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Asymptotic representation theory of the infinite quantum symmetric group

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Master's Thesis, Spring 2019

This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 30 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Let τ be an extremal tracial state on an AF-algebra $A = \overline{\bigcup_n A_n}$. Vershik and Kerov proved in 1981 that τ is a certain limit of traces on the finite dimensional algebras A_n. In this thesis we give a new proof of their result that does not rely on the description of A as grupoid C^*-algebra. We also generalize the proof to a slightly wider class of inductive limit C^*-algebras that we call *fat AF-algebras.* Fat AF-algebras satisfy many of the same properties as AF-algebras. For instance one can construct Bratteli diagrams and corresponding multiplicity matrices. The quantum permutation groups S_n^+, $n \geq 4$, give rise to a fat AF-algebra $\mathcal{A}(S_\infty^+)$, called the group C^*-algebra associated to the infinite quantum symmetric group S_∞^+. We show that the multplicity matrix for the inclusion of S_n^+ into S_{n+1}^+ does not depend on n for $n \geq 4$, and give a closed formula for the entries. Using this and the theory developed for traces on fat AF-algebras, we give a partial description of extremal traces on $\mathcal{A}(S_\infty^+)$.

Acknowledgements

First and foremost I would like to thank Sergey Neshveyev for great supervision, and for introducing me to the topics of this thesis. Thank you for answering all my questions, and for guiding me through this semester.

I would like to thank Johannes, Audun, and the rest of "lektorgjengen" for the first years at the university. Not only for great collaboration, but also for making the days at Blindern (and evenings elsewhere) so enjoyable. "Pavlova, hei!"

I must also thank my fellow students during the last two years. Both the people I have collaborated with in courses, and those I have shared study hall with. Thank you for interesting conversations about mathematics and other topics.

Finally I would like to thank my girlfriend, Ranveig, for her unwavering support.

Contents

1. INTRODUCTION

When studying representations of a locally compact group G, it is natural to consider the unitary dual $\hat{G}$. For some groups this is a nice set, and one can hope to classify all the irreducible representations of the group. However, for many infinite groups this is not the case. Indeed, when G is not of Type I the canonical Borel structure on $\hat{G}$ is so "wild" that one cannot hope to understand the representation theory of G completely ([Fol16], chapter 7). Nonetheless, even in the "wild" case one can try to understand the set of finite factor representations of the group.

1.1. Theorem. *Let A be a C^*-algebra with tracial state τ, and corresponding GNS-representation $\pi_\tau : A \to B(H_\tau)$. Then $\pi_\tau(A)''$ is a finite factor if and only if τ is an extremal trace.*

It is well known that the unitary representations of a locally compact group G is in one to one correspondence with the nondegenerate $*$-representations of its group C^*-algebra $C^*(G)$ [Fol16]. Thus, in view of the above theorem, one can study the factor representations of G by trying to understand extremal tracial states on the group C^*-algebra.

When the group appears as an inductive limit of finite dimensional groups the group C^*-algebra becomes an AF-algebra. In 1981 Vershik and Kerov introduced the so-called "ergodic-method" for studying extremal tracial states in these situations [VK81]. They considered the AF-algebra $C^*(S_\infty) = \varinjlim C^*(S_n)$ and looked at the set T of paths in the Bratteli diagram of $C^*(S_\infty)$. A path is simply a sequence of edges $(t_n)_n$ in the Bratteli diagram where the source of t_n is equal to the range of t_{n-1} for all n. If $(t_n)_n$ is such a path there is a natural way to associate to each edge t_n a representation of $C^*(S_n)$ onto a Hilbert space H_{t_n}, and a therefore a trace χ^{t_n} on $B(H_{t_n})$. Vershik and Kerov proved that any extremal tracial state on $C^*(S_\infty)$ appears as a certain limit of these traces:

1.2. Theorem. *Assume that τ is an extremal tracial state on $C^*(S_\infty)$. Then there exists a path $(t_n)_n \in T$ such that*

$$\tau(a) = \lim_{n\to\infty} \frac{\chi^{t_n}(a)}{\dim H_{t_n}}$$

fo any $a \in C^*(S_k)$ and $k \geq 1$.

Their proof is based on identifying $C^*(S_\infty)$ with a certain C^*-algebra coming from an equivalence relation on the path space, and make use of the fact that extremal tracial states on this C^*-algebra corresponds to ergodic measures on T. Then they conclude by using that the sequence in the theorem above forms a backwards martingale on T. Applying their result, they were able to find an explicit formula for extremal tracial states on $C^*(S_\infty^+)$.

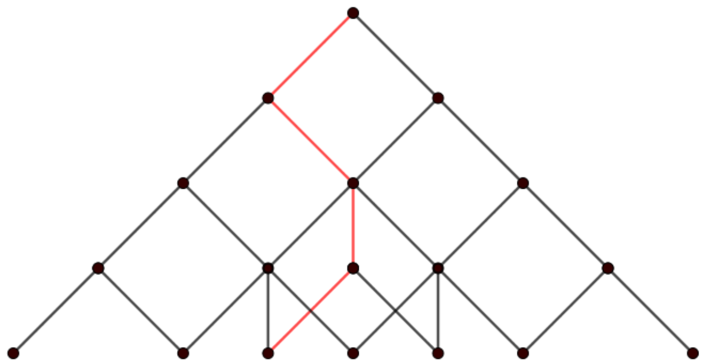

Figure 1.1: A path in the first few levels of the Bratteli diagram of $C^*(S_\infty)$.

This approach generalizes to inductive limits of second countable compact groups as well [EI16]. Here one has to take care of the fact that the group C^*-algebras can be infinite dimensional, and are not necessarily contained in each other. It is possible to fix this issue by considering the inductive limit of certain C^*-subalgebras of the group von Neumann algebras instead.

Quite recently the result of Vershik and Kerov has been extended to compact quantum groups [Sat18]. When given a family of compact quantum groups $\{G_n\}_{n\in\mathbb{N}}$, with G_n a quantum subgroup of G_{n+1}, it is not obvious what "$G = \varinjlim G_n$" is supposed to mean. However, it is possible to define a "group C^*-algebra" $C^*(G_n)$ for each n, and from this data one can construct an inductive limit C^*-algebra $\mathcal{A}(G)$ in a similar manner as for compact groups. Thus, although we do not have a compact quantum group G, it is possible to study the "inductive system" of compact quantum groups $\{G_n\}_n$ by analyzing $\mathcal{A}(G)$. This has been done in the paper [Sat18] for the case of unitary quantum groups $U_q(N)$.

In this thesis we consider the ideas above with the aim of applying them to compact quantum groups. In particular we study extremal tracial states on the group C^*-algebra $\mathcal{A}(S_\infty^+)$ associated to the infinite quantum symmetric group S_∞^+.

We give a new proof of theorem 1.2, using an approach where we do not need the construction of C^*-algebras coming from equivalence relations on the path space. Furthermore, we generalize this result in a way that can be applied to the inductive limit C^*-algebra $\mathcal{A}(S_\infty^+)$. In this case $C^*(S_n^+) = c_0 - \bigoplus_{i=0}^\infty B(H_{(n,i)})$, for $n \geq 4$, for some finite dimensional Hilbert space $H_{(n,i)}$. It turns out that the multiplicity a_{ij} of $B(H_{(n,i)})$ into $B(H_{(n+1,j)})$ does not depend on n, and we give a closed formula for the matrix $A = (a_{ij})_{ij}$. For many of the entries a_{ij} the formula is quite "ugly", but it does give us some immediate information. For instance, A is upper triangular with ones one the diagonal. We use this to show that the only extremal tracial state on $\mathcal{A}(S_\infty^+)$ coming from a "bounded path", is the trivial trace.

1.1 Outline of the thesis

Chapter 2. We start of by recalling some fundamental facts about von Neumann algebras and traces on them. In particular we prove theorem 1.1. Moreover, we introduce the notion of trace preserving conditional expectations, which is of great importance in Chapter 3.

Chapter 3. The goal of this chapter is to give a new proof of theorem 1.2. First we recall the notion of Bratteli diagrams and introduce the notion of paths in a precise manner. In this part of the chapter we also explain the multiplicity matrices corresponding to a Bratteli diagram, and how we can describe traces on the AF-algebra in terms of this matrix. Then we turn to the proof, which is very much based on properties of von Neumann algebras and trace preserving conditional expectations. Lastly we generalize this proof to a special case of inductive limit C^*-algebras which we call *fat AF-algebras*.

Chapter 4. In this chapter we give an informal and brief introduction to compact quantum groups. The goal is to explain the theory to the extent that we can define and study the group C^*-algebra associated to the "inductive limit" of compact quantum groups. We also introduce our main example of compact quantum groups, namely the quantum permutation groups S_n^+.

Chapter 5. We consider the group C^*-algebra $\mathcal{A}(S_\infty^+)$ associated to the "inductive system" of quantum permutation groups S_n^+. This relies heavily on the fact that representations of S_n^+, $n \geq 4$, satisfy the same fusion rule as the representations of SO(3) [Ban99]:

$$U_s \otimes U_t = U_{|s-t|} \oplus U_{|s-t|+1} \oplus \cdots \oplus U_{s+t}.$$

Because of this $\mathcal{A}(S_\infty^+)$ has a constant multiplicity matrix A, and we find a closed expression for A, using a relation between the characters of SO(3) and Chebyshev polynomials. We finish by discussing what consequences this has for the tracial states on $\mathcal{A}(S_\infty^+)$.

2. Preliminaries

Throughout this text we will assume that the reader is familiar with the basic theory of C^*-algebras and their representation theory. Nevertheless it will be convenient to recall some of the notions that will be important in the following chapters. We do this quite informally and mostly without proofs.

2.1 Conventions

If H is a Hilbert space, we denote by $B(H)$ the C^*-algebra of bounded operators on H. We write $(\cdot \mid \cdot)$ for the inner product on H, and always assume that it is antilinear in the second variable.

When $\pi : A \to B(H_\pi)$ is the GNS-representation of a C^*-algebra A corresponding to a positve linear functional ϕ, we will write $(\pi_\phi, H_\phi, \xi_\phi)$ for the corresponding GNS-triple. That is, $\pi_\phi = \pi$, $H_\phi = H_\pi$ and ξ_ϕ is the corresponding cyclic vector.

If A is a C^*-algebra and $\{S_i\}_{i \in I}$ is a family of subsets of A, then $C^*(S_i \mid i \in I)$ denotes the C^*-subalgebra of A generated by the sets S_i. Similarly, if A is von Neumann algebra we write $W^*(S_i \mid i \in I)$ for the von Neumann algebra generated by the sets S_i.

When given a family $\{A_i\}_{i \in I}$ of C^*-algebras we can form the algebraic direct sum $\bigoplus_{i \in I} A_i$, and the cartesian product $\prod_i A_i$. Between these two sets we have two different C^*-algebras with the norm $\|(a_i)_i\|_\infty = \sup_i \|a_i\|_i$ where $\|\cdot\|_i$ denotes the norm on A_i. Namely

$$c_0 - \bigoplus_{i \in I} A_i = \{(a_i)_i \in \prod_i A_i \mid \lim_i \|a_i\|_i = 0\}$$
$$\ell^\infty - \bigoplus_{i \in I} A_i = \{(a_i)_i \in \prod_i A_i \mid \sup_i \|a_i\|_i < \infty\}$$

We will use this notation throughout the text.

Lastly we note that we will often omit index sets from our notation. Hopefully the index sets being used are in all situations clear from the context.

2.2 Von Neumann algebras and traces

A *von Neumann algebra* is by definition a C^*-subalgebra of $B(H)$ (for some Hilbert space H) which contains the identity $1 \in B(H)$ and is closed in the strong operator topology. An easy way to obtain a von Neumann algebra is as follows: Let $S \subset B(H)$ be a subset which is closed under taking adjoints. Then the *commutant*

$$S' = \{a \in B(H) \,|\, ab = ba \ \forall b \in S\}$$

is quite easily seen to be a von Neumann algebra. The *double commutant* $S'' = (S')'$ is then a von Neumann algebra containing S. As the reader probably knows, one of the most important results in the whole theory of C^*-algebras is the *double commutant theorem*. Recall that a C^*-subalgebra A of $B(H)$ is called nondegenerate if AH is dense in H.

2.1. Theorem (von Neumann's double commutant theorem). *Assume that A is a nondegenerate C^*-subalgebra of $B(H)$. Then A is dense in A'' in the strong operator topology. In particular, if M is a von Neumann algebra, then $M = M''$.*

Proof. We assume that the reader is familiar with this result. A nice exposition of the proof can be found in for instance [Fol16]. □

Let us recall some terminology on linear functionals.

2.2. Definition. A functional ϕ on a C^*-algebra A is called *positive* if $\phi(a^*a) \geq 0$ for all $a \in A$. We have the following definitions for a postive linear functional ϕ:

- ϕ is called *faithful* if $\phi(a^*a) \neq 0$ whenever $a \neq 0$.
- If $\|\phi\| = 1$ we say that ϕ is a *state*. When A is unital this condition is equivalent to $\phi(1) = 1$.
- ϕ is called *tracial* if $\phi(ab) = \phi(ba)$ for all $a, b \in A$. We sometimes say that ϕ is a *trace*.

If $M \subset B(H)$ is a von Neumann algebra we say that a bounded functional ϕ is *normal* if the restriction of ϕ to the unit ball in M is weakly operator continuous. We denote by M_* the set of normal functionals on M. If H is a Hilbert space and $\xi, \zeta \in H$, denote by $\omega_{\xi,\zeta} \in B(H)^*$ the functional $T \mapsto (T\xi \,|\, \zeta)$. It can be shown that ([Bla06], III.2)

$$M_* = \overline{\mathsf{span}\{\omega_{\xi,\zeta}|_M \,|\, \xi, \zeta \in H\}}^{\|\cdot\|}.$$

Now we turn to some special classes of von Neumann algebras. Recall that two projections p, q in a C^*-algebra A are said to be *(Murray-von Neumann) equivalent* if there exists a partial isometry $u \in A$ such that $u^*u = p$ and $uu^* = q$. A projection p is said to be *finite* if there is no projection $q \neq p$ with $q \leq p$ equivalent to p.

2.3. Definition. A von Neumann algebra M is called *finite* if the identity in M is a finite projection.

Assume now that M is a von Neumann algebra with a faithful tracial state τ, and identity 1. Let $p \in M$ be a projection equivalent to 1, with $p \leq 1$. Then there is by definition a partial isometry $u \in M$ such that $u^*u = p$ and $uu^* = 1$. Thus

$$\tau(1-p) = 1 - \tau(u^*u) = 1 - \tau(uu^*) = 0.$$

Since τ is faithful we conclude that $p = 1$. Thus 1 is a finite projection, and M is a finite von Neumann algebra. In fact the converse is also true.

2.4. Theorem. *A von Neumann algebra M is finite if and only if it has a normal faithful tracial state.*

Proof. See [SZ79] theorem 7.11. □

It is a consequence of Kaplansky's density theorem that if τ is a tracial state on a C^*-algebra A, then $\omega_{\xi_\tau,\xi_\tau}$ defines a normal tracial state on $M = \pi_\tau(A)''$, which we still denote by τ. One can show that this induced trace is in fact faithful as well (and thus that $\pi_\tau(A)''$ is finite). We will sometimes use the follwing fact without mention:

2.5. Proposition. *Let B be C^*-algebra with C^*-subalgebra $A \subset B$ and assume that τ is a tracial state on B. Write $\psi = \tau|_A$. Then we have an isomorphism $\pi_\psi(A)'' \cong \pi_\tau(A)''$.*

Proof. Consider $K = \overline{\pi_\tau(A)\xi_\tau} \subset H_\tau$, and let p be the orthogonal projection onto K. Then the map

$$\pi_\psi(A)\xi_\psi \to \pi_\tau(A)\xi_\tau, \quad \pi_\psi(a)\xi_\psi \mapsto \pi_\tau(a)\xi_\tau,$$

extends to a unitary $U : H_\psi \to K$. Moreover U intertwines π_ψ and the representation $p\pi_\tau|_A$ on K. In particular $\pi_\psi(A) \cong p\pi_\tau(A)$. Using this identification it suffices to show that the map

$$\pi_\tau(A)'' \to \pi_\psi(A)'', \quad T \mapsto T|_K = pT$$

is an isomorphism. The only nontrivial part is injectivity. To show this, assume that $pT = 0$. Then $T\xi_\tau = pT\xi_\tau = 0$, which in turn implies that $(T^*T\xi_\tau \mid \xi_\tau) = 0$. By the faithfulness of $\omega_{\xi_\tau,\xi_\tau}$ on $\pi_\tau(B)''$, we conclude that $T = 0$. □

The following construction is often needed in the theory. With M and τ as above, define a map $M\xi \to M\xi$ by $x\xi_\tau \to x^*\xi_\tau$. Since $\|x\xi_\tau\| = \tau(x^*x) = \tau(xx^*) = \|x^*\xi_\tau\|$, this map is indeed well-defined and isometric, and it extends to an antilinear involution $J : H_\tau \to H_\tau$. We call J the *modular conjugation* of M. We will not elaborate on this here, but a key fact about the modular conjugation is that $JM'J = M$.

Now we arrive at the first theorem mentioned in the introduction. Recall that

2.6. Definition. A von Neumann algebra M is called a *factor* if its center is trivial:

$$Z(M) = \{a \in M \mid ab = ba, \ \forall b \in M\} = \mathbb{C}1.$$

2.7. Theorem. *Let A be a C^*-algebra with tracial state τ. Then $\pi_\tau(A)''$ is a finite factor if and only if τ is an extremal trace.*

Proof. That $M = \pi_\tau(A)''$ is finite is contained in the discussion above. For simplicity we write $\xi = \xi_\tau$.

Assume first that M has nontrival center $Z(M)$. Then we can find a nontrivial projection $p \in Z(M)$. It is easily verified that the linear functionals $\tau_1 = \tau(p\cdot)/\tau(p)$ and $\tau_2 = \tau((1-p)\cdot)/\tau(1-p)$ are traces.[1] Moreover, for any $x \in M$

$$\tau(x) = \tau(px) + \tau((1-p)x) = \tau(p)\tau_1(x) + \tau(1-p)\tau_2(x).$$

Hence, τ is not extremal. This shows the "if" part.

To show the other implication, recall that we have a bijection

$$\{x \in M' \mid 0 \le x \le 1\} \leftrightarrow \{\phi \in A^* \mid 0 \le \phi \le \tau\}$$

given by the map $x \mapsto (\pi_\tau(\cdot)x\xi \mid \xi)$. Thus, if τ is not extremal we can a find a nonscalar $x \in M'$, $0 \le x \le 1$, defining a trace $a \mapsto (ax\xi \mid \xi)$ on M. Using this, and the properties of the modular conjugation J mentioned above, we get for $a, b \in M$

$$\begin{aligned}(JxJa\xi \mid b\xi) &= (Jxa^*\xi \mid b\xi) \\ &= (b^*\xi \mid xa^*\xi) \\ &= (ab^*\xi \mid x\xi) \\ &= (b^*a\xi \mid x\xi) = (xa\xi \mid b\xi).\end{aligned}$$

Thus since $M\xi$ is dense in H, we must have that $JxJ = x$. But $JM'J = M$, whence $x = JxJ \in M' \cap M = Z(M)$. □

We conclude this section with the notion of *conditional expectation*.

2.8. Definition. Let A be a C^*-algebra with C^*-subalgebra B. A *conditional expectation* is a positive linear map $E : A \to B$ such that

(i) $E(b) = b$ for all $b \in B$

(ii) $E(ba) = bE(a)$ and $E(ab) = E(a)b$ for all $a \in A, b \in B$.

Suppose we are given a conditional expectation $E : A \to B$, and a tracial state τ on A. Then we say that E is *trace-preserving* if $\tau = \tau \circ E$.

2.9. Proposition. *Assume that M is a von Neumann algebra with faithful normal tracial state τ, and that $N \subset M$ is a von Neumann subalgebra. Then there is a unique trace preserving conditional expectation $E : M \to N$.*

[1] $\tau(p\cdot)$ is simply convenient notation for the functional $x \mapsto \tau(px)$.

Proof. Although it can be proven without much more theory, this is a special case of a more general result. See [Tak03], chapter 9, theorem 4.2. □

We will be using this in the following situation: Let A be a C^*-algebra with tracial state A, and suppose $B \subset A$ is a C^*-subalgebra. Then, by the discussion above, the induced trace on $\pi_\tau(A)''$ (still denoted by τ) is faithful and normal, and so there is a unique τ-preserving conditional expectation $E : \pi_\tau(A)'' \to \pi_\tau(B)''$.

3. Tracial states on AF-algebras

Recall that a C^*-algebra A is called *AF* (or an *AF-algebra*) if it is the closure of an increasing union of finite dimensional C^*-algebras A_n, i.e,

$$A_1 \subset A_2 \subset A_3 \subset \cdots \subset \overline{\bigcup_n A_n} = A.$$

Troughout this chapter we will assume that the embeddings $A_n \subset A_{n+1}$ are unital.

The goal of this chapter is to prove theorem 1.2 mentioned in the introduction. To do this we first need the notion of a Bratteli diagram for an AF-algebra and some related notions.

3.1 Bratteli diagrams and traces

A Bratteli diagram for an AF-algebra $A = \overline{\bigcup_n A_n}$ is a way of encoding the embeddings $A_n \subset A_{n+1}$ in a certain graph. This construction is more or less based on the results in theorem 3.1 below.

3.1. Theorem. *Let $M_n\mathbb{C}$ denote the C^*-algebra of $n \times n$ matrices with complex entries.*

(a) Any irreducible representation of $M_n\mathbb{C}$ is unitarily equivalent to the canonical (identity) representation on $\mathbb{C}^n$.

(b) If A is a finite dimensional C^*-algebra, then there exist numbers $n_1, ..., n_m \in \mathbb{N}$ such that

$$A \cong \bigoplus_{i=1}^{m} M_{n_i}\mathbb{C}.$$

Equivalently, we can choose a complete family $\mathsf{Irr}(A)$ of representatives of mutually inequivalent irreducible representations of A, and write

$$A \cong \bigoplus_{\pi \in \mathsf{Irr}(A)} B(H_\pi).$$

Remark. Assume that $\pi : M_n\mathbb{C} \to B(H)$ is a nondegenerate representation of $M_n\mathbb{C}$. Assertion (a) in the theorem tells us that there is a number $k \in \mathbb{N}$ such that π is unitarily equivalent to the representation

$$M_n\mathbb{C} \to B(K \oplus (\mathbb{C}^n)^{\oplus k}), \quad T \mapsto 0 \oplus T^{\oplus k}.$$

for some Hilbert space K such that $\dim K + nk = \dim H$. △

Proof. Both assertions above can be obtained from the representation theory of C^*-subalgebras of $K(H)$ (the compact operators on a Hilbert space H). See for instance the theorems I.10.5, I.10.8 and III.1.1 in [Dav96]. However, it is not hard to prove directly:

By the GNS-construction, the Gelfand-Naimark theorem and the finite dimensionality of A, we can find mutually inequivalent irreducible representations $\pi_1, ..., \pi_m$ on finite dimensional Hilbert spaces $H_1, ...H_n$, such that $\bigcap_i \ker \pi_i = 0$. Thus, the representation $\pi = \bigoplus_{i=1}^n \pi_i$ is faithful. Moreover, by Schur's lemma

$$\Big(\bigoplus_{i=1}^n \pi_i(A)\Big)' = \bigoplus_{i=1}^n \mathbb{C}1_{H_i},$$

whence

$$A \cong \Big(\bigoplus_{i=1}^n \pi_i(A)\Big)'' = \Big(\bigoplus_{i=1}^n \mathbb{C}1_{H_i}\Big)' = \bigoplus_{i=1}^n B(H_i).$$

Now, if $\pi : A \to B(H_\pi)$ is an irreducible representation of A which is not equivalent to any of the representations $\pi_1, \pi_2, ..., \pi_n$, then

$$A = B(H_\pi) \oplus \bigoplus_{i=1}^n B(H_i)$$

by the same argument. This is clearly nonsense. We conclude that the representatives $\pi_1, \pi_2, ..., \pi_n$ exhaust all equivalence classes of irreducible representations of A. This shows (ii). It immediatly follows from this that the only irreducible representation (up to unitary equivalence) of $M_n\mathbb{C}$ is the identity representation $M_n\mathbb{C} \to M_n\mathbb{C}$. □

Let A be a finite dimensional C^*-algebra. The notation $A \cong \bigoplus_{i=1}^m M_{n_i}\mathbb{C}$ can become quite messy when one considers AF-algebras. We will therefore assume that we have fixed a complete family $\mathsf{Irr}(A)$ of representatives of (mutually inequivalent) irreducible representations of A, and write

$$A \cong \bigoplus_{\pi \in \mathsf{Irr}(A)} B(H_\pi),$$

Assume that $B \cong \bigoplus_{\rho \in \mathsf{Irr}(B)} B(H_\rho)$ is a finite dimensional C^*-algebra as well, and that $A \subset B$. Consider an irreducible representation ρ of B. Then the restriction $\rho|_A$ is a representation of A, and hence it has a decompositon into irreducible representations of A. We can write this as

$$\rho|_A \sim 0 \oplus \bigoplus_{\pi \in \mathrm{Irr}(A)} \pi^{\oplus a(\pi,\rho)},$$

where $a(\pi, \rho)$ is the number of representations equivalent to π that appear in in the decomposition of $\rho|_A$. By the remark after theorem 3.1 it follows that the representation $B(H_\pi) \hookrightarrow B(H_\rho)$ coming from the inclusion $A \subset B$ is equivalent to the representation

$$B(H_\pi) \ni T \mapsto 0 \oplus T^{\oplus a(\pi,\rho)} \in B(H_\rho),$$

when we identify $B(H_\rho) \cong B(K \oplus H_\pi^{\oplus a(\pi,\rho)})$, for some Hilbert space K. We conclude that the embedding $A \hookrightarrow B$ is completely determined (up to unitary equivalence) by the numbers $a(\pi, \rho)$, and that we may think of $a(\pi, \rho)$ as the number of times $B(H_\pi)$ embeds in $B(H_\rho)$. We call the numbers $a(\pi, \rho)$ the *multiplicity* of π in ρ, and if $a(\pi, \rho) \neq 0$, we write $\pi \subset \rho|_{A_n}$.

3.2. Definition. Let $A = \overline{\bigcup_n A_n}$ be an AF-algebra. A *Bratteli diagram* for A is a graph described as follows:

- The vertices is the set $\bigsqcup_n \mathrm{Irr}(A_n)$, and a vertice π has the label $\dim \pi = \dim H_\pi$.
- If $\pi \in \mathrm{Irr}(A_n)$, $\rho \in \mathrm{Irr}(A_{n+1})$ and $\pi \subset \rho|_{A_n}$, the number of edges between π and ρ is $a(\pi, \rho)$. There are no other edges.

Remark that an AF-algebra A always has several Bratteli diagrams. However, they are all "equivalent" in a certain sense (see **[d96]** theorem III.3.5). We will often abuse terminology and simply speak about "the Bratteli diagram of A".

3.3. Example. If G is a finite group let $\hat{G}$ denote the (finite) set of equivalence classes of irreducible representations of G. The group C^*-algebra of G can be written as

$$C^*(G) = \bigoplus_{[\pi] \in \hat{G}} B(H_\pi).$$

If H is a subgroup of G, we can define multiplicities $a(\rho, \pi)$, $[\pi] \in \hat{G}$, $[\rho] \in \hat{H}$, as for finite dimensional C^*-algebras. It is then clear, from the formula above, that the multiplicities given by the inclusion $C^*(H) \hookrightarrow C^*(G)$ are determined by the multiplicities given by the inclusion $H \subset G$. Thus, if we have a sequence of finite groups $G_1 \subset G_2 \subset \cdots$, the branching rule for the AF-algebra $C^*(\varinjlim G_n)$ is completely determined by the inclusions $G_n \subset G_{n+1}$. $\triangle$

3.4. Example. Let S_n, $n \in \mathbb{N}$, be the symmetric group on n-letters, i.e, the group of bijections $\{1, ..., n\} \to \{1, ..., n\}$. We may view $\sigma \in S_n$ as an element of S_{n+1}, by defining $\sigma(n+1) = n+1$. The group S_∞ is by definition the inductive limit of the increasing sequence $S_1 \subset S_2 \subset S_3 \subset \cdots$. By the previous example, the Bratteli diagram for $C^*(S_\infty)$ is given by the system of inclusions $S_1 \subset S_2 \subset S_3 \subset \cdots$. This is well known, and given by the Young lattice, which describes the inclusion of Young diagrams ([Sag01], chapter 2). See figure 3.1.

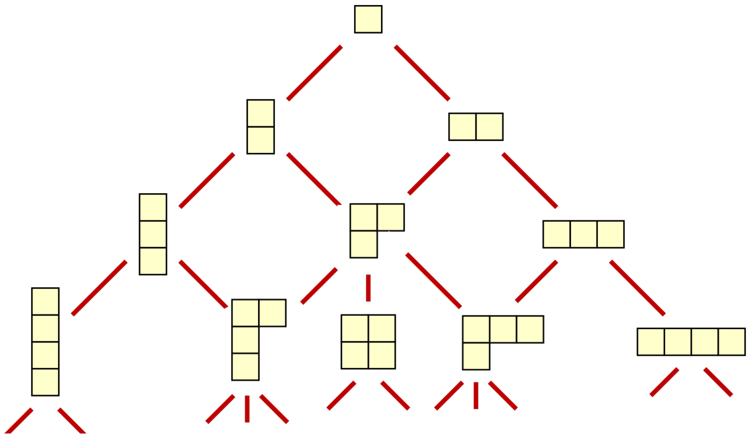

Figure 3.1: The Young lattice, describing the inclusion of Young diagrams.

Thus, the Bratteli diagram of $C^*(S_\infty)$ has the exact same form (see figure 1.1). $\triangle$

In the rest of this section we assume that $A = \overline{\bigcup_n A_n}$ is an AF-algebra. As before we may for each n write

$$A_n = \bigoplus_{\pi \in \mathrm{Irr}(A_n)} B(H_\pi), .$$

If τ is a tracial state on A it is described by the Bratteli diagram of A and the traces χ^π on $B(H_\pi)$. To explain this it will be convenient to change the notation a little. Namely, we order the sets $\mathrm{Irr}(A_n)$, so that we can write

$$A_n = \bigoplus_{i=1}^{m_n} B(H_{(n,i)}).$$

Let $a_{ij}^{(n)}$ denote the multiplicity of $B(H_{(n,i)})$ in $B(H_{(n+1,j)})$. Since each $B(H_{(n,i)})$ has a unique trace $\chi^{(n,i)}$ such that $\chi^{(n,i)}(1) = \dim H_{(n,i)}$ we see that

$$\tau|_{A_n} = \sum_{i=1}^{m_n} \nu_i^{(n)} \chi^{(n,i)}$$

for some numbers $\nu_i^{(n)} \geq 0$. Applying τ to a rank one projection $p \in B(H_{(n,i)})$, we obtain that $\nu_i^{(n)} = \tau(p)$. Moreover

$$\tau|_{A_{n+1}}(p) = \sum_{j=1}^{m_{n+1}} \nu_j^{(n+1)} a_{ij}^{(n)}.$$

Thus the sequence $(\nu^{(n)})_n$ of vectors $\nu^{(n)} = (\nu_1^{(n)}, ..., \nu_{m_n}^{(n)})$ is given by the matrix equations

$$A^{(n)} \nu^{(n+1)} = \nu^{(n)},$$

where $A^{(n)} = (a_{ij}^{(n)})_{i,j}$ is the matrix of multiplicities.

We finish this section by introducing *paths* in the Bratteli diagram of A. If e is an edge from $\pi \in \mathrm{Irr}(A_n)$ to $\rho \in \mathrm{Irr}(A_{n+1})$, let $s(e) = \pi$ (the *range* of e) and $r(e) = \rho$ (the *source* of e).

3.5. Definition. A *path* in the Bratteli diagram of A is a sequence of edges $(e_n)_n$ (the index set might be finite or infinite) such that $r(e_n) = s(e_{n+1})$.

It will be convenient to fix some notation regarding paths.

3.6. Notation. Let $e = (e_n)_n$ be a path in the Bratteli diagram of A.

- $\dim e_n = \dim r(e_n)$: the dimension of the Hilbert space corresponding to the representation $r(e_n) \in \mathrm{Irr}(A_{n+1})$.
- $\chi^{e_n} = \chi^{r(e_n)}$: the trace on $B(H_{r(e_n)})$.
- Suppose $\pi \in \mathrm{Irr}(A_n)$ and $\rho \in \mathrm{Irr}(A_m)$ for some $n, m \in \mathbb{N}$. Then $\dim(\pi, \rho)$ denotes the number of paths starting at π and terminating at ρ. With this convention, we might write $\dim(\pi, e_k) = \dim(\pi, r(e_k))$.

3.2 The main theorem and some motivation

As promised in the introduction we will prove the following theorem:

3.7. Theorem. *Assume that τ is an extremal tracial state on an AF-algebra $A = \overline{\bigcup_n A_n}$, with unital embeddings $A_n \subset A_{n+1}$. Then there exists a path $t = (t_n)_{n\in\mathbb{N}}$ in the Bratteli diagram of A such that*

$$\tau(a) = \lim_{n\to\infty} \frac{\chi^{t_n}(a)}{\dim t_n},$$

for any $a \in A_k$ and $k \geq 1$.

This was originally proven for the special case of $A = C^*(S_\infty)$ [VK81]. The method used has been called "the ergodic method", because they make use of the fact that extremal traces on $C^*(S_\infty)$ corresponds to certain ergodic measures on the set of paths. Moreover, the proof quite easily generalizes to AF-algebras. A central ingredient is the *backwards martingale theorem*.

3.8. Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$ a decreasing sequence of σ-algebras. Write $\mathcal{A}_\infty = \bigcap_n \mathcal{A}_n$. A *backwards martingale* (with respect to $(\mathcal{A}_n)_n$) is a sequence $(\phi_n)_n$ with $\phi_n \in L^1(X, \mathcal{A}_n, \mu)$ such that $\mathbb{E}(\phi_n \mid \mathcal{A}_{n+1}) = \phi_{n+1}$.

Here $\mathbb{E}(\phi_n \mid \mathcal{A}_{n+1})$ is the *conditional expectation of ϕ given $\mathcal{A}_{n+1}$*. Recall that this is by definition the unique (μ-a.e) $\mathcal{A}_{n+1}$-measurable function such that

$$\int_X g\phi_n \, d\mu = \int_X g\mathbb{E}(\phi_n \mid \mathcal{A}_{n+1}) \, d\mu, \tag{3.2.1}$$

for all $g \in L^\infty(X, \mathcal{A}_{n+1}, \mu)$. The uniqness and existence is a consequence of the Radon Nikodym theorem. (For more on this see for instance [MW99], chapter 9).

3.9. Theorem (The backwards martingale theorem). *Assume $(\phi_n)_n$ is a backwards martingale. Then $\phi_n \xrightarrow{n} \mathbb{E}(\phi_1 \,|\mathcal{A}_\infty)$ pointwise μ-a.e.*

Proof. This theorem is probably well known to anyone familiar with stochastic analysis, and we will not prove it here. For a proof, se for instance [Dur96], chapter 4. □

In our proof of theorem 3.7, we will also make use of the backwards martingale theorem. As it turns out "generalized martingales" appear quite naturally when one consider sequences of von Neumann algebras, which is is due to the existence of trace preserving conditional expectations. To explain this, assume we are given a von Neumann algebra M, with a decreasing sequence of von Neumann subalgebras $M \supset M_1 \supset M_2 \supset \cdots$. Suppose τ is a normal faithful tracial state on M, and denote by E_n the τ-preserving conditional expectations $E_n : M \to M_n$. Fix $a \in M$. Thinking of M as a "noncommutative measure space" with integration τ the condition 3.2.1 translates to

$$\tau(bE_n(a)) = \tau(bE_{n+1}(a))$$

for all $b \in M_{n+1}$. By looking at the definition of trace preserving conditional expectations it is easily verified that both sides of this equation is equal to $\tau(ba)$, so that it makes sense to call $E_n(a)$ a martingale. A natural question then, is whether the sequence $(E_n(a))_n$ converges towards $E_\infty(a)$, where E_∞ denotes the conditional expectation $E_\infty : M \to \bigcap_n M_n$. The following theorem shows that there is in fact such a convergence.

3.10. Theorem. *For all $x \in M$, the sequence $(E_n(x))_n$ converges towards $E_\infty(x)$ in the strong operator topology.*

Proof. Fix $x \in M$. First we show that $E_n(x) \to E_\infty(x)$ in the weak operator topology. Since the unit ball in M is weak operator compact, and the sequence is bounded, it suffices to show that $(E_n(x))_n$ has a unique cluster point in the weak operator topology (and that this is $E_\infty(x)$). For any $z \in \bigcap_n M_n$, we observe that

$$\tau(zx) = \tau(E_n(zx)) = \tau(zE_n(x)),$$

for all $n \in \mathbb{N}$. Thus, if y is a cluser point for $(E_n(x))_n$, we can for any $\varepsilon > 0$ find $n \in \mathbb{N}$ such that

$$|\tau(zx) - \tau(yx)| = |\tau(zE_n(x)) - \tau(zy)| < \varepsilon.$$

We conclude that $\tau(zE_\infty(x)) = \tau(zx) = \tau(zy)$ for any $z \in \bigcap_n M_n$. Clearly $y \in \cap_n M_n$, so we can choose $z = (E_\infty(x) - y)^* \in \bigcap_n M_n$. But this implies that

$$\tau((E_\infty(x) - y)^*(E_\infty(x) - y)) = 0,$$

whence $E_\infty(x) = y$ by the faithfulness of τ.

Under the realization of M as a subalgebra of $B(H)$, the tracial state τ is given by $\tau(x) = (x\xi \mid \xi)$ where $\xi \in H$ is a cyclic vector for M. For each $n \in \mathbb{N}$ let $H_n = \overline{M_n\xi}$ and $p_n : H \to H_n$ be the projection. It is not hard to check that $E_n(x)\xi = p_n x\xi$, for all $n \in \mathbb{N}$. Moreover we have that

$$E_n(x)\xi = p_n x\xi \to px\xi$$

where p is the projection onto $\bigcap_n H_n$. Since $E_n(x) \to E_\infty(x)$ in the weak operator topology, we deduce that $E_n(x)\xi \to E_\infty(x)\xi$ weakly. Thus $px\xi = E_\infty(x)\xi$.

Now, the faithfulness of τ is equivalent to ξ being cyclic for M'. Thus, if $\zeta \in H$ and $\varepsilon > 0$, we may find $y \in M'$ such that $\|y\xi - \zeta\| < \varepsilon/3\|x\|$. Moreover, by applying y to both sides of the limit above, we can choose n_0 such that $\|E_n(x)y\xi - E_\infty(x)y\xi\| < \varepsilon/3$ for $n \geq n_o$. We obtain that

$$\begin{aligned} \|E_n(x)\zeta - E_\infty(x)\zeta\| &\leq \|E_n(x)\zeta - E_n(x)y\xi\| + \|E_n(x)y\xi - E_\infty(x)y\xi\| \\ &\quad + \|E_\infty(x)y\xi - E_\infty(x)\zeta\|. \\ &\leq 2\|x\|\|\zeta - y\xi\| + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

where we used the fact that $\|E_n(x)\| \leq \|x\|$. □

Let $A = \overline{\bigcup_n A_n}$ be an AF-algebra, with extremal tracial state τ. The proof of theorem 3.7 will be inspired by the following observation: Consider $M = \pi_\tau(A)''$, and let $M_n = \pi_\tau(A_n)' \cap M$. The induced trace on M (still denoted by τ) is then normal and faithful, and we are therefore in the same situation as in the proposition above. Moreover $\bigcap_n M_n = Z(M) = \mathbb{C}1$ (by theorem 2.7), and it is easily seen that $E_\infty(x) = \tau(x)1$ for all $x \in M$. Thus, by the propositon above $E_n(x)$ converges towards $\tau(x)1$ in the strong operator topology. This gives a hint that the sequence $(E_n(x))_n$ is worth investigating.

3.3 Proof of the main theorem

For the rest of this section we fix an AF-algebra $A = \overline{\bigcup_n A_n}$ with a tracial state τ. We still assume that all embeddings $A_n \subset A_{n+1}$ are unital. Since the induced trace on $M = \pi_\tau(A)''$ (still denoted by τ) is normal and faithful, we have for each $n \in \mathbb{N}$ a unique τ-preserving conditional expectation

$$E_n : M \to \pi_\tau(A_n)' \cap M = M_n,$$

and a $\tau|_{\pi_\tau(A_n)}$-preserving center valued conditional expectation

$$F_n : \pi_\tau(A_n) \to Z(\pi_\tau(A_n)).$$

The following observation will be important.

3.11. Proposition. *Fix $n_0 \in \mathbb{N}$ and an element $a \in A_{n_0}$. Then $E_n(\pi_\tau(a)) = F_n(\pi_\tau(a))$ for all $n \geq n_0$.*

Proof. For simplicity we will in this proof write A_n for $\pi_\tau(A_n)$, and a for $\pi_\tau(a)$. For $x \in M$ consider the operator

$$I_n(x) = \int_{U(A_n)} uxu^* \, du.$$

Here $U(A_n)$ denotes the compact group of unitaries in A_n, and du the corresponing (unimodular) Haar measure. Let $v \in U(A_n)$. Then

$$vI_n(x)v^* = \int_{U(A_n)} vuxu^*v^* \, du = \int_{U(A_n)} uxu^* \, du = I_n(x)$$

since the Haar measure is left and right invariant. Since the unitaries in A_n generate A_n, we conclude that $I_n(x) \in A_n' \cap M = M_n$. Similarily, $I_n(a) \in Z(A_n)$ when $a \in A_n$. It is easy to check that the maps $I_n : M \to M_n$ and $I_n|_{A_n} : A_n \to Z(A_n)$ are trace preserving conditional expectations. By uniqueness $E_n(a) = I_n(a) = F_n(a)$, when $a \in A_{n_0}$, and $n \geq n_0$. □

3.12. Proposition. *With assumptions as above*

$$E_n(\pi_\tau(a)) = \sum_{\pi \in \mathsf{Irr}(A_n)} \frac{\chi^\pi(a)}{\dim \pi} 1_\pi \,,$$

where 1_π is the image of the identity in $B(H_\pi)$ under π_τ.

Proof. Writing A_n on the form

$$A_n = \bigoplus_{\pi \in \mathsf{Irr}(A_n)} B(H_\pi)$$

we obtain that $\tau|_{A_n} = \sum_{\pi \in \mathsf{Irr}(A_n)} \alpha_\pi \chi^\pi$, as explained in the previous section. Moreover

$$E_n(\pi_\tau(a)) \in Z(\pi_\tau(A_n)) = \bigoplus_{\pi \in \mathsf{Irr}(A_n)} \mathbb{C}1_\pi,$$

where $1_\pi \in B(H_\pi)$ is the identity (thought of as an element in M under the representation π_τ). It follows that there are complex numbers $\lambda_\pi(a)$ such that

$$E_n(\pi_\tau(a)) = \sum_{\pi \in \mathsf{Irr}(A_n)} \lambda_\pi(a) 1_\pi,$$

Now, since E_n is τ-preserving we easily see that $\lambda_\pi(a) = \chi^\pi(a)/\dim \pi$. In conclusion

$$E_n(\pi_\tau(a)) = \sum_{\pi \in \mathsf{Irr}(A_n)} \frac{\chi^\pi(a)}{\dim \pi} 1_\pi.$$

□

For simplicity we will abuse notation a little and henceforth write $E_n(a)$ instead $E_n(\pi_\tau(a))$. In view of the proposition above, $E_n(a)$ is an element in the von Neumann algebra $Z = W^*(Z(\pi_\tau(A_k)) \mid k \geq 1)$. Moreover

3.13. Proposition. *There is a measure μ_τ on $\Omega = \prod_k \mathsf{Irr}(A_k)$ such that $Z \cong L^\infty(\Omega, \mu_\tau)$.*

Proof. For each $n \in \mathbb{N}$, consider the product space

$$\Omega_n = \mathsf{Irr}(A_1) \times \mathsf{Irr}(A_2) \times \cdots \times \mathsf{Irr}(A_n)$$

as a discrete measurable space. Then we can define a probabillity measure μ_n on Ω_n by

$$\mu_n(\{\pi_1, \pi_2, ..., \pi_n\}) = \tau(1_{\pi_1} 1_{\pi_2} \cdots 1_{\pi_n}).$$

We use the Kolmogorov consistency theorem to define a measure on Ω with the product σ-algebra (see for instance [Dur96], appendix A). Thus we need to check that if $P_n : \Omega_{n+1} \to \Omega_n$ is the canonical projection, then $\mu_n = (P_n)_* \mu_{n+1}$. But this is clearly true, since

$$\begin{aligned} \mu_{n+1}(P_n^{-1}(\{\pi_1, \pi_2, ..., \pi_n\})) &= \mu_{n+1}\Big(\bigcup_{\pi \in \mathsf{Irr}(A_{n+1})} \{\pi_1, \pi_2, ..., \pi_n, \pi\} \Big) \\ &= \sum_{\pi \in \mathsf{Irr}(A_{n+1})} \tau(1_{\pi_1} 1_{\pi_2} \cdots 1_{\pi_n} 1_\pi) \\ &= \tau(1_{\pi_1} 1_{\pi_2} \cdots 1_{\pi_n}) = \mu_n(\{\pi_1, \pi_2, ..., \pi_n\}), \end{aligned}$$

where we used that $\sum_{\pi \in \mathsf{Irr}(A_{n+1})} 1_\pi$ is the identity in A_{n+1}. It follows from the Kolmogorov consistency theorem that the family $\{\mu_n\}_n$ defines a measure μ_τ on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ denotes the product σ-algebra. We set $L^\infty(\Omega, \mu_\tau) = L^\infty(\Omega, \mathcal{F}, \mu_\tau)$, where $\mathcal{F}$ is the completion of $\mathcal{B}(\Omega)$ with respect to μ_τ.

Now, let $Z_n = W^*(Z(\pi_\tau(A_k)) \,|\, 1 \leq k \leq n)$. Then, by the definition of the measures μ_n we have a unitary $U_n : L^2(\Omega_n, \mu_n) \to \overline{Z_n \xi_\tau}$ induced by the map

$$\mathbb{1}_{\{\pi_1, ..., \pi_n\}} \mapsto 1_{\pi_1} \cdots 1_{\pi_n} \xi_\tau,$$

where $\mathbb{1}_{\{\pi_1, ..., \pi_n\}}$ denotes the characteristic function on $\{\pi_1, ..., \pi_n\}$. $L^\infty(\Omega_n, \mu_n)$ is generated by the elements $\mathbb{1}_{\{\pi_1, ..., \pi_n\}}$, and Z_n is generated by the elemens $1_{\pi_1} \cdots 1_{\pi_n}$. Thus

$$U_n L^\infty(\Omega_n, \mu_n) U_n^* = Z_n|_{\overline{Z_n \xi_\tau}} = \{x|_{\overline{Z_n \xi_\tau}} : x \in Z_n\}.$$

Letting $n \to \infty$ we get a unitary $L^2(\Omega, \mu_\tau) \to \overline{Z\xi_\tau}$, such that

$$U L^\infty(\Omega_n, \mu_n) U^* = Z_n|_{\overline{Z\xi_\tau}}.$$

Since the algebras $L^\infty(\Omega_n, \mu_n)$ generate $L^\infty(\Omega, \mu_\tau)$, and the algebras Z_n generate Z, passing to the limit yields the wanted isomorphism. □

Remark. The product σ-algebra on Ω coincides with the Borel σ-algebra generated by the product topology on Ω. This explains the notation $\mathcal{B}(\Omega)$ in the proof above. Concretely $\mathcal{B}(\Omega)$ is generated by the sets $\prod_k X_k$, where $X_k = \mathsf{Irr}(A_k)$ for all but finitely many k. △

Using this proposition we can now realize $E_n(a)$ as a function on Ω. Fix $\pi \in \mathsf{Irr}(A_n)$ and define $C(\pi) = \{(x_k)_k \in \Omega \,|\, x_n = \pi\}$. Then

3.14. Corollary. *Under the isomorphism $Z \cong L^\infty(\Omega, \mu_\tau)$ we have*

$$E_n(a) = \sum_{\pi \in \mathsf{Irr}(A_n)} \frac{\chi^\pi(a)}{\dim \pi} \mathbb{1}_{C(\pi)},$$

when $a \in A_{n_0}$ and $n \geq n_0$.

Proof. By the definition of the isomorphism $Z \cong L^\infty(\Omega, \mu_\tau)$, we see that 1_π is sent to the characteristic function $\mathbb{1}_{C(\pi)}$. Thus, by 3.12 we obtain the claim. □

Define for each $n \in \mathbb{N}$ the von Neumann algebra $Z_{n,\infty}$ generated by the $Z(\pi_\tau(A_n))$ for $k \geq n$:

$$Z_{n.\infty} = W^*(Z(\pi_\tau(A_k)) \,|\, k \geq n).$$

Note that $E_n(a) \in Z(\pi_\tau(A_n)) \subset Z_{n,\infty}$ and that $Z_{n,\infty} \subset Z_{m,\infty}$ for $n > m$. Because of this we see that $(E_n(a))_n$ is a "generalized martingale" as explained in the previous section. In fact, even more is true:

3.15. Proposition. *When viewed as functions in $L^\infty(\Omega, \mu_\tau)$ the sequence $(E_n(a))_n$ is a backwards martingale. Hence $E_n(a)$ converges towards $E_\infty(a)$ pointwise μ_τ-a.e.*

Remark. Recall that E_∞ denotes the τ-preserving conditional expectation $E_\infty : M \to \bigcap_n M_n = Z(M)$. △

Proof. Write $\Omega_{n,\infty} = \prod_{n \leq k} \mathsf{Irr}(A_k)$, and let $P_{n,\infty} : \Omega \to \Omega_{n,\infty}$ be the canonical projection. By construction $Z_{n,\infty} \cong L^\infty(\Omega_{n,\infty}, \mathcal{F}_{n,\infty}, \mu_{n,\infty})$, where $\mu_{n,\infty}$ is the measure induced by the family $\{\mu_k\}_{n \leq k}$ as in the proof of proposition 3.13. Moreover, for any $X \in \mathcal{B}(\Omega_{n,\infty})$

$$\mu_\tau\Big(\big(\prod_{k=1}^{n-1} \mathsf{Irr}(A_k) \big) \times X \Big) = ((P_{n,\infty})_* \mu_\tau)(X) = \mu_{n,\infty}(X).$$

Thus, we may identify

$$Z_{n,\infty} \cong L^\infty(\Omega_{n,\infty}, \mathcal{F}_{n,\infty}, \mu_{n,\infty}) \cong L^\infty(\Omega, \mathcal{F}_n, \mu_\tau),$$

where $\mathcal{F}_n$ is the completion of the σ-algebra

$$\Big\{ \big(\prod_{k=1}^{n-1} \mathsf{Irr}(A_k) \big) \times X \;\Big|\; X \in \mathcal{B}(\Omega_{n,\infty}) \Big\},$$

with respect to μ_τ.

In particular the functions $(E_n(a))_n$ are measurable with respect to the decreasing sequence of σ-algebras $\mathcal{F} \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$. Since the integral on Ω is by construction given by τ, we conclude that $(E_n(a))_n$ is a backwards martingale (see section 3.2). By the backwards martingale theorem $E_n(a)$ converges towards $E_\infty(a)$, pointwise μ_τ-a.e. □

Now we are finally ready to prove theorem 3.7, which we restate here.

3.16. Theorem. *Assume that τ is an extremal tracial state on A. Then there exists a path $t = (t_n)_n$ in the Bratteli diagram of A such that*

$$\tau(a) = \lim_{n\to\infty} \frac{\chi^{t_n}(a)}{\dim t_n}$$

for any $a \in A_{n_0}$ and $n_0 \geq 1$.

Proof. By definition a path $t = (t_n)_n$ in the Bratteli diagram of A is a sequence of edges t_n in the Bratteli diagram. Thus, the range $r(t_n)$ sits inside $\mathsf{Irr}(A_n)$, for each $n \in \mathbb{N}$. Therefore it makes sense to consider the point $(r(t_k))_{k\in\mathbb{N}} \in \prod_k \mathsf{Irr}(A_k)$, and applying $E_n(a)$ to this point we clearly get

$$E_n(a)(r(t_k))_k = \frac{\chi^{t_n}(a)}{\dim t_n},$$

by corollary 3.14. By the previous proposition we conclude that

$$\lim_{n\to\infty} \frac{\chi^{t_n}(a)}{\dim t_n} = E_\infty(a)$$

for some choice of path $t = (t_n)_n$. Assuming now that τ is extremal we obtain

$$E_\infty(a) \in Z(\pi_\tau(A)'') = \mathbb{C}1,$$

by theorem 2.7. Thus $E_\infty(a) = \tau(a)1$ since the conditional expectations are trace-preserving. Hence we in fact have that

$$\lim_{n\to\infty} \frac{\chi^{t_n}(a)}{\dim t_n} = \tau(a)1.$$

$\square$

3.17. Corollary. *The coefficents ν_π appearing in the decomposition*

$$\tau|_{A_n} = \sum_{\pi\in\mathsf{Irr}(A_n)} \nu_\pi \chi^\pi$$

are given by

$$\nu_\pi = \lim_{n\to\infty} \frac{\dim(\pi, t_n)}{\dim t_n}.$$

Proof. Suppose $\rho \in \mathsf{Irr}(A_k)$, for some $k > n$ and that e is a rank one projection in $B(H_\pi)$, for $\pi \in \mathsf{Irr}(A_n)$. Then it is not hard to see that $\chi^\rho(e) = \dim(\pi, \rho)$. Recalling that $\nu_\pi = \tau(e)$ we obtain the wanted identity. $\square$

3.4 Generalization to fat AF-algebras

We will now generalize theorem 3.7 to what we will call *fat AF-algebras*. This is based on the construction in [EI16]. Recall that if A is a C^*-algebra, we denote by $M(A)$ its multiplier algebra. Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of C^*-algebras, with

$A_n \subset M(A_{n+1})$, and assume that $A_n A_{n+1}$ is dense in A_{n+1}. We will also assume that all A_n's are on the form

$$A_n = c_0 - \bigoplus_{\alpha \in I_n} B(H_\alpha),$$

where I_n is a countable index set, and $\dim H_\alpha < \infty$ for each $\alpha \in I_n$. Note that $M(A_n) = \ell^\infty - \bigoplus_{\alpha \in I_n} B(H_\alpha)$. Then

$$M(A_1) \subset M(A_2) \subset M(A_3) \subset \cdots,$$

and moreover $A_k \subset M(A_n)$ whenever $n \geq k$. Thus we can define a C^*-subalgebra of $M(A_n)$ by

$$\tilde{A}_n = \sum_{k=1}^{n} A_k \subset M(A_n).$$

Clearly $\tilde{A}_n \subset \tilde{A}_{n+1}$, and we can define the C^*-algebra $\tilde{A} = \varinjlim \tilde{A}_n$. We call $\tilde{A}$ the *fat AF-algebra associated to the family* $\{A_n\}_{n \in \mathbb{N}}$. Since all the Hilbert spaces H_α, $\alpha \in I_n$ are finite dimensional for any n, there is no problem in defining Bratteli diagrams for fat AF-algebras in the same way as for AF-algebras. We may therefore ask whether 3.7 holds for these C^*-algebras as well. To adopt the notation of the previous section we write χ^α for the trace on $B(H_\alpha)$, 1_α for the identity in $B(H_\alpha)$ and $\dim \alpha = \dim H_\alpha$.

Let τ be a tracial state on $\tilde{A}$. To generalize theorem 3.7, we need to add an assumption on the trace. Namely, we assume that $\tau_n = \tau|_{A_n}$ is a state for each $n \in \mathbb{N}$. We denote by $\hat{\tau}_n$ the canonical extension of τ_n to $M(A_n)$. That is if $x \in M(A_n)$, and $(e_i)_i$ is an approximate unit in A_n, then $\hat{\tau}_n(x) = \lim_i \tau_n(xe_i)$.

3.18. Lemma. *With assumpitons as above $\tau_n = \hat{\tau}_{n+1}|_{A_n}$ for all $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$, write B for the unitization of $\tilde{A}_{n+1} \subset M(A_{n+1})$ and consider the canonical extension ϕ of $\tau|_{\tilde{A}_{n+1}}$ to B. By our assumptions this is a state, and it clearly extends τ_{n+1}. Now, if $(e_i)_i$ is an approximate unit in A_{n+1} and $a \in B$ we have

$$\begin{aligned} |\phi(a) - \phi(ae_i)| &= |\phi(a(1-e_i)| \\ &\leq \phi(aa^*)^{1/2}\phi((1-e_i)^2)^{1/2} \\ &\leq \phi(aa^*)^{1/2}\phi(1-e_i)^{1/2} \to 0 \end{aligned}$$

by the Cauchy-Shcwartz inequality. Thus

$$\phi(a) = \lim_i \phi(ae_i) = \lim_i \tau_{n+1}(ae_i) = \hat{\tau}_{n+1}(a),$$

whence $\phi = \hat{\tau}|_B$. But then $\tau_n = \phi|_{A_n} = \hat{\tau}_{n+1}|_{A_n}$. □

Let $Z = W^*(Z(\pi_\tau(A_n))\,|n \geq 1)$. Then, as in the previous case

3.19. Proposition. *The measures on $\Omega_n = I_1 \times I_2 \times \cdots \times I_n$ defined by*

$$\mu_n(\{\alpha_1, \alpha_2, ..., \alpha_n\}) = \tau(1_{\alpha_1} 1_{\pi_2} \cdots 1_{\alpha_n})$$

define a measure μ_τ on $\Omega = \prod_k I_k$, and $Z \cong L^\infty(\Omega, \mu_\tau)$.

Proof. The difference from the proof of 3.13 is that it is now less obvious that the measures are compatible. However, our assumption on τ fixes the issue. First we note that the elements $\sum_{\alpha \in F} 1_\alpha \in A_{n+1}$ for finite subsets $F \subset I_{n+1}$ gives an approximate unit in A_{n+1}. Therefore, we can write $\tilde{\tau}_{n+1}(x) = \sum_{\alpha \in I_{n+1}} \tau_{n+1}(x 1_\alpha)$ for any $x \in M(A_{n+1})$. It follows that

$$\begin{aligned}(P_n)_* \mu_{n+1}(\{\alpha_1, ..., \alpha_n\}) &= \sum_{\alpha \in I_{n+1}} \mu_{n+1}(\{\alpha_1, ..., \alpha_n, \alpha\}) \\ &= \sum_{\alpha \in I_{n+1}} \tau(1_{\alpha_1} \cdots 1_{\alpha_n} 1_\alpha) \\ &= \hat{\tau}_{n+1}(1_{\alpha_1} \cdots 1_{\alpha_n}),\end{aligned}$$

By the lemma above we conclude

$$\begin{aligned}(\pi_n)_* \mu_{n+1}(\{\alpha_1, ..., \alpha_n\}) &= \hat{\tau}_{n+1}(1_{\alpha_1} \cdots 1_{\alpha_n}) \\ &= \tau(1_{\alpha_1} \cdots 1_{\alpha_n}) = \mu_n(\{\alpha_1, ..., \alpha_n\}).\end{aligned}$$

Hence we can apply the Kolmogorov consistency theorem also in this case, and the rest of the proof is similar to the proof of 3.13. □

Now, we consider the trace-preserving conditional expectation $E_n : M \to \pi_\tau(A_n)' \cap M$ and the center valued conditional expectation $F_n : \pi_\tau(A_n)'' \to Z(\pi_\tau(A_n)'')$. With a completely anologous proof as for 3.11, we get that

3.20. Proposition. *If $a \in A_{n_0}$ we have that $E_n(\pi_\tau(a)) = F_n(\pi_\tau(a))$ for all $n \geq n_0$. This implies that*

$$E_n(\pi_\tau(a)) = \sum_{\alpha \in I_n} \frac{\chi^\alpha(a)}{\dim \alpha} \mathbb{1}_{C(\alpha)} \in L^\infty(\Omega, \mu_\tau),$$

where $C(\alpha) = \{(x_k)_k \in \Omega \mid x_n = \alpha\}$.

With the same reasoning as in the previous section the sequence $(E_n(a))_n$ is a backwards martingale, and we obtain the generalized version of 3.7.

3.21. Theorem. *Let τ be an extremal tracial state on $\tilde{A}$, such that $\tau|_{A_n}$ is a state for each $n \in \mathbb{N}$. Then there is a path $t = (t_n)_n$ in the Bratteli diagram of $\tilde{A}$ such that*

$$\tau(a) = \lim_{n \to \infty} \frac{\chi^{t_n}(a)}{\dim t_n}$$

for any $a \in A_k$ and $k \geq 1$.

4. Compact quantum groups

4.1 Definition and two examples

4.1. Definition. A compact quantum group is a pair (A, Δ), where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism, such that the following conditions hold

(i) $(\iota \otimes \Delta) \circ \Delta = (\Delta \otimes \iota) \circ \Delta$

(ii) the spaces $(A \otimes 1)\Delta(A)$ and $(1 \otimes A)\Delta(A)$ are dense in $A \otimes A$.

Remark. Here ι denotes the identity map $A \to A$, and

$$(A \otimes 1)\Delta(A) = \mathsf{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\}.$$

$(1 \otimes A)\Delta(A)$ is defined similarly. △

The following example will serve as motivation for definition 4.1.

4.2. Example. Let G be a compact topological group, with multiplication map $\mu : G \times G \to G$. Then the space $C(G)$ of continuous functions on G is a unital commutative C^*-algebra. Let $\Delta : C(G) \to C(G \times G) = C(G) \otimes C(G)$ be the unital $*$-homomorphism dual to μ. That is, $\Delta(f) = f \circ \mu$. The associativity of the group law can be encoded in the following commutative diagram.

$$\begin{tikzcd} & G \times G \times G \arrow[dl, "\mu \times \iota"'] \arrow[dr, "\iota \times \mu"] & \\ G \times G \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd}$$

Dualizing this diagram we get

$$\begin{tikzcd}
 & C(G) \otimes C(G) \otimes C(G) & \\
C(G) \otimes C(G) \arrow[ur, "\iota\otimes\Delta"] & & C(G) \otimes C(G) \arrow[ul, "\Delta\otimes\iota"'] \\
 & C(G) \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

which is exactly the first condition in 4.1. To show the second condition we note that $(C(G) \otimes 1)\Delta(C(G))$ is a unital $*$-algebra spanned by functions of the form $(x, y) \mapsto f_1(x)f_2(xy)$, where $f_1, f_2 \in C(G)$. Thus, by the Stone Weierstrass theorem it suffices to show that these functions separate points of $G \times G$, and this follows from the cancellation property of the group law. For instance, if $y \neq y'$ are points in G, then $xy \neq xy'$ for any $x \in G$. Thus, we can find $f \in C(G)$ such that $f(xy) \neq f(xy')$. Clearly then $(1 \otimes 1)\Delta(f_1)(x, y) \neq (1 \otimes 1)\Delta(f_1)(x, y')$. The other cases are similar. We conclude that $(C(G) \otimes 1)\Delta(C(G))$ is dense in $C(G) \otimes C(G)$, and by similar argumentation that $(1 \otimes C(G))\Delta(C(G))$ is dense as well. $\triangle$

Remark. If (A, Δ) is a compact quantum group with commutative A, then one can show the spectrum of A is in fact a (compact) group (see [NT13]). Thus (A, Δ) is of the form in the example above. For any compact quantum group $G = (A, \Delta)$ we will write $C(G)$ for A. Although $C(G)$ is not in general a space of functions on some space G, it is convenient to think of it this way. Indeed, many of the notions that appear in the setting of compact quantum groups are more or less direct generalizations of facts true for genuine function algebras. $\triangle$

The following proposition gives rise to many examples of compact quantum groups.

4.3. Proposition. *Assume A is a unital C^*-algebra generated by elements u_{ij}, $1 \leq i, j \leq n$, such that the matrices $(u_{ij})_{i,j}$ and $(u_{ij}^*)_{i,j}$ are invertible in $M_n(A)$. Assume that $\Delta : A \to A \otimes A$ is a unital $*$-homomorphism such that*

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Then (A, Δ) is a compact quantum group.

Proof. See proposition 1.1.4. in [NT13]. $\square$

4.4. Example. Let $C(S_n^+)$ be the universal enveloping C^*-algebra of generators $(u_{ij})_{i,j}$, $1 \leq i, j \leq n$, such that

- The matrix $U = (u_{ij})_{i,j}$ is unitary,
- $u_{ij} = u_{ij}^* = u_{ij}^2$
- $\sum_i u_{ij} = 1$ for all i and $\sum_j u_{ij} = 1$ for all j.

It is not apriori clear if $C(S_n^+)$ is well defined as a C^*-algebra since the norm may not make sense. However, since U is unitary this definition is fine. Indeed, let (π, H_π) be a $*$-representation of the $*$-algebra generated by the u_{ij}. Then $(\pi(u_{ij}))_{i,j}$ is a unitary in $\mathsf{M}_n(B(H_\pi))$, so that $\|\pi(u_{ij})\| \leq 1$. Since this holds for any $*$-representation π, the norm on $C(S_n^+)$ is well defined.

Now, we simply define a unital $*$-homomorphism

$$\Delta : C(S_n^+) \to C(S_n^+) \otimes C(S_n^+)$$

by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Then, by 4.3, the pair $(C(S_n^+), \Delta)$ is a compact quantum group. For $n = 1, 2, 3$ one can show that $C(S_n^+) \cong C(S_n)$, while for $n \geq 4$ the C^*-algebra $C(S_n^+)$ is infinite dimensional and noncommutative ([BBC07], theorem 6.2). △

We end this section with the notion of quantum subgroup.

4.5. Definition. A quantum subgroup of a compact quantum group G, is a compact quantum group H together with a surjective $*$-homomorphism $\theta : C(G) \to C(H)$, such that $(\theta \otimes \theta) \circ \Delta_G = \Delta_H \circ \theta$.

4.6. Example. S_n^+ is a quantum subgroup of S_{n+1}^+. Indeed, let $U^{(n+1)} = (u_{ij}^{(n+1)})_{i,j}$ and $U^{(n)} = (u_{ij}^{(n)})_{i,j}$ be the matrices of generators for $C(S_{n+1}^+)$ and $C(S_n^+)$ respectivly. Let θ be the map defined $\theta(u_{ij}^{(n+1)}) = v_{ij}$, where $V = (v_{ij})_{i,j}$ is the matrix

$$V = \begin{bmatrix} U^{(n)} & 0 \\ 0 & 1_{S_n^+} \end{bmatrix}$$

It is easily checked that θ is a surjective $*$-homomorphism satisfying $(\theta \otimes \theta) \circ \Delta = \Delta \circ \theta$. △

4.2 The Haar state

Let G be a compact group with (unimodular) Haar measure ν. Then integration against the Haar-measure defines a linear functional ℓ_ν on $C(G)$. Applying $\iota \otimes \ell_\nu$ to $\Delta(f)$ for some $f \in C(G)$, and using the left-invariance of ν we obtain

$$(\iota \otimes \ell_\nu)\Delta(f)(t) = \int_G \Delta(f)(t,s)\, d\nu(s) = \int_G f(ts)\, d\nu(s) = \int_G f(s)\, d\nu(s) = \ell_\nu(f)$$

for any $t \in G$. Similarily $(\ell_\nu \otimes \iota)\Delta(f) = \ell_\nu(f)1$. As it turns out every compact quantum group admits such a functional ℓ_ν. To be precise:

4.7. Theorem. *Every compact quantum group (A, Δ) has a unique state h such that*

$$(\iota \otimes h)\Delta(a) = h(a)1 = (h \otimes \iota)\Delta(a)$$

for all $a \in A$.

Proof. Se theorem 1.2.1. in [NT13]. □

4.3 Representation theory

Much of the representation theory of compact quantum groups is analogous to the classical Peter Weyl theory. For instance any unitary irreducible representation is finite dimensional. As we mostly care about irreducible representations in this text, we will for this reason not even introduce infinite dimensional representations.

4.8. Definition. A representation of a compact quantum group G on a finite dimensional Hilbert space H_U is an invertible element $U \in B(H_U) \otimes C(G)$ such that

$$(\iota \otimes \Delta)(U) = U_{12}U_{13}.$$

Remark. The notation U_{12} means that U is embedded into the first two factors of $B(H_U) \otimes C(G) \otimes C(G)$. U_{13} is defined similarly. To illustrate this, consider an elementary tensor $T \otimes a \in B(H) \otimes C(G)$. Then $(T \otimes a)_{12} = T \otimes a \otimes 1$ and $(T \otimes a)_{13} = T \otimes 1 \otimes a$. △

If the element U is unitary, we say that U is a *unitary representation*.

4.9. Example. Consider the quantum permutation group S_n^+, and let $U = (u_{ij})_{i,j}$ be the matrix in $\mathsf{M}_n(C(S_n^+))$ consisting of the generators for $C(S_n^+)$. Take H to be any n-dimensional Hilbert space. Fix a basis $(e_i)_i$ for H, and denote by m_{ij} the operator defined by $m_{ij}e_k = \delta_{jk}e_i$. Then we can identify $\mathsf{M}_n(C(S_n^+))$ with $B(H) \otimes C(S_n^+)$, and U becomes

$$U = \sum_{i,j} m_{ij} \otimes u_{ij} \in B(H_U) \otimes C(S_n^+).$$

By the definition of Δ we obtain

$$(\iota \otimes \Delta)(U) = \sum_{i,j} m_{ij} \otimes \sum_k u_{ik} \otimes u_{kj} = U_{12}U_{13}.$$

We conclude that U is a unitary representation of S_n^+, and call it the *fundamental representation* of S_n^+. △

Remark. The operators m_{ij} in the example above are called the *matrix units* corresponding to the basis $(e_i)_i$. △

Fix two representations $U \in B(H_U) \otimes C(G)$ and $V \in B(H_V) \otimes C(G)$. The *direct sum* $U \oplus V$ is simply the obvious element in $B(H_U \oplus H_V) \otimes C(G)$, and the *tensor product* $U \otimes V$ is the representation $U_{13}V_{23}$ in $B(H_U \otimes H_V) \otimes C(G)$. An *intertwiner* between representations U and V is a linear operator $T : H_U \to H_V$ such that

$$(T \otimes 1)U = V(T \otimes 1).$$

The space of interwtiners between U and V is denoted by $\mathsf{Mor}(U, V)$, and $\mathsf{End}(U) = \mathsf{Mor}(U, U)$. If $\mathsf{End}(U) = \mathbb{C}$ we say that U is *irreducible*. U and V are said to be *equivalent* (resp. *unitarily equivalent*) if there is an invertible (resp. unitary) element in $\mathsf{Mor}(U, V)$. We denote by $\mathsf{Irr}(G)$ the set equivalence classes of (finite dimensional) unitary irreducible representations. Often we will write U_α for a representative of $\alpha \in \mathsf{Irr}(G)$, and fix a family $(U_\alpha)_{\alpha\in\mathsf{Irr}(G)}$ of such representatives.

As in classical representation theory for compact groups any finite dimensional representation is equivalent to a unitary representation and can be written as a direct sum of irreducible representations [NT13]. More or less as a direct consequence of the definitions above, we obtain a version of Schur's lemma for compact quantum groups.

4.10. Theorem (Schur's lemma). *Assume that U and V are unitary irreducible representations. Then $\mathsf{Mor}(U, V) = 0$, or U and V are unitarlity equivalent and $\dim \mathsf{Mor}(U, V) = 1$.*

If H is a finite dimensional Hilbert space, let $j : B(H) \to B(H^*)$ be the map sending an operator to its dual.

4.11. Definition. Given a finite dimensional representation U, the contragredient representation U^c is defined by

$$U^c = (j \otimes \iota)(U^{-1}) \in B(H^*) \otimes C(G).$$

With some work one can show that U^c is in fact an invertible element, so that it is a well defined representation (see for instance [NT13]). As one might expect, the original representation U is equivalent to $U^{cc} = (U^c)^c$, and moreover U^c is irreducible if and only if U is. It follows from Schur's lemma that $\mathsf{Mor}(U, U^{cc})$ is spanned by a single invertible operator.

4.12. Definition. Let U be an irreducible unitary representation. Then the quantum dimension of U is the number

$$\dim_q U = \mathsf{Tr}(\rho),$$

where ρ is the unique positive element in $\mathsf{Mor}(U, U^{cc})$ such that $\mathsf{Tr}(\rho) = \mathsf{Tr}(\rho^{-1})$.

Recall that if H is a Hilbert space and $\xi, \zeta \in H$, we denote by $\omega_{\xi,\zeta} \in B(H_U)^*$ the functional $T \mapsto (T\xi \mid \zeta)$. If $U \in B(H_U) \otimes C(G)$ is a representation, a *matrix coefficent* for U is an element of the form $(\omega_{\xi,\zeta} \otimes \iota)(U)$ for some pair $\xi, \zeta \in H_U$. Fix a basis $(e_i)_i$ for H_U, and let m_{ij} be the corresponding matrix units. Write U in matrix form as

$$U = \sum_{i,j} m_{ij} \otimes u_{ij}.$$

Then we see that $u_{ij} = (\omega_{e_i,e_j} \otimes \iota)(U)$. Similarily to the classical case, we have orthogonality relations for matrix coefficents.

4.13. Theorem. *Let U be an irreducible unitary representation, with corresponding matrix form u_{ij} with respect to some basis for H_U. Let $\rho \in \mathsf{Mor}(U, U^{cc})$ be the element in 4.12. Then*

(i) $h(u_{kl}u_{ij}^*) = \dfrac{\delta_{ki}\omega_{e_j,e_l}(\rho)}{\dim_q U}$ *and* $h(u_{ij}^* u_{kl}) = \dfrac{\delta_{jl}\omega_{e_k,e_i}(\rho^{-1})}{\dim_q U}$

(ii) If V is an irreducible unitary representation not equivalent to U, then $h(v_{kl}u_{ij}^*) = h(u_{ij}^*v_{kl}) = 0$.

Proof. See theorem 1.4.3. in [NT13]. □

4.4 Quantum group C*-algebras

Let G be a compact quantum group, and fix a family of representatives U_α, $\alpha \in \mathrm{Irr}(G)$. Denote by $\mathbb{C}[G]$ the linear span of matrix coefficents of finite dimensional unitary representations of G. For each $\alpha \in \mathrm{Irr}(G)$, fix a basis in H_α and let $(u_{ij}^\alpha)_{i,j}$ be the matrix form of U_α with respect to this basis. Then, by theorem 4.13, the u_{ij}^α form a linear basis for $\mathbb{C}[G]$. $\mathbb{C}[G]$ becomes a Hopf $*$-algebra with counit ε and antipode S defined on basis elements by $\varepsilon(u_{ij}^\alpha) = \delta_{ij}$ and $S(u_{ij}^\alpha) = (u_{ji}^\alpha)^*$. The comultiplication is simply $\Delta|_{\mathbb{C}[G]}$. Although we will not spend much time on studying $\mathbb{C}[G]$ in this text, it is of great importance in the theory. We have a quantum analog of the Peter-Weyl theorem:

4.14. Theorem. *$\mathbb{C}[G]$ is dense in $C(G)$.*

Proof. See corollary 1.5.5. in [NT13]. □

Write $\mathcal{U}(G) = \mathbb{C}[G]^*$. Then $\mathcal{U}(G)$ is a $*$-algebra with product

$$\omega\eta = (\omega \otimes \eta)\Delta.$$

and involution

$$\omega^*(a) = \overline{\omega((Sa)^*)}, \quad a \in \mathbb{C}(G)$$

When U is a unitary finite dimensional representation of G we get a $*$- representation π_U of $\mathcal{U}(G)$ defined by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. These induced $*$-representations of $\mathcal{U}(G)$ gives a $*$-homomorphism

$$\Phi : \mathcal{U}(G) \to \prod_\alpha B(H_\alpha)$$

defined by

$$\omega \mapsto (\pi_{U_\alpha}(\omega))_\alpha$$

Now, take $\omega_{ij}^\alpha \in \mathcal{U}(G)$ to be the functional defined by $\omega_{ij}^\alpha(u_{k\ell}^\beta) = \delta_{i,k}\delta_{j,\ell}\delta_{\alpha,\beta}$. Then $\Phi(\omega_{ij}^\alpha) \mapsto m_{ij}^\alpha$; the (i,j)'th matrix unit in $B(H_{U_\alpha})$. This shows that Φ is surjective. Moreover, if $\Phi(\omega) = 0$, then $\omega(u_{ij}^\alpha) = 0$ on every basis element u_{ij}^α, whence $\omega = 0$. We conclude that Φ is injective as well, so that it is in fact an isomorphism.

Inside $\prod_\alpha B(H_\alpha)$ there are some interesting C^*-algebras. Namely we have

$$C^*(G) = c_0 - \bigoplus_{\alpha \in \mathrm{Irr}(G)} B(H_\alpha)$$

and

$$W^*(G) = \ell^\infty - \bigoplus_{\alpha \in \mathrm{Irr}(G)} B(H_\alpha) = M(C^*(G)).$$

We call $C^*(G)$ the *group C^*-algebra* associated to G and $W^*(G)$ the *group von Neumann algebra* associated to G.

4.15. Proposition. *Suppose (H, θ) is a quantum subgroup of G. Then $\theta(\mathbb{C}[G]) = \mathbb{C}[H]$.*

Proof. Consider a representation U_α, $\alpha \in \mathrm{Irr}(G)$. Then $(\iota \otimes \theta)(U_\alpha)$ is a finite dimensional representation of H, and so it decompoes into a direct sum of irreducible representations. We conclude that $\theta(\mathbb{C}[G]) \subset \mathbb{C}[H]$. Moreover, $\theta(\mathbb{C}[G])$ is the set of matrix coefficents for the representations V_β, $\beta \in \mathrm{Irr}(H)$ such that V_β is a subrepresentation of $(\iota \otimes \theta)(U_\alpha)$ for some $\alpha \in \mathrm{Irr}(G)$.

Now, take V_β, $\beta \in \mathrm{Irr}(H)$, and write it in matrix form v^β_{kl}. Consider the linear functional $h(v^\beta_{kl}\theta(\cdot))$. Since θ is surjective and $\mathbb{C}[G]$ is dense in $C(G)$ there must be a representation U_α such that $h(v^\beta_{kl}\theta(u^\alpha_{ij})) \neq 0$, for at least one of the matrix coefficents u^α_{ij}. By theorem 4.13 we conclude that V_β is a subrepresentation of $(\iota \otimes \theta)(U_\alpha)$. Then, by the first argument $v^\beta_{kl} \in \Theta(\mathbb{C}[G])$. □

The representation $(\iota \otimes \theta)(U_\alpha)$ in the proof above, is called the *restriction* of U_α to H. We will often denote the restriction by $U_\alpha|_H$. If V_β, $\beta \in \mathrm{Irr}(H)$, is a subrepresentation of $U_\alpha|_H$, we write $\beta \subset \alpha$. We have the following relation between the group von Neumann algebras of H and G.

4.16. Proposition. *If (H, θ) is a quantum subgroup of G, θ induces an inclusion*

$$\Theta : \mathcal{U}(H) \hookrightarrow \mathcal{U}(G), \quad \omega \mapsto \omega \circ \theta$$

such that $\Theta(W^*(H)) \subset W^*(G)$.

Proof. The injectivity of Θ follows immediately from the previous proposition. To see that $\Theta(W^*(H)) \subset W^*(G)$, we note that under the isomorphisms $\mathcal{U}(G) \cong \prod_\alpha B(H_\alpha)$ and $\mathcal{U}(H) \cong \prod_\beta B(K_\beta)$, Θ becomes the map $(\rho_\beta(\omega))_\beta \mapsto (\pi_\alpha(\omega \circ \theta))_\alpha$. Moreover

$$\begin{aligned}\pi_\alpha(\omega \circ \theta) &= (\iota \otimes \omega)(\iota \otimes \theta)(U_\alpha) \\ &= (\iota \otimes \omega)(\bigoplus_{\beta \subset \alpha} V_\beta^{\oplus a(\beta,\alpha)}) \\ &= \bigoplus_{\beta \subset \alpha} \rho_\beta(\omega)^{\oplus a(\beta,\alpha)}.\end{aligned}$$

The norm of this last expression is given by $\max_{\beta \subset \alpha} \|\rho_\beta(\omega)\|$. Hence, if $(\rho_\beta(\omega))_\beta \in W^*(H)$, then

$$\sup_\alpha \|\pi_\alpha(\omega \circ \theta)\| = \sup_\alpha \max_{\beta \subset \alpha} \|\rho_\beta(\omega)\| \leq \|(\rho_\beta(\omega))_\beta\|_\infty < \infty$$

whence $(\pi_\alpha(\omega \circ \theta))_\alpha \in W^*(G)$. □

Suppose now that we are given a family of compact quantum groups $\{G_n\}_n$ with each G_n a quantum subgroup of G_{n+1}. Then one can ask what the "inductive limit $G = \varinjlim G_n$" should be. We do not have inclusions $G_n \subset G_{n+1}$ (we have not even defined what "$\subset$" means in this case), but rather the surjective maps $\theta_n : C(G_{n+1}) \to C(G_n)$. The result above shows that the θ_n's give rise to inclusions $W^*(G_n) \hookrightarrow W^*(G_{n+1})$. Moreover since $C^*(G_n) \subset W^*(G_n) \subset W^*(G_k)$ for $k \geq n$, we can consider the C^*-algebras

$$\mathcal{A}(G) = \sum_{k=1}^{n} C^*(G_k) \subset W^*(G_n),$$

and define the fat AF-algebra $\mathcal{A}(G_\infty) = \varinjlim \mathcal{A}_n(G)$ as in the previous chapter. We call $\mathcal{A}(G)$ the *group C^*-algebra associated to G*.

The branching rule is then described as before by how $U_\alpha|_{G_{n-1}} = (\iota \otimes \theta)(U_\alpha)$, $\alpha \in \mathrm{Irr}(G_n)$, decomposes into a direct sum

$$U_\alpha|_{G_{n-1}} = \bigoplus_{\beta \subset \alpha} V_\beta^{\oplus a(\beta,\alpha)}.$$

4.5 Fusion rules and the representation ring

Let G be a compact quantum group with a fixed family $\{U_\alpha\}_{\alpha \in \mathrm{Irr}(G)}$. This forms an abelian monoid with addition $\oplus$ and identity element the zero representation. Thus we can form the Grothendieck group: Let D be the set of formal differences $U_\alpha - U_\beta$, $\alpha, \beta \in \mathrm{Irr}(G)$. We impose the equivalence relation $\sim$ on D defined by

$$U_\alpha - U_\beta \sim U_{\alpha'} - U_{\beta'} \quad \Leftrightarrow \quad U_\alpha \oplus U_{\beta'} = U_{\alpha'} \oplus U_\beta$$

The Grothendieck group is then the set $K(G) = D/\sim$ consisting of equivalence classes $[U_\alpha - U_\beta]$ with addition defined by

$$[U_\alpha - U_\beta] + [U_{\alpha'} - U_{\beta'}] = [U_\alpha \oplus U_{\alpha'} - U_\beta \oplus U_{\beta'}].$$

Note that the family of representations is naturally embedded into this ring by

$$U_\alpha \mapsto [U_\alpha] = [U_\alpha - 0].$$

Now take $\alpha, \beta \in \mathrm{Irr}(G)$. By complete reducibility we can always find a finite set $\alpha_1, \alpha_2, ..., \alpha_k \in \mathrm{Irr}(G)$ such that

$$U_\alpha \otimes U_\beta = U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_k}.$$

We call this a *fusion rule* since it tells us how the "fusion" of U_α and U_β behaves. Because of this we can form a ring $R(G)$ from the Grothendieck group $K(G)$. Namely, we take the tensor product of representations

$$[U_\alpha] \cdot [U_\beta] = [U_\alpha \otimes U_\beta],$$

and extend to all elements by imposing the usual distribution laws. We call $R(G)$ the *representation ring of* G. With this construction it makes perfect sense to consider elements $p(U_\alpha) \in R(G)$ for any polynomial $p \in \mathbb{Z}[x]$, and $\alpha \in \mathrm{Irr}(G)$. This will be usefull in the next section.

5. The infinite quantum symmetric group

In this chapter we investigate the tracial states on the group C^*-algebra $\mathcal{A}(S_\infty^+)$ associated to S_∞^+. That is, $\mathcal{A}(S_\infty^+) = \varinjlim \mathcal{A}_n(S_\infty^+)$ is the fat AF-algebra associated to the group C^*-algebras $C^*(S_n^+)$. We remark that we do not care about S_n^+ for $n = 1, 2, 3$ here, since theorem 5.1 does not hold for these cases. However this does not affect the inductive limit.

5.1 The irreducible representations of S_n^+

For each $n \geq 4$ let $U^{(n)}$ be the fundamental representation of S_n^+ (see example 4.9).

5.1. Proposition (Banica). *Fix $n \geq 4$. For each $s = 0, 1, 2, \ldots$ there is a unitary irreducible representation $U_s^{(n)} \in B(H_{(n,s)}) \otimes C(S_n^+)$ of S_n^+ where $U_0^{(n)}$ is the trivial representation and $U^{(n)} = U_1^{(n)} \oplus U_0^{(n)}$. These representations exhaust all unitary irreducible representations of S_n^+ (up to unitary equivalence). Moreover we have the following fusion rule*

$$U_s^{(n)} \otimes U_t^{(n)} = U_{|s-t|}^{(n)} \oplus U_{|s-t|+1}^{(n)} \oplus \cdots \oplus U_{s+t}^{(n)}.$$

Proof. This is theorem 4.1. in [Ban99]. □

For each $n \geq 4$ we fix a family $(U_j^{(n)})_{j \geq 0}$ of representatives of the unitary irreducible representations given in the theorem above. The Hilbert space corresponding to $U_j^{(n)}$ is written $H_{(n,j)}$. The group C^*-algebra of S_n^+ is then equal to

$$C^*(S_n^+) = c_0 - \bigoplus_j B(H_{(n,j)}),$$

and we can write a_{ij} for the multiplicity of $U_i^{(n)}$ in $U_j^{(n+1)}$. That is

$$U_j^{(n+1)}|_{S_n^+} = \bigoplus_i (U_i^{(n)})^{\oplus a_{ij}} = \sum_i a_{ij} U_i^{(n)},$$

where the last expression is just to simplify notation. We will see that the a_{ij} does not depend n for $n \geq 4$, or in other words the multiplicity matrix $A^{(n)} = A = (a_{ij})_{i,j}$ is constant.

Suppose τ is an extremal tracial state on $\mathcal{A}(S_\infty^+)$, such that $\tau|_{C^*(S_n^+)}$ is a state for each n. Similarily to the case in section 3.1, τ is given by a sequence $(\nu^{(n)})_n$ of vectors such that $A\nu^{(n+1)} = \nu^{(n)}$. Thus, to investigate the tracial states on $\mathcal{A}(S_\infty^+)$ the main object of study will be the matrix A. Given the information in theorem 5.1, we can find a recursive formula for the entries of the matrix.

5.2. Proposition. *For $n \geq 4$ the multiplicites a_{ij} does not depend on n, and they satisfy the follwing relations:*

- $a_{0,0} = a_{0,1} = a_{1,1} = 1$
- $a_{i,0} = 0$ *for all* $i \geq 1$, *and* $a_{i,1} = 0$ *for all* $i \geq 1$.
- $a_{0,j} = a_{1,j-1} - a_{0,j-2}$ *for* $j \geq 2$
- $a_{i,j} = a_{i-1,j-1} + a_{i,j-1} + a_{i+1,j-1} - a_{i,j-2}$ *for* $i \geq 1$ *and* $j \geq 2$.

Remark. The first two bullet points above gives the first two columns in the multiplicity matrix. Given this information the two recurrence relations determine the rest of the matrix. △

Proof. That $a_{0,0} = 1$ and that $a_{i.0} = 0$ for all $i \geq 1$ is obvious since $\dim U_0^{(n)} = 1$ for all n.

Let $\theta_n : C(S_{n+1}^+) \to C(S_n^+)$ denote the surjective map given in example 4.6. The fundamental representations satisfy the following relation

$$U^{(n+1)}|_{S_n^+} = (\iota \otimes \theta_n)(U^{(n+1)}) = U^{(n)} \oplus U_0^{(n)}. \tag{5.1.1}$$

Thus, by theorem 5.1 we get that

$$U^{(n+1)}|_{S_n^+} = U_1^{(n)} \oplus 2U_0^{(n)},$$

from which we easily deduce that

$$U_1^{(n+1)}|_{S_n^+} = U_0^{(n)} \oplus U_1^{(n)}.$$

In other words $a_{0,1} = a_{1,1} = 1$.

Now we turn to the recursive formulas. Setting $t = 1$ in theorem 5.1 we obtain that

$$U_s^{(n)} \otimes U_1^{(n)} = U_{s-1}^{(n)} \oplus U_s^{(n)} \oplus U_{s+1}^{(n)}. \tag{5.1.2}$$

Combining (5.1.1) with (5.1.2), we get

$$\begin{aligned} U_{s+1}^{(n+1)}|_{S_n^+} \oplus U_s^{(n+1)}|_{S_n^+} \oplus U_{s-1}^{(n+1)}|_{S_n^+} &= (U_1^{(n)} \oplus U_0^{(n)}) \otimes U_s^{(n+1)}|_{S_n^+} \\ &= (U_1^{(n)} \otimes U_s^{(n+1)}|_{S_n^+}) \oplus U_s^{(n+1)}|_{S_n^+}, \end{aligned}$$

which yields

$$U_{s+1}^{(n+1)}|_{S_n^+} = (U_1^{(n)} \otimes U_s^{(n+1)}|_{S_n^+}) \ominus U_{s-1}^{(n+1)}|_{S_n^+}.$$

Writing this in terms of representations of S_n^+ and using the fusion rule 5.1.2 again we obtain

$$\begin{aligned} U_j^{(n+1)}|_{S_n^+} &= (U_1^{(n)} \otimes U_{j-1}^{(n+1)}|_{S_n^+}) \ominus U_{j-2}^{(n+1)}|_{S_n^+} \\ &= a_{0,j-1}U_1^{(n)} \oplus (\sum_{i\geq 1} a_{i,j-1}(U_{i-1}^{(n)} \oplus U_i^{(n)} \oplus U_{i+1}^{(n)})) \ominus \sum_{i\geq 0} a_{i,j-2}U_i^{(n)}, \end{aligned}$$

for all $j \geq 2$. Collecting the $U_i^{(n)}$ we get

$$U_j^{(n+1)}|_{S_n^+} = (a_{1,j-1} - a_{0,j-2})U_0^{(n)} \oplus \sum_{j\geq 1}(a_{i-1,j-1} + a_{i,j-1} + a_{i+1,j-1} - a_{i,j-2})U_i^{(n)}.$$

This gives the two last formulas in the proposition. □

5.3. Corollary. *$a_{i,i} = 1$ for all $i \geq 0$ and $a_{i,j} = 0$ whenever $i > j$.*

Proof. This is immediate from the previous proposiiton. □

5.4. Example. Using the recursive relations it is quite easy to compute a finite upper left block of the multiplicity matrix. For instance here is the upper left 9×9 block:

$$(a_{i,j})_{i,j=1}^9 = \begin{pmatrix} 1 & 1 & 0 & 1 & 2 & 3 & 7 & 15 & 32 \\ 0 & 1 & 2 & 2 & 4 & 9 & 18 & 39 & 87 \\ 0 & 0 & 1 & 3 & 5 & 10 & 23 & 51 & 115 \\ 0 & 0 & 0 & 1 & 4 & 9 & 20 & 48 & 114 \\ 0 & 0 & 0 & 0 & 1 & 5 & 14 & 35 & 89 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 20 & 56 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

△

5.2 Chebyshev polynomials and a closed formula

The reader familiar with representation theory of compact Lie groups, has probably noticed that the fusion rule in proposition 5.1 is equal to the fusion rule for irreducible representations of SO(3).[1] Recall that for each $s \in \frac{1}{2}\mathbb{Z}_+$ there is an irreducible representation V_s of SU(2) of dimension $\dim V_s = 2s + 1$. It is well known that these representations satisfy the following fusion rule:

$$V_s \otimes V_t = V_{s-t} \oplus V_{|s-t|+2} \oplus \cdots \oplus V_{s+t}.$$

The irreducible representations of SO(3) correspond to the subfamily $\{V_s\}_{s\in\mathbb{Z}_+}$, and by applying the fusion rule to this subfamily we clearly get the same fusion rule as in 5.1.

The character χ_s of the representation V_s is determined by

$$\chi_s\Big(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\Big) = e^{i2s\theta} + e^{i2(s-1)\theta} + \cdots + e^{-i2s\theta} = \frac{\sin((2s+1)\theta)}{\sin\theta}.$$

since any element in SU(2) is conjugate to a matrix of the above form. As it turns out, this is closely related to the so-called *Chebyshev polynomials.*

5.5. Definition (and some facts). The *Chebyshev polynomials of the second kind* [2] (henceforth reffered to only as Cheybyshev polynomials) is a family $\{P_n\}_{n\in\mathbb{Z}_+}$ of polynomials defined by the following recursive relation

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 2x \\ P_{n+1}(x) &= 2xP_n(x) - P_{n-1}(x). \end{aligned}$$

Moreover, they satisfy the following properties:

(i) $P_n(\cos\theta) = \dfrac{\sin((n+1)\theta)}{\sin\theta}$

(ii) $\int_{-1}^{1}(1-x^2)^{1/2}P_n(x)P_k(x)\,dx = \begin{cases} 0 & \text{if } k \neq n \\ \pi/2 & \text{if } k = n \end{cases}$

(iii) $P_{2n} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}4^k x^{2k}$

Proof. The statements (i) and (ii) are very well known, and can be found in for instance [MH02], while the last statement is contained in corollary 2 on page 7 in [WBR07]. □

Thus the characters of SU(2) are related to the Chebyshev polynomials in the following way:

$$\chi_s\Big(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\Big) = P_{2s}(\cos\theta). \tag{5.2.1}$$

[1] The representation theory of SU(2) and SO(3) can be found in almost any book Lie groups or compact groups. A very nice introduction is [HH03], and this book covers all we need in this text.

[2] These polynomials are often denoted by U_n in the litterature, but we have reserved this letter for representations.

We care mostly about the characters of SO(3), so we will from now on take $s \in \mathbb{Z}_+$ only. Using that $P_2(x) = 4x^2 - 1$, and the formula in 5.5 we may write

$$P_{2s}(x) = \sum_{k=0}^{s} (-1)^{s-k} \binom{s+k}{2k} (P_2(x)+1)^k, \quad s \in \mathbb{Z}_+. \tag{5.2.2}$$

Thus P_{2s} can be expressed as a polynomial in P_2. Equivalently, in terms of characters of SO(3), there is for each $j \in \mathbb{N}$ a polynomial $h_j \in \mathbb{Z}[x]$ of degree j such that $\chi_j = h_j(\chi_1)$.

5.6. Observation. *Fix $j \in \mathbb{N}$. Let h_j denote the polynomial of degree j satisfying $h_j(\chi_1) = \chi_j$, and let $A = (a_{ij})_{i,j}$ be the multiplicity matrix for S_∞^+. Then*

$$h_j(x+1) = \sum_{i=0}^{j} a_{ij} h_i(x).$$

Proof. Since the representations $\{U_j^{(n)}\}_{j \in \mathbb{Z}_+}$ of S_n^+ satisfy the same fusion rule as the representations of SO(3), we conclude that $U_j^{(n)} = h_j(U_1^{(n)})$ as well (where we work in the representattion ring of S_n^+). Note moreover that this yields

$$U_j^{(n)}|_{S_{n-1}^+} = h_j(U_1^{(n)})|_{S_{n-1}^+} = h_j(U_1^{(n-1)} \oplus U_0^{(n-1)}),$$

by the branching rules (5.2). On the other hand

$$U_j^{(n)}|_{S_{n-1}^+} = \sum_{i=0}^{j} a_{ij} U_i^{(n-1)} = \sum_{k=0}^{n} a_{ij} p_i(U_1^{(n-1)}).$$

Combining the two expression for $U_j^{(n)}|_{S_{n-1}^+}$ we get the claim. $\square$

Because of this we can now use the orthogonality of Chebyshev polynomials to compute a closed formula for the multiplicity matrix.

5.7. Theorem. *The multiplicity matrix $A = (a_{ij})_{i,j}$ is determined by*

$$a_{ij} = \begin{cases} \frac{c_{ij}}{4^i} + \sum_{k=i+1}^{j} \left(\binom{2k}{k-i} - \binom{2k}{k-i-1} \right) \frac{c_{kj}}{4^k} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

where the c_{kj}'s are given by writing $h_j(P_2(x)+1) = c_{0j} + c_{1j}x^2 + \cdots + c_{jj}x^{2j}$. Explicitly

$$c_{kj} = 4^k \sum_{l=k}^{j} (-1)^{j-l} \binom{j+l}{2l} \binom{l}{k}.$$

Proof. Let $(\cdot\,|\,\cdot)$ denote the inner product on $\mathcal{L}^2([-1,1],d\mu)$, $d\mu = (1-x^2)^{1/2}dx$, i.e,

$$(f\,|\,g) = \int_{-1}^{1}(1-x^2)^{1/2}f(x)g(x)\,dx, \quad f,g \in \mathcal{L}^2([-1,1],d\mu).$$

Now, using the observation (5.6) above we can write

$$h_j(P_2(x)+1) = \sum_{i=1}^{j} a_{ij}h_i(P_2(x)) = \sum_{i=1}^{j} a_{ij}P_{2i}(x).$$

Then the orthogonality of the Chebyshev polynomials (5.5) tells us that

$$\frac{\pi}{2}a_{ij} = (\tilde{h}_j\,|\,P_{2i}),$$

where $\tilde{h}_j(x) = h_j(P_2(x)+1)$.

Step 1. Writing x^{2k} in terms of P_{2i}. We start by considering the intergral

$$(x^{2k}\,|\,P_{2i}) = \int_{-1}^{1}(1-x^2)^{1/2}x^{2k}P_{2i}(x)\,dx.$$

Since P_{2i} looks much nicer in polar coordinates, we set $x = \cos\theta$. Then

$$\int_{-1}^{1}(1-x^2)^{1/2}x^{2k}P_{2i}(x)\,dx = \int_{0}^{\pi}\sin\theta\cos^{2k}\theta\sin((2i+1)\theta),$$

where we use that $(1-x)^{-1/2} = \arccos'(x)$ and that $\sin(\theta)P_{2i}(\cos\theta) = \sin((2i+1)\theta)$. One may rewrite $\cos^{2k}\theta$ to obtain

$$\cos^{2k}\theta = 4^{-k}\binom{2k}{k} + \frac{2}{4^k}\sum_{l=0}^{k-1}\binom{2k}{l}\cos(2(k-l)\theta).$$

Ignoring the summation for now, we assume that $i \geq 1$ and compute

$$\begin{aligned}
&\int_{0}^{\pi}\cos(2(k-l)\theta)\sin\theta\sin((2i+1)\theta)\,d\theta \\
&= \frac{1}{2}\int_{0}^{\pi}\cos(2(k-l)\theta)\big(\cos(2i\theta) - \cos((2(i+1)\theta)\big)\,d\theta \\
&= \frac{\pi}{4}(\delta_{2i}^{2(k-l)} - \delta_{2i+2}^{2(k-l)}).
\end{aligned}$$

Implementing this in the computation of $(x^{2k}\,|\,P_{2i})$ we obtain

$$\begin{aligned}
(x^{2k}\,|\,P_{2i}) &= 4^{-k}\binom{2k}{k}(1\,|\,P_{2i}) + \frac{2\pi}{4^{k+1}}\sum_{l=0}^{k-1}\binom{2k}{l}(\delta_{2i}^{2(k-l)} - \delta_{2i+2}^{2(k-l)}) \\
&= \begin{cases} \frac{2\pi}{4^{k+1}}\left(\binom{2k}{k-i} - \binom{2k}{k-i-1}\right) & \text{if } 1 \leq i \leq k-1 \\ \frac{2\pi}{4^{k+1}} & \text{if } i = k \\ 0 & \text{if } i \geq k \end{cases}
\end{aligned}$$

where the first term vanishes in all cases, since $(1 \mid P_{2i}) \neq 0$ if and only if $i = 0$.

Lastly we compute $(x^{2k} \mid P_0) = (x^{2k} \mid 1)$. The integral of interest becomes

$$\begin{aligned}\int_0^\pi \cos(2(k-l)\theta)\sin^2\theta\, d\theta &= \frac{1}{2}\int_0^\pi \cos(2(k-l)\theta)\big(1-\cos((2\theta)\big)\, d\theta \\ &= -\frac{\pi}{4}\delta_{2i+2}^{2(k-l)}.\end{aligned}$$

This yields

$$\begin{aligned}(x^{2k} \mid P_0) &= 4^{-k}\binom{2k}{k}(1 \mid P_0) - \frac{2\pi}{4^{k+1}}\sum_{l=0}^{k-1}\binom{2k}{l}\delta_{2i+2}^{2(k-l)} \\ &= \frac{2\pi}{4^{k+1}}\left(\binom{2k}{k} - \binom{2k}{k-1}\right).\end{aligned}$$

which is the exact same formula as for $i = 1, 2, ..., k-1$ but with $i = 0$ instead.

Step 2. Computing a_{ij} in terms of the c_{kj}'s. Write $\tilde{h}_j(x) = c_{0j} + c_{1j}x^2 + \cdots + c_{jj}x^{2j}$. Then, as we saw above

$$\frac{\pi}{2}a_{ij} = (\tilde{h}_j \mid P_{2i}) = \sum_{k=0}^{j} c_{kj}(x^{2k} \mid P_{2i}).$$

Using the formulas above we obtain

$$a_{ij} = \frac{c_{ij}}{4^i} + \sum_{k=i+1}^{j}\left(\binom{2k}{k-i} - \binom{2k}{k-i-1}\right)\frac{c_{kj}}{4^k},$$

which is what we wanted to prove!

Step 3. Computing c_{kj}. The only thing remaining is to find an explicit expression for c_{kj}. Using the equation (5.2.2) we see that

$$\begin{aligned}h_j(P_2(x)+1) &= \sum_{l=0}^{j}(-1)^{j-l}\binom{s+l}{2l}(P_2(x)+2)^l \\ &= \sum_{l=0}^{j}(-1)^{j-l}\binom{j+l}{2l}(4x^2+1)^l \\ &= \sum_{l=0}^{j}(-1)^{j-l}\binom{j+l}{2l}\sum_{k=0}^{l}\binom{l}{k}4^k x^{2k}\end{aligned}$$

Extracting the coeffecient in front of x^{2k} yields

$$c_{kj} = 4^k\sum_{l=k}^{j}(-1)^{j-l}\binom{j+l}{2l}\binom{l}{k}.$$

□

5.3 Traces on $\mathcal{A}(S_\infty^+)$

Assume that τ is an extremal tracial state on $\mathcal{A}(S_\infty^+)$ such that $\tau|_{C^*(S_n)}$ is state for each $n \geq 4$. Recall from chapter 3 that τ must satisfy the following properties:

- For each $n \geq 4$ there is a vector $\nu^{(n)} = (\nu_0^{(n)}, \nu_1^{(n)}, ...)$ of nonnegative numbers such that
$$\tau|_{C^*(S_n^+)} = \sum_{i=0}^{\infty} \nu_i^{(n)} \chi^{(n,i)}, \tag{5.3.1}$$
where $\chi^{(n,i)}$ is the trace on $B(H_{(n,i)})$. This sequence of vectors $(\nu^{(n)})_n$ satisfies $A\nu^{(n+1)} = \nu^{(n)}$ and $\sum_i \nu_i^{(n)} = 1$.
- There is a sequence $(i_n)_n$ of nonnegative integers such that
$$\tau(a) = \lim_{n\to\infty} \frac{\chi^{(n,i_n)}(a)}{\dim U_{i_n}^{(n)}}, \tag{5.3.2}$$
for each $k \geq 4$ and $a \in C^*(S_k^+)$. Moreover
$$\nu_i^{(k)} = \frac{\dim(U_i^{(k)}, U_{i_n}^{(n)})}{\dim U_{i_n}^{(n)}}, \tag{5.3.3}$$
where $\dim(U_i^{(k)}, U_{i_n}^{(n)})$ denotes the number of paths in the Bratteli diagram from $U_i^{(k)}$ to $U_{i_n}^{(n)}$.

Considering these properties, and the expression for the multiplicity matrix A, the only obvious trace on $\mathcal{A}(S_\infty)$ is the one corresponding to the trivial representation. This is given by the sequence where $\nu^{(n)} = (1, 0, 0, ...)$ for all n. It is of course possible that there there are no other traces. Equivalently, it might happen that there are no other sequences $(\nu^{(n)})_n$ of nonnegative vectors satisfying $A\nu^{(n+1)} = \nu^{(n)}$. This is certainly the case in the similar (but a lot easier) case of the following proposition.

5.8. Proposition. *Suppose $B = (b_{ij})_{i,j}$ is an $n \times n$ matrix of the form*
$$\begin{bmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 1 & b_{23} & \cdots & b_{2n} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$
where $b_{i,i+1} > 0$ for $i = 1, ..., n-1$. Assume that $(\nu^{(k)})_{k\in\mathbb{N}}$ is a nontrivial sequence of nonnegative vectors $\nu^{(k)} = (\nu_1^{(k)}, \nu_2^{(k)}, ..., \nu_n^{(k)})$ such that $B\nu^{(k+1)} = \nu^{(k)}$. Then $\nu^{(n)} = (1, 0, 0, ..., 0)$ for all $n \in \mathbb{N}$.

Proof. It is not hard to see that the inverse of B is

$$B^{-1} = I - N + N^2 - \cdots + (-1)^{n-1}N^{n-1},$$

where I is the identity matrix and $N = B - I$. Let us rewrite the assumption on $\nu^{(k)}$ as

$$\nu^{(k+1)} = B^{-1}\nu^{(k)} = B^{-k}\nu^{(1)}.$$

Now

$$\begin{aligned} B^{-2} = I - N + N^2 - N^3 + \cdots &+ (-1)^{n-1}N^{n-1} \\ -N + N^2 - N^3 + \cdots &+ (-1)^{n-1}N^{n-1} \\ +N^2 - N^3 + \cdots &+ (-1)^{n-1}N^{n-1} \\ \ddots \qquad &\vdots \\ &(-1)^{n-1}N^{n-1}, \end{aligned}$$

from which it should be clear that $B^{-2} = \sum_{j=0}^{n-1}(-1)^j\alpha_j(2)N^j$, for some positive integers $\alpha_j(2)$. It follows by induction on k that

$$B^{-k} = \sum_{j=0}^{n-1}(-1)^j\alpha_j(k)N^j,$$

for some positive integers $\alpha_j(k)$ such that $\alpha_j(k) \to +\infty$ for $j = 1, 2, ..., n$. From this we see that $\nu^{(k)}$ is of the form

$$\nu^{(k)} = B^{-k}\nu^{(1)} = \Big(\sum_{j=0}^{n-1}(-1)^j\alpha_j(k)N^j\Big)\nu^{(1)} = \begin{bmatrix} v_1^{(k)} \\ \vdots \\ v_{n-2}^{(k)} \\ v_{n-1}^{(1)} - \alpha_1(k)b_{n-1,n}v_n^{(1)} \\ v_n^{(1)} \end{bmatrix},$$

because I is the only matrix with nonzero elements on the diagonal, and N is the only matrix with nonzero elements on the second diagonal.

Now, if the number $\nu_n^{(1)}$ is positive, we conclude that the vector $\nu^{(k)}$ has a negative element for k large enough (since $\alpha_1(k) \to +\infty$ and $b_{n-1,n} > 0$), which contradicts our assumption. Thus $\nu_n^{(1)} = 0$, and we can simply reduce the situation to the the upper left $(n-1) \times (n-1)$ block of B and do the same proof again. After a finite number of steps we conclude that $\nu^{(n)} = (1, 0, 0, ..., 0)$ for all n. □

Our matrix A is an infinite dimensional version of the matrix B in the proposition above. This is easily checked by using one of the theorems 5.7 or 5.2. In fact, in this case the recursive formula is probably the easiest:

$$\begin{aligned} a_{i,i+1} &= a_{i-1,i} + a_{i,i} + a_{i+1,i} - a_{i,i-3} \\ &= 1 + a_{i-1,1} \\ &= i + a_{0,1} = i + 1 \end{aligned}$$

since $a_{i,j} = 0$ when $j > i$, $a_{i,i} = 1$ for all i and $a_{0,1} = 1$.

We obtain the following result:

5.9. Proposition. *Let τ be an extremal tracial state on $\mathcal{A}(S_\infty^+)$, such that $\tau|_{C^*(S_n^+)}$ is a state for each n. Let $(i_n)_n$ be the sequence determining τ as in equation (5.3.2). Then, either τ is trivial or $i_n \to \infty$.*

Proof. By the branching rule the sequence $(i_n)_n$ is increasing, simply because $U_j^{(n)}|_{S_{n-1}^+}$ does not contain any representation $U_i^{(n-1)}$ for $i > j$. Thus, if the sequence is bounded, then $i_n \to \ell$ for some $\ell \geq 0$. It follows from this, and the branching rule again, that $\dim(U_j^{(m)}, U_{i_n}^{(n)}) = 0$ for all $m \leq n$ whenever $j > \ell$. We conclude then by 5.3.3 that $\tau|_{C^*(S_m^+)}$ is given by a finite vector $(\nu_0^{(m)}, \nu_1^{(m)}, ..., \nu_\ell^{(m)})$. But this means that τ is given by a sequence of nonnegative vectors of length ℓ such that

$$A(\ell)\nu^{(n+1)} = \nu^{(n)},$$

where $A(\ell)$ denotes the upper left $\ell \times \ell$-block of A. By the observation preceeding this proposition, $A(\ell)$ is of the same form as B in proposition 5.8. We conclude that τ is trivial. □

This proposition tells us that we can exclude all the bounded sequences $(i_n)_n$ in the search for tracial states on $\mathcal{A}(S_\infty^+)$. Although this is the last result of this thesis, there are still several interesting questions to ask. Does there exist a nicer way to write the formula in theorem 5.7? Is the only trace on $\mathcal{A}(S_\infty^+)$ (satisfying our assumptions) the trivial one? We suspect that the answer to the last question is yes. If one wants to prove this, a generalization of proposition 5.8 to infinte matrices will do the job. However, since the proof above clearly depends on the matrix being finite dimensional, a more general result will surely need some adjustments to the proof. There are probably many other ways to explore these questions as well, but for now we leave further speculation to the reader.

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