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CHAPTER 4 – THE CONCEPTION OF MATHEMATICS KNOWLEDGE FOR TEACHING FROM AN INTERNATIONAL PERSPECTIVE - THE CASE OF THE TEDS-M-STUDY

ABSTRACT

Mathematics knowledge for teaching plays an important role in school practice. However, research has mainly evaluated empirically the nature of this knowledge in the last two decades. The question, how teacher education and teaching practice contribute to mathematics knowledge for teaching has only been explored more recently. This chapter aims to discuss different descriptions and conceptualizations of the conception of mathematics knowledge for teaching from an international study using the large-scale study Teacher Education and Development Study in Mathematics (TEDS-M 2008) as example. In the first part of the paper, the main currently discussed theoretical models for the construct of mathematics knowledge for teaching are described. In the second part of the paper, the ways to empirically evaluate the different facets of this knowledge for teaching are described using items and descriptions from TEDS-M. The paper closes with prospects for further research.

THEORETICAL FRAMEWORKS FOR MATHEMATICS KNOWLEDGE FOR TEACHING

The knowledge of mathematics teachers and how it is accomplished during teacher education has been conceptualized differently over time as well as across research paradigms and countries. A first important model that characterized mathematics teachers' knowledge started from classroom practices and was focused on knowledge acquisition by observation in a kind of apprenticeship (Zeichner, 1980). During the 1990s, the cognitive basis of teachers' pedagogical practices started to emerge and first small-scale comparative studies were carried out (Kaiser, 1995; Pepin, 1999).

More recently, research has focused even more strongly on the knowledge base of mathematics teachers' classroom practice. Several large scale studies developed theoretical frameworks focusing on mathematics knowledge for teaching, as well as a few more qualitatively oriented studies. In the following, we first present selected theoretical frameworks and then describe the international comparative study TEDS as an example of these kinds of studies.

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A milestone within the recent developments of conceptions of mathematics knowledge for teaching is the seminal work by Shulman (1986, 1987), in which he developed theoretical categories for the knowledge base of teachers analyzing the specifics of these knowledge categories for the teaching profession. He distinguished the following categories of the knowledge base of teachers:

- “content knowledge;
- general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
- curriculum knowledge, with particular grasp of the materials and programs that serve as ‘tools of the trade’ for teachers;
- pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of the teachers, their own special form of professional understanding;
- knowledge of learners and their characteristics;
- knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and
- knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.”

(Shulman, 1987, p. 8)

Shulman (1987) emphasizes that among those categories “pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching.” He describes pedagogical content knowledge as “blending of content and pedagogy” and as the “category most likely to distinguish the understanding of the content specialist from that of the pedagogue” (p. 8).

In further work, Shulman concentrates on three of these categories, all related to content, namely “subject matter content knowledge”, “pedagogical content knowledge” and “curricular knowledge” (1986, p. 9). He describes subject matter content knowledge as “the amount and organization of knowledge per se in the mind of the teacher” and puts the understanding of the structures of the subject in the foreground in contrast to the pure knowledge of facts and concepts. Pedagogical content knowledge “is pedagogical knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*.” Pedagogical content knowledge is defined as “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (p. 9). The third category of content knowledge, namely curricular knowledge, refers to the curriculum and the programs for teaching specific subjects. Shulman (1986) describes curriculum and its associated materials as “the *materia medica* of pedagogy, the pharmacopeia from which the teacher draws those tools of teaching that present or exemplify particular content and remediate or evaluate the adequacy of student accomplishments” (p. 10).

The question about the specific features of the professional knowledge for teachers and how to distinguish it from other forms of professional knowledge has created many discussions. Shulman (1987) writes: “But the key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (p. 15).

Although Shulman’s work was ground-breaking and can be described as milestone in the development of the theory of teachers’ professional knowledge, critique was developed emphasizing that the knowledge facets were not sufficiently defined in order to allow operationalized empirical research. Especially the distinction between Shulman’s concept of subject matter content knowledge and pedagogical content knowledge remains unclear according to these critical positions (Ball et al. 2008). We will come back to this critique while describing the approach developed by Ball and others in order to overcome this weakness. Another researcher, Anne Meredith, criticizes that the pedagogical content knowledge as defined by Shulman (1986, 1987) “seems to imply one type of pedagogy rooted in particular representations of prior knowledge” (1995, p. 176). She continues that Shulman’s concept of pedagogical content knowledge “is perfectly adequate if mathematical knowledge is seen as absolute, incontestable, unidimensional and static. On the other hand, teachers who conceive of subject knowledge as multidimensional, dynamic and generated through problem solving may require and develop very different knowledge for teaching” (p. 184).

The critique of Shulman’s work led to other conceptualizations on teachers’ knowledge. Fennema and Franke (1992) in their famous handbook chapter discuss that the critical word, *transform,* in Shulman’s approach neglects the complexity of the interaction between teachers and students. “This transformation is not simple, nor does it occur at one point in time. Instead, it is continuous and must change as the students who are being taught change. In other words, teachers’ use of their knowledge must change as the context in which they work change.” (p. 162) Based on this critique they modify the model by Shulman by emphasizing that teachers’ knowledge is characterized by its “interactive and dynamic nature” (p. 162). They distinguish the following components of teacher knowledge: knowledge of the content of mathematics, knowledge of pedagogy, knowledge of students’ cognitions, and teachers’ beliefs. “It also shows each component in context.” (p. 162)

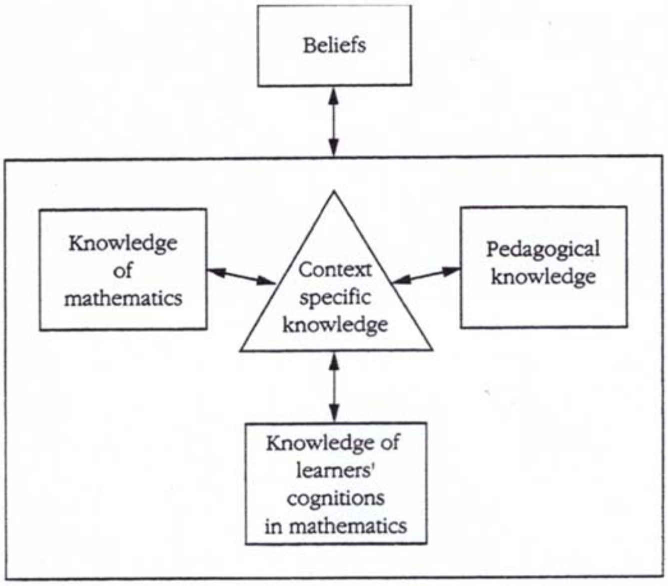


Fig. 1: Teachers' Knowledge Developing in Context (Fennema & Franke 1992, p. 162)

Based on the description of teachers' knowledge as situated, which was a new perspective at this time, they explain the main characteristics of their model as follows: "The center triangle of our model indicates the teachers' knowledge and beliefs in context or as situated. The context is the structure that defines the components of knowledge and beliefs that come into play. Within a given context, teachers' knowledge of content interacts with knowledge of pedagogy and students' cognitions and combines with beliefs to create a unique set of knowledge that drives classroom behavior." (p. 162)

Departing from the critique that the two knowledge facets, subject matter knowledge and pedagogical content knowledge, are not distinguished precisely enough for measurement purposes, two US-American research projects based at the University of Michigan developed another modification of the Shulman model. The *Mathematics Teaching and Learning to Teach Project* (MTLT) and the *Learning Mathematics for Teaching Project* (LMT) define and distinguish between different knowledge facets functional for mathematics teaching. Widely discussed, especially in the US-American community, the MTLT project studies the interplay of mathematics and pedagogy in the teaching of elementary school mathematics. By looking closely at the mathematical and pedagogical work of teaching such as managing discussions, asking questions, interpreting students' thinking, the project aims to identify mathematical insight, appreciation, and knowledge that matters for teaching. In addition, the project aims to analyze and articulate ways in which it might be entailed in practice. The MTLT project developed the construct of Mathematical Knowledge for Teaching (MKT) as a part of quality mathematics instruction and defines MKT as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill, Rowan, & Ball, 2005, p. 373).

MKT embraces the two knowledge facets by Shulman, namely the subject matter knowledge and the pedagogical content knowledge, but differentiates them in various sub-facets (fig. 3). Subject matter knowledge includes both the mathematical knowledge that is common to individuals working in diverse professions and the mathematical knowledge that is specialized to teaching. It contains new strands that lie outside Shulman's conceptualization, namely common content knowledge (CCK) and specialized content knowledge (SCK). Common content knowledge (CCK) is, according to Hill et al. (2008), what Shulman likely meant by his original subject matter knowledge. It is this content knowledge that is used in the work of teaching in the same way as it is used in many other professions or occupations that also use mathematics. Specialized content knowledge (SCK) is a newer conceptualization and describes the mathematical knowledge that allows teachers to engage in particular teaching tasks, such as how to represent mathematical ideas or provide mathematical explanations. The third sub-facet, horizon content knowledge (HCK), is defined more as an awareness of the large mathematical landscape in which the present experience and instruction is situated than as practical knowledge. The second facet, pedagogical content knowledge, refers to Shulman's conceptualization. It contains knowledge of content and students (KCS) and is focused on teachers' understanding of how students learn a particular content. The second sub-facet, the knowledge of content and curriculum (KCC), refers to the arrangement of the mathematical topics within the curriculum and ways of using curriculum resources and materials. Knowledge of content and teaching (KCT) covers the knowledge about both mathematics and teaching such as the introduction of new concepts (for details see Hill et al. 2008, Ball et al. 2008).

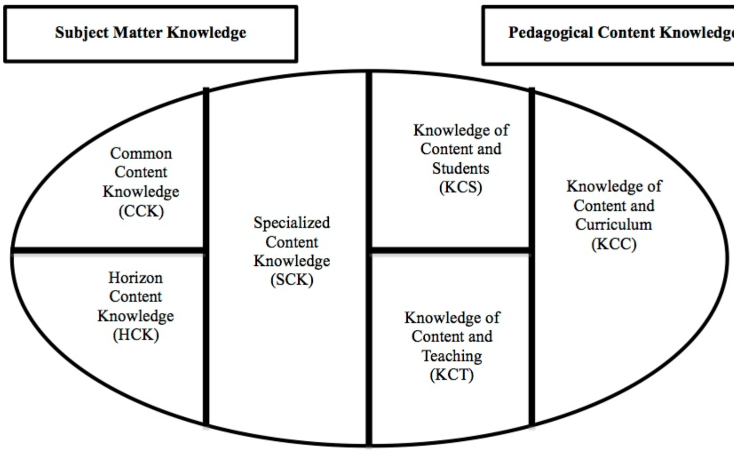


Fig. 2: Mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008, p. 403)

One central achievement of these two projects is the development of instruments to measure teachers’ mathematical knowledge based on a series of multiple choice items. Although this measurement instrument originates from the U.S., it was applied in several other countries (Hill et al., 2007) and revealed severe cultural differences. For example, Ng (2012) studied Indonesian teachers’ performances on geometry items and concluded that the initial measures may not be valid because of national differences between the United States and Indonesia in how shapes are classified. In contrast, Cole (2012) found that most items could be used validly in Ghana despite evidence of cultural incongruence in teaching practices between the U.S. and Ghana.

Another huge step forward achieved by these two projects is the identification of the relationship between teacher knowledge and students’ achievements in mathematics and the evidence that teachers with weak knowledge transmit this to their students (Hill, Rowan, & Ball, 2005).

However, this new framework on mathematical knowledge for teaching has a few significant weaknesses such as not including teachers’ beliefs, despite clear evidence from empirical research that teachers’ beliefs about the nature of mathematics or about the genesis of mathematical knowledge strongly influences their teaching (Schoenfeld 2011). Another problem is the proximity of various sub-facets such as the sub-facet specialized content knowledge and the sub-facets of knowledge of content and teaching and knowledge of content and students, which hardly can be differentiated (theoretically and empirically).

Looking back, Ball et al. (2008) reflect on the uncertainties and possible weaknesses of their model as follows:

“We are not yet sure, whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category on its own right. We also provisionally include a third category within subject matter knowledge, what we call ‘horizon’ knowledge. …. Again we are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories. We hope to explore these ideas theoretically, empirically, and also pragmatically as the ideas are used in teacher education or in the development of curriculum materials for use in professional developments.” (p. 403)

The theoretical approach of the Knowledge Quartet - developed by research groups at the University of Cambridge - arose out of research into teachers’ mathematical content knowledge. The approach refers to the theoretical conceptualization by Shulman, but takes up characteristics of the Fennema and Franke model by categorizing classroom situations where mathematical knowledge surfaces in teaching situations. “The purpose of the research from which the Knowledge Quartet emerged was to develop an empirically-based conceptual framework for lesson review discussions *with a focus on the mathematics content* of the lesson and the role of the trainee's mathematics subject matter knowledge (SMK) and pedagogical content knowledge (PCK). … The focus of this particular research was therefore to identify ways that teachers' mathematics content knowledge – both SMK and PCK – can be observed to 'play out' in practical teaching." (Turner & Rowland, 2011, p. 197) Based on videotaped lessons of novice and trainee teachers the following four categories – foundation, transformation, connection and contingency - were distinguished in order to analyze the interplay of SMK and PCK. The first category, *foundation*, "is rooted in the foundation of the teacher's theoretical background and beliefs. It concerns their knowledge, understanding and ready recourse to what was learned at school, and at college/university … It differs from the other three units in the sense that it is about knowledge 'possessed'" (Turner & Rowland, 2011, p. 200). The other three categories "focus on knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself" (Turner & Rowland, 2011, p. 200). The category, *transformation,* departs from Shulman's notion of the transformation of content knowledge into pedagogically powerful knowledge forms and mainly refers to the usage of instructional material, teacher demonstrations, and the choice of representations and examples. According to Turner and Rowland (2011), the category, *connection,* refers to "the coherence of the planning or teaching displayed across an episode, lesson or a series of lessons" (p. 201) and is characterized by making connections between procedures or concept, decisions about sequencing and so on. The last category, *contingency*, describes "the teachers' response to classroom events that were not anticipated in the planning" (Turner & Rowland, 2011, p. 202). This contingent action describes the adaptive and adequate response on children's ideas, the teachers' deviation from the agenda.

Although the Knowledge Quartet departs from the approach by Shulman with the strong and exclusive focus on SMK and PCK, it does not explicitly include curricular knowledge due to the traditionally minor role of curricular reflections in mathematics education in Britain.

The next theoretical approach developed by the German project, Cognitively Activating Instruction (COACTIV), also refers to the approach by Shulman, as it describes teaching as professional activity and knowledge as the core of professionalism. Departing from the theoretical approach of professional competence as defined by Weinert (2001), Baumert and Kunter (2013) describe competence as "the personal capacity to cope with specific situational demands" (p. 27). COACTIV uses a non-hierarchical model of professional competence as generic structural model, which is specified in fig. 3.

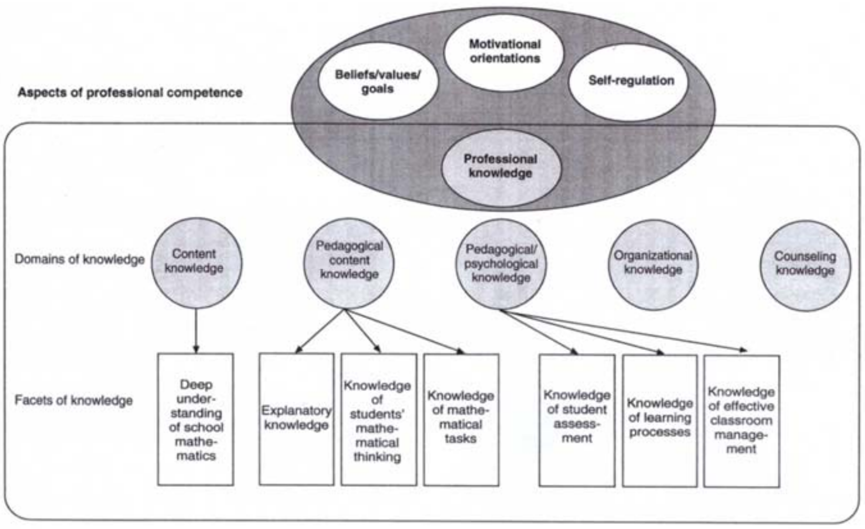


Fig. 3: The COACTIV model of professional competence, with the aspect of professional knowledge specified for the context of teaching (Baumert & Kunter, 2013, p. 29)

The model distinguishes "between four aspects of competence (knowledge, beliefs, motivation, and self-regulation), each of which comprises more specific domains derived from the available research literature. These domains are further differentiated into facets, which are operationalized by concrete indicators." (Baumert & Kunter, 2013, p. 28) Concerning the mathematical content knowledge, COACTIV focuses on profound mathematical understanding of the mathematics taught at school, although theoretically four different levels of understanding of mathematics are distinguished starting with academic research knowledge as highest knowledge and ending with the mathematical everyday knowledge all adults should have. Pedagogical content knowledge is described by Baumert and Kunter (2013, p. 33) with three dimensions:

- "Knowledge of the didactic and diagnostic potential of task, their cognitive demands and the prior knowledge they implicitly require, their effective orchestration in the classroom, and the long-term sequencing of learning content in the curriculum
- Knowledge of student cognitions (misconceptions, typical errors, strategies) and ways of assessing student knowledge and comprehension processes
- Knowledge of explanations and multiple representations"

In addition, the model includes facets of general pedagogical knowledge such as pedagogical knowledge on effective classroom management and instructional planning. Moreover, the model covers various kinds of beliefs such as epistemological beliefs on the body of knowledge, beliefs about learning in a school subject area, and so on.

Like the LMT-study, the related empirical study provides insight into the strong relationship between teacher professional competency and students' achievements in mathematics.

However, the COACTIV-study has several weaknesses, namely the extended differentiation of the various competency facets, especially the general pedagogical knowledge, which were not covered in the main study, but only in the extension study, COACTIV-R.

From an international perspective the question arises whether the various facets of the professional competency and the professional knowledge of teachers can be distinguished empirically in different cultural context. Going back to the original conceptualization of PCK by Shulman as an amalgam of content and pedagogical knowledge, An, Kulm, and Wu (2004) compared the pedagogical content knowledge of mathematics teachers (PCK) between Chinese and U.S. groups and focused on fractions, ratio, and proportion. They found out that in contrast to the U.S. teachers, the Chinese had gained much of their knowledge through school-based in-service training led by expert teachers and continuous professional development activities, especially by observing each other's lessons and jointly discussing them (An, Kulm, & Wu, 2004; Paine, 1997; Paine & Ma, 1993).

These different studies point to the difficulty to reach international agreement on a definition of mathematical knowledge for teaching and how to acquire it. As Pepin (1999) pointed out, these differences also reflect differences in the meaning of mathematics didactics or pedagogy (for example called *Mathematikdidaktik* in German). Continental traditions are based on educational, philosophical, and theoretical reflections including normative descriptions of the teaching-and-learning-processes. In contrast, reflections on the knowledge transformation, its student-related simplification throughout the process to teaching knowledge, called *elementarization* in German, can hardly be found in English-speaking countries. From the beginning in English-speaking countries, research on mathematics knowledge and teacher education (Kaiser, 1999, 2002) was more outcome-based and thus, to a large extent, based on empirical studies in order to identify and determine influential factors as predictors of successful teaching and learning. As Westbury (2000) pointed out, the dominant features of the U.S. curriculum tradition was of organizational nature, referring to schools as institutions, where teachers were expected to be agents for an optimal school system.

Given these cultural differences an intriguing question is, whether it is possible to conceptualize the professional mathematics knowledge of teachers in a comparative study. The first enterprise in this respect was the international comparative study TEDS-M (Teacher Education and Development Study in Mathematics[i]), carried out in 2008 under the auspices of the International Association for the Evaluation of Educational Achievement (IEA) with 23,000 participants coming from 17 countries, the study was based on representative samples. Its aim was to understand how national policies and institutional practices influence the outcomes of mathematics teacher education. Referring to Shulman's model of the professional knowledge of teachers, the achievements in the knowledge facets, mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) were defined as outcomes and measures for the efficiency of teacher education.

Not surprisingly, the development of MCK and MPCK assessments was controversial in TEDS-M. Although international agreement was reached with respect to their core dimensions, national specifications had to be left out, as is common in comparative large-scale assessments. In the following, the nature of the TEDS-M tests for the lower-secondary study is analyzed in detail in order to show exemplarily what it means to measure mathematical knowledge for teaching. The objectives are, firstly, to increase the understanding of the nature of MCK and MPCK, which are still fuzzy domains (for an overview on the most recent discussion see Depaepe, Verschaffel, & Kelchtermans, 2013) by extending our work on the TEDS-M assessment of primary teachers (Döhrmann, Kaiser, & Blömeke, 2012). Secondly, we aim to provide a substantive background for interpretations of the TEDS-M test results by examining whether some educational traditions may be more accurately reflected in the test items than others. For this purpose, the TEDS-M items that have been released by the IEA are presented and analyzed.

OBJECTIVES AND DESIGN OF TEDS-M

The main research questions of TEDS-M were:

> "What is the level and depth of the mathematics and related teaching knowledge attained by prospective primary and lower secondary teachers? How does this knowledge vary across countries?" (Tatto et al., 2008, p. 13)

Similarly to the COACTIV-study, TEDS-M is based on the competency approach by Weinert (2001), who described the professional competencies of teachers as the specific ability to cope with the professional demands of teaching and is strongly related to action-oriented approaches:

> "The theoretical construct of action competence comprehensively combines those intellectual abilities, content-specific knowledge, cognitive skills, domain-specific strategies, routines and subroutines, motivational tendencies, volitional control systems, personal value orientation, and social behaviors into a complex system. Together, this system specifies the prerequisites required to fulfill the demands of a particular professional position." (p. 51)

Departing from the theoretical approach by Shulman (1987), like most projects and theoretical approaches described above, TEDS-M describes MCK and MPCK as essential cognitive components underlying teacher performance in the classroom, complemented by general pedagogical knowledge, personality traits and beliefs.

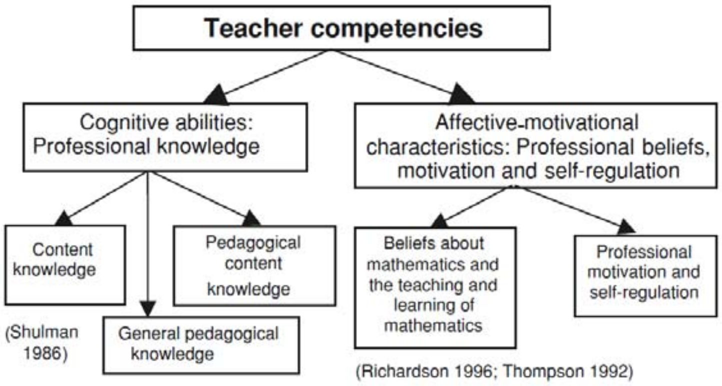


Fig. 4: Conceptual model of teachers' professional competencies (Döhrmann, Kaiser, & Blömeke, 2012, p. 327)

MCK and MPCK were assessed with paper-and-pencil-tests (Tatto et al., 2008). The underlying conceptual framework was the result of a long and intense discussion between the participating countries, in which international acceptance was accomplished. In order to achieve this, national specifications of what was meant by MCK or MPCK had – by necessity – to be left out.

As TEDS-M is the first international large-scale study on teacher education, the theoretical conceptualization of MCK and MPCK as well as developing proficiency tests necessitated extensive work and an enormous amount of time previous to the implementation of the study. In 2002, representatives from the countries participating in TEDS-M met for the first time to discuss their nationally and culturally shaped conceptions on the professional knowledge of mathematics teachers. The result emerging from this process was a definition of MCK and MPCK that predominantly focused teachers' tasks rather than normative – often implicit – curricular requirements. Thus, a teacher's mathematical knowledge was expected to cover at least the mathematical content of the grades the teacher will teach from a higher and reflected level. In addition, a teacher was considered to be able to integrate the educational context as well as to connect the mathematics content to following, higher levels of education. Therefore, the conceptualization of MCK considers the content areas used by TIMSS 2007 where the international discussions lead to the distinction of the four subdomains of number, algebra, geometry and data as essential for school mathematics. The MCK and MPCK tests were oriented towards these subdomains. The overall reliability, validity and credibility of the items have already been demonstrated (Blömeke, Suhl, Kaiser, & Döhrmann, 2012; Blömeke, Suhl, & Kaiser, 2011; Senk et al., 2012). Now, we can look beyond what was accomplished in order to meet further research needs.

In order to provide insight into the nature of the TEDS-M tests, special aspects of the items released and their requirements are featured. This detailed item analyses is partly based on ACER documents, as these provide the percentage of correct answers as indicators of the countries' range of proficiency. In addition, we provide background information about the items and an analysis from a mathematics education point of view. The complete set of TEDS-M Lower-secondary items released by the IEA together with coding guides is available: http://www.acer.edu.au/research/projects/iea-teacher-educationdevelopment-study-teds-m/. As displayed in the diagram on the teachers' professional competencies, beliefs play an important role in TEDS-M. Using well-known scales, various kinds of epistemological beliefs were evaluated such as the beliefs on the genesis of mathematics knowledge and its nature. Due to the focus of this paper on the mathematical knowledge for teaching, this aspect is not described in this paper, but was covered in the evaluation.

ANALYZING TEDS-M ITEMS

We start by analyzing the items that are supposed to assess MCK, and refer to the subdomains of algebra, geometry, number and data. After that, an analysis follows that covers the items that are supposed to assess MPCK, and which refer to the knowledge displayed prior to a lesson in terms of planning but also enacted knowledge in terms of student-teacher interaction. Several item examples are presented and discussed in detail.

Assessing Mathematics Content Knowledge (MCK)

The MCK test consists of 76 items in a multiple-choice and constructed-response format. It covers topics dominating mathematics education all over the world and which mainly come from algebra, number and geometry. Data and probability items are scarcely represented in the test. This reflects their low importance in the mathematics curricula of schools and teacher education in the participating countries. As a consequence, the number of items for the subdomains algebra, number and geometry were nearly uniformly distributed in the test with four items per subdomain. In addition to these subdomains, three cognitive domains were defined: knowing, applying and reasoning (according to TIMSS). The cognitive as well as the content domains constituted a heuristic tool for the item development.

All items were categorized into levels of difficulty arising from the item's curricular level. In detail, the *novice* level of difficulty indicates mathematics content that is typically taught at the grades the future teacher will teach. The *intermediate* level of difficulty indicates content that is typically taught one or two grades beyond the highest grade the future teacher will teach and finally, the *advanced* level of difficulty indicates content that is typically taught three or more years beyond the highest grade the future teacher will teach (Tatto et al., 2008, p. 37).

Subdomain Algebra:

The algebra items of the MCK test mostly belong to the field of functions. Proportional relations as well as linear, quadratic and exponential functions, and the absolute value function are typical mathematical topics in most secondary schools all over the world. These are also represented in the test. The required competencies include, for example, identifying the graph of a given function class, identifying the quantitative relation within a given realistic context, and determining a given function as appropriate to model the relation as well as judging the adequacy of given examples for the definitions of a continuous function.

In the test for the future secondary school teachers, patterns play a less role than in the primary test (Döhrmann, Kaiser, & Blömeke, 2012). Only one MCK item refers to patterns (and also 3 MPCK items). Here, the test persons have to compare and determine various patterns of growth. Another item refers to sequences and requires higher skills which are usually taught in university. For this item, the future teachers need to know the concept of convergence and the limit of a sequence.

Skills concerning functions are also partly needed for items in the subdomain geometry. Here, some items refer to equations and require, for example, to use equations to represent and solve a given contextualized problem as well as determine the set of solutions for a given equation especially on the set of complex numbers.

Overall the subdomain algebra emphasizes the concept of functions as well as the language of formulae and their application to contextualized problems and problems within mathematics. Structural algebraic concepts such as groups or rings are not covered by the items.

The following item example shows one algebra item concerning functions on an intermediate level.

Prove the following statement:

If the graphs of linear functions

$$\mathrm{f}(x) = ax + b \text{ and } \mathrm{g}(x) = cx + d$$

intersect at a point P on the x-axis, the graph of their sum function

$$(\mathrm{f}+\mathrm{g})(x)$$

must also go through P.

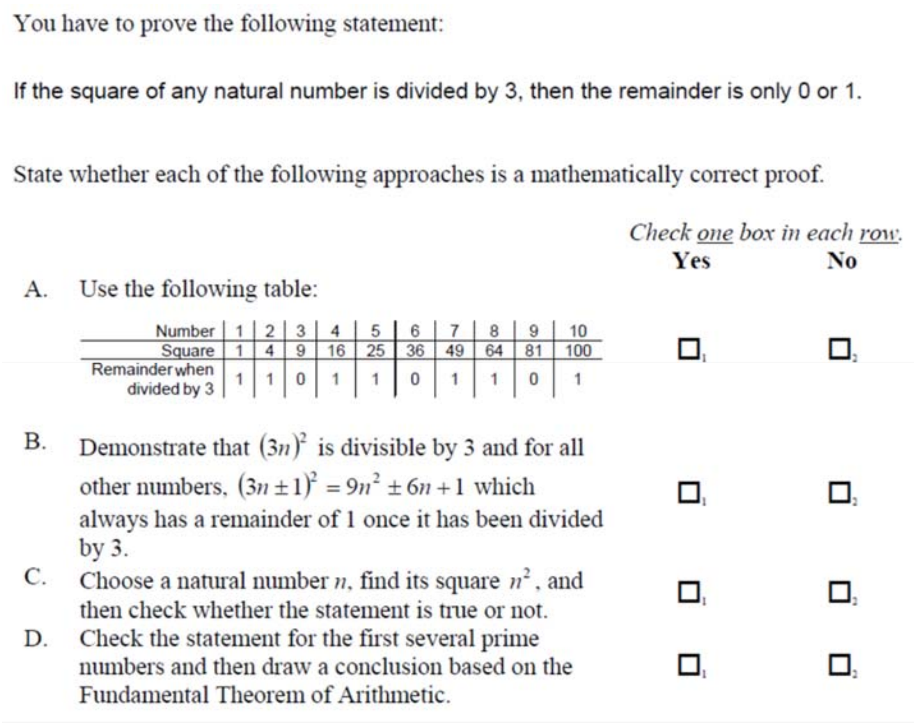
For this item, participants have to prove that the graph of the sum of two linear functions f(x) and g(x) also goes through the point P on the x-axis if both functions f(x) and g(x) intersect the x-axis at that point. This proof could be realized with or without using the function expressions of f and g. A complete and accurate answer would be for example the following sentences: “Suppose f(x) and g(x) intersect at point (p, 0) on the x-axis. Then f(p) = 0, g(p) = 0. Then (f + g)(p) = f(p) + g(p) = 0 + 0 = 0. Therefore f+g also goes across point (p, 0).”

Knowledge about linear functions is essential for solving this task as well as knowledge about the intersection of two functions and the sum function which is constituted by summing both function values (actually, for any functions, this statement is true). Only 10 percent of the future secondary school teachers tested in TEDS-M were able to adequately and completely formulate this proof. Another 8 percent of those teachers received partial credit because their proof was valid to some extent but incomplete. Internationally, this task showed an enormous deviation. 99.7 percent of the future teachers in Chile did not achieve any solution while 69 percent of the future teachers in Taiwan solved the task completely. Prior to the test, this task was classified as intermediate level of difficulty but empirically it showed to be more complex. This may attribute to difficulties in formulating adequate proof.

Subdomain Number:

In the subdomain, number, there is only one item that was classified as novice level. This item requires only a simple operation, but knowledge about the concept of the arithmetic mean is essential as well. The other items in this subdomain refer to mathematics topics that are usually not taught in secondary school. The subdomain number mostly focuses on the cognitive domains knowing and reasoning while algebra predominantly focuses on applying knowledge. The future mathematics teachers need, among other things, to judge if statements about irrational numbers are true, assign the solution of equations to the set of numbers it belongs to, and judge the adequacy of given examples as a proof of a statement about number theory.

The test requirements in this subdomain are quite high. While computations with numbers are not requested, most of the items deal with statements about numbers and its characteristics. Here, the test persons have to apply and compare the properties of numbers and number systems. The following item shows an item-example of the subdomain number that was classified as intermediate level and belongs to the cognitive domain reasoning.



You have to prove the following statement:

If the square of any natural number is divided by 3, then the remainder is only 0 or 1.

State whether each of the following approaches is a mathematically correct proof.

Check one box in each row.

		Yes	No
A.	Use the following table:		
	see table below	☐$_1$	☐$_2$
B.	Demonstrate that $(3n)^2$ is divisible by 3 and for all other numbers, $(3n \pm 1)^2 = 9n^2 \pm 6n + 1$ which always has a remainder of 1 once it has been divided by 3.	☐$_1$	☐$_2$
C.	Choose a natural number n, find its square n^2, and then check whether the statement is true or not.	☐$_1$	☐$_2$
D.	Check the statement for the first several prime numbers and then draw a conclusion based on the Fundamental Theorem of Arithmetic.	☐$_1$	☐$_2$

Number	1	2	3	4	5	6	7	8	9	10
Square	1	4	9	16	25	36	49	64	81	100
Remainder when divided by 3	1	1	0	1	1	0	1	1	0	1

In this task, different ideas are presented to the future teachers who need to judge whether the given arguments represent correct or incorrect proofs. Only the idea given in B illustrates a mathematically correct proof while the ideas presented in A and C are only based on examples and thus they are not mathematically correct proofs. In addition, the idea presented in D is unsuitable to prove the given statement. Besides basic knowledge about number theory, the future teachers predominantly need abilities to prove mathematical statements as well as knowledge about the criteria that contribute to a correct and complete proof in order to solve this task correctly.

Regarding the international average of all countries participating in TEDS-M, the items A and C describing possible arguments for the statement (see item above) were solved very differently. Item A which is based on ten examples was solved correctly by 45 percent of the future teachers while item C, which is only based on one example, was detected as an incorrect proof by 57 percent. Again, there was an enormous range that varied from 18 percent correct answers in Malaysia to 84 percent in Taiwan concerning item A. Regarding item C, the range varied from 18 percent in Chile to 92 percent correct answers in Taiwan. Identifying item B as a correct proof was easiest to the future secondary school teachers. On average, 62 percent of the test persons succeeded ranging from 25 percent in Chile to 91 percent in Taiwan. Item D was solved correctly by 54 percent in average. In every country, more than one fourth of the teachers correctly answered this task (Chile: 28%) but no country reached more than 80 percent correct answers (Taiwan: 79%). This may be due to the implicit reference to the proving concept of induction that shows through the item as well as its reference to the fundamental theorem of number theory which is a powerful theorem in the field of number theory but its inappropriateness for the given proof is not immediately obvious.

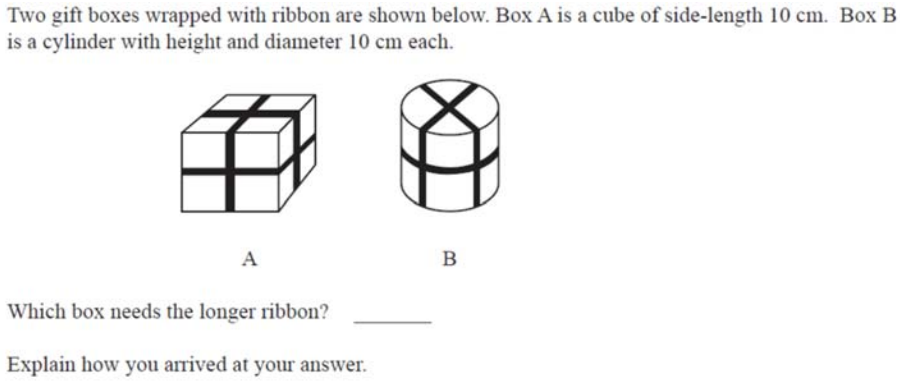
Subdomain Geometry:

Only a few items in the subdomain geometry refer to geometric measurement of two- and three-dimensional objects. For one task, the future teachers have to determine the area of a pictured irregular shape. For another, they have to estimate the surface area and the volume of a represented three-dimensional object. For a third one, they have to compare the properties of three-dimensional objects. These tasks were assigned to a basic level of difficulty prior to the testing and may as well be content of a secondary school-mathematics textbook. The most difficult task of the entire test belongs to the subdomain geometry though and refers to the axiom of the uniqueness of a parallel line. In order to solve this task, the future teachers need to decide if statements are equivalent to the axiom of the uniqueness of a parallel line. This requires mathematical knowledge that is usually taught in university. The task includes four items, three of which were classified as the most difficult items of the entire MCK-test as measured by the international Rasch difficulty.

In addition, two more tasks that were assigned to a high level of difficulty refer to university mathematics. One of these tasks, which refers to analytical geometry, demands the future teachers to interpret the solutions of a linear function geometrically, while they have to interpret the properties of a geometrical function in another task. The high level of difficulty that was assigned to the tasks prior to the testing was also approved empirically, but its solution required a more general understanding of geometry than the task on axiom of the uniqueness of a parallel line which involved more specialized knowledge.

Two tasks refer to the topic of transformations. In one of these tasks, the future teachers have to determine the number of symmetry axes for different shapes, in the other task they had to identify the transformation that was applied to an object. The other items refer to relations between lines and angles in geometrical figures. Here, for example, the future teachers need to judge if a statement is true by using the theorem of intersecting lines.

The following task illustrates the requirement to compare the properties of three-dimensional objects.



Two gift boxes wrapped with ribbon are shown below. Box A is a cube of side-length 10 cm. Box B is a cylinder with height and diameter 10 cm each.

Which box needs the longer ribbon? _______

Explain how you arrived at your answer.

Here, it is required to identify whether the ribbon of the cube or the cylinder is longer and to explain the choice adequately. Answers are regarded as correct and complete if box A is identified and the choice is explained by correctly calculating both ribbon lengths as shown in the following example: “Box A requires 3*40cm = 120cm ribbon. Box B requires 2*40cm = 80cm and the length of the circumference 2*π*5cm ≈ 31cm. In total, Box B needs 111cm ribbon and thus less than Box A.” But teachers could also focus their reasoning on comparing the circumferences of the circle and the square regarding that the remaining lengths are equal for both boxes. An example for this explanation would be the following: “Box A needs more ribbon because the circumference of a circle with a diameter of 10 cm is smaller than the circumference of a square with 10 cm side lengths and all remaining sizes are equal.”

Incomplete answers as well as answers with minor errors are valued as partial solutions. Answers are considered wrong if either the lengths of both ribbons are calculated incorrectly or teachers expect the same length for both ribbons. In addition, answers are considered wrong if a correct answer did not include an explanation because this was the main focus for this task. Finally, answers are considered wrong if misconceptions become apparent (for example with recourse to area or volume calculation).

In advance to the test, this task was classified as novice level of difficulty but the international average proved that it was more difficult regarding completely correct answers. Only 33 percent of the future teachers reached a complete and correct solution ranging from three percent correct solutions in Chile and the Philippines to 75 percent in Taiwan. Almost all the teachers chose the approach to calculate both ribbon lengths and compare the results. Apart from basic additions of the cube’s side lengths, this only requires knowledge about calculating the circle’s circumference by multiplying the diameter and π. But this is probably the greatest difficulty in most country and resulted in less complete and correct answers than expected. Substantial parts of teachers who reason conceptually by comparing the square’s and the circle’s circumference and refer to equality of all other sides only appeared with about five to ten percent in Singapore, Taiwan, Russia, Germany and Switzerland.

On average of all TEDS-M countries, 21 percent of the future teachers succeeded in developing a partial solution ranging from five percent in Chile to 38 percent in Germany. In Chile, only eight percent developed at least an explanatory approach while future teachers in Germany easily managed to develop at least a partial solution. In Germany a correct or partial correct solution was achieved by 79% of the future secondary school teachers.

Subdomain Data:

Only four MCK-items and two MPCK-items refer to the subdomain data. Two of the four MCK-items deal with probability, the two other items with statistics. Here, the future teachers have to calculate the probability of an event in a Laplace experiment or interpret and compare data with regard to their standard deviation.

For one item the future teachers have to calculate the conditional probability of an event. This item was classified as advanced level of difficulty in advance to the test and proved to be empirically difficult as well. Internationally, the subject area "data" is not part of every school curriculum and the item requires knowledge that goes beyond basic abilities in the field of probability.

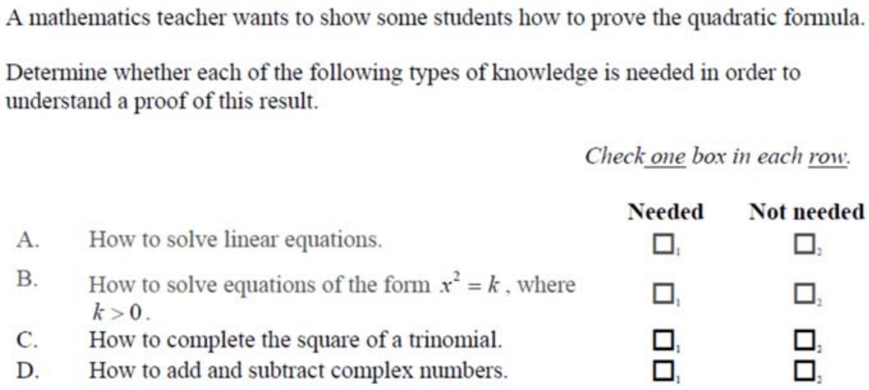
One item of this subdomain that combines MCK and MPCK is presented in the following description of the MPCK test.

Assessing Mathematics Pedagogical Content Knowledge (MPCK)

The part of the test that is supposed to assess MPCK consists of only 27 items. The lower number of items results especially from the difficulties to obtain an internationally accepted consensus about MPCK that is universally required by future mathematics teachers. Compared to MCK this was an even greater challenge. In this regard, theories and developments are affected even stronger by traditions and culture. The conceptualization of MPCK therefore was oriented towards the teacher's core task of teaching. For TEDS-M, two subdomains of mathematics pedagogical content knowledge were differentiated: (a) *Curricular knowledge and knowledge of planning for mathematics teaching and learning* and (b) *knowledge of enacting mathematics for teaching and learning*.

Subdomain Curricular knowledge and knowledge of planning for mathematics teaching and learning:

Three tasks which were categorized to this subdomain refer to curricular knowledge. The future teachers had to identify consequences for the planning of teaching due to a thematic change of the curriculum and determine the required precognition for a mathematical content. The item example below shows a task of this category. Another task that required the future teachers to decide if some given story-problems are an adequate representation of mathematical content was categorized as knowledge of planning for mathematics teaching and learning.



A mathematics teacher wants to show some students how to prove the quadratic formula.

Determine whether each of the following types of knowledge is needed in order to understand a proof of this result.

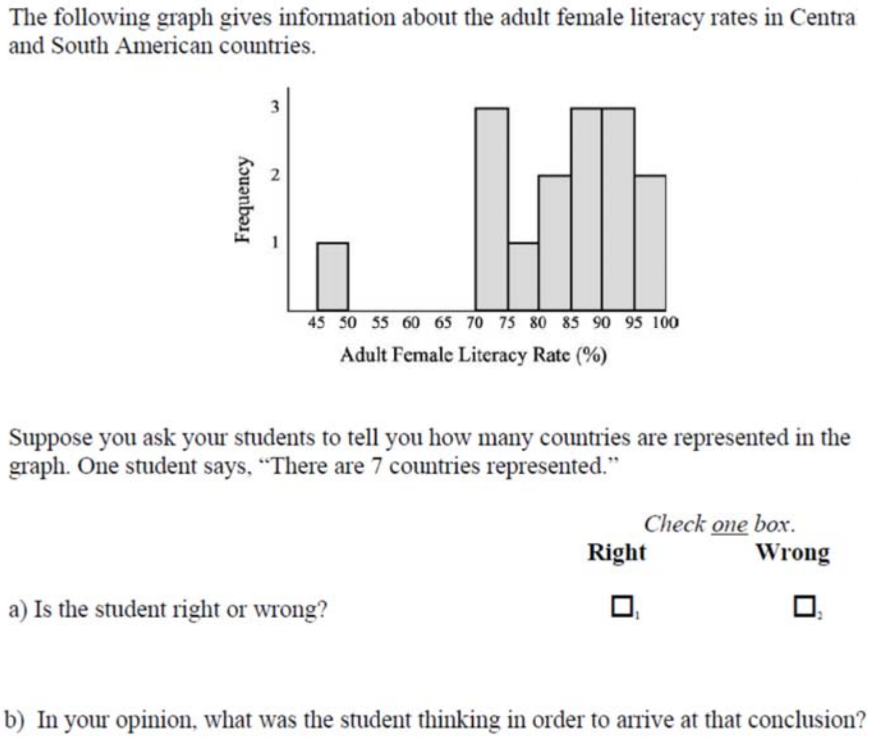
Check one box in each row.

		Needed	**Not needed**
A.	How to solve linear equations.	☐$_1$	☐$_2$
B.	How to solve equations of the form $x^2 = k$, where $k > 0$.	☐$_1$	☐$_2$
C.	How to complete the square of a trinomial.	☐$_1$	☐$_2$
D.	How to add and subtract complex numbers.	☐$_1$	☐$_2$

For this task, the future teachers have to determine the required precognition for proving the quadratic formula. The knowledge described in A, B and C is needed, but the knowledge described in D is not. Mathematical knowledge of the quadratic formula and methods to prove this formula are essential in order to solve this task correctly. The future teachers easily solved items A, B and D. On international average, item A was answered correctly by 77 percent of the future teachers, item B by 76 percent and item D by 62 percent. Item C was merely solved correctly by 48 of the test persons. Probably, the term trinomial is not very familiar and is not directly associated with solving quadratic equations. Merely 35 percent of the future teachers in Singapore, who reached the third highest score in the MCK test in total, answered item C correctly while at least 85 percent of them answered the other three items correctly. Then again, 80 percent of the future teachers on the Philippines correctly solved item C but only achieved low scores in the MCK test in total and only 31 percent of them correctly solved item D. This may indicate cultural differences regarding the familiarity of the term trinomial as well as knowledge about proving the quadratic formula.

Subdomain Knowledge of enacting mathematics for teaching and learning:

15 items were categorized as knowledge of enacting mathematics for teaching and learning. For one item, the future teachers have to argue why the level of difficulty for two tasks with a similar context is different. The other items refer to students' solutions. Here, the future teachers have to evaluate given verbal or illustrated solutions. They need to decide if some given statements are appropriate responses to a student's solution or they have to analyze a student's solution with regard to typical students' misconceptions. One task that combines this subdomain with the MCK subdomain data is shown below.



The following graph gives information about the adult female literacy rates in Centra and South American countries.

Suppose you ask your students to tell you how many countries are represented in the graph. One student says, "There are 7 countries represented."

Check one box.

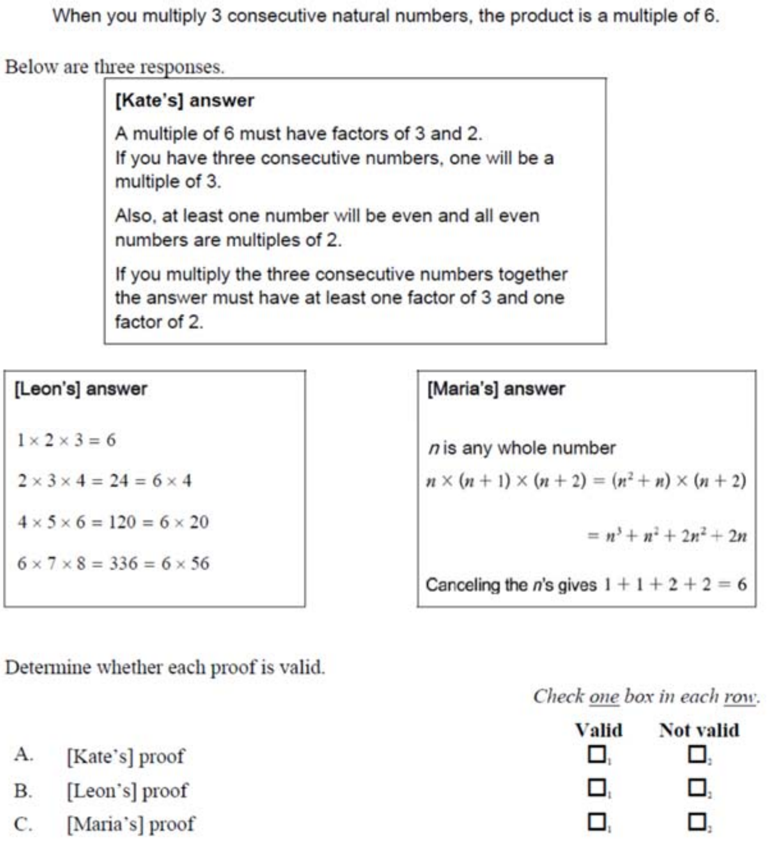
	Right	**Wrong**
a) Is the student right or wrong?	☐1	☐2

b) In your opinion, what was the student thinking in order to arrive at that conclusion?

The first item requires the future teachers to judge whether the given interpretation of the bar chart is correct or not (MCK). In the following MPCK item, they need to analyze the reason for the student's interpretation. In order to give a correct answer to the first item, the concept of frequency needs to be known and the future teachers have to understand the bar chart as a frequency distribution. The correct answer is "wrong" because the bar chart represents 15 countries instead of only seven. Regarding the MPCK item, responses are accepted that indicate that the student thought each bar represents one country. This task is a positive example for successfully linking the MCK and the MPCK items. Both items refer to the same context while the MPCK item can be answered correctly even if the MCK item was answered incorrectly.

Both items were classified as novice level of difficulty in advance to the test. This task requires mathematical knowledge that is attained in secondary school in most of the countries and may already be part of primary school mathematics education. On international average, 72 percent of the future teachers correctly answered the MCK item while 70 percent gave a correct response to the MPCK item. Thus, these items proved to be empirically easy as well. The least correct answers for both items were given by future teachers in Georgia. Here, the MCK item reached 40 percent correct responses and the MPCK item 19 percent. The highest solution frequency regarding the MCK item was achieved by future teachers in Singapore (95% correct responses) while future teachers in Switzerland reached the highest solution frequency for the MPCK item (91% correct responses).

For another task the future teachers have to judge if three student responses are valid proofs of a statement of number theory (see below).



When you multiply 3 consecutive natural numbers, the product is a multiple of 6.

Below are three responses.

[Kate's] answer

A multiple of 6 must have factors of 3 and 2.
If you have three consecutive numbers, one will be a multiple of 3.

Also, at least one number will be even and all even numbers are multiples of 2.

If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

[Leon's] answer

$1 \times 2 \times 3 = 6$

$2 \times 3 \times 4 = 24 = 6 \times 4$

$4 \times 5 \times 6 = 120 = 6 \times 20$

$6 \times 7 \times 8 = 336 = 6 \times 56$

[Maria's] answer

n is any whole number

$$n \times (n + 1) \times (n + 2) = (n^2 + n) \times (n + 2)$$

$$= n^3 + n^2 + 2n^2 + 2n$$

Canceling the n's gives $1 + 1 + 2 + 2 = 6$

Determine whether each proof is valid.

Check one box in each row.

		Valid	**Not valid**
A.	[Kate's] proof	☐$_1$	☐$_2$
B.	[Leon's] proof	☐$_1$	☐$_2$
C.	[Maria's] proof	☐$_1$	☐$_2$

With the approval of Healy and Hoyles, the task was adapted and used in the TEDS-M study in 2008 (Healy & Hoyles, 1998). It refers to the domain number and requires interactive knowledge on a novice level. Three different student solutions are presented that involve different ideas of proof and need to be verified regarding validity. To begin with, the participating future teachers need to understand and analyze the student solutions.

Kate’s solution involves a complete and generally valid proof that can be classified as correct. In contrast, Leon’s solution only uses examples because he only analyzes four special cases. Thus, his proof is not generally valid. Maria initially uses an adequate approach to prove the statement but undertakes an invalid reduction and, thus, does not give valid proof either.

The three items presented proved to be of different empirical difficulty. As classified in advance, identifying Kate’s proof as invalid was easy. On international average, 74% identified this student solution as invalid, ranging from 51% in Botswana to 97% in Taiwan.

The future secondary school teachers had more difficulties in classifying Maria’s solution. On average, of all TEDS-M participating countries, 59% of the future teachers classified the proof as invalid ranging from 44% in Chile to 92% in Taiwan. Interestingly, classifying Leon’s solution caused the greatest difficulties across the 15 participating countries of the TEDS-M study. Only 45% rejected this student solution. In Botswana, merely 3% of the future mathematics teachers identified that the proof is based on examples. Obviously, the formalized procedure was confused with general validity.

SUMMARY, DISCUSSION AND CONCLUSIONS

The mathematical requirements in the TEDS-M secondary level test are much higher than in the primary level test (Döhrmann, Kaiser, & Blömeke, 2012). There are tasks in all three subdomains, algebra, number and geometry, which require MCK that is usually acquired in university courses. Such tasks only sporadically appear in the primary level test while the secondary level test’s subdomains, number and geometry, include several of these items. Again, the subdomain, algebra, contains additional tasks that can be solved by secondary school students as well.

Similar to the MCK primary level test, the subdomain, data, is merely represented by few items. Statistics and probability are unequally implemented into the mathematics curricula of schools and teacher education in the participating countries while algebra, number and geometry belong to the standard repertoire of mathematics education all over the world (cf. KMK, 2004; NCTM, 2000; NGA & CCSSO, 2010; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997;).

The secondary level test more strongly focusses on the conceptual understanding of mathematics and understanding of mathematical structures as compared to the primary level test. Computations are of lower relevance while argumentations and proofs are strongly fostered. Like in the primary level test, heuristic problem solving, modelling of non-routine problems and the use of technology are areas that were mostly left out of the test. This leads to the conclusion that more traditional notions of mathematics influenced the conceptualization of the cognitive domains of MCK in TEDS-M.

It is by definition probably impossible to design MPCK items without any mathematical content. However, it must be acknowledged that in order to answer some of the MPCK items in the TEDS-M test correctly, mathematical knowledge is required. The MPCK item presented above is a representative example for this effect. In addition, the items' different levels of difficulties can hardly be explained by mathematics pedagogical content knowledge and skills. Instead, they may be caused by different *mathematical* requirements. The MPCK test focuses on an analysis and evaluation of students' responses while other *didactical* requirements of the framework, such as identifying the key ideas in learning programs, establishing appropriate learning goals, choosing assessment formats and predicting typical students' responses, were less often considered.

As already stated for the primary level test, it also extends to the secondary level test of TEDS-M that the conceptualization of MPCK was oriented towards the teacher's core task of teaching. The test refers to various abilities and skills that are essential to concretely plan and realize mathematical lessons. These abilities and skills can be described as an internationally accepted common core of MPCK that is universally required by future mathematics teachers. This also includes analyzing and evaluating students' responses.

National characteristics of MPCK from individual participating countries had to be excluded, of course. The framework, for example, did not include didactical concepts, the promoting of process-related competencies based on mathematical contents, strategies for dealing with children's heterogeneity, theoretical knowledge about preschool age mathematical knowledge development or the knowledge about research in mathematics pedagogy. As already indicated by findings from the primary school study, the conceptualization of MPCK in TEDS-M is thus guided by curriculum theory and educational psychology which dominates in English-speaking countries. In contrast, continental European traditions rather focus on subject-related reflections, called *Didaktik* in German or *didactique* in French. Subject-related didactics describe the pedagogical transformation of disciplinary content to teaching content, taking into account the whole teaching-and-learning-process (Pepin, 1999). These differences in basic orientations of the countries participating in TEDS-M need to be explored in further studies although it may be difficult to test corresponding knowledge and skills on a large scale.

NOTES

[i] TEDS-M was funded by the IEA, the National Science Foundation (REC 0514431), and the participating countries. In Germany, the German Research Foundation funded TEDS-M (DFG, BL 548/3-1). The instruments are copyrighted by the TEDS-M International Study Center at MSU (ISC). The views expressed in this paper are those of the authors and do not necessarily reflect the views of the IEA, the ISC, the participating countries or the funding agencies.

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