Truth in mathematics

Øystein Linnebo

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1 Introduction

Many questions in the philosophy of mathematics are concerned with the nature and role of the concept of truth in mathematics. I shall here be concerned with four such questions.

One question is whether the concept of truth is needed in a philosophical account of mathematics. Do we need to attribute truth and falsity to mathematical statements in the way that we do to empirical statements? Quite clearly, mathematics operates with strict standards of correctness. For instance, philosophers of every orientation agree that it is correct to say that $7 + 5 = 12$ and incorrect to say that $7 + 5 = 13$. But it is not obvious that this notion of correctness needs to be identified with truth. Many philosophers have argued that the notion of correctness that is operative in mathematics falls short of full-fledged truth and that this notion is better understood in terms of acceptability by certain agreed standards. I shall argue that such arguments should be resisted and that the concept of truth is indeed needed in mathematics.

Another question concerns the nature of mathematical truth. On a classical conception of truth, the truth of a mathematical sentence requires the existence of mathematical objects for its singular terms to denote and its quantifier to range over. This forms the core of an influential argument developed by Frege for the existence of abstract mathematical objects. In order to avoid this ontology of abstract objects and the philosophical puzzles to which it gives rise, other philosophers have proposed various non-classical conceptions of mathematical truth, which allow mathematical sentences to be truth without there being any mathematical objects. I shall argue that these non-classical conceptions are problematic, whereas the classical conception enjoys substantial support.

A third question concerns the relation between the existence of mathematical objects and the objectivity of mathematical truth. According to a traditional platonist view, the former explains the latter. It is because there exist mind-independent mathematical objects with mind-independent properties that mathematical sentences have their objective truth-values. I discuss some considerations, due to Frege and others, that favor the opposite direction of explanation: from mathematical truth to mathematical objects.

The final question is to what extent the truth-values of mathematical statements are objective. Do mathematical statements have their truth-values independently of our current choice of axioms and our ability to prove theorems that follow from these axioms? This question has great significance for the methodology of mathematics. The search for new axioms for some branch of mathematics is typically motivated by the belief that some of the statements that are left undecided by our current axioms have an objective truth-value that can and should be made explicit by the adoption of further axioms.

2 Why mathematics needs a concept of truth

Our first question is whether a concept of truth is needed in a philosophical account of mathematics. Can we make sense of mathematical language and practice without invoking the notions of truth and falsity?

Let's begin with the slightly narrower question whether the notion of correctness that governs mathematical practice needs to be identified with the notion of truth. In other areas of discourse, there are good reasons to take truth to be the principal notion of theoretical correctness. Set aside practical notions of correctness that are concerned with the demands of instrumental rationality, ethics, or etiquette. Then it seems that a statement about, say, the furniture in my room is correct just in case it is true. Why should mathematics be an exception to this identification of theoretical correctness with truth? Much of the resistance to this identification stems from the difficulty of making sense of what would make a mathematical statement true or false. A statement about my furniture is made true by my furniture (or lack thereof) and its properties. But it is far less obvious what would make a mathematical statement true. The platonist's answer that mathematical statements are made true by an abstract realm of mathematical objects seems needlessly speculative. Many philosophers have found it more prudent to analyze mathematical discourse and practice in terms of notions that are less problematic. Among the least problematic facts about mathematics is that mathematicians prove theorems and that there is wide agreement in the mathematical community about the standards for assessing the correctness of proofs. So why not try to use these relatively unproblematic facts as a starting point for our philosophical analysis?

These reflections motivate a *formalist* approach to mathematics, according to which the highest notion of correctness that is found in mathematics, or is needed in its analysis, is theoremhood in some axiomatic theory. For instance, on this view the correctness of the statement that there are infinitely many prime numbers consists in its being a theorem of the relevant axiomatic theory, for instance Peano Arithmetic.

An extreme version of this view is *game formalism*, which regards mathematics as a meaningless game with symbols.[1] The axioms describe the strings of symbols that count as legitimate "initial positions", and the rules of inference characterize the permissible "moves" that allow one to progress from some strings of symbols to another. A mathematical theory is thus much like the game of chess, except that its moves are made with strings of symbols rather than pieces on a board. In particular, a string of mathematical symbols is just as devoid of meaning as a configuration of chess pieces. So such a string admits of truth or falsity no more than a configuration of chess pieces.

There are milder versions of formalism as well. Instead of denying that mathematical language has any sort of meaning, one can limit oneself to the weaker claim that all philosophically important notions can be analyzed in terms of purely syntactic properties of mathematical language, making no appeal to any semantic properties. This weaker claim can also be combined with a liberal view of what counts as a syntactic property.[2]

Probably the most famous defense of the need for a notion of truth in mathematics is due to Frege, who writes as follows:

> Why can arithmetical equations be applied? Only because they express thoughts.

[1] See (Resnik, 1980) and (Shapiro, 2000b) for discussion.

[2] An example of a liberal view of this sort is the program canvassed in (Carnap, 1934), where for instance the omega-rule is counted as syntactic. (This is the infinitary arithmetical rule which says that, from premises $\phi(\bar{n})$ for every natural number n, one may infer $\forall n\, \phi(n)$.)

> How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now it is applicability alone which elevates arithmetic from a game to the rank of a science ((Frege, 1903), sec. 91; transl. in (Frege, 1952)).

Unlike a configuration of chess pieces, the string of symbols that make up a mathematical theorem says something. And it is because what the theorem says is true that it is applicable in scientific arguments and explanations.

In order to explain and evaluate Frege's argument, it will be useful to consider an example of how pure mathematics can contribute to a scientific explanation. So let's consider Euler's famous problem of the Königsberg bridges. The great 18th century mathematician Euler wondered whether it is possible to choose a route through the city of Königsberg that involves crossing each of its seven bridges once and only once. He showed how the problem can be formulated in the abstract terms of what is now known as graph theory. Each land mass can be represented by a single node, and each bridge, by an edge connecting two nodes. A simple mathematical analysis of the resulting graph then demonstrates that, given the arrangement of rivers and bridges in Königsberg, there can be no route of the desired kind. This establishes the empirical claim that no matter what route Euler chooses through the city, he will not manage to cross each bridge precisely once. How does Euler's mathematical argument succeed in assuring us of the truth of this empirical conclusion? The most obvious answer is that this is accomplished because the argument is sound: it is a logically valid argument from true premises. But this answer presupposes a notion of truth that is applicable to empirical and mathematical premises alike, and which is such that the relevant theorem of graph theory does indeed count as true.

An interesting objection to Frege's argument is suggested by the work of Hartry Field, using ideas that go back to the great mathematician David Hilbert.[3] Consider a valid argument whose premises are either theorems of pure mathematics or nominalistic truths about

[3]See respectively (Field, 1984) and (Hilbert, 1926). Of course, Field is not a formalist but an error theorist: he accepts that mathematical statements have truth-values but takes all atomic such statements to be false. But formalists and error theorists are united in their opposition to Frege's argument.

the physical world, and whose conclusion C is another nominalistic statement. In order to explain why the conclusion C is guaranteed to be true, there is no need to ascribe truth to the mathematical theorems, as Frege's explanation does. It suffices that the mathematical theorems have a weaker property known as *conservativeness.* Let T_1 be a theory about the physical world formulated in a nominalistic language $\mathcal{L}_1$. Let $\mathcal{L}_2$ be a language that extends $\mathcal{L}_1$ by adding some mathematical vocabulary, and let T_2 be a theory in $\mathcal{L}_2$ that extends T_1. Then T_2 is said to be conservative over T_1 just in case every consequence of T_2 that is formulated in the restricted language $\mathcal{L}_1$ is also a consequence of T_1. Field then observes that in order to establish that the conclusion C is true, it suffices to show that the mathematical premises are conservative over the nominalistic statements.

However, a lot of hard work remains before this strategy will enable us to dispense with all ascriptions of truth to mathematical statements. Firstly, we would have to provide a purely nominalistic formulation of all of science. Field shows how this can be done for Newtonian gravitational theory, which is a good start. But it remains uncertain whether all of contemporary science can be nominalized in this way.[4]

Secondly, the conservativeness (and thus also consistency) of the various theories of pure mathematics would have to be established. And this would have to be done in a way that avoids ascribing truth to these theories. Here another observation inspired by Frege is relevant. Sometimes our best evidence for the consistency of a theory is also evidence for the truth of the theory.[5] Consider for instance our conviction that ZFC set theory is consistent. This conviction is not based primarily on the empirical fact that no contradiction has yet been found. Rather it is based on our having a reasonably clear grasp of the intended model for ZFC, namely the iterative hierarchy of sets. Our conviction that ZFC is consistent rests on the observation that its axioms are true in this intended model.

A third worry concerns the fact that Field's project is a reconstructive one. Science isn't actually done in the way that Field suggests. Our best scientific theories aren't formulated in a nominalistic language but freely make use of large amounts of mathematics. So even if the

[4]See (Burgess and Rosen, 1997), part II.

[5]See (Frege, 1903), sect. 144 (repr. in (Frege, 1952)), as well as the Frege-Hilbert correspondence, repr. in (Frege, 1984). See (Field, 1991) for a defense of the opposite view.

first two worries could be addressed, the challenge would still remain of making sense of actual scientific practice.[6] How do actual scientific arguments, with their liberal mix of mathematical and physical vocabularies and considerations, succeed in establishing true conclusions? Here Frege's argument still has force and gives us a reason to ascribe truth to premises formulated in a vocabulary that is partly or wholly mathematical.

The third worry points to another argument for the truth of mathematical theorems, which has received much attention in recent years. This argument can be formulated as follows.

P1. Mathematicians' attitude towards mathematical theorems is largely one of taking-to-be-true.

P2. Mathematicians' attitude towards mathematical theorems is largely appropriate.

C. Mathematical theorems are by and large true.

The argument is clearly valid, since the attitude of taking-to-be-true is appropriately taken only towards claims that are in fact true. But are the premises true?

The second premise, P2, is often defended by observing that mathematics is a tremendously successful science, concerning which mathematicians possess great expertise. So there is reason to believe that whatever attitude mathematicians take to their theorems is indeed appropriate. This argument is often given a "naturalistic" development, to the effect that philosophers should defer to the opinions of working scientists, whose track record is better than that of philosophers.[7] Although this form of naturalism would certainly suffice to support P2, it is not necessary. Many philosophers who are deeply opposed to this form of naturalism nevertheless believe that the scientific legitimacy of mathematics should serve as a datum for philosophical theorizing and not be explained away. Kant and Frege are two famous examples. This philosophical orientation too supports P2.

The prospects for denying P1 seem better. The task of determining what attitude mathematicians take to their theorems falls to philosophers or psychologists, not to mathematicians themselves. So here it would be inappropriate simply to defer to mathematicians' opinion. Some philosophers have objected to P1 by invoking a form of *fictionalism*, according to which

[6] Cf. (Burgess and Rosen, 1997) and (Maddy, 1997).

[7] See (Burgess and Rosen, 1997), (Lewis, 1991), pp. 57-9 and (Maddy, 1997).

mathematicians' attitude towards mathematical theorems is some form of acceptance that falls short of taking-to-be-true. For instance, mathematicians accept the claim that there are infinitely many primes but don't actually take it to be true.

However, this objection to P1 is problematic. Is there really room for two kinds of favorable attitude—acceptance and taking-to-be-true—to live side by side? There is no phenomenological evidence that mathematicians' acceptance of a mathematical theorem is any less wholehearted than that involved in their taking various empirical claims to be true. Nor does the view enjoy any solid psychological evidence detectable from an objective, third-person point of view. For instance, mathematicians are not any more disposed to retract their claims when challenged than physicists are. This stands in stark contrast to cases that do involve two kinds of favorable attitude. For instance, meteorologists often find it convenient to talk about Coriolis forces, although they know perfectly well that such forces are purely fictional and invoked only as an innocent trick to simplify arguments and calculations. In such cases meteorologists are consciously aware that their "acceptance" of the relevant claims falls short of a commitment to their truth. And this awareness is externally detectable as a tendency to retract or reformulate the relevant claims when challenged.

I conclude that a strong case can be made that the notion of truth is indeed needed in mathematics. However, as it stands, this argument places few or no constraints on the nature of the notion of truth that is found in mathematics. For all we know so far, this notion may be very minimal or otherwise very different from the notion of truth that is applicable to ordinary empirical statements. To investigate this, I turn in the next section to the question of the semantics for mathematical language.

3 Different conceptions of mathematical truth

In light of the discussion in the previous section, I shall henceforth make the following assumption:

> **Mathematical Truth.** Most sentences accepted as mathematical theorems are true (regardless of their syntactic and semantic structure).

So far, this is not very surprising. Successful sciences discover truths, and mathematics is no exception. But in 1884, Frege observed that this assumption, when combined with another assumption about the correct semantics for the language of mathematics, has the surprising implication that there exist mathematical objects (Frege, 1953). I shall refer to this as the *Fregean argument*.

The second assumption can be formulated as follows.

> **Classical Semantics.** The singular terms of the language of mathematics purport to refer to mathematical objects, and its first-order quantifiers purport to range over such objects.

The word 'purport' needs to be explained. When a sentence S purports to refer or quantify in a certain way, this means that for S to be true, S must succeed in referring or quantifying in this way.

The Fregean argument goes as follows. Consider sentences that are accepted as mathematical theorems and that contain one or more mathematical singular terms. By Mathematical Truth, most of these sentences are true.[8] Let S be one such sentence. By Classical Semantics, the truth of S requires that its singular terms succeed in referring to mathematical objects. Hence there must be mathematical objects.

In fact, if there are any mathematical objects, they would presumably be abstract (that is, non-spatiotemporal and causally inefficacious); otherwise mathematicians' attitude towards such objects would be thoroughly misguided.[9] This makes the following conclusion appropriate:

> **Anti-nominalism.** There are abstract mathematical objects.

Let's now take a closer look at the assumption of Classical Semantics. The claim is that the language of mathematics functions semantically much like language in general has traditionally been assumed to function; that is, the semantic function of singular terms and quantifiers is respectively to refer to objects and to range over objects. This is a broadly

[8] Note that this step uses the parenthetical precisification in Mathematical Truth, without which it would be possible for most sentences accepted as mathematical theorems to be true yet all sentences of the form mentioned in the text to be false.

[9] For critical discussion of the distinction between concrete and abstract, see (Rosen, 2008).

empirical claim about the workings of a semi-formal language used by the community of professional mathematicians.[10] Note also that Classical Semantics is compatible with most traditional views on semantics. In particular, it is compatible with all the standard views on the meanings of sentences, namely that they are truth-values, propositions, or sets of possible worlds.

Classical Semantics enjoys strong *prima facie* plausibility. For the language of mathematics certainly appears to have the same semantic structure as ordinary non-mathematical language.[11] As (Burgess, 1999) observes, the following two sentences appear to have the same simple semantic structure of a predicate being ascribed to a subject (p. 288):

(1) Evelyn is prim

(2) Eleven is prime

This appearance is also borne out by the semantic theories developed by linguists and semanticists. These theories do not exempt mathematical language from the general semantic analyses that they propose.

However, since the philosophical stakes are high, it is unsurprising that Classical Semantics has been challenged. Perhaps the apparent similarities between mathematical and non-mathematical language are deceptive and that mathematical language would be better analyzed in some alternative way. I shall now consider four non-classical conceptions of how mathematical sentences get their truth-values.

The first non-classical conception seeks to reduce mathematical truth to theoremhood. This is an example of what (Benacerraf, 1973) calls a "combinatorial" account of truth. The idea is that a mathematical sentence is true just in case it is a theorem—in some appropriate sense. But what is the appropriate notion of theoremhood?

A notion of theoremhood that is tied to some fixed formal system S is unlikely to be appropriate for the present purpose. To see this, note first that any useful formal system

[10] In the widely adopted terminology of (Burgess and Rosen, 1997), pp. 6-7, Classical Semantics is a *hermeneutic* claim; that is, it is a descriptive claim about how a certain language is actually used, not a normative claim about how this language ought to be used.

[11] This is why (Benacerraf, 1973) argues there is a strong presumption in favor of a unified semantics, which treats mathematical and non-mathematical language in analogous ways.

has to be recursively axiomatizable. Gödel's incompleteness theorems thus imply that, if S is consistent and rich enough to express a certain amount of arithmetic, it will be incomplete; in particular, S will be unable to prove the arithmetical statement $\mathrm{Con}(S)$ which expresses that S is consistent. However, acceptance of a formal system S implicitly involves acceptance of its consistency and thus also of $\mathrm{Con}(S)$. For one would never accept a formal system unless one thought it was consistent.

This informal argument can be made technically more precise as follows. Let S' be the formal system that results from S by adding variables and quantifiers of one order higher than those found in S, as well as the ordinary rules and axioms governing these quantifiers. We can then define in S' a compositional truth predicate applicable to all sentences of S.[12] This enables us to give a proof in S' of $\mathrm{Con}(S)$, something we could not do in the weaker system S. Since it is unreasonable to accept the standards of proof associated with S without also being willing to accept those associated with S', this shows that it is unreasonable to tie mathematical truth to theoremhood in S or any other fixed formal system.

So if mathematical truth is to be reduced to theoremhood, then the relevant notion of theoremhood should not be tied to some fixed formal system. As it turns out, mathematicians often speak about proof without having any specific formal system in mind. A proof in this informal sense is not tied to a fixed formal system but is seen as a sound mathematical argument whose assumptions can be recognized as acceptable axioms by all competent mathematicians. Perhaps mathematical truth can be reduced to the corresponding, informal notion of theoremhood. However, some serious difficulties remain. The informal notion of provability is poorly understood.[13] Moreover, no satisfactory proof-based semantics has been developed for the language of mathematics, despite several decades of attempts by philosophers and logicians such as Dummett and Prawitz.[14]

A second non-classical semantics is inspired by one of the traditional formalist approaches to mathematics. Let's consider the case of arithmetic, where this approach has its greatest

[12] Alternatively, one could add such a truth-predicate as a new primitive symbol, governed by the appropriate axioms.

[13] See, however, (Leitgeb, 2009).

[14] See (Dummett, 1991) and (Prawitz, 2006), as well as the other contributions to the special issue of *Synthese* where the latter appears.

plausibility. The semantics in question denies that the numerals play any direct referential role, claiming instead that they function as symbols on which computations are carried out in accordance with certain "rewrite rules". For instance, one such rule says that $m + S(n)$ may be rewritten as $S(m+n)$; there are other rules governing the other symbols. An arithmetical equation is then said to be true just in case it can be turned into a tautology by means of these rewrite rules.[15]

One problem with this semantics concerns the quantifiers. How are they to be analyzed? Perhaps the most natural approach is to analyze the quantifiers as infinite conjunctions and disjunctions. However, this involves an extreme idealization of what mathematicians and ordinary people mean by their quantifiers. A semantics based on such an extreme idealization effectively gives up on one of the main goals of semantic theorizing, namely to explain the semantic competence that is underlying people's understanding of the language in question. Another problem is that the proposed semantics adopts a non-standard interpretation of the identity predicate '='. Ordinarily, the identity predicate is concerned with the identity of the referents of the terms flanking it; but in mathematics, the predicate is, on the present view, concerned with these terms themselves. This dichotomy is unnatural and poorly motivated. In fact, the proposed semantics seems to conflate the meaning of the identity predicate with our procedures for verifying identity facts.

A third non-classical semantics is associated with the *modal structuralism* proposed in (Putnam, 1967) and developed in (Hellman, 1989). Let's again consider the case of arithmetic. Modal structuralists reject the platonist view that the language of arithmetic is concerned with a particular system of abstract natural numbers. Rather, they understand arithmetical sentences as making assertions about what is necessarily the case in any system of objects that are structured in the way that the natural numbers are supposed to be structured. This structure is described up to isomorphism by the axioms of second-order Peano Arithmetic. Let PA2 is the conjunction of these axioms. Then $\mathrm{PA2}(X, f, a)$ expresses the claim that the relevant structure is instantiated by the collection X of objects (playing the role of the natural

[15]This semantics departs from, but improves on, the view known as *term formalism*, which holds that the numerals denote themselves, understood as either types or tokens. This ascription of reference has bizarre and unintended consequences, such as the truth of sentences such as '0 has topological genus one' and '1 is nearly vertical'.

numbers), the function f (playing the role of the successor function that maps one number n to its immediate successor $n+1$), and the designated object a (playing the role of zero). The modal structuralist analysis of an arithmetical sentence A can now be formalized as follows:

$$\Box \forall X \forall f \forall a [\mathrm{PA2}(X, f, a) \rightarrow A(X, f, a)] \tag{1}$$

Is this an adequate account of arithmetical language as used by professional mathematicians and competent lay people? It will be objected that neither mathematicians nor lay people have in mind these sorts of modalized generalizations when they use arithmetical language. Although I believe this objection has force, it is not obviously conclusive. For instance, (Pettigrew, 2008) proposes that the ordinary symbols for the collection of natural numbers, the successor function, and zero can be understood as "designated free variables", whose values are implicitly assumed to satisfy the antecedent of (1). This provides the resources for a significant response to the above objection.

The final non-classical account of how mathematical sentences get their truth-values is a version of fictionalism. Where the previous version of fictionalism was based on a non-standard view of *the attitude* that mathematicians take to their utterances, this version is based on a non-standard view of *the content* of these utterances. The idea is that in mathematical discourse, language is used in some non-literal way. For instance, Stephen Yablo argues that, although the literal content of a mathematical sentence often involves a variety of exotic mathematical objects, its "real content" is often just that the world is in a particular kind of state, namely a state that makes it appropriate to make the relevant mathematical assertion.[16] For instance, the real content of "the number of planets is 8" is just that there are eight planets. Mathematical objects are invoked primarily because they provide an easier way of conveying the real content. For instance, the real content of "the number of planets is prime" could not be conveyed as easily without talking about mathematical objects.

This version of fictionalism has faced a variety of objections.[17] I shall mention two. The first objection concerns systematicity. Non-literal language use is often quite unsystematic.

[16] See (Yablo, 2005). However, as Yablo is acutely aware, the specification of the real content is often a delicate matter; see (Yablo, 2001).

[17] See (Stanley, 2001).

For instance, metaphors can be highly creative, and the metaphorical effect is fragile and can be destroyed by the replacement of an expression with another with the same semantic value. By contrast, the language of mathematics is one of the more systematic parts of natural language. So *prima facie*, the comparison of mathematical language with metaphorical language is implausible. However, systematicity need not pose an insuperable problem for all non-literal approaches to mathematical language. Although metaphors and many other forms of non-literal language use are highly unsystematic, there is nothing inherent in the idea of non-literal use that conflicts with systematicity. Just as there are systematic rules governing literal meaning, there could in principle be languages with a parallel set of systematic rules governing non-literal meaning.

This brings us to the second objection. Is there really room for two kinds of systematic meaning—literal and non-literal—to live side by side? There is no phenomenological evidence that in mathematics, language is being used in a special, non-literal way. When I say that there are infinitely many primes, it seems to me that I mean precisely what my words mean. Nor is there any robust psychological evidence available from an objective, third-person point of view to support the view that mathematical language tends to be used in some non-literal way. For instance, mathematicians are no more disposed to paraphrase or restate their claims when challenged than other scientists are. By contrast, such evidence is not hard to come by in paradigmatic cases of non-literal language use. For instance, when language is used metaphorically, speakers tend to be aware that certain words' literal meanings are being exploited to create a special, non-literal effect. For this reason, they are also disposed to paraphrase or explain their meaning when challenged.

I conclude that classical semantics is so-called with good reason, as there is substantial evidence in favor of it, and that the non-classical alternatives face serious problems.

4 Objects and objectivity in mathematics

I shall now describe the two forms of realism about mathematics and examine the relation between them. One form is concerned with the independent existence of mathematical objects, and the other, with the objectivity of mathematical truths.

The first form of realism can be expressed as follows.

> **Mathematical Platonism.** There are abstract mathematical objects, and these objects exist independently of intelligent agents and their language, thought, and practices.

Mathematical Platonism goes beyond the thesis of Anti-Nominalism, which we encountered in Section 3. For the latter thesis says just that there are abstract mathematical objects, whereas the former adds a claim about the independent existence of these objects. This independence claim is meant to cash out an analogy between mathematics and the natural sciences. Just as electrons and planets exist independently of us, so do numbers and sets. And just as statements about electrons and planets are made true or false by the objects with which they are concerned and these objects' perfectly objective properties, so are statements about numbers and sets. Mathematical truths are therefore discovered, not invented.

The second form of realism about mathematics can be formulated as follows.

> **Truth-value realism.** Every well-formed mathematical statement has a unique and objective truth-value which is independent of whether it can be known by us or proved from our current mathematical theories.

So truth-value realism is clearly a *metaphysical* view. But unlike mathematical platonism, it is not an *ontological* view. For although truth-value realism claims that mathematical statements have unique and objective truth-values, it is not committed to the distinctively platonistic idea that these truth-values flow from an ontology of independently existing mathematical objects.

What is the relation between these two forms of realism about mathematics? Mathematical platonism clearly motivates truth-value realism by providing an account of how mathematical statements get their truth-values. But further assumptions would be needed for the former view to entail the latter. Even if there are mathematical objects, referential and quantificational indeterminacy may deprive mathematical statements of a unique and objective truth-value (Putnam, 1980), (Field, 1998). Conversely, truth-value realism does not by itself entail mathematical platonism (or even the weaker thesis of anti-nominalism). For we have seen that there are accounts of how mathematical sentences can come to possess unique and objective truth-values which avoid positing a realm of mathematical objects. In fact, many nominalists endorse truth-value realism, at least about more basic branches of mathematics, such as arithmetic.

Nominalists of this type are committed to the slightly odd-sounding view that, although the ordinary mathematical statement

(3) There are primes numbers between 10 and 20.

is true, there are in fact no mathematical objects and thus in particular no numbers. But there is no contradiction here. We must distinguish between the language $\mathcal{L}_M$ in which mathematicians make their claims and the language $\mathcal{L}_P$ in which philosophers make theirs. The statement (3) is made in $\mathcal{L}_M$. But the nominalist's assertion that (3) is true but that there are no abstract objects is made in $\mathcal{L}_P$. The nominalist's assertion is thus perfectly coherent provided that (3) is translated non-homophonically from $\mathcal{L}_M$ into $\mathcal{L}_P$. And when nominalists claim that the truth-values of sentences of $\mathcal{L}_M$ are fixed in a way that doesn't appeal to mathematical objects, it is precisely this sort of non-homophonic translation they have in mind. (The modal structuralism outlined in Section 3 provides an example.) This shows that the theses of mathematical platonism and anti-nominalism will have their intended effect only if they are expressed in the language $\mathcal{L}_P$ used by us philosophers. If the theses were expressed in the language $\mathcal{L}_M$ used by mathematicians, then nominalists would be able to accept them while still denying that there are mathematical objects.

We have seen that it is possible to accept one form of realism about mathematics without the other. I now survey some of the main philosophical views on the relation between the two forms of realism.

A traditional platonistic view accepts both forms of realism but regards the platonist form as more fundamental. On this view, the objective truth-values of mathematical statements are underpinned and explained by the independent existence of mathematical objects. It is because these objects exist and have their properties independently of us that every meaningful mathematical question has an objective answer. Mathematics is in this respect like the natural sciences, whose statements also have objective truth-values that are determined by the objects concerned and these objects' perfectly objective properties.

However, this traditional platonist view gives rise to serious epistemological problems. The proposed order of explanation may well be appropriate in the natural sciences, where it makes sense to talk about first identifying certain objects and then examining these objects in order to determine their mind-independent properties. But it is doubtful that it makes sense to talk about identifying an abstract mathematical object independently of determining the truth-values of statements about it. If we have any kind of "access" to such objects at all, this seems to proceed via the truth-values of claims about these objects.[18]

These considerations suggest that mathematical platonists should proceed in the opposite direction and regard truth-value realism as more fundamental than mathematical platonism. This reversal of explanatory direction is often associated with Frege and his "context principle", the most famous occurrence of which is found in the *Foundations of Arithmetic*.

> How, then, are the numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a sentence that words have any meaning, our problem becomes this: To define the sense of a sentence in which a number word occurs. ((Frege, 1953), §62)[19]

The proposal is thus that the question about our "access" to mathematical objects should be transformed to a question about the meaning of complete sentences concerned with these objects.

When implementing this proposal, Frege argued that it is particularly important to explain the meaning of identities flanked by number terms of the form '$\#F$', which abbreviates "the number of Fs". He suggests that the meaning of such identities can be explained by what has become known as Hume's Principle:

(HP) $$\#F = \#G \leftrightarrow F \approx G$$

[18] A more general epistemological problem was articulated by (Benacerraf, 1973). How can concrete beings like us gain knowledge of abstract objects with which we cannot be causally influenced? It remains controversial whether this problem of "epistemic access" is genuine; see (Burgess and Rosen, 1997). The problem described in the text is more specific, as it concerns the explanatory direction associated with traditional platonism.

[19] My translation differs slightly from that of J.L. Austin in that I render Frege's original 'Satz' as 'sentence' rather than 'proposition'. This should be uncontroversial, as the context in which a syntactic item such as a word occurs is clearly that of a sentence, not a proposition.

where the right-hand side abbreviates the formalization in pure second-order logic of the claim that the Fs and the Gs can be one-to-one correlated. This principle, and abstraction principles more generally, have since played a central role in Fregean approaches to mathematics (Wright, 1983), (Hale and Wright, 2001).

The Fregean tradition regards questions about complete sentences as explanatorily more fundamental than questions about individual singular terms. An important aspect of this orientation is the view encapsulated in Kreisel's famous dictum that 'the problem is not the existence of mathematical objects but the objectivity of mathematical statements'.[20] That is, the sentence-level question of truth-value realism is more fundamental than the question of mathematical platonism, which is concerned with the reference of singular terms. An influential proponent of this orientation is Dummett, who has long urged that the debate about platonism should be replaced by, or at least transformed into, a debate about truth-value realism, because the latter debate is more tractable and of greater importance to philosophy and mathematics than the former ((Dummett, 1978a), pp. 228-232 and (Dummett, 1991), pp. 10-15)). Mathematical objects are on this view at best a kind of byproduct of the objectivity of mathematical sentences. Insofar as we can "identify" any mathematical objects at all, this goes via their properties.

This emphasis on the close connection between mathematical objects and mathematical objectivity is shared by some non-Fregean approaches as well. One example are non-eliminative structuralist views, which hold that there are mathematical objects but that these are nothing but positions in abstract mathematical structures. By tying mathematical objects to their structures, such views link the question about the existence of mathematical objects and the question about truth in a structure. Indeed, Shapiro goes as far as to say that eliminative and non-eliminative structuralism "are equivalent" and "[i]n a sense [...] say the same thing, using different primitives" (Shapiro, 1997), pp. 96-7.

[20]As reported in (Dummett, 1978b), p. xxxviii; see also (Dummett, 1981), p. 508. The remark of Kreisel's to which Dummett is alluding appears to be (Kreisel, 1958), p. 138, fn. 1 (which, if so, is rather less memorable than Dummett's paraphrase).

5 Defenses of truth-value realism

Truth-value realism claims that every meaningful mathematical sentence has a unique and objective truth-value that is independent of whether it can be known by us or proved from our current mathematical theories. I now discuss two strategies for defending this claim.

One strategy aims to show that in many core branches of mathematics, our theories and conceptions determine a structure that is unique up to isomorphism (in the sense that any two models that satisfy one of these theories and conceptions are isomorphic). A theory or conception with this property is said to be *categorical*. Since the truth-value of a sentence is the same in any two isomorphic models, any categorical theory or conception will thus ensure that every sentence of the relevant language has a unique and objective truth-value.

How can a mathematical structure be determined uniquely up to isomorphism? One popular answer appeals to the categoricity theorems of second-order logic, which say that categorical characterizations of many of the basic structures of mathematics are available, provided that the second-order quantifiers are given the standard interpretation as ranging over all subsets of the domain (Shapiro, 2000a). In particular, on the standard interpretation of second-order logic, categorical characterizations are available of the structures of the natural numbers and the real numbers, as well as of some important initial segments of the iterative hierarchy of sets.

However, the reliance on second-order logic is a cause for concern. The argument assumes that there is a unique standard interpretation of the second-order quantifiers and that we have succeeded in giving our second-order quantifiers this particular interpretation. But what assurance do we have that these assumptions are satisfied? One might respond that the second-order quantifiers are perfectly well understood and can therefore legitimately be taken at face value. But this response is problematic in the present dialectical situation. For the ordinary language of arithmetic is just as well understood as that of pure second-order logic. So if it is legitimate to take the latter at face value and dismiss worries about non-standard interpretations, presumably the same goes for the former. But then the detour via second-order logic becomes redundant! In short, why should someone who is genuinely worried about the categoricity of arithmetic be any less worried about the interpretation of our second-order quantifiers? It is not a good answer that the second-order quantifiers belong to pure logic whereas the primitive expressions of arithmetic do not. For regardless of the logicality of these expressions, they will have to be interpreted, and mathematical logic shows that a rich variety of non-standard interpretations are available.

For a defense of the categoricity of arithmetic to do the requisite philosophical work, the resources that it employs must be less problematic than those of arithmetic. Several philosophers have recently argued that a schematic form of Peano arithmetic fits the bill. Let me outline the argument as developed by Charles Parsons.[21] The key observation is that the principle of mathematical induction has a schematic character. When we learn that any property had by 0 and inherited from one natural number to the next is had by any natural number, the notion of property that is used is not tied to what is expressible in a fixed language, nor to any other fixed domain of properties. Rather, whenever we become convinced that a formula $\phi(x)$ can be meaningfully applied to the natural numbers, we accept the corresponding induction axiom:

$$\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall x(\mathbb{N}(x) \rightarrow \phi(x))$$

Assume now that two arithmetically competent people engage in discourse and become convinced that the other's arithmetical vocabulary is meaningful. The schematic principle of induction then licenses each person to use induction axioms that contain the other's arithmetical vocabulary. The two people can thus go through the steps of the ordinary categoricity argument and prove that their respective 'copies' of the natural numbers are isomorphic.[22] Each person can thus convince herself that 'her' natural numbers are isomorphic to those of any other person with whom discourse is possible. This restricted form of categoricity appears to be sufficient for most purposes.

The previous paragraphs have focussed on the case of arithmetic. What about analysis and set theory? The concerns about the reliance on (standardly interpreted) second-order logic remain largely unchanged. However, it remains unclear whether a schematic approach,

[21] See (McGee, 1997), (Lavine, 1994), and (Parsons, 2008), section 49.

[22] Strictly speaking, we also need a schematic form definition by primitive recursion. But this is no more problematic than the schematic form of induction.

with less problematic assumptions, can be made to work beyond arithmetic.

The second strategy for defending truth-value realism aims to show that there are non-arbitrary ways of extending incomplete mathematical theories. Gödel's incompleteness theorems provide a simple example. Provided that it is consistent, PA neither proves nor refutes the formalization, Con(PA), of the claim that it is consistent. But as we saw in Section 3, it is far more plausible to extend PA by adding Con(PA) than by adding the negation of this formula. In particular, Con(PA) becomes provable when second-order logic is added to PA. Emboldened by this observation, Gödel sought to generalize. Perhaps all cases of incompleteness can be eliminated by adding expressive and inferential resources of higher and higher order, or (equivalently, Gödel thought) by adding axioms that require the iterative hierarchy of sets to extend higher and higher.[23] Unfortunately, it has turned out that some of the most interesting cases of incompleteness stubbornly resist this attempted elimination. An example is Cantor's famous Continuum Hypothesis, which is neither provable nor refutable from standard ZFC set theory and remains so even when large cardinal axioms are added. In contemporary set theory the discussion has therefore moved on to more "extrinsic" considerations for and against possible extensions of our mathematical theories. For instance, the evidence that is adduced concerns the extent to which a proposed extension is natural, explanatory, theoretically fruitful, and extends already established patterns.

Perhaps such considerations will one day convince the mathematical community to adopt certain new axioms.[24] If so, will this development support truth-value realism by providing an example of how mathematical statements can have objective truth-values despite not being provable from our current theories? The answer will depend on how the convergence on the new axioms is best explained.

One explanation is that the convergence results from an implicit decision by mathematical community to sharpen their concepts and theories in a particular way. A comparison may be instructive. Suppose the chess community was required to explore ways of extending the rules of chess. A convergence on one optimal way of doing so might well result. But the best

[23] See (Gödel, 1946), p. 151. For discussion of this point, as well as the material in the rest of this paragraph, see (Koellner, 2006).

[24] This is in no way clear, as there is substantial resistance, as illustrated for instance by (Feferman, 1999).

explanation of this convergence would not be that the chosen additions were true or valid all along but rather that the resulting extension has attractive properties that moved the community to favor this extension of their game over alternative ones.

Another explanation of a convergence on new axioms sees it as a gradual uncovering of an implicit conception that the mathematical community has of the mathematical structure in question. A comparison may again be instructive. Perhaps the principle of induction was implicit in mathematicians' conception of the natural numbers prior to its first systematic and widespread adoption as an axiom in the 17th century. Earlier mathematicians may for instance have been implicitly committed to the view that the natural numbers are exhausted by 0 and numbers that can be reached from 0 by iterated application of the successor function. If so, then the convergence on the principle of induction as an axiom would only support a weak form of truth-value realism because the truth of the principle was already present, if not exactly in mathematicians' explicit theory of the natural numbers, then at least in their implicit conception of this structure.

Thus, even if there were to be a convergence on new axioms for set theory, more work would be needed to show that this phenomenon supports truth-value realism at all (unlike the first alternative explanation outlined above) or that it supports more than a weak form of truth-value realism (unlike the second alternative explanation).

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