EXPOSING BOUNDARY POINTS OF STRONGLY PSEUDOCONVEX SUBVARIETIES IN COMPLEX SPACES

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Abstract. We prove that all locally exposable points in a Stein compact in a complex space can be exposed along a given curve to a given real hypersurface. Moreover, the exposing map for a boundary point can be sufficiently close to the identity map outside any fixed neighborhood of the point. We also prove a parametric version of this result for bounded strongly pseudoconvex domains in $\mathbb{C}^n$. For a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ and a given boundary point of it, we prove that there is a global coordinate change on the closure of the domain which is arbitrarily close to the identity map with respect to the C^1-norm and maps the boundary point to a strongly convex boundary point.

1. Introduction

Let X be a complex space. We assume throughout this paper that all complex spaces are reduced, irreducible, and paracompact. Let X_{sing} be the set of singular points of X and let $X_{\text{reg}} = X\backslash X_{\text{sing}}$ be the set of smooth points of X.

Definition 1.1. Let X be a complex space, let $K \subset X$ be a compact set, and let $\zeta \in K$ be a point in X_{reg}. We will say that ζ is *locally exposable* if there exists an open set $U \subset X$ containing ζ and ρ a C^2-smooth strictly plurisubharmonic function on U such that

(i) $\rho(\zeta) = 0$ and $\mathrm{d}\rho(\zeta) \neq 0$, and
(ii) $\rho < 0$ on $(K \cap U) \setminus \{\zeta\}$.

Our main concern in this paper is to show that locally exposable points are globally exposable (see Definition 1.2).

The first result of the present paper is the following.

Theorem 1.1. *Let X be a complex space, let $K \subset X$ be a Stein compact, and let $\zeta \in K \cap X_{\text{reg}}$ be locally exposable. Let $H \subset X\backslash(K \cup X_{\text{sing}})$ be a locally closed C^2-smooth subset of a real hypersurface in X. Let $\gamma : [0,1] \to X_{\text{reg}}$ be a smoothly embedded curve in X with $\gamma(0) = \zeta$, $\gamma(1) \in H$, and $\gamma(t) \in X\backslash(K \cup H)$ for $t \in (0,1)$.*

Then for any (small) neighborhood V of γ, and any $\epsilon > 0$, there exist an open neighborhood U of K, an arbitrarily small neighborhood $V' \subset V$ of ζ, and a biholomorphic map $f : U \to f(U) \subset X$ such that the following holds:

(a) *$f(V') \subset V$ and $f(\zeta) = \gamma(1)$,*
(b) *$\operatorname{dist}(f(z), z) < \epsilon$ for $z \in K\backslash V'$, and*
(c) *$f(K) \cap H = \{\gamma(1)\}$.*

where $dist(\cdot,\cdot)$ is a fixed distance on X. In the case $X = \mathbb{C}^n$ and $K \subset \mathbb{C}^n$ is polynomially convex, f can be taken to be a holomorphic automorphism of $\mathbb{C}^n$.

By saying that K is a Stein compact we mean that K has a Stein neighbourhood basis in which K is holomorphically convex.

Definition 1.2. The map f will be said to *expose* the point ζ with respect to H, and $f(\zeta)$ is said to be exposed. If $K \subset B_R(0) \subset \mathbb{C}^n$ and $H = \{z \in \mathbb{C}^n : \|z\| = R\}$, for any $R > 0$, we will say that $f(\zeta)$ is globally exposed.

From the proof, we will see that Theorem 1.1 can be generalized in various directions as follows:

- one can easily expose finitely many points simultaneously,
- the last statement can be generalized as: if X is a Stein manifold and has density property (for definition see [12]) and K is $\mathcal{O}(X)$-convex, then f can be taken in $\mathrm{Aut}_{\mathrm{hol}}X$,
- if X is a 1-convex space, the same statement still holds if we assume that ζ is outside the exceptional set. This can be proved by Remmert's reduction and interpolation for the exposing maps constructed in Theorem 1.1. (Exposing boundary points in this setting was proposed by Franc Forstnerič in [6].)

The importance of Theorem 1.1 in the case that K is the closure of a domain with a strictly pseudoconvex boundary point ζ, is that it tells us that K° resembles a strictly convex domain in a concrete geometric sense. Convex domains are exceptionally well behaved from a complex analysis point of view. As a comparison, the Levi problem was to show that such a domain K° resembles a strictly convex domain in a much weaker function theoretic sense.

Our next result concerns in which ways a locally exposable point can be locally exposed with global maps. It is quite simple using Andersén-Lempert theory to show that a locally defined exposing map in $\mathbb{C}^n$ can be approximated by holomorphic automorphisms, but as a consequence one looses completely control of the behaviour on most of K.

Theorem 1.2. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary. Then for any $\zeta \in \partial D$ and any $\epsilon > 0$, there is an injective holomorphic map $F : \overline{D} \to \mathbb{C}^n$ such that $F(\zeta) = \zeta$ is a strictly convex boundary point of $F(D)$, $\partial F(D)$ and ∂D are tangent at ζ, and $||F - \mathrm{id}||_{C^1(\overline{D})} < \epsilon$.*

In §2 we will construct a strongly pseudoconvex domain such that the map F in Theorem 1.2 can not be taken in $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ for some boundary points.

Furthermore we are interested in parametric version of the previous two theorems.

Theorem 1.3. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with smooth boundary. Let $H \subset \mathbb{C}^n$ be a smooth closed real hypersurface with $\overline{D} \cap H = \emptyset$. Let $\gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n$ be a fiber-preserving continuous map such that for all $\zeta \in \partial D$:*

1) $\gamma_\zeta = \gamma(\zeta, \cdot) : I \to \mathbb{C}^n$ *is a smooth embedding;*
2) $\gamma_\zeta(0) = \zeta, \gamma_\zeta(1) \in H$ *and* $\gamma_\zeta|_{(0,1)} \subset \mathbb{C}^n \backslash (\overline{D} \cup H)$.

(We implicitly make the identification of $\{\zeta\} \times \mathbb{C}^n$ with $\mathbb{C}^n$.) Then for any neighborhood V of $\gamma(\partial D \times [0,1])$ and any $\epsilon > 0$, for any sufficiently small neighborhood $V' \subset V$ of $\{(\zeta, \zeta); \zeta \in \partial D\}$ in $\partial D \times \mathbb{C}^n$ there exists a smooth fiber-preserving map $f : \partial D \times \overline{D} \to \partial D \times \mathbb{C}^n$ such that the following holds for each $\zeta \in \partial D$:

(a) $f_\zeta := f(\zeta, \cdot) : \overline{D} \to \mathbb{C}^n$ *is injective and holomorphic,*
(b) $f_\zeta(V'_\zeta) \subset V_\zeta$ *and* $f_\zeta(\zeta) = \gamma_\zeta(1)$,
(c) $||f_\zeta(z) - z|| < \epsilon$ *for* $z \in \overline{D} \backslash V'_\zeta$,

(d) $f_\zeta(\overline{D}) \cap H = \{\gamma_\zeta(1)\}$,

where $V_\zeta := V \cap (\{\zeta\} \times \mathbb{C}^n)$. *If in addition* $\overline{D}$ *is polynomially convex, we can take* f *a smooth map from* $\partial D \times \mathbb{C}^n$ *to itself such that* $f|_{\mathbb{C}^n_\zeta} \in Aut(\mathbb{C}^n)$ *for all* $\zeta \in \partial D$.

Theorem 1.4. *Let* $D \subset\subset \mathbb{C}^n$ *be a strongly pseudoconvex domain with* C^2*-smooth boundary. For any* $\epsilon > 0$, *there is a continuous map* $f : \partial D \times \overline{D} \to \mathbb{C}^n$ *such that for all* $\zeta \in \partial D$ *the following hold:*

1) $f_\zeta = f(\zeta, \cdot) : \overline{D} \to \mathbb{C}^n$ *is a holomorphic injective map,*
2) $f_\zeta(\zeta) = \zeta$ *is a strictly convex boundary point of* $f_\zeta(D)$,
3) $\partial f_\zeta(D)$ *and* ∂D *are tangential at* ζ, *and*
4) $||f_\zeta - \mathrm{id}||_{C^1(\overline{D})} < \epsilon$.

For a further discussion of parametric exposing of points, see Section 5.

2. Transforming a boundary point to a strongly convex one

In this section, we consider transforming a boundary point of a strongly pseudoconvex domain to a strictly convex one, with certain control of the behavior of the involved transformation. The aim is to prove Theorem 1.2 and Theorem 1.4. We also construct a strongly pseudoconvex domain in $\mathbb{C}^2$ for which the transformation furnished by Theorem 1.2 can not be taken in $Aut(\mathbb{C}^2)$.

We need the following lemmas. The first lemma is to prove the existence of peak functions with certain estimates. The key tool is the existence of embedding of strongly pseudoconvex domains into strongly convex domains with certain boundary conditions, established by the second author in [4].

Lemma 2.1. *Let* D *be a bounded strictly pseudoconvex domain in* $\mathbb{C}^n$ *with* C^2*-smooth boundary. Then for any* $\zeta \in \partial D$, *there is a holomorphic function* f *defined on some neighborhood of* $\overline{D}$ *which satisfies the following two conditions:*
1) $f(\zeta) = 1$, *and* $|f(z)| < 1$ *for* $z \in \overline{D} \backslash \{\zeta\}$;
2) the estimate

$$|f(z)| \leq e^{-c||z-\zeta||^2}$$

holds on $\overline{D}$ *for some constant* $c > 0$.

Proof. For the case that D is the ball $B := \{z \in \mathbb{C}^n; |z_1+r|^2+|z_2|^2+\cdots+|z_n|^2 < r^2\}$ and $\zeta = 0$, a simple calculation shows that $f(z) := e^{z_1}$ satisfies the conditions. By the result in [4] mentioned above, the general case can be reduced to the case that $D = B$. □

For the proof of Theorem 1.4, we need a parametric version of Lemma 2.1.

Lemma 2.2. *Let* $D \subset \mathbb{C}^n$ *be a bounded strongly pseudoconvex domain with smooth boundary. Then there is a smooth map* $p : \partial D \times \overline{D} \to \mathbb{C}$ *such that:*
1) For each $\zeta \in \partial D$, $p_\zeta(\cdot) := p(\zeta, \cdot)$ *is a holomorphic function on* $\overline{D}$;
2) $p_\zeta(\zeta) = 1$ *and* $|p_\zeta(z)| < 1$ *for* $z \in \overline{D} \backslash \{\zeta\}$ *for all* $\zeta \in \partial D$;
3) there exists a constant c *such that* $|p_\zeta(z)| \leq e^{-c||z-\zeta||^2}$ *for* $(\zeta, z) \in \partial D \times \overline{D}$.

Proof. By a result in [4], there is a proper holomorphic embedding σ from some neighborhood of $\overline{D}$ into some neighborhood of a bounded strongly convex domain $W \subset \mathbb{C}^N$ for some N such that $\sigma(\partial D) \subset \partial W$ and $\sigma(\overline{D})$ and ∂W intersect transversely. Let ρ be a defining function for W, for each $\xi \in \partial W$ set

$\Lambda_\xi(w) := \sum_{j=1}^N \partial\rho/\partial w_j(\xi)\cdot(w_j-\xi_j)$, and set $g_\xi(w) = e^{\Lambda_\xi(w)}$. Then g satisfies 1-3 for D replaced by W. So we may set $p(\zeta,z) = g_{\sigma(\zeta)}(\sigma(z))$. □

The last lemma we need is the following

Lemma 2.3. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary. Let F be a diffeomorphism from some neighborhood of $\overline{D}$ to its image in $\mathbb{C}^n$. Assume that $\{F_j\}(j\geq 1)$ is a sequence of smooth maps defined on some neighborhood of $\overline{D}$ such that $||F_j - F||$ converges uniformly on $\overline{D}$ to 0 in C^1-norm. Then F_j is injective on $\overline{D}$ for j large enough.*

Proof. We prove by contradiction. If it were not the case, then, without loss of generality, we can assume that for each j there exist a_j and b_j in $\overline{D}$ with $a_j\neq b_j$ such that $F_j(a_j) = F_j(b_j)$. We may assume $a_j\to a$ and $b_j\to b$ as $j\to\infty$. Then it is necessary that $a=b$ since F is injective. We have $a\in D$ or $a\in\partial D$. Since ∂D is C^2, taking a C^2-smooth coordinate change near a if necessary, we may assume that there is a neighborhood U of a such that $U\cap\overline{D}$ is convex. Let γ_j be the line segment in $\mathbb{C}^n$ given by $t\to(1-t)a_j+tb_j,\ t\in[0,1]$, then $\gamma_j\subset\overline{D}$ for j large enough. We have

$$\begin{aligned} F_j(b_j)-F_j(a_j) &= \int_0^1 \frac{dF_j(\gamma_j(t))}{dt}dt \\ &= (\int_0^1 dF_j(\gamma_j(t))dt)(b_j-a_j), \end{aligned} \tag{1}$$

which can not be 0 for j large enough since $\int_0^1 dF_j(\gamma_j(t))dt\to dF(a)$ and $dF(a)$ is nonsingular. Contradiction. □

Proof of Theorem 1.2: We can assume that $\zeta=0$ and the local defining function $\rho(z)$ of D near 0 can be expanded as

$$\rho(z) = Re(z_n+Q(z)) + \sum_{i,j} a_{ij}z_i\bar{z}_j + \cdots,$$

where $Q(z)=\sum_{i,j} q_{ij}z_iz_j$ is a symmetric quadratic form. Let f be a holomorphic function defined in some neighborhood of $\overline{D}$ that is furnished by Lemma 2.1. We will show that there are positive integers M and $N_j, j=1,\cdots,M$ such that the transformation $F(z)=(\tilde{z}_1,\cdots,\tilde{z}_n)$ given by

$$\begin{cases} \tilde{z}_k = z_k, \text{ for } k=1,\cdots,n-1, \\ \tilde{z}_n = z_n + \sum_{j=1}^M \frac{1}{M}Qf^{N_j} \end{cases} \tag{2}$$

is that expected by this theorem.

First note that

$$||Q'f^N|| \lesssim ||z||e^{-Nc||z||^2},$$

and hence goes to zero uniformly on $\overline{D}$ as $N\to\infty$. Note also that

$$||QNf'f^{N-1}|| \lesssim N||z||^2e^{-Nc||z||^2}. \tag{3}$$

Let σ be the function on $(0,+\infty)$ given by $x\mapsto xe^{-cx}$. It is clear that $\sigma(0)=0$ and $\lim_{x\to+\infty}\sigma(x)=0$, and σ attains its maximum $\frac{1}{ce}$ at $x=1/c$. So the right hand side of (3) attains its maximum $\frac{1}{ce}$ at $||z||^2=\frac{1}{Nc}$. For any $\epsilon>0$ and a sufficiently large integer M with $\frac{1}{Mec}<\epsilon/2$, by the above discussion, we can find $r_1>r_2>\cdots>r_M>0$ and positive numbers $N_1,\cdots,N_M$ such that $||(Qf^{N_j})'||<$

$\frac{\epsilon}{2M}$ on $\{z \in \overline{D}; ||z|| \geq r_j \text{ or } ||z|| \leq r_{j+1}\}$. Define $\varphi_M = \sum_{j=1}^{M} \frac{1}{M} Q f^{N_j}$, then φ_M is a holomorphic function defined on some neighborhood of $\overline{D}$, and $||\varphi_M'|| < \epsilon$ on $\overline{D}$. By taking N_j large enough, we can also require that $|\varphi_M(z)| < \epsilon$ for $z \in \overline{D}$. Let F_M be the map defined as in (2). By Lemma 2.3, F_M is injective on $\overline{D}$ for M and N_j large enough. It is clear that $||F_M - \mathrm{id}||_{C^1}$ converges to 0 uniformly on $\overline{D}$, and $F_M(\zeta)$ is a strongly convex boundary point of $F_M(D)$.

□

Proof of Theorem 1.4: For $\zeta \in \partial D$, denote by n_ζ the unit outward-pointing normal vector of ∂D at ζ. Let $L_\zeta = \{xn_\zeta + iyn_\zeta; x, y \in \mathbb{R}\}$ and let $\pi_\zeta : \mathbb{C}^n \to L_\zeta$ be the orthogonal projection. For $z \in \mathbb{C}^n$, let $t_\zeta(z) = x + iy$ if $\pi_\zeta(z - \zeta) = xn_\zeta + iyn_\zeta$ and let $z'_\zeta = (z - \zeta) - \pi_\zeta(z - \zeta)$. Note that $z \mapsto (z'_\zeta, t_\zeta(z))$ is a linear isomorphism form $\mathbb{C}^n$ to $(T^{\mathbb{C}}_\zeta \partial D) \times \mathbb{C}$ which maps ζ to the origin, where $T^{\mathbb{C}}_\zeta \partial D = T_\zeta \partial D \cap iT_\zeta \partial D$ is the complex tangent space of ∂D at ζ. Let ρ be a defining function of D which is strictly plurisubharmonic on some neighborhood of $\overline{D}$. After normalization, near each $\zeta \in \partial D$, ρ can be expanded as follows:

$$\rho(z) = Re(t_\zeta(z)) + Q_\zeta(z - \zeta) + \mathcal{L}_\zeta(z - \zeta) + o(||z - \zeta||^2), \tag{4}$$

where L_ζ is the Levi form of ρ at ζ and Q is the complex Hessian of ρ at ζ.

By some basic results from calculus, the right hand side of (4) can be viewed as a smooth function defined on some neighborhood of $\partial D \times \overline{D}$ in $\partial D \times \mathbb{C}^n$. Let $p : \partial D \times \overline{D} \to \mathbb{C}$ be the smooth map furnished by Lemma 2.2. Then, by the same estimate as in the proof of Theorem 1.2, we can show that there are positive integer M and $N_j, j = 1, \cdots, M$ such that the map $F(\zeta, z) : \partial D \times \overline{D} \to \mathbb{C}^n$ given by

$$\begin{cases} F(\zeta, z)'_\zeta = z'_\zeta \\ t_\zeta(F(\zeta, z)) = t_\zeta(z) + \sum_{j=1}^{M} \frac{1}{M} Q_\zeta(z - \zeta) p_\zeta^{N_j}(z) \end{cases}$$

satisfies the required conditions. □

We now construct a strongly pseudoconvex domain $D \subset \mathbb{C}^2$ such that the map F in Theorem 1.2 can not be taken in $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ for some $p \in \partial D$.

Example 2.1. Let $D := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |\frac{1}{z}|^2 + |w|^2 < 3\}$. Then D is a bounded strongly pseudoconvex domain with smooth boundary. The intersection A of D and the z-axis is an annulus. Let p be an inner boundary point of A. It is clear that there exists a compact set $V \subset D$ such that the polynomial hull of V contains an open neighborhood of p. So if $F_j \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^2)$ is a sequence that converges to id uniformly on $\overline{D}$, then F_j converges to id uniformly on a neighborhood of p. So $F_j(p)$ can not be a strictly convex boundary point of $F_j(D)$ for j large enough.

3. The ball model case in $\mathbb{C}^n$

The main technical problem in proving Theorem 1.1 which differs from the results in [3] is to prove an exposing result for balls, which can later be used to pass from local to global exposing by approximately gluing. For $r \in \mathbb{R}$, we denote the point $(0, \cdots, 0, r)$ in $\mathbb{C}^n$ by p_r. For $r > 0$ and $a \in \mathbb{C}^n$, $B_r(a)$ denotes the ball in $\mathbb{C}^n$ centered at a with radius r, and we denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$ as usual. For $r, s \in \mathbb{R}, r < s$, we let $l_{r,s}$ denote the closed line segment between p_r and p_s. The main aim of this section is to prove the following theorem which is a key step in the proof of Theorem 1.1.

Theorem 3.1. *Let r, s be positive numbers with $s > r + 1$. Then for any open neighborhood V of $l_{1,s-r}$, any open neighborhood U of $l_{1,s-r} \cup \overline{B_r(p_s)}$, and any sufficiently small $\epsilon > 0$, there exits a sequence of maps $\phi_{\nu,t} : \overline{\mathbb{B}^n} \to \mathbb{C}^n$, $(t \in [0,1])$, of injective holomorphic maps such that the following hold:*

(i) *$\phi_{\nu,t}$ are smooth in t and $\phi_{\nu,0} = Id$,*
(ii) *$\phi_{\nu,t} \to Id$ uniformly on $\overline{\mathbb{B}^n} \backslash B_\epsilon(p_1)$ for $t \in [0,1]$, as $\nu \to \infty$,*
(iii) *$\phi_\nu(p_1) = p_{s+r}$ for all $\nu \in \mathbb{N}$, where $\phi_\nu := \phi_{\nu,1}$,*
(iv) *$\phi_\nu(B_\epsilon(p_1) \cap \overline{\mathbb{B}^n}) \subset V \cup B_r(p_s) \cup \{p_{s+r}\}$,*
(v) *$\phi_{\nu,t}(\overline{\mathbb{B}^n} \cap B_\epsilon(p_1)) \subset U$ for all ν and t,*
(vi) *the images $\phi_{\nu,t}(\overline{\mathbb{B}^n})$ are polynomially convex for all ν and t.*

Lemma 3.2. *Let $a < b$ be points on the real line with $b - a > 2$, and let l be the closed line segment between $a + 1$ and $b - 1$. Let $\delta_\nu \to 0$ and let Ω_ν be the δ_ν neighborhood of $\overline{D_1(a)} \cup l \cup \overline{D_1(b)}$, where $D_s(a)$ is the disk in $\mathbb{C}$ centered at a with radius s. There exist injective holomorphic maps $f_\nu, g_\nu : \overline{\mathbb{D}} \to \mathbb{C}$ such that the following holds:*

(i) *$f_\nu(\overline{\mathbb{D}} \cap \mathbb{R}) \subset \mathbb{R}, g_\nu(\overline{\mathbb{D}} \cap \mathbb{R}) \subset \mathbb{R}$, and $f_\nu(\mathbb{D}) = g_\nu(\mathbb{D}) \subset \Omega_\nu$ for all ν,*
(ii) *$f_\nu(-1) = g_\nu(-1) = a - 1, f_\nu(1) = g_\nu(1) = b + 1$ and $f_\nu(0) = a, g_\nu(0) = b$, for all ν,*
(iii) *$f_\nu(z) \to z + a$ uniformly on some fixed neighborhood of $\overline{\mathbb{D}} \backslash D_\epsilon(1)$ for any small $\epsilon > 0$;*
(iv) *$g_\nu(z) \to z + b$ uniformly on some fixed neighborhood of $\overline{\mathbb{D}} \backslash D_\epsilon(-1)$ for any small $\epsilon > 0$;*
(v) *$f_\nu(z) = g_\nu(m_\nu(z))$ for all ν, where*
$$m_\nu(z) = \frac{z - r_\nu}{1 - r_\nu z}$$
with $r_\nu \to 1$ as $\nu \to \infty$.

Proof. For simplicity we assume $a = -2, b = 2$. We will construct maps f_ν that satisfy (i)-(iii) and $f_\nu(\overline{\mathbb{D}})$ is invariant under the transformation $z \mapsto -z$. By symmetry, if we put $g_\nu(z) = -f_\nu(-z)$, then all conditions about f_ν and g_ν in the lemma hold. The convergence of r_ν to 1 in (v) will be guaranteed by the convergence of f_ν as presented in (iv).

We now construct such f_ν. By Mergelyan's Theorem there exists a sequence of embeddings $\phi_\nu : \overline{D_1(-2)} \cup l \cup \overline{D_1(2)} \to \mathbb{C}$ such that $\phi_\nu \to$ id on $\overline{D_1(-2)}$, $\phi_\nu(l)$ shrinks to the point -1, as $\nu \to \infty$, and such that ϕ_ν is are as close as we like to a translation and a scaling on $\overline{D_1(2)}$, such that the diameter of $\phi_\nu(D_1(2))$ shrinks to zero. We may also interpolate to get $\phi_\nu(-2) = -2, \phi'_\nu(-2) = 1$). Replacing ϕ_ν by the map given by $z \mapsto \frac{\phi_\nu(z) + \overline{\phi_\nu(\bar{z})}}{2}$ if necessarily, we can assume that $\phi_\nu(z) = \overline{\phi_\nu(\bar{z})}$, and hence ϕ_ν maps real numbers to real numbers Let $W_\nu = D_1(-2) \cup l(\delta'_\nu) \cup D_1(2)$, where $l(\delta'_\nu)$ is the δ'_ν-neighborhood of l and $0 < \delta'_\nu < \delta_\nu$. If the δ'_ν are chosen small enough, then $\phi_\nu(W_\nu)$ converges to the ball $D_1(-2)$ in the sense of Goluzin.

Let $\psi_\nu : \mathbb{D} \to \phi_\nu(W_\nu)$ be the Riemann map with $\psi_\nu(0) = -2, \psi'_\nu(0) > 0$, we have that $\psi_\nu(z) \to z - 2$ uniformly on $\overline{\mathbb{D}}$ (see [7], Theorem 2, p. 59.). Morover, $\psi_\nu(z) = \overline{\psi_\nu(\bar{z})}$. Then it is clear that $\tilde{f}_\nu := \phi_\nu^{-1} \circ \psi_\nu$ satisfy (i), (ii), and (iii).

□

Lemma 3.3. *Let $0 < r < 1$ and define $\varphi_r(z) = \frac{z-r}{1-rz}$. Then either $|\varphi_r(z)| < |z|$ or $Re(\varphi_r(z)) < 0$ for all $z \in \mathbb{D}$.*

Proof. Note first that any circle $\Gamma_a = \{|z| = a\}$ with $a > 0$ is mapped by φ_r to a circle symmetric with respect to the real line, and $\varphi_r(a) < a$. A straightforward computation shows that $|\varphi_r(a) - \varphi_r(-a)| = 2a\frac{1-r^2}{1-a^2r^2}$ and so the radius of $\varphi_r(\Gamma_a)$ is less than a. Since also $\varphi_r(a) < a$ this implies that $\varphi_r(\Gamma_a) \cap \Gamma_a$ is either empty or is contained in $\{Re(z) < 0\}$. □

Lemma 3.4. *Let r_j, s_j be real numbers for $j = 1, 2$ with $s_j > r_j + 1$. Then there exists a sequence*

$$\phi_\nu : \overline{\mathbb{B}^n} \cup l_{1,s_1} \cup \overline{B_{r_1}(p_{s_1})} \to \mathbb{C}^n \tag{5}$$

of injective holomorphic maps such that the following hold:

(i) $\phi_\nu \to Id$ *uniformly on* $\overline{\mathbb{B}^n}$,

(ii) $\phi_\nu(z) \to \psi(z) = p_{s_2} + \frac{r_2}{r_1}(z - p_{s_1})$ *uniformly on* $\overline{B_{r_1}(p_{s_1})}$,

(iii) ϕ_ν *converges uniformly on* l_{1,s_1} *to a smooth embedding of* l_{1,s_1} *onto* l_{1,s_2}.

Moreover, the maps ϕ_ν are of the form $\phi_\nu(z) = (h_\nu(z_n) \cdot z', f_\nu(z_n))$, where h_ν, f_ν are holomorphic functions in one variable. Finally, the maps ϕ_ν can be chosen to match ψ to any given order at the point $p_{s_1+r_1}$.

Proof. By Mergelyan's theorem there exists one variable functions $f_\nu(z_n)$ satisfying (i)-(iii) on the intersection with the z_n-axis. Again by Mergelyan's theorem there exist a sequence $h_\nu(z_n)$ converging to the constant function r_2/r_1 on the disk of radius r_1 centred at s_1, to 1 on the closed unit disk, and some fixed real function increasing/decreasing from r_2/r_1 to 1 on the line segment connecting the two disks. □

We now give the proof of Theorem 3.1.

Proof. We prove it first without the parameter t, *i.e.*, we prove the existence of the sequence $\phi_{\nu,1}$. By Lemma 3.4, it is enough to prove this for some special case of r, s. We take the special case that $r = 1.5, s = 3$. Let f_ν be maps furnished by Lemma 3.2 with $a = 0, b = 3.5$. We define $\phi_\nu(z) := (z_1, \cdots, z_{n-1}, f_\nu(z_n))$. Then the properties (ii), (iii), (v) and (vi) hold (we have that (vi) holds because f_ν^{-1} is approximable by entire maps).

We now prove (iv). Since ϵ can be chosen arbitrarily small, the only place where the inclusion in (iv) can fail is near the point $p_{4.5}$. Consider first the sequence

$$\tilde{\phi}_\nu(z) = (z_1, \cdots, z_{n-1}, g_\nu(z_n)).$$

Then $\tilde{\phi}_\nu(z) \to z + p_b$ uniformly on $\overline{\mathbb{B}^n} \setminus B_\epsilon(p_{-1})$, and since $g'_\nu(1)$ is real (and goes to 1) we have that $\tilde{\phi}(\mathbb{B}^n \setminus B_\epsilon(p_{-1}))$ is eventually contained in $B_{1.5}(p_3)$. Now let $z = (z_1, ..., z_n)$ be a point in $\overline{\mathbb{B}^n} \cap B_\epsilon(p_1)$. If $\mathrm{Re}(m_\nu(z_n)) \leq 0$ then $\phi_\nu(z)$ is far away from the point $p_{4.5}$ since $\phi_\nu(z) = (z_1, ..., z_{n-1}, g_\nu(m_\nu(z_n)))$. If $\mathrm{Re}(m_\nu(z_n)) > 0$ it follows from Lemma 3.3 that $(z_1, ...z_{n-1}, m_\nu(z_n)) \in \mathbb{B} \setminus B_\epsilon(p_{-1})$, and so $\phi_\nu(z) \in B_{1.5}(p_3)$.

To construct isotopies we simply define

$$\phi_{\nu,t}(z) = \frac{1}{t}\phi_\nu(tz).$$

Note first that we can choose $\epsilon > 0$ arbitrarily small, and that by the construction, $\mathbb{B}^n \cap B_\epsilon(p_1)$ gets mapped by ϕ_ν into a relatively compact subset $\tilde{U}$ of U which is independent of ϵ. Write

$$\phi_\nu(z) = A_\nu(z) + G_\nu(z),$$

where $A_\nu \to \mathrm{Id}$ and $\|G_\nu(z)\| \leq \delta_\nu \|z\|^2$ for $\|z\| < 1-\epsilon$ where $\delta_\nu \to 0$. Consider a point $z \in B_\epsilon(p_1) \cap \mathbb{B}^n$. If $t < 1-\epsilon$ then $\phi_{\nu,t}(z) = A_\nu z + \frac{1}{t} G_\nu(tz)$, with $\|G_\nu(tz)\| \leq \delta_\nu t^2 \|z\|^2$, so if ν is large we are in U. If $t \geq 1-\epsilon$ we consider two cases. If $tz \notin B_\epsilon(p_1)$ we may assume that ϕ_ν is as close to the identity as we like, and so we are still in U. If $tz \in B_\epsilon(p_1)$ then ϕ_ν maps tz into $\tilde{U}$, and $\frac{1}{t}\tilde{U} \subset U$ provided ϵ was chosen small enough. □

4. Exposing a single boundary point

The aim of this section is to prove Theorem 1.1. Recall that a pair (A, B) of compact sets in a complex space X is called a *Cartan pair* if $A, B, A \cap B, A \cup B$ all admit Stein neighborhood bases in X and $\overline{A \backslash B} \cap \overline{B \backslash A} = \emptyset$. The following lemma due to Forstnerič will be used to (approximately) glue locally defined exposing maps to define global ones.

Lemma 4.1 ([5][6])**.** *Assume that X is a complex space and X' is a closed complex subvariety of X containing the singular locus X_{sing}. Let (A, B) be a Cartan pair in X such that $C := A \cap B \subset X \backslash X'$. For any open set $\tilde{C} \subset X$ containing C there exist open sets $A' \supset A, B' \supset B, C' \supset C$ in X, with $C' \subset A' \cap B' \subset \tilde{C}$, satisfying the following property. For every number $\eta > 0$ there exists a number $\epsilon_\eta > 0$ such that for each holomorphic map $\gamma : \tilde{C} \to X$ with $dist_{\tilde{C}}(\gamma, Id) < \epsilon_\eta$ there exist biholomorphic maps $\alpha = \alpha_\gamma : A' \to \alpha(A') \subset X$ and $\beta = \beta_\gamma : B' \to \beta(B') \subset X$ satisfying the following properties:*

- (i) *$\gamma \circ \alpha = \beta$ on C',*
- (ii) *$dist_{A'}(\alpha, Id) < \eta$ and $dist_{B'}(\beta, Id) < \eta$, and*
- (iii) *α and β are tangent to the identity map to any given finite order along the subvariety X' intersected with their respective domain.*

Moreover, the maps α_γ and β_γ can be chosen to depend continuously on γ such that $\alpha_{Id} = Id$ and $\beta_{Id} = Id$.

We start by embedding a neighbourhood of the curve γ suitably into $\mathbb{C}^n$, where we will define local exposing maps.

Lemma 4.2. *There exists an open Stein neighbourhood W of γ, an open neighbourhood $U_\zeta \subset W$ of ζ,and a holomorphic embedding $\phi : W \to \mathbb{C}^n$ ($n = \dim(X)$) such that the following holds:*

- (i) *$\phi(\zeta) = 0$,*
- (ii) *$\phi((U_\zeta \cap K) \setminus \{\zeta\}) \subset \{z \in \mathbb{C}^n : 2\mathrm{Re}(z_n) + \|z\|^2 < 0\}$*

Proof. First let $M \subset X$ be a totally real manifold (with boundary) of dimension n that contains the curve γ, and choose a smooth embedding $g : M \to \mathbb{R}^n \subset \mathbb{C}^n$. Then M admits a Stein neighbourhood W_0 such that g may be approximated in C^1-norm on M by holomorphic maps $\phi_0 : W_0 \to \mathbb{C}^n$. So ϕ_0 may be taken to be an embedding of an open Stein neighbourhood W of γ into $\mathbb{C}^n$. By assumption there is a (local) strictly pseudoconvex hypersurface $\Sigma \subset \mathbb{C}^n$ and an open set U_ζ such that Σ touches $\phi_0(U_\zeta \cap K)$ only at the point $\phi_0(\zeta)$. Let $F \in \mathrm{Aut}_{\mathrm{hol}}\mathbb{C}^n$ such that $F(\Sigma)$ is strictly convex near $F(\phi_0(\zeta))$. After a translation and scaling, the map $\phi = F \circ \phi_0$ will ensure the conclusions of the lemma. □

We may now modify γ such that near the origin, the curve $\tilde{\gamma} := \phi(\gamma)$ coincides with the line segment $l_{0,r}$ for some $r > 0$, and such that $\tilde{\gamma}$ is perpendicular to the (local) hypersurface $\tilde{H} := \phi(W \cap H)$ where they intersect. We keep the notation γ for the modified curve. Let γ_ζ denote the piece $\phi^{-1}(l_{0,r})$ of γ.

Lemma 4.3. *There exists a a compact set $K' \subset X$ with $K \subset \mathrm{int}(K')$ such that for any $\epsilon > 0$ there exists an open neighbourhood Ω of $K' \cup \gamma_\zeta$ and a holomorphic embedding $\psi : \Omega \to X$ such that the following holds:*

(i) $\mathrm{dist}(z, \psi(z)) < \epsilon$ *for all* $z \in K'$,
(ii) $\psi(\gamma_\zeta) \subset V \cap W$,
(iii) $\psi(\gamma_\zeta) \cap H = \psi(\phi^{-1}(p_r))$ *and the intersection is perpendicular to H in the coordinates furnished by ϕ.*

Proof. We will first define a map $\tilde{h}$ accomplishing (i)-(iii) in the local coordinates furnished by ϕ, and then we will (approximately) glue the map $\phi^{-1} \circ \tilde{h} \circ \phi$ to the identity map on K'.

We start by defining a suitable Cartan pair. Writing $\phi = (\phi_1, ..., \phi_n)$ note that the pluriharmonic function $\rho := \mathrm{Re}(\phi_n)$ extends to a plurisubharmonic function on an open neighbourhood of K which agrees with ρ on an open neighbourhood of ζ, and which is negative and uniformly bounded away from zero away from where they agree. We keep the notation ρ for the extended function. Then for a sufficiently small smoothly bounded strictly pseudoconvex open neighbourhood Ω' of $K \cup \gamma_\zeta$ and sufficiently small $\tau > 0$, the sets

$$A_\tau = \overline{\Omega}' \cap \{\rho \geq -2\tau\} \text{ and } B_\tau = \overline{\Omega}' \cap \{\rho \leq -\tau\}$$

define a Cartan pair (A_τ, B_τ). Let $\tilde{A}_\tau = \phi(A_\tau)$ and $\tilde{B}_\tau = \phi(B_\tau \cap W)$. If Ω' and τ are chosen sufficiently small we have that $A_\tau \cap B_\tau \subset W$ and $\tilde{A}_\tau \cap \tilde{B}_\tau \subset B_{r/2}(0)$. Set $\tilde{C}_\tau = B_{r/2}(0)$. Fix K' a Stein compact such that K' contains a neighbourhood of K and $K' \subset B_\tau \cup \phi^{-1}(B_{r/2})$.

Now let D be an open neighbourhood of $\overline{B}_{r/2}(0) \cup l_{0,r}$ with $\tilde{g} : D \to \mathbb{C}^n$ a smooth embedding such that $\tilde{g} = \mathrm{id}$ near $\overline{B}_{r/2}(0)$, $\tilde{g}$ stretches $l_{0,r}$ to cover $\tilde{\gamma}$, $\tilde{g}^{-1}(\tilde{H})$ is perpendicular to $l_{0,r}$, and such that $\tilde{g}$ is $\overline{\partial}$-flat to order one along $l_{0,r}$. Then $\tilde{g}$ is uniformly approximable on $\overline{B}_{r/2}(0) \cup l_{0,r}$ in $\mathcal{C}^1$-norm with jet interpolation at the point p_r. So there exists a holomorphic embedding $\tilde{h} : \overline{B}_{0,r/2}(0) \cup l_{0,r} \to \mathbb{C}^n$, as close to the identity as we like on $\overline{B}_{r/2}(0)$, with the image $\tilde{h}(l_{0,r})$ as close as we like to $\tilde{\gamma}$ and with $\tilde{h}^{-1}(\tilde{H})$ perpendicular to $l_{0,r}$. Set $h := \phi^{-1} \circ \tilde{h} \circ \phi$. Now if $\tilde{h}$ is close enough to the identity on $B_{r/2}$ we have that Lemma 4.1 furnishes maps α and β such that the map ψ defined as $\psi := h \circ \alpha$ on A_τ and $\psi = \beta$ on B_τ will satisfy the claims of the lemma.

□

Proof of Theorem 1.1: We use the extended function ρ from the proof of the previous lemma to define a Cartan pair:

$$A_\tau = K \cap \{\rho \geq -2\tau\} \text{ and } B_\tau = K \cap \{\rho \leq -\tau\}$$

Then for τ small we have that ϕ maps A_τ into $B_1(p_{-1}) \cup \{0\}$, and $C_\tau = A_\tau \cap B_\tau$ gets mapped into the ball $B_1(p_{-1})$. Choose a small $\delta > 0$ such that the ball $B_\delta(p_{r-\delta})$ touches $\tilde{h}^{-1}(H)$ only at the point p_r. Now Theorem 3.1 applied to the dumbbell $B_1(p_{-1}) \cup l_{0,r-2\delta} \cup B_\delta(p_{r-\delta})$ furnishes local exposing maps $\tilde{\phi}_\nu : \overline{B}_1(p_{-1}) \to \mathbb{C}^n$ such that the maps $\phi_\nu := \phi^{-1} \circ \tilde{\phi}_\nu \circ \phi$ may be approximately glued to the identity map over a neighbourhood of C_τ to conclude the proof of the first part of Theorem 1.1 by setting $f = \psi \circ \phi_\nu$.

For the last part we now assume that $K \subset \mathbb{C}^n$ is polynomially convex. Let ρ be a non-singular strictly plurisubharmonic function on an open neighbourhood U_ζ of ζ such that $\rho(\zeta) = 0$ and $\rho < 0$ on $(K \cap U_\zeta) \setminus \{\zeta\}$. Then for any sufficiently small Runge and Stein neighbourhood Ω of K the function ρ extends to a plurisubharmonic function on Ω which is strictly negative on $\Omega \setminus U_\zeta$, and so $\Omega' = \Omega \cap \{\rho < 0\}$ is a Runge and Stein domain with $\zeta \in b\Omega'$. Moreover, choosing $\overline{\Omega}$ to have a Runge and Stein neighbourhood basis, we have that $\overline{\Omega}'$ will have a Runge and Stein neighbourhood basis, and so by Theorem 1.3 in [3] there exists $F \in \mathrm{Aut}_{\mathrm{hol}}\mathbb{C}^n$ such that $F(K \setminus \{\zeta\}) \subset \mathbb{B}^n$ and $F(\zeta) = p_1$. Since we can conjugate by F we may as well assume that $K \setminus \{\zeta\} \subset \mathbb{B}^n$ and that $\zeta = p_1$.

My slightly modifying γ we may assume that γ agrees with $l_{1,1+r}$ near p_1 and that it is perpendicular to H where they intersect. First let D be an open neighbourhood of $\overline{\mathbb{B}^n} \cup l_{1,1+r}$ and $g : [0,1] \times D \to \mathbb{C}^n$ be an isotopy of embeddings such that $g_0 = \mathrm{id}$, g stretches $l_{1,1+r}$ along γ, $g_t = \mathrm{id}$ near $\overline{\mathbb{B}^n}$ for all t, $g^{-1}(H)$ is perpendicular to $l_{1,1+r}$, and g_t is $\overline{\partial}$-flat on $l_{1,1+r}$ to order one. Then the whole parameter family g_t may be approximated by a parameter family $h : [0,1] \times \mathbb{C}^n$ of holomorphic maps in C^1-norm on $\overline{\mathbb{B}^n} \cup l_{1,1+r}$ with jet interpolation at the end point of $l_{1,1+r}$. So h_t is a family of holomorphic embeddings on some open neighbourhood of $\overline{\mathbb{B}^n} \cup l_{1,1+r}$, and by the Andersén-Lempert theory h_1 may be approximated by a holomorphic automorphism G with interpolation at the end point of $l_{1,1+r}$.

Let $\tilde{H} = G^{-1}(H)$ which is now perpendicular to $l_{1,1+r}$. Choose $s > 0$ such that $B_s(p_{1+r-s})$ touches $\tilde{H}$ only at a single point. Now Theorem 3.1 applied to the dumbbell $\mathbb{B}^n \cup l_{1,1+r} \cup B_s(p_{1+r-s})$ furnishes a one-parameter family of maps ϕ_t such that $G \circ \phi_1$ is the exposing map we are after, except that ϕ_1 is not an automorphism. However, by (vi), $\phi_t(\overline{\mathbb{B}^n})$ is polynomially convex for each t, and so ϕ_1 is approximable by holomorphic automorphisms. The proof is complete.

□

5. Regularity of exposing maps with parameters

In this section we will prove Theorem 1.3. Before we proceed with the proof, we give a further discussion on parametric exposing. The terminology of being exposed will refer to being globally exposed in the sense of Definition 1.2, and R may vary with the parameter.

Definition 5.1. Let D be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Then we say that D satisfies:
1) *Property (E)*, if there exists a continuous map $F : \partial D \times \bar{D} \to \mathbb{C}^n$ such that for each fixed $p \in \partial D$ the map $F_\zeta := F(\zeta, \cdot) : \bar{D} \to \mathbb{C}^n$ is an exposing of D at ζ; or
2) *Property (AE)*, if there exists a continuous map $F : \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that for each fixed $\zeta \in \partial D$ we have that $F_\zeta \in \mathrm{Aut}_{\mathrm{hol}}\mathbb{C}^n$ is an exposing of D at ζ; or
3) *Property (ASE)*, if there exists a continuous map $F : \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that for each fixed $\zeta \in \partial D$ we have that $F_\zeta \in \mathrm{Aut}_{\mathrm{diff}}\mathbb{C}^n$ and $F_\zeta(D)$ is exposed at $F_\zeta(\zeta)$.

In the case 2) we say that F_ζ is ambient exposing, and in case 3) we say that F_ζ is ambient smoothly exposing. Note that Property (ASE) has nothing to do with the complex structure of D and $\mathbb{C}^n$. In [2], the following question was proposed:

Question 1. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Does D satisfies Property (E)?*

The following theorem, which can be viewed as a parametric version of Theorem 1.1, partially answers the above question.

Theorem 5.1. *Let $D \subset \mathbb{C}^n (n > 1)$ be a bounded strongly pseudoconvex domain with smooth boundary. If D satisfies Property (ASE), then for any positive numbers ϵ, δ, there exists a continuous map $f : \partial D \times \overline{D} \to \mathbb{C}^n$ such that for each $\zeta \in \partial D$:*

1) *$f_\zeta(\cdot) = f(\zeta, \cdot) : \overline{D} \to \mathbb{C}^n$ is holomorphic and injective;*
2) *$f_\zeta(D)$ is exposed at $f_\zeta(\zeta)$; and*
3) *$||f_\zeta(z) - z|| < \epsilon$ for $z \in \overline{D} \backslash B_\delta(\zeta)$, where $B_\delta(\zeta) = \{z \in \mathbb{C}^n ; ||z - \zeta|| < \delta\}$.*

In particular, D satisfies Property (E). If in addition $\overline{D}$ is polynomially convex, then f can be taken to be a smooth map $f : \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that $f_\zeta \in Aut(\mathbb{C}^n)$ for all $\zeta \in \partial D$. In particular, D satisfies Property (AE).

When trying to apply Theorem 1.3 to Question 1, we meet a topological obstruction. Even we originally just interested in exposing but not ambient exposing, it turns out that Property (ASE) is needed in our argument. It is clear that Property (ASE) is a necessary condition for Property (AE), but we don't know if it is also necessary for Property (E).

For the special case when $\overline{D}$ is diffeomorphic to the closed unit ball, we can prove the following

Theorem 5.2. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that either D is diffeomorphic to the unit ball B^n and $n \geq 3$, or the closure $\overline{D}$ of D is diffeomorphic to the closed unit ball and $n = 2$. Then D satisfies Property (E). If in addition $\bar{D}$ is polynomially convex, then D satisfies Property (AE).*

Remark 5.1. It is known that any smoothly embedded S^{n-1} in $\mathbb{R}^n$ bounds a domain that is diffeomorphic to the unit ball except $n \neq 4$ (see [10]). The case for $n = 4$ is still open. So for $n \neq 2$, the condition in Theorem 5.2 that D is diffeomorphic to B^n can be replaced by that ∂D is diffeomorphic to S^{2n-1}.

5.1. **A parametric version of Forstnerič's splitting lemma.** We first introduce some notations. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary and δ be a sufficiently small positive number. For $\zeta \in \partial D$ we define:

$$\begin{aligned} A_\zeta &= \{z \in \overline{D} : ||z - \zeta|| \leq \delta\}, \\ B_\zeta &= \{z \in \overline{D} : ||z - \zeta|| \geq \delta/2\}, \\ C_\zeta &= A_\zeta \cap B_\zeta. \end{aligned}$$

For a closed subset X in $\mathbb{C}^n$ and $a > 0$, we set $X(a) := \{z \in \mathbb{C}^n : ||z - w|| \leq a \text{ for some } w \in X\}$ the a-neighborhood of X.

The following result is the main result in the thesis of Lars Simon ([11])

Lemma 5.3. *Let D and δ as above. If $\tau > 0$ is small enough (this depends on δ), then for any $\eta > 0$ there exists $\epsilon > 0$ such that the following holds. If $\mu > 5\tau$ and if γ_ζ is a family of injective holomorphic maps $\gamma_\zeta : C_\zeta(\mu) \to \mathbb{C}^n$ satisfying*

- *$\gamma : \{(z, \zeta) : z \in C_\zeta(\mu), \zeta \in \partial D\} \to \mathbb{C}^n$, $(z, \zeta) \mapsto \gamma_\zeta(z)$ is continuous,*

- *$dist_{C_\zeta(\mu)}(\gamma_\zeta, Id) < \epsilon$ for all $\zeta \in \partial D$,*

then there exist families $\{\alpha_\zeta\}_{\zeta\in\partial D}$ and $\{\beta_\zeta\}_{\zeta\in\partial D}$ of injective holomorphic maps $\alpha_\zeta : A_\zeta(2\tau) \to \mathbb{C}^n$ and $\beta_\zeta : B_\zeta(2\tau) \to \mathbb{C}^n$ with the following properties:

1. *for all $\zeta \in \partial D$ we have $\gamma_\zeta = \beta_\zeta \circ \alpha_\zeta^{-1}$ on $C_\zeta(\tau)$,*
2. *$dist_{A_\zeta(2\tau)}(\alpha_\zeta, Id) < \eta$ and $dist_{B_\zeta(2\tau)}(\beta_\zeta, Id) < \eta$,*
3. *the maps α and β are continuous, where*

$$\alpha : \{(z,\zeta) \in \mathbb{C}^n \times \partial D : z \in A_\zeta(2\tau)\} \to \mathbb{C}^n,\ (z,\zeta) \mapsto \alpha_\zeta(z),$$
$$\beta : \{(z,\zeta) \in \mathbb{C}^n \times \partial D : z \in B_\zeta(2\tau)\} \to \mathbb{C}^n,\ (z,\zeta) \mapsto \beta_\zeta(z).$$

Moreover, the maps α_ζ may be constructed to match the identity to any given order at ζ, and if we are given a parameter family $\gamma_{\zeta,t}$ with $t \in [0,1]$ we may obtain $\alpha_{\zeta,t}$ and $\beta_{\zeta,t}$ also jointly continuous in (ζ, t).

We note that the two last points were not a part of the statement in [11], but they follow from the proof.

5.2. **Exposing along normal directions.** Let $\Omega \subset \mathbb{C}^n$ be a domain with a strictly plurisubharmonic defining function ρ, and fix $p \in \partial\Omega$. By choosing a smooth family orthogonal frames in $T_\zeta\partial\Omega$ for ζ near p we may construct a smooth family of locally injective holomorphic maps $g_\zeta(z)$ such that $g_\zeta(\Omega)$ has a defining function

$$\rho_\zeta(z) = 2\mathrm{Re}(z_1 + Q_\zeta(z)) + \mathcal{L}_\zeta(z) + \text{h.o.t} \tag{6}$$

near the origin. Define $G_\zeta(z) = (z_1 - Q_\zeta(z), z_2, ..., z_n)$, so that $H_\zeta = G_\zeta \circ g_\zeta$ maps $\partial\Omega$ to a strictly convex surface near the origin. We may cover $\partial\Omega$ by finitely many open sets U_j such that we have such families of maps H^j_ζ defined on each U_j, and they all coincide modulo a unitary change of coordinates in $\{z_1 = 0\}$, sending one frame to another. More precisely, for maps g^i_ζ and g^j_ζ we have that $g^i_\zeta(z) = Vg^j_\zeta(z)$, where $Vz = (z_1, Uz')$. For the corresponding defining functions we get that $\rho^i_\zeta(z_1, z') = \rho^j_\zeta(z_1, U^{-1}z')$. Furthermore we have

$$G^j_\zeta(z) = (z_1 - Q_\zeta(z), z_2, ..., z_n), \tag{7}$$

and

$$G^i_\zeta(z) = (z_1 - Q_\zeta((z_1, U^{-1}(z'))), z_2, ...z_n). \tag{8}$$

For a point w near ζ, writing $z = g^j_\zeta(w)$, we see that

$$G^i_\zeta(g^i_\zeta(w)) = G^i_\zeta(z_1, Uz') = (z_1 - Q_\zeta(z), Uz'), \tag{9}$$

and in particular we see that the map $g_\zeta^{-1}G_\zeta g_\zeta$ is independent of the choice of orthonormal fram on the tangent space.

Choose a small $a > 0$, let l_a denote the line segment between 0 and a in the z_1-axis, and let $\eta_\zeta = (H^j_\zeta)^{-1}(l_a)$. For small $b, c > 0$ we let $B(b,\zeta)$ denote the ball $(H^j_\zeta)^{-1}(B_b(a))$, and $V_\zeta = (H^j_\zeta)^{-1}(l_a(c))$, the inverse image of the open c-tube around l_a.

Theorem 5.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain with a strictly plurisubarmonic defining function ρ and with objects defined above. Then for $\epsilon_1, \delta_1 > 0$ and $a, b, c > 0$ small enough there exists a continuous map $F : \partial\Omega \times \overline{\Omega} \subset \mathbb{C}^n$ such that the following holds.*

(i) *For each* $\zeta \in \partial\Omega$ *the map* $F_\zeta = F(\zeta, \cdot) : \overline{\Omega} \to \mathbb{C}^n$ *is a holomorphic embedding,*
(ii) $F_\zeta(\zeta) = (H^j_\zeta)^{-1}(a, 0, \cdot, 0) =: q_\zeta$,
(iii) $F_\zeta(\overline{\Omega} \cap B_{\delta_1}(\zeta)) \subset V_\zeta \cup B(b, \zeta) \cup \{q_\zeta\}$, *and*
(iv) $\|F(\zeta, \cdot) - \mathrm{id}\|_{\overline{\Omega}\setminus B_{\delta_1}(\zeta)} < \epsilon_1$.

Moreover, if $\overline{\Omega}$ *is polynomially convex we may achieve that* $F_\zeta \in \mathrm{Aut}_{\mathrm{hol}}\mathbb{C}^n$ *for each* ζ.

Proof. We first define local maps achieving (i)-(iii), and then these will be approximately glued to the identity map using Lemma 5.3 above.

By scaling we may assume that all (local) images $H^j_\zeta(\Omega)$ are contained in the ball $B = \{z : 2\mathrm{Re}z_1 + \|z\|^2 < 0\}$ of radius one centered at the point $(-1, 0, ..., 0)$. Let $\phi_{\nu,t}$ be the maps from Theorem 3.1 defined for the ball B, the line segment l_a, the ball of radius b centred at the point a, and the set $V = l_a(c)$. Similarly to the above discussion, since $\phi_{\nu,t}$ are one variable maps, the maps defined by $\tilde{F}_\zeta(z) := (H^j_\zeta)^{-1} \circ \phi_\nu \circ H^j_\zeta$ do not depend on the choice of orthonormal frame on the tangent space, and so are well defined independently of j, and $\tilde{F}_\zeta$ is a locally defined map (near each ζ) achieving (i)-(iii) for ν large enough.

Now choose $\delta > 0$ small enough, set $\gamma_\zeta = \tilde{F}_\zeta|_{C_\zeta(\mu)}$, and let $\alpha_\zeta, \beta_\zeta$ be the maps from Lemma 5.3. Then the maps F_ζ defined by $\tilde{F}_\zeta \circ \alpha_\zeta$ on $A_\zeta(2\tau)$ and β_ζ on $B_\zeta(2\tau)$ are globally defined on Ω and will satisfy (i)-(iv) as long as all constants involved are small enough.

For the last part, assuming polynomial convexity, note that F_ζ is isotopic to the identity map through holomorphic embeddings by setting $\tilde{F}_{\zeta,t} := (H^j_\zeta)^{-1} \circ \phi_{t,\nu} \circ H^j_\zeta$, including the t-parameter from Theorem 3.1, and invoking the last statement in Lemma 5.3. Following the arguments in Section 5 in [3] each image $F_{\zeta,t}(\overline{\Omega})$ will be polynomially convex, and so by the Andersén-Lempert theorem with parameters and jet-interpolation, the map $F_{\zeta,1}$ may be approximated by a family of holomorphic automorphisms. Note that in the usual statements of the Andersén-Lempert theorems, isotopies are required to by of class C^2, but it is quite simple to work with C^0-isotopies instead, see [1] for details. □

5.3. **Proof of Theorem 1.3.** Let η_ζ denote the arcs from the previous section with $a << 1$. We leave it to the reader to convince himself/herself that the arcs γ_ζ may be modified so that each of them parametrises η_ζ over an interval $[0, s]$, and so that γ_ζ is perpendicular to H. Let $\psi_t : \partial\Omega \times \overline{\Omega} \cup (\bigcup_\zeta \{\zeta\} \times \eta_\zeta) \to \partial\Omega \times \mathbb{C}^n$ be a fiberpreserving smooth map, $t \in [0, 1]$, that is the identity near $\partial\Omega \times \overline{\Omega}$, and stretches each η_ζ to cover γ_ζ. According to [9] Section 5 we have that ψ_t may be approximated by an isotopy of injective holomorphic maps such that $\psi_1(\eta_\zeta)$ is still perpendicular to H, keeping the notation ψ_t for the approximation. Let W_ζ denote $\psi^{-1}_{\zeta,1}(H)$. Then each W_ζ is transverse to η_ζ, and there is a $b > 0$ such that the ball of radius b centred at $a - b$ is tangent to $(H^j_\zeta)^{-1}(W_\zeta)$ for all ζ, where H^j_ζ is as in the previous section. Letting F_ζ be maps as in Theorem 5.4, the maps $\psi_{\zeta,1} \circ F_\zeta$ will satisfy the claims of the theorem. Finally, as before if $\overline{\Omega}$ is polynomially convex, the maps $\psi_{\zeta,1}$ may be taken to be holomorphic automorphisms.

5.4. **Proof of Theorem 5.1 and Theorem 5.2.** In this subsection, we give the proofs of Theorem 5.1 and Theorem 5.2, which are based on Theorem 1.3.

Proof. (of Theorem 5.1) For $R > 0$, let B_R be the ball in $\mathbb{C}^n$ centered at the origin with radius R, and let $S_R = \partial B_R$ be the boundary of B_R. We want to prove that there is a smooth family of curves $\gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n$ which satisfies the conditions 1)-4) in Theorem 1.3, with H replaced by S_R for $R >> 1$.

By assumption, there is a smooth map $F : \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that for each $\zeta \in \partial D$, $F_\zeta(\cdot) := F(\zeta, \cdot) : \mathbb{C}^n \to \mathbb{C}^n$ is a diffeomorphism and $F_\zeta(D)$ is exposed at $F_\zeta(\zeta)$.

We assume that all F_ζ are orientation preserving. For any $\zeta \in \partial D$ there is a natural isotopy $H_\zeta : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ from F_ζ to the identity, given by $F_{\zeta,t} = F_\zeta(t,z) := \frac{1}{t}F_\zeta(tz)$ for $t \neq 0$, and smoothly extended to 0 by setting $F_{\zeta,0}(z) = z$.

This isotopy gives us a smooth family of vector fields $X_\zeta(t,z)$ on $\mathbb{C}^n$. In other words, $F_{\zeta,t}$ is generated by X_ζ. It is clear that $X_\zeta(t,z)$ is even smooth jointly with respect to ζ, t, and z.

Let $r > 0$ such that $F_{\zeta,t}(\overline{D}) \subset B_r(0) := \{z \in \mathbb{C}^n : ||z|| < r\}$ for all $\zeta \in \partial D$ and $t \in [0,1]$. Let $R >> r$ be a positive number. Let ψ be a positive smooth function on $\mathbb{C}^n$ such that $\psi \equiv 1$ on B_r and $\psi \equiv 0$ on $\mathbb{C}^n \backslash B_R$. Let $X'_\zeta(t,z) = \psi X_\zeta(t,z)$, then X' has compact support.

Let $G_\zeta : \mathbb{C}^n \to \mathbb{C}^n$ be the diffeomorphism of $\mathbb{C}^n$ generated by $X'_\zeta(t,z), t \in [0,1]$. Then $G_\zeta = F_\zeta$ on $\overline{D}$ and $G_\zeta = Id$ on $\mathbb{C}^n \backslash B_R$.

For $\zeta \in \partial D$, let n_ζ be the unit out-pointing normal vector of $\partial G_\zeta(D)$ at $G_\zeta(\zeta)$. Then there is a unique $r_\zeta \in (0, +\infty)$ such that $G_\zeta(\zeta) + r_\zeta n_\zeta \in S_R$. We define a smooth family of curves $\gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n$ by setting $\gamma(\zeta, t) = (\zeta, G_\zeta^{-1}(G_\zeta(\zeta) + r_\zeta t n_\zeta))$.

Then the theorem follows by applying Theorem 1.3 with γ and $H = S_R$. □

Proof. (of Theorem 5.2) For $n = 2$, we already assume that $\overline{D}$ is diffeomorphic to the closed unit ball. For $n > 2$, we want to show that $\overline{D}$ is diffeomorphic to the closed unit ball too. By the Collar Theorem (see Theorem 6.1 in [8]), ∂D is simply connected. Let $z_0 \in D$ and let B be an open ball centered at z_0 with boundary $S \subset D$. Let $W = \overline{D} \backslash B$, then the triad $(W; S, \partial D)$ is an h-cobordism. Note that the relative homology $H_*(W, S) = 0$. By the h-Cobordism Theorem (see Theorem 9.1 in [10]), W is diffeomorphic to $S \times [0,1]$ and hence $\overline{D}$ is contractible. So $\overline{D}$ is diffeomorphic the the unit ball (See Proposition A in §9 in [10]). By a result in differential topology (see Theorem 3.1 in [8]), there is a diffeomorphism σ of $\mathbb{C}^n$ such that $\sigma(\overline{D})$ is the closed unit ball. So the Theorem follows from Theorem 5.1. □

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