EXTENSIONS TO THE BOUNDARY OF RIEMANN MAPS ON VARYING DOMAINS IN THE COMPLEX PLANE

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Abstract. We give a short proof of the convergence to the boundary of Riemann maps on varying domains. Our proof provides a uniform approach to several ad-hoc constructions that have recently appeared in the literature.

1. Introduction

In the following $\triangle$ denotes the unit disk in $\mathbb{C}$, and $\triangle_r(z_0)$ denotes the disk of radius r centred at the point z_0.

Theorem 1.1. *Let $\Gamma \subset \partial\triangle$ be an open interval, let $U \subset \mathbb{C}$ be an open neighbourhood of $\overline{\Gamma}$, and suppose that $\{D_j\}_{j=1}^{\infty}$ is a sequence of simply connected domains containing $\triangle$, each bounded by a closed Jordan curve and contained in $\triangle_R$ for some fixed $R > 0$, and assume that $D_j \cap U = \triangle \cap U$ for all j. Assume further that $D_j \to \triangle$ with respect to kernel convergence, and for each j let $f_j : D_j \to \triangle$ be the Riemann map with $f_j(0) = 0$ and $f_j'(0) > 0$. Then $f_j \to \mathrm{id}$ uniformly on compact subsets of $\triangle \cup \Gamma$.*

The definition of kernel convergence will be recalled in section 3.

A special case of Theorem 1.1 appeared in [5] as an ingredient in the proof of the fact that any bordered Riemann surface in $\mathbb{C}^2$ embeds properly into $\mathbb{C}^2$. It appeared again later in [2] and [1] as ingredients in techniques for globally exposing points on certain Stein compacts in $\mathbb{C}^n$ and in Stein spaces, and recently yet another improvement appeared in [4], as an ingredient in the construction of certain exotic proper embeddings of the unit disk into $\mathbb{B}^2$. However, in all cases, the sequences D_j are ad hoc constructions intended for very specific purposes, and all of them are obtained by adding to D certain shrinking tubes around special arcs attached to D. The proofs in these special cases do not adapt to the general setting of Theorem 1.1.

Remark 1.2. This result holds in much more generality. For instance, if $\triangle$ in the theorem is replaced by any simply connected Jordan domain D, and if $\Gamma \subset \partial D$ is a Jordan arc, elementary arguments involving Riemann mappings reduce the situation to the setting of the theorem. Inspecting the proof below also makes it clear that one may consider much more general domains, whose boundaries only contain a common Jordan arc, and without the assumption imposed by U, but one might have to consider one sided convergence. One may also prove similar results for domains D_j which are not simply connected, but which are all conformally equivalent to a given

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domain D. By introducing additional arguments one can get convergence in C^k-norm, granted that the arc Γ (in the general case) is of class $C^{k,\alpha}$ for some $\alpha > 0$.

2. Lindelöfs maximum principle

We recall with a proof the following result (see e.g. Goluzin [6], page 33).

Lemma 2.1. *(Lindelöf) Let $D \subset \mathbb{C}$ be a domain, let $z_0 \in D$, and suppose for $r > 0$ and $m \in \mathbb{N}$, there is an interval $I \subset \partial\triangle_r(z_0)$ of length at least $\frac{2\pi r}{m}$, such that $I \subset \mathbb{C} \setminus \overline{D}$. Then for any $f \in \mathcal{O}(D)$ with $|f(z)| \leq M$ for all $z \in D$, satisfying*

$$\limsup_{(\triangle_r(z_0)\cap D)\ni z\to\partial D} |f(z)| < \epsilon,$$

we have that $|f(z_0)| < (\epsilon M^{m-1})^{1/m}$.

Proof. Without loss of generality we may assume that $z_0 = 0$. Consider the domain

$$\tilde{D} := \bigcap_{k=0}^{m-1} (e^{2\pi i k/m} \cdot D), \tag{2.1}$$

the intersection of m rotated copies of D. Then $\tilde{D}$ is invariant under rotation by the angle $2\pi/m$, and its connected component containing 0, is strictly contained in $\triangle_r(0)$. Define the function $\tilde{f}$ on $\tilde{D}$ by

$$\tilde{f}(z) := \prod_{k=0}^{m-1} f(e^{2\pi i k/m} \cdot z). \tag{2.2}$$

Then we have that

$$\limsup_{\tilde{D}\ni z\to\partial\tilde{D}} |f(z)| \leq M^{m-1} \cdot \epsilon, \tag{2.3}$$

and so by the maximum principle, $|\tilde{f}(0)| = |f(0)|^m \leq M^{m-1} \cdot \epsilon$. □

3. Proof of Theorem 1.1

Before we embark on the proof, we recall the definition of *kernel convergence*: if $\{D_j\}_{j\in\mathbb{N}}$ is a sequence of domains (connected open sets) such that $\triangle_\delta(0) \subset D_j$ for all j, for some fixed $\delta > 0$, we define the kernel of the sequence $\{D_j\}$ to be the largest domain D with the property that any compact set $K \subset D$ is contained in D_j for all $j \in \mathbb{N}$ sufficiently large. We say that D_j converges to its kernel, if any subsequence of $\{D_j\}$ has the same kernel D. In the setting of Theorem 1.1, it is known that $f_j \to \mathrm{id}$ uniformly on compact subsets of $\triangle$ (see [6], Theorem 1, page 55). It is also known that each f_j extends to a homeomorphism between the closures of the domains (see [6], Theorem 4, page 44).

Now as the first step towards proving Theorem 1.1 we show that for any compact subset $I \subset \Gamma$ the family $f_j|_I$ is equicontinuous. If not, then by passing to a subsequence we may assume that there is a sequence of shrinking intervals $\gamma_j \subset \Gamma$ with end points $a_j, b_j \to e^{i\theta}$ for which the length of the intervals $f_j(\gamma_j) \subset \partial\triangle$ is bounded away from zero. After composing with rotations, we may therefore assume that $f_j(\gamma_j) \supset \{e^{i\alpha} : -\delta < \alpha < \delta\} =: \eta$ for some $\delta > 0$.

We next consider the sequence of inverse maps $g_j = f_j^{-1}$. Then since $g_j(\eta) \subset \gamma_j$, we have that $g_j|_\eta \to e^{i\theta}$ uniformly as $j \to \infty$. Hence given $\epsilon > 0$ it follows that for large enough j we have $|g_j(w) - e^{i\theta}| < \epsilon$ for all $w \in \eta$. Choose $r > 0$ and $0 < x_0 < 1$ such that for all $x_0 < x < 1$ we have that $\triangle_r(x) \cap \partial\triangle \subset \eta$. After increasing x_0 if necessary we may also assume that the length of $\partial\triangle_r(x) \setminus \triangle$ is greater than πr for all $x_0 < x < 1$. Hence as points in $(\triangle_r(x) \cap \triangle)$ converge to $\partial\Delta$, they necessarily converge to η, and it follows that

$$\limsup_{(\triangle_r(x)\cap\triangle)\ni z\to\partial\triangle} |g_j(z) - e^{i\theta}| < \epsilon.$$

Therefore by Lemma 2.1, we have that $|g_j(x) - e^{i\theta}| < (\epsilon R)^{1/2}$ for all $x_0 < x < 1$ and for all j sufficiently large. Since this holds for any $\epsilon > 0$ we have that $g_j(x) \to e^{i\theta}$ uniformly as $j \to \infty$, for all $x_0 < x < 1$. But as remarked above, the kernel convergence implies that the maps g_j (up to the compositions with rotations used above) converge uniformly on compact sets in $\triangle$ to the identity map, leading to a contradiction.

Hence after passing to a subsequence we may assume that $f_j|_I$ is a Cauchy sequence. So for any $\epsilon > 0$ we have that $|f_k(w) - f_l(w)| < \epsilon$ for all $w \in I$ and for all sufficiently large $k, l \in \mathbb{N}$. We can find an open set $U \subset \Delta$ with $\overline{U} \cap \partial\Delta = I$, such that for all $z \in U$ there exists $r = r(z) > 0$ such that the disk $\Delta_r(z)$ intersects $\partial\Delta$ in K and its boundary $\partial\Delta_r(z)$ contains an interval of length πr lying outside of Δ. By applying Lemma 2.1 to the functions $f_k - f_l$, it follows that $|f_k(z) - f_l(z)| < \epsilon^{1/2}$ for all $z \in U$, hence f_j is a Cauchy sequence on an extension of K into $\triangle$. The theorem now follows from the fact that $f_j \to \mathrm{id}$ uniformly on compact sets in $\triangle$. □

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