Dept. of Math., University of Oslo
Research Report in Mechanics No.1
ISSN 0801–9940 March 2019

Approximate end and maximum moment formulations for slender columns in frames with sway

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ABSTRACT

Based on a study of isolated restrained columns, and two-column panel frames, the often complicated mechanics of moment formations in the members are studied using elastic second-order theory for the purpose of establishing a base on which procedures for moments due to sidesway can be evaluated. A distinction is made between moment developments in "supporting (bracing) sway columns" and "supported (braced) sway columns". The primary objective is to derive improved slender column design expressions that are suitable in approximate design code formats for columns in frames with sway. Both shear, end moments and maximum moments, applicable over the full range of axial loads, are considered. Proposals are presented that will allow more economical designs of framed columns with sway. Extended applications to the general case of load combinations that include both gravity and sideways loading are also made, and discussed with reference to formulations found in the literature and to two major design codes for structural concrete and steel structures.

KEYWORDS

Buckling, columns (supports), design, elasticity, frames, stability, second-order theory, sidesway, design moment formulations, structural engineering.

1 Introduction

1.1 General

Columns and frames in most structures, such as buildings and bridges, exhibit behaviour that are bounded on one extreme by completely elastic behaviour and on the other by completely inelastic behaviour. Nevertheless, design procedures for bending and stability of slender concrete and steel columns (beam-columns) are most often based on elastic concepts and analyses. Elastic methods that consider second-order effects of axial forces are therefore important in practical analysis and design of such structures. This includes approximate methods, that have been, and still are, important in parallel with, or as a complement to, more exact methods.

Sidesway of one-storey structures, or stories of multi-storey structures, caused by lateral loads and possible unsymmetry in the structure and loadings, can be handled by existing second-order analysis computer programs. Also, approximate methods whereby overturning (global) second-order effects of vertical loads are accounted for by storey (system) sway displacement magnification factors, are available. For an overview of such factors found in the literature and in structural codes, see for instance Hellesland (2009a, 2009b). Common practice with approximate methods in existing design codes is to magnify column end moments (for second-order effects) by such storey-sway factors. However, unlike the sidesway, which increases percentage wise by the same amount at all interacting column axes on the same level (storey), the corresponding increase of the larger end moment of the individual columns of the storey will be smaller (Hellesland 2019). It will also be different in the different columns due to different axial load levels and end restraints of the columns.

The reduced increase in the larger end moment, as compared to the sway increase itself, is not recognized in present approximate design procedures of major codes (e.g., ACI 318 (ACI 2014), AISC 360 (AISC 2016), Eurocode 2 (CEN 2004)and Eurocode 3 (CEN 2005), etc.). Some, but limited, attention to the matter can be found in the literature. The AISC Commentary dealt with a reduction in the maximum moment due to sidesway already in 1969, and later in the 1978 edition of the Commentary. The same reduction factor expression was suggested for the sway action of all columns of the storey, and it was allowed to approximate it by a constant factor (0.85). Both the expression (derived for the special case of a cantilever column fixed at one end) and the constant are unconservative in a general case. In later commentary revisions, this reduction factor is omitted.

Hellesland (1976), recapitulated in a slightly different form in Hellesland (2009a), derived an expression for end moments in individual column axes for special cases (columns with equal end moments, or with one end moment known). LeMessurier (1977) derived a similar expression for a cantilever column, and proposed an extension to a general column of a sway frame. However, the extension is incorrect in that it yields the same moment magnifier for moments at both ends of all columns of the storey. Hellesland and MacGregor (1981) presented an extension of the previous work by Hellesland (1976). Lui (1992) also suggested an approach allowing for different moment magnification factors in different column axes. The approach seems not to be well founded and breaks down in the general case (as discussed in Hellesland (1992)).

This aspect of different end moment magnifiers in the different column axes, and other behavioural aspects, was considered in detail in another, companion report (Hellesland 2019), in which the response of rotationally restrained columns and frame panels with sidesway were studied, over the full range of axial loads and thus considering both supporting (bracing) and supported (braced) sway columns. In that report, emphasis was on column mechanics and the identification and establishments of simple, novel closed form expressions defining characteristic points in the axial load-moment solution space, useful in teaching and design work, and as a complement to full second-order analyses.

1.2 Objectives

In light of the findings in Hellesland (2019), main attention of the present report is directed towards the formation of moments at the ends, and between ends, of slender columns in elastic frames with sway, and to establish extended and improved methods for prediction of such moments, that are suitable in approximate design code formats. The full range of axial loads are considered, thus covering both suporting and supported columns. More specifically, the main objectives are 1) to derive appropriate moment multiplication factors that, when applied to first-order column end moments due to sidesway, account for local (member) second-order effects, and, not least 2) to establish improved multiplication factors for the prediction of maximum column moments between ends of columns with sway, and to consider possible extensions to moments from load combinations of gravity and sideways loads.

1.3 Superposition, problem definition, scope

Subject to certain conditions, elastic analyses allow the use of the invaluable principle of superposition, that allows results from different load cases to be added directly together to give total results. As is well known, this principle is valid for elastic first-order analysis results, and also for results of second-order analyses provided the axial forces in the various members in the structure are the same in all load cases to be added together. When this is so, the total moments obtained from second-order analysis at member ends can be written

$$M_1 = M_{1b} + M_{1s} \quad \text{and} \quad M_2 = M_{2b} + M_{2s} \tag{1}$$

for a load combination consisting of two load cases labelled b and s, here taken to be due to gravity loading (selfweight etc.) and lateral (sideways) loading, respectively. The moments in Eq. (1) are total moments (including first-order and second-order effects) as computed according to second-order theory.

In slender compression members of the structure, the maximum moment of compression members may develop between ends. For such members without transverse loading, the exact maximum moment, M_{max}, can be expressed in terms of an end moment, for instance the one at the end with the larger moment sum. Then, if denoting this end as end 2, M_{max} can be written as follows:

$$M_{max} = B_{tmax}\, M_2 = B_{tmax}\,(M_{2b} + M_{2s}) \tag{2}$$

where the absolute value of the maximum moment multiplier, B_{tmax}, in second-order analysis may be written in a well known form (e.g., Galambos 1968, Hellesland 2019) by

$$B_{tmax} = \sqrt{\frac{1 + \mu_t^2 - 2\mu_t \cos pL}{\sin^2 pL}} \tag{3}$$

when $\mu_t > \cos pL$, and $B_{tmax} = 1$ otherwise. Here, $pL = L\sqrt{N/EI}$, and $\mu_t = -M_1/M_2$ is the ratio between end moments (Eq. (1)), positive when end moments act in opposite directions at the two ends, and negative otherwise.

For the superposition principle to be valid in second-order analyses (equal axial forces in all load cases to be added together in a load combination), second-order computer analyses will have to be carried out for one full load combination at a time (with all loads in the combination applied to the structure), rather than split up into the two parts M_b and M_s. With the powerful computers available today, this is certainly feasible, but nevertheless a drawback of second-order analyses in many cases of day-to-day design work.

If second-order analyses programmes are not available, or not found practicable, advantage may be taken of approximate first-order based elastic methods for which the uperposition principle is valid. Such methods may also be preferred in applications to structures with strongly nonlinear material properties for which equivalent, representative elastic stiffnesses are not always easy to determine.

Most structural design codes offer approximate methods expressed by magnified first-order moments. To consider transition to such methods, Eq. (2) may first be rewritten in terms of first-order moments, here denoted M_{0b} and M_{0s} for the two load cases mentioned above. Expressed in terms of these at end $j = 2$, Eq. (2) becomes

$$M_{max} = B_{tmax}(B_{e2b}\, M_{02b} + B_{e2s}\, B_s\, M_{02s}) \tag{4}$$

where, at the considered end 2, $B_{e2b} = M_{2b}/M_{02b}$ and $B_{e2s} = M_{2s}/(B_s M_{02s})$. These two factors reflect local second-order effects, while B_s reflects global second-order sway effects (to be discussed later). In the formulation above, it is tacitly implied that all effects of first-order sidesway displacements are included in M_{0s} (also possible sidesway effect due to gravity loading on non-symmetrical structures). Eq. (4) provides a formulation suitable for discussing simplifications. Emperically based approximations, often implied tacitly in publications on the topic, are to set $B_{e,b} = 1$ and $B_{e,s} = 1$ and to replace B_{tmax} (in terms of the total end moment ratios, $\mu_t = -M_1/M_2$) by a factor B_{max} expressed in terms of first-order end moment ratios $\mu_0 = -M_{01}/M_{02}$. Then, Eq. (2) is replaced by a first-order based approximation that may be expressed by

$$M_{max} = B_{max}\, M_{02} = B_{max}\, (M_{02b} + B_s M_{02s}) \tag{5}$$

The accuracy of this expression is dependent on the accuracy of B_{max} in describing local (member) second-order effects.

The "evolution" to this type expression took many years, and many turns and twists. Details of these and alternative formulations are reviewed and discussed in Hellesland (2008). The form above was first presented by Lai and MacGregor (1983) and incorporated into the Canadian code in 1984 (CSA 1984), and into the sway frame provision of the ACI 318 code in 1985.

The scope and efforts of the present report is primarily aimed at formulation of improved moment expressions for the sway action, i.e. to the second portion of the maximum moment expression in Eq. (5), and corresponding end moment formulations. Even though it is common practice in building structures to design each column for the maximum moment along the column, reliable end moment predictions are also of interest, in particular for the design of adjacent structural elements (beams, foundations, etc.). Also formulations for the gravity and sway combination are examined, and proposals presented.

2 Frame mechanics – Moment formation

Fig. 1(a) shows a one storey, laterally loaded three bay frame that relies on the columns for its lateral resistance and stability. The principles of the presentation below would not be altered, however, if the frame had been partially braced by an external bracing in the form of a truss, flexible shear wall, or similar. In the absence of axial forces in the columns, the lateral load (H) gives rise to a first-order sway displacement (Δ_0) that is equal at all column tops when axial beam deformations are neglected. The lateral load is at this stage resisted by (first-order) column shears (V_0) that are proportional to the relative lateral stiffness of the columns. When axial loading, that may be different in the different columns, is applied, the sway displacement of the frame increases to $\Delta = B_s\Delta_0$ due to second-order effects. In this process, the shears redistribute, from their first-order values (V_{0i}) to their final values (V_i) in the respective column axes i.

The sway displacement magnifier (B_s) is is defined by

$$B_s = \frac{\Delta}{\Delta_0} \tag{6}$$

It reflects global second-order (overturning "$N\Delta$") effects of all interconnected columns on the same level (storey) of the frame system. It includes, in the general case, also local second-order ("$N\delta$") effects in the individual columns (due to the axial load action on the column deflection away from the chord through the column ends). A brief review of a general B_s expression is given in Section 3.

For the sake of the illustration, a pin-ended Column 4 is included in the figure. It does not contribute towards providing lateral support (bracing) of the frame. Rather, it has a "driving" or overturning effect on the frame displacement, and needs lateral support itself in order to remain stable laterally, as indicated in the figure by its negative shear force. It "leans" on the rest of the frame for its lateral stability. So does Column 3, that also has a negative shear, but to a lesser extent. This leaves Column 1 and 2 to provide lateral frame stability. Column 1 has no axial load and is consequently not affected by local second-order effects.

The formation of moments in the columns will depend on the the axial loading (second-order effects) and will be discussed with reference to Fig. 1(b), where possible moment distributions along the individual column axes are illustrated.

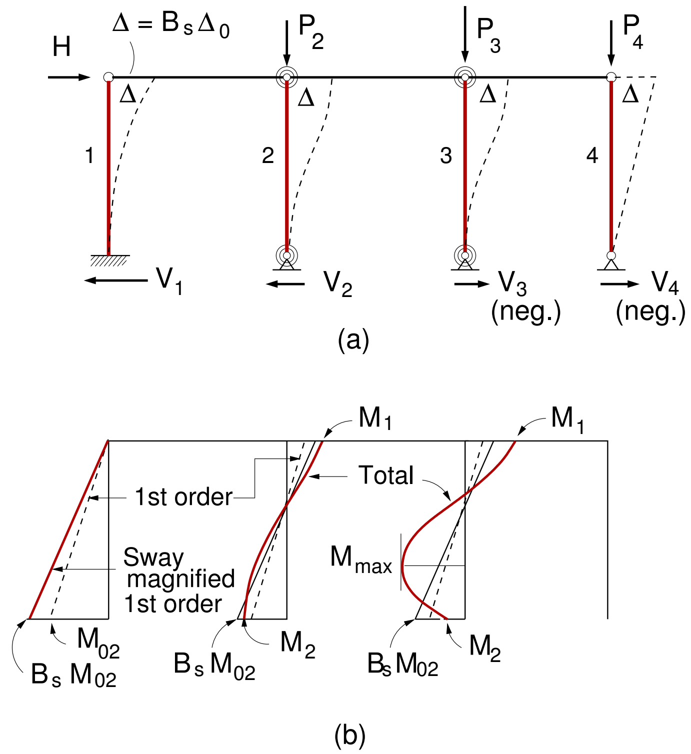

Figure 1: *(a) Laterally loaded, braced three-bay frame; (b) Possible moment diagrams in laterally supporting and supported columns. (From Hellesland et al. (2013)).*

No local second-order effects.
In Column 1, with no axial force and therefore no local second-order effects, the moment diagram stays linear, and the final moments are directly proportional to the sway displacement:

$$M_0^* = B_s M_0 \tag{7}$$

Although this final moment varies linearly along the column axis in the way a first-order moment does, it does not comply with the definition of a first-order moment ("obtained from equilibrium based on the undeformed geometry"), since it includes global second-order effects (through B_s in the present presentation). For the sake of precision and distinction, it may be labelled "sway-magnified first-order moment" and denoted M_0^* (Hellesland 2008, Hellesland et al. 2013).

These moments are also indicated in the moment diagrams for Column 2 and 3. The pin-ended Column 4 has no shear and no moments due to sidesway. If it was not perfectly straight, its axial force would have caused imperfection moments between ends (first- and second-order).

Moderate local second-order effects.
Column 2 has a positive shear, and thus contributes to the bracing of the frame, and it has a moment diagram typical for columns with moderate local second-order effects (due to combination of axial load level, slenderness and rotational end restraints). The end moment has decreased somewhat below the sway-magnified first-order moment at end 2 (with the stiffer end restraint), and has increased somewhat at end 1. In other cases, also the moment at end 1 may decrease.

The end moments at end $j = 1$ and $j = 2$ can be given in terms of the respective sway-magnified first-order end moments as

$$M_j = B_j(B_s M_{0j}) \qquad j = 1, 2 \tag{8}$$

where B_1 and B_2 are end moment factors at end 1 and 2, respectively. They reflect the local second-order effects as manifested at the ends. They may be greater or smaller than unity. In the figure illustration, the maximum moment in Column 2 is at end 2. Thus $B_{max} = B_2 = 1$ in this case.

Significant local second-order effects.
In columns with significant local second-order effects, the maximum moment may form away from the column end. It may be expressed by

$$M_{max} = B_{max}(B_s M_{02}) \tag{9}$$

where M_{02} is the first-order moment at end 2, which, in line with conventional practice, will be defined as the end at which the first-order moment has its largest absolute value. The maximum moment multiplier B_{max} reflects local second-order effects as manifested at the maximum moment location.

Shear formulation.
Similarly, the final shear, which is of importance for deriving moment expressions, may be expressed by

$$V = B_v(B_s V_0) \tag{10}$$

3 Global second-order effects

Although the sidesway displacement ($\Delta = B_s \Delta_0$) is given (assumed to be known) in this study, it is appropriate for the sake of perspective and completeness, and useful for later discussions, with a brief review of a suitable approximate magnifier expression for unbraced frames, or partially braced frames such as that in Fig. 2.

The storey (system) sway magnifier B_s is a function of the lateral stiffness of all interconnected columns of a single storey frame (or on the same level (storey) of a multistorey frame), and possible external bracing force S_B (per unit displacement). It can be determined from horizontal equilibrium between the applied external horizontal load and the shears (given by Eq. (23)). Solving for B_s gives

$$B_s = \frac{1}{1 - \alpha_{ss}} \tag{11}$$

where α_{ss} is the storey (system) stability index defined by

$$\alpha_{ss} = \frac{\sum(\gamma_n N/L)}{(H/\Delta_0)} \quad \text{or} \quad \alpha_{ss} = \frac{\sum(\gamma_n N/L)}{\sum(\gamma_s N_{cs}/L) + S_B} \tag{12 a, b}$$

Here, the ratio H/Δ_0, between a horizontal load H applied to the top of the storey and the **corresponding** first-order displacement Δ_0 due to this H, is the first-order lateral stiffness of the storey (system), and γ_n and γ_s are flexibility factors, reflecting local second-order effects, given and discussed in Section 7.1. The summations are over all interacting columns. Possible external bracings, for instance due to a truss such as in Fig. 3, or similar, are normally included in the computed first-order stiffness (H/Δ_0) in Eq. (12a). In multistorey structures, Δ_0 is the interstorey first-order displacement and H the corresponding storey shear.

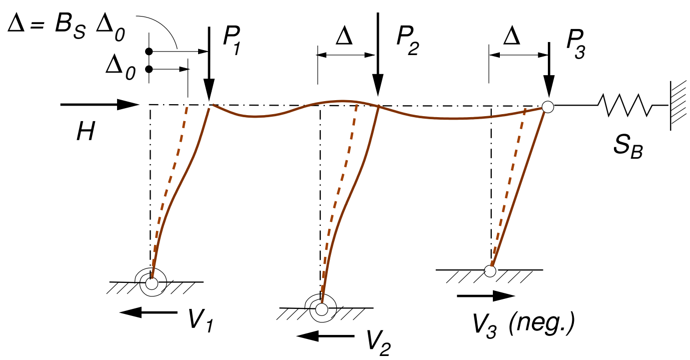

Figure 2: *Partially braced frame with sidesway (from Hellesland (2009a))*

The two B_s expressions are equivalent provided the pseudo-critical loads (in the latter expression) are calculated with the same first-order restraints implicit in the H/Δ_0 calculation. In the derivation, Δ_0 was assumed to be equal in all axes (implying axial beam deformations to be neglible).

These forms were presented by Hellesland (2009a) and applied to 1) system instability problems ($\alpha_{ss} = 1$), to the prediction of 2) effective lengths, 3) sway displacements and 4) end moments. With the γ_n factor defined as above, covering the full range of axial loads, and thus both laterally supporting and supported columns (and potential sway-braced column interaction), the B_s expression in

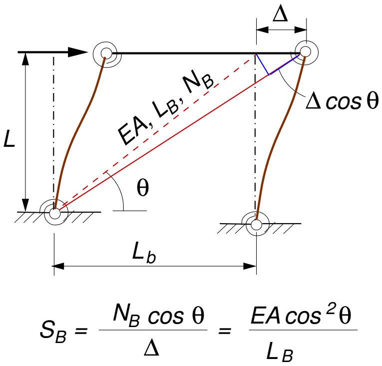

Figure 3: *Diagonal bracing truss.*

Eq. (12a) above represented a novel formulation when it was presented. In earlier work, also by the author (Hellesland 1976, 1981, 2007), no distinction was made between a γ_s and a γ_n factor. The single gamma factor used was simply taken equal to that defined by the present γ_s.

Several authors and codes give similar sway magnifiers. They can all be derived from simplifications of the α_{ss} in Eq. (12) above. For instance, in regular moment frames, in building structures with similar stiffness and loading of columns, axial load levels are relatively low so that γ_n can safely be approximated by γ_s. Also, with constant column lengths, L can be placed outside the summation signs, and, furthermore, the $\sum(\gamma_n N)$ can be replaced by the mean effect $\overline{\gamma}\sum N$. Stevens (1967) suggested an effect that can be expressed by $\overline{\gamma} = 10/9$ $(= 1.11)$. Lai and MacGregor (1983) suggested values for different floor levels, and with $\overline{\gamma} = 1.15$ suggested as an overall conservative value. The AISC code (2016, Appendix 8) implies $\overline{\gamma} = 1/0.85 = 1.18$. The ACI code (ACI 2014) and the Eurocode 2 (CEN 2004) and Eurocode 3 (CEN 2005) imply $\overline{\gamma} = 1$, which is unconservative.

For practical frames, B_s will normally be lower than 1.5, corresponding to α_{ss} values lower than 0.33. This is well below the loading causing system (global) instability (at $\alpha_{ss} = 1$). For loads giving a braced critical load index α_b (Eq. 17b) of an individual column that exceeds α_{ss}, premature system instability will be induced by local column instability (in an approximate braced buckling mode of the column).

4 Local second-order effects

The local member second-order ($N\delta$) effects can be quantified by the ratios of results obtained from second-order analyses of a column with a specified sidesway

$\Delta = B_s\Delta_0$ and given axial loads N and results from the same analyses for $N = 0$ (first-order, determined from an analysis that is based on the original, geometry). These factors are

$$B_{max} = M_{max}/(B_s M_{02}) \tag{13}$$

$$B_1 = M_1/(B_s M_{01}) \tag{14}$$

$$B_2 = M_2/(B_s M_{02}) \tag{15}$$

$$B_v = V/(B_s V_0) \tag{16}$$

Above, M_{max} is expressed as a function of the first-order moment at end 2, which per definition is taken as the end with the larger first-order end moment (absolute value).

For frames with sway due to lateral and axial loading only, these coefficients depend on (1) the end restraints, which then uniquely define the first-order moment gradient, and on (2) the axial load level defined for instance by the nondimensional load parameters α_s, α_b or α_E, as defined by

$$\alpha_s = \frac{N}{N_{cs}}\ ;\ \ \alpha_b = \frac{N}{N_{cb}}\ ;\ \ \alpha_E = \frac{N}{N_E} \qquad (17\,\mathrm{a-c})$$

Here, N_{cs} and N_{cb} are the free-to-sway and the fully braced critical load, respectively, of the column when considered in isolation from the rest of the frame, but with rotational restraints reflecting (in an approximate sense) the interaction with the real frame of which it may be a part. Except when the frame consists of a single column, these are strictly pseudo-critical loads, but are useful in column characterisation and discussion. N_E is the socalled Euler load (critical load of a pin-ended column), which is a convenient reference load parameter in several contexts.

For an elastic, framed member of length L, uniform axial load level and uniform sectional stiffness EI along the length, the critical loads above can in the conventional manner be defined by

$$N_{cs} = \frac{N_E}{\beta_s^2} \quad ; \quad N_{cb} = \frac{N_E}{\beta_b^2} \quad ; \quad N_E = \frac{\pi^2 EI}{L^2} \qquad (18\,\mathrm{a-d})$$

where β is an effective length factor, equal to β_s and β_b for the free-to-sway and the braced case, respectively. As defined above, the load indices are interrelated. For instance, $\alpha_s = \alpha_E\,\beta_s^2$, $\alpha_b = \alpha_E\,\beta_b^2$, $\alpha_b = \alpha_s\,(\beta_b/\beta_s)^2$.

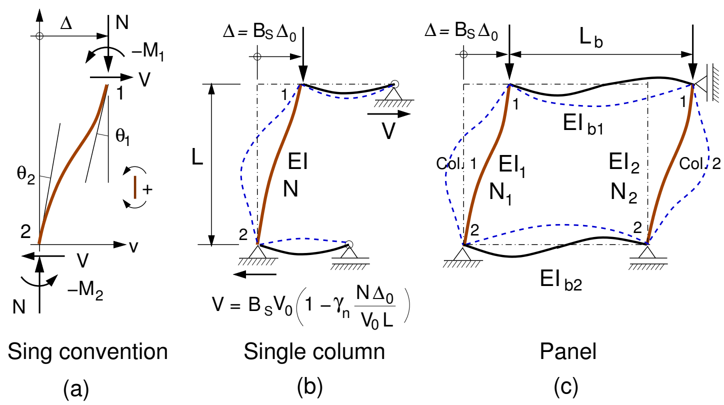

Figure 4: *(a) Sign convention; (b) Single column model; (c) Two-column panel model. Dashed curves show possible deflection modes at member instability.*

5 Local second-order analysis of sway-displaced columns

For the verification of the proposals of the present paper, results were obtained using the second-order analyses presented in Hellesland (2019). Columns considered are either single restrained columns, as shown in Fig. 4(b), or columns that are part of a panel frame, such as shown in Fig. 4(c). The columns are initially straight and have lengths L and uniform section stiffnesses EI. The deflection shapes indicated by the solid lines in the figure, are those due to an initial, imposed top (joint) displacement $\Delta = B_s\Delta_0$. For this Δ to remain constant for increasing axial loading, the column shears V (lateral loading) will have to decrease to compensate for the increased overturning effect of the vertical loading acting on the relative joint displacement. No gravity load induced moments (such as from loading on beams) are included. The dashed lines are deflection shapes developing as the critical axial loading is approached. For the panel, other deflection shapes may result depending on the relative stiffness and axial loading in the members.

The restrained single column may be the complete structure, or it can be considered isolated from the two-column panel, or from a greater frame. In the latter case, the rotational end restraints should reflect the rotational interaction at the joints with restraining beams ("horizontal interaction"), and possible other columns framing into the considered joint ("vertical interaction").

The rotational end restraints can conveniently be represented by rotational restraint stiffnesses (or spring stiffnesses) labelled k_1 and k_2 (equal to the moments required to give a unit rotation), or in nondimensional form by κ_j at end $j = 1$

and $j = 2$ defined by

$$\kappa_j = \frac{k_j}{(EI/L)} \qquad j = 1, 2 \tag{19}$$

Alternatively, similar factors, such as the well kown G factors may be used. Unlike the κ factors, that are nondimensional stiffness factors, the G factors are nondimensional, *scaled* flexibility factors. In their generalized forms (Hellesland and Bjorhovde 1996a, 1996b) they can be defined by

$$G_j = b_o \, \frac{(EI/L)}{k_j} \quad \left(= \frac{b_o}{\kappa_j} \right) \qquad j = 1, 2 \tag{20}$$

where b_o is simply a reference (datum) factor by which the relative restraint flexibilities are scaled. Conventional datum values, as adopted for instance in AISC (2016), ACI (2014), are $b_o = 6$ and $b_o = 2$ for unbraced and braced frames, respectively. It should be noted that $b_o = 6$ is used throughout this paper, unless otherwise noted.

In the present paper, end restraints are due to beams. Then, k_j is equal to the rotational stiffness $k_{bj} = (\sum bEI_b/L_b)_j$ of all beams connected to the joint considered. For rigidly connected beams with negligible axial forces and antisymmetrical, double curvature bending, the bending stiffness coefficient b becomes $b = 6$. In this case, $b_o = 6$ will cancel out in Eq. (20), and the well known, conventional G factor expression for the lateral loading case can be obtained. Other familiar values of b are 2, 3 and 4, obtained for restraining beams bent in symmetrical single curvature, beams pinned at the far end and beams fixed at the far end, respectively.

6 Elastic column response in frames with sway

6.1 Single columns – Stationary restraints

Characteristics study.
In a another study (Hellesland 2019), response characteristics were presented for single columns and panels with various restraint combinations. Also presented were closed form expressions defining a number of key characteristics, or ”landmarks” in the moment versus axial load “map”, useful for for enabling a quick establishment of moment-axial load relationship of laterally displaced columns. Moment results from that study, to be used for the verification of approximated moment proposals, and points of interest for the present paper, will be briefly recaptured and discussed where appropriate.

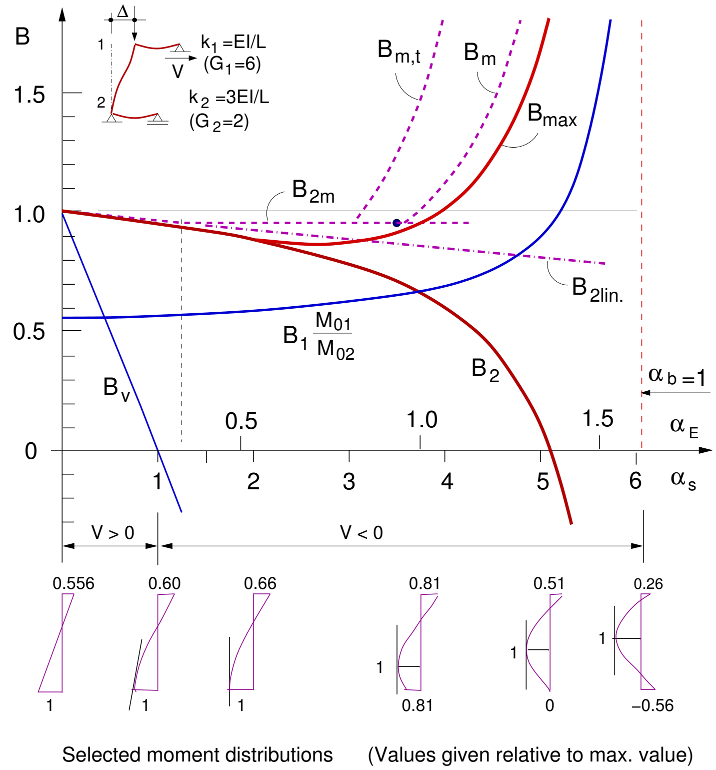

Figure 5: *Moments and shear versus axial load level in column with unequal, relative flexible end restraints* ($\beta_s = 1.932$, $\beta_b = 0.785$, $\alpha_E = 0.268\alpha_s$)

General.
Typical moment and shear responses versus increasing axial load, computed with the second-order analysis, are shown in Fig. 5, Fig. 6, Fig. 7 and Fig. 8 for columns with unequal, rotational end restraints with different degree of rotational stiffness. The columns are illustrated by the insert in the upper, left hand part of the figures. The column tops are initially displaced laterally by an amount $B_s\Delta_0$ ("sway-magnified first-order displacement"), and then kept constant at this value (in a real case by the action of the overall frame of which the column may be considered isolated from).

The moments and shear are shown nondimensionally in terms of the respective B factors, Eqs. (13) to (16). Axial forces are given nondimensionally in terms of axial load indices α_s and α_E (Eq. (17)). All moment results in the figure are given in terms of B_sM_{02}. Therefore, $B_1(= M_1/B_sM_{01})$ is represented by $B_1 \cdot M_{01}/M_{02}$ in the figures.

The curves labelled B_{2lin}, representing a linear approximation of B_2, and the maximum moment factor approximations, B_m, $B_{m,t}$ and B_{2m}, will be discussed later (Section 9 and 11). So will the dots (blue) on the B_{2m} curves.

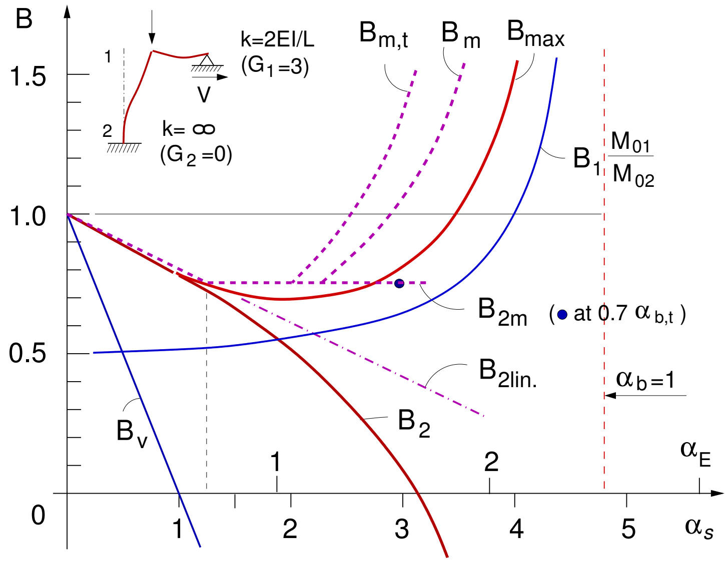

Figure 6: *Moments and shear versus axial load level in column with unequal end restraints, $G_1 = 3$, $G_2 = 0$ ($\beta_s = 1.373$, $\beta_b = 0.626$)*

The elastic critical loads N_{cb} of the fully braced columns, with the end restraints given in the figure inserts, correspond to $\alpha_b = 1$ in the figures, or $\alpha_E = 1/\beta_b^2$. For an elastic column, the braced critical load is independent of whether the column is fully braced at zero or at a non-zero end displacement. Both moments and shears approach infinity (in either the positive or negative direction) as $\alpha_b = 1$ is approached.

System instability of real frame.
The results in the figures are independent of the sway magnification factor B_s, as stated previously. However, if the considered column was part of a larger frame, it is worth noting that system (total frame) instability (reflected in practice by B_s reaching large, unacceptable values) may result well before the present "local instability" is reached (at "$\alpha_b = 1$" in the figure). System instability may in some cases of very slender columns be initiated by "local instability" (buckling between ends), in which case α_b approaches 1.0 at finite B_s values.

End moments.
For the laterally loaded columns, the largest absolute value of the first-order end moment is obtained at the end with the largest rotational restraint stiffness. This end is conventionally denoted end 2 and the moment M_2 (B_2). So also here. As seen, M_2 (or B_2) decreases continuously with increasing load level. At some point, it becomes zero and changes direction. End moments at the two ends become equal at $\alpha_E = 1$ (Hellesland 2019).

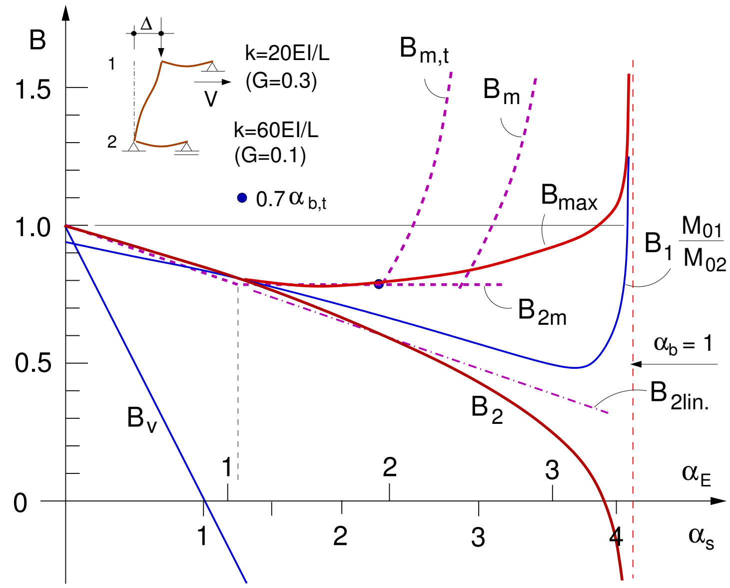

Figure 7: *Moments and shear versus axial load level in column with unequal, very stiff end restraints ($\beta_s = 1.065$, $\beta_b = 0.532$, $\alpha_E = 0.88\alpha_s$)*

In cases with relatively low to moderate restraint stiffness at end 1 (Fig. 5, Fig. 6), the end moment M_1, reflected by $B_1 \cdot M_{01}/M_{02}$, typically increases slowly at first with increasing axial load, and then more sharply towards infinity at $\alpha_b = 1$.

In the case with unequal, very stiff restraints (Fig. 7), both end moments decrease markedly at first with increasing axial load. Then, when the end restraints at the two ends are not too different, as in the present case, B_1 and B_2 follow each other fairly closely up till rather high load levels at which B_1 reaches a minimum, and then starts increasing sharply towards plus infinity as $\alpha_b = 1$ is approached.

Maximum moment.
The maximum moment (M_{max}, B_{max}) is initially equal to the larger end moment (M_2, B_2). Following an initial decrease along the B_2 path, B_{max} starts forming away from end 2. This happens, depending on restraints, for relative axial load levels within the alternative, equivalent ranges defined by (Hellesland 2019)

$$0.25 \geq \alpha_E \leq 1.0 \quad \text{or} \quad 1.0 \geq \alpha_s \leq \beta_s^2 \tag{21}$$

Thereafter, following a continued small decrease with increasing load, maximum moments increase and approach infinity for axial loads approaching the braced critical load.

Shears.
The value of the shear, or B_v, required to maintain the column displacement

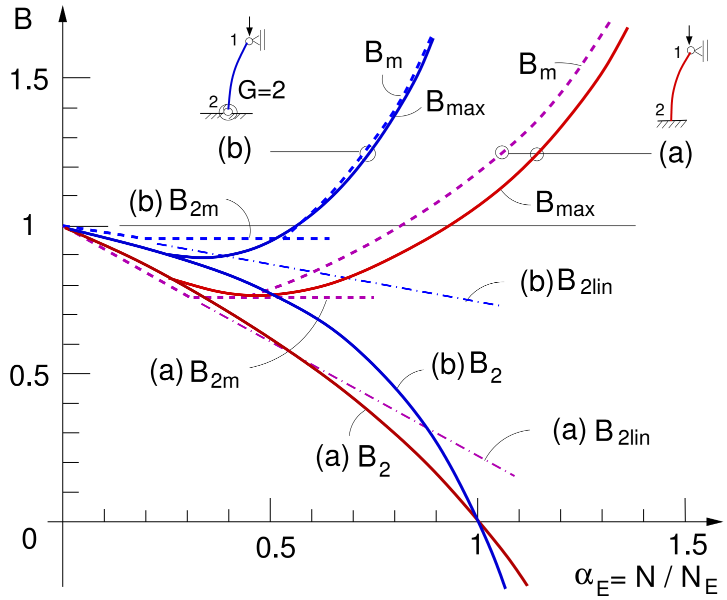

Figure 8: *Moments versus axial load level for two different cantilever columns: (a) Fixed at the base ($\beta_s = 2$ and $\beta_b = 0.7$); (b) Partially fixed at the base ($\beta_s = 2.635$ and $\beta_b = 0.843$); $\alpha_s = \alpha_E \beta_s^2$*

$\Delta = B_s \Delta_0$, decreases from 1.0 for zero axial load (at $\alpha_s = 0$, or $\alpha_E = 0$) to zero at the free-sway critical load ($\alpha_s = 1$, or $\alpha_E = 1/\beta_s^2$) and becomes negative as α_s increases further. At $\alpha_s = 1$, the function of the column changes. Columns with $\alpha_s < 1$ ($V \leq 0$) are capable of resisting external lateral loads, and is referred to as laterally "supporting sway columns". On the other hand, columns with $\alpha_s > 1$ ($V < 0$) need support (negative shears) to maintain the specified sidesway, and is referred to as laterally "supported sway columns".

6.2 Panel columns – Non-stationary restraints

In the preceding section, end restraints of the single columns were given as a constant (stationary, invariant) value with increasing axial loading. For frames with more than one column, end restraints will not be stationary in the general case, due to differences in axial load levels and stiffnesses of the columns. This can be seen in Fig. 9, where a panel of two columns, rigidly connected to beams at the top and bottom, is considered. It is illustrated in the insert at the top left of the figure. Also shown by an insert in the figure is a single column, isolated from the panel by assuming hinged supports at the first-order inflection points (near midlengths) of the panel beams.

EI/L values of the panel members are EI/L, $1.1EI/L$, $0.333EI/L$ and $1.667EI/L$ for Column 1 (left hand), Column 2 (right hand), top beam and bottom beam, respectively. The bottom beam is considerably stiffer than the upper beam, and will attract the larger first-order end moments. Axial forces are neglected in the beams. The columns have the same axial force N. Then, because of the stiffness difference, the load index in Column 1 (left hand) becomes 1.1 times that in Column 2 ($\alpha_{E1} = 1.1\alpha_{E2}$). Thus, Column 1 is the more flexible of the two panel columns, and the one at which system instability will be initiated.

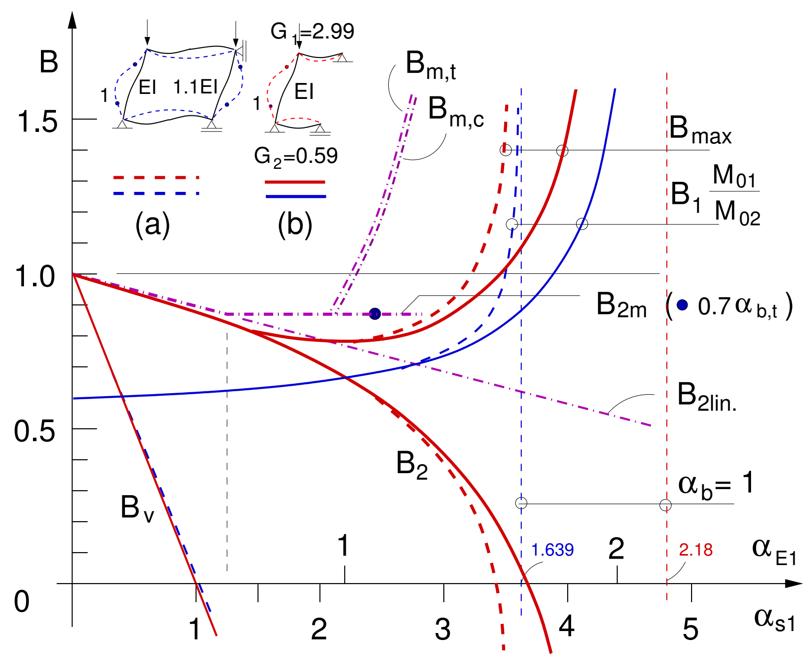

Figure 9: *Moments and shears for two cases: (a) Column 1 (see insert, left hand) of Panel 1 (dashed lines); (b) Column 1 in the panel considered in isolation with approximate restraints (solid lines).* B_{2lin} *are approximate secant predictions of* B_2.

The results in Fig. 9 are for the most flexible Column 1 in the panel (broken lines) and for the isolated Column 1 (solid lines). Results are plotted versus α_{E1} (α_E for Column 1). The $\alpha_{s1}(= \alpha_{E1}\beta_{s1}^2)$ abscissa, added for convenience and information, is computed with the free-sway critical load of the isolated Column 1 defined above (with effective length factor $\beta_{s1} = 1.483$).

The critical loads of the panel columns are lower than those of the isolated columns with stationary restraints. This is due to the reduced restraints offered by the panel beams as the axial column loading increases towards the braced critical load. This is indicated in Fig. 4, and in the insert in Fig. 9, by the unwinding of the beams from nearly antisymmetrical double curvature ($k \approx 6EI_b/L_b$) to nearly symmetrical single curvature bending ($k \approx 2EI_b/L_b$).

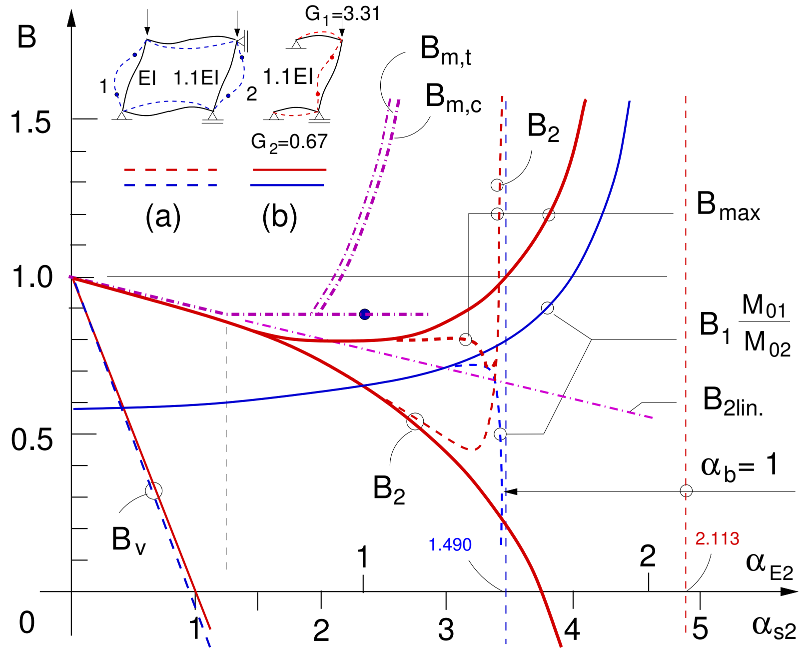

Figure 10: *Moments and shears for two cases: (a) Column 2 (see insert, right hand) of Panel 1 (dashed lines); (b) Column 1 in the panel considered in isolation with approximate restraints (solid lines). B_{2lin} are approximate secant predictions of B_2.*

Similar results for the stiffer Column 2 of the panel, and for Column 2 considered in isolation (with $\beta_{s2} = 1.522$), are shown in Fig. 10. Associated with the reduction in beam restraints, a rather sudden reversal of end moments is seen to take place in the stiffer Column 2 for loads close to the critical load.

The isolated column is seen to describe the panel end moment response almost exactly up to fairly high axial load levels. More details and discussion of response characteristics of this and another panel (with a stiffer Column 2) are given elsewhere (Hellesland 2019).

7 Shear formulation

7.1 Generalized shear expression

The shear expression $V = B_v B_s V_0$ (from Eq. (10)) with the linear approximation of the shear factor B_v defined by

$$B_v = 1 - \alpha_s \tag{22}$$

provides an excellent shear approximation for supporting sway columns ($\alpha_s < 1$), as seen in Figs. 5, 6 and 7 for single columns and in Figs. 9 and 10 for panel columns. However, it becomes increasingly inaccurate (giving too small negative values) with increasing α values typical for highly loaded, supported (braced) sway columns. As a consequence, sway magnifier expressions and end moments calculated based on Eq. (22) may become very inaccurate for highly loaded columns.

In an effort to extend the range of applications to include effects of highly loaded columns, Hellesland (2009a) established an approximate shear formulation that covered the full transition of a column from free-sway to nearly fully braced. It was applied to derive the sway magnifier in Section 3, and it will be applied in the general moment formulation in Section 8. It is briefly reviewed below.

The general shear factor B_v can be defined by

$$B_v = 1 - \frac{\gamma_n N \Delta_0}{V_0 L} \tag{23 a}$$

or

$$B_v = 1 - \alpha_s \frac{\gamma_n}{\gamma_s} \tag{23 b}$$

The second expression above is obtained from the first expression by first noting that the free-sway critical load N_{cs} can be solved for from Eq. (23a) at $B_v = 0$ (free-sway condition), and written

$$N_{cs} = \frac{V_0 L}{\gamma_s \Delta_0} \tag{24}$$

Eq. (23b)) can then be formed by recalling that $\alpha_s = N/N_{cs}$ and noting that γ_n, which is a load dependent factor, will be denoted γ_s at the free-sway condition, such that

$$\gamma_n = \gamma_s \quad \text{at} \quad \alpha_s = 1 \tag{25}$$

The novelty of Eq. (23) is represented by the γ_n factor and the distinction between γ_n and γ_s. Both are column flexibility factors representing the increased column flexibility caused by the column axial load acting on the deflection of the column away from the chord through the column ends ($N\delta$ effects). This increased flexibility in turn increases the global sway. The γ_s factor will be a more well known factor than the general γ_n factor and is discussed in more detail in Section 7.2.

The general flexibility factor γ_n is defined by Hellesland (2009a) in terms of γ_s and two additional flexibility terms:

$$\gamma_n = \gamma_s + \Delta\gamma_1 + \Delta\gamma_2 \quad (\geq \gamma_s) \tag{26 a}$$

$$\Delta\gamma_1 = 0.12\,(\gamma_s - 1)(\alpha_s - 1) \quad ; \quad \Delta\gamma_2 = 0.6\,\alpha_{s,b}\left(\frac{\alpha_s - 1}{\alpha_{s,b}}\right)^8 \qquad (26\,\mathrm{b,c})$$

Here, the free-sway load index ($\alpha_s = N/N_{cs}$) is defined previously, and $\alpha_{s,b}$ is the same index at the fully braced critical load ($N = N_{cb}$). Thus,

$$\alpha_{s,b} = \frac{N_{cb}}{N_{cs}} = \left(\frac{\beta_s}{\beta_b}\right)^2 \qquad (27)$$

The constants 0.6 and 8 in Eq. (26c) are strictly not constants, but vary with rotational end restraints, and in particular with the difference in restraints at the two ends. For the sake of simplicity, fixed values were chosen following comparisons with exact results for a wide range of restraints.

The variation in γ_n for loads between zero (γ_0) and the free-sway buckling load (γ_s) is very small. For instance, for a cantilever column fixed at the base, $\gamma_0 = 1.2$ (1.19 according to the two first terms of Eq. (26a)) and $\gamma_s = 1.216$. The suggested lower limitation on γ_n ($\geq \gamma_s$) in Eq. (26a) is not necessary, but may be adopted when this represents a simplification. For pin-ended columns ($N_{cs} = 0$), $\gamma_n = 1$. This value may also be taken in the rare case of a column with axial tensile loading.

The full γ_n formulation (Eq. (26)) can be reduced to the first term (γ_s) for practical sway frames with reasonably similar columns and loads in the various axes. In other cases, it is adequate to include the two first terms. Still other cases, in particular when axial loads are close to the local braced critical load, may call for the full formulation.

7.2 The flexibility factor at the free-sway condition

The γ_n factor may take on large values as the braced critical load is approached. At $\alpha_s = 1$, $\gamma_n = \gamma_s$. The γ_s factor varies between 1 and 1.216 (1.22) for columns with positive end restraints, and becomes up to about 1.34 in cases with negative restraints at one end (Hellesland 2009a, 2009b), such as in single curvature regions of multistory unbraced frames where columns may have negative end restraints (Hellesland 2009b).

Several diagrams have been established for easy determination of γ_s for positive end restraints (Hellesland 1976, LeMessurier 1977), and also for cases with both positive and negative restraints (Hellesland 2009a, 2009b). Furthermore, general γ_s expressions are available, notably one first given by Rubin (1973). A more detailed summary is provided in Hellesland (2009b).

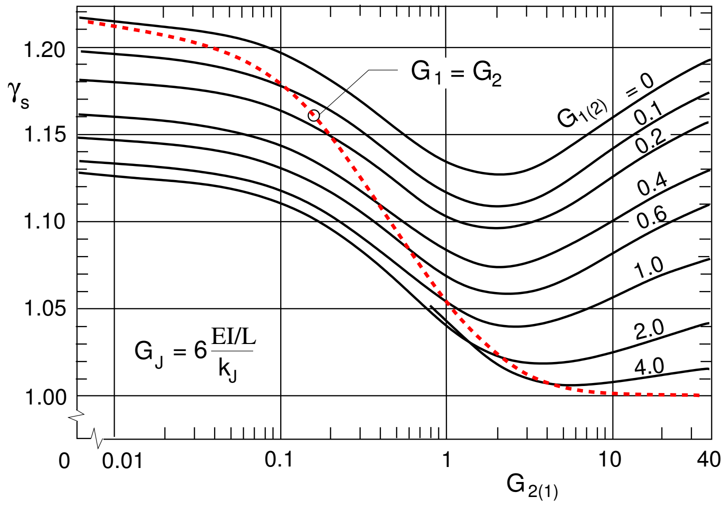

Figure 11: *The flexibility factor γ_s at the free-sway condition versus positive rotational end restraints in terms of G factors (reproduced from Hellesland (1976)).*

A rather simple approximate expression (Hellesland 2000) that is adequate in the present context, is given by

$$\gamma_s = 1 + 0.11\,\frac{1 + [1 - (0.5\,G_{max})^{\,p}]^{\,3}}{(1 + 0.5\,G_{min})^{\,2}} \tag{28}$$

where $p = 1$ for $G_{\max} \leq 2$ and $p = -1$ for $G_{\max} > 2$. $G_{\max}$ is the larger and $G_{\min}$ the smaller of the G factors at the two column ends.This expression can be extended to include cases with negative end restraints. Eq. (28) breaks down into a case (pin-ended) that will be considered later (in conjunction with Eq. (34).

Eq. (28) was initially proposed by the author in 1981 (during a research stay at the University of Alberta, Edmonton), based on observation of the variation of γ_s with changing restraints as illustrated in Fig. 11 (from Hellesland (1976)).

8 Moments due to sidesway - General

Good end moment formulations require good descriptions of the shear V. Here, the shear description in Section 7.1 is adopted. From moment equilibrium of a laterally loaded (displaced) column (Fig. 4a), $M_1 + M_2 + N\Delta + VL = 0$, where $\Delta = B_s\Delta_0$ and V is the shear given by Eq. (23(b)), the end moment sum may be written

$$\frac{M_1 + M_2}{B_s(M_{01} + M_{02})} = 1 - \frac{\gamma_n - 1}{\gamma_s} \cdot \alpha_s \tag{29}$$

From this equation, end moments can be computed directly in cases in which there is only one unknown end moment. There is two such cases. These are: 1) columns pinned at end 1 ($M_{01} = M_1 = 0$), and 2) columns with equal end restraints ($M_1 = M_2$). In such cases, Eq. (29) reduces to

$$\frac{M_2}{B_s M_{02}} = 1 - \frac{\gamma_n - 1}{\gamma_s} \cdot \alpha_s \tag{30}$$

End moment predictions with this equation are shown in Fig. 12 (from Hellesland

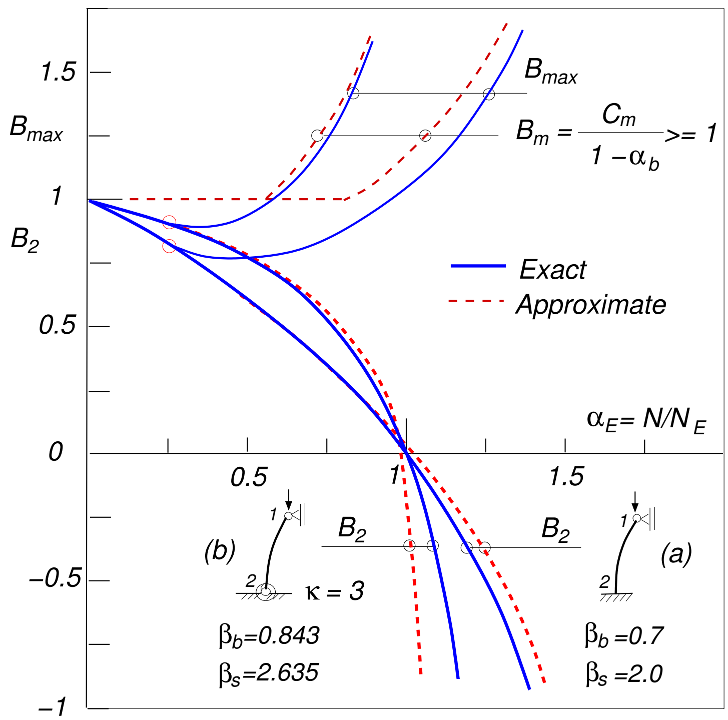

Figure 12: *Moment factors versus axial load levels for cantilever columns with two different base restraints (from Hellesland (2009a)).*

(2009a)) for two cantilever columns with an imposed sidesway at the top. The top is pinned ($G_1 = \infty$) and the base is either fully fixed ($G_2 = 0$) or partially fixed ($G_2 = 2$, $k_2 = 3EI/L$), respectively. The agreement with theory (full lines, labelled "Exact" in the figure) is seen to be good, in particular in the positive B_2 range. Maximum moment predictions (B_m) will be discussed later.

In the general case, with unequal end restraints at the two ends, Eq. (29) is not directly useful. The distribution of the moment sum to the two ends must be established before individual end moments can be calculated. Efforts at accomplishing this have not been successful, and remains a task for future research.

The simpler task of establishing moment expressions that are valid over a more limited load range, is pursued here. In typical moment frames, most of the columns will be supporting sway columns with $\alpha_s < 1$. As shown earlier (Eq. (21)), the maximum moment is always located at an end (with the stiffest rotational restraint) in such columns. It is therefore of considerable interest to establish approximate expressions for the end moments in this range, and somewhat beyond.

9 Simplified end moment formulations

9.1 Secant formulation

End moments will generally be discussed in terms of the end moment factors B_1 and B_2 for the simplified case in which γ_n is approximated by its value, γ_s, at the free-sway critical load. This implies a linear shear variation versus axial load (Eq. (22)), and also a linear variation of the moment sum (Eq. (29)). The linear shear variation is within 1 % of correct shears for laterally supporting (bracing) columns ($\alpha_s < 1$).

Linear moment relations for the individual end moments, taken as secant approximations to the end moment curves at load levels in the range $\alpha_s = 0$ to 1.0, were in Hellesland (2019) expressed by

$$B_{1lin} = \frac{M_1}{B_s M_{01}} = 1 - (1 - B_{1s})\,\alpha_s \tag{31}$$

and

$$B_{2lin} = \frac{M_2}{B_s M_{02}} = 1 - (1 - B_{2s})\,\alpha_s \tag{32}$$

where, B_{1s} and B_{2s} are the moment factor values at $\alpha_s = 1$ shown in Fig. 13 for a wide combination of end restraints.

Results for B_{2s} and B_{1s} coincide in the case with equal end restraints, and are shown by the dash-dot borderline (labelled $G_1 = G_2$). Results for B_{1s}, shown by dashed lines in the figure, and B_{2s}, shown by solid lines, are located above and below the borderline, respectively. G_2 is by definition taken to represent the end with the stiffer restraint (with the smaller G value). Corresponding B_{1s} and B_{2s} curves terminates therefore at the dash-dot curve. At $G_2 = 0$ (fixed end), B_{2s} may have values between 0.79 and 0.82, and B_{1s} between about 0.82 and 1.05.

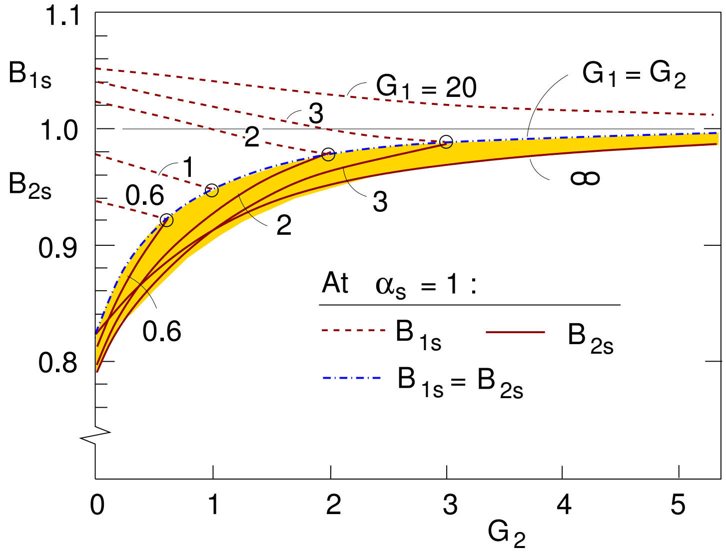

Figure 13: *End moment factors B_{1s} and B_{2s} at $\alpha_s = 1$ versus end restraints in terms of G factors (from Hellesland (2019)).*

9.2 B_{2s} factors in two special cases

For the two cases of a column pinned at end 1 and a symmetrically restrained column ($M_1 = M_2$), Eq. (29) transforms to

$$B_2 = \frac{M_2}{B_s M_{02}} = 1 - (1 - \frac{1}{\gamma_s})\,\alpha_s \tag{33}$$

Evaluated at $\alpha_s = 1$, B_2 becomes $B_{2s} = 1/\gamma_s$. For the two special cases considered, for which γ_s is can be found in the literature, B_{2s} can now be defined as follows:

1) *Column pinned at end 1*; $\gamma_s = \gamma_{s,pin1}$:

$$B_{2s} = \frac{1}{\gamma_{s,pin1}} \quad \text{with} \quad \gamma_{s,pin1} = 1 + \frac{0.216}{(1 + 0.5\,G_2)^2} \tag{34 a, b}$$

2) *Column with equal end restraints*; $\gamma_s = \gamma_{s,equal}$:

$$B_{2s} = \frac{1}{\gamma_{s,equal}} \quad \text{with} \quad \gamma_{s,equal} = 1 + \frac{0.216}{(1 + G_2)^2} \tag{35 a, b}$$

The γ_s expression given by Eq. (34b), was derived by Hellesland (1976), and along different lines and in a different form ($C_L = 1 - \gamma_s$) by LeMessurier (1977). Also, Eq. (28) breaks down into this expression for the pin-ended case. The equal restraint case consists of two pin-ended columns of length $L/2$. Thus, Eq. (35b) can be obtained from Eq. (34b) by replacing L by $L/2$. The factor 0.216 may clearly be rounded off to 0.22, or even 0.2, in practical applications. The latter (0.2) is the correct value of the flexibility factor at zero axial load level (γ_0).

$B_{2s} = 1/\gamma_s$ for the two cases defined above (Eq. (34) and Eq. (35)), can be identified in Fig. 13 by the solid line labelled "∞" and by the dash-dot line labelled "$G_1 = G_2$", respectively. The equal-ended case can be seen to provide an upper bound on the B_{2s} results, and the pin-ended case a practical lower bound, except at low G_2 values (less than about 1.5) where slightly, but insignificantly, smaller B_{2s} values result.

9.3 Approximate B_{2s} factors for arbitrary retraints

"Combination" B_{2s} factor for arbitrary retraints.
A possible solution might be to adopt an approximate "combination" expression for γ_s, that yields results enclosed by the "limiting" cases given by Eq. (34) and Eq. (35)). Such an expression might be written as follows:

$$B_{2s} = \frac{1}{\gamma_{s,comb}} \quad \text{with} \quad \gamma_{s,comb} = 1 + \frac{0.216}{(1 + \frac{G_1 G_2}{2G_1 - G_2})^2} \tag{36}$$

Conservative B_{2s} factor for arbitrary retraints.
Alternatively, since the equal-ended case provides an upper bound on the B_{2s} results, a conservative (a little too big) estimate of the maximum end moment will result by taking B_{2s} in Eq. (32) according to Eq. (35).

"Average" B_{2s} factor for arbitrary retraints.
Finally, since the B_{2s} results lie within a reasonably narrow band (shaded in Fig. 13), another alternative in the general case is obtained by adopting an "average" B_{2s} value located approximately in the middle of the shaded band. Such a factor may be given by

$$B_{2s} = \frac{1}{\gamma_{s,aver}} \quad \text{with} \quad \gamma_{s,aver} = 1 + \frac{0.24}{(1 + 0.75G_2)^2} \tag{37 a}$$

or, when rewriting, by

$$B_{2s} = 1 - \frac{0.24}{0.24 + (1 + 0.75G_2)^2} \tag{37 b}$$

This expression is independent of G_1 (reflecting the relative insensitivity to G_1) and gives B_{2s} values within $\pm$ 2.5% of the correct value. This accuracy is generally quite acceptable, and suggests the use of B_{2lin} (Eq. (32)) expressed with B_{2s} taken according to this latter alternative in the general case.

9.4 Approximate B_{1s} factor

"Rigorous" estimate of B_{1s} factor for moment M_{1s}.
When B_{2s} is known, the moment factor B_{1s} at column end 1 can be solved for from Eq. (29) for α_s=1 (at which $\gamma_n = \gamma_s$). By substituting $M_1 = B_{1s}B_sM_{01}$ and $M_2 = B_{2s}B_sM_{02}$ into Eq. (29), B_{1s} can be solved for. It becomes

$$B_{1s} = \left(B_{2s} - \frac{1-\mu_0}{\gamma_s}\right)\frac{1}{\mu_0} \tag{38}$$

where

$$\mu_0 = -\frac{M_{01}}{M_{02}} = -\frac{G_2+3}{G_1+3} \tag{39}$$

is the ratio between the first-order end moments. Such defined, μ_0 becomes positive for single first-order curvature bending and negative for double curvature bending.

The ratio expression above in terms of end restraints can easily be established (e.g., see Hellesland 2019). The γ_s factor in Eq. (38) is a function of both G_1 and G_2. Sufficiently accurate values can be obtained from Eq. (28).

The accuracy of B_{1s} from Eq. (38) is dependent on the B_{2s} approximation. Taking B_{2s} according to the "average", as given by Eq. (37), B_{2s} will sometimes be smaller, and sometimes larger than the correct value. An overestimation of B_{2s} will lead to an underestimation of B_{1s}, and vice versa.

Example 1: $G_2 = 1$, $G_1 = 2$: $\mu_0 = -0.8$, $\gamma_s = 1.049$ (Eq. (28)). According to Fig. 13, $B_{2s} \approx 0.925$ and $B_{1s} \approx 1.0$. Predictions: $B_{2s} = 0.927$ based on the "average" value, Eq. (37); $B_{1s} = (B_{2s} - 1.716)/(-0.8)) = 2.145 - 1.25B_{2s} = 0.996$ according to Eq. (38). The predicted values are almost identical to the correct results.

Example 2: $G_2 = 1$, $G_1 = 1$: $\mu_0 = -1.0$, $\gamma_s = 1.055$. According to Fig. 13, or Eq. (35), $B_{2s} = 0.946$ and $B_{1s} = 0.946$. Predictions: $B_{2s} = 0.927$ based on the "average" value, Eq. (37); $B_{1s} = 1 - 1.896B_{2s} = 0.969$ from Eq. (38). These predictions underestimate B_{2s} by about 2%, and overestimates B_{1s} by about 2.4 %.

These accuracies are quite acceptable, and the application of Eq. (38) is straigthforward. However, considering the lesser importance of obtaining very accurate B_{1s} predictions, the simpler, more approximate approach below may be justified.

Simplified estimate of B_{1s} for moment M_{1s}.
B_1 for columns with flexible restraints at end 1 stays almost stationary with increasing axial load at low and moderate load levels (Fig. 5), and follows, as observed earlier, B_2 for stiffer restraints at end 1 (Fig. 7). On this base, a simpler approximation is proposed, defined by Eq. (31) with

$$B_{1s} = 1 \qquad \text{for} \quad G_1 > 1.25 \tag{40 a}$$

$$B_{1s} = 1 - \frac{0.22}{0.22 + (1 + G_1)^2} \qquad \text{for} \quad G_1 \leq 1.25 \tag{40 b}$$

This equation is a rewrite of Eq. (35), in which G_2 is replaced by G_1. Thus, for $G_1 \leq 1.25$, B_{1s} is directly given by the curve labelled $G_1 = G_2$ in Fig. 13.

Examples: For the examples considered above, this simplified approach, Eq. (40), gives $B_{1s} = 1.0$ in Example 1, and $B_{1s} = 0.948$ in Example 2. These predictions are close to those by the more rigorous approach above, and is sufficiently accurate.

9.5 B_{2lin} compared with exact results

Linear end moment predictions at end 2, defined by B_{2lin}, Eq. (32), with B_{2s} given by the "average" γ_s, Eq. (37), are shown in Figs. 5, 6 and 7 and 8 for single restrained columns, and in Figs. 9 and 10 for the two columns of a panel. It can be seen that the linearized end moment approximation generally provides good B_2 predictions not only for supporting sway columns ($\alpha_s < 1$), but also well beyond this range in the cases considered.

10 Rising moment branch predictions

10.1 Common formulation

The exact maximum moment, $M_{max} = B_{max} B_s M_{02}$ (Eq. (9)), is here approximated by

$$M_{max} = B_m B_s M_{02} \tag{41}$$

where the approximate member magnification factor is denoted B_m to distinguish it from the exact B_{max}. The rising moment branch is commonly computed using variations of B_m below:

$$B_m = B_b \geq \lim B_m \tag{42}$$

$$B_b = \frac{1 + A\,\alpha_b}{1 - \alpha_b}\,C_m \tag{43 a}$$

$$C_m = 0.6 + 0.4\mu_0 \tag{43 b}$$

$$\mu_0 = -\frac{B_s M_{01}}{B_s M_{02}} = -\frac{M_{01}}{M_{02}} \tag{43 c}$$

Here, α_b and N_{cb} are the braced critical load index (Eq. (17b)) and the critical load (Eq. (18b)) of the column considered braced, respectively. C_m is a factor correcting for non-uniform first-order moment gradient along the column, and μ_0 is the first-order end moment ratio (positive when the member has single first-order curvature bending, and negative otherwise). A is a factor, typically about 0.25 for pin-ended columns with uniform first-order bending, but commonly neglected ($A = 0$). So also in computations of this study.

Conventional ("present practice") application ($\lim B_m = 1$**;** $A = 0$).
With the lower limit taken as $\lim B_m = 1$, and $A = 0$, the approximate B_m formulation above can be found in most structural design codes (such as ACI 318 and AISC 360 (limited to "braced moments"), Eurocodes 2 and 3). In AISC and ACI, B_m is denoted B_1 and δ, respectively. The C_m expression, proposed by Austin (1961) for steel structures, was initially limited to 0.4 to guard against lateral-torsional buckling. With such phenomena covered by separate provisions in AISC, the 0.4 limit was later omitted in AISC. And also in the ACI flexural bending code provisons.

The $\lim B_m$ will be adressed below.

Conventional end restraint assumptions.
For a given, single column, the load index α_b is to be computed with the given restraints (stationary). For a general framed column, being part of a larger frame, the bending mode of the beams may not be stationary, but change with increasing load level as discussed in conjunction with the panels in Figs. 9 and 10 (non-stationary restraints). For regular frames it is common in column design practice to assume that beams bend into symmetrical, single curvature (with rotational bending stiffness $2EI_b/L_b$) at braced frame instability, and to use such restraints in the calculation of α_b. This is considered a prudent approach, and is in accordance with most codes of practice. This assumption will also be adopted in the comparisons below.

B_m – **Effective length factor approach**.
Eq. (3) for B_{tmax} is directly applicable to a pin-ended column with applied ("first-

order") end moments, since the end moments in this case are identical to the total end moments. B_m in Eq. (42) was initially derived as an approximation of Eq. (3) for the pin-ended column case (Galambos 1968). An extension to restrained columns were obtained by replacing the member length L in the expression by the effective length $\beta_b L$, and the end moments M_1 and M_2 by the first-order end moments M_{01} and M_{02}, or, rather, by the sway-magnified first-order end moments $B_s M_{01}$ and $B_s M_{02}$ in the considered case. This approach is known as the "effective length factor approach" (Winter 1954). The approximation and its faults are discussed in considerable detail by Lai, MacGregor and Hellesland (1983).

10.2 Comparisons with single and panel columns (B_m, $B_{m,t}$ and $B_{m,c}$)

Single column results.
Approximate predictions of the rising maximum moment branch according to Eq. (42), are shown, in terms of B_m, in Figs. 5, 6 7 and 8 for single restrained columns.

Two sets of B_m predictions are shown in each figure.
1) B_m: For the single columns (Figs. 5 to 8), the curves labelled just B_m are computed with the restraints given in the inserts of the figures (G (or k) based on $k_b = 6EI_b/L_b (= 3EI_b/(L_B/2))$.
2) $B_{m,t}$: The curves labelled $B_{m,t}$ (Figs. 5 to 7) are computed assuming that the columns have been isolated from a greater frame with rotational beam stiffnesses of $k_b = 2EI_b/L_b$ (single, symmetrical curvature bending). These are 1/3rd of the restraints given in the figures.

As seen, the B_m predictions by Eq. (42) are rather conservative. This is primarily due to the C_m approximation, that tends to become very conservative (too large) for columns with significant double curvature bending, as in the present cases.

There are no exact results to which the $B_{m,t}$ predictions can be compared. The $B_{m,t}$ predictions was simply included for comparative reasons, to get an indication of the extent the reduced end restraints have on the raising moment branch predictions.

Panel column results.
Two sets of predictions for columns of a panel frame are shown in Figs. 9 and

10. The $B_{m,t}$ curves are computed as described above (with beam stiffnesses $k_b = 2EI_b/L_b$). The $B_{m,c}$ curves are computed by Eq. (42) with $A = 0$ and α_b taken as the exact critical load index of the panel. For instance, in terms of α_E in Fig. 9, $\alpha_b = \alpha_E/(\alpha_E)_c$, where $(\alpha_E)_c = 1.639$ (see figure). The end moment ratios used (μ_0, Eq. (39)), are those obtained from the computer analysis ($\mu_0 = -0.600$ and -0.585, respectively, for Column 1 (left) and 2 (right) of the panel).

The $B_{m,t}$ curves are seen to be very close to the $B_{m,c}$ curves. This implies that the assumed single, symmetrical curvature bending of the beam restraints ($k_b = 2EI_b/L_b$) in the $B_{m,t}$ computation is close to the exact one for this panel. The difference will be more marked in cases with greater difference between the columns of the panel.

The predictions of B_m ($B_{m,t}$ and $B_{m,c}$) according to Eq. (42) are seen to give very conservative predictions also for the panel.

It has proven difficult to develop reasonably accurate approximations for maximum moments in the general case. An attempt is made below to provide some improvement to presently common regular design work procedures.

10.3 Thoughts on limitation on axial load indices in design

Unbraced frames. The sway magnifier B_s is in some codes limited to about 1.5. This requires the corresponding storey (system) sway stability index, α_{ss} (Eq. (12)), to be less than 0.33. In unbraced frames ($S_B = 0$ in Eq. (12b)), this requires the lateral bracing to be provided solely by the columns. Most of the columns must then have α_s values below 0.33 on the average. But an individual column may have a larger value. In order to get some indication of what the maximum α_s value in an individual column can possibly be, consider an approximation of α_{ss} in Eq. (12b) given by (Hellesland 1995)

$$\alpha_{ss} \approx (\alpha_{s1} + \alpha_{s2} \cdots)/n \qquad (44)$$

Example 1, $B_s = 1.5$, 10 columns: 9 with $\alpha_s = 0.25$ allow one with $\alpha_s = 1.05$.
Example 2, $B_s = 1.5$, 10 columns: 9 with $\alpha_s = 0.15$ allow one with $\alpha_s = 1.98$.
Example 3, $B_s = 2.0$, 10 columns: 9 with $\alpha_s = 0.25$ allow one with $\alpha_s = 2.75$.

Eq. (44) becomes increasingly inaccurate with increasing difference between the individual α_s values in the summation. Nevertheless, some rough conclusions based on the estimates above are justified: Even with the very low, to low, $\alpha_s = 0.15 - 0.25$ values above in 90 % of the columns, the remaining 10% must have α_s values less than about 1 to 2 in order for B_s to remain below 1.5. And, even for the very high sway magnifier of $B_s = 2.0$, α_s is not likely to ever exceed 3.0.

Partially braced frames. For partially braced frames with sway ($S_B > 0$ in Eq. (12b)), the column indices can be greater than in the unbraced case above without inflicting a storey (system) sway stability index, α_{ss} above 1.5.

However, there is many good reasons in design to limit axial load levels to reasonable levels, well below the critical braced column loads. For one thing, moments may change and increase quickly near critical loads. Complicated mechanics of the frame action at high axial loads (unwinding type phenomena, moment reversals etc.) combined with the sensitivity of frame action to uncertainties in restraints assessments, including the effect of possible beam yielding on restraints, and column stiffness itself, calls for utmost caution.

Column load levels, in terms of critical axial loads for braced cases, or better in terms of the system critical loads, should probably be limited to values below $0.7\alpha_b$ or so in regular design work.

11 Maximum column moment proposals

11.1 Maximum moment proposal 1

Alternative (1a) – Low to moderately high axial load levels.
For most practical load levels, it is acceptable for design purposes to compute the maximum moment according to $M_{max} = B_m B_s M_{02}$, Eq. (41), with the simplification

$$B_m = 1.0 \tag{45}$$

It has been found (Hellesland 2019), from elastic second-order analyses of single columns with practical (and invariant) end restraints, that this B_m approximation is conservative for load indices given, in terms of the free-sway and the braced critical load indices, respectively, by

$$\alpha_s < 3.5\,(3.0) \quad \text{or} \quad \alpha_b < \ 0.5\,(0.8) \tag{46}$$

If some 5 to 10% underestimation of moments were accepted, the limits above could be increased somewhat. For columns in sway frames, most columns will have load indices well below the values indicated above (see Sect. 10.3).

The numbers in parantheses were found for columns in panels (frames) subjected to a sidesway displacement. The restraints of such columns are non-stationary in that the restraining beams, typically bent initially into a double curvature shape, will change into a more flexible bending shape, typically towards more single curvature bending, as the axial load is increased towards the braced instability load. As a consequence of this, the critical braced load indices will be lowered, and the rising, maximum moment branch will be pressed upwards, towards larger values (eventually infinity) at smaller load indices than in the single column case.

The limit in terms of the free-sway load index, α_s, is seen in the figures to be affected much less than the corresponding limit in terms α_b by the changing of the end restraints described above. Consequently, α_s is the better suited parameter of the two, to indicate range of applicability of simplified maximum moment expressions.

Alternative (1b) – Present practice, for any load level..
In partially braced frames with sidesway, axial load indices may exceed those in Eq. (46). Care should be exercised if allowing columns to approach their critical braced loads, as moments may change and increase quickly near critical loads. To complement Alternative (1a) above, it is therefore prudent, at very high load levels, to include a rising moment branch. Common practice is, as mentioned before, to calculate B_m according to Eq. (42) defined with $\lim B_m = 1$ and $A = 0$. Thus,

$$B_m = B_b \geq 1.0 \tag{47}$$

11.2 Maximum moment proposal 2

To simplify presentation and discussion, proposal 2 alternatives for maximum design moment are illustrated for a specific restraint case in Fig. 14.

Alternative (2a) – Low to moderately high load levels
In the figures (Figs. 5-10) it is seen that maximum moment factors, B_{max}, are less than 1.0 up to fairly high load indices, and that they are well below 1.0 in cases with relatively stiff end restraints. For columns in unbraced sway frames, $\alpha_s = 3$ is not likely to ever be exceeded (see Sect. 10.3). Taking advantage of this, a more economical design than that in Alternative 1 above can be achieved for moderate to relatively high load indices.

For columns with axial load indices $\alpha_s < 3$, it is proposed to take the maximum moment factor according to a “maximum moment modified B_{2lin}”, labelled B_{2m},

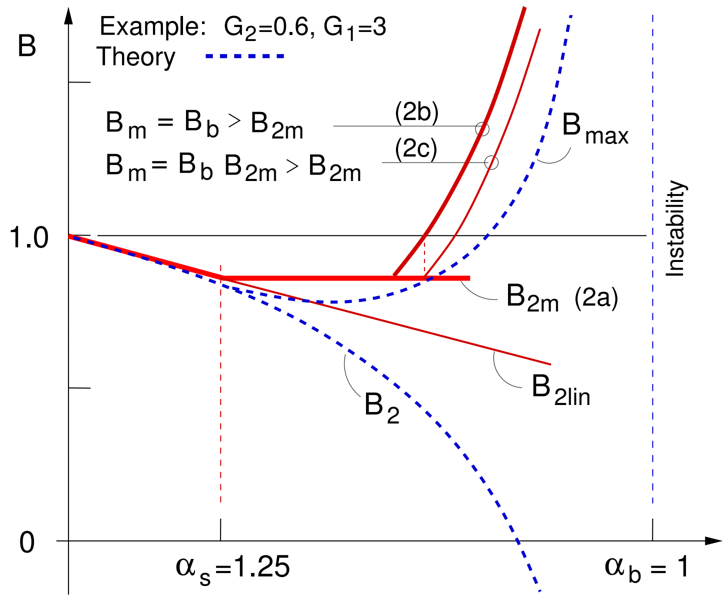

Figure 14: *Maximum moment proposal alternatives (2a), (2b) and (2c), as illustrated for a specific case ($G_1 = 3$, $G_2 = 0.6$).*

and defined below and in Fig. 14. Thus,

$$B_m = B_{2m} \tag{48}$$

where

$$B_{2m} = 1 - g_2 \cdot \alpha_s \geq 1 - g_2 \cdot 1.25 \tag{49 a}$$

$$g_2 = 1 - B_{2s} = \frac{0.24}{0.24 + (1 + 0.75G_2)^2} \tag{49 b}$$

This B_{2m} factor is defined by a bilinear curve. It follows the descending secant approximation B_{2lin} (Eq. (32)) for axial loads between $\alpha_s = 0$ and $\alpha_s = 1.25$, and, thereafter, it is kept constant and equal to the value at $\alpha_s = 1.25$. The g_2 function above correponds to Eq. (37).

The rotational end restraints to be used here are those for the column considered free-to-sway. Typically, in lieu of more accurate values, beam restraints of $k = k_b = 6EI_bL_b$ and $G_2 = 6(EI/L)/k$, corresponding to antisymmetrical beam bending, may be assumed.

Comparisons. Predictions by B_{2m}, Eq. (49a), are included in Figs. 5-10. It can be seen to give good, yet conservative, estimates of the maximum moment in columns with axial loads up to and often well beyond the value of α_s at which maximum moments form away from end 2 of the column.

The value of $\alpha_s = 1.25$ used in the lower limitation of Eq. (49a) was in an initial research phase (Hellesland and MacGregor 1981) set conservatively to 1.0. It can in many cases be increased also beyond the present value of 1.25. A value of 1.5 have been tried, and found to be acceptable in all cases except for those with very stiff restraints, such as in Fig. 7. Therefore, the lower 1.25-limit was chosen here.

B_{2m}, Eq. (49), is shown in Figs. 5-10. It can be seen that it provides conservative B_{max} predictions for $\alpha_s < 3$, for all cases investigated, but for the column that is fully fixed at the base and pinned at the top, Fig. 8(a). In this latter, rather special case, B_{2m} can be seen to underestimate B_{max} with at most about 12% at the "limit" of $\alpha_s = 3$ ($\alpha_E = 3/2^2 = 0.75$). This is considered acceptable for this theoretical case. For a slightly lower and more realistic fixity at the base. the underestimation will be significantly reduced.

Bullets are included on the B_{2m} curve at load indices corresponding to 70 % of the braced critical load ($\alpha_{b,t} = 0.7$) computed with single curvature beam bending (horizontal tangent at beam midlength). The purpose of this inclusion is simply to indicate a load level in terms of the braced critical load that might have been chosen as an application limit instead of that in terms of α_s. It can be seen in the figures that $\alpha_{b,t} = 0.7$ sometimes represents a lower axial load, and somtimes a higher, than $\alpha_s = 3$. On the overall, $\alpha_s = 3$ represents a more consistent limit, it seems, and is chosen as the prime indicator here.

Alternative (2b) – Very high axial load level.
To cover load levels at which the maximum moment rises above the value of B_{2m}, as given above, a rising moment branch approximation defined by

$$B_m = B_b \geq B_{2m} \tag{50}$$

is adopted. Here B_b is, as before, given by Eq. (43a). In predictions in this report, $A = 1$ is used.

This B_m is equal to the "present practice" formulation, except for the lower limit, which now is lowered from 1.0 to by B_{2m}. As discussed previously, and seen in Figs. 5-10, a rising moment branch defined in terms of B_b is very conservative for columns with significant double curvature bending.

At one stage, also $B_{2lin} = 1-(1-B_{2s})\,\alpha_s$, Eq. (32), was considered as a lower limit on B_m in Eq. (50), instead of B_{2m}. It provided improved predictions at moderate to high axial loads for a wide range of end restraint combinations. But this was not so in cases with very stiff end restraints, for which the limit was found to be unconservative at higher load indices. It was therefore not considered further.

Alternative (2c) – Very high axial load levels.

A less conservative magnifier than that defined above by Eqs. (47) and (50)) is illustrated in Fig. 14 by the curve marked (2c). It is given by a product defined by

$$B_m = B_b \cdot B_{2m} \geq B_{2m} \tag{51}$$

In this formulation, the rising branch of the proposal alternative (2b) above is lowered by the multiplication with B_{2m}. Predictions by Eq. (51) are not included in the figures (Figs. 5-10), but the effect is that the conservativeness of the rising moment branch predictions is reduced, yet still quite conservative in most cases investigated.

An exception is found for the case of the cantilever column pinned at the top ($G_1 = \infty$) and fully fixed (clamped) at the base ($G_2 = 0$). This case is shown in Fig. 15, where it can be seen that Eq. (51) gives somewhat unacceptably unconservative predictions (Column (a), the lower dashed curve). However, before concluding, two aspects of fixed-pinned cantilever columns may be worth considering:

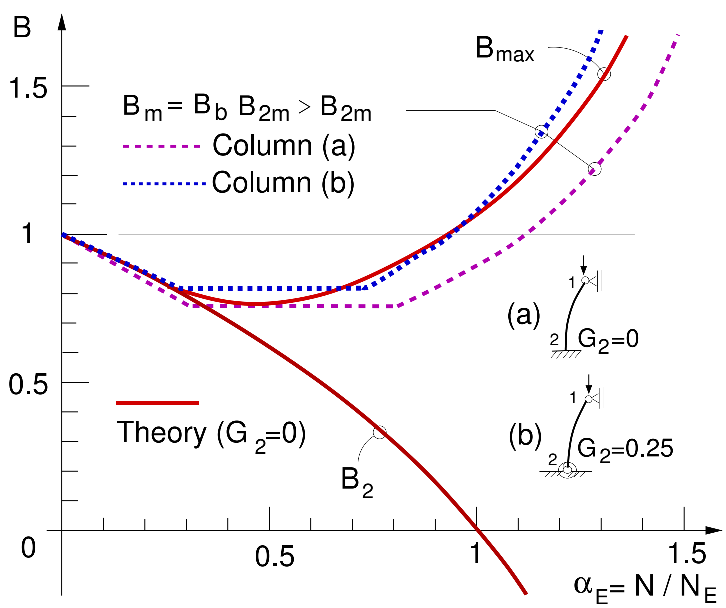

Figure 15: *Exact maximum moment results ($G_2 = 0$) and predictions according to proposal alternative (2c) for two cases: (a) fully fixed-pinned cantilever column; (b) cantilever column with a small base restraint reduction.*

(i) For a column hinged to a frame, an ideal, moment-free pin (at the column top in our case) is in practice difficult to achieve. There will normally be some moment restraint offered at the hinge. If only a small restraint of, say $G_1 = 10$ or so, is present, the apparent unconservativeness in the present case will be more

than compensated for by this alone.
(ii) Full fixity (at the column base in our case) is also difficult to achieve in practice. A small reduction in the intended fixity will reduce the mentioned unconservativeness.

To illustrate item (ii), consider Fig. 15, where alternative (2c) predictions are shown for a Column (a) with theoretically full fixity at the base ($G_2 = 0$) and a Column (b) with the slightly more flexible base restraint of $G_2 = 0.25$ ($k_2 = 24EI/L$). Whereas the predictions for Column (a) are unconservative (lying below the exact B_{max} results), the predictions for Column (b), with the small restraint reduction (giving, approximately, $\beta_s = 2.10$, $\beta_b = 0.74$, $\alpha_b = 0.546\alpha_E$), are seen to be in close agreement with exact results.

Validity conclusion: Separately, or in sum, items (i) and (ii) justify the use Eq. (51) also for practical pin-ended cantilever columns with very stiff base restraints. Consequently, alternative (2c), Eq. (51), can be considered applicable to any practical column.

Alternatively, if one does not want to rely on this "postulate", predictions of pin-ended cantilever columns can be carried out with slightly unfavourable (reduced) base restraint stiffnesses. At present, it is tentatively suggested to replace the theoretical G_2 by $G_2 + \Delta G_2$, where $\Delta G_2 = 0.25(1 - 0.5G_2) \geq 0$. This relation implies there is no need for modifications for cases with $G_2 \geq 2$ ($k_2 \leq 3EI/L$). This can be confirmed by the cantilever case with the base restraint $G_2 = 2$ in Fig. 8.

Summary and conclusions on maximum moment proposals.
The framed column proposals above for columns with sway due to lateral loads are all considered feasible in design.

From a practical design point of view, the proposal (1) alternatives (a and b) are believed to be the most suitable in early design phases, when details of column end restraints are yet to be determined.

The proposal (2) alternatives (a, b and c) will allow more economical designs than the proposal (1) alternatives, but are believed to be most suitable in final design phases, following preliminary design, and in possible design check situations.

The alternative (2c) formulation has the advantage over alternative (2b) that it is less conservative in. Also, and conceptually more important, it can be applied in a more rational manner in cases with load combinations that also include moments from gravity loading (see Eq. (53a)). The latter aspect is important.

12 Maximum moments for combinations of lateral and gravity loads

12.1 Maximum column moment proposal for load combinations

For framed columns with first-order end moments from both lateral loading (M_{0s}) and gravity loading (M_{0b}), the sway modified first-order moment sum at the two ends are defined by

$$M_{01}^* = M_{01b} + B_s M_{01s} \quad \text{and} \quad M_{02}^* = M_{02b} + B_s M_{02s} \tag{52}$$

where M_{02}^* is taken, per definition, to be the moment at the end with the larger moment sum (absolute value), and M_{01}^* the moment at the end with the smaller end moment sum.

Proposal 2, alternative (2c) offers the most rational maximum column moment formulation. Adopting this alternative, it is proposed to calculate maximum column moments as follows:

(i) When the larger moment sum occurs at the end with the larger sway moment M_{0s} (i.e., at the end with the stiffer end restraint), it can be calculated from

$$M_{max} = B_b \left(M_{0b} + B_{2m} B_s M_{0s}\right)_2 \tag{53a}$$

(ii) When the larger moment sum occurs at the end with the smaller sway moment M_{0s} (i.e., at the most flexible end restraint), as above but with $B_{2m} = 0$. Thus,

$$M_{max} = B_b \left(M_{0b} + B_s M_{0s}\right)_2 \tag{53b}$$

Here,
B_{2m} is defined by Eq. (49);
B_b is defined by Eq. (43a), but now subject to the restriction $B_b \geq 1$;
C_m is defined by Eq. (43b), but now with μ_0 in Eq. (43c) replaced by

$$\mu_0 = -\frac{M_{01}^*}{M_{02}^*} = -\frac{(M_{0b} + B_s M_{0s})_1}{(M_{0b} + B_s M_{0s})_2} \tag{54}$$

The moment ratio is taken positive when the member has single first-order curvature bending, and negative otherwise. Eq. (53aa) breaks down to Eq. (51) when $M_{0b} = 0$, and into the conventional, common practice formulation for gravity load moments when $M_{0s} = 0$.

It is believed that this proposal is most relevant in a final design phase, following preliminary design carried out with $B_{2m} = 0$. It is a drawback of the proposal that it is necessary to check if the larger end moment sum and the larger sway moment occur at the same or different column ends. The advantage is a more economical design.

The relevant ACI code provisions, reviewed below, are similar to the proposal above when B_{2m} is taken equal to 1.0.

12.2 Simplified maximum column moment proposal for load combinations

Simplification-1. A more conservative alternative than that above is obtained by combining maximum moments computed separately for each load case in the load combination, such that

$$M_{max} = B_{b,b}M_{0b} + B_{b,s}B_{2m}B_sM_{0s} \tag{55}$$

where M_{0b} and M_{0s} are the larger first-order end moments in the respective load cases; it should be noted that they may occur at different column ends in the two load cases.

$B_{b,b}$ and $B_{b,s}$, both to be taken greater or equal to 1.0, are the local member moment magnifiers defined by Eq. (43a). For the gravity load case, $B_{b,b}(\geq 1)$ is to be computed using the C_m factor defined with the end moment ratio $\mu_0 = -M_{01b}/M_{02b}$. Similarly, for the sway load case, $B_{b,s}(> 1)$ is to be computed with the ratio $\mu_0 = -M_{01s}/M_{02s}$. According to the principle of superposition, these moment magnifiers ($B_{b,b}$, $B_{b,s}$) should be computed for the same axial loads in the columns, i.e., the axial loads from the load combination. Axial loads from the lateral load case are often small, thereby allowing rough load estimates to be made.

The maximum moment in one load case may not occur at the same section as the maximum moment in the other load case. Clearly then, the summation of these maximum moments may lead to a very conservative prediction, and one that may be considerably more conservative than obtained from Eq. (53a).

Simplification-2. In the study of the sway column response in Sect. 11.2 above, it was concluded in conjunction with the proposal (2a) alternative that the accurate maximum moment factor B_{max} was less than B_{2m} for a wide range of low to moderately high axial load levels. In such cases, it is justified to take $B_{b,s} = 1$. Then,

$$M_{max} = B_{b,b} M_{0b} + B_{2m} B_s M_{0s} \tag{56}$$

In view of the general conservativeness of adding the two maximum moment contributions, this formulation is probably also conservative at high axial load levels. The author is not aware of investigations into this.

Simplification-3. A further simplification is obtained by taking $B_{2m} = 1$. Thus,

$$M_{max} = B_{b,b} M_{0b} + B_s M_{0s} \tag{57}$$

This last simplification makes Eq. (57) more conservative than Eq. (56). Such a formulation was proposed by Ford et al. (1981), and incorporated into the 1983 edition of ACI 318. This form was was also adopted by AISC, and it is still retained as an alternative today (AISC 2016); see also Eq. (59). In a discussion of the paper by Ford et al. (1981), Hellesland and MacGregor (1982) suggested a similar provision, but with the maximum member magnifier $B_{b,s}$ included in the sway portion of the expression.

12.3 Comparable structural code formulations

Comparable expressions, presently in the two major codes ACI 318-14 (ACI 2014) for concrete structures and AISC 360 (AISC 2016) for steel structures, are given by

$$\text{ACI}: \qquad M_{max} = \delta\,(M_{2ns} + \delta_s\, M_{2s}) \tag{58}$$

$$\text{AISC}: \qquad M_{max} = B_1\, M_{nt} + B_2\, M_{lt} \tag{59}$$

where the notation is that used in the respective codes. In ACI, M_{ns} is the first-order moment "due to loads that cause no appreciable sidesway", and M_s the first-order moment "due to loads that cause appreciable sidesway". Correspondingly, in AISC, M_{nt} is the first-order moment "assuming there is no lateral translation", and M_{lt} is the first-order moment "caused by lateral translation only". The difference between these definitions is minor in most practical cases with reasonable symmetry, and consequently with little sidesway due to gravity loads.

B_2 is similar to the present B_s. B_1 and δ are defined in similar ways in the two codes, both given by B_b in Eq. (43a) with $A = 0$, but with an important difference in the manner the moment ratio μ_0 in the moment gradient factor, C_m (Eq. (43b)), is defined: B_1 in AISC is computed with μ_0 defined by the ratio of

M_{nt} end moments, whereas δ in ACI is computed with μ_0 defined by the ratio of end moment sums $(M_{ns} + \delta_s\, M_s)$.

From a column mechanics point of view, the ACI approach is clearly the most rational of the two, and generally the least conservative one.

13 Conclusions

Development of shears, end moments and maximum moments between ends of framed columns have been studied using second-order theory. Main attention was on columns with imposed sidesway displacements. These allowed the study, over the full range of axial loads, of any type of columns in frames with sidesway, i.e. of both "supporting sway columns" ($\alpha_s < 1$) and "supported (braced) sway columns" ($\alpha_s > 1$).

An explicit, closed form shear force expression, that gives excellent shear predictions for columns with low to moderate and high axial loads, have been reviewed and applied, and the accuracy of a linearized approximation discussed.

Considerable attention has also been given to end moments caused by sidesway. Linearized, secant approximations have been derived that give excellent moment predictions for supporting columns, and also for supported columns with rather high axial load levels.

Major attention was on development of maximum moment approximations for columns with sidesway, that are suitable in typical design code formats. Alternative design moment expressions have been proposed and dicussed. They are simple to incorporate in regular design work, and will reduce the conservativeness of current procedures, thus allowing more economical designs than presently available structural code expressions.

Extensions of the proposals, for columns with sway, to the general case of columns with moments from both gravity and lateral loading, have been made, and discussed with reference to formulations found in the literature and to two major design codes for structural concrete and steel structures.

ACKNOWLEDGMENTS

This research was initiated during a research stay by the first author at the Univ. of Alberta (UA), Edmonton, Canada, already in 1981. A preliminary draft, entitled "Mechanics and design of columns in sway frames", co-authored with the now deceased Professor J. G. MacGregor at UA, was neither completed nor published. The present paper includes part of the initial draft, but is significantly revised and extended. The input by Prof. MacGregor in the early phase is greatly appreciated. So is the initial computer analyses of the panels by S.M.A. Lai.

NOTATION

B_b	= conventional rising moment branch approximation;
B_m	= approximate maximum moment magnification factor;
B_{max}	= maximum moment magnification factor applied to first-order moments;
B_{tmax}	= maximum moment magnification factor in second-order analysis;
B_s	= system (storey) sway magnification factor;
B_1, B_2	= first-order end moment magnification factors;
EI, EI_b	= cross-sectional stiffness of columns, and beams;
G_j	= relative rotational restraint flexibility at member end j;
H	= applied lateral storey load (sum of column shears and bracing force);
L, L_b	= lengths of considered column and of restraining beam(s);
M_{0j}, M_j	= moment in first-order and second-order analysis, at end j;
N	= axial (normal) force;
N_{cr}	= critical load in general $(= \pi^2 EI/(\beta L)^2)$
N_{cb}, N_{cs}	= critical load of columns considered fully braced, and free-to-sway, respectively;
N_E	= Euler buckling load of a pin-ended column $(= \pi^2 EI/L^2)$
S_B	= lateral stiffness of external bracing(s) ;
V_0, V	= first-order, and total (first+second-order) shear force in a column;
k_j	= rotational restraint stiffness (spring stiffness) at end j
α_{cr}	= member (system) stability index $(= N/N_{cr})$;
α_b, α_s	= load index of column considered fully braced, and free-to-sway, respectively;
α_{ss}	= system (storey) stability index;
α_E	= nominal axial load index of a column $(= N/N_E)$;
β	= effective length factor (at system instability);
β_b, β_s	= effective length factor corresponding to N_{cb} and N_{cs}, respectively;
Δ_0, Δ	= first-order, and total lateral displacement;
γ, γ_n	= flexibility factor in general, and load (N-) dependent flexibility factor;
γ_s, γ_0	= flexibility factor at free-sway, and at zero axial load, respectively:
κ_j	= relative rotational restraint stiffness at end j $(=k_j/(EI/L))$.

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