UiO : Department of Mathematics
University of Oslo

On quantum channels and reduced twisted group C*-algebras

Sveinung Knutsen Nøding

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

The purpose of this thesis is to look at certain abelian groups and their reduced twisted group C^*-algebras. We consider a class of quantum channels on these algebras, associated with normalised positive definite functions, and study their contractive properties with respect to the trace-norm metric and the Bures metric.

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Contents

List of Figures

CHAPTER 1

Introduction

Quantum channels are studied in quantum information theory. From a physical point of view they can be viewed as communication channels transmitting quantum information, but we will not be concerned with this. From our point of view, quantum channels are completely positive trace-preserving linear maps on a unital C^*-algebra. The concepts discussed in this introduction will be formalised in Chapter 2. In quantum information theory the C^*-algebra of choice for quantum channels is usually the matrices, but other C^*-algebras could be considered as well.

This brings us to our next topic, reduced twisted group C^*-algebras. Group algebras assign an operator algebra, in our case a C^*-algebra, to a locally compact group G, such that representations of the algebra are related to representations of the group. In our case, we also consider a 2-cocycle σ on a discrete group G which in many cases makes the resulting algebra $C_r^*(G,\sigma)$ non-abelian. Specifically, $C_r^*(G,\sigma)$ is generated by the σ-projective left regular representation λ_σ of G on $\ell^2(G)$. It is known that $C_r^*(G,\sigma)$ has a canonical faithful tracial state τ, satisfying that $\tau(\lambda_\sigma(g)) = 0$ for all $g \in G$ different from the unit. This is related to our previous paragraph in that we want to study certain quantum channels on various reduced twisted group C^*-algebras $(C_r^*(G,\sigma),\tau)$.

Contractive maps are used in numerous circumstances, such as Picard's existence theorem from differential equations and the Implicit Function Theorem. They are interesting because they allow us to use Banach Fixed-Point Theorem to show that the map has a unique fixed point. This is useful in for instance studying convergence of iterated systems. The reason that we are concerned with contractions in this thesis, is that we are interested in the contractive properties of the quantum channels discussed above. More generally, one wants to use that a map is contractive to establish the following property.

Definition 1.0.1. Let (X,d) be a metric space. We say that a map $T : X \to X$ has the *attractor property* if there exists some $x_0 \in X$ satisfying that

$$T^n(x) \to x_0 \text{ as } n \to \infty$$

for every $x \in X$, in which case it follows readily that T has a unique fixed point, namely x_0.

We say that a map T on a metric space (X,d) is *locally* contractive if $d(T(x),T(y)) < d(x,y)$ for all $x,y \in X$, $x \neq y$. Furthermore we say that T is *strictly contractive* if there exists $C \in [0,1)$ such that $d(T(x),T(y)) \leq Cd(x,y)$

for all $x, y \in X$. Let us consider two examples of how the contractivity of a map T lets us conclude that it has the attractor property.

Example 1.0.2. Let (X, d) be a complete metric space. If $T : X \to X$ is a strict contraction, then T has the attractor property. This is the most common example, and is just a formulation of Banach Fixed-Point Theorem, which we mentioned above.

Example 1.0.3. Let (X, d) be a compact metric space. If $T : X \to X$ is locally contractive, then T has the attractor property. This is known as the Nemytzki-Edelstein Theorem, see [Vas07].

For a historical perspective we briefly return to the physical interpretation of quantum channels as an imperfect transmission of information. In [Rag02], M. Raginsky modelled errors in physically realisable quantum computers as strictly contractive quantum channels with respect to the trace-norm distance $d_1(\sigma, \rho) = \tau(|\sigma - \rho|)$. If a quantum channel is strictly contractive, repeated iteration of that channel on a piece of quantum information will converge to the channel's unique fixed point, effectivly erasing the original information. Later, in [FR17], D. Farenick and M. Rahaman studied locally contractive quantum channels with respect to the Bures metric d_B^τ.

The main work of this thesis is structured as follows. In each of four examples where G is an abelian group we consider $C_r^*(G, \sigma)$ and attempt to find an isomorphism from $C_r^*(G, \sigma)$ into some other more familiar C^*-algebra X. To each normalised positive-definite function φ on G one may associate a quantum channel Q_φ on $C_r^*(G, \sigma)$ such that

$$Q_\varphi(\lambda_\sigma(g)) = \varphi(g)\lambda_\sigma(g)$$

for all $g \in G$. We then use this class of quantum channels and the isomorphism to induce quantum channels on X. Finally we attempt to characterise the contractive properties of these new channels.

1.1 Outline

The text is organised in the following manner:

Chapter 2 We set notation and introduce various preliminaries, including reduced twisted group C^*-algebras, quantum channels and contractions. This includes results and tools that we will use frequently in considering the examples.

Chapter 3 $G = \mathbb{Z}_N$. This is the simplest example, where G is finite and the corresponding algebra $C_r^*(\mathbb{Z}_N)$ is commutative. We give an explicit isomorphism of $C_r^*(\mathbb{Z}_N)$ with $\mathbb{C}^N$ and use this to construct a class of quantum channels on $\mathbb{C}^N$. Next we show that these channels correspond to a subset of the doubly stochastic matrices and investigate their contractive properties. Here we give a condition for such a quantum channel being strictly contractive w.r.t. d_1.

Chapter 4 $G = \mathbb{Z}_N \times \mathbb{Z}_N$. In this example $C_r^*(G, \sigma)$ is no longer commutative. We exhibit an explicit isomorphism from $C_r^*(G, \sigma)$ to $M_N(\mathbb{C})$ and use it to construct a class of quantum channels on $M_N(\mathbb{C})$. We then show that this class is in fact the Weyl-covariant channels studied in quantum information theory. Finally we consider their contractive properties.

Chapter 5 $G = \mathbb{Z}$. In this example G is no longer finite, but the corresponding algebra $C_r^*(\mathbb{Z})$ is still commutative. We give an explicit isomorphic map from $C_r^*(\mathbb{Z})$ to the continuous functions $C(\mathbb{T})$ on the unit circle and describe the resulting class of quantum channels on $C(\mathbb{T})$. It would have been interesting to also study the contractive properties of the resulting quantum channels on $C(\mathbb{T})$, but time did not permit this.

Chapter 6 $G = \mathbb{Z}^2$. In the final example neither G is finite nor $C_r^*(\mathbb{Z}^2, \sigma)$ is commutative. The example splits into two cases, depending on whether a parameter θ is rational or irrational. We describe two isomorphisms, but there was not enough time to investigate quantum channels on these algebras.

CHAPTER 2

Preliminaries

Throughout this thesis the ground field for all vector spaces and algebras is the complex field $\mathbb{C}$, unless explicitly stated otherwise. In addition, any group G is also assumed to be discrete unless otherwise stated.

2.1 C^*-algebras and tracial states

The basic notion that we are concerned with is that of a C^*-algebra. We start by recalling this and some related concepts. This should mostly be familiar, so we do not dwell on it in detail. A discussion of these topics can be found in standard texts such as [Mur14], [Ped79] or [Bla06].

Definition 2.1.1. An *algebra* is a vector space A together with a bilinear map

$$A \times A \to A, \quad (a, b) \mapsto ab$$

such that

$$a(bc) = (ab)c$$

for all $a, b, c \in A$.

A *normed algebra* is an algebra A together with a norm $\|\cdot\|$ on A such that $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$. A *Banach algebra* is a complete normed algebra. If a Banach algebra has a unit it is called *unital*.

Definition 2.1.2. An *involution* on an algebra A is a conjugate linear map $a \mapsto a^*$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. An algebra with an involution is called a $*$-algebra.

Definition 2.1.3. A C^**-algebra* is a Banach algebra A with an involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for all $a \in A$. This relation is referred to as the C^*-identity. A C^*-algebra is called *unital* if it contains a *unit* 1, i.e. $1a = a1 = a$ for all $a \in A$ and $\|1\| = 1$. If $aa^* = a^*a$, a is called *normal*. If $a = a^*$, a is called *self-adjoint*. Furthermore, a is called unitary if $a^*a = aa^* = e$, where e is the identity element of A.

An example of a C^*-algebra is the set of bounded operators on some Hilbert space $\mathcal{H}$, which we denote $\mathcal{B}(\mathcal{H})$. We will also consider the group of unitary operators $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$.

Theorem 2.1.4. *Let A be a C^*-algebra and $a \in A$. Then the following are equivalent:*

(i) $a = b^2$ *for some* $b = b^*$,

(ii) $a = b^*b$ *for some* $b \in A$.

We say that any $a \in A$ *that satisfies this is a* positive element *of* A.

We write A^+ for the set of all elements of a satisfying Theorem 2.1.4.

Definition 2.1.5. A linear map $\varphi : A \to B$ between two C^*-algebras A and B is called *positive* if $\varphi(A^+) \subseteq B^+$.

For a linear functional φ on a C^*-algebra A this is equivalent to $\varphi(a^*a) \geq 0$ for all $a \in A$.

Definition 2.1.6. A *state* φ on a C^*-algebra A is a positive linear functional on A of norm 1.

Definition 2.1.7. A positive linear functional on a C^*-algebra A is called *tracial* whenever $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.

We recall that the above condition is equivalent to $\tau(xy) = \tau(yx)$ for all $x, y \in A$. [Mur14] page 179.

Definition 2.1.8. A positive linear functional on a C^*-algebra A is called *faithful* if $\tau(a^*a) = 0$ implies $a = 0$.

By *trace* we will mean a positive faithful tracial functional, not necessarily a state.

2.2 L^p-spaces

For both reduced group C^*-algebras and the twisted case, we need to know about ℓ^2-spaces. Though there is a more general notion of L^p-spaces that we mention briefly, the former case is our main interest in this thesis. This is a standard construction, see e.g. [MW13].

Definition 2.2.1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $0 < p < \infty$. The collection of all measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}} < \infty,$$

where functions which are equal almost everywhere are identified, is denoted $L^p(\Omega, \mathcal{A}, \mu)$, or sometimes $L^p(\Omega)$.

It is well known that $L^2(\Omega)$ is a Hilbert space. We will not be concerned with other values for p. Rather, our main interest will be the case where both $p = 2$ and $\Omega = G$ is a discrete group equipped with the counting measure. This space is denoted by $\ell^2(G)$, and is equivalently described as the space of all functions $\xi : G \to \mathbb{C}$ such that

$$\sum_{g \in G} |\xi(g)|^2 < \infty.$$

It is possible to show that $\ell^2(G)$ is a Hilbert space with inner product $\langle \xi, \zeta \rangle = \sum_{g \in G} \xi(g)\overline{\zeta(g)}$, where $\xi, \zeta \in \ell^2(G)$. It also has a basis that we will make use of. For each $g \in G$, let δ_g be the function

$$\delta_g(h) = \begin{cases} 1 & \text{for } h = g \\ 0 & \text{otherwise,} \end{cases}$$

where $h \in G$. We also write δ for δ_e, with e being the identity element of G. The following is a well known result.

Proposition 2.2.2. *The set $\{\delta_g \mid g \in G\}$ is an orthonormal basis for $\ell^2(G)$.*

2.3 The reduced twisted group C^*-algebra

Before we go on to defining the reduced twisted group C^*-algebras, it is useful to recollect the reduced group C^*-algebras. A discussion on these algebras can be found in [Dav96]. The rest of our work in this section is based on [BC06], but presented in more detail at points.

Recall that the *unitary operators* $\mathcal{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is the subgroup of $\mathcal{B}(\mathcal{H})$ of operators U such that $UU^* = U^*U = I$, which means that $U^* = U^{-1}$. An equivalent description is that U is a surjective bounded operator that preserves the inner product.

We further recall that the *left regular representation* λ of a group G is the function $\lambda : G \to \mathcal{U}(\ell^2(G))$ given by $\lambda(g) = \lambda_g$, where λ_g acts on $\ell^2(G)$ by $(\lambda_g \xi)(h) = \xi(g^{-1}h)$ for all $h \in G, \xi \in \ell^2$.

Definition 2.3.1. Let G be a group. The *reduced group C^*-algebra* of G is defined as

$$C_r^*(G) = \overline{\operatorname{span}\{\lambda_g \mid g \in G\}}^{\|\cdot\|},$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{B}(\ell^2(G))$.

2-cocycles

The construction of a reduced twisted group C^*-algebra is very similar to the above, but to define it we need the following concept.

Definition 2.3.2. Let G be a group. A map $\sigma : G \times G \to \mathbb{T}$ is called a 2-*cocycle* on G (with values in the unit circle $\mathbb{T}$) if it satisfies

$$\sigma(g, h)\sigma(gh, k) = \sigma(h, k)\sigma(g, hk)$$

for all $g, h, k \in G$. Furthermore, we call σ *normalised* if also

$$\sigma(g, e) = \sigma(e, g) = 1$$

for all $g \in G$, where e is the identity element of G. Unless otherwise noted, any 2-cocycle is assumed to be normalised. Multiplication of 2-cocycles is defined pointwise.

Lemma 2.3.3. *For each normalised σ we have that $\sigma(g, g^{-1}) = \sigma(g^{-1}, g)$ for all $g \in G$.*

Proof. Since σ is a 2-cocycle we have that

$$\sigma(g, h)\sigma(gh, k) = \sigma(h, k)\sigma(g, hk)$$

for all $g, h, k \in G$. Inserting $h = g^{-1}$ and $k = g$ this leaves us with

$$\sigma(g, g^{-1})\sigma(gg^{-1}, g) = \sigma(g^{-1}, g)\sigma(g, g^{-1}g),$$

which is equivalent to

$$\sigma(g, g^{-1})\sigma(e, g) = \sigma(g^{-1}, g)\sigma(g, e).$$

Since σ is also normalised we have that $\sigma(e, g) = \sigma(g, e) = 1$, yielding the result. ■

Having introduced 2-cocycles we are ready to give the twisted counterpart to the left regular representation.

Definition 2.3.4. Let G be a group and σ a 2-cocycle. The corresponding *twisted left regular representation* λ_σ of G is the function $\lambda_\sigma : G \to \mathcal{B}(\ell^2(G))$ given by $(\lambda_\sigma(g)\xi)(h) = \sigma(g, g^{-1}h)\xi(g^{-1}h)$ for all $h \in G$.

In fact, $\lambda_\sigma(g) \in \mathcal{U}(\ell^2(G))$, just as in the left regular representation case. An important difference between the left regular representation λ and λ_σ is that if G is abelian, then $\lambda(g)$ and $\lambda(h)$ commute for all $g, h \in G$. This is not true in general for λ_σ.

Projective unitary representations

Now we have the tools to define the reduced twisted group C^*-algebra, but a priori it will not be clear that it is in fact a C^*-algebra. Therefore we postpone the definition until after a discussion of σ-projective unitary representations, which will help us show this.

Definition 2.3.5. Let G be a group and σ a 2-cocycle. A *σ-projective unitary representation* U of G on a (non-zero) Hilbert space $\mathcal{H}$ is a map $g \mapsto U(g)$ from G into $\mathcal{U}(\mathcal{H})$ such that

$$U(g)U(h) = \sigma(g, h)U(gh)$$

for all $g, h \in G$.

Lemma 2.3.6. *For a σ-projective unitary representation U of G on $\mathcal{H}$ we have that $U(e) = I_{\mathcal{H}}$ and*

$$U(g)^* = \overline{\sigma(g, g^{-1})}U(g^{-1})$$

for all $g \in G$.

Proof. By definition we have that

$$U(g)U(e) = \sigma(g, e)U(ge) = U(g) \text{ for all } g \in G$$

because σ is normalised. Similarly $U(e)U(g) = U(g)$, hence $U(e) = I_{\mathcal{H}}$.

Consider now

$$U(g)U(g^{-1}) = \sigma(g, g^{-1})U(gg^{-1}) = \sigma(g, g^{-1})I_{\mathcal{H}},$$

which means that

$$U(g)\left[\overline{\sigma(g, g^{-1})}U(g^{-1})\right] = I_{\mathcal{H}}.$$

Thus $\overline{\sigma(g, g^{-1})}U(g^{-1})$ is a right inverse of $U(g)$. Similarly we have that

$$U(g^{-1})U(g) = \sigma(g^{-1}, g)I_{\mathcal{H}},$$

which means that

$$\left[\overline{\sigma(g, g^{-1})}U(g^{-1})\right] U(g) = I_{\mathcal{H}}.$$

Hence $\overline{\sigma(g, g^{-1})}U(g^{-1})$ is a left inverse also. This means that $U(g)^{-1} = \overline{\sigma(g, g^{-1})}U(g^{-1})$, but since $U(g)$ is unitary by definition, $U(g)^* = U(g)^{-1}$ and the claim follows. ∎

The reason we are interested in this currently is that we can use the preceding lemma to find the adjoints of elements in $C_r^*(G, \sigma)$. This is possible due to the next result.

Proposition 2.3.7. *The twisted left regular representation λ_σ is a σ-projective unitary representation. In particular*

$$\lambda_\sigma(g)\lambda_\sigma(h) = \sigma(g, h)\lambda_\sigma(gh) \tag{2.1}$$

for all $g, h \in G$.

Proof. Since $\lambda_\sigma(g)$ is unitary for all $g \in G$ it remains to check Equation (2.1). Let us compare how each side of the equation acts on $\ell^2(G)$. Suppose that $\xi \in \ell^2(G)$ and $g, h, k \in G$. Then

$$\begin{aligned}(\lambda_\sigma(g)\lambda_\sigma(h)\xi)(k) &= \sigma(g, g^{-1}k)(\lambda_\sigma(g)\xi)(g^{-1}k)\\ &= \sigma(g, g^{-1}k)\sigma(h, h^{-1}g^{-1}k)\xi(h^{-1}g^{-1}k)\end{aligned}$$

by definition. We also have that

$$\begin{aligned}\sigma(g, h)(\lambda_\sigma(gh)\xi)(k) &= \sigma(g, h)\sigma(gh, (gh)^{-1}k)\xi((gh)^{-1}k)\\ &= \sigma(g, h)\sigma(gh, h^{-1}g^{-1}k)\xi(h^{-1}g^{-1}k).\end{aligned}$$

Setting $u = h^{-1}g^{-1}k$ and using that σ is a cocycle we get that

$$\begin{aligned}\sigma(g, h)\sigma(gh, h^{-1}g^{-1}k) &= \sigma(g, h)\sigma(gh, u) = \sigma(h, u)\sigma(g, hu)\\ &= \sigma(h, h^{-1}g^{-1}k)\sigma(g, hh^{-1}g^{-1}k)\\ &= \sigma(g, g^{-1}k)\sigma(h, h^{-1}g^{-1}k).\end{aligned}$$

Hence $\lambda_\sigma(g)\lambda_\sigma(h)$ and $\sigma(g, h)\lambda_\sigma(gh)$ act identically on $\ell^2(G)$, so we are done. ∎

Corollary 2.3.8. *It holds that*

$$\lambda_\sigma(g)^* = \overline{\sigma(g, g^{-1})}\lambda_\sigma(g^{-1}),$$

for all $g \in G$.

Proof. Follows immediately from Lemma 2.3.6 since λ_σ is a σ-projective unitary representation by Proposition 2.3.7. ∎

The reduced twisted group C^*-algebra

Having done some work, we get to define the main object of our study. Note that the following definition is very similar to the familiar case of Definition 2.3.1, excepting the 2-cocycle.

Definition 2.3.9. Let G be a group and σ a 2-cocycle. The *reduced twisted group C^*-algebra* of G is defined as

$$C_r^*(G,\sigma) = \overline{\operatorname{span}\{\lambda_\sigma(g) \mid g \in G\}}^{\|\cdot\|}, \tag{2.2}$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{B}(\ell^2(G))$, and λ_σ is the twisted left regular representation from Definition 2.3.4.

The preceding definition is important, as we will be preoccupied with these algebras throughout this thesis. Sometimes we will want to consider the span without the closure. We label this $\mathbb{C}(G,\sigma) = \operatorname{span}\{\lambda_\sigma(g) \mid g \in G\}$.

Example 2.3.10. Let σ be the *trivial 2-cocycle*, i.e. the 2-cocycle on G such that $\sigma(g,h) = 1$ for all $g,h \in G$. Then Equation (2.2) holds as usual, but

$$\left(\lambda_\sigma(g)\xi\right)(h) = \sigma(g,g^{-1}h)\xi(g^{-1}h) = \xi(g^{-1}h) = \left(\lambda(g)\xi\right)(h),$$

which means that $C_r^*(G,\sigma) = C_r^*(G)$ in this case. This shows that the reduced group C^*-algebra is a special case of the reduced twisted group C^*-algebra.

As mentioned, it is not apparent that $C_r^*(G,\sigma)$ is a C^*-algebra, so we have to check this.

Proposition 2.3.11. *$C_r^*(G,\sigma)$ is a C^*-algebra.*

Proof. We first show that (i) $\mathbb{C}(G,\sigma)$ is a $*$-subalgebra of $\mathcal{B}(\ell^2(G))$ and then (ii) that $C_r^*(G,\sigma)$ is a C^*-algebra.

(i) By definition, $\mathbb{C}(G,\sigma)$ is a subspace of $\mathcal{B}(\ell^2(G))$. We have to check that it is closed under multiplication and taking adjoints. Suppose that $x,y \in \mathbb{C}(G,\sigma)$. Then there exists some finite subset $F \subseteq G$ and constants $c_g, d_g \in \mathbb{C}, g \in F$ such that $x = \sum_{g\in F} c_g\lambda_\sigma(g)$ and $y = \sum_{h\in F} d_h\lambda_\sigma(h)$. Their product becomes

$$\begin{aligned} xy &= \left(\sum_{g\in F} c_g\lambda_\sigma(g)\right)\left(\sum_{h\in F} d_h\lambda_\sigma(h)\right) \\ &= \sum_{g,h\in F} c_g d_h \lambda_\sigma(g)\lambda_\sigma(h) \\ &= \sum_{g,h\in F} c_g d_h \sigma(g,h)\lambda_\sigma(gh) \end{aligned}$$

by Proposition 2.3.7. This is clearly in $\mathbb{C}(G,\sigma)$. We also calculate

$$x^* = \left(\sum_{g\in F} c_g\lambda_\sigma(g)\right)^* = \sum_{g\in F} \overline{c}_g \lambda_\sigma^*(g) = \sum_{g\in F} \overline{c}_g \overline{\sigma(g,g^{-1})}\lambda_\sigma(g^{-1})$$

by Corollary 2.3.8. This is also in $\mathbb{C}(G,\sigma)$, which is hence a $*$-subalgebra of $\mathcal{B}(\ell^2(G))$.

(ii) We have that $\mathbb{C}(G,\sigma)$ is a subalgebra of $\mathcal{B}(\ell^2(G))$, so we can equip it with the operator norm restricted to $\mathbb{C}(G,\sigma)$. Since it is also a $*$-algebra its closure in operator norm $C_r^*(G,\sigma)$ is a Banach $*$-algebra. The C^*-identity $\|a^*a\| = \|a\|^2$ holds for all $a \in C_r^*(G,\sigma))$ since $\mathcal{B}(\ell^2(G))$ is a C^*-algebra. Thus $C_r^*(G,\sigma)$ is also a C^*-algebra, as desired.

■

In Chapter 4 we will want to establish a $*$-isomorphism between some $C_r^*(G',\sigma')$ and the $N \times N$-matrices. To do so we will need a trace on $C_r^*(G',\sigma')$ to show that our candidate for the $*$-isomorphism is injective. Luckily there is always a natural choice of trace for each $C_r^*(G,\sigma)$, given by $\tau(x) = \langle x\delta, \delta\rangle$. Before we prove that τ is in fact a trace, we show the following useful property.

Lemma 2.3.12. *For each $g \neq e$ in G we have that $\tau(\lambda_\sigma(g)) = 0$, while $\tau(\lambda_\sigma(e)) = 1$.*

Proof. By the definition of τ we have that $\tau(\lambda_\sigma(g)) = \langle \lambda_\sigma(g)\delta, \delta\rangle$. Furthermore

$$(\lambda_\sigma(g)\delta)\,(h) = \sigma(g, g^{-1}h)\delta(g^{-1}h)$$

for all $g, h \in G$. By definition $\delta(g^{-1}h)$ is equal to 1 whenever $g^{-1}h = e$, and zero otherwise. This corresponds to $h = g$, which means that $\delta(g^{-1}h) = \delta_g(h)$. In this case

$$\sigma(g, g^{-1}h) = \sigma(g, e) = 1$$

also, as σ is normalised. Hence $\lambda_\sigma(g)\delta = \delta_g$, which means that

$$\tau(\lambda_\sigma(g)) = \langle \lambda_\sigma(g)\delta, \delta\rangle = \langle \delta_g, \delta\rangle.$$

As we saw in Proposition 2.2.2, this is an inner product of orthonormal elements. Hence the claim follows. ■

Now we have the tools to show that τ is a trace on $C_r^*(G,\sigma)$. In our upcoming proof of the $*$-isomorphism, it will not be enough that τ is a tracial state. There we want to use τ to make a norm on $C_r^*(G',\sigma')$, and to do so it is also necessary that τ is faithful.

Proposition 2.3.13. *There is a faithful tracial state τ on $C_r^*(G,\sigma)$ given by $\tau(x) = \langle x\delta, \delta\rangle$. We call this the canonical trace on $C_r^*(G,\sigma)$.*

Proof. It is well known that τ is a state, see [Dav96] . We have to show that (i) that τ is tracial and (ii) that τ is faithful.

(i) First we show that τ is tracial. This means that $\tau(xy) = \tau(yx)$ for all $x, y \in C_r^*(G,\sigma)$. First suppose that $g, h \in G$. By Proposition 2.3.7 we have that

$$\tau\left(\lambda_\sigma(g)\lambda_\sigma(h)\right) = \tau\left(\sigma(g,h)\lambda_\sigma(gh)\right) = \sigma(g,h)\langle \lambda_\sigma(gh)\delta, \delta\rangle.$$

Furthermore,

$$\langle \lambda_\sigma(gh)\delta, \delta\rangle = \sum_{k\in G} \left(\lambda_\sigma(gh)\delta\right)(k)\ \overline{\delta(k)},$$

but the only non-zero part of this sum corresponds to $k = e$, so it is equal to

$$(\lambda_\sigma(gh)\delta)(e)\,\overline{\delta(e)} = \sigma(gh, h^{-1}g^{-1}gh)\delta(gh),$$

which is non-zero only if $gh = e$, in which case it is equal to one. In this case we have $h = g^{-1}$. Hence we get that

$$\tau(\lambda_\sigma(g)\lambda_\sigma(h)) = \tau\left(\lambda_\sigma(g)\lambda_\sigma(g^{-1})\right) = \sigma(g, g^{-1})1.$$

Similarly

$$\tau(\lambda_\sigma(h)\lambda_\sigma(g)) = \sigma(g^{-1}, g),$$

but $\sigma(g^{-1}, g) = \sigma(g, g^{-1})$ by Lemma 2.3.3 since σ is normalised. Thus

$$\tau(\lambda_\sigma(g)\lambda_\sigma(h)) = \tau(\lambda_\sigma(h)\lambda_\sigma(g)) \tag{2.3}$$

for all $g, h \in G$.

Now suppose that $x, y \in \mathbb{C}(G, \sigma)$. By linearity and Equation (2.3) we get that $\tau(xy) = \tau(yx)$, because x and y are finite linear combinations of elements in $\mathbb{C}(G, \sigma)$.

Finally suppose that $x, y \in C_r^*(G, \sigma)$. Since $\mathbb{C}(G, \sigma)$ is norm dense in $C_r^*(G, \sigma)$ there exist sequences $\{x_i\}_i, \{y_i\}_i$ in $\mathbb{C}(G, \sigma)$ converging to x and y respectively. Because multiplication is continuous this means that $\{x_i y_i\}_i$ is a sequence converging to xy. Hence, by the continuity of τ,

$$\tau(xy) = \tau\left(\lim_i x_i y_i\right) = \lim_i \tau(x_i y_i) = \lim_i \tau(y_i x_i) = \tau(yx)$$

by the above argument, as each x_i and y_i is in $\mathbb{C}(G, \sigma)$. Thus τ is tracial.

(ii) Next we check that τ is faithful. Recall that this means that if $\tau(x^*x) = 0$, then necessarily $x = 0$. Suppose, then, that $\tau(x^*x) = 0$ for some $x \in C_r^*(G, \sigma)$. This means that $\langle x\delta, x\delta\rangle = 0$, which again means that $x\delta = 0$. Let $a, b \in G$ and consider

$$\begin{aligned}\langle x\delta_a, \delta_b\rangle &= \langle x\lambda_\sigma(a)\delta, \lambda_\sigma(b)\delta\rangle = \langle \lambda_\sigma(b)^* x\lambda_\sigma(a)\delta, \delta\rangle \\ &= \tau(\lambda_\sigma(b)^* x\lambda_\sigma(a)) = \tau(\lambda_\sigma(a)\lambda_\sigma(b)^* x)\end{aligned}$$

since τ is tracial. Furthermore

$$\tau(\lambda_\sigma(a)\lambda_\sigma(b)^* x) = \langle \lambda_\sigma(a)\lambda_\sigma(b)^* x\delta, \delta\rangle = 0$$

since $x\delta = 0$. Since this means that $\langle x\delta_a, \delta_b\rangle = 0$ for all $a, b \in G$, we get that $x = 0$ and τ is faithful.

■

We will make more use of τ later, but is also helpful straight away. Earlier we defined $C_r^*(G, \sigma) = \overline{\operatorname{span}\{\lambda_\sigma(g) \mid g \in G\}}^{\|\cdot\|}$, but could it be that some of the $\lambda_\sigma(g)$'s are redundant, and we could make do with less? The following result tells us that the answer is no.

Lemma 2.3.14. *The elements of $\{\lambda_\sigma(g) \mid g \in G\}$ are linearly independent.*

Proof. Let F be a finite subset of G and write $x = \sum_{h \in F} c_h \lambda_\sigma(h)$. We claim that $c_g = \tau(x\lambda_\sigma(g)^*)$ for all $g \in F$. Since τ is a tracial state we get that

$$\begin{aligned}\tau(x\lambda_\sigma(g)^*) &= \tau(\lambda_\sigma(g)^* x) = \langle \lambda_\sigma(g)^* x\delta, \delta\rangle \\ &= \langle x\delta, \lambda_\sigma(g)\delta\rangle = \langle x\delta_e, \delta_g\rangle \\ &= \langle \sum_{h\in F} c_h \lambda_\sigma(h)\delta_e, \delta_g\rangle = \langle \sum_{h \in F} c_h \delta_h, \delta_g\rangle \\ &= c_g\end{aligned}$$

since $g \in F$ and because the δ_g's are an orthonormal basis for $\ell^2(G)$. If $g \notin F$ we get 0 instead. This means that the coefficients c_g are uniquely determined by x, which means that $\{\lambda_\sigma(g) \mid g \in G\}$ is linearly independent. ∎

Initially we will be interested in reduced twisted group C^*-algebras arising from finite groups. For these simpler cases this result gives us a basis for the algebra.

Similar 2-cocycles

Suppose that we have two distinct 2-cocycles σ and σ' on some group G. Are then, informally speaking, $C^*_r(G, \sigma)$ and $C^*_r(G, \sigma')$ necessarily distinct? The following is taken from [BC06], and the answer is no.

If $b : G \to \mathbb{T}$ is a map with $b(e) = 1$, we get a 2-cocycle db on G given by

$$db(g, h) = b(g)b(h)\overline{b(gh)}.$$

If σ and σ' are 2-cocycles on G, we say that σ' is *similar* to σ when there exists b such that $\sigma' = (db)\sigma$.

Proposition 2.3.15. *Let σ and σ' be similar 2-cocycles on G. Then $C^*_r(G, \sigma) \simeq C^*_r(G, \sigma')$.*

We will not work with similar 2-cocycles directly, but rather cite some other source to establish classes of similar 2-cocycles such that we get to employ Proposition 2.3.15.

Bicharacters and 2-cocycles

We round out this discussion on the reduced twisted group C^*-algebra with an examination on the relationship between 2-cocycles and bicharacters. The reason that we do this is that it will give us an easier way of checking that a function is a 2-cocycle.

Definition 2.3.16. A *character* on a group G is a group homomorphism $G \to \mathbb{T}$.

Definition 2.3.17. A map $\phi : G \times G \to \mathbb{T}$ is called a *bicharacter* if the map $g \mapsto \phi(g, h)$ is a character for all $h \in G$, and similarly the map $g \mapsto \phi(h, g)$ is a character for all $h \in G$.

Proposition 2.3.18. *Any bicharacter on a group G is a normalised 2-cocycle on G.*

Proof. Let ω be a bicharacter on G. For any $g, h, k \in G$, we have that

$$\omega(g,h)\omega(gh,k) = \omega(g,h)\omega(g,k)\omega(h,k) \text{ and}$$
$$\omega(h,k)\omega(g,hk) = \omega(h,k)\omega(g,h)\omega(g,k)$$

because ω is a bicharacter. These are equal, hence ω is a 2-cocycle. Furthermore $\omega(g,e) = \omega(g,ee) = \omega(g,e)\omega(g,e)$ for any $g \in G$. Since $\omega(g,e) \in \mathbb{T}$ this means that $\omega(g,e) = 1$. Similarily $\omega(e,g) = 1$ for any $g \in G$. Thus ω is a normalised 2-cocycle. ■

2.4 Quantum channels

The two following definitions are standard, see [Wat18] and [Bla06].

Definition 2.4.1. Suppose that $\varphi : A \to B$ is a linear map between C^*-algebras. Then φ is called *completely positive* if, for any $k \in \mathbb{N}$ and positive matrix (in the sense of Theorem 2.1.4)

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \in M_k(A),$$

the matrix

$$\begin{bmatrix} \varphi(a_{11}) & \cdots & \varphi(a_{1k}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{k1}) & \cdots & \varphi(a_{kk}) \end{bmatrix} \in M_k(B),$$

is also positive.

For the work we do in this thesis, demanding that quantum channels are completely positive is not strictly necessary. In our applications mere positivity would suffice, but we stick to the standard definition above.

Definition 2.4.2. Consider a pair (A, τ), where A is a unital C^*-algebra and τ a faithful tracial positive linear functional on A. Then a *quantum channel* on (A, τ) is a τ-preserving, completely positive linear map Q from A into itself.

Suppose that we have two isomorphic C^*-algebras and a quantum channel on one of them. Loosely speaking, can we then induce a quantum channel on the other? This problem will arise often in this thesis, and the answer is luckily yes:

Proposition 2.4.3. *Let (A, τ) and B be a unital C^*-algebras, where A has a faithful tracial positive linear functional τ. Suppose that Φ is a $*$-isomorphism from A to B and Q is a quantum channel on (A, τ). Then $\tilde{Q} = \Phi Q \Phi^{-1}$ is a quantum channel on $(B, \tau \circ \Phi^{-1})$.*

Proof. We have to show that $\tilde{Q} = \Phi Q \Phi^{-1}$ is completely positive and preserves $\tau \circ \Phi^{-1}$. Let $b \in B$. The latter part holds because

$$\begin{aligned} \tau \circ \Phi^{-1}\left(\tilde{Q}(b)\right) &= \tau \Phi^{-1} \Phi Q \Phi^{-1}(b) \\ &= \tau\left(Q\Phi^{-1}(b)\right) \\ &= \tau \circ \Phi^{-1}(b), \end{aligned}$$

where the last equality holds since Q is a quantum channel on (A, τ), hence τ-preserving. Thus $\tilde{Q}$ is $\tau \circ \Phi^{-1}$-preserving.

It remains to show that $\tilde{Q}$ is completely positive. We show in detail that $\tilde{Q}$ is positive, and then sketch complete positivity. Let $b_1 \in B$ be a positive element, i.e. $b_1 = b_2^* b_2$ for some $b_2 \in B$. Then

$$\tilde{Q}(b_1) = \tilde{Q}(b_2^* b_2) = \Phi Q \Phi^{-1}(b_2^* b_2) = \Phi Q \left(\Phi^{-1}(b_2)^* \Phi^{-1}(b_2)\right)$$

since Φ is an $*$-homomorphism. Write $c = \Phi^{-1}(b_2)^* \Phi^{-1}(b_2)$. Clearly, c is a positive element of A, which means that Qc is positive, since Q is positive. This means that there exists some $a \in A$ such that $Qc = a^* a$. Hence

$$\Phi Q c = \Phi(a^* a) = \Phi(a)^* \Phi(a),$$

which is a positive element of B. Hence $\tilde{Q}$ is positive.

By [Bla06] [page 152, II.6.9.3(i)], any $*$-homomorphism is completely positive. Hence Φ is completely positive. Then $\tilde{Q}$ is completely positive as a composition of completely positive maps. ■

Quantum channels on the reduced twisted group algebra

In accordance with the project description we want to study a certain class of quantum channels, which we discuss here.

Definition 2.4.4. [Dav96] A function φ on a group G (into $\mathbb{C}$) is called *positive definite* if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \varphi\left(s_j^{-1} s_i\right) \geq 0 \tag{2.4}$$

for all $n \geq 1, \alpha_i \in \mathbb{C}, s_i \in G$, where $1 \leq i \leq n$. Furthermore we call φ *normalised* if $\varphi(e) = 1$.

We will apply the following result frequently. It was first shown in the non-twisted case, i.e. $\sigma = 1$, by U. Haagerup.

Proposition 2.4.5 ([BC09], or [BC06] for more details)**.** *Let φ be a normalised positive definite function on G, and σ a 2-cocycle. Then there is a quantum channel Q_φ on $(C_r^*(G, \sigma), \tau)$ satisfying*

$$Q_\varphi(\lambda_\sigma(g)) = \varphi(g)\lambda_\sigma(g) \text{ for all } g \in G, \tag{2.5}$$

where τ is the canonical trace.

Corollary 2.4.6. *Suppose that B is a $\mathbb{C}^*$-algebra and $\Phi : (C_r^*(G, \sigma) \to B$ is a $*$-isomorphism. Further assume that φ be a normalised positive definite function on G, and σ a 2-cocycle. Then*

$$\tilde{Q}_\varphi = \Phi Q_\varphi \Phi^{-1}$$

is a quantum channel on $(B, \tau \circ \Phi^{-1})$.

Proof. Since Q_φ is a quantum channel on $C_r^*(G, \sigma)$ by Proposition 2.4.5, then $\tilde{Q}_\varphi$ is a quantum channel on $(B, \tau \circ \Phi^{-1})$ by Proposition 2.4.3. ■

Later we will want to study these quantum channels in four cases. It turns out that the definition of positive definiteness is not the most useful for describing the Q_φ's in these situations. We therefore give an alternate characterisation of them by Bochner's theorem.

Recall that if G is a locally compact abelian group, the set of all continuous characters on G can be made into a locally compact abelian group. This group is called the *dual group* of G and is denoted $\widehat{G}$. When $g \in G$ and $\gamma \in \widehat{G}$, we write $\langle g, \gamma \rangle \coloneqq \gamma(g)$.

Definition 2.4.7. [[MW13], adapted to our situation.] Let Ω be a locally compact Hausdorff space. A Borel measure μ is said to be *regular* if for each Borel set B and $\epsilon > 0$, there is a compact set K and an open set O such that $K \subset B \subset O$ and $\mu\left(O \backslash K\right) < \epsilon$.

Theorem 2.4.8 (Bochner's theorem, [Loo53], page 142, section 36A)**.** *Let G be a discrete abelian group and φ a function on G. Then φ is positive definite if and only if there exists a regular Borel measure μ on $\widehat{G}$ such that*

$$\varphi(g) = \int_{\widehat{G}} \langle g, \gamma \rangle d\mu(\gamma).$$

Weyl-covariant channels

In Chapter 4 we will want to compare the quantum channels arising from Proposition 2.4.5 with the Weyl-covariant channels on $M_N(\mathbb{C})$. This section is about the latter, and our discussion is taken from section 4.1.2 in [Wat18], but the notation is adapted to our purposes.

Let $N \in \mathbb{N}$ and recall that $\mathbb{Z}_N$ is the group $\{0, 1, 2, \ldots, N-1\}$ with addition modulo N, which we denote by $\dot{+}$. For each pair $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$ we also let $E_{i,j}$ be the linear operator on $\mathbb{C}^N$ given by $E_{i,j}(\boldsymbol{e}_k) = \boldsymbol{e}_i$ if $k = j$, and $\mathbf{0}$ otherwise.

Definition 2.4.9. Let $N \in \mathbb{N}$, and write $\zeta = e^{i\frac{2\pi}{N}}$ for notation. The *discrete Weyl operators* is the collection of linear operators on $\mathbb{C}^N$ on the form

$$W_{m_1,m_2} = \sum_{k \in \mathbb{Z}_N} \zeta^{m_1 k} E_{m_2 \dot{+} k, k},$$

where $(m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$.

The discrete Weyl operators are unitary. We denote the standard trace on $M_N(\mathbb{C})$ by Tr.

Definition 2.4.10. A linear map ϕ from $(M_N(\mathbb{C}), \mathrm{Tr})$ to itself is called a *Weyl-covariant map* if

$$\phi\left(W_{m_1,m_2} X W_{m_1,m_2}^*\right) = W_{m_1,m_2} \phi\left(X\right) W_{m_1,m_2}^*$$

for each $X \in M_N(\mathbb{C})$ and $(m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$. If ϕ is also a quantum channel it is called a *Weyl-covariant channel.*

In some spots this definition won't give the most useful characterisation of Weyl-covariant channels. The next two results will be used for checking whether a map ϕ is Weyl-covariant, and characterising a Weyl-covariant channel.

Theorem 2.4.11. *Let ϕ be a linear map from $(M_N(\mathbb{C}), \mathrm{Tr})$ to itself. Then ϕ is a Weyl-covariant map if there exist coefficients $\alpha_{m_1,m_2} \in \mathbb{C}$, where $(m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$, such that*

$$\phi(W_{m_1,m_2}) = \alpha_{m_1,m_2} W_{m_1,m_2}$$

for all $(m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$.

By a *probability vector p* over a group G we mean a function $p : G \to [0, 1]$ such that $\sum_{g \in G} p(g) = 1$.

Proposition 2.4.12. *Let ϕ be a Weyl-covariant channel. Then there exists a probability vector p over $\mathbb{Z}_N \times \mathbb{Z}_N$ such that*

$$\phi(A) = \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1, k_2) W_{k_1,k_2} A W^*_{k_1,k_2}$$

for all $A \in M_N(\mathbb{C})$.

2.5 Contractive channels

The discussion in this section is taken from [FR17].

A density matrix is a positive semidefinite matrix D whose trace is 1. In other words, the space of density matrices over $\mathbb{C}^N$ is given by

$$\{D \in M_N(\mathbb{C}) \mid D = B^*B \text{ for some } B \in M_N(\mathbb{C}), \mathrm{Tr}(D) = 1\}.$$

These matrices are important in mathematical physics and quantum theory. They are a quantum mechanical analogue to phase-space probability measures in classical statistical mechanics. In quantum mechanics, they are frequently used for describing mixed quantum states. In mathematics they are used to describe the state space of $M_N(\mathbb{C})$. A useful property is that quantum channels map density matrices to density matrices.

We mention this here because we are interested in the following generalisation of density matrices.

Definition 2.5.1. Suppose that A is a unital C^*-algebra, and τ is a faithful tracial state on A. We then let $\mathcal{D}_\tau(A)$ denote the *τ-density space of A*, which we define by

$$\mathcal{D}_\tau(A) = \{a \in A^+ \mid \tau(a) = 1\}.$$

Proposition 2.5.2. *The τ-density space of A is both convex and closed in norm.*

Proposition 2.5.3. *If Q is a quantum channel on (A, τ), then*

$$Q(\mathcal{D}_\tau(A)) \subseteq \mathcal{D}_\tau(A).$$

Proof. Since Q is a quantum channel on (A, τ) it is τ-preserving. In addition, the complete positivity of Q implies positivity of Q, hence Q maps A^+ into A^+. ∎

Later we will study whether or not certain quantum channels on $\mathcal{D}_\tau(A)$ are contractions. By this we mean that we are interested in studying when each of the following conditions are satisfied for a given quantum channel.

Definition 2.5.4. Suppose that $f : X \to X$ is a map on a metric space (X, d). Then

1. f is *nonexpansive*, if $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$, for all $x_1, x_2 \in X$.
2. f is *locally contractive*, if $d(f(x_1), f(x_2)) < d(x_1, x_2)$, for all distinct $x_1, x_2 \in X$.
3. f is *strictly contractive*, if there exists a constant $0 \leq C < 1$ such that $d(f(x_1), f(x_2)) \leq Cd(x_1, x_2)$, for all $x_1, x_2 \in X$. The number C is called the *contraction constant* of f.

We sometimes refer to whether or not a function f is simply a *contraction*. This is a vague reference to the above conditions that we use when we do not wish to be precise.

For this to make sense in our case, we will have to specify some metric d on $\mathcal{D}_\tau(A)$. There are multiple possible metrics on $\mathcal{D}_\tau(A)$. We will use the upcoming Bures metric and trace-metric.

The Bures metric

Definition 2.5.5. We define the *Bures distance* d_B^τ on $\mathcal{D}_\tau(A)$ by

$$d_B^\tau(a, b) = \sqrt{1 - \tau(|a^{\frac{1}{2}} b^{\frac{1}{2}}|)},$$

where $a, b \in \mathcal{D}_\tau(A)$.

Theorem 2.5.6. *The function d_B^τ is a metric on $\mathcal{D}_\tau(A)$, for every unital C^*-algebra A and faithful trace τ on A.*

Hence we refer to d_B^τ as the Bures metric instead. While we are concerned that d_B^τ is a metric on $\mathcal{D}_\tau(A)$, it is more generally also a metric on A^+. Having settled on a metric, let us return to contractivity.

Proposition 2.5.7. *If $\mathcal{E} : A \to A$ is a quantum channel, then the function*

$$f_\mathcal{E} : \mathcal{D}_\tau(A) \to \mathcal{D}_\tau(A)$$

defined by $f_\mathcal{E}(a) = \mathcal{E}(a)$, for $a \in \mathcal{D}_\tau(A)$, is a nonexpansive continuous affine function on $(\mathcal{D}_\tau(A), d_B^\tau)$.

This, in particular, means that if we restrict a quantum channel to $\mathcal{D}_\tau(A)$, then it is nonexpansive. The question then becomes whether a given channel is subject to stricter contractivity properties.

The authors of [FR17] refer to a channel $\mathcal{E} : A \to A$ as a *Bures contraction* if $f_\mathcal{E}$ is a locally contractive map on the metric space $(\mathcal{D}_\tau(A), d_B^\tau)$. We will not use this terminology.

Let us consider an example of a contraction.

Example 2.5.8. The *completely depolarising channel* $\Omega : A \to A$ defined by

$$\Omega(x) = \frac{\tau(x)}{\tau(1)} 1,$$

for $x \in A$, is a quantum channel that is strictly contractive w.r.t $d_\mathcal{B}^\tau$.

Proof. The map Ω is trace preserving since

$$\tau(\Omega(x)) = \frac{\tau(x)}{\tau(1)}\tau(1) = \tau(x).$$

We do not show complete positivity, but note that

$$\Omega\left(\mathcal{D}_\tau(A)\right) = \{\frac{\tau(x)}{\tau(1)}1\} = \{\frac{1}{\tau(1)}\}1$$

since $\tau(x) = 1$ for all $x \in \mathcal{D}_\tau(A)$. The image $\Omega\left(\mathcal{D}_\tau(A)\right)$ is thus a one point set, implying that $d^\tau_B(\Omega(a), \Omega(b)) = 0$ for all $a, b \in \mathcal{D}_\tau(A)$. Hence Ω is strictly contractive w.r.t. $d^\tau_{\mathcal{B}}$. ■

There is an additional point of interest here, specific to our setting. Suppose that $A = C^*_r(G, \sigma)$, for some group G and 2-cocycle σ, equipped with the canonical trace τ. Furthermore, let $\varphi_0 = G \to \mathbb{C}$ be the positive definite function given by

$$\varphi_0(g) = \begin{cases} 1 \text{ for } g = e, \\ 0 \text{ otherwise,} \end{cases}$$

for all $g \in G$. Then $\Omega = Q_{\varphi_0}$, where Q_{φ_0} is the quantum channel on (A, τ) from Proposition 2.4.5.

We mention the following result.

Proposition 2.5.9. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are quantum channels on A, and if at least one of them is locally contractive w.r.t. d^τ_B, then so is $t\mathcal{E}_1 + (1-t)\mathcal{E}_2$, for every $t \in (0, 1)$.*

Example 2.5.10. (Continued) Let us consider this result in the setting of Example 2.5.8, letting φ be a normalised positive definite function on G. Then Q_φ is a quantum channel in the same way as before, and

$$Q^t = (1-t)Q_\varphi + tQ_{\varphi_0}$$

is locally contractive w.r.t. d^τ_B for each $t \in (0, 1)$. As φ is normalised,

$$Q_\varphi(\lambda_\sigma(0)) = Q_\varphi(\lambda_\sigma(0)) = \varphi(0)\lambda_\sigma(0) = \lambda_\sigma(0).$$

Hence, choosing t close to 0, Q^t agrees with Q_φ on the identity and is perturbed arbitrarily little elsewhere, while becoming locally contractive w.r.t. d^τ_B. Furthermore, $(1-t)\varphi + t\varphi_0$ is normalised positive definite, and $Q^t = Q_{(1-t)\varphi + t\varphi_0}$. The last equality holds because, for $g \in G$,

$$\begin{aligned} Q^t(g) &= (1-t)Q_\varphi(g) + tQ_{\varphi_0}(g) \\ &= (1-t)\varphi(g)\lambda_\sigma(g) + t\varphi_0(g)\lambda_\sigma(g), \end{aligned}$$

and

$$\begin{aligned} Q_{(1-t)\varphi + t\varphi_0}(g) &= \left((1-t)\varphi + t\varphi_0\right)(g)\lambda_\sigma(g) \\ &= (1-t)\varphi(g)\lambda_\sigma(g) + t\varphi_0(g)\lambda_\sigma(g) \end{aligned}$$

by Equation (2.5).

The trace metric

We turn to the other metric of interest. Let (A, τ) be a unital C^*-algebra and τ a faithful tracial state on A. The *trace norm* $\|\cdot\|_1$ on A is defined

$$\|a\|_1 = \tau(|a|),$$

for all $a \in A$. It gives rise to the following metric in the usual manner.

Definition 2.5.11. The *trace metric* d_1 on (A, τ) is given by

$$d_1(a, b) = \|a - b\|_1$$

for all $a, b \in A$.

This implies that $d_1(a, b) = \tau(|a - b|)$ for all $a, b \in A$.
Raginsky shows a variant of Proposition 2.5.9 for d_1, i.e.

Proposition 2.5.12 ([Rag02](page 6))**.** *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are quantum channels on A, and if at least one of them is strictly contractive w.r.t. d_1, then so is $t\mathcal{E}_1 + (1-t)\mathcal{E}_2$, for every $t \in (0, 1)$.*

This gives that $Q^t = Q_{(1-t)\varphi + t\varphi_0}$ is a strict contraction w.r.t. d_1 for any normalised positive definite φ and $t \in (0, 1)$, analogously to Example 2.5.10.

In Chapter 1 we considered the attractor property in Definition 1.0.1. Suppose that a map T on $\mathcal{D}_\tau(A)$ is a contraction w.r.t d_B^τ. This does not necessarily mean that it is a contraction w.r.t. d_B^τ, and vice versa. On the other hand, what about the attractor property?

Proposition 2.5.13. *The metric spaces $(\mathcal{D}_\tau(A), d_1)$ and $(\mathcal{D}_\tau(A), d_B^\tau)$ are homeomorphic.*

Since homeomorphisms preserve topological properties, this means that for some $x_0 \in \mathcal{D}_\tau(A)$, $T^n(x) \to x_0$ as $n \to \infty$ w.r.t. d_1 if and only if the same holds w.r.t. d_B^τ. In other words, whether or not T has the attractor property is independent of our choosing d_1 or d_B^τ.

In our upcoming examples, we will study the contractivity properties of the quantum channels Q_φ from Proposition 2.4.5 on $\mathcal{D}_\tau(A)$, where $A = C_r^*(G, \sigma)$ and τ is the canonical trace from Proposition 2.3.13 on $C_r^*(G, \sigma)$.

CHAPTER 3

The first example

In this chapter we consider the reduced twisted group C^*-algebra of $G = \mathbb{Z}_N, N \in \mathbb{N}$ and related quantum channels, including those arising from Corollary 2.4.6. Finally we study the contractive properties of these channels

All 2-cocycles on $\mathbb{Z}_N$ are similar to the trivial 2-cocycle by [Kle65][Theorem 7.1]. By Proposition 2.3.15 this means that $C_r^*(\mathbb{Z}_N, \sigma) \simeq C_r^*(\mathbb{Z}_N, 1)$ regardless of our choice of 2-cocycle σ. As we saw in Example 2.3.10 this further means that $C_r^*(\mathbb{Z}_N, \sigma) \simeq C_r^*(\mathbb{Z}_N)$. Hence we need only consider the reduced group C^*-algebra $C_r^*(\mathbb{Z}_N)$ in this chapter, and it will thus not be necessary to work with 2-cocycles.

It is possible to use Gelfand theory to see directly that $C_r^*(\mathbb{Z}_N)$ is $*$-isomorphic to $\mathbb{C}^N$. We will describe an explicit isomorphism Φ to apply Corollary 2.4.6 to get quantum channels on $\mathbb{C}^N$.

3.1 The isomorphism $C_r^*(\mathbb{Z}_N) \simeq \mathbb{C}^N$

The purpose of this section is to exhibit an explicit $*$-isomorphism Φ from $C_r^*(\mathbb{Z}_N)$ to $\mathbb{C}^N$. Our strategy in this regard is to split the task into three steps, where each step is a $*$-isomorphism that takes us a part of the way. In the end we compose these to arrive at Φ.

Let I be the $N \times N$ identity matrix and

$$S = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

or in other words, $S = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{N-1} & \boldsymbol{e}_0 \end{bmatrix}$, where $\boldsymbol{e}_0 = (1, 0, \ldots, 0)^T$. Note that S is unitary. We let $C^*(I, S)$ denote the C^*-subalgebra of $M_N(\mathbb{C})$ generated by I and S.

Lemma 3.1.1. *The linear map which is determined by $\lambda(m) \mapsto S^m$ is a $*$-isomorphism from $C_r^*(\mathbb{Z}_N)$ to $C^*(I, S)$. We write $C_r^*(\mathbb{Z}_N) = C^*(I, S)$.*

Proof. The map is well defined as it is linear and defined on a basis by Lemma 2.3.14 and $C_r^*(\mathbb{Z}_N)$ being finite dimensional. Also, it is clearly a homomorphism and it is $*$-preserving since $(S^m)^* = S^{-m}$, while $\lambda(m)^* = \lambda(-m)$ by

Corollary 2.3.8 since σ is trivial by assumption. It is surjective as it is onto both generating elements.

Further, let $x, y \in C_r^*(\mathbb{Z}_N)$ with $x \neq y$. Then there exists constants c_k, d_k for $k \in \mathbb{Z}_N$ such that

$$x = \sum_{k \in \mathbb{Z}_N} c_k \lambda(k) \mapsto \sum_{k \in \mathbb{Z}_N} c_k S^k$$

and

$$y = \sum_{k \in \mathbb{Z}_N} d_k \lambda(k) \mapsto \sum_{k \in \mathbb{Z}_N} d_k S^k.$$

Injectivity of the maps follows since the S^k's are linearly independent. ■

Next we note that since S is unitary, it is in particular normal. Hence we can apply the spectral theorem for normal matrices. This means that S can be diagonalised by some unitary matrix W. To determine the factorisation we need to find the eigenvalues of S.

Lemma 3.1.2. *The eigenvalues of S are $\{e^{i\frac{2\pi}{N}k} \mid k \in \mathbb{Z}_N\}$.*

Proof. We calculate the eigenvalues by solving the characteristic equation $\det(\lambda I - S) = 0$ for λ. Clearly,

$$\det(\lambda I - S) = \begin{bmatrix} \lambda & 0 & \cdots & 0 & -1 \\ -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{bmatrix}$$

$$= \lambda \det \begin{bmatrix} \lambda & 0 & \cdots & 0 & 0 \\ -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \end{bmatrix} \pm \det \begin{bmatrix} -1 & \lambda & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

by expanding the determinant along the top row. The first term is λ^N. If N is even, the $\pm$ above is a $+$, but the determinant is -1 as there are $N-1$ terms along the diagonal. On the other hand, if N is odd, the $\pm$ becomes $-$, but the determinant is $+1$. Either way we have

$$\det(\lambda I - S) = \lambda^N - 1 = 0,$$

yielding the result. ■

The eigenvectors of S are the vectors on the form

$$\boldsymbol{b}_m = \left(1, e^{i\frac{2\pi}{N}m}, \ldots, e^{i\frac{2\pi}{N}m(N-1)}\right)$$

for $m \in \mathbb{Z}_N$. Hence we can write

$$S = W \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}(N-1)} \end{bmatrix} W^*, \tag{3.1}$$

where W is the unitary matrix containing the normalised eigenvectors of S, i.e.

$$W = \frac{1}{\sqrt{N}} \begin{bmatrix} \boldsymbol{b}_0^T & \boldsymbol{b}_1^T & \cdots & \boldsymbol{b}_{N-1}^T \end{bmatrix}.$$

Using this diagonalisation of S we can show that $C^*(I, S)$ is isomorphic to the diagonal $N \times N$-matrices on $\mathbb{C}$. Let us denote these matrices by $MD_N(\mathbb{C})$.

Lemma 3.1.3. *The map $\psi : C^*(I, S) \to MD_N(\mathbb{C})$ defined by*

$$x \mapsto W^* x W$$

for all $x \in C^*(I, S)$ is a $*$-isomorphism.

Proof. Let $x, y \in C^*(I, S)$. First we show that ψ takes values in the diagonal matrices. Let $x \in C^*(I, S)$. We can then write

$$x = \sum_{k=0}^{N-1} \alpha_k S^k$$

for some $\alpha_k \in \mathbb{C}, k \in \mathbb{Z}_N$. Then

$$\begin{aligned}
\psi(x) &= W^* x W \\
&= W^* \left(\sum_{k=0}^{N-1} \alpha_k S^k \right) W \\
&= \sum_{k=0}^{N-1} \alpha_k W^* S^k W \\
&= \sum_{k=0}^{N-1} \alpha_k W^* \left(W \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}(N-1)} \end{bmatrix} W^* \right)^k W \\
&= \sum_{k=0}^{N-1} \alpha_k \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}k} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}k(N-1)} \end{bmatrix},
\end{aligned}$$

which is a diagonal matrix. Next, ψ is linear because

$$\psi(\alpha x + y) = W^*(\alpha x + y)W = \alpha W^* x W + W^* y W = \alpha\psi(x) + \psi(y)$$

for all $\alpha \in \mathbb{C}$. Moreover, the map is a $*$-homomorphism since

$$xy \mapsto W^* xy W = (W^* x W)(W^* y W),$$

and

$$x^* \mapsto W^* x^* W = (W^* x W)^*$$

as W is unitary.

$$\begin{tikzcd} C_r^*(\mathbb{Z}_N) \arrow[r, "="] & C^*(I,S) \arrow[r, "\simeq"] \arrow[rr, bend right, "\Phi"'] & MD_N(\mathbb{C}) \arrow[r, "\simeq"] & \mathbb{C}^N \end{tikzcd}$$

Figure 3.1: The $*$-isomorphism Φ

We see that ψ is injective as follows. Suppose $x \neq y$ for $x, y \in C^*(I,S)$. Then

$$\psi(x) - \psi(y) = W^*xW - W^*yW = W^*(x-y)W \neq 0$$

because $x - y \neq 0$ and W is unitary.

In addition, we have that $\dim(C^*(I,S)) = \dim(C_r^*(\mathbb{Z}_N)) = N$, due to Lemma 2.3.14. Since ψ is injective its rank is N, which means that ψ is surjective by the rank-nullity theorem. ■

The aforementioned diagonal matrices are again $*$-isomorphic $\mathbb{C}^N$ through the map

$$D \mapsto \sum_{m \in \mathbb{Z}_N} D\boldsymbol{e}_m. \tag{3.2}$$

In other words, D is sent to the vector of its eigenvalues. The composition of the three isomorphisms discussed up to this point finally gives us the desired isomorphism from $C_r^*(\mathbb{Z}_N)$ to $\mathbb{C}^N$. See Figure 3.1.

Theorem 3.1.4. *The linear map Φ from $C_r^*(\mathbb{Z}_N) = C^*(I,S)$ to $\mathbb{C}^N$ determined by*

$$\Phi(S^m) = \boldsymbol{b}_m \tag{3.3}$$

for all $m \in \mathbb{Z}_N$ is a $*$-isomorphism.

Proof. Let $m \in \mathbb{Z}_N$, then $S^m \in C^*(I,S)$. Note that

$$\begin{aligned} S^m &= \left(W \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}(N-1)} \end{bmatrix} W^* \right)^m \\ &= W \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}(N-1)} \end{bmatrix}^m W^* \end{aligned}$$

by Equation (3.1) and W unitary. Applying the isomorphism ψ in Lemma 3.1.3 we get

$$\begin{aligned} S^m \mapsto & W^*W \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}(N-1)} \end{bmatrix}^m W^*W \\ = & \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\frac{2\pi}{N}m} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{i\frac{2\pi}{N}m(N-1)} \end{bmatrix}. \end{aligned}$$

Applying the isomorphism from Equation (3.2) to this we get

$$\left(1, e^{i\frac{2\pi}{N}m}, \dots, e^{i\frac{2\pi}{N}m(N-1)}\right).$$

This means that Φ is equal to the composition of those maps on the elements that generate $C^*(I, S)$, and is thus equal on all of $C^*(I, S)$ by linearity. Hence Φ is the composition of $*$-isomorphisms, and is therefore a $*$-isomorphism itself. ■

Note that $\{\frac{1}{N}\boldsymbol{b}_m \mid m \in \mathbb{Z}_N\}$ is an orthonormal basis for $\mathbb{C}^N$.

3.2 Quantum channels on $C_r^*(\mathbb{Z}_N)$

In this chapter quantum channels on both $C_r^*(\mathbb{Z}_N)$ and $\mathbb{C}^N$ are considered. For this to make sense we must specify which faithful tracial positive linear functionals they are equipped with, as per Definition 2.4.2. Throughout this chapter τ denotes the canonical trace on $C_r^*(\mathbb{Z}_N)$ from Proposition 2.3.13, and $\mathrm{Tr} : \mathbb{C}^N \to \mathbb{C}$ is defined by $\mathrm{Tr}(z) = \sum_{i=0}^{N-1} z_i$ for all $z \in \mathbb{C}^N$. This is a faithful tracial positive linear functional, we omit the proof. In the context of quantum channels we write $C_r^*(\mathbb{Z}_N)$ and $\mathbb{C}^N$ as shorthand for the pairs $(C_r^*(\mathbb{Z}_N), \tau)$ and $\left(\mathbb{C}^N, \mathrm{Tr}\right)$ respectively.

We want to study the quantum channels Q_φ that we get from Proposition 2.4.5, but it is hard to grapple with what it means that φ is a positive definite function on $\mathbb{Z}_N$. Therefore our first order of business is to use Bochner's theorem to characterise φ in terms of a stochastic vector.

Proposition 3.2.1. *Let φ be a normalised positive definite function on $\mathbb{Z}_N$. Then φ is associated with a stochastic vector $p \in \mathbb{R}^N$ in the sense that there exists such a p such that*

$$\varphi(m) = \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k) \tag{3.4}$$

for all $m \in \mathbb{Z}_N$. We write $\varphi = \varphi_p$. Conversely, if p is a stochastic vector, φ_p is normalised positive definite.

Proof. Since $\mathbb{Z}_N$ is abelian and φ is a map on $\mathbb{Z}_N$ Theorem 2.4.8 implies that φ is a positive definite function on $G = \mathbb{Z}_N$ if and only if

$$\varphi(g) = \int_{\widehat{G}} \langle g, \gamma \rangle d\mu(\gamma), \forall g \in G$$

for some regular Borel measure μ on $\widehat{G}$. In our case we have $G = \mathbb{Z}_N \simeq \widehat{\mathbb{Z}}_N$ by the map

$$k \mapsto \gamma_k, \quad k \in \mathbb{Z}_N,$$

where

$$\gamma_k(m) = e^{i\frac{2\pi}{N}km}$$

for all $m \in \mathbb{Z}_N$. Hence we get that

$$\begin{aligned} \varphi(m) &= \int_{\widehat{\mathbb{Z}_N}} \langle m, \gamma \rangle d\mu(\gamma) = \int_{\mathbb{Z}_N} \langle m, \gamma_k \rangle d\mu(k) \\ &= \sum_{k=0}^{N-1} \gamma_k(m) p(k) = \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k) \end{aligned} \tag{3.5}$$

for all $m \in \mathbb{Z}_N$, and p is the vector with k'th entry $p(k) = \mu(\{k\})$. Since μ is a measure, necessarily $p(k) \geq 0$. As we additionally demand that φ is normalised, i.e. that $\varphi(0) = 1$, we get that

$$\varphi(0) = \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}k0} p(k) = \sum_{k=0}^{N-1} p(k) = 1,$$

which means that p must be a stochastic vector.

Conversely, if p is a stochastic vector, then the map φ_p defined by

$$\varphi_p(m) = \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k)$$

for all m in $\mathbb{Z}_N$, is on the form mandated by Bochner's theorem, and is thus positive definite. ■

We could make a slight alteration to the above result by not demanding that φ be normalised. Then φ corresponds to a vector $r \in \mathbb{R}^N$ with non-negative entries, but not necessarily to a stochastic vector.

Now that our grip on φ is more firm, we get to give a general expression for Q_φ.

Corollary 3.2.2. *Let φ_p be a normalised positive definite function on $\mathbb{Z}_N$. Then there exists a quantum channel Q_{φ_p} on $C_r^*(\mathbb{Z}_N)$ such that*

$$Q_{\varphi_p}(\lambda(m)) = \varphi_p(m)\lambda(m) = \left(\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k) \right) \lambda(m) \tag{3.6}$$

for all $m \in \mathbb{Z}_N$.

Proof. Since φ_p is positive definite, Q_{φ_p} exists by Proposition 2.4.5 and is on the form $Q_{\varphi_p}(\lambda(m)) = \varphi(m)\lambda(m)$. The rest follows from Proposition 3.2.1. ■

$$\begin{tikzcd}
C_r^*(\mathbb{Z}_N) \arrow[r, shift left, "\Phi"] \arrow[d, "Q_\varphi"'] & \mathbb{C}^N \arrow[l, dashed, shift left, "\Phi^{-1}"] \arrow[d, "\tilde{Q}_\varphi"] \\
C_r^*(\mathbb{Z}_N) \arrow[r, "\Phi"] & \mathbb{C}^N
\end{tikzcd}$$

Figure 3.2: The induced map $\tilde{Q}_\varphi$ for $G = \mathbb{Z}_N$

Induced channels on $\mathbb{C}^N$

At this point a question arises. We have a collection of quantum channels on $C_r^*(\mathbb{Z}_N)$ as well as an isomorphism $C_r^*(\mathbb{Z}_N) \simeq \mathbb{C}^N$. Could we turn these into quantum channels on $\mathbb{C}^N$ as well, using Proposition 2.4.3? What then, are the properties of these new channels? We start answering these questions by expressing the new channels in terms of p.

Theorem 3.2.3. *Let p be a stochastic vector in $\mathbb{R}^N$ and let φ_p be the normalised positive definite function associated with p, in the sense of Proposition 3.2.1. Then the quantum channel $\tilde{Q}_{\varphi_p}$ on $\mathbb{C}^N$ arising from Corollary 2.4.6 is on the form*

$$\tilde{Q}_{\varphi_p}(\boldsymbol{z}) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle p(k)\boldsymbol{b}_m \tag{3.7}$$

for all $\boldsymbol{z} \in \mathbb{C}^N$.

Proof. Let $\boldsymbol{z} \in \mathbb{C}^N$. We want to calculate $\tilde{Q}_{\varphi_p}(\boldsymbol{z}) = \Phi Q_{\varphi_p}\Phi^{-1}(\boldsymbol{z})$. Since

$$\{\frac{1}{N}\boldsymbol{b}_m \mid m \in \mathbb{Z}_N\}$$

is an orthonormal basis for $\mathbb{C}^N$ we can write

$$\boldsymbol{z} = \frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle \boldsymbol{b}_m.$$

Hence, by linearity of Φ and Equation (3.3),

$$\Phi^{-1}(\boldsymbol{z}) = \frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle S^m.$$

Next we use the identification of $\lambda(m)$ with S^m from Lemma 3.1.1 together with Equation (3.6) we get that

$$\begin{aligned} Q_{\varphi_p}\left(\Phi^{-1}(\boldsymbol{z})\right) &= Q_{\varphi_p}\left(\frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle S^m\right) \\ &= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k)\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle S^m \end{aligned}$$

by the linearity of Q_{φ_p}. Finally we have

$$\begin{aligned} \tilde{Q}_{\varphi_p}(\boldsymbol{z}) &= \Phi\left(Q_{\varphi_p}\left(\Phi^{-1}(\boldsymbol{z})\right)\right) \\ &= \Phi\left(\frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k)\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle S^m\right) \\ &= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} \langle \boldsymbol{z}, \boldsymbol{b}_m\rangle p(k)\boldsymbol{b}_m, \end{aligned}$$

by definition of $\tilde{Q}_{\varphi_p}$ and Φ. ■

Note that this is a somewhat cumbersome formula. In Section 3.4 we derive a more convenient description, but let us first take a detour to consider what quantum channels on $\mathbb{C}^N$ look like in general.

3.3 General quantum channels on $\mathbb{C}^N$

Our goal in this section is to show that the quantum channels on $\left(\mathbb{C}^N, \mathrm{Tr}\right)$ correspond to the stochastic $N \times N$-matrices, where

$$\mathrm{Tr}\,(z) = \sum_{i=0}^{N-1} z_i.$$

Lemma 3.3.1. *Let Q be a quantum channel on $\left(\mathbb{C}^N, \mathrm{Tr}\right)$. Then the standard matrix Q' of Q is a stochastic matrix.*

Proof. Let $z = (z_0, z_1, \ldots, z_{N-1})$ be a positive element in $\mathbb{C}^N$, which by Theorem 2.1.4 means that $z = w^*w$ for some $w \in \mathbb{C}^N$. If $w = (w_0, w_1, \ldots, w_{N-1})$, then $w^* = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1})$ and $w^*w = \left(|w_1|^2, |w_2|^2, \ldots, |w_{N-1}|^2\right)$, which means that z has non-negative entries.

Since Q is completely positive it is in particular a positive map. Since z is positive this means that Qz is positive, and has non-negative entries. Suppose that the (i, j)'th entry of Q' was negative. Then the i'th entry of $Q\boldsymbol{e}_j$ would be negative. Since $\boldsymbol{e}_j$ is positive, this would mean that Q is not positive, yielding a contradiction. Hence Q' has non-negative entries.

Next we show that the sum of each column in Q' must be 1. Let $q_{i,j}$ be the (i, j)'th entry of Q'. Then $\mathrm{Tr}(Q\boldsymbol{e}_j)$ is equal to the column sum of the j'th column of Q'. Since $\mathrm{Tr}(\boldsymbol{e}_j) = 1$ for all $j \in \mathbb{Z}_N$ and Q is Tr-preserving, this means that the column sum of each column of Q' is 1. Since Q' has non-negative entries this means that Q' is a stochastic matrix. ■

Lemma 3.3.2. *Let P be a $N \times N$-dimensional stochastic matrix with entries $p_{i,j}$. Then the corresponding linear operator $\tilde{P}$ on $\left(\mathbb{C}^N, \mathrm{Tr}\right)$ is a quantum channel.*

Proof. Let $j \in \mathbb{Z}_N$. The map $\tilde{P}$ is Tr-preserving by linearity of Tr because

$$\mathrm{Tr}\left(\tilde{P}\boldsymbol{e}_j\right) = \sum_{i=0}^{N-1} p_{i,j} = 1$$

since P is stochastic.

It remains to show that $\tilde{P}$ is completely positive. Since $\mathbb{C}^N$ is abelian it is sufficient to check positivity. Let $z \in \mathbb{C}^N$ be a positive element. By Theorem 2.1.4 $z = w\bar{w}$ for some $w \in \mathbb{C}^N$. This is clearly equivalent with z having non-negative entries. Since the standard matrix of $\tilde{P}$ has non-negative entries, $\tilde{P}z$ must also be positive. Hence $\tilde{P}$ is a quantum channel on $\left(\mathbb{C}^N, \mathrm{Tr}\right)$. ∎

Put together, these two lemmas provide us with what we wanted. Let us summarise our discussion in a proposition.

Proposition 3.3.3. *Let $Q : \mathbb{C}^N \to \mathbb{C}^N$ be a linear map. Then Q is a quantum channel on $(\mathbb{C}^N, \mathrm{Tr})$ if and only if its standard matrix is stochastic.*

Proof. Immediate from Lemma 3.3.1 and Lemma 3.3.2. ∎

Even though we have now achieved our goal in this section, one question remains. While Tr may be a natural choice of trace for $\mathbb{C}^N$, it is not the one suggested by Proposition 2.4.3, namely $\tau \circ \Phi^{-1}$. This discrepancy could cause problems, for are the quantum channels on $(\mathbb{C}^N, \mathrm{Tr})$ the same as the quantum channels on $(\mathbb{C}^N, \tau \circ \Phi^{-1})$? Fortunately the difference turns out to be simply a matter of scaling, relieving us of this issue.

Lemma 3.3.4. *Let τ be the canonical trace on $C_r^*(\mathbb{Z}_N)$ and Φ the isomorphism from Theorem 3.1.4. Then*

$$\tau \circ \Phi^{-1} = \frac{1}{N}\mathrm{Tr}.$$

Proof. Let $\boldsymbol{z} \in \mathbb{C}^N$. Since $\{\frac{1}{N}\boldsymbol{b}_m \mid m \in \mathbb{Z}_N\}$ is an orthonormal basis for $\mathbb{C}^N$ we can write $\boldsymbol{z} = \frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle \boldsymbol{b}_m$. Then

$$\begin{aligned}
\tau \circ \Phi^{-1}(\boldsymbol{z}) &= \tau \circ \Phi^{-1}\left(\frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle \boldsymbol{b}_m\right) \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle \tau \circ \Phi^{-1}(\boldsymbol{b}_m) \\
&= \frac{1}{N}\sum_{m=0}^{N-1}\langle \boldsymbol{z}, \boldsymbol{b}_m\rangle \tau(\lambda(m)).
\end{aligned}$$

Now, we know from Lemma 2.3.12 that $\tau(\lambda(m))$ is 1 if $m = 0$, and 0 otherwise. Thus

$$\tau \circ \Phi^{-1}(\boldsymbol{z}) = \frac{1}{N}\langle \boldsymbol{z}, \boldsymbol{b}_0\rangle = \frac{1}{N}\sum_{i=0}^{N-1} z_i = \frac{1}{N}\mathrm{Tr}(\boldsymbol{z}).$$

∎

Proposition 3.3.5. *Let Q be a map on $\mathbb{C}^N$. Then Q is a quantum channel on* $(\mathbb{C}^N, \mathrm{Tr})$ *if and only if it is a quantum channel on* $(\mathbb{C}^N, \tau \circ \Phi^{-1})$.

Proof. We merely have to note that Q is Tr-preserving if and only if it is $\tau \circ \Phi^{-1}$-preserving since these are equal up to a constant by Lemma 3.3.4 and Q is linear. ∎

3.4 The standard matrix of $\tilde{Q}_\varphi$

In this section we want to arrive at a more illuminating description of $\tilde{Q}_{\varphi_p}$ than the one from Equation (3.7), and see how this fits into the larger picture from Section 3.3. To that end we want to find the standard matrix for $\tilde{Q}_\varphi$ for $\varphi = \varphi_p$. Let $0 \leq j < N$, then

$$\begin{aligned}
\tilde{Q}_\varphi(\boldsymbol{e}_j) &= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}mk} \langle \boldsymbol{e}_j, \boldsymbol{b}_m \rangle p(k) \boldsymbol{b}_m \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{N}m} \\ \vdots \\ e^{i\frac{2\pi}{N}m(N-1)} \end{bmatrix} e^{-i\frac{2\pi}{N}mj} \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}mk} p(k) \\
&= \sum_{k=0}^{N-1} p(k) \sum_{m=0}^{N-1} \frac{1}{N} \begin{bmatrix} 1 \\ e^{i\frac{2\pi}{N}m} \\ \vdots \\ e^{i\frac{2\pi}{N}m(N-1)} \end{bmatrix} e^{-i\frac{2\pi}{N}mj} e^{i\frac{2\pi}{N}mk}
\end{aligned}$$

because $\boldsymbol{b}_m = \left(1, e^{i\frac{2\pi}{N}m}, \ldots, e^{i\frac{2\pi}{N}m(N-1)}\right)$ and $\langle \boldsymbol{e}_j, \boldsymbol{b}_m \rangle = e^{-i\frac{2\pi}{N}mj}$. We have to consider

$$\begin{bmatrix} 1 \\ e^{i\frac{2\pi}{N}m} \\ \vdots \\ e^{i\frac{2\pi}{N}m(N-1)} \end{bmatrix} e^{-i\frac{2\pi}{N}mj} e^{i\frac{2\pi}{N}mk}.$$

What does each term in this vector look like? Numbering the terms by q we get that the q'th term is

$$e^{i\frac{2\pi}{N}mq} e^{-i\frac{2\pi}{N}mj} e^{i\frac{2\pi}{N}mk} = e^{i\frac{2\pi}{N}m(q-j+k)}.$$

Now, there are two cases. Either $q - j + k$ is equal to 0 or N, in which case the q'th term is 1. Otherwise

$$\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}m(q-j+k)} = \sum_{m=0}^{N-1} \left(e^{i\frac{2\pi}{N}(q-j+k)}\right)^m = \frac{1 - \left(e^{i\frac{2\pi}{N}(q-j+k)}\right)^N}{1 - e^{i\frac{2\pi}{N}(q-j+k)}} = 0.$$

What this means is a bit opaque, so let us consider an example. Letting $j = 0$ we get that

$$\tilde{Q}_\varphi(\boldsymbol{e}_0) = \sum_{k=0}^{N-1} p(k) \sum_{m=0}^{N-1} \frac{1}{N} \begin{bmatrix} e^{i\frac{2\pi}{N}m(0-0+k)} \\ e^{i\frac{2\pi}{N}m(1-0+k)} \\ \vdots \\ e^{i\frac{2\pi}{N}m(N-1-0+k)} \end{bmatrix} = \sum_{k=0}^{N-1} p(k)\boldsymbol{e}_{N\dot{-}k},$$

where the $\cdot$ denotes subtraction modulo N, i.e. $\boldsymbol{e}_{N\dot{-}0} = \boldsymbol{e}_0$. This is equal to

$$\begin{bmatrix} p(0) \\ p(N-1) \\ p(N-2) \\ \vdots \\ p(1) \end{bmatrix}.$$

Incrementing j by 1 we get that the 0'th term of $\tilde{Q}_\varphi(\boldsymbol{e}_1)$ is $p(1)$. Continuing in this manner we get that the standard matrix of $\tilde{Q}_\varphi$ is

$$\begin{bmatrix} p(0) & p(1) & p(2) & \dots & p(N-1) \\ p(N-1) & p(0) & p(1) & \dots & p(N-2) \\ p(N-2) & p(N-1) & p(0) & \dots & \vdots \\ \vdots & \vdots & & \ddots & p(1) \\ p(1) & p(2) & \dots & p(N-1) & p(0) \end{bmatrix}. \tag{3.8}$$

Note that this is a doubly stochastic matrix. Reiterating, this means that each induced quantum channel Q_φ on $C_r^*(\mathbb{Z}_N)$ corresponds to a doubly stochastic matrix on the above form through the aforementioned isomorphism. We give our results for this chapter so far in the following theorem.

Theorem 3.4.1. *Let $\tilde{Q}_{\varphi_p}$ be the quantum channel on $(\mathbb{C}^N, \tau \circ \Phi^{-1})$ from Theorem 3.2.3 corresponding to the stochastic vector p. Then $\tilde{Q}_{\varphi_p}$ is also a quantum channel on $(\mathbb{C}^N, \mathrm{Tr})$ and its standard matrix is on the form Equation (3.8).*

Proof. That $\tilde{Q}_{\varphi_p}$ is a quantum channel on $(\mathbb{C}^N, \mathrm{Tr})$ follows directly from Proposition 3.3.5, while the standard matrix is established in the above discussion. ■

3.5 Contractive channels on $\mathbb{C}^N$

Our final task in this chapter is to study the contractive properties of the quantum channels $\tilde{Q}_\varphi$ discussed above. We are interested in which circumstances the conditions stated in Definition 2.5.4 hold for these channels. The metric space of interest is $(\mathbb{C}^N, d)$, where d is either the Bures metric d_B^{Tr} from Definition 2.5.5 or the metric $d_1 : \mathbb{C}^N \to \mathbb{R}$ defined by

$$d_1(\boldsymbol{z}, \boldsymbol{w}) = \mathrm{Tr}(|z_i - w_i|),$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^N$. Let us start with the latter.

Note: As per our definitions, we are interested in whether or not certain functions are contractions. Nevertheless we sometimes talk about matrices being contractions. If we say that a matrix A is a contraction, our precise meaning is that the function $f_A : \mathbb{C}^N \to \mathbb{C}^N$ defined by $f_A(\boldsymbol{z}) = A\boldsymbol{z}$ is a contraction.

Contractions with relation to d_1

From Theorem 3.4.1 we know that each $\tilde{Q}_{\varphi_p}$ has a doubly stochastic standard matrix on the form Equation (3.8). Let us name this matrix A_p. Before we tackle this class of matrices we consider what is known more generally. We mention the following result, which is adapted to our notation.

Proposition 3.5.1 ([HR16](Proposition 4.1))**.** *Let $\phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ be a linear map. If ϕ is positive and* Tr*-preserving, then ϕ is non-expansive w.r.t. d_1.*

Applied to our situation, we get

Corollary 3.5.2. *Let φ_p be a normalised positive definite function on $\mathbb{Z}_N$. Then $\tilde{Q}_{\varphi_p}$ is non-expansive w.r.t. d_1.*

Proof. We have from Proposition 3.3.5 that $\tilde{Q}_{\varphi_p}$ is Tr-preserving, so the result follows by Proposition 3.5.1. ∎

The following theorem is an amalgamation of [KNR11][Theorem 2.1] and [KM14][Proposition 3.2], adapted to our notation.

Theorem 3.5.3. *Let $S = [s_{ij}]_{ij}$ be an $N \times N$ stochastic matrix. Then the following are equivalent:*

(i) *S is strictly contractive (w.r.t. d_1),*

(ii) *S is locally contractive (w.r.t. d_1),*

(iii) *all entries of $S^T S$ are positive,*

(iv) *for all $k, l \in \mathbb{Z}_N$ there exists $j \in \mathbb{Z}_N$ such that $s_{jk}s_{jl} > 0$, i.e. s_{jk} and s_{jl} are both non-zero.*

As mentioned, we know from Theorem 3.4.1 that

$$A_p = \begin{bmatrix} p(0) & p(1) & p(2) & \dots & p(N-1) \\ p(N-1) & p(0) & p(1) & \dots & p(N-2) \\ p(N-2) & p(N-1) & p(0) & \dots & \vdots \\ \vdots & \vdots & & \ddots & p(1) \\ p(1) & p(2) & \dots & p(N-1) & p(0) \end{bmatrix}. \tag{3.9}$$

Hence A_p is completely determined by the stochastic vector p. Therefore it seems reasonable that there should be some condition on p that makes A_p strictly contractive. To get to this condition we will describe (iii) from Theorem 3.5.3 in terms of p.

Proposition 3.5.4. *All entries of $A_p^T A_p$ are positive if and only if*

$$\sum_{i=1}^{N-1} p(i)p(i \dotplus j) > 0 \tag{3.10}$$

for all $j \in \mathbb{Z}_N$.

Proof. We compute that $A_p^T A_p =$

$$\begin{bmatrix} p(0) & p(N-1) & \dots & p(1) \\ p(1) & p(0) & \dots & p(2) \\ \vdots & \vdots & \ddots & \vdots \\ p(N-1) & p(N-2) & \dots & p(0) \end{bmatrix} \begin{bmatrix} p(0) & p(1) & \dots & p(N-1) \\ p(N-1) & p(0) & \dots & p(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ p(1) & p(2) & \dots & p(0) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=0}^{N-1} p(i)^2 & \sum_{i=0}^{N-1} p(i)p(i \dotplus 1) & \dots & \sum_{i=0}^{N-1} p(i)p(i \dotplus N \dot{-} 1) \\ & \sum_{i=0}^{N-1} p(i)^2 & \dots & \vdots \\ & & \ddots & \sum_{i=0}^{N-1} p(i)p(i \dotplus 1) \\ & & & \sum_{i=0}^{N-1} p(i)^2 \end{bmatrix}.$$

This matrix is symmetric, the entries below the diagonal have been removed to conserve space. Each term from Equation (3.10) appears once in any row or column of this matrix. The remaining entry of any such row or column is then $\sum_{i=0}^{N-1} p(i)^2$, but this is always positive since p is stochastic. By Theorem 3.5.3 A_p is strictly contractive if and only if this matrix has positive entries, yielding the result. ∎

Proposition 3.5.5. *The quantum channel $\tilde{Q}_{\varphi_p}$ is a strict contraction on $\mathbb{C}^N$ if p contains at most $\frac{N}{2} - 1$ (rounded up) entries that are 0. In other words, if more than half of the entries in p are positive.*

Proof. We want to check that Equation (3.10) holds for all $j \in \mathbb{Z}_N$. To do this we first reformulate the problem. For a given j, Equation (3.10) holds if and only if there exists some $i \in \mathbb{Z}_N$ such that $p(i)p(i \dotplus j) > 0$. This is true if and only if both $p(i) \neq 0$ and $p(i + j) \neq 0$. Since we only care about whether a given entry in p is positive or not, we replace p with a new vector $p \in \mathbb{R}^N$ that stores this information. That is, $p(i) = 1$ if $p(i) > 0$, and $p(i) = 0$ otherwise, for all $i \in \mathbb{Z}_N$.

We introduce the claim

$$P(j): \text{ there exists } i \in \mathbb{Z}_N \text{ such that } p'(i) = p'(i \dotplus j) = 1. \tag{3.11}$$

for each $j \in \mathbb{Z}_N$. Note that $P(0)$ always holds, regardless of the choice of p.

The claim $P(j)$ is equivalent to Equation (3.10) holding, not for all j, but for a single one. Then the problem of whether $\tilde{Q}_{\varphi_p}$ is a strict contraction is equivalent to the claim

$$P: P(k) \text{ holds for all } k \in \mathbb{Z}_N. \tag{3.12}$$

Figure 3.3 illustrates an example where P holds, but it is neither required nor wholly explained at this point.

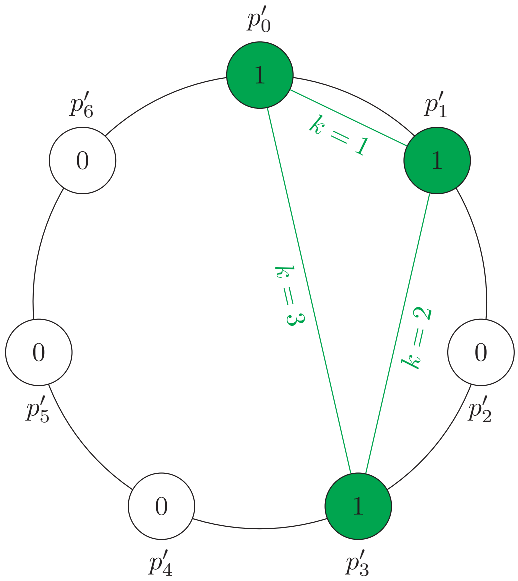

Figure 3.3: Counterexample to the converse

We claim that if

$$\sum_{i=0}^{N-1} p'(i) \geq \left\lfloor \frac{N}{2} \right\rfloor + 1, \tag{3.13}$$

then P holds. Note that Equation (3.13) says that more than half of the entries in p' are 1.

Suppose that P does not hold. Since P does not hold there exists some $j \in \mathbb{Z}_N$ such that $P(j)$ does not hold. This means that for any $i \in \mathbb{Z}_N$, at least one of $p'(i)$ and $p'(i \dot{+} j)$ is equal to zero. This means that at least half the entries in p' are zero, which means that Equation (3.13) does not hold. Hence the claim follows contrapositively.

■

Is it possible to improve upon this result? By this we mean to ask whether (i) it is possible to demand fewer positive entries in p, or (ii) whether the converse of the proposition holds.

For (i), suppose we demanded half instead, which is at most one less than before. Then for $N = 2$ we get e.g. $p = [1, 0]$, for which $p'(0)p'(1) = 0$, hence A_p is not strictly contractive w.r.t. d_1 by Theorem 3.5.3. For (ii), regard Example 3.5.6.

Example 3.5.6. Suppose that $N = 7$ and p is such that $p' = [1, 1, 0, 1, 0, 0, 0, 0]$. We are interested in whether A_p is strictly contractive w.r.t. d_1. Following the proof from Proposition 3.5.5 this is equivalent to checking P from Equation (3.12).

In extension of the explanation from the proof, we can regard p' as a circle with $N = 7$ nodes where we measure the distance k between two nodes along the circle edge. A distance of k is equivalent to a distance of $N-k$. This corresponds to the fact that e.g.

$$p'(i)p'(i\dot{+}5) = p'(i\dot{+}2)p'(i) = p'(i)p'(i\dot{+}2),$$

which is true since $5\dot{+}2 = 0$.

From the above observation we see that it is sufficient to check $P(k)$ from Equation (3.11) for $k = 1, 2, 3$ to establish P. This is easily done in this case, see Figure 3.3 . In words: $j = 0$ satisfies $P(1)$, $j = 1$ satisfies $P(2)$ and $j = 0$ satisfies $P(3)$.

We have seen that P is satisfied for our choice of p, hence A_p is strictly contractive w.r.t. d_1. The point of this example is that

$$\left\lfloor \frac{N}{2} \right\rfloor + 1 = \left\lfloor \frac{7}{2} \right\rfloor + 1 = 4,$$

while it is not true that p has 4 positive entries, it has 3. This means that we could not have applied Proposition 3.5.5 to show that A_p is strictly contractive w.r.t. d_1, even though it is.

The converse of Proposition 3.5.5 is that if fewer than half of the entries in p are positive, then A_p is not strictly contractive w.r.t. d_1. The above example shows that this is not true. Hence we do not know what happens if the condition of Proposition 3.5.5 is not satisfied.

Suppose that we have some strict contraction $Q : M_N(\mathbb{C}) \to M_N(\mathbb{C})$. Then we know that there exists some $C < 1$ such that $d_1(Q(A), Q(B)) \leq C d_1(A, B)$ for all $A, B \in M_N(\mathbb{C})$. There are still questions to ask about Q. For instance, what is the best contraction constant for Q? By best we mean the lowest possible contraction constant. The next result gives us some insight into this question.

Proposition 3.5.7 ([KNR11])**.** *Let $A \in M_N(\mathbb{C})$ be on the form*

$$A = [a_0 \ a_1 \ \dots \ a_{N-1}]$$

. If A satisfies Theorem 3.5.3 the best contraction constant C_A for A w.r.t. d_1 is

$$C_A = \frac{1}{2} \max \left\{ \|a_k - a_l\|_1 \ : \ k, l \in \mathbb{Z}_N \right\}. \tag{3.14}$$

Applied to our case, we get

Corollary 3.5.8. *Suppose that A_p satisfies Theorem 3.5.3. Then the best contraction constant C_p for the quantum channel $\tilde{Q}_{\varphi_p}$ w.r.t. d_1 is*

$$C_p = \frac{1}{2} \max \left\{ \sum_{j=0}^{N-1} |p(j) - p(j\dot{+}k)| \ : \ k \in \mathbb{Z}_N \right\}.$$

Proof. We have that A_p is on the form $A_p = [a_0 \ a_1 \ \dots \ a_{N-1}]$, where $a_k = [p(k) \ p(N\dot{-}1\dot{+}k \dots p(1\dot{+}k)]$. Then A_p satisfies Proposition 3.5.7 by assumption, and the best contraction constant for A_p is

$$\begin{aligned} C_p &= \frac{1}{2} \max \left\{ \|a_k - a_l\|_1 \ : \ k, l \in \mathbb{Z}_N \right\} \\ &= \frac{1}{2} \max \left\{ \|a_0 - a_k\|_1 \ : \ k \in \mathbb{Z}_N \right\} \end{aligned}$$

by the structure of the a_k's, briefly $a_k(i \dot{+} j) = a_{k \dot{-} j}(i)$. Using the definition of $\|\cdot\|_1$ and the of a_k's this is again equal to

$$\begin{aligned} C_p &= \frac{1}{2} \max \left\{ \sum_{j=0}^{N-1} |a_0(j) - a_k(j)| \;:\; k \in \mathbb{Z}_N \right\} \\ &= \frac{1}{2} \max \left\{ \sum_{j=0}^{N-1} |p(j) - p(j \dot{+} k)| \;:\; k \in \mathbb{Z}_N \right\}. \end{aligned}$$

■

Contractions with relation to d_B^τ

Let us turn to the Bures metric next. For a unital C^*-algebra we defined d_B^{Tr} on the its Tr-density space. Currently the C^*-algebra under consideration is $\mathbb{C}^N$, so the Tr-density space becomes

$$\begin{aligned} \mathcal{D}_{\mathrm{Tr}}(\mathbb{C}^N) &= \left\{ a \in \mathbb{C}_+^N \mid \mathrm{Tr}(a) = 1 \right\} \\ &= \left\{ a \in \mathbb{R}^N \mid a \text{ is a stochastic vector} \right\}. \end{aligned}$$

In other words, we are interested in quantum channels on the N-dimensional stochastic vectors. We recall from Definition 2.5.5 that the Bures metric on $\mathcal{D}_{\mathrm{Tr}}(\mathbb{C}^N)$ is given by

$$d_B^\tau(a,b) = \sqrt{1 - \mathrm{Tr}(|a^{\frac{1}{2}} b^{\frac{1}{2}}|)}$$

for all $a, b \in \mathcal{D}_\tau(\mathbb{C}^N)$. Furthermore

$$\mathrm{Tr}(|a^{\frac{1}{2}} b^{\frac{1}{2}}|) = \sum_{i=0}^{N-1} a(i)^{\frac{1}{2}} b(i)^{\frac{1}{2}}, \tag{3.15}$$

where $a(i)$ is the i'th entry of b. Note that this is always a number in the interval $[0,1]$. Otherwise d_B^τ would not be a metric.

The question is then how this relates to $\tilde{Q}_{\varphi_p}$ (or A_p). Since each $\tilde{Q}_{\varphi_p}$ is a quantum channel we already have from Proposition 2.5.7 that it is at least non-expansive.

Let $a, b \in \mathcal{D}_{\mathrm{Tr}}(\mathbb{C}^N)$. For A_p to be locally contractive we need that

$$d_B^{\mathrm{Tr}}(A_p a, A_p b) < d_B^{\mathrm{Tr}}(a,b),$$

which is equivalent to

$$\sqrt{1 - \mathrm{Tr}(|(A_p a)^{\frac{1}{2}} (A_p b)^{\frac{1}{2}}|)} < \sqrt{1 - \mathrm{Tr}(|a^{\frac{1}{2}} b^{\frac{1}{2}}|)}, \tag{3.16}$$

which is again equivalent to

$$\mathrm{Tr}(|(A_p a)^{\frac{1}{2}} (A_p b)^{\frac{1}{2}}|) > \mathrm{Tr}(|a^{\frac{1}{2}} b^{\frac{1}{2}}|).$$

Let us calculate each of these quantities. The right hand side is known from Equation (3.15), and it is possible to compute that the right hand side is

$$\mathrm{Tr}(|(A_p a)^{\frac{1}{2}} (A_p b)^{\frac{1}{2}}|) = \sum_{j=0}^{N-1} \left(\left(\sum_{i=0}^{N-1} p(i-j) a(i) \right)^{\frac{1}{2}} \left(\sum_{i=0}^{N-1} p(i-j) b(i) \right)^{\frac{1}{2}} \right), \tag{3.17}$$

which means that we are interested in finding a condition on p such that

$$\sum_{j=0}^{N-1}\left(\left(\sum_{i=0}^{N-1} p(i-j)a(i)\right)^{\frac{1}{2}}\left(\sum_{i=0}^{N-1} p(i-j)b(i)\right)^{\frac{1}{2}}\right) > \sum_{i=0}^{N-1} a(i)^{\frac{1}{2}} b(i)^{\frac{1}{2}}. \quad (3.18)$$

We have not been able to find such a condition, and consider two examples instead.

Example 3.5.9. Suppose $p = [1\ 0]$, then $A_p = I_2$. Hence A_p is not a strict contraction w.r.t. d_B^{Tr}.

Let us pick some other seemingly harmless p, and attempt to get a strict contraction.

Example 3.5.10. Suppose $p = [\frac{1}{2}\ \frac{1}{2}\ 0]$, then

$$A_p = \begin{bmatrix} p(0) & p(1) & p(2) \\ p(2) & p(0) & p(1) \\ p(1) & p(2) & p(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Let a and b be stochastic vectors in $\mathbb{R}^3$. Then, by Equation (3.17),

$$\begin{aligned} \mathrm{Tr}(|(A_p a)^{\frac{1}{2}}(A_p b)^{\frac{1}{2}}|) &= \sum_{j=0}^{2}\left(\left(\sum_{i=0}^{2} p(i-j)a(i)\right)^{\frac{1}{2}}\left(\sum_{i=0}^{2} p(i-j)b(i)\right)^{\frac{1}{2}}\right) \\ &= \left(\frac{1}{2}a_0 + \frac{1}{2}a_1\right)^{\frac{1}{2}}\left(\frac{1}{2}b_0 + \frac{1}{2}b_1\right)^{\frac{1}{2}} + \left(\frac{1}{2}a_1 + \frac{1}{2}a_2\right)^{\frac{1}{2}}\left(\frac{1}{2}b_1 + \frac{1}{2}b_2\right)^{\frac{1}{2}} \\ &\quad + \left(\frac{1}{2}a_0 + \frac{1}{2}a_2\right)^{\frac{1}{2}}\left(\frac{1}{2}b_0 + \frac{1}{2}b_2\right)^{\frac{1}{2}}. \end{aligned} \quad (3.19)$$

By Equation (3.18) we want to compare this to

$$a_0^{\frac{1}{2}} b_0^{\frac{1}{2}} + a_1^{\frac{1}{2}} b_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} b_2^{\frac{1}{2}}. \quad (3.20)$$

If $a = b$, then Equation (3.19) is equal to Equation (3.20). What happens otherwise is less clear.

CHAPTER 4

The second example

In this chapter we consider the twisted reduced group C^*-algebra of $\mathbb{Z}_N \times \mathbb{Z}_N$. To do so we need to define some suitable 2-cocycles. Our first order of business is to establish that it makes sense to define our algebra in the desired way. Afterwards we show that the algebra is isomorphic to the $N \times N$ matrices and study the relationship between the quantum channels from Proposition 2.4.5 and the Weyl-covariant channels from Definition 2.4.10. Finally we regard the contractive properties of these channels.

4.1 Setting

Let $G = \mathbb{Z}_N \times \mathbb{Z}_N$ for some $N \in \mathbb{N}$ and let $k \in \{0, 1, \ldots, N-1\}$. Throughout Chapter 4 σ_k will denote the cocycle on G given by

$$\sigma_k((m_1, m_2), (n_1, n_2)) = e^{ik\frac{2\pi}{N}m_1 n_2} \tag{4.1}$$

for all $(m_1, m_2), (n_1, n_2) \in G$. We write σ as shorthand for σ_1. As stated we want to show that $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_n, \sigma) \simeq M_N(\mathbb{C})$, where $M_N(\mathbb{C})$ is the space of all $N \times N$ matrices over the complex numbers. The elements of G will be denoted both on the form $g \in G$ and $(m_1, m_2) \in G$ throughout this chapter. Our first task is to show that each σ_k is actually a 2-cocycle.

Proposition 4.1.1. *The map σ_k is a bicharacter on G, hence a 2-cocycle.*

Proof. Fix $(k_1, k_2) \in G$ and define $\gamma : G \to \mathbb{T}$ by

$$\gamma\left((m_1, m_2)\right) = \sigma_k((m_1, m_2), (k_1, k_2)).$$

This is a homomorphism because

$$\begin{aligned}
\gamma\left((m_1, m_2)(n_1, n_2)\right) &= \sigma_k\left((m_1 + n_1, m_2 + n_2), (k_1, k_2)\right) \\
&= e^{ik\frac{2\pi}{N}(m_1+n_1)k_2} = e^{ik\frac{2\pi}{N}m_1 k_2} e^{ik\frac{2\pi}{N}n_1 k_2} \\
&= \gamma\left((m_1, m_2)\right)\gamma\left((n_1, n_2)\right)
\end{aligned}$$

by definition of γ and σ_k. The map γ' defined by

$$\gamma'\left((m_1, m_2)\right) = \sigma_k((k_1, k_2), (m_1, m_2))$$

is similarly a homomorphism. Hence σ_k is a bicharacter on G. By Proposition 2.3.18 σ_k is thus a normalised 2-cocycle on G. ∎

Since each σ_k is a cocycle it makes sense to define $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ for each k also. Having defined this algebra, we want to define an explicit $*$-isomorphism with the $N \times N$ matrices over $\mathbb{C}$. Our strategy will be to find a suitable linear function that maps basis elements of $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ into $M_N(\mathbb{C})$. We have done most of the work finding a basis for the former already, but let us state it here.

Lemma 4.1.2. *The set* $\{\lambda_{\sigma_k}((m_1, m_2)) \mid (m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N\}$ *is a basis for* $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ *for each* $k \in \{0, 1, \dots, N-1\}$.

Proof. For any $N \in \mathbb{N}$, $\{\lambda_{\sigma_k}((m_1, m_2)) \mid (m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N\}$ is a finite set that spans $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma)$, by Definition 2.3.9. It is a basis since it is also linearly independent by Lemma 2.3.14. ∎

All 2-cocycles on G are similar to σ_k for some $k \in \{0, 1, \dots, N-1\}$. This can be shown from [Kle65](Theorem 7.1), but is also mentioned explicitly in e.g. [Li+19](Example 3.1). By Proposition 2.3.15 it is thus sufficient to study each $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$, we need not consider any other 2-cocycles.

4.2 The isomorphism for $k = 1$

Throughout this section we will only consider σ_k for the case $k = 1$. In it we will establish an explicit isomorphism $\Phi : C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma) \to M_N(\mathbb{C})$, but before we define Φ it will be useful to discuss a basis for $M_N(\mathbb{C})$. The case for general k is postponed to Section 4.4.

Two useful matrices

Having found a basis for $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ our next task is to give matrices which we can map these basis elements onto. One natural choice could be to employ the matrices with a single 1 in the (i, j)'th spot, and all zeros otherwise. This is clearly a basis, but it turns out to be impractical for our purposes. We have to go about this in a more circuitous manner.

Let $N \in \mathbb{N}$ and denote $w = e^{i\frac{2\pi}{N}}$. Consider the $N \times N$-matrices

$$D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & w & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & w^{N-1} \end{bmatrix} \quad \text{and } S = \begin{bmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{N-1} & \boldsymbol{e}_0 \end{bmatrix},$$

where $\{\boldsymbol{e}_k\}_0^{N-1}$ is the standard basis for $\mathbb{C}^N$ with $\boldsymbol{e}_0 = (1\ 0\ \cdots 0)^T$. Both S and D were discussed in Chapter 3 as well. These matrices correspond to linear operators in $\mathcal{L}(\mathbb{C}^N)$ given by $\tilde{D}(\boldsymbol{e}_k) = e^{ik\frac{2\pi}{N}}\boldsymbol{e}_k$ and $\tilde{S}(\boldsymbol{e}_k) = \boldsymbol{e}_{k\dot{+}1}$, where $\dot{+}$ denotes addition modulo N.

The following lemma is clearly true; the proof is omitted.

Lemma 4.2.1. *D and S are unitary matrices with $D^N = S^N = I_N$.*

We will see later that the elements of $M_N(\mathbb{C})$ on the form $e^{-i\frac{2\pi}{N}m_1 m_2} D^{m_1} S^{m_2}$ for $(m_1, m_2) \in G$ constitute a basis. This is a convenient motivation for how we will define our isomorphism Φ, but in our presentation the truth of this claim follows from Φ being an isomorphism rather than the other way around.

Next we discuss a commutation relation for D and S that we will make use of.

Lemma 4.2.2. *For the above matrices we have the following commutation rule*

$$DS = e^{i\frac{2\pi}{N}} SD.$$

Proof. It suffices to show this for the corresponding linear transformations applied to an arbitrary basis vector $\boldsymbol{e}_k$. We see that

$$\left(\tilde{S}\tilde{D}\right)(\boldsymbol{e}_k) = \tilde{S}\left(\tilde{D}\boldsymbol{e}_k\right) = \tilde{S}\left(e^{ik\frac{2\pi}{N}}\boldsymbol{e}_k\right) = e^{ik\frac{2\pi}{N}}\tilde{S}(\boldsymbol{e}_k) = e^{ik\frac{2\pi}{N}}\boldsymbol{e}_{k\dot{+}1},$$

and

$$\left(\tilde{D}\tilde{S}\right)(\boldsymbol{e}_k) = \tilde{D}\boldsymbol{e}_{k\dot{+}1} = e^{i(k\dot{+}1)\frac{2\pi}{N}}\boldsymbol{e}_{k\dot{+}1} = e^{i(k+1)\frac{2\pi}{N}}\boldsymbol{e}_{k\dot{+}1} = e^{i\frac{2\pi}{N}}e^{ik\frac{2\pi}{N}}\boldsymbol{e}_{k\dot{+}1}.$$

Hence $\left(\tilde{D}\tilde{S}\right)(\boldsymbol{e}_k) = e^{i\frac{2\pi}{N}}\left(\tilde{S}\tilde{D}\right)(\boldsymbol{e}_k)$ for all k. ∎

Lemma 4.2.3. *Let $n, m \in \mathbb{N}$. For the above matrices we have the following commutation rule*

$$S^n D^m = e^{-i\frac{2\pi}{N}mn} D^m S^n.$$

Proof. By Lemma 4.2.2 we have that $SD = e^{-i\frac{2\pi}{N}} DS$. To commute $S^n D$ we have to apply the lemma n times, hence $S^n D = e^{-i\frac{2\pi}{N}n} DS^n$. Thus $S^n D^m = e^{-i\frac{2\pi}{N}n} DS^n D^{m-1}$, and by doing this m times we get the result. ∎

An explicit isomorphism

Now we get to define our candidate for the $*$-isomorphism. For the remainder of this section we let Φ be the linear function from $C_r^*(G, \sigma)$ to $M_N(\mathbb{C})$ determined by

$$\Phi\left(\lambda_\sigma((m_1, m_2))\right) = e^{-i\frac{2\pi}{N}m_1 m_2} D^{m_1} S^{m_2}. \tag{4.2}$$

This is well defined as Φ is a linear function defined on a basis, see Lemma 4.1.2. As we have mentioned, we want to show that Φ is a $*$-isomorphism. Much of the necessary work was done in previous sections, but let us still split the proof in two parts. First we show that Φ is a $*$-homorphism.

Lemma 4.2.4. *The map Φ defined above is a $*$-homomorphism.*

Proof. We start by showing that Φ is a homomorphism. As Φ is linear it is sufficient to check this for the basis elements. Let $(m_1, m_2), (n_1, n_2) \in G$, then

$$\begin{aligned} &\Phi\left(\lambda_\sigma((m_1, m_2))\lambda_\sigma((n_1, n_2))\right) \\ &\quad = \sigma((m_1, m_2), (n_1, n_2))\Phi\left(\lambda_\sigma((m_1 + n_1, m_2 + n_2))\right) \end{aligned}$$

by Proposition 2.3.7. By definition this is again equal to

$$\begin{aligned} &e^{i\frac{2\pi}{N}m_1 n_2} e^{-i\frac{2\pi}{N}(m_1+n_1)(m_2+n_2)} D^{m_1+n_1} S^{m_2+n_2} \\ &\quad = e^{-i\frac{2\pi}{N}(m_1 m_2 + m_2 n_1 + n_1 n_2)} D^{m_1+n_1} S^{m_2+n_2}. \end{aligned}$$

On the other hand we have that

$$\begin{aligned}\Phi\left(\lambda_\sigma((m_1, m_2)\right)\Phi\left(\lambda_\sigma((n_1, n_2)\right) &= e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}e^{-i\frac{2\pi}{N}n_1n_2}D^{n_1}S^{n_2}\\ &= e^{-i\frac{2\pi}{N}(m_1m_2+n_1n_2)}D^{m_1}S^{m_2}D^{n_1}S^{n_2}.\end{aligned}$$

By applying the commutation rule from Lemma 4.2.3 we get that this again equals

$$\begin{aligned}&e^{-i\frac{2\pi}{N}(m_1m_2+n_1n_2)}D^{m_1}e^{-i\frac{2\pi}{N}m_2n_1}D^{n_1}S^{m_2}S^{n_2}\\ &= e^{-i\frac{2\pi}{N}(m_1m_2+m_2n_1+n_1n_2)}D^{m_1+n_1}S^{m_2+n_2},\end{aligned}$$

as in the first case. Hence Φ is a homomorphism.

Next, we have to show that $\Phi\left(\lambda_\sigma((m_1, m_2))\right)^* = \Phi\left(\lambda_\sigma((m_1, m_2))^*\right)$. We calculate each side separately. Note that $S^* = S^{-1}$ and $D^* = D^{-1}$, so the left hand side becomes

$$\begin{aligned}\Phi\left(\lambda_\sigma((m_1, m_2))\right)^* &= \left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right)^*\\ &= e^{i\frac{2\pi}{N}m_1m_2}\left(S^T\right)^{m_2}\left(\overline{D}\right)^{m_1}\\ &= e^{i\frac{2\pi}{N}m_1m_2}S^{-m_2}D^{-m_1}\\ &= e^{i\frac{2\pi}{N}m_1m_2}e^{-i\frac{2\pi}{N}(m_1)(m_2)}D^{-m_1}S^{-m_2}\\ &= D^{-m_1}S^{-m_2}\end{aligned}$$

by the commutation rule from Lemma 4.2.3.

For the right hand side we recall that λ_σ is a σ-projective unitary representation by Proposition 2.3.7. Then Lemma 2.3.6 implies that

$$\begin{aligned}\Phi\left(\lambda_\sigma((m_1, m_2))^*\right) &= \Phi\left(e^{i\frac{2\pi}{N}m_1m_2}\lambda_\sigma((-m_1, -m_2))\right)\\ &= e^{i\frac{2\pi}{N}m_1m_2}e^{-i\frac{2\pi}{N}(-m_1)(-m_2)}D^{-m_1}S^{-m_2}\\ &= D^{-m_1}S^{-m_2}.\end{aligned}$$

Hence Φ is also $*$-preserving, and we are finished. ■

Next is the main result of this section. In addition to being our current goal, we will also need it later when we study Weyl-covariant channels.

Theorem 4.2.5. *Suppose $G = \mathbb{Z}_N \times \mathbb{Z}_N$ for some $N \in \mathbb{N}$. Let*

$$\Phi : C_r^*(G, \sigma) \to M_N(\mathbb{C})$$

be the linear transformation given by

$$\Phi(\lambda_\sigma(m_1, m_2)) = e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}$$

for all $m_1, m_2 \in G$. Then Φ is a $*$-isomorphism.

Proof. From Lemma 4.2.4 we already know that Φ is a $*$-homomorphism. The main job that we have to do in this proof is establishing that Φ is injective. To do so we will make use of the canonical trace τ defined in Proposition 2.3.13.

Since τ is a faithful trace we can define a norm on $C_r^*(G, \sigma)$ by $\|x\|_\tau = \tau(x^*x)^{\frac{1}{2}}$, since $\|x\|_\tau = 0$ implies $x = 0$. Had τ not been faithful, we would merely have had a seminorm. Similarily, $\|A\|_{\mathrm{tr}} = \mathrm{tr}(A^*A)^{\frac{1}{2}}$ is a norm on $M_N(\mathbb{C})$, where tr is the (normalised) standard matrix trace. If we can show that Φ is isometric in regard to these norms, it will be an injection.

We want to show that $\tau = \mathrm{tr} \circ \Phi$. It is sufficient to show this for $\lambda_\sigma((m_1, m_2))$ for all $(m_1, m_2) \in G$ since those elements are a basis and both τ and $\mathrm{tr} \circ \Phi$ are linear. Let $(m_1, m_2) \neq (0, 0)$. Then $\tau(\lambda_\sigma((m_1, m_2))) = 0$ by Lemma 2.3.12, while

$$\begin{aligned}\mathrm{tr} \circ \Phi(\lambda_\sigma((m_1, m_2))) &= \mathrm{tr}\left(e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2}\right) \\ &= e^{-i\frac{2\pi}{N}m_1m_2} \mathrm{tr}(D^{m_1} S^{m_2}).\end{aligned}$$

Let us calculate $D^{m_1} S^{m_2}$. Recall that D and S have corresponding linear operators $\tilde{D}$ and $\tilde{S}$ defined by $\tilde{D}(\boldsymbol{e}_k) = e^{ik\frac{2\pi}{N}} \boldsymbol{e}_k$ and $\tilde{S}(\boldsymbol{e}_k) = \boldsymbol{e}_{k\dot{+}1}$. Hence the k'th column of $D^{m_1} S^{m_2}$ is given by

$$\tilde{D}^{m_1} \tilde{S}^{m_2} \boldsymbol{e}_k = \tilde{D}^{m_1} \boldsymbol{e}_{k\dot{+}m_2} = e^{i(k+m_2)m_1\frac{2\pi}{N}} \boldsymbol{e}_{k\dot{+}m_2}.$$

Now, if $m_2 \neq 0$, this means that each diagonal entry of $D^{m_1} S^{m_2}$ is 0, so the trace is 0. If $m_2 = 0$, then $m_1 \neq 0$, so we get that

$$\mathrm{tr}\left(D^{m_1} S^{m_2}\right) = \sum_{k=0}^{N-1} e^{ikm_1\frac{2\pi}{N}} = 0.$$

In either case we have that $\tau(\lambda_\sigma((m_1, m_2))) = \mathrm{tr} \circ \Phi(\lambda_\sigma((m_1, m_2)))$ for all $(m_1, m_2) \neq (0, 0)$. If $(m_1, m_2) = (0, 0)$ we simply get $\tau((0, 0)) = 1$ and $\mathrm{tr} \circ \Phi(\lambda_\sigma((m_1, m_2))) = e^0 = 1$. Hence $\tau = \mathrm{tr} \circ \Phi$.

Since $\tau = \mathrm{tr} \circ \Phi$, Φ is norm preserving w.r.t. $\|\cdot\|_\tau$ and $\|\cdot\|_{\mathrm{tr}}$. An isometric linear transformation is injective. As Φ is then injective, $\dim \ker(\Phi) = 0$, which means that

$$N^2 = \dim C_r^*(G, \sigma) = \dim \ker(\Phi) + \dim \mathrm{im}(\Phi) = \dim \mathrm{im}(\Phi).$$

Because $\dim M_N(\mathbb{C}) = N^2$ also, and Φ is linear, this means that Φ is surjective as well. Hence Φ is a $*$-isomorphism and we are finished. ∎

Corollary 4.2.6. *The set*

$$\{e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2} \mid (m_1, m_2) \in \mathbb{Z}_N \times \mathbb{Z}_N\}$$

is a basis for $M_N(\mathbb{C})$.

Proof. By Theorem 4.2.5, Φ is an isomorphism and

$$\phi\left(\{\lambda_\sigma\left(m_1, m_2\right) \mid (m_1, m_2) \in G\}\right) = \{e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2} \mid (m_1, m_2) \in G\}.$$

Since the former set is a basis by Lemma 4.1.2, the latter set is then a basis for $M_N(\mathbb{C})$. ∎

4.3 Quantum channels on $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma)$

In this section we are interested in examining the relationship between the quantum channels on $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma)$ arising from positive-definite functions φ, and the Weyl-covariant channels on $M_N(\mathbb{C})$. As in Chapter 3 we use Bochner's theorem to characterise φ.

Proposition 4.3.1. *Let φ be a normalised positive definite function on $\mathbb{Z}_N$. Then φ is associated with a probability vector p over $\mathbb{Z}_N \times \mathbb{Z}_N$ in the sense that there exists such a p such that*

$$\varphi(m_1, m_2) = \sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)} p(k_1, k_2), \tag{4.3}$$

for all $m_1, m_2 \in \mathbb{Z}_N$. We write $\varphi = \varphi_p$. Conversely, if p is such a probability vector, φ_p is normalised positive definite.

Proof. As G is abelian, we know by Theorem 2.4.8 that φ is positive definite if and only if there exists a regular Borel measure μ on $\widehat{G}$ such that

$$\varphi(g) = \int_{\widehat{G}} \langle g, \gamma\rangle d\mu(\gamma). \tag{4.4}$$

It is well known that

$$(k_1, k_2) \mapsto \gamma_{(k_1,k_2)},$$

where

$$\gamma_{(k_1,k_2)}(m_1, m_2) = e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)}$$

is an isomorphism of G with $\widehat{G}$. Hence Equation (4.4) becomes

$$\begin{aligned}\varphi(m_1, m_2) &= \int_G \langle (m_1, m_2), \gamma_{(k_1,k_2)}\rangle d\mu((k_1, k_2))\\ &= \int_G e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)} d\mu((k_1, k_2))\end{aligned}$$

which means that

$$\varphi(m_1, m_2) = \sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)} p(k_1, k_2),$$

where $p(k_1, k_2) = \mu(\{k_1, k_2\})$. Since μ is a measure p has non-negative values. As we additionally demand that φ is normalised we get that

$$\varphi(0, 0) = \sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1 0+k_2 0)} p(k_1, k_2) = \sum_{(k_1,k_2)\in G} p(k_1, k_2) = 1$$

which means that p is a probability vector over $\mathbb{Z}_N \times \mathbb{Z}_N$.

Conversely, if p is a probability vector over $\mathbb{Z}_N \times \mathbb{Z}_N$, then the map φ_p defined by

$$\varphi_p(m_1, m_2) := \sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}km} p(k)$$

for all $m_1, m_2 \in \mathbb{Z}_N$ is positive definite by Theorem 2.4.8. ∎

$$\begin{tikzcd}
C_r^*(G,\sigma) \arrow[rr, shift left, "\Phi"] \arrow[dd, "Q_\varphi"'] & & M_N(\mathbb{C}) \arrow[ll, dashed, shift left, "\Phi^{-1}"] \arrow[dd, "\tilde{Q}_\varphi"] \\
& & \\
C_r^*(G,\sigma) \arrow[rr, "\Phi"] & & M_N(\mathbb{C})
\end{tikzcd}$$

Figure 4.1: The induced map $\tilde{Q}_\varphi$ for $G = \mathbb{Z}_N \times \mathbb{Z}_N$

As was the case in Chapter 3, we could relax our condition that φ_p be normalised. Then the corresponding p is just a map $p : \mathbb{Z}_N \times \mathbb{Z}_N \to [0, \infty)$, not necessarily a probability vector.

Corollary 4.3.2. *Let φ_p be a normalised positive definite function on $\mathbb{Z}_N \times \mathbb{Z}_N$. Then there exists a quantum channel Q_{φ_p} on $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma)$ such that*

$$Q_{\varphi_p}(\lambda(m_1, m_2)) = \left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} p(k_1, k_2) \right) \lambda_\sigma(m_1, m_2) \tag{4.5}$$

for all $m_1, m_2 \in \mathbb{Z}_N$.

Proof. Since φ_p is positive definite, Q_{φ_p} exists by Proposition 2.4.5 and is on the form $Q_{\varphi_p}(\lambda_\sigma(m_1, m_2)) = \varphi(m_1, m_2)\lambda_\sigma(m_1, m_2)$. Then Equation (4.5) follows from Equation (4.3). ∎

Induced channels on $M_N(\mathbb{C})$

Let φ be a (normalised) positive definite function on G. By Proposition 2.4.5 we know that there exists a quantum channel Q_φ on $(C_r^*(G,\sigma), \tau)$ satisfying

$$Q_\varphi(\lambda_\sigma(g)) = \varphi(g)\lambda_\sigma(g) \text{ for all } g \in G. \tag{4.6}$$

From Theorem 4.2.5 we know an explicit $*$-isomorphism $\Phi : C_r^*(G,\sigma) \to M_N(\mathbb{C})$. Hence each Q_φ induces a quantum channel $\tilde{Q}_\varphi = \Phi \circ Q_\varphi \circ \Phi^{-1}$ on $(M_N(\mathbb{C}), \tau \circ \Phi^{-1})$ by Corollary 2.4.6, but it can be shown that $\tau \circ \Phi^{-1} = \frac{1}{N}\mathrm{Tr}$, where Tr is the standard matrix trace. This was essentially done in the proof of Theorem 4.2.5, where we showed that $\tau = \mathrm{tr} \circ \Phi$. Hence $\tilde{Q}_\varphi$ is a quantum channel on $(M_N(\mathbb{C}), \mathrm{Tr})$ as well, analogously to Proposition 3.3.5. See Figure 4.1.

Our current goal is to show that each $\tilde{Q}_\varphi$ is a Weyl-covariant channel. These were discussed in Section 2.4. To do so we first need to characterise the Q_φ's.

Theorem 4.3.3. *Let p be a probability vector over $\mathbb{Z}_N \times \mathbb{Z}_N$ and let φ_p be the normalised positive definite function associated with p, in the sense of*

Proposition 4.3.1. Then the quantum channel $\tilde{Q}_{\varphi_p}$ on $M_N(\mathbb{C})$ arising from Corollary 2.4.6 is on the form

$$\tilde{Q}_\varphi\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right) = \left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)}p(k_1,k_2)\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2} \tag{4.7}$$

for all $m_1, m_2 \in \mathbb{Z}_N$.

Proof. That $\tilde{Q}_{\varphi_p}$ exists was established in the above discussion. Next we want an expression for $\tilde{Q}_\varphi(A)$ when $\varphi = \varphi_p$. That is, we want to calculate $\Phi\circ Q_\varphi\circ\Phi^{-1}(A)$. We know how Φ^{-1} works on matrices on the form $e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}$, so let us start with that. Since these matrices constitute a basis for $M_N(\mathbb{C})$ by Corollary 4.2.6 this is in fact sufficient. Calculating we see by definition of Φ that

$$\begin{aligned}\Phi\circ Q_\varphi\circ\Phi^{-1}\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right) &= \Phi\circ Q_\varphi\left(\lambda_\sigma(m_1,m_2)\right)\\ &= \Phi\left(\varphi(m_1,m_2)\lambda_\sigma(m_1,m_2)\right)\end{aligned}$$

by Equation (4.6), which is again equal to

$$\left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)}p(k_1,k_2)\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}$$

by definition of Φ and Equation (4.3). In conclusion,

$$\tilde{Q}_\varphi\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right) = \left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2m_2)}p(k_1,k_2)\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}.$$

■

Equation (4.7) is useful because it completely describes $\tilde{Q}_{\varphi_p}$ due to Corollary 4.2.6.

As stated, we want to show that $\tilde{Q}_\varphi$ is a Weyl-covariant channel. We can do this by using Theorem 2.4.11, but $\tilde{Q}_\varphi$ is defined on

$$\{e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2} \mid (m_1,m_2)\in\mathbb{Z}_N\times\mathbb{Z}_N\}.$$

Rather, we are interested in how it acts on the discrete Weyl operators. Luckily it turns out that these operators are in fact identical to our matrix basis elements.

Lemma 4.3.4. *The set of linear operators corresponding to the set of matrices*

$$\{e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2} \mid (m_1,m_2)\in\mathbb{Z}_N\times\mathbb{Z}_N\}$$

is the set of discrete Weyl-operators.

Proof. We have to show that

$$\sum_{k \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}km_1} E_{m_2 \dot{+} k} = e^{-i\frac{2\pi}{N}m_1 m_2} \tilde{D}^{m_1} \tilde{S}^{m_2}$$

for all $m_1, m_2 \in \mathbb{Z}_N$. We write $\tilde{D}$ and $\tilde{S}$ because the left hand side is an operator, and the right hand side strictly speaking would be a matrix otherwise. To do this we apply each part to the standard basis vector $\boldsymbol{e}_n$ for some $n \in \mathbb{Z}_N$. Doing this with the discrete Weyl-operators we get

$$\left(\sum_{k \in \mathbb{Z}_N} e^{i\frac{2\pi}{N}km_1} E_{m_2 \dot{+} k} \right) \boldsymbol{e}_n = e^{i\frac{2\pi}{N}nm_1} \boldsymbol{e}_{m_2 \dot{+} n}$$

because $E_{m_2 \dot{+} k, k} \boldsymbol{e}_n = \boldsymbol{e}_{m_2 \dot{+} n}$ whenever $k = n$, and $\mathbf{0}$ otherwise. On the other hand

$$\begin{aligned} e^{-i\frac{2\pi}{N}m_1 m_2} \tilde{D}^{m_1} \tilde{S}^{m_2} \boldsymbol{e}_n &= e^{-i\frac{2\pi}{N}m_1 m_2} \tilde{D}^{m_1} \boldsymbol{e}_{m_2 \dot{+} n} \\ &= e^{-i\frac{2\pi}{N}m_1 m_2} e^{i\frac{2\pi}{N}m_1 (m_2 \dot{+} n)} \boldsymbol{e}_{m_2 \dot{+} n} \\ &= e^{i\frac{2\pi}{N}m_1 n} \boldsymbol{e}_{m_2 \dot{+} n}, \end{aligned}$$

which completes the proof. ■

This means that we can consider the basis for $M_N(\mathbb{C})$ to be the discrete Weyl-operators.

Corollary 4.3.5. *Identifying each discrete Weyl-operator W_{m_1,m_2} with its standard matrix we may write*

$$\tilde{Q}_{\varphi_p}(W_{m_1,m_2}) = \left(\sum_{(k_1,k_2) \in G} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} p(k_1, k_2) \right) W_{m_1,m_2}. \tag{4.8}$$

Proof. This follows from applying Lemma 4.3.4 to Equation (4.7). ■

Now we know how $\tilde{Q}_\varphi$ acts on the discrete Weyl-operators, so we can check that it is a Weyl-covariant channel.

Proposition 4.3.6. *The map $\tilde{Q}_\varphi$ is a Weyl-covariant channel.*

Proof. We want to calculate $\tilde{Q}_\varphi(W_{m_1,m_2})$ for each $m_1, m_2 \in \mathbb{Z}_N$. By Corollary 4.3.5 we have that

$$\tilde{Q}_{\varphi_p}(W_{m_1,m_2}) = \left(\sum_{(k_1,k_2) \in G} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} p(k_1, k_2) \right) W_{m_1,m_2}$$

Setting

$$\alpha_{m_1,m_2} = \sum_{(k_1,k_2) \in G} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} p(k_1, k_2)$$

for all $m_1, m_2 \in \mathbb{Z}_N$ we see that $\tilde{Q}_\varphi$ is a Weyl-covariant map by Theorem 2.4.11. ■

Theorem 4.3.7. *Let $T : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ be a linear map. Then T is a Weyl-covariant channel if and only if there exists a (normalised) positive definite function φ on $\mathbb{Z}_N \times \mathbb{Z}_N$ such that $T = \tilde{Q}_\varphi$ on $M_N(\mathbb{C})$.*

Proof. Suppose first that there exists such a φ. Then T is a Weyl-covariant channel by Proposition 4.3.6.

For the other direction, suppose that T is a Weyl-covariant channel. We want to check that there exists some positive definite function φ such that T and $\tilde{Q}_\varphi$ act identically on the basis elements. Since T is Weyl-covariant Proposition 2.4.12 implies there exists a probability vector p over $\mathbb{Z}_N \times \mathbb{Z}_N$ such that

$$T(A) = \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) W_{k_1,k_2} A W^*_{k_1,k_2}$$

for all $A \in M_N(\mathbb{C})$. In particular, for $m_1, m_2 \in \mathbb{Z}_N$,

$$\begin{aligned} T\left(e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2}\right) &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) W_{k_1,k_2} e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2} W^*_{k_1,k_2} \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) e^{-i\frac{2\pi}{N}k_1k_2} D^{k_1} S^{k_2}\; e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2}\; e^{i\frac{2\pi}{N}k_1k_2} S^{-k_2} D^{-k_1} \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) D^{k_1} S^{k_2} D^{m_1} S^{m_2} S^{-k_2} D^{-k_1} e^{-i\frac{2\pi}{N}m_1m_2}. \end{aligned} \tag{4.9}$$

Next we apply the commutation rule from Lemma 4.2.3 to get

$$S^{k_2} D^{m_1} = e^{-i\frac{2\pi}{N}m_1k_2} D^{m_1} S^{k_2}$$

and

$$D^{k_1} S^{m_2} = e^{i\frac{2\pi}{N}k_1m_2} S^{m_2} D^{k_1}.$$

Returning to Equation (4.9), this means that

$$\begin{aligned} &T\left(e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2}\right) \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) D^{k_1} \left(S^{k_2} D^{m_1}\right) S^{m_2} S^{-k_2} D^{-k_1} e^{-i\frac{2\pi}{N}m_1m_2} \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) D^{k_1} \left(e^{-i\frac{2\pi}{N}m_1k_2} D^{m_1} S^{k_2}\right) S^{m_2} S^{-k_2} D^{-k_1} e^{-i\frac{2\pi}{N}m_1m_2} \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) D^{m_1} \left(D^{k_1} S^{m_2}\right) D^{-k_1} e^{-i\frac{2\pi}{N}m_1k_2} e^{-i\frac{2\pi}{N}m_1m_2} \\ &= \sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) D^{m_1} \left(e^{i\frac{2\pi}{N}k_1m_2} S^{m_2} D^{k_1}\right) D^{-k_1} e^{-i\frac{2\pi}{N}m_1k_2} e^{-i\frac{2\pi}{N}m_1m_2} \\ &= \left(\sum_{k_1,k_2 \in \mathbb{Z}_N} p(k_1,k_2) e^{-i\frac{2\pi}{N}(m_1k_2 + k_1m_2)}\right) e^{-i\frac{2\pi}{N}m_1m_2} D^{m_1} S^{m_2}. \end{aligned} \tag{4.10}$$

Now we are nearly done, but two of the indices are wrong. Swapping variable names for k_1 and k_2 in Equation (4.10) we get that

$$\begin{aligned}
&T\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right)\\
&\quad=\left(\sum_{k_1,k_2\in\mathbb{Z}_N}p(k_1,k_2)e^{-i\frac{2\pi}{N}(m_1k_2+k_1m_2)}\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\\
&\quad=\left(\sum_{k_1,k_2\in\mathbb{Z}_N}p(k_2,k_1)e^{-i\frac{2\pi}{N}(m_1k_1+k_2m_2)}\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\\
&\quad=\tilde{Q}_{\varphi_{p'}}\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right),
\end{aligned}$$

where $p'(k_1,k_2)=p(k_2,k_1)$ for all $k_1,k_2\in\mathbb{Z}_N$. In the last step we used Equation (4.8) and the fact that for each p, $\varphi_{p'}$ is a positive definite function. Hence T and $\tilde{Q}_{p'}$ are equal on each basis element, and therefore on $M_N(\mathbb{C})$. Thus the proof is finished. ■

4.4 The case for general k

In Theorem 4.2.5 we showed that $C_r^*(G,\sigma)\simeq M_N(\mathbb{C})$, but what we actually wanted was $C_r^*(G,\sigma_k)\simeq M_N(\mathbb{C})$ for all $k\in\mathbb{Z}_N$. The latter is what we will be concerned with in this section, but it turns out to only hold for some k. We do not show this directly, but rather show that $C_r^*(G,\sigma_k)$ is $*$-isomorphic to $C_r^*(G,\sigma)$ for the relevant $k\in\mathbb{Z}_N$.

General remarks

Let us start with some more general remarks and the return to the situation at hand later. In this subsection we assume that G is a group, σ a 2-cocycle on G and $\alpha: G\to G$ a homomorphism with $\alpha(e)=e$. Furthermore $\sigma_\alpha: G\times G\to\mathbb{T}$ is a map given by

$$\sigma_\alpha(g,h)=\sigma(\alpha(g),\alpha(h)) \tag{4.11}$$

We check that this is in fact also a 2-cocycle, which lets us define $C_r^*(G,\sigma_\alpha)$.

Lemma 4.4.1. *The map σ_α defined above is a 2-cocycle.*

Proof. The proof follows by computation. Let $g,h,k\in G$. Then, by definition

$$\begin{aligned}
\sigma_\alpha(g,h)\sigma_\alpha(gh,k)&=\sigma(\alpha(g),\alpha(h))\;\sigma(\alpha(gh),\alpha(k))\\
&=\sigma(\alpha(g),\alpha(h))\;\sigma(\alpha(g)\alpha(h),\alpha(k))
\end{aligned}$$

since α is a homomorphism. Using that σ is a 2-cocycle we get that this is again equal to

$$\begin{aligned}
\sigma(\alpha(h),\alpha(k))\;\sigma(\alpha(g),\alpha(h)\alpha(k))&=\sigma(\alpha(h),\alpha(k))\;\sigma(\alpha(g),\alpha(hk))\\
&=\sigma_\alpha(h,k)\sigma_\alpha(g,hk).
\end{aligned}$$

Hence σ_α is a 2-cocycle. It is normalised because

$$\sigma_\alpha(g,e) = \sigma(\alpha(g),\alpha(e)) = \sigma(\alpha(g),e) = 1,$$

since $\alpha(e) = e$ by assumption and σ is normalised. ■

We denote by $\mathrm{Aut}(G)$ the group of automorphisms on G, i.e. bijective homomorphisms from G to itself. From now on, in this subsection, we also assume that $\alpha \in \mathrm{Aut}(G)$.

Next we define the map $U_\alpha : \ell^2(G) \to \ell^2(G)$ by $(U_\alpha \xi)(g) = \xi\left(\alpha^{-1}(g)\right)$. It has the following properties.

Lemma 4.4.2. *The adjoint U_α^* of U_α is $U_{\alpha^{-1}}$, and U_α is unitary.*

Proof. The inverse α^{-1} is defined since $\alpha \in \mathrm{Aut}(G)$. Let $\xi, \zeta \in \ell^2(G)$. Then

$$\langle U_\alpha \xi, \zeta\rangle = \sum_{g\in G} (U_\alpha\xi)(g)\overline{\zeta(g)} = \sum_{g\in G} \xi\left(\alpha^{-1}(g)\right)\overline{\zeta(g)},$$

and

$$\langle \xi, U_{\alpha^{-1}}\zeta\rangle = \sum_{h\in G} \xi(h)\overline{(U_{\alpha^{-1}}\zeta)(h)} = \sum_{h\in G} \xi(h)\overline{\zeta(\alpha(h))}.$$

These are the same since we can substitute $h = \alpha^{-1}(g)$ in the second sum. This shows that $U_\alpha^* = U_\alpha^{-1}$.

It is clear that U_α is unitary because, for all $g \in G$, we have that

$$(U_\alpha^* U_\alpha \xi)(g) = (U_\alpha\xi)(\alpha(g)) = \xi\left(\alpha^{-1}(\alpha(g))\right) = \xi(g)$$

and

$$(U_\alpha U_\alpha^* \xi)(g) = (U_\alpha\xi)\left(\alpha^{-1}(g)\right) = \xi\left(\alpha(\alpha^{-1}(g))\right) = \xi(g),$$

which means that both $U_\alpha^* U_\alpha$ and $U_\alpha U_\alpha^*$ are equal to the identity operator. ■

Lemma 4.4.3. *Let $g \in G$ and consider the corresponding $\lambda_{\sigma_\alpha}(g) \in C_r^*(G,\sigma_\alpha)$ and $\lambda_\sigma(g) \in C_r^*(G,\sigma)$. We have that*

$$U_\alpha \lambda_{\sigma_\alpha}(g) U_\alpha^* = \lambda_\sigma(\alpha(g)).$$

Proof. The proof follows by computation. Let $\xi \in \ell^2(G)$ and $h \in G$. Then by definition of the twisted left regular representation

$$(\lambda_\sigma(\alpha(g))\xi) = \sigma\left(\alpha(g), \alpha(g)^{-1}h\right)\left(\alpha(g)^{-1}h\right),$$

while

$$\begin{aligned}
(U_\alpha \lambda_{\sigma_\alpha}(g) U_\alpha^*)(h) &= (\lambda_{\sigma_\alpha}(g)U_\alpha^*)\left(\alpha^{-1}(h)\right) \\
&= \sigma_\alpha\left(g, g^{-1}\alpha^{-1}(h)\right)(U_\alpha^*)\left(g^{-1}\alpha^{-1}(h)\right) \\
&= \sigma\left(\alpha(g), \alpha\left(g^{-1}\alpha^{-1}(h)\right)\right)\left(\alpha\left(g^{-1}\alpha^{-1}(h)\right)\right) \\
&= \sigma\left(\alpha(g), \alpha(g^{-1})h\right)\left(\alpha(g^{-1})h\right) \\
&= \sigma\left(\alpha(g), \alpha(g)^{-1}h\right)\left(\alpha(g)^{-1}h\right)
\end{aligned}$$

because α is a homomorphism. Hence $U_\alpha \lambda_{\sigma_\alpha}(g) U_\alpha^* = \lambda_\sigma(\alpha(g))$. ■

Our preceding construction lets us give an explicit $*$-isomorphism between $C_r^*(G, \sigma_\alpha)$ and $C_r^*(G, \sigma)$ as follows.

Proposition 4.4.4. *The map $\pi_\alpha : C_r^*(G, \sigma_\alpha) \to C_r^*(G, \sigma)$ given by*

$$\pi_\alpha(x) = U_\alpha x U_\alpha^*$$

for $x \in C_r^*(G, \sigma_\alpha)$ is a $*$-isomorphism.

Proof. By Lemma 4.4.3, π_α maps $C_r^*(G, \sigma_\alpha)$ into $C_r^*(G, \sigma)$. As U_α is unitary by Lemma 4.4.2 we have that for all $x_1, x_2 \in C_r^*(G, \sigma_\alpha)$

$$\pi_\alpha(x_1)\pi_\alpha(x_2) = (U_\alpha x_1 U_\alpha^*)(U_\alpha x_2 U_\alpha^*) = U_\alpha x_1 x_2 U_\alpha^* = \pi_\alpha(x_1 x_2).$$

Also, for $x \in C_r^*(G, \sigma_\alpha)$,

$$\pi_\alpha(x)^* = (U_\alpha x U_\alpha^*)^* = (U_\alpha^*)^* x^* U_\alpha^* = \pi_\alpha(x^*).$$

Hence π_α is a $*$-homomorphism. Furthermore it is clearly linear since U_α is.

Note next that for $y \in C_r^*(G, \sigma)$, π_α has an inverse π_α^{-1} given by

$$\pi_\alpha^{-1}(y) = U_\alpha^* y U_\alpha$$

because U_α is unitary. This maps $C_r^*(G, \sigma)$ into $C_r^*(G, \sigma_\alpha)$ by Lemma 4.4.3. Hence π_α is a $*$-isomorphism. ■

Keeping the last two results in mind, we return to the matter at hand.

Our case

Recall that we are working with the case $G = \mathbb{Z}_N \times \mathbb{Z}_N$ and are studying the 2-cocycle σ_k on G given by $\sigma_k((m_1, m_2), (n_1, n_2)) = e^{i\frac{2\pi}{N}km_1n_2}$, see Section 4.1. To proceed we start by, for each $k \in \mathbb{Z}_N$, defining a map $\alpha_k : G \to G$ by

$$\alpha_k((m_1, m_2)) = (m_1, km_2).$$

This is clearly a homomorphism that sends $\mathbf{0}$ to $\mathbf{0}$.

The key point in this argument is that we know from elementary number theory that $\alpha_k \in \operatorname{Aut}(G)$ if and only if k and N are relatively prime. Keeping this in mind we consider the 2-cocycle σ_{α_k} from Equation (4.11) for $\sigma = \sigma_1$ and $\alpha = \alpha_k$. We get

$$\begin{aligned} \sigma_{\alpha_k}((m_1, m_2), (n_1, n_2)) &= \sigma(\alpha_k(m_1, m_2), \alpha_k(n_1, n_2)) \\ &= \sigma((m_1, km_2), (n_1, kn_2)) \\ &= e^{i\frac{2\pi}{N}m_1kn_2} \\ &= \sigma_k((m_1, m_2), (n_1, n_2)). \end{aligned}$$

Hence in fact $\sigma_{\alpha_k} = \sigma_k$. Let us summarise our results in a proposition.

Proposition 4.4.5. *Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}_N$ be such that k and N are relatively prime. Then $\pi_{\alpha_k} : C_r^*(G, \sigma_k) \to C_r^*(G, \sigma)$ given by*

$$\pi_{\alpha_k}(x) = U_{\alpha_k} x U_{\alpha_k}^*$$

is a $*$-isomorphism.

Proof. Because $\alpha_k \in \mathrm{Aut}(\mathbb{Z}_N \times \mathbb{Z}_N)$ this follows from Proposition 4.4.4 since $\sigma_{\alpha_k} = \sigma_k$. ■

Theorem 4.4.6. *Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}_N$. Then $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ and $M_N(\mathbb{C})$ are $*$-isomorphic if and only if k is relatively prime to N.*

Proof. Suppose that k and N are relatively prime. Then $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ is $*$-isomorphic to $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma)$ by Proposition 4.4.5, which is $*$-isomorphic to $M_N(\mathbb{C})$ by Theorem 4.2.5.

For k and N not relatively prime, [Li+19] show that $\mathbb{Z}_N \times \mathbb{Z}_N$ has multiple σ_k-projective representations. Had $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ been isomorphic to $M_N(\mathbb{C})$, there could only have been one. ■

As we do not get an isomorphism for the case where k and N are not relatively prime, we restrict ourselves to the case where they are.

Quantum channels on $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$ revisited

Having found an isomorphism we once more let φ be a (normalised) positive definite function on G and turn to studying the quantum channel $Q_\varphi^{\sigma_k}$ on $C_r^*(G, \sigma_k)$ arising from Proposition 2.4.5. By Proposition 2.4.3 and the preceding discussion, $Q_\varphi^{\sigma_1} = \pi_{\alpha_k} Q_\varphi^{\sigma_k} \pi_{\alpha_k}^{-1}$ is a quantum channel on $C_r^*(G, \sigma)$.

Our current goal is to mimic the construction of $\tilde{Q}_\varphi$ on $M_N(\mathbb{C})$ from Section 4.3. To do so we first check how $Q_\varphi^{\sigma_1}$ acts on $C_r^*(G, \sigma)$. Since $Q_\varphi^{\sigma_1}$ is linear it suffices to apply it to the basis elements. Hence we calculate, for $g \in G$,

$$\begin{aligned} Q_\varphi^{\sigma_1}(\lambda_\sigma(g)) &= \pi_{\alpha_k} Q_\varphi^{\sigma_k} \pi_{\alpha_k}^{-1}(\lambda_\sigma(g)) \\ &= \pi_{\alpha_k} Q_\varphi^{\sigma_k}\left(U_{\alpha_k}^* \lambda_\sigma(g) U_{\alpha_k}\right) \\ &= \pi_{\alpha_k} Q_\varphi^{\sigma_k}\left(\lambda_{\sigma_k}(\alpha_k^{-1}(g))\right) \end{aligned}$$

by Lemma 4.4.3. Using Proposition 2.4.5 we get that this again is equal to

$$\pi_{\alpha_k}\left(\varphi(\alpha^{-1}(g))\lambda_{\sigma_k}(g)\right) = \varphi(\alpha_k^{-1}(g)) U_{\alpha_k} \lambda_{\sigma_k}(\alpha^{-1}(g)) U_{\alpha_k}^* = \varphi(\alpha_k^{-1}(g))\lambda_\sigma(g),$$

by Lemma 4.4.3. Thus

$$Q_\varphi^{\sigma_1}(\lambda_\sigma(g)) = \varphi(\alpha_k^{-1}(g))\lambda_\sigma(g). \tag{4.12}$$

It follows that $Q_\varphi^{\sigma_1}$ is the quantum channel associated with the positive definite function $\varphi \circ \alpha_k^{-1}$.

Since both π_{α_k} and Φ are $*$-isomorphisms, so is their composition. Hence $\tilde{Q}_\varphi^{\sigma_k} = \Phi \pi_{\alpha_k} Q_\varphi^{\sigma_k} \pi_{\alpha_k}^{-1} \Phi^{-1}$ is a quantum channel on $M_N(\mathbb{C})$ by Proposition 2.4.3. See Figure 4.2.

We want an explicit expression for $\tilde{Q}_\varphi^{\sigma_k}$, but repeating our previous arguments would be tedious. Instead we notice that $\tilde{Q}_\varphi^{\sigma_k} = \Phi Q_\varphi^{\sigma_1} \Phi^{-1}$. Since $Q_\varphi^{\sigma_1}$ is a quantum channel on $C_r^*(G, \sigma)$, we can use our earlier work from Section 4.3.

Recall from Corollary 4.2.6 that elements on the form $e^{-i\frac{2\pi}{N}m_1 m_2} D^{m_1} S^{m_2}$, where $m_1, m_2 \in \mathbb{Z}_N$, constitute a basis for the $N \times N$ matrices. Let $m_1, m_2 \in \mathbb{Z}_N$,

$$\begin{tikzcd}
C_r^*(G,\sigma_k) \arrow[r, shift left, "\pi_{\alpha_k}"] \arrow[d, "Q_\varphi^{\sigma_k}"'] & C_r^*(G,\sigma_1) \arrow[l, dashed, shift left, "\pi_{\alpha_k}^{-1}"] \arrow[r, shift left, "\Phi"] \arrow[d, "Q_\varphi^{\sigma_1}"'] & M_N(\mathbb{C}) \arrow[l, dashed, shift left, "\Phi^{-1}"] \arrow[d, "\tilde{Q}_\varphi^{\sigma_k}"] \\
C_r^*(G,\sigma_k) \arrow[r, "\pi_{\alpha_k}"] & C_r^*(G,\sigma_1) \arrow[r, "\Phi"] & M_N(\mathbb{C})
\end{tikzcd}$$

Figure 4.2: The induced map $\tilde{Q}_\varphi^{\sigma_k}$

then by definition of Φ (Equation (4.2))

$$\begin{aligned}
\tilde{Q}_\varphi^{\sigma_k}\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right) &= \Phi Q_\varphi^{\sigma_1}\Phi^{-1}\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right)\\
&= \Phi Q_\varphi^{\sigma_1}\left(\lambda_\sigma(m_1,m_2)\right)\\
&= \Phi\left(\varphi(\alpha_k^{-1}(m_1,m_2))\lambda_\sigma(m_1,m_2)\right)
\end{aligned}$$

by Equation (4.12). Using linearity and the definition of Φ once more, this again is equal to $\varphi(\alpha_k^{-1}(m_1,m_2))e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}$. Since α_k is a bijection there is for each $(m_1,m_2)\in G$ a unique element $l_{k,m_2}\in\mathbb{Z}_N$ such that $\alpha_k^{-1}(m_1,m_2)=(m_1,l_{k,m_2})$. Hence we can write

$$\begin{aligned}
\varphi\left(\alpha_k^{-1}(m_1,m_2)\right)e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2} &= \varphi(m_1,l_{k,m_2})e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\\
= \left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2kl_{k,m_2})}p(k_1,k_2)\right) e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}
\end{aligned}$$

by Equation (4.3). Concluding, this means that

$$\begin{aligned}
&\tilde{Q}_\varphi^{\sigma_k}\left(e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}\right)\\
&\quad= \left(\sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2l_{k,m_2})}p(k_1,k_2)\right) e^{-i\frac{2\pi}{N}m_1m_2}D^{m_1}S^{m_2}.
\end{aligned}\tag{4.13}$$

Theorem 4.4.7. *Let φ be a normalised positive definite function on G. Then the map $\tilde{Q}_\varphi^{\sigma_k}: M_N(\mathbb{C})\to M_N(\mathbb{C})$ discussed above is a Weyl-covariant channel.*

Proof. By Equation (4.13) we have that $\tilde{Q}_\varphi^{\sigma_k}$ satisfies Theorem 2.4.11 with

$$\alpha_{m_1,m_2} = \sum_{(k_1,k_2)\in G} e^{i\frac{2\pi}{N}(k_1m_1+k_2l_{k,m_2})}p(k_1,k_2)$$

for all $m_1,m_2\in G$. Hence the claim follows. ∎

Since $\tilde{Q}_\varphi^{\sigma_k}$ is a Weyl-covariant channel, and Weyl-covariant channels are described by some $\tilde{Q}_\varphi$ in Theorem 4.3.7, we get to forget about the case $k \neq 1$ when k and N are relatively prime.

4.5 Contractive channels on $M_N(\mathbb{C})$

We end this chapter by discussing the contractive properties of the quantum channels $\tilde{Q}_\varphi$ that we have considered so far. Our main piece of information about them is from Theorem 4.3.7: Studying the contractive properties of the channels $\tilde{Q}_\varphi$ is equivalent to studying the contractive properties of Weyl-covariant channels. As in Section 3.5 we are interested in the d_1- and d_B^τ-metrics.

The Tr-density space of $M_N(\mathbb{C})$ is

$$\mathcal{D}_{\mathrm{Tr}}(M_N(\mathbb{C})) = \left\{a \in M_N(\mathbb{C})^+ \mid \mathrm{Tr}(a) = 1\right\},$$

which is the usual space of density matrices, discussed in [Wat18] (density operators).

Contractions with relation to d_1

Let Q be a quantum channel on $M_N(\mathbb{C})$. We are interested in whether or not it is a contraction w.r.t. the d_1-metric. The d_1-metric on $M_N(\mathbb{C})$ is defined by

$$d_1(A, B) = \mathrm{Tr}(|A - B|)$$

for all $A, B \in M_N(\mathbb{C})$. Here Tr is the standard matrix trace. We mention the following result, which is adapted to our notation.

Proposition 4.5.1 ([HR16] (Proposition 4.1))**.** *Let $\phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ be a linear map. If ϕ is positive and* Tr*-preserving, then ϕ is non-expansive w.r.t. d_1.*

Applied to our situation, we get

Corollary 4.5.2. *Let φ be a normalised positive definite function on $\mathbb{Z}_N \times \mathbb{Z}_N$. Then $\tilde{Q}_\varphi$ is non-expansive w.r.t. d_1.*

Proof. Since $\tilde{Q}_{\varphi_p}$ is a quantum channel on $(M_N(\mathbb{C}), \mathrm{Tr})$ the result follows by Proposition 4.5.1. ■

That Q is non-expansive is all well and good, but we would like to know more. Specifically, for which φ is $\tilde{Q}_\varphi$ a strict contraction? We follow [HR16] in introducing the *contraction coefficient* η^{Tr} w.r.t. d_1 of a quantum channel ϕ on $M_N(\mathbb{C})$ as

$$\begin{aligned}\eta^{\mathrm{Tr}}(\phi) &= \sup\left\{\frac{\|\phi(a) - \phi(b)\|_1}{\|a - b\|_1} \mid a, b \in \mathcal{D}_{\mathrm{Tr}}(M_N(\mathbb{C})), a \neq b\right\} \\ &= \sup\left\{\frac{\|\phi(a)\|_1}{\|a\|_1} \mid a \in M_N(\mathbb{C}) \text{ with } a = a^*, \mathrm{Tr}(a) = 0 \text{ and } a \neq 0\right\}.\end{aligned}$$

This is the lowest number C such that ϕ is a strict d_1-contraction w.r.t. it. They mention that η^{Tr} is a quantum generalisation of the Dobrushin coefficient of ergodicity, and that

$$\eta^{\mathrm{Tr}}(\phi) = \frac{1}{2} \sup \left\{ \|\phi(E) - \phi(F)\|_1 : E, F \text{ rank 1 projections in } M_N(\mathbb{C}), E \perp F \right\}.$$

Furtherfore, we have the following result.

Proposition 4.5.3 ([HR16] (Proposition 4.2))**.** *Let ϕ be a positive* Tr*-preserving map on $M_N(\mathbb{C})$. Then the following are equivalent.*

(i) $\eta^{\mathrm{Tr}}(\phi) < 1$,

(ii) $\mathrm{Tr}(\phi(E)\phi(F)) > 0$ *for all rank one projections $E, F \in M_N(\mathbb{C})$, $E \perp F$.*

(iii) $\mathrm{Tr}(\phi(A)\phi(B)) > 0$ *for all positive (semidefinite) $A, B \in M_N(\mathbb{C})$.*

Our goal now is to use Proposition 4.5.3 to put some condition on p such that $\tilde{Q}_{\varphi_p}$ is a strict contraction w.r.t. d_1 whenever that condition holds, similar to the role Proposition 3.5.5 played in Chapter 3. We have not been able to accomplish this, and settle for an example.

Example 4.5.4. Let M be an $N \times N$-matrix. Since the discrete Weyl-matrices form a basis for $M_N(\mathbb{C})$, we can write $M = \sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} W_{m_1,m_2}$ for some $c_{m_1,m_2} \in \mathbb{C}$. Suppose $p(0,0) = 1$, and $p(i,j) = 0$ otherwise. Then

$$\begin{aligned}
\tilde{Q}_{\varphi_p}(M) &= \tilde{Q}_{\varphi_p}\left(\sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} W_{m_1,m_2} \right) \\
&= \sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} \tilde{Q}_{\varphi_p}\left(W_{m_1,m_2}\right) \\
&= \sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} \left(\sum_{(k_1,k_2) \in G} e^{i\frac{2\pi}{N}(k_1 m_1 + k_2 m_2)} p(k_1,k_2) \right) W_{m_1,m_2}
\end{aligned}$$

by Equation (4.8). Since p is non-zero only when $k_1 = k_2 = 0$, this becomes

$$\begin{aligned}
\tilde{Q}_{\varphi_p}(M) &= \sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} \left(e^{i\frac{2\pi}{N}(0 m_1 + 0 m_2)} p(0,0) \right) W_{m_1,m_2} \\
&= \sum_{m_1,m_2 \in \mathbb{Z}_N} c_{m_1,m_2} W_{m_1,m_2} \\
&= M.
\end{aligned}$$

Hence $\tilde{Q}_{\varphi_p}$ is the identity on $M_N(\mathbb{C})$ in this case, and is thus not a strict contraction.

Contractions with relation to d_B^τ

Computations with the Bures metric are generally more difficult than computations with the trace metric. In this spirit we were not able to use our knowledge of $\tilde{Q}_{\varphi_p}$ to improve upon the conditions from Section 2.5.

CHAPTER 5

The third example

In this chapter we consider the reduced twisted group C^*-algebra of $\mathbb{Z}$, and related quantum channels.

All 2-cocycles on $\mathbb{Z}$ are similar to the trivial one [Kle65], so by Proposition 2.3.15 and Example 2.3.10 we only need to consider the reduced group C^*-algebra $C_r^*(\mathbb{Z})$ of $\mathbb{Z}$.

We have $G = \mathbb{Z}$. Our work in this chapter is twofold. First we want to establish a $*$-isomorphism Φ from $C_r^*(\mathbb{Z})$ to $C(\mathbb{T})$, the continuous functions from the unit circle $\mathbb{T}$ to $\mathbb{C}$. Secondly we want to study quantum channels on $C_r^*(\mathbb{Z})$.

5.1 The isomorphism $C_r^*(\mathbb{Z}) \simeq \mathbb{C}(\mathbb{T})$

If we merely wanted to know that $C_r^*(\mathbb{Z})$ is isomorphic to $\mathbb{C}(\mathbb{T})$ we could apply Gelfand theory to get that $C_r^*(\mathbb{Z}) = C(\widehat{\mathbb{Z}}) = C(\mathbb{T})$, but we want an explicit isomorphism to apply Corollary 2.4.6 and get an explicit expression for $\tilde{Q}_\varphi$.

To arrive at our desired $*$-isomorphism Φ we have to define some other maps first. Let h be the normalised Lebesgue measure on $[0, 2\pi)$ and write $L^2(\mathbb{T})$ for the L^2-space w.r.t. h. Furthermore, let $e_m \in C(\mathbb{T})$ be defined by

$$e_m(t) = e^{imt}, \quad 0 \leq t < 2\pi,$$

where $m \in \mathbb{Z}$. It is clear that $\{e_m\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. We then define a map

$$\begin{aligned} \mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}) \quad &\text{by} \\ g \mapsto \widehat{g}, \end{aligned} \tag{5.1}$$

where

$$\widehat{g}(m) = \langle g, e_m \rangle = \int_0^{2\pi} g(t) e^{-imt} dh(t)$$

for $m \in \mathbb{Z}$. For $\mathcal{F}$ to be well defined we have to show that $\widehat{g}$ is in $\ell^2(\mathbb{Z})$ for each $g \in L^2(\mathbb{T})$. This follows since

$$\int_0^{2\pi} g(t) dh(t) < \infty$$

because $g \in L^2(\mathbb{T})$. We denote that inner products on $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ by $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle \cdot, \cdot \rangle_{\ell^2}$ respectively, or simply $\langle \cdot, \cdot \rangle$ if it is clear from context which inner product we refer to.

Lemma 5.1.1. *The map $\mathcal{F}$ from Equation* (5.1) *is a unitary operator.*

Proof. We have to show that $\mathcal{F}$ is (i) bijective and (ii) conserves inner products, i.e. that

$$\langle \mathcal{F}f, \mathcal{F}g\rangle_{\ell^2} = \langle f, g\rangle_{L^2} \tag{5.2}$$

for all $f, g \in L^2(\mathbb{T})$.

(i) Firstly, $\mathcal{F}$ is linear since for $\alpha \in \mathbb{C}, n \in \mathbb{Z}$ and $f, g \in C(\mathbb{T})$,

$$\begin{aligned}\mathcal{F}(\alpha f + g)(n) &= (\widehat{\alpha f + g})(n) = \langle \alpha f + g, e_n\rangle \\ &= \alpha\langle f, e_n\rangle + \langle g, e_n\rangle = \alpha\mathcal{F}(f)(n) + \mathcal{F}(g)(n).\end{aligned}$$

Next we note that for each $m \in \mathbb{Z}$,

$$\mathcal{F}(e_m)(n) = \widehat{e_m}(n) = \langle e_m, e_n\rangle = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.}\end{cases}$$

Hence $\mathcal{F}$ maps the basis elements of $L^2(\mathbb{T})$ onto the basis elements of $\ell^2(\mathbb{Z})$, and is thus bijective.

(ii) We proceed by computation. Let $f, g \in C(\mathbb{T})$. By definition this means that

$$\begin{aligned}\langle \mathcal{F}f, \mathcal{F}g\rangle_{\ell^2} &= \langle \widehat{f}, \widehat{g}\rangle_{\ell^2} \\ &= \sum_{m\in\mathbb{Z}} \widehat{f}(m)\overline{\widehat{g}} \\ &= \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle\overline{\langle g, e_m\rangle}.\end{aligned} \tag{5.3}$$

Let us keep this in mind and consider the right hand side of Equation (5.2). There we see that

$$\begin{aligned}\langle f, g\rangle_{L^2} &= \int_0^{2\pi} f(t)\overline{g(t)}dh(t) \\ &= \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} \langle f, e_m\rangle e_m(t)\right)\left(\sum_{n\in\mathbb{Z}} \langle g, e_n\rangle e_n(t)\right) \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, e_m\rangle\langle g, e_n\rangle \int_0^{2\pi} e_{m-n}(t)dh(t) \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, e_m\rangle\langle g, e_n\rangle \int_0^{2\pi} e^{i(m-n)t}dh(t).\end{aligned}$$

The integral $\int_0^{2\pi} e^{i(m-n)t}dh(t)$ is equal to 1 for $m = n$ and 0 otherwise. Hence we get that

$$\langle f, g\rangle_{L^2} = \sum_{m\in\mathbb{Z}} \langle f, e_m\rangle\langle g, e_m\rangle, \tag{5.4}$$

which is exactly what we got in Equation (5.3). Thus Equation (5.2) is satisfied and $\mathcal{F}$ is unitary.

■

Next we define a map

$$F : \mathcal{B}(L^2(\mathbb{T})) \to \mathcal{B}(\ell^2(\mathbb{Z})) \;\text{ by}\\ T \mapsto \mathcal{F}T\mathcal{F}^*. \tag{5.5}$$

Lemma 5.1.2. *The map F from Equation* (5.5) *is an isometric $*$-homomorphism.*

Proof. Let $T, S \in \mathcal{B}(L^2(\mathbb{T}))$ and note that

$$F(T)F(S) = \mathcal{F}T\mathcal{F}^*\mathcal{F}S\mathcal{F}^* = \mathcal{F}TS\mathcal{F}^* = F(TS),$$

since $\mathcal{F}$ is unitary by Lemma 5.1.1. Similarly,

$$F(T)^* = (\mathcal{F}T\mathcal{F}^*)^* = \mathcal{F}T^*\mathcal{F}^* = F(T^*).$$

Hence F is a $*$-homomorphism.

For F to be isometric we have to show that

$$\|F(T)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} = \|T\|_{\mathcal{B}(L^2(\mathbb{T}))}.$$

By definition, we have

$$\begin{aligned} \|F(T)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} &= \|\mathcal{F}T\mathcal{F}^*\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} \\ &= \sup\left\{\|\mathcal{F}T\mathcal{F}^*\xi\|_{\ell^2} \;:\; \xi \in \ell^2(\mathbb{Z}), \|\xi\|_{\ell^2} = 1\right\} \\ &= \sup\left\{\|\mathcal{F}Tf\|_{\ell^2} \;:\; f \in L^2(\mathbb{T}), \|f\|_{L^2} = 1\right\}. \end{aligned}$$

Since $\mathcal{F}$ is unitary it preserves the norm, so $\|\mathcal{F}Tf\|_{\ell^2} = \|Tf\|_{L^2}$. Hence

$$\|F(T)\|_{\mathcal{B}(\ell^2(\mathbb{Z}))} = \sup\left\{\|Tf\|_{L^2} : f \in L^2(\mathbb{T}), \|f\|_{L^2} = 1\right\} = \|T\|_{\mathcal{B}(L^2(\mathbb{T}))},$$

and F is isometric.

■

Before proceeding, we define a final new map

$$M : C(\mathbb{T}) \to \mathcal{B}(L^2(\mathbb{T})) \;\text{ by}\\ f \mapsto M_f, \tag{5.6}$$

where $M_f(g) = fg$ for $g \in L^2(\mathbb{T})$. This is well defined because $fg \in L^2(\mathbb{T})$ for such f and g. In addition, M_f is clearly linear and bounded by the norm of f.

Lemma 5.1.3. *The map M from Equation* (5.6) *is an isometric $*$-homomorphism.*

Proof. Let $f, h \in C(\mathbb{T})$. The map M is a homomorphism because $M_{fh}(g) = fhg = M_f M_h g$ for each $g \in L^2(\mathbb{T})$, i.e. $M(fh) = M(f)M(h)$. Furthermore $M_{\overline{f}}$ is the adjoint of M_f because

$$\begin{aligned} \langle M_{\overline{f}}g_1, g_2\rangle &= \langle \overline{f}g_1, g_2\rangle \\ &= \int_{\mathbb{T}} \overline{f}g_1\overline{g_2}dh = \int_{\mathbb{T}} g_1\overline{fg_2}dh \\ &= \langle g_1, M_f g_2\rangle \end{aligned}$$

$$\begin{tikzcd} C(\mathbb{T}) \arrow[r, "M"] \arrow[rr, bend right, "\psi = F \circ M"'] & \mathcal{B}(L^2(\mathbb{T})) \arrow[r, "F"] & \mathcal{B}(\ell^2(\mathbb{Z})) \end{tikzcd}$$

Figure 5.1: The isometric $*$-homomorphism ψ

for all $g_1, g_2 \in L^2(\mathbb{T})$. Then clearly $M(f)^* = M(f^*)$.

It remains to show that M is isomorphic. We check that $\|M_f\| \leq \|f\|$ for $f \in C(\mathbb{T})$. This holds because

$$\|M_f\|_{\mathcal{B}(L^2(\mathbb{T}))} = \sup \left\{ \|M_f(g)\|_{L^2} : g \in L^2(\mathbb{T}), \|g\|_{L^2} = 1 \right\},$$

while

$$\|M_f(g)\|_{L^2} = \|fg\|_{L^2} \leq \|f\|_\infty \|g\|_{L^2} = \|f\|_\infty$$

for such g. Hence

$$\begin{aligned} \|M_f\|_{\mathcal{B}(L^2(\mathbb{T}))} &= \sup \left\{ \|fg\|_{L^2} : g \in L^2(\mathbb{T}), \|g\|_{L^2} = 1 \right\} \\ &\leq \sup \left\{ \|f\|_\infty \|g\|_{L^2} : g \in L^2(\mathbb{T}), \|g\|_{L^2} = 1 \right\} \\ &= \|f\|_\infty \sup \left\{ \|g\|_{L^2} : g \in L^2(\mathbb{T}), \|g\|_{L^2} = 1 \right\} \\ &= \|f\|_\infty. \end{aligned}$$

The converse holds since there exists a sequence $\{g_n\}$ in $L^2(\mathbb{T})$ such that $\|g_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|fg_n\|_{L^2} = \|f\|_\infty$. ∎

Let ψ be the composition of M and F, see Figure 5.1. By Lemma 5.1.2 and Lemma 5.1.3 ψ is also an isometric $*$-homomorphism.

Lemma 5.1.4. *The image of $C(\mathbb{T})$ under ψ is equal to $C_r^*(\mathbb{Z})$, i.e.*

$$\psi\left(C(\mathbb{T})\right) = C_r^*(\mathbb{Z}).$$

Proof. We know that $\psi(f)$ is in $\mathcal{B}(\ell^2(\mathbb{Z}))$ for each f in $C(\mathbb{T})$. To prove the claim, we check that ψ maps the basis elements e_m of $C(\mathbb{T})$ onto the basis elements $\lambda(m)$ of $C_r^*(\mathbb{Z})$. To do this we compute how both $\lambda(m)$ and $\psi(e_m)$ acts on each $\xi \in \ell^2(\mathbb{Z})$. Since $\mathcal{F}$ is surjective as we saw in Lemma 5.1.1 there always exists $g \in L^2(\mathbb{T})$ such that $\mathcal{F}(g) = \widehat{g} = \xi$. Therefore it suffices to check this for $\widehat{g} \in \ell^2(\mathbb{Z})$.

Let $m, n \in \mathbb{Z}$. Naturally,

$$\left(\lambda(m)\widehat{g}\right)(n) = \widehat{g}(m^{-1}n) = \widehat{g}(n-m).$$

On the other hand,

$$\psi(e_m) = F(M(e_m)) = F(M_{e_m}) = \mathcal{F} M_{e_m} \mathcal{F}^*,$$

and

$$\mathcal{F} M_{e_m} \mathcal{F}^* \widehat{g} = \mathcal{F} M_{e_m} g = \mathcal{F}(e_m g) = \widehat{e_m g}.$$

Applying this to n and using the definition of $\mathcal{F}$ we get

$$\begin{aligned}\widehat{e_m g}(n) = \langle e_m g, e_n\rangle &= \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{imt}e^{-int}dh(t)\\ &= \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-i(n-m)t}dh(t)\\ &= \langle g, e_{n-m}\rangle = \widehat{g}(n-m).\end{aligned}$$

Hence

$$\psi(e_m) = \lambda(e_m) \tag{5.7}$$

and the proof is complete. ■

If we restrict ψ to the preimage $\psi^{-1}(C_r^*(\mathbb{Z}))$ we can define an inverse map $\Phi = \psi^{-1}$.

Proposition 5.1.5. *The map $\Phi : C_r^*(\mathbb{Z}) \to C(\mathbb{T})$ discussed above is an isometric $*$-isomorphism.*

Proof. By Lemma 5.1.2 and Lemma 5.1.3, ψ is an isometric $*$-homomorphism. It is also linear and hence injective as it is an isometry. When we further restrict ψ to its image, it becomes surjective as well. Hence Φ is an isometric $*$-isomorphism. ■

Having shown the requisite isomorphism, we proceed to discussing quantum channels once again.

5.2 Quantum channels on $C_r^*(\mathbb{Z})$

We mimic our previous constructions, attempting to find a relationship between the quantum channels Q_φ on $C_r^*(\mathbb{Z})$ and quantum channels on $C(\mathbb{T})$. Hence we let φ be a (normalised) positive definite function on $\mathbb{Z}$. By Theorem 2.4.8 there exists a regular Borel measure μ on $\widehat{\mathbb{Z}}$ such that

$$\varphi(m) = \int_{\widehat{G}} \langle m, \gamma_\theta\rangle d\mu(\gamma_\theta)$$

for all $m \in \mathbb{Z}$. By [Loo53][page 140, theorem 35E] we have that $\widehat{\mathbb{Z}} \simeq \mathbb{T}$ through the homeomorphism $\theta \mapsto \gamma_\theta$, where $\gamma_\theta(m) = e^{im\theta}$. Hence we can write

$$\varphi(m) = \int_{\mathbb{T}} \langle m, \gamma_\theta\rangle d\mu(\theta) = \int_{\mathbb{T}} \gamma_\theta(m) d\mu(\theta) = \int_{\mathbb{T}} e^{im\theta} d\mu(\theta) \tag{5.8}$$

for all $m \in \mathbb{Z}$.

As in the previous examples we may use Proposition 2.4.5 to get a quantum channel Q_φ on $C_r^*(\mathbb{Z})$ satisfying

$$Q_\varphi(\lambda(m)) = \varphi(m)\lambda(m) \tag{5.9}$$

for all $m \in \mathbb{Z}$. Here we consider $C_r^*(\mathbb{Z})$ to be equipped with the canonical trace τ.

Using the isomorphism Φ from Proposition 5.1.5, Proposition 2.4.3 implies there is a quantum channel $\tilde{Q}_\varphi$ on $(C(\mathbb{T}), \tau')$, where $\tau' = \tau \circ \Phi^{-1}$. It is given

$$\begin{tikzcd} C_r^*(\mathbb{Z}) \arrow[r, shift left, "\Phi"] \arrow[d, "Q_\varphi"'] & C(\mathbb{T}) \arrow[l, dashed, shift left, "\Phi^{-1}=\psi"] \arrow[d, "\tilde{Q}_\varphi"] \\ C_r^*(\mathbb{Z}) \arrow[r, "\Phi"] & C(\mathbb{T}) \end{tikzcd}$$

Figure 5.2: The induced map $\tilde{Q}_\varphi$ for $G = \mathbb{Z}$

by $\tilde{Q}_\varphi = \Phi Q_\varphi \Phi^{-1}$. Our current goal is to describe $\tilde{Q}_\varphi(f)$ in terms of the regular Borel measure μ associated to φ, where $f \in C(\mathbb{T})$. As $\{e_m\}_{m\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$ it suffices to do this for each e_m.

Lemma 5.2.1. *For each $m \in \mathbb{Z}$ it holds that $\tilde{Q}_\varphi(e_m) = \varphi(m)e_m$.*

Proof. By definition

$$\begin{aligned} \tilde{Q}_\varphi(e_m) &= \Phi Q_\varphi \Phi^{-1}(e_m) \\ &= \Phi Q_\varphi \psi(e_m) = \Phi Q_\varphi(\lambda(m)) \end{aligned}$$

by Equation (5.7). Using Equation (5.9) this is equal to

$$\Phi(\varphi(m)\lambda(m)) = \varphi(m)\Phi(\lambda(m)) = \varphi(m)e_m. \tag{5.10}$$

■

This gives us a nice description of $\tilde{Q}_\varphi$ in terms of φ. But as stated, we want to describe it in terms of μ. To this end we define a new function $P_\mu : C(\mathbb{T}) \to C(\mathbb{T})$ by

$$\left(P_\mu(g)\right)(t) = \int_{\mathbb{T}} g(t+s)d\mu(s) \tag{5.11}$$

for $g \in C(\mathbb{T})$ and $t \in \mathbb{T}$. If we can describe $\tilde{Q}_\varphi$ in terms of P_μ, we will have achieved our goal.

Proposition 5.2.2. *Let $\tilde{Q}_\varphi$ and P_μ be the previously defined functions. Then $\tilde{Q}_\varphi = P_\mu$.*

Proof. Linearity of P_μ is clear as integration is linear. Furthermore, P_μ is bounded because

$$\begin{aligned} \|P_\mu(g)\|_\infty &= \sup\left\{\left|\left(P_\mu(g)\right)(t)\right| \ : \ t \in \mathbb{T}\right\} \\ &= \sup\left\{\left|\int_{\mathbb{T}} g(t+s)d\mu(s)\right| \ : \ t \in \mathbb{T}\right\} \end{aligned}$$

and

$$\left|\int_{\mathbb{T}} g(t+s)d\mu(s)\right| \leq \left|\int_{\mathbb{T}} \|g\|_\infty d\mu(s)\right| = \mu(\mathbb{T})\|g\|_\infty.$$

Thus $\|P_\mu(g)\|_\infty \leq \mu(\mathbb{T})\|g\|_\infty$, which implies $\|P_\mu\| \leq \mu(\mathbb{T})$. Hence it suffices to check that $\tilde{Q}_\varphi$ and P_μ are equal on each basis element e_m. Let $m \in \mathbb{Z}$ and $t \in \mathbb{T}$. Then by definition of P_μ

$$\begin{aligned}
(P_\mu(e_m))(t) &= \int_{\mathbb{T}} e_m(t+s)d\mu(s) \\
&= \int_{\mathbb{T}} e^{im(t+s)}d\mu(s) \\
&= \left(\int_{\mathbb{T}} e^{ims}d\mu(s)\right) e^{imt} \\
&= \left(\int_{\mathbb{T}} e^{ims}d\mu(s)\right) e_m(t),
\end{aligned}$$

which means that $P_\mu(e_m) = \left(\int_{\mathbb{T}} e^{ims}d\mu(s)\right) e_m$, but by Lemma 5.2.1,

$$\tilde{Q}_\varphi(e_m) = \varphi(m)e_m = \left(\int_{\mathbb{T}} e^{ims}d\mu(s)\right) e_m$$

as well. The last equality holds due to Equation (5.8).

Note that this makes P_μ a quantum channel. ■

5.3 Contractive channels on $C(\mathbb{T})$

We briefly discuss the contractive properties of the quantum channels $\tilde{Q}_\varphi = P_\mu$ previously mentioned. We equip $C(\mathbb{T})$ with the trace $\tau' = \tau \circ \Phi^{-1}$. The metric space in question is

$$\mathcal{D}_{\tau'}(C(\mathbb{T})) = \left\{f \in C(\mathbb{T})^+ \mid \tau'(f) = 1\right\},$$

equipped with either the Bures metric $d_{\mathcal{B}}^{\tau'}$ or the trace metric d_1. We consider an example.

Example 5.3.1. In this example we examine the contractive properties of P_μ in the case where μ is the normalised Lebesgue-measure. We want to compute $P_\mu(f)$ for all $f \in C(\mathbb{T})$, but consider $f = e_m$ first. We get

$$\begin{aligned}
p_\mu(e_m)(t) &= \int_{\mathbb{T}} e_m(t+s)d\mu(s) \\
&= \int_{\mathbb{T}} e^{im(t+s)}d\mu(s) \\
&= \begin{cases} 1 \text{ for } m = 0 \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

Had μ been an arbitrary regular Borel measure this would not necessarily be true. Since $\operatorname{span}\{e_m \mid m \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$, this means that $P_\mu(f) = \tau'(f) \cdot 1$ for all $f \in C(\mathbb{T})$. In particular, $P_\mu(f) = 1$ for all $f \in \mathcal{D}_{\tau'}(C(\mathbb{T}))$. Since $P_\mu(f)$

sends any function from $\mathcal{D}_{\tau'}(C(\mathbb{T}))$ to the constant function 1, it is a strict contraction on $(\mathcal{D}_{\tau'}(C(\mathbb{T})), d)$, regardless of choice of metric d.

This also lets us give an alternate description of the trace τ'. Let $m \in \mathbb{Z}$. We know from Lemma 2.3.12 that

$$\tau'(e_m) = \tau \circ \Phi(e_m) = \tau(\lambda(m)) = \begin{cases} 1 \text{ for } m = 0 \\ 0 \text{ otherwise} \end{cases},$$

meaning

$$\tau'(e_m) = \int_{\mathbb{T}} e_m d\mu.$$

Applying density again, this means that

$$\tau'(f) = \int_{\mathbb{T}} f d\mu$$

for all $f \in C(\mathbb{T})$.

We mention that the positive definite function on $\mathbb{Z}$ corresponding to the normalised Lebesgue measure μ on $\mathbb{T}$ is φ_0 from Example 2.5.10, i.e.

$$\varphi_0(m) = \begin{cases} 1 \text{ for } m = 0 \\ 0 \text{ otherwise} \end{cases},$$

for all $m \in \mathbb{Z}$. This follows from the fact that

$$Q_{\varphi_0}(\lambda(m)) = \varphi_0(m) Q(\lambda(m)) = \begin{cases} 1 \text{ for } m = 0 \\ 0 \text{ otherwise} \end{cases}.$$

In other words, $P_\mu = \tilde{Q}_{\varphi_0}$. Also, $\tilde{Q}_{\varphi_0}$ is the completely depolarising channel on $(C_r^*(\mathbb{Z}), \tau)$, see Example 2.5.8.

CHAPTER 6

The fourth example

We have arrived at our final example, where we consider the reduced twisted group C^*-algebra of $\mathbb{Z}^2$ and related quantum channels. As opposed to last chapter we get more 2-cocycles than just the trivial one. In Chapter 4 we defined σ_k, and there was a distinction where either k was either relatively prime with N or not. This lead to two separate cases for $C_r^*(\mathbb{Z}_N \times \mathbb{Z}_N, \sigma_k)$. In this chapter there is a very similar phenomenon.

To reiterate, in this chapter we have $G = \mathbb{Z}^2$. As in Chapter 4 we need some 2-cocycle to talk about the reduced twisted group C^*-algebra. To this end we let $\theta \in [0, 1)$ and define a map

$$\sigma_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$$

by

$$\sigma_\theta \left((m_1, m_2), (n_1, n_2)\right) = e^{i2\pi\theta m_1 n_2}.$$

This is our candidate for a 2-cocycle on $\mathbb{Z}^2$. Notice that this is very similar to Equation (4.1) from Chapter 4. Let us start by showing that this is in fact a 2-cocycle.

Proposition 6.0.1. *The map σ_θ is a bicharacter on $\mathbb{Z}^2$, hence a 2-cocycle.*

The proof is the same as for Proposition 4.1.1:

Proof. Fix $(k_1, k_2) \in \mathbb{Z}^2$ and define $\gamma : \mathbb{Z}^2 \to \mathbb{T}$ by

$$\gamma\left((m_1, m_2)\right) = \sigma_\theta((m_1, m_2), (k_1, k_2)).$$

This is a homomorphism because

$$\begin{aligned} \gamma\left((m_1, m_2)(n_1, n_2)\right) &= \sigma_k\left((m_1 + n_1, m_2 + n_2), (k_1, k_2)\right) \\ &= e^{i\theta \frac{2\pi}{N}(m_1+n_1)k_2} = e^{i\theta \frac{2\pi}{N} m_1 k_2} e^{i\theta \frac{2\pi}{N} n_1 k_2} \\ &= \gamma\left((m_1, m_2)\right)\gamma\left((n_1, n_2)\right) \end{aligned}$$

by definition of γ and σ_θ. The map γ' defined by

$$\gamma'\left((m_1, m_2)\right) = \sigma_k((k_1, k_2), (m_1, m_2))$$

is similarly a homomorphism. Hence σ_θ is a bicharacter on $\mathbb{Z}^2$. By Proposition 2.3.18 σ_θ is thus a normalised 2-cocycle on $\mathbb{Z}^2$. ■

Hence $C_r^*(\mathbb{Z}^2, \sigma_\theta)$ is a reduced twisted group C^*-algebra. Moreover these are the only such for $\mathbb{Z}^2$, since each 2-cocycle on $\mathbb{Z}^2$ is similar to σ_θ for some $\theta \in [0, 1)$. This follows from [Kle65](Theorem 7.1), but is also mentioned in [BB70](Theorem 4.1). As in the earlier examples Proposition 2.3.15 then gives that each other reduced twisted group C^*-algebra on $\mathbb{Z}^2$ is isomorphic to $C_r^*(\mathbb{Z}^2, \sigma_\theta)$.

In accordance with [Dav96] we write $\mathcal{A}_\theta := C_r^*(\mathbb{Z}^2, \sigma_\theta)$.

As before we are interested in establishing $*$-isomorphisms of $\mathcal{A}_\theta$ and studying the quantum channels Q_φ. In this chapter our work splits into two cases: either θ is irrational or θ is rational. This is somehow analogous to k being relatively prime with N or not in Chapter 4.

6.1 The parameter θ is irrational

We mention an passant that it can be shown that $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries U and V satisfying

$$UV = e^{i2\pi\theta} VU. \tag{6.1}$$

For a discussion on this, see [Bla06] (II.8.3.3(i), II.10.4.12, II.10.7.5). Furthermore the next result follows from [GVF01] (Section 12.2).

Proposition 6.1.1. *The following are equivalent:*

(i) *$\mathcal{A}_\theta$ is simple, i.e.* 0 *an $\mathcal{A}_\theta$ are its only closed ideals.*

(ii) *θ is irrational.*

(iii) *$\mathcal{A}_\theta$ has exactly one tracial state, namely the canonical tracial state τ.*

Our goal, as ever, is to establish some $*$-isomorphism from $\mathcal{A}_\theta$. In previous chapters we found explicit isomorphisms that were useful for studying $\tilde{Q}_\varphi$. In this case we are not so fortunate, and settle for referencing that one exists.

A $*$-homomorphism of $\mathcal{A}_\theta$ on $\ell^2(\mathbb{Z})$ is obtainable by defining

$$\begin{aligned} S :&\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \text{ by} \\ &(S\xi)(m) = \xi(m-1) \end{aligned}$$

and

$$\begin{aligned} M_\theta :&\ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \text{ by} \\ &(M_\theta\xi)(m) = e^{i2\pi\theta m}\xi(m), \end{aligned}$$

where $\xi \in \ell^2\mathbb{Z}$ and $m \in \mathbb{Z}$. These are infinite dimensional analogues to the matrices S and D from Chapter 4.

Lemma 6.1.2. *The operators S and M_θ defined above are unitary and satisfy Equation* (6.1)*.*

Proof. The adjoints S^* and M_θ^* of S and M_θ are given by $(S^*\xi)(m) = \xi(m+1)$ and $(M_\theta^*\xi)(m) = e^{-i2\pi\theta m}\xi(m)$. Thus S and M_θ are clearly unitary, as they satisfy $U^*U = UU^* = I$. The condition Equation (6.1) follows by a short computation. Let $\xi \in \ell^2\mathbb{Z}$ and $m \in \mathbb{Z}$, then

$$\begin{aligned}(SM_\theta\xi)(m) &= (M_\theta\xi)(m-1) = e^{i2\pi(m-1)}\xi(m-1), \text{ and}\\ (M_\theta S\xi)(m) &= e^{i2\pi m}(S\xi)(m) = e^{i2\pi m}\xi(m-1).\end{aligned}$$

∎

This lets us achieve part of what we wanted:

Theorem 6.1.3. *There exists a surjective $*$-homomorphism from $\mathcal{A}_\theta$ to $C^*(S, M_\theta)$. It is injective whenever θ is irrational.*

Proof. The existence of a surjective $*$-homomorphism follows from Lemma 6.1.2 together with the universal property of $\mathcal{A}_\theta$. Injectivity when θ is irrational follows from the fact that $\mathcal{A}_\theta$ is simple by Proposition 6.1.1. ∎

6.2 The parameter θ is rational

Having dealt with the case where θ is irrational, we now move onto the case where θ is rational. More precisely, let $\theta = \frac{p}{q}$, where $p, q \in \mathbb{N}, p < q$ and p and q are relatively prime. For the case where $p = 0$ we get $\mathcal{A}_0 = C_r^*(\mathbb{Z}^2) \simeq C(\mathbb{T}^2)$. This could be done in the same way as for $G = \mathbb{Z}$, or more generally using that $C_r^*(G) = C(\widehat{G})$ by Gelfand theory.

Otherwise we set $U = \lambda_{\sigma_{\alpha_\theta}}(1,0), V = \lambda_{\sigma_{\alpha_\theta}}(0,1)$. Then both U and V are unitary by Corollary 2.3.8. Also, Equation (6.1) is satisfied because

$$\begin{aligned}UV &= \lambda_{\sigma_\theta}(1,0)\lambda_{\sigma_\theta}(0,1)\\ &= \sigma_\theta\left((1,0),(0,1)\right)\lambda_{\sigma_\theta}(1,1)\\ &= e^{i2\pi\theta}\lambda_{\sigma_\theta}(1,1)\end{aligned}$$

and

$$\begin{aligned}VU &= \lambda_{\sigma_\theta}(0,1)\lambda_{\sigma_\theta}(1,0)\\ &= \sigma_\theta\left((0,1),(1,0)\right)\lambda_{\sigma_\theta}(1,1)\\ &= \lambda_{\sigma_\theta}(1,1)\end{aligned}$$

by Proposition 2.3.7 and the definition of σ_θ. It can then be shown that there exists an injective $*$-homomorphism from $\mathcal{A}_\theta$ into $C(\mathbb{T}^2, M_q(\mathbb{C}))$ that sends U to $\tilde{U}$ and V to $\tilde{V}$, where

$$\begin{aligned}\tilde{U}(z_1, z_2) &= z_1 D^p,\\ \tilde{V}(z_1, z_2) &= z_2 S,\end{aligned}$$

where D and S are as given in Section 4.2. This result follows from [GVF01] (Proposition 12.2, page 527).

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