mathbb{R}_+,\mathbb{C})$. Moreover,*

$$\frac{e^{-2T\lambda}}{1-e^{-2T\lambda}}|f|_2^2 \leq |\mathcal{A}f|_2^2 \leq \frac{1}{1-e^{-2T\lambda}}|f|_2^2$$

for any $f \in L^2([0,T),\mathbb{C})$.

Proof. This proof can be found in Benth and Krühner [3, Lemma A.1]. □

In the following Proposition 4.3, we calculate a Riesz basis of the space $\mathrm{ran}(\mathcal{A})$ and its biorthogonal system. The Riesz basis will be given as the image of an orthonormal basis of $L^2([0,T),\mathbb{C})$. Consequently, its biorthogonal system is given by the image of $(\mathcal{A}^{-1})^*$, which we calculate in the Lemma below:

Lemma 4.2. *The dual $(\mathcal{A}^{-1})^*$ of the inverse of $\mathcal{A} : L^2([0,T),\mathbb{C}) \to \mathrm{ran}(\mathcal{A})$ is given by*

$$\begin{aligned}
(\mathcal{A}^{-1})^* &: L^2([0,T),\mathbb{C}) \to \mathrm{ran}(\mathcal{A}),\\
(\mathcal{A}^{-1})^* f(x) &= (1-e^{-2\lambda T})e^{-\lambda x}\left(e^{2\lambda \mathrm{cut}(x)} f(\mathrm{cut}(x))\right)\\
&= (1-e^{-2\lambda T})e^{2\lambda \mathrm{cut}(x)} \mathcal{A}f(x), \quad x \geq 0\,.
\end{aligned}$$

Proof. Let $f, g \in L^2([0,T],\mathbb{C})$ and define $h(x) := (1-e^{-2\lambda T})e^{2\lambda \mathrm{cut}(x)}\mathcal{A}f(x)$ for any $x \geq 0$. Then we have

$$\begin{aligned}
(h, \mathcal{A}g)_2 &= \int_0^\infty h(y)\overline{\mathcal{A}g(y)}dy \\
&= (1-e^{-2\lambda T})\sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{2\lambda(x-nT)}(e^{-\lambda x}f(x-nT))(e^{-\lambda x}\overline{g(x-nT)})dx \\
&= (1-e^{-2\lambda T})\sum_{n=0}^\infty e^{-2\lambda nT}\int_{nT}^{(n+1)T} f(x-nT)\overline{g(x-nT)}dx \\
&= \int_0^T f(y)\overline{g(y)}dy\,.
\end{aligned}$$

On the other hand,

$$((\mathcal{A}^{-1})^* f, \mathcal{A}g)_2 = (f,g)_2 = \int_0^T f(y)\overline{g(y)}dy\,.$$

Since g is arbitrary, we have $h = (\mathcal{A}^{-1})^* f$ as claimed. □

In the next Proposition we introduce a Riesz basis on the closed subspace $\mathrm{ran}(\mathcal{A})$ of $L^2(\mathbb{R}_+,\mathbb{C})$ and identify its biorthogonal system $\{e_n^*\}_{n\in\mathbb{Z}}$. Linked to this basis is a projector $\mathcal{P}_\mathcal{A}$ which we also introduce and provide some properties of. We remark that parts of the next proposition can be found in Benth and Krühner [3, Lemma 4.22].

Proposition 4.3. *Define*

$$e_n(x) := \frac{1}{\sqrt{T}}\exp\left(\left(\frac{2\pi i n}{T}-\lambda\right)x\right), \quad x \geq 0, n\in\mathbb{Z}.$$

Then $\{e_n\}_{n\in\mathbb{Z}}$ *is a Riesz basis on the closed subspace* $\mathrm{ran}(\mathcal{A})$ *of* $L^2(\mathbb{R}_+,\mathbb{C})$ *and*

$$F := \{f \in L^2(\mathbb{R}_+,\mathbb{C}) : f(x) = 0, x\in[0,T)\}$$

is a closed vector space compliment of $\mathrm{ran}(\mathcal{A})$. *The continuous linear projector* $\mathcal{P}_\mathcal{A}$ *with range* $\mathrm{ran}(\mathcal{A})$ *and kernel* F *has operator norm* $\sqrt{\frac{1}{1-e^{-2\lambda T}}}$ *and we have*

$$\mathcal{P}_\mathcal{A} f(x) = f(x), \quad x\in[0,T], f\in L^2(\mathbb{R}_+,\mathbb{C}).$$

The biorthogonal system $\{e_n\}_{n\in\mathbb{Z}}^*$ *for the Riesz basis* $\{e_n\}_{n\in\mathbb{Z}}$ *is given by*

$$e_n^*(x) = \left(1-e^{-2\lambda T}\right)e^{2\lambda\mathrm{cut}(x)}e_n(x), \quad x\geq 0.$$

Proof. Recall that the range of $\mathcal{A}$ is a closed subspace of $L^2(\mathbb{R}_+,\mathbb{C})$ due to Lemma 4.1. Furthermore, $\{b_n\}_{n\in\mathbb{Z}}$ with

$$b_n(x) := \frac{1}{\sqrt{T}}\exp\left(\frac{2\pi i n}{T}x\right), \quad n\in\mathbb{Z}, x\in[0,T)$$

is an orthonormal basis of $L^2([0,T),\mathbb{C})$. Observe, that $e_n = \mathcal{A}b_n$ and hence $\{e_n\}_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathrm{ran}(\mathcal{A})$.

Define the continuous linear operators

$$\mathcal{M}_\lambda : L^2([0,T),\mathbb{C}) \to L^2([0,T),\mathbb{C}), \mathcal{M}_\lambda f(x) := e^{\lambda x} f(x),$$
$$\mathcal{C} : L^2(\mathbb{R}_+,\mathbb{C}) \to L^2([0,T),\mathbb{C}), f \mapsto f|_{[0,T)}$$

and $\mathcal{P}_\mathcal{A} := \mathcal{A}\mathcal{M}_\lambda\mathcal{C}$. Observe, that $\mathcal{M}_\lambda\mathcal{C}\mathcal{A}$ is the identity operator on $L^2([0,T),\mathbb{C})$ and hence $\mathcal{P}_\mathcal{A}^2 = \mathcal{P}_\mathcal{A}$. Therefore, $\mathcal{P}_\mathcal{A}$ is a continuous linear projection with kernel F and range $\operatorname{ran}(\mathcal{A})$.

Let $f \in L^2(\mathbb{R}_+,\mathbb{C})$ be orthogonal to any element of the kernel of $\mathcal{P}_\mathcal{A}$. Then $f(x) = 0$ Lebesgue-$a.e.$ for any $x \geq T$. Hence, we have

$$\begin{aligned}
|\mathcal{P}_\mathcal{A} f|_2^2 &= \sum_{n\in\mathbb{N}} \int_{nT}^{nT+T} (e^{-\lambda x} e^{\lambda(x-nT)})^2 |f(x-nT)|^2 dx \\
&= \sum_{n\in\mathbb{N}} e^{-2n\lambda T} |f|_2^2 \\
&= \frac{1}{1-e^{-2\lambda T}} |f|_2^2
\end{aligned}$$

and it follows that $\|\mathcal{P}_\mathcal{A}\|_{\text{op}} = \sqrt{\frac{1}{1-e^{-2\lambda T}}}$.

According to Lemma 4.2, we have

$$\begin{aligned}
e_n^*(x) &= (\mathcal{A}^{-1})^* b_n(x) \\
&= (1-e^{-2\lambda T}) e^{-\lambda x} \left(e^{2\lambda \operatorname{cut}(x)} b_n(\operatorname{cut}(x))\right) \\
&= \left(1-e^{-2\lambda T}\right) e^{2\lambda \operatorname{cut}(x)} e_n(x)\,,
\end{aligned}$$

for any $n \in \mathbb{Z}$, $x \geq 0$, as required. □

The statements collected in this section have so far been about the space $L^2(\mathbb{R}_+,\mathbb{C})$. However, our main interest is the space H_α, which has a natural and simple isometry to $\mathbb{C} \times L^2(\mathbb{R}_+,\mathbb{C})$. In the next theorem we translate the $L^2(\mathbb{R}_+,\mathbb{C})$-statements above to H_α, and thus concluding the first part of this Section. But before stating the theorem, we introduce an operator which will turn out to be convenient here and in the sequel: Define

$$\Theta : H_\alpha \to \mathbb{C} \times L^2(\mathbb{R}_+,\mathbb{C}), f \mapsto (f(0), w_\alpha f')\,, \tag{8}$$

where $w_\alpha(x) := e^{x\alpha/2}$ for $x \geq 0$. Then Θ is an isometry of Hilbert spaces with the inverse given by

$$\Theta^{-1} : \mathbb{C} \times L^2(\mathbb{R}_+,\mathbb{C}) \to H_\alpha, (z,f) \mapsto z + \int_0^{(\cdot)} w_\alpha^{-1}(y) f(y) dy\,. \tag{9}$$

We use the operator Θ and its inverse to prove:

Theorem 4.4. *The system $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ defined in* (5)-(6) *is a Riesz basis of a closed subspace H_α^T of H_α. Indeed, H_α^T is the space generated by $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$. Moreover, there is a continuous linear projector $\Pi : H_\alpha \to H_\alpha$ with range H_α^T and operator norm $\sqrt{\frac{1}{1-e^{-2\lambda T}}}$ such that*

$$\Pi h(x) = h(x), \quad h \in H_\alpha, x \in [0,T].$$

Consequently, $\Pi\mathcal{U}_t h(x) = \mathcal{U}_t \Pi h(x) = h(x+t)$ for any $t \in [0,T]$ and any $x \in [0,T-t]$.

The biorthogonal system $\{g_*^*, \{g_n^*\}_{n\in\mathbb{Z}}\}$ *is given by*

$$g_*^*(x) = g_*(x) = 1$$
$$g_n^*(x) = \int_0^x e^{-y\frac{\alpha}{2}} e_n^*(y)dy$$

where e_n^* *is given in Proposition 4.3 for any* $n \in \mathbb{Z}$, $x \geq 0$.

Proof. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the Riesz basis from Proposition 4.3, V the linear vector space generated by $\{e_n\}_{n\in\mathbb{Z}}$ (which is in fact $\mathrm{ran}(\mathcal{A})$) and $\mathcal{P}_\mathcal{A}$ the projector from that proposition. Then $\{(1,0), \{(0,e_n)\}_{n\in\mathbb{Z}}\}$ is a Riesz basis of $\mathbb{C}\times V$. Furthermore, $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ is a Riesz basis of $\Theta^{-1}(\mathbb{C}\times V)$ because $g_* = \Theta^{-1}(1,0)$ and $g_n = \Theta^{-1}(0,e_n)$. Define $\Pi := \Theta^{-1}(\mathrm{Id},\mathcal{P}_\mathcal{A})\Theta$. Then Π is a linear projector with the same bound as $\mathcal{P}_\mathcal{A}$ where

$$(\mathrm{Id},\mathcal{P}_\mathcal{A})(z,f) := (z, \mathcal{P}_\mathcal{A} f), \quad z\in\mathbb{C}, f\in L^2(\mathbb{R}_+,\mathbb{C})\,.$$

Let $h \in H_\alpha$ and $x \in [0,T]$. Then $\mathrm{cut}(y) = y$ for any $y \in [0,x]$. We have from the definition of the various operators that

$$\begin{aligned}
\Pi h(x) &= \Theta^{-1}(\mathrm{Id},\mathcal{P}_\mathcal{A})(h(0), \exp(\alpha\cdot/2)h')(x) \\
&= \Theta^{-1}\left((h(0), (\exp((\lambda+\alpha/2)\cdot)h')|_{[0,T)}(\mathrm{cut}(\cdot))\exp(-\lambda\cdot)\right)(x) \\
&= h(0) + \int_0^x e^{-(\lambda+\alpha/2)y} e^{(\lambda+\alpha/2)\mathrm{cut}(y)} h'(\mathrm{cut}(y))\,dy \\
&= h(0) + \int_0^x h'(y)\,dy = h(x)\,.
\end{aligned}$$

Hence, $\Pi h(x) = h(x)$ for any $x\in[0,T]$. □

In the next proposition we compute the action of the shifting semigroup $\{\mathcal{U}_t\}_{t\geq 0}$ on the Riesz basis of Theorem 4.4 and the dual semigroup on the biorthogonal system.

Proposition 4.5. *For the Riesz basis* $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ *in* (5)-(6) *and its biorthogonal system* $\{g_*^*, \{g_n^*\}_{n\in\mathbb{Z}}\}$ *derived in Theorem 4.4, it holds*

(1) $\mathcal{U}_t g_n = e^{\lambda_n t} g_n + g_n(t) g_*$ *and*
(2) $\mathcal{U}_t^* g_n^* = e^{\overline{\lambda_n} t} g_n^*$,

for any $n\in\mathbb{Z}$.

Proof. Claim (1) follows from a straightforward computation. For claim (2), we compute

$$\begin{aligned}
\mathcal{U}_t^* g_n^* &= g_*\langle \mathcal{U}_t^* g_n^*, g_*\rangle_\alpha + \sum_{k\in\mathbb{Z}} g_k^*\langle \mathcal{U}_t^* g_n^*, g_k\rangle_\alpha \\
&= g_*\langle g_n^*, \mathcal{U}_t g_*\rangle_\alpha + \sum_{k\in\mathbb{Z}} g_k^*\langle g_n^*, \mathcal{U}_t g_k\rangle_\alpha \\
&= e^{\overline{\lambda_n} t} g_n^*
\end{aligned}$$

for any $n\in\mathbb{Z}$, $t\geq 0$. Thus, the Proposition follows. □

Proposition 4.5 shows that the system $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ is close to form a set of eigenvectors for the shift operator $\mathcal{U}_t$. On the other hand, the biorthogonal system $\{g_n^*\}_{n\in\mathbb{Z}}$ is a set of eigenvectors for the adjoint operator $\mathcal{U}_t^*$, but $\mathcal{U}_t^* g_* = g_* + \sum_{n\in\mathbb{Z}} g_n(t) g_n^*$. This explicit and simple relationship between the shift operator and the Riesz basis is very attractive in our further analysis.

Let $k \in \mathbb{N}$ and introduce the finite dimensional subspace $H_\alpha^{T,k}$

$$H_\alpha^{T,k} := \text{span}\{g_*, g_{-k}, \dots, g_k\}\,. \tag{10}$$

Here, $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ is the Riesz basis defined in (5)-(6) on the closed subspace H_α^T (recall Theorem 4.4). $H_\alpha^{T,k}$ will be the space where we will study finite dimensional approximations of the SPDE (1). To this end, define the projection operator

$$\Pi_k : H_\alpha^T \to H_\alpha^{T,k}, h \mapsto h(0)g_* + \sum_{n=-k}^{k} g_n \langle h, g_n^* \rangle_\alpha, \tag{11}$$

where the biorthogonal system to our Riesz basis $\{g_*^*, \{g_n^*\}_{n\in\mathbb{Z}}\}$ is given in Theorem 4.4.

Proposition 4.6. *For the operator Π_k defined in* (11), $\|\Pi_k\|_{op}$ *is bounded uniformly in $k \in \mathbb{N}$ and $\Pi_k h \to h$ when $k \to \infty$ for any $h \in H_\alpha^T$.*

Proof. Let $h \in H_\alpha^T$. Since $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ is a Riesz basis of H_α^T we have

$$h = g_* \langle h, g_* \rangle_\alpha + \sum_{n\in\mathbb{Z}} g_n \langle h, g_n^* \rangle_\alpha\,,$$

and hence we get $\Pi_k h \to h$ for $k \to \infty$.

We prove that the operator norm of Π_k is uniformly bounded in $k \in \mathbb{N}$. Recall from Theorem 4.4 and (9) $g_n = \Theta^{-1}(0, \mathcal{A}b_n), n \in \mathbb{Z}$ and $g_* = \Theta^{-1}(1,0)$, where $\mathcal{A}$ is defined in (4) and $\{b_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2([0,T],\mathbb{C})$. Without loss of generality, we assume $h(0) = 0$ for $h \in H_\alpha^T$, and find that

$$\Pi_k h = \sum_{n=-k}^{k} g_n \langle h, g_n^* \rangle_\alpha = \sum_{n=-k}^{k} \mathcal{T} b_n (\mathcal{T}^{-1}h, b_n)_2 = \mathcal{T} \sum_{n=-k}^{k} b_n (\mathcal{T}^{-1}h, b_n)_2\,.$$

Here, $\mathcal{T}f := \Theta^{-1}(0, \mathcal{A}f) \in H_\alpha$ for $f \in L^2([0,T],\mathbb{C})$, which is a bounded linear operator. Hence, since $\sum_{n=-k}^{k} b_n(\mathcal{T}^{-1}h, b_n)_2$ is the projection of $\mathcal{T}^{-1}h \in L^2([0,T],\mathbb{C})$ down to its first $2k+1$ coordinates,

$$\|\Pi_k h\|_\alpha \le \|\mathcal{T}\|_{\text{op}} \left| \sum_{n=-k}^{k} b_n(\mathcal{T}^{-1}h, b_n)_2 \right|_2 \le \|\mathcal{T}\|_{\text{op}} |\mathcal{T}^{-1}h|_2$$

But since $\mathcal{T}^{-1}$ also is a bounded operator, it follows that $\|\Pi_k\|_{\text{op}} \le \|\mathcal{T}\|_{\text{op}} \|\mathcal{T}^{-1}\|_{\text{op}}$. $\square$

In the analysis of approximative solutions of SPDE (1) in the space $H_\alpha^{T,k}$, the Lie commutator $[\Pi_k, \mathcal{U}_t]$ plays a crucial role. We recall that $[\Pi_k, \mathcal{U}_t] = \Pi_k \mathcal{U}_t - \mathcal{U}_t \Pi_k$. In the next proposition, we derive an explicit formula for the Lie commutator, as well as showing an essential convergence result that will be applied in Section 5 in the analysis of approximations of the SPDE (1).

Proposition 4.7. *Let $k \in \mathbb{N}$ and $t \ge 0$. It holds that $[\Pi_k, \mathcal{U}_t] = \mathcal{C}_{k,t}$ where*

$$\mathcal{C}_{k,t} : H_\alpha^T \to \mathit{span}\{g_*\}, h \mapsto \langle h, c_{k,t} \rangle_\alpha g_*.$$

for

$$c_{k,t} := \sum_{|n|>k} g_n(t) g_n^*.$$

Moreover, $\sup_{s\in[0,t]} \|\mathcal{C}_{k,s}h\|_\alpha \to 0$ *for* $k \to \infty$ *and any* $h \in H_\alpha^T$.

Proof. Let $h \in H_\alpha^T$. Benth and Krühner [3, Lemma 3.2] yields that convergence in H_α implies local uniform convergence. From Proposition 4.7 we know $h - \Pi_k h \to 0$, and thus it holds

$$\sup_{s\in[0,t]} |h(s) - \Pi_k h(s)| \to 0\,,$$

for $k \to \infty$. Hence, we find

$$\sup_{s\in[0,t]} \left| \sum_{|n|>k} g_n(s) \langle h, g_n^* \rangle_\alpha \right| = \sup_{s\in[0,t]} |h(s) - \Pi_k h(s)| \to 0\,,$$

for $k \to \infty$. Therefore, $\sup_{s\in[0,t]} \|\mathcal{C}_{k,s}h\|_\alpha \to 0$ for $k \to \infty$.

Let $n \in \mathbb{Z}$. Then, by Proposition 4.5

$$\begin{aligned}
[\Pi_k, \mathcal{U}_t] g_n &= \Pi_k (e^{\lambda_n t} g_n + g_n(t) g_*) - 1_{\{|n|\le k\}} \mathcal{U}_t g_n \\
&= 1_{\{|n|\le k\}} e^{\lambda_n t} g_n + g_n(t) g_* - 1_{\{|n|\le k\}} (e^{\lambda_n t} g_n + g_n(t) g_*) \\
&= 1_{\{|n|>k\}} g_n(t) g_* \\
&= \mathcal{C}_{k,t} g_n
\end{aligned}$$

for any $t \ge 0$. Moreover,

$$[\Pi_k, \mathcal{U}_t] g_* = \Pi_k g_* - \mathcal{U}_t g_* = 0 = \mathcal{C}_{k,t} g_*.$$

The proof is complete. □

The next result concerns convergence of stochastic integrals of the Lie commutator:

Proposition 4.8. *Let X be a stochastic process with values in H_α^T such that $X(t) = Y(t) + M(t)$ for some square integrable process Y of finite variation and a square integrable martingale M. Then,*

$$\lim_{k\to\infty} \int_0^t [\Pi_k, \mathcal{U}_{t-s}]\, dX(s) = 0\,,$$

where the convergence is in $L^2(\Omega, H_\alpha)$, the space of H_α-valued random variables Z with $\mathbb{E}[\|Z\|_\alpha^2] < \infty$.

Proof. Recall from Proposition 4.7 that $[\Pi_k, \mathcal{U}_{t-s}] = \mathcal{C}_{k,t-s}$.

Let $\langle\langle M, M\rangle\rangle(t) = \int_0^t Q_s d\langle M, M\rangle(s)$ be the quadratic variation processes of the martingale M given in Peszat and Zabczyk [28, Theorem 8.2][1]. Then, Peszat and Zabczyk [28, Theorem 8.7(ii)] yields

$$\mathbb{E}\left[\| \int_0^t \mathcal{C}_{k,t-s} dM(s) \|_\alpha^2 \right] = \mathbb{E}\left[\int_0^t \mathrm{Tr}(\mathcal{C}_{k,t-s} Q_s \mathcal{C}_{k,t-s}^*) d\langle M, M\rangle(s) \right].$$

[1] In Peszat and Zabczyk [28], $\langle\langle \cdot, \cdot \rangle\rangle$ is called the operator angle bracket process, while $\langle \cdot, \cdot \rangle$ is the angle bracket process and the operator-valued process Q is introduced in that statement as well.

Recall that for $h \in H_\alpha^T$, we find $\mathcal{C}_{k,t}h = \langle h, c_{k,t}\rangle_\alpha g_*$. Thus,

$$\langle h, \mathcal{C}_{k,t}^* g_*\rangle_\alpha = \langle \mathcal{C}_{k,t}h, g_*\rangle_\alpha = \langle h, c_{k,t}\rangle_\alpha \,,$$

which gives that $\mathcal{C}_{k,t}^* g_* = c_{k,t}$, with $c_{k,t}$ defined in Proposition 4.7. For $g \in H_\alpha^T$ orthogonal to g_* we have

$$\langle h, \mathcal{C}_{k,t}^* g\rangle_\alpha = \langle \mathcal{C}_{k,t}h, g\rangle_\alpha = \langle h, c_{k,t}\rangle_\alpha \langle g_*, g\rangle_\alpha = 0$$

for any $h \in H_\alpha^T$ and hence $\mathcal{C}_{k,t}^* g = 0$. We get

$$\begin{aligned}\mathrm{Tr}(\mathcal{C}_{k,t-s}Q_s\mathcal{C}_{k,t-s}^*) &= \langle \mathcal{C}_{k,t-s}Q_s\mathcal{C}_{k,t-s}^* g_*, g_*\rangle_\alpha \\ &= \langle Q_s c_{k,t-s}, c_{k,t-s}\rangle_\alpha \\ &\leq \|c_{k,t-s}\|_\alpha^2 \mathrm{Tr}(Q_s)\,.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t \mathcal{C}_{k,t-s}dM(s)\right\|_\alpha^2\right] &= \mathbb{E}\left[\int_0^t \mathrm{Tr}(\mathcal{C}_{k,t-s}Q_s\mathcal{C}_{k,t-s}^*)d\langle M,M\rangle(s)\right] \\ &\leq \sup_{s\in[0,t]}\|c_{k,s}\|_\alpha^2 \mathbb{E}\left[\int_0^t \mathrm{Tr}(Q_s)d\langle M,M\rangle(s)\right] \\ &= \sup_{s\in[0,t]}\|c_{k,s}\|_\alpha^2 \mathbb{E}\left[\|M(t)-M(0)\|_\alpha^2\right] \\ &\to 0\end{aligned}$$

for $k \to \infty$. Similarily, we get

$$\left\|\int_0^t \mathcal{C}_{k,t-s}dY(s)\right\|_\alpha^2 \leq \sup_{s\in[0,t]}\|c_{k,s}\|_\alpha^2 \left(\int_0^t \|dY\|_\alpha(s)\right)^2 \to 0$$

as $k \to 0$, where $\|dY\|_\alpha$ denotes the total variation measure associated with dY (see Dinculeanu [16, Definition §2.1]). The claim follows. □

Our next aim is to identify the convergence speed of approximations in $H_\alpha^{T,k}$ of certain smooth elements $f \in H_\alpha^T$, that is, how close is $\Pi_k f$ to f in terms of number of Riesz basis functions. We show a couple of technical results first.

Lemma 4.9. *Let $f \in H_\alpha^T$. Then, we have*

$$\frac{e^{-2\lambda T}}{1-e^{-2\lambda T}}\left(|f(0)|^2 + \sum_{n\in\mathbb{Z}}|\langle f, g_n^*\rangle_\alpha|^2\right) \leq \|f\|_\alpha^2 \leq \frac{1}{1-e^{-2\lambda T}}\left(|f(0)|^2 + \sum_{n\in\mathbb{Z}}|\langle f, g_n^*\rangle_\alpha|^2\right).$$

Proof. Theorem 4.4 states that $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ is a Riesz basis of H_α^T. Moreover, it is given by $g_* = \Theta^{-1}(1,0)$, $g_n = \Theta^{-1}(0,e_n)$ for any $n \in \mathbb{Z}$ where Θ is the isometry given in (9) and $\{e_n\}_{n\in\mathbb{Z}}$ is the Riesz basis given in Proposition 4.3. Moreover, Lemma 4.1 yields that $e_n = \mathcal{A}b_n$ for any $n \in \mathbb{Z}$ where $\{b_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2([0,T],\mathbb{C})$ and $\|\mathcal{A}\|_{\mathrm{op}}^2 \leq \frac{1}{1-e^{-2\lambda T}}$. Thus, we can construct a Hilbert space with orthonormal basis $\{b_*, \{b_n\}_{n\in\mathbb{Z}}\}$ and a bounded linear operator $\mathcal{B}$ with $\|\mathcal{B}\|_{\mathrm{op}}^2 \leq \frac{1}{1-e^{-2\lambda T}}$, such that $g_* = \mathcal{B}b_*$,

$g_n = \mathcal{B}b_n$. Thus, we have

$$\begin{aligned}\|f\|_\alpha^2 &= \|g_*\langle f, g_*\rangle_\alpha + \sum_{n\in\mathbb{Z}} g_n \langle f, g_n^*\rangle_\alpha\|_\alpha^2 \\ &= \|\mathcal{B}b_*\langle f, g_*\rangle_\alpha + \sum_{n\in\mathbb{Z}} \mathcal{B}b_n \langle f, g_n^*\rangle_\alpha\|_\alpha^2 \\ &\le \frac{1}{1-e^{-2\lambda T}} \left(|\langle f, g_*\rangle_\alpha|^2 + \sum_{n\in\mathbb{Z}} |\langle f, g_n^*\rangle_\alpha|^2 \right)\end{aligned}$$

where $\{g_*, \{g_n^*\}_{n\in\mathbb{Z}}\}$ denotes the biorthogonal system to $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ given in Theorem 4.4. The lower inequality simply uses the lower inequality of Lemma 4.1 instead. □

It is more convenient to work with an orthonormal basis rather than a Riesz basis. It is known that for any Riesz basis on a Hilbert space there is a scalar product whose norm is equivalent to the original norm such that the Riesz basis is an orthonormal basis relative to the new scalar product, cf. [35, Section 1, Theorem 9]. It is interesting to note that in our setup the scalar product can be identified explicitly, as the next result shows.

Proposition 4.10. *Define for fixed parameters* $\lambda, \alpha > 0$

$$\langle f, h\rangle_{\lambda,\alpha} := f(0)\overline{h}(0) + \int_0^T e^{(2\lambda+\alpha)x} f'(x)\overline{h}'(x)dx, \quad f, h \in H_\alpha^T.$$

Then $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$ *defines a scalar product on* H_α^T *whose norm* $\|\cdot\|_{\lambda,\alpha}$ *is equivalent to the norm* $\|\cdot\|_\alpha$ *and we have that* $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ *is an orthonormal basis of* $(H_\alpha^T, \|\cdot\|_{\lambda,\alpha})$. *We also have*

$$\frac{e^{-2\lambda T}}{1-e^{-2\lambda T}}\|f\|_{\lambda,\alpha}^2 \le \|f\|_\alpha^2 \le \frac{1}{1-e^{-2\lambda T}}\|f\|_{\lambda,\alpha}^2, \quad f \in H_\alpha^T.$$

Proof. The definition of $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$ makes sense for all elements of H_α^T because their absolutely continuous derivatives are locally square-integrable. Also, $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$ is obviously a hermitian form and hence generates a semi-norm $\|\cdot\|_{\lambda,\alpha}$ on H_α^T.

We have $\langle g_*, g_*\rangle_{\lambda,\alpha} = 1$. Let $n \in \mathbb{Z}$. Then we have $\langle g_*, g_n\rangle_{\lambda,\alpha} = 0$ which shows that g_* is normed and orthogonal to $\{g_n\}_{n\in\mathbb{Z}}$ for the hermitian form $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$. Let $m \in \mathbb{Z}$. A direct calculation reveals,

$$\langle g_n, g_m\rangle_{\lambda,\alpha} = \frac{1}{T}\int_0^T e^{(2\lambda+\alpha+\lambda_n+\overline{\lambda}_m)x}dx = 1_{\{n=m\}}.$$

Thus, $\{g_n\}_{n\in\mathbb{Z}}$ is an orthonormal system relative to the hermitian form $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$. Let $f \in H_\alpha^T$. Then we know that $f = f(0)g_* + \sum_{n\in\mathbb{Z}}\langle f, g_n^*\rangle_\alpha g_n$ and, hence

$$\|f\|_{\lambda,\alpha}^2 = |f(0)|^2 + \sum_{n\in\mathbb{Z}} |\langle f, g_n^*\rangle_\alpha|^2.$$

Lemma 4.9 yields

$$\frac{e^{-2\lambda T}}{1-e^{-2\lambda T}}\|f\|_{\lambda,\alpha}^2 \le \|f\|_\alpha^2 \le \frac{1}{1-e^{-2\lambda T}}\|f\|_{\lambda,\alpha}^2, \quad f \in H_\alpha^T.$$

In particular, the norm $\|\cdot\|_\alpha$ is equivalent to $\|\cdot\|_{\lambda,\alpha}$. Thus, $(H_\alpha^T, \|\cdot\|_{\lambda,\alpha})$ is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$. □

The next technical result connects the inner product of elements in H_α^T with the bi-orthogonal basis functions to a simple Fourier-like integral on $[0,T]$:

Lemma 4.11. *Assume $f \in H_\alpha^T$. Then, for any $n \in \mathbb{Z}$,*

$$\langle f, g_n^* \rangle_\alpha = \frac{1}{\sqrt{T}} \int_0^T f'(x) \exp\left(\left(-\frac{2\pi i n}{T} + \lambda + \alpha/2\right) x\right) \, dx$$

Proof. Both $\langle\cdot, g_n^*\rangle_\alpha$ and $\langle\cdot, g_n\rangle_{\lambda,\alpha}$ are the coefficient functionals for the n-th vector of the Riesz basis $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ where $\langle\cdot,\cdot\rangle_{\lambda,\alpha}$ is the scalar product given in Proposition 4.10. Thus, $\langle\cdot, g_n^*\rangle_\alpha = \langle\cdot, g_n\rangle_{\lambda,\alpha}$. We conclude

$$\langle f, g_n^* \rangle_\alpha = \langle f, g_n\rangle_{\lambda,\alpha} = \frac{1}{\sqrt{T}} \int_0^T f'(x) e^{\left(-\frac{2\pi i n}{T} + \lambda + \alpha/2\right) x} \, dx.$$

□

With these results at hand, we can prove a convergence rate of order $1/\sqrt{k}$ for sufficiently smooth functions in H_α^T.

Proposition 4.12. *Assume $f \in H_\alpha^T$ is such that $f|_{[0,T]}$ is twice continuously differentiable. Then, we have*

$$\|f - \Pi_k f\|_\alpha^2 \leq \frac{C_1}{k},$$

for any $k \in \mathbb{N}$, where

$$C_1 = \frac{T\left|f'(T)e^{T(\lambda+\alpha/2)} - f'(0)\right|^2 + (\int_0^T |f''(x)| e^{x(\lambda+\alpha/2)}\, dx)^2}{\pi^2(1-e^{-2\lambda T})},$$

and we recall the projection operator Π_k from (11).

Proof. Lemma 4.9 yields

$$\|f - \Pi_k f\|_\alpha^2 = \|\sum_{|n|>k} g_n \langle f, g_n^*\rangle_\alpha\|_\alpha^2 \leq C \sum_{|n|>k} |\langle f, g_n^*\rangle_\alpha|^2$$

where $C := (1-e^{-2\lambda T})^{-1}$. Define $h_n(x) := \exp(\xi_n x)$, $x \geq 0$, where we denote $\xi_n = -\frac{2\pi i}{T} n + \lambda + \frac{\alpha}{2}$. Then, by Lemma 4.11 and integration-by-parts we find

$$\begin{aligned}
|\langle f, g_n^*\rangle_\alpha|^2 &= T^{-1} \left| \int_0^T f'(x) h_n(x) dx \right|^2 \\
&= T^{-1} \frac{1}{|\xi_n|^2} \left| f'(T)h_n(T) - f'(0)h_n(0) - \int_0^T f''(x) h_n(x)\, dx \right|^2 \\
&\leq \frac{2}{T} \frac{1}{|\xi_n|^2} A_f,
\end{aligned}$$

for any $n \in \mathbb{Z}\backslash\{0\}$, where the constant A_f is

$$A_f := \left|f'(T)e^{T(\lambda+\alpha/2)} - f'(0)\right|^2 + (\int_0^T |f''(x)| e^{x(\lambda+\alpha/2)}\, dx)^2.$$

Moreover, we have

$$\sum_{|n|>k} \frac{1}{|\xi_n|^2} = 2\sum_{n>k} \frac{1}{|\xi_n|^2} \leq \frac{T^2}{2\pi^2 k}.$$

Putting the estimates together, we get

$$\|f - \Pi_k f\|_\alpha^2 \leq A_f \frac{CT}{\pi^2 k},$$

as claimed. □

We can find a similar convergence rate for the series $c_{k,t}$ defined in Proposition 4.7, a result which becomes useful later:

Lemma 4.13. *Let $c_{k,t}$ be given as in Proposition 4.7. Then,*

$$\|c_{k,t}\|_\alpha^2 \leq \frac{C_2}{k},$$

for any $k \in \mathbb{N}$, where $C_2 = T/\pi^2(1 - \exp(-2\lambda T))$.

Proof. We appeal to Lemma 4.9, using $\{g_n^*\}_{n\in\mathbb{Z}}$ as the Riesz basis with biorthogonal system $\{g_n\}_{n\in\mathbb{Z}}$, to find

$$\begin{aligned}
\|c_{k,t}\|_\alpha^2 &= \|\sum_{|n|>k} g_n(t) g_n^*\|_\alpha^2 \\
&\leq C \sum_{|n|>k} |g_n(t)|^2 \\
&= \frac{C}{T} \sum_{|n|>k} \frac{1}{|\lambda_n|^2} \left|e^{\lambda_n t} - 1\right|^2 \\
&\leq \frac{2C}{T}(1 + e^{-(2\lambda+\alpha)t}) \sum_{|n|>k} \frac{1}{|\lambda_n|^2} \\
&\leq \frac{CT}{\pi^2} \frac{1}{k},
\end{aligned}$$

for $C = (1 - \exp(-2\lambda T))^{-1}$. Hence, the result follows. □

With these results at hand we are now in the position to study arbitrage-free approximations of the forward dynamics in (1).

5. Arbitrage-free approximation of forward term structure models

In this section we find an arbitrage-free approximation of a forward term structure model (1)– stated in the Heath-Jarrow-Morton-type setup with the Musiela parametrization – which lives in the finite dimensional state space $H_\alpha^{T,k}$. We furthermore derive the convergence speed of the approximation, and extend the results to account for forward contracts delivering the underlying commodity over a period which is the case for electricity and gas.

Consider the SPDE (1) with a mild solution $f \in H_\alpha$ given by (2). We recall from (5)-(6) and Theorem 4.4 the Riesz basis $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$ on the space H_α^T with the biorthogonal system $\{g_*, \{g_n^*\}_{n\in\mathbb{Z}}\}$. Furthermore, we recall from (10) and (11) the projection Π_k of H_α^T on $H_\alpha^{T,k}$, and the operator $\mathcal{C}_{k,t}$ for $k\in\mathbb{N}$, $t\geq 0$ defined in Proposition 4.7.

Let us define the continuous linear operator $\Lambda_k : H_\alpha \to H_\alpha^{T,k}$ by

$$\Lambda_k = \Pi_k \Pi \tag{12}$$

for any $k\in\mathbb{N}$, where Π is the projection from H_α onto H_α^T given in Theorem 5.1. The following theorem is one of the main results of the paper:

Theorem 5.1. *For $k\in\mathbb{N}$, let f_k be the mild solution of the SPDE*

$$df_k(t) = \partial_x f_k(t)dt + \Lambda_k\beta(t)dt + \Lambda_k\Psi(t)dL(t), \quad t\geq 0, f_k(0) = \Lambda_k f_0 \,. \tag{13}$$

Then, we have

(1) $\mathbb{E}\left[\sup_{x\in[0,T-t]}|f_k(t,x)-f(t,x)|^2\right]\to 0$ *for $k\to\infty$ and any $t\in[0,T]$,*
(2) *f_k takes values in the finite dimensional space $H_\alpha^{T,k}$, moreover, f_k is a strong solution to the SPDE* (13)*, i.e. $f_k\in\mathrm{dom}(\partial_x)$, $t\mapsto\partial_x f_k(t)$ is P-a.s. Bochner-integrable and*
$$f_k(t) = f_k(0) + \int_0^t (\partial_x f_k(s) + \Lambda_k\beta(s))ds + \int_0^t \Lambda_k\Psi(s)dL(s)\,,$$
(3) *and,*
$$f_k(t) = S_k(t) + \sum_{n=-k}^{k}\left(e^{\lambda_n t}\langle f_k(0), g_n^*\rangle_\alpha + \int_0^t e^{\lambda_n(t-s)}dX_n(s)\right)g_n\,,$$
where $S_k(t) = \delta_0(f_k(t))$ and $X_n(t) := \int_0^t\langle\Pi\beta(s)ds + \Pi\Psi(s)dL(s), g_n^*\rangle_\alpha$ for any $n\in\mathbb{Z}$, $t\geq 0$.

Remark 5.2. Assume that the model f is stated in the arbitrage-free framework, that is, that P is such that $\{F(t,\tau)\}_{t\in[0,\tau]}$ is a local P-martingale for any $\tau>0$. Then the dynamics of f are given by

$$df(t) = \partial_x f(t)dt + \Psi(t)dL(t),$$

i.e. $\beta = 0$ and L is a local martingale. Consequently, the dynamics of f_k in Theorem 5.1 are given by

$$df_k(t) = \partial_x f_k(t)dt + \Lambda_k\Psi(t)dL(t).$$

Thus the forward prices $F_k(t,\tau) := f_k(t,\tau-t)$ in the approximation models are local martingales as well. Indeed, the set of local martingale measures for the approximation models is larger than the set of local martingale measures for the initial model. In particular, one can work with the same pricing measure for the initial and the approximation models. Note that the existence of local martingale measures is connected to economically meaningful notions of no-arbitrage, cf. the fundamental work of Delbaen and Schachermayer [15, Theorem 1.1] and the related work of Cuchiero, Klein and Teichmann [13, Theorem 1.1]. From these considerations we conclude that $\{f_k\}_{k\in\mathbb{N}}$ satisfies requirements (i) to (iii) set out in Section 1. For requirement (iv), we will prove in the next statement, Corollary 5.3 below, that the solution essentially is a superposition of OU-process driven by some martingales.

Proof of Theorem 5.1. (1) Define

$$f_\Pi(t) := \mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))), \quad t \geq 0.$$

Since f_k is a mild solution, we have

$$\begin{aligned}
f_k(t) &= \mathcal{U}_t \Pi_k \Pi f_0 + \int_0^t \mathcal{U}_{t-s}\Pi_k(\Pi\beta(s)ds + \Pi\Psi(s)dL(s)) \\
&= \Pi_k \mathcal{U}_t \Pi f_0 + \int_0^t \Pi_k \mathcal{U}_{t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s)) \\
&\quad - \mathcal{C}_{k,t}\Pi f_0 - \int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s)) \\
&= \Pi_k \left(\mathcal{U}_t \Pi f_0 + \int_0^t \mathcal{U}_{t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))) \right) \\
&\quad - \mathcal{C}_{k,t}\Pi f_0 - \int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s)) \\
&= \Pi_k(f_\Pi(t)) - \mathcal{C}_{k,t}\Pi f_0 - \int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))
\end{aligned}$$

for any $t \geq 0$. From Benth and Krühner [3, Lemma 3.2] the sup-norm is dominated by the H_α-norm. Thus, there is a constant $c > 0$ such that

$$\mathbb{E}\left[\sup_{x\in[0,T-t]} |\Pi_k(f_\Pi(t,x)) - f_\Pi(t,x)|^2\right] \leq c\mathbb{E}\left[\|(\Pi_k - \mathcal{I})f_\Pi(t)\|_\alpha^2\right]$$

for any $t \geq 0$ where $\mathcal{I}$ denotes the identity operator on H_α. The dominated convergence theorem yields that the right-hand side converges to 0 for $k \to \infty$. Lemma 4.13 yields

$$\sup_{x\in[0,T-t]} |\mathcal{C}_{k,t} f_\Pi(0,x)| \leq c\|\mathcal{C}_{k,t} f_\Pi(0)\|_\alpha \leq c\|c_{k,t}\|_\alpha \|f_\Pi(0)\|_\alpha \to 0\,,$$

for $k \to \infty$. Proposition 4.8 states that

$$\mathbb{E}\left[\left\|\int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\right\|_\alpha^2\right] \to 0\,,$$

for $k \to 0$. Hence, we have

$$\mathbb{E}\left[\sup_{x\in[0,T-t]} |f_k(t,x) - f_\Pi(t,x)|^2\right] \to 0\,,$$

for $k \to \infty$ and any $t \in [0,T]$. Since $f_\Pi(t,x) = f(t,x)$ for any $t \in [0,T]$, $x \in [0, T-t]$ the first part follows.

(2) Note first that $\partial_x g_n(x) = \exp(\lambda_n x)/\sqrt{T} = \lambda_n g_n(x) + g_*(x)/\sqrt{T}$, and hence $\partial_x g_n \in H_\alpha^{T,k}$ whenever $|n| \leq k$. Thus, $H_\alpha^{T,k}$ is invariant under the generator ∂_x, and its restriction

to $H_\alpha^{T,k}$ is continuous and bounded. We find that f_k takes values only in $H_\alpha^{T,k}$ because

$$f_k(t) = \Pi_k\left(\mathcal{U}_t\Pi f_0 + \int_0^t \mathcal{U}_{t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\right)\\ - \mathcal{C}_{k,t}\Pi f_0 - \int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\,,$$

where all summands are clearly in $H_\alpha^{T,k}$.

(3) As $f_k(t) \in H_\alpha^{T,k}$, we have the representation

$$f_k(t) = \langle f_k(t), g_*^*\rangle_\alpha g_* + \sum_{n=-k}^{k} \langle f_k(t), g_n^*\rangle_\alpha g_n\,.$$

Since $g_*^* = 1$, we find that $\langle f_k(t), g_*^*\rangle_\alpha = f_k(t,0)$. Thus, from the mild solution of (13) we find, using Proposition 4.5

$$\begin{aligned} f_k(t) &= S_k(t) + \sum_{n=-k}^{k}\left\langle \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s}(\Lambda_k\beta(s)ds + \Lambda_k\Psi(s)dL(s)), g_n^*\right\rangle_\alpha g_n \\ &= S_k(t) + \sum_{n=-k}^{k}\langle f_k(0), \mathcal{U}_t^* g_n^*\rangle_\alpha g_n \\ &\quad + \sum_{n=-k}^{k}\int_0^t \langle \Lambda_k\beta(s)ds + \Lambda_k\Psi(s)dL(s), \mathcal{U}_{t-s}^* g_n^*\rangle_\alpha g_n \\ &= S_k(t) + \sum_{n=-k}^{k} e^{\lambda_n t}\langle f_k(0), g_n^*\rangle_\alpha g_n \\ &\quad + \sum_{n=-k}^{k}\int_0^t e^{\lambda_n(t-s)}\langle \Lambda_k\beta(s)ds + \Lambda_k\Psi(s)dL(s), g_n^*\rangle_\alpha g_n\,. \end{aligned}$$

Observe that for any $f \in H_\alpha$,

$$\Lambda_k f = \Pi_k(\Pi f) = (\Pi f)(0)g_* + \sum_{m=-k}^{k}\langle \Pi f, g_m^*\rangle_\alpha g_m\,,$$

and since $\{g_*, \{g_n\}_{n\in\mathbb{Z}}\}$, $\{g_*^*, \{g_n^*\}_{n\in\mathbb{Z}}\}$ are biorthogonal systems

$$\langle \Lambda_k f, g_n^*\rangle_\alpha = (\Pi f)(0)\langle g_*, g_n^*\rangle_\alpha + \sum_{m=-k}^{k}\langle \Pi f, g_m^*\rangle_\alpha\langle g_m, g_n^*\rangle_\alpha = \langle \Pi f, g_n^*\rangle_\alpha 1_{\{|n|\le k\}}\,.$$

Hence, the claim follows. □

Another view on Theorem 5.1 is that all processes in the k-th approximation of f can be expressed in terms of the factor processes $X_*, X_{-k}, \ldots, X_k$, as stated below.

Corollary 5.3. *Under the assumptions and notations of Theorem 5.1, we have for $k \in \mathbb{N}$,*

$$f_k(t,x) = S_k(t) + \sum_{n=-k}^{k} U_n(t) g_n(x)\,,$$

for any $0 \le t < \infty$ and $x \ge 0$. Here,

$$S_k(t) = S_k(0) + X_*(t) + \sum_{n=-k}^{k} \left(g_n(t) U_n(0) + \int_0^t g_n(t-s) dX_n(s) \right)\,,$$

with,

$$\begin{aligned} X_n(t) &:= \left\langle \int_0^t (\Pi\beta(s)ds + \Pi\Psi(s)dL(s)), g_n^* \right\rangle_\alpha\,, \\ X_*(t) &:= \left\langle \int_0^t (\Pi\beta(s)ds + \Pi\Psi(s)dL(s)), g_* \right\rangle_\alpha\,, \\ U_n(t) &:= e^{\lambda_n t} \langle f_k(0), g_n^* \rangle_\alpha + \int_0^t e^{\lambda_n (t-s)} dX_n(s) \end{aligned}$$

for $n \in \{-k, \ldots, k\}$.

Proof. The first equation is a restatement of (3) in Theorem 5.1. Proposition 4.5 yields

$$\langle \mathcal{U}_t h, g_* \rangle_\alpha = \langle h, g_* \rangle_\alpha + \sum_{n=-k}^{k} g_n(t) \langle h, g_n^* \rangle_\alpha$$

for any $h \in H_\alpha^{T,k}$ because $h = \langle h, g_* \rangle_\alpha g_* + \sum_{n=-k}^{k} \langle h, g_n^* \rangle_\alpha g_n$. Thus, since $g_* = 1$ and $g_n(0) = 0$ we have

$$\begin{aligned} S_k(t) &= f_k(t,0) \\ &= \langle f_k(t), g_* \rangle_\alpha \\ &= \langle \mathcal{U}_t f_k(0), g_* \rangle_\alpha + \int_0^t \langle \mathcal{U}_{t-s}(\Lambda_k \beta(s)\, ds + \Lambda_k \Psi(s)\, dL(s)), g_* \rangle_\alpha \\ &= \langle f_k(0), g_* \rangle_\alpha + \sum_{n=-k}^{k} g_n(t) \langle f_k(0), g_n^* \rangle_\alpha \\ &\qquad + \int_0^t \langle \Lambda_k \beta(s)\, ds + \Lambda_k \Psi(s)\, dL(s), g_* \rangle_\alpha \\ &\qquad + \sum_{n=-k}^{k} \int_0^t g_n(t-s) \langle \Lambda_k \beta(s) + \Lambda_k \Psi(s)\, dL(s), g_n^* \rangle_\alpha\,. \end{aligned}$$

As in the proof of Theorem 5.1, we have $\langle \Lambda_k f, g_n^* \rangle_\alpha = \langle \Pi f, g_n^* \rangle_\alpha$ for any $f \in H_\alpha$. Similarly, $\langle \Lambda_k f, g_* \rangle_\alpha = \langle \Pi f, g_* \rangle_\alpha$ for $n \in \mathbb{Z}$ with $|n| \le k$. The result follows. ☐

The processes $S_k, U_{-k}, \ldots, U_k$ in Corollary 5.3 capture at any time t the whole state of the market in the approximation model. I.e., the spot price and the forward curve are simple functions of these state variables. As we will see in Corollary 5.6 below, the forward prices of contracts with delivery periods can be expressed in these state variables as well. Note that if we assume $\langle \Pi\beta, g_n^*\rangle_\alpha$ and $\langle \Pi\Psi, g_n^*\rangle_\alpha$ to be constant (non-random), then $(X_{-k},\dots,X_k)$ is a $2k+1$-dimensional Lévy process and $U_{-k},\dots,U_k$ are Ornstein-Uhlenbeck processes. This corresponds to the spot price model suggested in Benth, Kallsen and Meyer-Brandis [2].

From the proof of Corollary 5.3 we find that $S_k(0) = \langle f_k(0), g_*\rangle_\alpha$. But then

$$S_k(0) = \langle \Lambda_k f_0, g_*\rangle_\alpha = \langle \Pi f_0, g_*\rangle_\alpha = (\Pi f_0)(0) = f_0(0)\,.$$

Obviously, $f_0(0)$ is equal to today's spot price, so we obtain that the starting point of the process $S_k(t)$ in the approximation f_k is today's spot price. Since we have $f_k(t,0) = S_k(t)$, S_k is the approximative spot price process associated with f_k. For $U_n(0)$, $n\in\mathbb{Z}$, invoking Lemma 4.11 shows that

$$\begin{aligned} U_n(0) &= \langle \Pi f_0, g_n^*\rangle_\alpha \\ &= \frac{1}{\sqrt{T}}\int_0^T (\Pi f_0)'(y)\exp((\lambda+\alpha/2)x)\exp\left(\frac{2\pi i}{T}nx\right)dy\,. \end{aligned}$$

This is the Fourier transform of the initial forward curve f_0 (or, rather its derivative scaled by an exponential function). In any case, both $S_k(0)$ and $U_n(0)$ are given by (functionals of) the initial forward curve f_0.

Next, we would like to identify the convergence speed of our approximation, that is, the rate for the convergence in part (1) of Theorem 5.1.

Proposition 5.4. *Assume that $x\mapsto f(t,x)$ is twice continuously differentiable and let f_k be the mild solution of the SPDE*

$$df_k(t) = \partial_x f_k(t)dt + \Lambda_k\beta(t)dt + \Lambda_k\Psi(t)dL(t),\quad t\geq 0, f_k(0) = \Lambda_k f_0\,.$$

Then, we have

$$\mathbb{E}\left[\sup_{x\in[0,T-t]}|f_k(t,x)-f(t,x)|^2\right] \leq \frac{A(T)}{k}\,,$$

for any $k>1$, where

$$\begin{aligned} A(T) := \frac{3T(1+\alpha^{-1})}{(1-e^{-2\lambda T})}&\left\{\|\Pi f_0\|_\alpha^2 + \int_0^T \mathbb{E}[\mathrm{Tr}(\Psi(s)Q\Psi^*(s))]ds + \left(\int_0^T \mathbb{E}\left[\|\beta(s)\|_\alpha\right]ds\right)^2\right\} \\ &+ \frac{3(1+\alpha^{-1})}{\pi^2(1-e^{-2\lambda T})}\left\{T\mathbb{E}\left[|\partial_x f_\Pi(t,T)e^{T(\lambda+\alpha/2)} - \partial_x f_\Pi(t,0)|^2\right]\right. \\ &\qquad\left. + \left(\int_0^T \mathbb{E}\left[|\partial_x^2 f_\Pi(t,x)|\right]e^{x(\lambda+\alpha/2)}dx\right)^2\right\}\,. \end{aligned}$$

Remark 5.5. In the preceding proposition one might have expected a convergence rate of order $1/k^2$ which would be the rate in the corresponding Galerkin approximation, cf. Kruse [25, Theorem 1.1] (Note that we state the error in the squared norm-distance instead of the usual norm-distance). However, different from the typical Galerkin approximation, we included a correction term to retain the derivative operator in the approximation instead of discretising it. The convergence speed of the correction term towards zero is analysed in Lemma 4.13 and is only of order $1/k$.

Proof of Proposition 5.4. In the proof of Theorem 5.1 we have shown that

$$f_k(t) = \Pi_k(f_\Pi(t)) - \mathcal{C}_{k,t}\Pi f_0 - \int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\,,$$

where $f_\Pi(t) := \mathcal{U}_t\Pi f_0 + \int_0^t \mathcal{U}_{t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s)))$ for any $t \geq 0$. By Proposition 4.12 we have

$$\|f_\Pi(t) - \Pi_k(f_\Pi(t))\|_\alpha^2 \leq \frac{C_1(t)}{k}$$

where $C_1(t)$ is the random variable defined by

$$C_1(t) = \frac{T|\partial_x f_\Pi(t,T)e^{T(\lambda+\alpha/2)} - \partial_x f_\Pi(t,0)|^2 + (\int_0^T |\partial_x^2 f_\Pi(t,x)|e^{x(\lambda+\alpha/2)}\,dx)^2}{\pi^2(1-e^{-2\lambda T})}\,.$$

Proposition 4.7 and Lemma 4.13 yield

$$\|\mathcal{C}_{k,t}h\|_\alpha^2 = \|\langle h, c_{k,t}\rangle_\alpha g_*\|_\alpha^2 = |\langle h, c_{k,t}\rangle_\alpha|^2 \leq \|h\|_\alpha^2\|c_{k,t}\|_\alpha^2 \leq \|h\|_\alpha^2\frac{C_2}{k}\,,$$

for the constant $C_2 = T/\pi^2(1-e^{-2\lambda T})$. Then, we have

$$\begin{aligned}\|f_k(t) - f_\Pi(t)\|_\alpha^2 &\leq 3\|\Pi_k(f_\Pi(t)) - f_\Pi(t)\|_\alpha^2 + 3\|\mathcal{C}_{k,t}\Pi f_0\|_\alpha^2 \\ &\qquad + 3\|\int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\|_\alpha^2 \\ &\leq \frac{3C_1(t)}{k} + \frac{3C_2}{k}\|\Pi f_0\|_\alpha^2 \\ &\qquad + 3\|\int_0^t \mathcal{C}_{k,t-s}(\Pi\beta(s)ds + \Pi\Psi(s)dL(s))\|_\alpha^2.\end{aligned}$$

By Lemma 3.2 in Benth and Krühner [3], the uniform norm is bounded by the H_α-norm with a constant $c = \sqrt{1+\alpha^{-1}}$. Hence, taking expectations, yield

$$\begin{aligned}&\mathbb{E}\left[\sup_{x\in[0,T-t]}|f_k(t,x) - f(t,x)|^2\right] \\ &\quad\leq c^2\mathbb{E}\left[\|f_k(t) - f_\Pi(t)\|_\alpha^2\right] \\ &\quad\leq \frac{3c^2}{k}\left(\mathbb{E}\left[C_1(t)\right] + C_2\|\Pi f_0\|_\alpha^2\right) \\ &\qquad + \frac{6c^2}{k}C_2\left(\int_0^T \mathbb{E}[\mathrm{Tr}(\Psi(s)Q\Psi^*(s))]ds + \left(\int_0^T \mathbb{E}\left[\|\beta(s)\|_\alpha\right]ds\right)^2\right)\,.\end{aligned}$$

The result follows. □

In electricity and gas markets forward contracts deliver over a future period rather than at a fixed time. The holder of the forward contract receives a uniform stream of electricity or gas over an agreed time period $[\tau_1, \tau_2]$. The forward prices of delivery period contracts can be derived from a "fixed-delivery time" forward curve model (see Benth et al. [7]) by

$$F(t,\tau_1,\tau_2) := \frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} f(t, s-t)\,, ds \tag{14}$$

where f is given by the SPDE (1). The following corollary adapts Theorem 5.1 to the case of forward contracts with delivery period.

Corollary 5.6. *Assume the conditions of Theorem 5.1 and define*

$$F_k(t,\tau_1,\tau_2) := \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} f_k(t,s-t)ds$$

for any $0\le t\le \tau_1\le\tau_2\le T$. *Then, we have*

$$F_k(t,\tau_1,\tau_2)\to F(t,\tau_1,\tau_2)$$

for $k\to\infty$ *in* $L^2(\Omega)$ *where* F *is given in* (14). *Furthermore,*

$$F_k(t,\tau_1,\tau_2) = S_k(t) + \sum_{n=-k}^{k} G_n(t,\tau_1,\tau_2)\left(e^{\lambda_n t}\langle g_n^*, f_k(0)\rangle_\alpha + \int_0^t e^{\lambda_n(t-s)}dX_n(s)\right),$$

for any $t\le\tau_1\le\tau_2\le T$ *where* $S_k(t)=\delta_0(f_k(t))$,

$$G_n(t,\tau_1,\tau_2) = \frac{\exp(\lambda_n(\tau_2-t)) - \exp(\lambda_n(\tau_1-t)) - \lambda_n(\tau_2-\tau_1)}{\lambda_n^2\sqrt{T}(\tau_2-\tau_1)}$$

and $X_n(t) := \int_0^t\langle \Pi\beta(s)ds + \Pi\Psi(s)dL(s), g_n^*\rangle_\alpha$.

Proof. Theorem 5.1 yields uniform L^2 convergence of the integrands appearing in F_k to the integrand appearing in F and hence the convergence follows. The representation of F_k follows immediately from part (3) of Theorem 5.1. □

We remark in passing that the temperature derivatives market (see e.g. Benth and Šaltytė Benth [8]) trades in forwards with a "delivery period" as well. In this market, the forwards are cash-settled against an index of the daily average temperature measured in a city over a given period. Temperature forward prices can be approximated using our approach.

Our forward price dynamics f in (1) may also be a model for *forward rates* in fixed-income theory (see for instance Filipovic [19], Peszat and Zabczyk [28] and Carmona and Tehranchi [12]). Indeed, this is the application area where much of the theoretical developments and interest for the HJMM dynamics comes from. We end this section with a discussion of forward rates in view of our approximations of (1) in Theorem 5.1.

In the fixed-income theory, it is customary to formulate the HJMM dynamics of forward rates directly in the risk neutral setting, which imposes a drift condition relating β with Ψ (see Filipovic [19], Peszat and Zabczyk [28] and Carmona and Tehranchi [12]). Let us take the set-up in Peszat and Zabczyk [28, Ch. 20], and restrict our attention to the Wiener case for simplicity, that is, we let $L=W$. Suppose that $\Psi(t)$ is defined via an H_α-valued stochastic process $\sigma(t,x), t,x\ge 0$ such that

$$\Psi(t)f(x) = \langle\sigma(t,x), f\rangle_\alpha\,.$$

Without going into details, we assume that σ is such that $\Psi(t)$ satisfies the required conditions (recall the assumptions in Section 2). From Remark 20.2 in Peszat and Zabczyk [28], the drift condition becomes

$$\beta(t,x) = \frac{1}{2}\langle\mathcal{Q}\sigma(t,x), \int_0^x\sigma(t,y)\,dy\rangle_\alpha\,.$$

We note here that $\sigma(t,y) \in H_\alpha$ for all $y \geq 0$, and hence the integral above is to be understood in the Bochner sense (which we assume is well-defined, here). By the definition of $\Psi(t)$, we have

$$\beta(t,x) = \frac{1}{2}\int_0^x \Psi(t)(\mathcal{Q}^*\sigma(t,y))(x)\,dy\,.$$

Now, from Theorem 5.1 we find an approximation f_k where the drift is $\beta_k(t) := \Lambda_k\beta(t)$ and volatility $\Psi_k(t) := \Lambda_k\Psi(t)$. Under suitable regularity conditions on σ, we find that

$$\Lambda_k\Psi(t)f = \langle \Lambda_k\sigma(t,\cdot,\cdot), f\rangle_\alpha$$

with the interpretation that the inner product is taken with respect to the third argument of σ and Λ_k acts on the second argument. Hence, with $\sigma_k(t,x,y) = \Lambda_k\sigma(t,\cdot,y)(x)$, we have that f_k is an arbitrage-free dynamics if the drift in the dynamics of f_k satisfies

$$\widehat{\beta}_k(t,x) := \frac{1}{2}\int_0^x \Psi_k(t)(\mathcal{Q}^*\sigma_k(t,\cdot,y))\,dy$$

But this is in general different from $\beta_k(t)$, and we conclude that our approach does not give an arbitrage-free approximative dynamics of the forward rate model.

6. Refinement to Markovian forward price models

In this Section we refine our analysis to Markovian forward price models, making the additional assumption that the coefficients β and Ψ depend on the state of the forward curve. More specifically, we assume that

$$\beta(t) = b(t,f(t)), \tag{15}$$

$$\Psi(t) = \psi(t,f(t)), \tag{16}$$

where $b : \mathbb{R}_+ \times H_\alpha \to H_\alpha$, $\psi : \mathbb{R}_+ \times H_\alpha \to L(H_\alpha)$ are measurable, Lipschitz continuous functions of linear growth in the sense

$$\|b(t,f) - b(t,g)\|_\alpha \leq C_b\|f-g\|_\alpha\,, \tag{17}$$

$$\|(\psi(t,f) - \psi(t,g))\mathcal{Q}^{1/2}\|_{\mathrm{HS}} \leq C_\psi\|f-g\|_\alpha\,, \tag{18}$$

and

$$\|b(t,f)\|_\alpha \leq C_b(1+\|f\|_\alpha)\,, \tag{19}$$

$$\|\psi(t,f)\mathcal{Q}^{1/2}\|_{\mathrm{HS}} \leq C_\psi(1+\|f\|_\alpha)\,, \tag{20}$$

for positive constants C_b, C_ψ. Under these conditions there exists a unique mild solution f of the semilinear SPDE

$$df(t) = (\partial_x f(t) + b(t,f(t)))dt + \psi(t,f(t-))dL(t), \quad f(0) = f_0 \tag{21}$$

with càdlàg paths, cf. Tappe [34, Theorem 4.5, Remark 4.6]. We would like to note that semilinear SPDEs of this type are treated in the book by Peszat and Zabczyk [28] and in Tappe [34]. Additionally, we assume that

$$b(t,h) = b(t,g), \tag{22}$$

$$\psi(t,h) = \psi(t,g)\,, \tag{23}$$

for any $h, g \in H_\alpha$ such that $h(x) = g(x)$ for any $x \in [0, T-t]$, i.e. the structure of the curve beyond our time horizon T does not influence the dynamics of the curve-valued process $f(t)$.

Before continuing our analysis of the arbitrage-free approximation in the Markovian case, we show a couple of useful lemmas. The first states a version of Doob's L^2 inequality for Volterra-like Hilbert space-valued stochastic integrals with respect to the Lévy process L, and is essentially collected from Filipović, Tappe and Teichmann [20].

Lemma 6.1. *Suppose that $\Phi \in \mathcal{L}^2_L(H_\alpha)$. Then,*

$$\mathbb{E}\left[\sup_{s\in[0,t]} \| \int_0^s \mathcal{U}_{s-r}\Phi(r)\, dL(r)\|_\alpha^2\right] \leq 4c_t^2 \int_0^t \mathbb{E}\left[\|\Phi(r)\mathcal{Q}^{1/2}\|_{HS}^2\right]\, dr\,,$$

for $c_t > 0$ being at most exponentially growing in t.

Proof. Note first that due to Benth and Krühner [3, Lemma 3.5] the C_0-semigroup $\{\mathcal{U}_t\}_{t\geq 0}$ is pseudo-contractive. Filipović, Tappe and Teichmann [20, Prop. 8.7][2] state that there is a Hilbert space extension H of H_α (i.e. H is a Hilbert space and H_α is its subspace and the norm of H_α equals the norm of H restricted to H_α where by slight abuse of notation we write $\|\cdot\|_\alpha$ for the norm on H as well) and a C_0-group $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ on H such that $\Pi_{H_\alpha}\mathcal{V}_t|_{H_\alpha} = \mathcal{U}_t$ for $t \geq 0$ where Π_{H_α} is the orthogonal projection from H to H_α. Then, we have

$$\begin{aligned}\sup_{s\in[0,t]} \| \int_0^s \mathcal{U}_{s-r}\Phi(r)\, dL(r)\|_\alpha &\leq \sup_{s\in[0,t]} \|\mathcal{V}_{s-t}\|_{\mathrm{op}} \| \int_0^s \mathcal{V}_{t-r}\Phi(r)\, dL(r)\|_\alpha \\ &\leq \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}} \sup_{s\in[0,t]} \| \int_0^s \mathcal{V}_{t-r}\Phi(r)\, dL(r)\|_\alpha\,.\end{aligned}$$

Thus, by Doob's maximal inequality, Thm. 2.2.7 in Prevot and Röckner [31], we find

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\in[0,t]} \| \int_0^s \mathcal{U}_{s-r}\Phi(r)\, dL(r)\|_\alpha^2\right] \\ &\qquad \leq \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}}^2 \mathbb{E}\left[\sup_{s\in[0,t]} \| \int_0^s \mathcal{V}_{t-r}\Phi(r)\, dL(r)\|_\alpha^2\right] \\ &\qquad \leq 4 \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}}^2 \mathbb{E}\left[\| \int_0^t \mathcal{V}_{t-r}\Phi(r)\, dL(r)\|_\alpha^2\right] \\ &\qquad = 4 \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}}^2 \int_0^t \mathbb{E}\left[\|\mathcal{V}_{t-r}\Phi(r)\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right]\, dr \\ &\qquad \leq 4 \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}}^2 \sup_{s\in[0,t]} \|\mathcal{V}_s\|_{\mathrm{op}}^2 \int_0^t \mathbb{E}\left[\|\Phi(r)\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right]\, dr\end{aligned}$$

This proves the Lemma by letting $c_t = \sup_{s\in[0,t]} \|\mathcal{V}_{-s}\|_{\mathrm{op}} \sup_{0\leq s\leq t} \|\mathcal{V}_s\|_{\mathrm{op}}$ and recalling that any C_0-group is bounded in operator norm by an exponentially increasing function in t. Hence, $c_t \leq c\exp(wt)$ for some constants $c, w > 0$. □

[2] This is a very useful consequence of the Szökefalvi-Nagy dilation theorem [32, Theorem I.8.1].

We remark in passing that the above result holds for any pseudo-contractive semigroup $\mathcal{S}_t, t \geq 0$.

The next lemma is a useful technical result on the distance between processes and the fixed point of an integral operator defined via the mild solution of (21). The lemma plays a crucial role in showing that certain arbitrage-free approximations of (21) converge to the right limit. Essentially, it states the error of an arbitrary element to a fix point of some operator V in terms of a constant times the error of the element to its next iteration.

Lemma 6.2. *Let h be an H_α-valued adapted càdlàg process with $\mathbb{E}[\int_0^t \|h(s)\|_\alpha^2 \, ds] < \infty$. Then we can define*

$$V(h)(t) := \mathcal{U}_t f_0 + \int_0^t \mathcal{U}_{t-s} b(s, h(s)) \, ds + \int_0^t \mathcal{U}_{t-s} \psi(s, h(s-)) \, dL(s) \,,$$

for any $t \geq 0$, $V(h)$ has a càdlàg modification and we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \|h(s) - f(s)\|_\alpha^2\right] \leq \frac{\pi^2}{6} \exp(4C_t) \mathbb{E}\left[\sup_{0\leq s\leq t} \|V(h)(s) - h(s)\|_\alpha^2\right] \,,$$

for any $t \geq 0$ where $C_t := 2C_b^2 t \sup_{s\in[0,t]} \|\mathcal{U}_s\|_{\mathrm{op}} + 8c_t^2 C_\psi^2$ and c_t is the constant given in Lemma 6.1.

Proof. The linear growth assumption (19) on b yields

$$\begin{aligned}
\mathbb{E}[\int_0^t \|\mathcal{U}_{t-s} b(s, h(s))\|_\alpha \, ds] &\leq C_b e^{wt} (t + \mathbb{E}[\int_0^t \|h(s)\|_\alpha \, ds]) \\
&\leq C_b e^{wt} (t + \sqrt{t} \mathbb{E}[\int_0^t \|h(s)\|_\alpha^2 \, ds]^{1/2}) \\
&< \infty \,.
\end{aligned}$$

Furthermore, from the linear growth condition (20) on ψ we have

$$\mathbb{E}[\int_0^t \|\mathcal{U}_{t-s} \psi(s, h(s))\|_\alpha^2 \, ds] \leq 2C_\psi^2 e^{2wt} \left(t + \mathbb{E}[\int_0^t \|h(s)\|_\alpha^2 \, ds]\right) < \infty \,.$$

Hence, $V(h)$ is well-defined, and it is an adapted process. Filipović, Tappe and Teichmann [20, Prop. 8.7] state that there is a Hilbert space extension H of H_α and a C_0-group $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ on H such that $\Pi_{H_\alpha} \mathcal{V}_t|_{H_\alpha} = \mathcal{U}_t$ for $t \geq 0$ where Π_{H_α} is the orthogonal projection from H to H_α. We have

$$V(h)(t) = \Pi_{H_\alpha} \mathcal{V}_t \left(f_0 + \int_0^t \mathcal{V}_{-s} b(s, h(s)) ds + \int_0^t \mathcal{V}_{-s} \psi(s, h(s-)) dL(s)\right)$$

for any $t \geq 0$. The expression in the bracket has a càdlàg modification l and then we see that $\{\Pi_{H_\alpha} \mathcal{V}_t(l(t))\}_{t\geq 0}$ is a càdlàg modification of $V(h)$ by the local uniform boundedness of $t \mapsto \|\mathcal{V}_t\|_{\mathrm{op}}$ and the strong continuity of $\mathcal{V}$. For the rest of the proof we mean by $V(h)$ a càdlàg modification.

Define $L^2(\Omega, D([0,t], H_\alpha))$ to be the vector space of H_α-valued adapted càdlàg stochastic processes g for which $\mathbb{E}[\sup_{s\in[0,t]} \|g(s)\|_\alpha^2] < \infty$ and define a norm $\|\cdot\|_t$ on it by

$$\|g\|_t^2 := \mathbb{E}[\sup_{s\in[0,t]} \|g(s)\|_\alpha^2].$$

From the above consideration we can define V on the entire normed space. By a straightforward estimation using again the linear growth of b and ψ, we find similarly that

$$\mathbb{E}[\sup_{s\in[0,t]} \|V(h)(s)\|_\alpha^2] \leq K_t \left(1 + \mathbb{E}\left[\int_0^t \|h\|_\alpha^2 \, ds\right]\right) < \infty \,,$$

for some constant $K_t > 0$. Thus, V is an operator on $L^2(\Omega, D([0,t], H_\alpha))$ and can be iterated.

Observe, that $V(f) = f$ which means that V has a fixed point. We now show that the nth iteration V^n of V is Lipschitz continuous with constant less than one for some $n \in \mathbb{N}$. If that is shown, then Banach's fixed point theorem yields that f is the unique fixed point and $V^n h \to f$ for $n \to \infty$. To this end, let $g \in L^2(\Omega, D([0,t], H_\alpha))$. Then, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t} \|V(h)(s) - V(g)(s)\|_\alpha^2\right] \\
&\qquad \leq 2\mathbb{E}\left[\sup_{0\leq s\leq t} \|\int_0^s \mathcal{U}_{s-r}\left(b(r,h(r)) - b(r,g(r))\right) \, dr\|_\alpha^2\right] \\
&\qquad\quad + 2\mathbb{E}\left[\sup_{0\leq s\leq t} \|\int_0^s \mathcal{U}_{s-r}\left(\psi(r,h(r-)) - \psi(r,g(r-))\right) \, dL(r)\|_\alpha^2\right] .
\end{aligned}$$

Consider the first term on the right hand side of the inequality. By the norm inequality for Bochner integrals and Lipschitz continuity of b in (17), we find

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq s\leq t} \|\int_0^s \mathcal{U}_{s-r}\left(b(r,h(r)) - b(r,g(r))\right) \, dr\|_\alpha^2\right] \\
&\qquad \leq \mathbb{E}\left[\sup_{0\leq s\leq t} \left(\int_0^s \|\mathcal{U}_{s-r}\|_{\text{op}} \|b(r,h(r)) - b(r,g(r))\|_\alpha \, dr\right)^2\right] \\
&\qquad \leq t\mathbb{E}\left[\sup_{0\leq s\leq t} \int_0^s \|\mathcal{U}_{s-r}\|_{\text{op}}^2 \|b(r,h(r)) - b(r,g(r))\|_\alpha^2 \, dr\right] \\
&\qquad \leq t \sup_{0\leq s\leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \mathbb{E}\left[\int_0^t \|b(r,h(r)) - b(r,g(r))\|_\alpha^2 \, dr\right] \\
&\qquad \leq tC_b^2 \sup_{0\leq s\leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \int_0^t \mathbb{E}\left[\|h(r) - g(r)\|_\alpha^2\right] \, dr \,,
\end{aligned}$$

where we have applied Cauchy-Schwartz' inequality. Recall that since $\mathcal{U}_t$ is a pseudo-contractive semigroup, we find for some $w > 0$, it holds that

$$\sup_{0\leq s\leq t} \|\mathcal{U}_s\|_{\text{op}}^2 \leq \exp(2wt) < \infty.$$

For the second term, we find by appealing to Lemma 6.1 and the Lipschitz continuity in (18) of ψ,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq t}\|\int_0^s \mathcal{U}_{s-r}\left(\psi(r,h(r-))-\psi(r,g(r-))\right)\,dL(r)\|_\alpha^2\right] \\
&\leq 4c_t^2\int_0^t \mathbb{E}\left[\|(\psi(r,h(r))-\psi(r,g(r)))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right]\,dr \\
&\leq 4c_t^2C_\psi^2\int_0^t \mathbb{E}\left[\|h(r)-g(r)\|_\alpha^2\right]\,dr
\end{aligned}$$

Then, we have from the definition of C_t that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq t}\|V^n(h)(s)-V^n(g)(s)\|_\alpha^2\right] \\
&\leq C_t\int_0^t \mathbb{E}\left[\|V^{n-1}(h)(s_1)-V^{n-1}(g)(s_1)\|_\alpha^2\right]\,ds_1 \\
&\leq C_t^n\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}}\mathbb{E}\left[\|h(s_n)-g(s_n)\|_\alpha^2\right]ds_n\dots ds_1 \\
&\leq \frac{C_t^n}{n!}\mathbb{E}\left[\sup_{0\leq s\leq t}\|h(s)-g(s)\|_\alpha^2\right],
\end{aligned}$$

for any $n\in\mathbb{N}$. Consequently, V^n is Lipschitz continuous with constant strictly less than 1 for some $n\in\mathbb{N}$. We have that

$$\|f-h\|_t=\lim_{n\to\infty}\|V^n(h)-h\|_t\,,$$

and

$$\|V^n(h)-h\|_t\leq\sum_{k=0}^{n-1}\|V^{k+1}(h)-V^k(h)\|_t\leq\|V(h)-h\|_t\sum_{k=0}^{n-1}\left(\frac{C_t^k}{k!}\right)^{1/2}.$$

From Cauchy-Schwartz' inequality we get

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(\frac{C_t^k}{k!}\right)^{1/2} &= \sum_{k=0}^{n-1}(k+1)^{-1}\left(\frac{(k+1)^2C_t^k}{k!}\right)^{1/2} \\
&\leq\left(\sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\right)^{1/2}\left(\sum_{k=0}^{n-1}\frac{(k+1)^2C_t^k}{k!}\right)^{1/2} \\
&\leq\frac{\pi}{\sqrt{6}}\left(\sum_{k=0}^{n-1}\frac{4^kC_t^k}{k!}\right)^{1/2} \\
&\leq\frac{\pi}{\sqrt{6}}\exp(2C_t)\,,
\end{aligned}$$

where we have used the elementary inequality $k+1\leq 2^k$, $k\in\mathbb{N}$. $\square$

Let us define the Lipschitz continuous functions $b_\Pi := \Pi \circ b$ and $\psi_\Pi := \Pi \circ \psi$. Then, Tappe [34, Theorem 4.5] yields a mild solution f_Π for the SPDE

$$df_\Pi(t) = (\partial_x f_\Pi(t) + b_\Pi(t, f_\Pi(t)))\, dt + \psi_\Pi(t, f_\Pi(t-))\, dL(t), \quad f_\Pi(0) = \Pi f_0 \,. \tag{24}$$

Furthermore, it will be convenient to use the notations

$$b_k(t,h) := \Lambda_k(b(t,h)), \tag{25}$$

$$\psi_k(t,h) := \Lambda_k(\psi(t,h)) \tag{26}$$

for any $h \in H_\alpha$, $t \geq 0$.

In the proof of Theorem 5.1 we compared the solution f to the projected solution Πf which are essentially the same due to properties of Π. Then we compared Πf to f_Π which again had been essentially the same. Finally, we compared $\Pi_k f_\Pi$ to solutions of the projected SPDE where the difference was given by a certain Lie-commutator. However, in the Markovian setting we want to change the dependencies of the coefficients as well, which complicates the proof of the approximation result.

Theorem 6.3. *Denote by $\widehat{f}_k$ be the mild solution of the SPDE*

$$d\widehat{f}_k(t) = (\partial_x \widehat{f}_k(t) + b_k(t, \widehat{f}_k(t)))\, dt + \psi_k(t, \widehat{f}_k(t-))\, dL(t), \quad \widehat{f}_k(0) = \Lambda_k f_0, t \geq 0 \,.$$

Then, $\widehat{f}_k \in H_\alpha^{T,k}$ is a strong solution, and we have

$$\mathbb{E}\left[\sup_{t\in[0,T], x\in[0,T-t]} |\hat{f}_k(t,x) - f(t,x)|^2\right] \to 0$$

for $k \to \infty$.

Proof. First we note that a unique mild solution $\widehat{f}_k$ with càdlàg paths to the SPDE exists due to Tappe [34, Theorem 4.5] and for the same reason there is a unique mild càdlàg solution f_k to the SPDE

$$df_k(t) = (\partial_x \widehat{f}_k(t) + b_k(t, \widehat{f}(t)))\, dt + \psi_k(t, \widehat{f}(t-))\, dL(t), \quad \widehat{f}_k(0) = \Lambda_k f_0, t \geq 0 \,.$$

where we like to note that we insert the original process f into the drift function b_k and the diffusion function ψ_k. Define

$$V_k(h)(t) := \mathcal{U}_t f_k(0) + \int_0^t \mathcal{U}_{t-s}(b_k(s, h(s))\, ds + \psi_k(s, h(s-))\, dL(s)) \,,$$

for any $k \in \mathbb{N}$, $t \geq 0$ and any adapted càdlàg process h in H_α with $\mathbb{E}[\int_0^t \|h(s)\|_\alpha^2\, ds] < \infty$. We have $\widehat{f}_k = V_k(\widehat{f}_k)$ and $f_k = V(f_\Pi)$. By Lemma 6.2, it holds

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \|f_\Pi(t) - \hat{f}_k(t)\|_\alpha^2\right] \leq \frac{\pi^2}{6} \exp(4C_t) \mathbb{E}\left[\sup_{0\leq s\leq t} \|f_k(s) - f_\Pi(s)\|_\alpha^2\right] \,,$$

for any $k \in \mathbb{N}$, $t \geq 0$ and C_t given in the lemma (recall from Section 2 that the operator norm of the shift semigroup $\mathcal{U}_t$ is uniformly bounded by the constant $C_{\mathcal{U}}$). By the definition of f_k and f_Π we find

$$\|f_k(s) - f_\Pi(s)\|_\alpha^2 \leq 3\|f_k(0) - f_\Pi(0)\|_\alpha^2 + 3\|\int_0^s \mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\,dr\|_\alpha^2 + 3\|\int_0^s \mathcal{U}_{s-r}(\psi_k(r, f_\Pi(r-)) - \psi_\Pi(r, f_\Pi(r-)))\,dL(r)\|_\alpha^2 .$$

Consider the first term on the right-hand side of the inequality. By the norm inequality for Bochner integrals, Cauchy-Schwartz' inequality and boundedness of the operator norm of $\mathcal{U}_t$ we find (for $s \leq t$)

$$\begin{aligned}
\|\int_0^s &\mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\,dr\|_\alpha^2 \\
&\leq \left(\int_0^s \|\mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\|_\alpha\,dr\right)^2 \\
&\leq t\int_0^t \|\mathcal{U}_{s-r}(b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r)))\|_\alpha^2\,dr \\
&\leq tC_{\mathcal{U}}^2\int_0^t \|b_k(r, f_\Pi(r)) - b_\Pi(r, f_\Pi(r))\|_\alpha^2\,dr \\
&\leq tC_{\mathcal{U}}^2\int_0^t \|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2\,dr
\end{aligned}$$

Here, $\mathcal{I}$ denotes the identity operator on H_α^T. Hence, using Lemma 6.1 and the fact that $\{\mathcal{U}_t\}_{t\geq 0}$ is pseudo-contractive,

$$\begin{aligned}
\mathbb{E}&\left[\sup_{0\leq s\leq t}\|f_k(s) - f_\Pi(s)\|_\alpha^2\right] \\
&\leq 3\|f_k(0) - f_\Pi(0)\|_\alpha^2 + 3tC_{\mathcal{U}}^2\int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2\right]\,dr \\
&\qquad + 3\mathbb{E}\left[\sup_{0\leq s\leq t}\|\int_0^s \mathcal{U}_{s-r}(\psi_k(r, f_\Pi(r-)) - \psi_\Pi(r, f_\Pi(r-)))\,dL(r)\|_\alpha^2\right] \\
&\leq 3\|f_k(0) - f_\Pi(0)\|_\alpha^2 + 3tC_{\mathcal{U}}^2\int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2\right]\,dr \\
&\qquad + 12c_t^2\int_0^t \mathbb{E}\left[\|(\psi_k(r, f_\Pi(r)) - \psi_\Pi(r, f_\Pi(r)))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right]\,dr \\
&\leq 3\|f_k(0) - f_\Pi(0)\|_\alpha^2 + 3tC_{\mathcal{U}}^2\int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2\right]\,dr \\
&\qquad + 12c_t^2\int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})\psi_\Pi(r, f_\Pi(r))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right]\,dr .
\end{aligned}$$

Denote by

$$K_t(k) := 3\|f_k(0) - f_\Pi(0)\|_\alpha^2 + 3tC_{\mathcal{U}}^2 \int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})b_\Pi(r, f_\Pi(r))\|_\alpha^2\right] dr$$
$$+ 12c_t^2 \int_0^t \mathbb{E}\left[\|(\Pi_k - \mathcal{I})\psi_\Pi(r, f_\Pi(r))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\right] dr\,,$$

for $k \in \mathbb{N}$. By standard norm inequalities and the fact that $\mathcal{Q}$ is trace class, we have

$$K_t(k) \le 6(1 + \|\Pi_k\|_{\mathrm{op}}^2)\|f_\Pi(0)\|_\alpha^2 + 6tC_{\mathcal{U}}^2(1 + \|\Pi_k\|_{\mathrm{op}}^2) \int_0^t \mathbb{E}\left[\|b_\Pi(r, f_\Pi(r))\|_\alpha^2\right] dr$$
$$+ 24c_t^2(1 + \|\Pi_k\|_{\mathrm{op}}^2)\|\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2 \int_0^t \mathbb{E}\left[\|\psi_\Pi(r, f_\Pi(r))\|_{\mathrm{op}}^2\right] dr\,,$$

which is seen to be bounded uniformly in $k \in \mathbb{N}$ from Proposition 4.6. Hence, we have $K_t(k) \to 0$ for $k \to \infty$ and any $t \ge 0$ by the dominated convergence theorem because $(\Pi_k - \mathcal{I})h \to 0$ for $k \to \infty$ and any $h \in H_\alpha^T$. Thus, we find

$$\mathbb{E}\left[\sup_{0\le t\le T} \|f_\Pi(t) - \hat{f}_k(t)\|_\alpha^2\right] \to 0\,,$$

for $k \to \infty$. We have $f_\Pi(t,x) = f(t,x)$ for any $t \in [0,T]$, $x \in [0, T-t]$ and from Lemma 3.2 in Benth and Krühner [3] the sup-norm is dominated by the H_α-norm, and therefore we have

$$\mathbb{E}\left[\sup_{t\in[0,T],x\in[T-t]} |\hat{f}_k(t,x) - f(t,x)|^2\right] \le c\mathbb{E}\left[\sup_{0\le t\le T} \|\hat{f}_k(t) - f_\Pi(t)\|_\alpha^2\right] \to 0\,,$$

for $k \to \infty$ where $c = \sqrt{1 + 1/\alpha}$. □

The philosophy in Theorem 6.3 is to take $f(t)$ as the actual forward curve dynamics, and study finite dimensional approximations $\widehat{f}_k(t)$ of it. By construction, $\widehat{f}_k$ solves a HJMM dynamics which yields that the approximating forward curves become arbitrage-free. From the main theorem, the approximations $\widehat{f}_k(t)$ converge uniformly to $f(t)$ for $x \in [0, T-t]$. As time t progresses, the times to maturity $x \ge 0$ for which we obtain convergence shrink. The reason is that information of f is transported to the left in the dynamics of the SPDE. We recall that the approximation of f is constructed by first localizing f to $x \in [0,T]$ for a fixed time horizon T by the projection operator Π down to H_α^T, and next creating finite-dimensional approximations of this.

Alternatively, we may use $f_\Pi(t)$ as our forward price model. Then, the finite dimensional approximation $f_k(t)$ will converge uniformly over all $x \in [0,T]$. In practice, there will be a time horizon for the futures market for which we have no information. For example, in liberalized power markets like NordPool and EEX, there are no futures contracts traded with settlement beyond 6 years. Hence, it is a delicate task to model the dynamics of the futures price curve beyond this horizon. The alternative is then clearly to restrict the modelling perspective to the dynamics with the maturities confined in $x \in [0,T]$. Indeed, in such a context the structural conditions (22) and (23) will be trivially satisfied as we restrict our model parameters in any case to the behaviour on $x \in [0,T]$.

We end our paper with a short discussion on a possible numerical implementation of $\widehat{f}_k(t)$, the finite-dimensional approximation of $f(t)$. Since $\widehat{f}_k(t) \in H_\alpha^{T,k}$, we can express it as

$$\widehat{f}_k(t) = \widehat{f}_{k,*}(t) + \sum_{n=-k}^{k} g_n \widehat{f}_{k,n}(t)\,,$$

where $\widehat{f}_{k,*}(t) = \widehat{f}_k(t,0)g_*$ and $\widehat{f}_{k,n}(t) = \langle \widehat{f}_k(t), g_n^* \rangle_\alpha$ are $\mathbb{C}$-valued functions. For any $h \in H_\alpha^{T,k}$ it follows that $b_k(t,h) \in H_\alpha^{T,k}$. Define for $n = -k, \ldots, k$ the functions

$$\overline{b}_{k,n} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} \to \mathbb{C}\,; \qquad (t, x_*, x_{-k}, \ldots, x_k) \mapsto \left\langle b_k(t, x_* g_* + \sum_{j=-k}^{k} x_j g_j), g_n^* \right\rangle_\alpha\,,$$

$$\overline{b}_{k,*} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} \to \mathbb{C}\,; \qquad (t, x_*, x_{-k}, \ldots, x_k) \mapsto \left\langle b_*(t, x_* g_* + \sum_{j=-k}^{k} x_j g_j), g_n^* \right\rangle_\alpha\,.$$

Furthermore, $\psi_k(t,h) \in L_{\mathrm{HS}}(H_\alpha, H_\alpha^{T,k})$. Thus, for any $g \in H_\alpha$ we have that $\psi_k(t,h)(g) \in H_\alpha^{T,k}$. We define the mappings

$$\overline{\psi}_{k,n} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} \to H_\alpha^*; (t, x_*, x_{-k}, \ldots, x_k) \mapsto \left\langle \psi_k(t, x_* g_* + \sum_{j=-k}^{k} x_j g_j)(\cdot), g_n^* \right\rangle_\alpha$$

$$\overline{\psi}_{k,*} : \mathbb{R}_+ \times \mathbb{C}^{2k+2} \to H_\alpha^*; (t, x_*, x_{-k}, \ldots, x_k) \mapsto \left\langle \psi_*(t, x_* g_* + \sum_{j=-k}^{k} x_j g_j)(\cdot), g_n^* \right\rangle_\alpha$$

for $n = -k, \ldots, k$. Now, since $\partial_x g_* = 0$ and $\partial_x g_n = \lambda_n g_n + g_*/\sqrt{T}$, we find from the SPDE of $\widehat{f}_k$ the following $2k+2$ system of stochastic differential equations (after comparing terms with respect to the Riesz basis functions),

$$\begin{aligned}
d\widehat{f}_{k,*}(t) &= \left(\frac{1}{\sqrt{T}} \sum_{n=-k}^{k} \widehat{f}_{k,n}(t) + \overline{b}_{k,*}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \ldots, \widehat{f}_{k,k}(t)) \right) dt \\
&\qquad + \overline{\psi}_{k,*}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \ldots, \widehat{f}_{k,k}(t-))(dL(t)) \\
d\widehat{f}_{k,-k}(t) &= \left(\lambda_{-k} \widehat{f}_{k,-k}(t) + \overline{b}_{k,-k}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \ldots, \widehat{f}_{k,k}(t)) \right) dt \\
&\qquad + \overline{\psi}_{k,-k}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \ldots, \widehat{f}_{k,k}(t-))(dL(t)) \\
&\cdots\cdot \\
&\cdots\cdot \\
d\widehat{f}_{k,k}(t) &= \left(\lambda_{k} \widehat{f}_{k,k}(t) + \overline{b}_{k,k}(t, \widehat{f}_{k,*}(t), \widehat{f}_{k,-k}(t), \ldots, \widehat{f}_{k,k}(t)) \right) dt \\
&\qquad + \overline{\psi}_{k,k}(t, \widehat{f}_{k,*}(t-), \widehat{f}_{k,-k}(t-), \ldots, \widehat{f}_{k,k}(t-))(dL(t))
\end{aligned}$$

In a matrix notation, defining $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_{2k+2}(t))'$ and

$$A = \begin{bmatrix} \frac{1}{\sqrt{T}} & \frac{1}{\sqrt{T}} & \frac{1}{\sqrt{T}} & \cdots & \frac{1}{\sqrt{T}} \\ 0 & \lambda_{-k} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_{-k+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix},$$

we have the dynamics

$$d\mathbf{x}(t) = (A\mathbf{x}(t) + \overline{\mathbf{b}}_k(t, \mathbf{x}(t)))\,dt + \overline{\psi}_k(t, \mathbf{x}(t-))(dL(t))\,,$$

with $\widehat{f}_{k,*} = x_1, \widehat{f}_{k,-k} = x_2, \dots, \widehat{f}_{k,k} = x_{2k+2}$. Using for example an Euler approximation, we can derive an iterative numerical scheme for this stochastic differential equation in $\mathbb{C}^{2k+2}$. We refer to Platen and Bruti-Liberati [29] for a detailed analysis of numerical solution of stochastic differential equations driven by jump processes.

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