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Dynamic Risk Measures Generated by Time-changed BSDEs with Jumps

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Det er ganske sandt, hvad Philosophien siger, at Livet maa forstaaes baglænds. Men derover glemmer man den anden Sætning, at det maa leves forlænds.

It is perfectly true, as the philosophers say, that life must be understood backwards. But they forget the other proposition, that it must be lived forwards.

SØREN KIERKEGAARD

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If there are any errors in this thesis, I am solely responsible for them. If anything worth reading could be found, the credit should be given to those mentioned above.

Contents

Introduction

Risk measurement is vital to the finance industry. Inadequate risk analysis can misprice derivatives, underestimate volatilities, and in extreme situations, lead to bankruptcies and collapse of the entire finance sector.

In insurance and financial mathematics, a risk measure is used to quantify the risks faced by an asset or a financial position within a certain time frame. It can also be used to calculate a monetary amount that has to be kept in reserve. Regulators can thus impose limits on the risks taken by financial institutions, such as banks and insurance companies. The Solvency II directive in EU and risk supervisions by Finanstilsynet in Norway can serve as such examples.

Risk measures are a subject that has been widely studied. The literature can trace back to [Art+99], where the authors proposed coherent risk measures as axiomatic tools to study riskiness of financial positions. By weakening the coherence conditions, [FS04] managed to represent convex risk measures in general probability spaces. Authors in [FR02] obtained the same result independently.

Yet all these above mentioned papers investigated risk measures in a static environment. In the years between 1999 and 2002, researchers in the field studied risk measures with a different set of axioms, and in doing so they generalised the study in a dynamic setting, as authors in [QS13] and [Ros06] pointed out.

This thesis focuses on dynamic risk measures generated by backward stochastic differential equations (BSDEs). The author in [Ros06] identified the link under the Brownian motion framework, whereas the author in [Roy06] generalised the connection in a setting where the BSDES are driven by Lévy noises. In either case, we try to derive a risk measure ρ from a given Lipschitz BSDE driver f and a terminal time T, so that we can measure the riskiness of a financial position X at time t. In other words, we use the solution of the BSDE, driven by either a Brownian motion or Lévy processes in general, together with a terminal condition X, to represent the risk measure ρ at time t. Authors in [QS13], among others, have investigated and presented some nice properties of dynamic risk measures in the setting of BSDEs with jumps.

In this thesis, we will study dynamic risk measures generated by BSDEs driven by time-changed Lévy noises. This is a more general framework, thanks to Change of Time Methods (CTMs). The main idea of CTMs is to perturb the time line of a complicated stochastic process with another stochastic process and obtain a relatively simpler process. In this way, it enables us to represent processes with a complicated structure, usually referred to as the *base*, by some well-known processes, such as the Brownian motion.

In the meantime, this extra stochastic component at the level of time perturbation in the underlying model gives us an alternative to the use of stochastic volatility. One may refer to [Swi16] for a general introduction to CTMs theory.

Another benefit of CTMs is that it enables us to better describe the price dynamics of financial derivatives. Compared with the classical Black-Scholes model which assumes constant volatility, CTMs allows us to achieve better description of the so-called "volatility smile", as empirical data suggest that volatility tends to vary with respect to the option strike price and expiration time. The use of CTMs here represents the transition from the real-time clock to the trading clock, providing us with a method to deal with the fact that volatility increases with the intensity of trading activities.

Lévy processes, being a rich class, are widely used as the base process in the literature, and in this thesis, we use time-changed Lévy noises to drive BSDEs. In particular, this thesis considers mainly time changes that are absolutely continuous with respect to the Lebesgue measure. One of the difficulties of working with this type of time change is that the resulting time-changed process may no longer be a Lévy process. In the meantime, this process still has many interesting properties: being conditionally a Lévy process and, under certain conditions, a martingale. Authors in [DS14] have presented many interesting results in this respect, and this thesis refers to [App09] for a general introduction to Lévy processes.

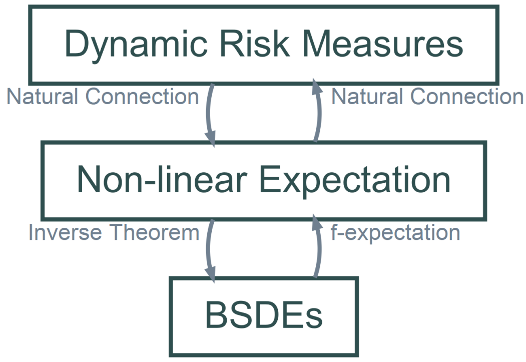

Figure 1: An illustration of connections between different components in the thesis

Figure 1 illustrates the three components discussed in this thesis and the connections among them. The connection between dynamic risk measures and non-linear expectations is both natural and well-established in the literature. On the other hand, the relationship between non-linear expectations and BSDEs is complicated. Classical results have shown that, under the Brownian motion framework, we can define a non-linear expectation by a so-called "g-expectation" described by the given BSDE. This type of expectation was later generalised in the setting of BSDEs with jumps by researchers such as Royer in [Roy06], and then named as "f-expectation" to differentiate from the classical Brownian motion case.

Yet if we start from opposite direction and try to represent a given non-linear expectation by a time-changed BSDE with jumps, things are much more difficult. It turns out that, in order to show this result, it requires establishing the Doob-Meyer Decomposition in the non-linear expectation martingale setting. This is particularly difficult since classical arguments for such decomposition rely heavily on the linearity of ordinary expectations. To our best knowledge, Peng and researchers in [Pen99] and [Coq+02] are the first ones to establish this result in the Brownian motion setting. Royer in [Roy06] is one of the first to generalise this result in the setting of BSDEs with jumps. This thesis dedicates the entire Chapter 4 to proving this important result in our time-changed setting, and this is, so far as we know, the first time this result is established under such a framework.

This thesis starts by discussing BSDEs with jumps in the classical setting, and then develops the subject further under the framework of time-changed Lévy noises with an absolutely continuous change of time component. We present in Chapter 3 a representation of convex and coherent risk measures derived from such time-changed BSDEs, and then prove the Inverse Theorem under the time-changed framework in Chapter 4. In this way, this thesis establishes a full connection between dynamic risk measures and time-changed BSDEs with jumps under a more general framework.

We organise this thesis as follows. In Chapter 1, we recall the classical setting of BSDEs with jumps, present definitions and properties we will discuss in the following thesis, and review the theory of risk measures as well BSDEs with jumps. Results reviewed in Chapter 1, recalled from [FS04], [Ros06], [AP11] [Roy06] and [QS13], are obtained in the classical setting.

The key part in Chapter 1 is to trace connections between dynamic risk measures and BSDEs established in the classical setting, the so-called "non-linear expectation". Inspired by the classical results, this thesis sets out to establish the same connection in a time-changed setting in Chapter 3.

Chapter 2 focuses on the discussion of CTMs. We start the chapter by reviewing the theory of CTMs as well as properties of time-changed Brownian motion noises, before we proceed to examine properties of the more general time-changed Lévy processes. As mentioned earlier, one of the difficulties of working with time-changed Lévy processes is that they may no longer stay Lévy processes. Chapter 2 presents two of the most widely used time change processes, namely, subordinators and absolutely continuous time change, and compare how the time-changed Lévy processes behave under them. Most of the theoretical results used in this Chapter are recalled from [BS10].

In the end of Chapter 2, we review the framework proposed in [DS14], where authors from the paper apply absolutely continuous time changes to a Brownian motion and a centred, pure jump Lévy process, and use these time-changed noises to drive BSDEs. Given the conditional stationary independent increments and absolute continuity, this time-changed framework has several "nice" properties, including the martingale property for the time-changed noises. We will continue our study of time-changed BSDEs under this framework.

One thing we particularly need to point out is, in setting up the framework, we need to make use of the filtration $\mathbb{G}$, and all the results we obtain are $\mathbb{G}$-adapted. This is a big, technical filtration that includes "anticipating-information", which is the entire history of the time-changed noises that we use to generate the BSDEs. In applications, we can still solve an optimal control problem with a classical performance functional, and this is achieved by projecting the results we obtain in filtration $\mathbb{G}$ onto filtration $\widetilde{\mathbb{F}}$, the smallest right-continuous filtration to which our random signed measure μ is adapted. Here μ is the mixture of a conditional Brownian measure and a centred doubly stochastic Poisson measure, both of which are used to construct the framework.

For a detailed implementation of this idea, we refer to Section 6 in [DS14].

BSDEs and CTMs come together in Chapter 3, where we develop further topics discussed in Chapter 1. All the important results of classical BSDEs, such as the existence and uniqueness of solution, the Comparison Theorem, are established under our new, time-changed framework in the first half of this chapter. One can indeed observe the correspondence between the two settings. Despite the difference in set-up, several proofs under the time-changed framework can be carried out by adapting arguments used in the classical setting to the new, time-changed framework.

The second half of Chapter 3 shows how to define a dynamic risk measure by a given time-changed BSDE. Similar as in the classical setting, the link is established via the so-called "f-expectation", a non-linear expectation characterised by the given time-changed BSDE. As mentioned earlier, we give a representation of convex and coherent risk measures in our time-changed setting, and this is one of the main results of this thesis.

Chapter 4 establishes the link between dynamic risk measures and time-changed BSDEs from the opposite direction, by proving that we can indeed represent a given non-linear expectation by a time-changed BSDE with jumps under rather general conditions. This is the so-called "Inverse Theorem", as shown in figure 1.

The interesting part of the Inverse Theorem is that it enables us to convert a problem of non-linear expectations into study time-changed BSDEs. In this way, we can represent a large class of non-linear expectations by solving time-changed BSDEs.

As mentioned earlier, this is a deep result, and it took leading researchers in the field quite some time to establish in the classical setting in the first place. In proving this theorem, we follow the work that has been done by Peng in [Pen99] and Royer in [Roy06], and generalise the results in our present, time-changed setting. The entire Chapter 4 is devoted to proving this one theorem, which says something about the importance and difficulty of this result.

The appendix gathers important elements of the theory of stochastic processes and calculus that we use throughout this thesis. The ❇ mark before some of the proofs signifies our original effort to introduce a genuine result, or extend and generalise results obtained in the classical setting under the time-changed framework.

Chapter 0

Notation and Basic Definitions

One of the first challenges one faces in the field of stochastic analysis is the mathematical notation, and studying BSDEs under two different settings in this thesis has only made it worse. It is because of this that we are giving the following list of set-ups for different parts of the thesis.

In the classical set-up for BSDEs with jumps, definitions of the relevant spaces are conventional. In comparison, the set-up for time-changed BSDEs in Section 0.2, proposed in [DS14], is more complicated, and we will give more explanations of this framework in Section 2.3.

0.1 Set-up for BSDEs with Jumps

We use the following important spaces in our set-up for BSDEs with jumps.

Definition 0.1.1. Throughout this thesis, we denote by $(\Omega, \mathcal{F}, P)$ a probability space, W a one-dimensional Brownian motion and $N(\mathrm{d}t, \mathrm{d}u)$ a Poisson random measure, defined in Definition A.0.13. We define $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. In the following notations, $\mathcal{E}$ is a general σ-algebra and $\mathbb{E} := \{\mathcal{E}_t, t \geq 0\}$ a filtration.

- Let $\nu(\mathrm{d}u)\mathrm{d}t$ be the compensator of $N(\mathrm{d}t, \mathrm{d}u)$, defined in Definition A.0.7, such that ν is a σ-finite measure on $\mathbb{R}^*$, equipped with its Borel field $\mathcal{B}(\mathbb{R}^*)$;
- Let $\widetilde{N}(\mathrm{d}t, \mathrm{d}u)$ be the compensated process, defined in Definition A.0.7, of $N(\mathrm{d}t, \mathrm{d}u)$;
- Let $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ be the natural filtration associated with W and N;
- Let Σ_p be the predictable σ-algebra, defined in Definition A.0.11, on $[0, T] \times \Omega$, where $T \in (0, \infty]$;
- Let $L^p(\Omega, \mathcal{E}, P)$, for all $p \in (0, \infty]$, be the set of functions f that are measurable with respect to a general σ-algebra $\mathcal{E}$, such that

$$\|f\|_p := \Big(\int_\Omega |f|^p \mathrm{d}P \Big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$, we define

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega}\{|f(\omega)|\} := \inf\{M \in [0,\infty] : |f(\omega)| \le M \ a.e.\} < \infty.$$

We denote by L^p when there is no ambiguity, and denote by $L^p(\mathcal{E}_T)$, for all $p \in (0,\infty]$, the set of random variables such that they are $\mathcal{E}_T$-measurable and p-integrable;

- Let $\mathcal{H}^p_{\mathcal{E}}(0,T)$ be the set of real-valued $\mathbb{E}$-predictable processes ϕ such that

$$\|\phi\|^p_{\mathcal{H}^p_{\mathcal{E}}(0,T)} := \mathrm{E}\Big[\Big(\int_0^T \phi_t^2 \mathrm{d}t\Big)^{\frac{p}{2}}\Big] < \infty.$$

  For a special case where $\beta > 0$ and $\phi \in \mathcal{H}^p_{\mathcal{E}}(0,T)$, we introduce the norm

$$\|\phi\|^2_{\beta,T} := \mathrm{E}\Big[\int_0^T e^{\beta t}\phi_t^2 \mathrm{d}t\Big];$$

- Let $\mathcal{H}^p_{\mathcal{E}}(0,T,\nu)$ be the set of processes l which are $\mathbb{E}$-predictable, that is, measurable

$$l : ([0,T]\times\Omega\times\mathbb{R}^*, \Sigma_p \otimes \mathcal{B}(\mathbb{R}^*)) \to (\mathbb{R},\mathcal{B}(\mathbb{R})); \quad (t,\omega,u) \to l_t(\omega,u)$$

  such that

$$\|l\|^p_{\mathcal{H}^p_{\mathcal{E}}(0,T,\nu)} := \mathrm{E}\Big[\Big(\int_0^T \|l_t\|^2_\nu \mathrm{d}t\Big)^{\frac{p}{2}}\Big] < \infty.$$

  For $\beta > 0$ and $l \in \mathcal{H}^p_{\mathcal{E}}(0,T,\nu)$, we set $\|l\|^2_{\nu,\beta,T} := \mathrm{E}\Big[\int_0^T e^{\beta s}\|l_s\|^2_\nu \mathrm{d}s\Big]$.

- Let L^p_ν be the set of Borelian functions $\ell : \mathbb{R}^* \to \mathbb{R}$ such that

$$\|\ell\|^p_{\mathcal{H}^p_{\mathcal{E}}(0,T,\nu)} := \int_{\mathbb{R}^*} |\ell(u)|^p \,\nu(\mathrm{d}u) < \infty.$$

  The set L^2_ν is a Hilbert space equipped with the scalar product

$$\langle\delta,\ell\rangle_\nu := \int_{\mathbb{R}^*} \delta(u)\ell(u)\nu(\mathrm{d}u), \quad \text{for all } \delta,\ell \in L^2_\nu \times L^2_\nu,$$

  and the norm

$$\|\ell\|^2_\nu := \|\ell\|^2_{2,\nu} = \int_{\mathbb{R}^*} |\ell(u)|^2 \,\nu(\mathrm{d}u);$$

- Let $S^p_{\mathcal{E}}(0,T)$ be the set of real-valued càdlàg $\mathbb{E}$-adapted processes ϕ with $\|\phi\|^p_{S^p_{\mathcal{E}}(0,T)} := \mathrm{E}[\sup_{0\le t\le T}|\phi_t|^p] < \infty$.

When T is a fixed time horizon and when there is no ambiguity, we denote $\mathcal{H}^p_{\mathcal{E}}(0,T)$ by $\mathcal{H}^p_{\mathcal{E}}$, $\mathcal{H}^p_{\mathcal{E}}(0,T,\nu)$ by $\mathcal{H}^p_{\mathcal{E},\nu}$ and $S^p_{\mathcal{E}}(0,T)$ by $S^p_{\mathcal{E}}$.

In the following definition of a driver, we make use of the spaces defined above.

Definition 0.1.2 (Lipschitz Driver)**.** A function f is called a *driver* if it satisfies the following conditions:

(i) $f : ([0,T] \times \Omega \times \mathbb{R}^2 \times L^2_\nu) \to \mathbb{R}$; $\quad (t,\omega,x,\pi,\ell(u)) \to f(t,\omega,x,\pi,\ell(u))$ is $\Sigma_p \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_\nu)$-measurable.

(ii) $f(t,0,0,0) \in \mathcal{H}^2$.

A driver f is called a *Lipschitz driver* if there exists a constant $C \geq 0$ such that $\mathrm{d}P \otimes \mathrm{d}t$-a.s., for each $(x_1,\pi_1,\ell_1),(x_2,\pi_2,\ell_2)$, we have

$$|f(t,\omega,x_1,\pi_1,\ell_1) - f(t,\omega,x_2,\pi_2,\ell_2)| \leq C(|x_1 - x_2| + |\pi_1 - \pi_2| + \|\ell_1 - \ell_2\|_\nu).$$

Definition 0.1.3 (BSDE with Jumps)**.** A solution of a *BSDE with jumps* with terminal time T, terminal condition ξ and driver f consists of a triple of processes (Y,π,l) satisfying

$$\begin{aligned} -\mathrm{d}Y_t &= f(t,Y_{t^-},\pi_t,l_t(u))\mathrm{d}t - \pi_t \mathrm{d}W_t - \int_{\mathbb{R}^*} l_t(u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u) \\ Y_T &= \xi \end{aligned}$$

where $Y \in S^2_{\mathcal{F}}(0,T)$ is a càdlàg optional process and $\pi \in \mathcal{H}^2_{\mathcal{F}}(0,T)$ (respectively $l \in \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$) is a $\mathbb{R}$-valued $\mathbb{F}$-predictable process defined on $\Omega \times [0,T]$ (respectively $\Omega \times [0,T] \times \mathbb{R}^*$) such that the stochastic integral with respect to W (respectively $\widetilde{N}$) is well defined. This solution is defined by $(Y(\xi,T),\pi(\xi,T),l(\xi,T))$.

Remark 0.1.4. We note that since the process $f(t,Y_{t^-},\pi_t,l_t(u))$ is $\mathbb{F}$-predictable, it satisfies $f(t,Y_{t^-},\pi_t,l_t(u)) = f(t,Y_t,\pi_t,l_t(u))$ $\mathrm{d}P \otimes \mathrm{d}t$-a.s.

0.2 Set-up for time-changed BSDEs with Jumps

Let $(\Omega, \mathcal{F}, P)$ be a complete probability measure space and we define $X = [0,T] \times \mathbb{R}$, with $T > 0$ being a finite time horizon. We will consider $X = ([0,T] \times \{0\}) \cup ([0,T] \times \mathbb{R}^*)$ where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

We denote by $\mathcal{B}_X$ the Borel σ-algebra on X, and by $\Delta \subset X$ an element in $\mathcal{B}_X$. We also denote by $\mathcal{B}_{[0,T]}$ the Borel σ-algebra on $[0,T]$ and by m the Lebesgue measure.

Let $\lambda = (\lambda^B, \lambda^H)$ be a two dimensional stochastic process such that each component $\lambda^i,\ i = B, H$ satisfies the following three conditions:

(i) $\lambda^i \geq 0$ P-a.s. for all $t \in [0,T]$;

(ii) $\lim_{h\to 0} P(\left|\lambda^i_{t+h} - \lambda^i_t\right| \geq \varepsilon) = 0$ for all $\varepsilon > 0$ and almost all $t \in [0,T]$;

(iii) $\mathrm{E}[\int_0^T \lambda^i \mathrm{d}t] < \infty$.

We denote $\mathcal{L}$ as the space of all processes $\lambda = (\lambda^B, \lambda^H)$ satisfying the above three conditions.

Now we define a random measure (Definition A.0.13) Λ on X by

$$\Lambda(\Delta) = \int_0^T \mathbb{1}_{\{(t,0)\in\Delta\}}(t)\lambda^B_t \mathrm{d}t + \int_0^T \int_{\mathbb{R}^*} \mathbb{1}_{\Delta}(t,z)\nu(\mathrm{d}z)\lambda^H_t \mathrm{d}t,$$

as the mixture of measures on disjoint sets. Here ν is a deterministic, σ-finite measure on the Borel sets of $\mathbb{R}^*$ satisfying

$$\int_{\mathbb{R}^*} z^2 \nu(\mathrm{d}z) < \infty.$$

We denote the σ-algebra generated by values of Λ by $\mathcal{F}^\Lambda$. And we let Λ^H denote the restriction of Λ to $[0,T]\times\mathbb{R}^*$ and Λ^B the restriction of Λ to $[0,T]\times\{0\}$. We then obtain $\Lambda(\Delta) = \Lambda^B(\Delta\cap[0,T]\times\{0\}) + \Lambda^H(\Delta\cap[0,T]\times\mathbb{R}^*)$, with $\Delta \subseteq X$.

Now we introduce the noises which drive the BSDEs.

Definition 0.2.1. B is a signed random measure on the Borel sets of $[0,T]\times\{0\}$ satisfying

(B1) $P(B(\Delta) \leq x \mid \mathcal{F}^\Lambda) = P(B(\Delta) \leq x \mid \Lambda^B(\Delta)) = \Phi\Big(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\Big)$, where $x \in \mathbb{R},\ \Delta \subseteq [0,T] \times \{0\}$;

(B2) $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent give $\mathcal{F}^\Lambda$ whenever Δ_1 and Δ_2 are disjoint sets.

Here Φ stands for the cumulative probability distribution function of a standard normal random variable.

H is a random measure on the Borel sets of $[0,T] \times \mathbb{R}^*$ satisfying

(H1) $P(H(\Delta) = k \mid \mathcal{F}^\Lambda) = P(H(\Delta) = k \mid \Lambda^H(\Delta)) = \frac{\Lambda^H(\Delta)^k}{k!} e^{-\Lambda^H(\Delta)}$, where $k \in \mathbb{N}, \Delta \subseteq [0,T] \times \mathbb{R}^*$;

(H2) $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent give $\mathcal{F}^\Lambda$ whenever Δ_1 and Δ_2 are disjoint sets.

In addition, we assume that

(BH) B and H are conditionally independet given $\mathcal{F}^\Lambda$.

Remark 0.2.2. Conditions (B1) and (H1) show that given Λ, B is a Gaussian random measure and H is a Poisson random measure. This also implies that if $\lambda^i,\ i = B, H$ are deterministic, then B is a Brownian motion and H is a Poisson random measure.

Remark 0.2.3. The existence of such conditional distributions as defined in the previous definition is a classical result in the literature. We refer to [Gri75] for more details on this.

Now we define a signed random measure

$$\widetilde{H}(\Delta) = H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}^*. \tag{0.1}$$

We now use this $\widetilde{H}$ to construct a random measure for the noises that drive BSDEs with jumps as defined in Definition 0.1.3.

Definition 0.2.4. We define a signed random measure μ on the Borel subsets of X by

$$\mu(\Delta) = B(\Delta \cap [0,T] \times \{0\}) + \widetilde{H}(\Delta \cap [0,T] \times \mathbb{R}^*), \quad \Delta \subseteq X. \tag{0.2}$$

Remark 0.2.5. From conditions (B1), (B2), (H1), (H2) and (BH) defined in Definition 0.2.1, we can conclude that conditional on $\mathcal{F}^\Lambda$, $\mu(\Delta_1)$ and $\mu(\Delta_2)$ are orthogonal for Δ_1 and Δ_2 disjoint. We refer to Definition 2.2 in [DS14] for details.

We define $\mathbb{F}^\mu = (\mathcal{F}^\mu_t)_{t\in[0,T]}$ as the filtration generated by $\mu(\Delta),\ \Delta \subset [0,t]\times\mathbb{R}$. By conditions (B1) and (H1) in Definition 0.2.1, we have for any $t \in [0,T]$:

$$\mathcal{F}^\mu_t = \mathcal{F}^B_t \vee \mathcal{F}^H_t \vee \mathcal{F}^\Lambda_t,$$

where $\mathcal{F}^B_t$ is generated by $B(\Delta \cap [0,T] \times \{0\})$, $\mathcal{F}^H_t$ by $B(\Delta \cap [0,T] \times \mathbb{R}^*$ and $\mathcal{F}^\Lambda_t$ by $\Lambda(\Delta),\ \Delta \in [0,t] \times \mathbb{R}$.

We set $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, where

$$\widetilde{\mathcal{F}}_t = \bigcap_{r>t} \mathcal{F}^\mu_r.$$

Finally, we set $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ where $\mathcal{G}_t = \mathcal{F}^\mu_t \vee \mathcal{F}^\Lambda$. This implies that $\mathcal{G}_T = \widetilde{\mathcal{F}}_T$ and $\mathcal{E}_0 = \mathcal{F}^\Lambda$, whereas $\mathcal{F}^\mu_0$ is trivial. We denote $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_T$.

Definition 0.2.6. We define the following two spaces in connection with time-changed BSDEs:

(i) Let $\mathcal{I}$ as a subspace of $L^2([0,T]\times\mathbb{R}\times\Omega, \mathcal{B}_X\times\widetilde{\mathcal{F}}, \Lambda\times P)$, then for a process $\phi \in \mathcal{I}$ where $\phi := (\phi_t(0), \phi_t(u))$ for $u \in \mathbb{R}^*$. Then we define:

$$\|\phi\|^2_{\mathcal{I}} = \mathrm{E}\bigg[\int_0^T \phi_s(0)^2\lambda^B_s \mathrm{d}s + \int_0^T\int_{\mathbb{R}^*}\phi_s(u)^2\nu(\mathrm{d}u)\lambda^H_s\mathrm{d}s\bigg] < \infty.$$

We define the (Itô type) non-anticipative stochastic integral for $\phi \in \mathcal{I}$ as $I : \mathcal{I} \mapsto L^2(\Omega, \mathcal{F}, P)$ by

$$I(\phi) := \int_0^T \phi_s(0)\mathrm{d}B_s + \int_0^T \int_{\mathbb{R}^*} \phi_s(u)\widetilde{H}(\mathrm{d}s, \mathrm{d}u).$$

For a more detailed explanation of this integral, we refer to (2.9) in [DS14].

(ii) Let Φ be the space of functions $\phi := (\phi(0), \phi_t(u))$, $\mathbb{R}^2 \mapsto \mathbb{R}$ with $u \in \mathbb{R}^*$ such that

$$|\phi(0)|^2 + \int_{\mathbb{R}^*} \phi(u)^2 \nu(\mathrm{d}u) < \infty.$$

Definition 0.2.7 (Time-changed BSDE with Jumps)**.** A solution of a *Time-changed BSDE with jumps* with terminal time T, terminal condition ξ and driver f consists of a triple of processes (Y, ϕ) satisfying

$$\begin{aligned} -\mathrm{d}Y_t &= f(t, \lambda_t, Y_t, \phi_t)\mathrm{d}t - \int_{\mathbb{R}} \phi_t(u)\mu(\mathrm{d}t, \mathrm{d}u) \\ &= f(t, Y_t, \phi_t(0), \phi_t(u))\mathrm{d}t - \phi_t(0)\mathrm{d}W_t - \int_{\mathbb{R}^*} \phi_t(u)\widetilde{H}(\mathrm{d}t, \mathrm{d}u) \\ Y_T &= \xi \end{aligned} \tag{0.3}$$

where $Y \in S^2_{\mathcal{E}}(0, T)$ is a càdlàg process and $\phi \in \mathcal{I}$, where $\mathcal{I}$ is defined in Definition 0.2.6, such that the stochastic integral with respect to μ defined in (0.2) is well defined. This solution is defined by $(Y(\xi, T), \phi(\xi, T))$.

In correspondence with Definition 0.1.2, we need to impose some conditions on the driver f in our BSDE driven by time-changed noises.

Definition 0.2.8 (Standard Parameters)**.** We call (ξ, f) *standard parametres* when $\xi \in L^2(\mathcal{F}_T)$ and $f : [0, T] \times \mathbb{R} \times \Phi \times \Omega \mapsto \mathbb{R}$ such that f satisfies the following conditions, for some constant $C_f > 0$:

(i) $f_t(\lambda, Y, \phi, \omega)$ is $\mathbb{G}$-adapted for all $\lambda \in \mathcal{L}, Y \in S^2_{\mathcal{G}}(0, T), \phi \in \mathcal{I}$;

(ii) $f_t(\lambda, 0, 0, \omega) \in \mathcal{H}^2_{\mathcal{G}}(0, T)$, and $\mathbb{G}$-adapted for all $\lambda \in \mathcal{L}$;

(iii) $\big|f_t((\lambda^B, \lambda^H), y_1, \phi(0)_1, l_1) - f_t((\lambda^B, \lambda^H), y_2, \phi(0)_2, l_2)\big| \leq$
$C_f\big(|y_1 - y_2| + |\phi(0)_1 - \phi(0)_2|\sqrt{\lambda^B} + \sqrt{\int_{\mathbb{R}^*} |l_1(u) - l_2(u)|^2 \nu(\mathrm{d}u)}\sqrt{\lambda^H}\big)$,
for all $(\lambda^B, \lambda^H) \in [0, \infty)^2, y_1, y_2 \in \mathbb{R}$, and $(\phi(0)_1, l_1), (\phi(0)_2, l_2) \in \Phi, \mathrm{d}t \times \mathrm{d}P$-a.s.

Chapter 1

Risk Measures and BSDEs with Jumps

We start this chapter by recalling the definition and properties of static risk measures, and then further generalise them in the dynamic setting. One can easily observe the correspondence of their properties in the two different settings. In the meantime, since the dynamic risk measure is a generalisation of the static ones, it has some extra properties related to the filtration that static risk measures do not have. We give a discussion about this in Section 1.1, based on work done in [FS04] [FS02] [AP11] and [Ros06].

Section 1.2 recalls the definition and theory of backward stochastic differential equations (BSDEs) with jumps. We review several important definitions and results related to BSDEs with jumps here, including the Existence and Uniqueness of Solution Theorem, linear BSDEs with jumps and exponential local martingales. The most important theorem in the section is the Comparison Theorem for BSDEs with Jumps. This section summarises work done in [AP11], [Ros06] and [QS13].

Section 1.3 establishes the connection between BSDEs with jumps and dynamic risk measures in the classical setting. The connection is realised through the so-call "f-expectation", a non-linear expectation associated with the driver of the given BSDE with jumps. This is realised by first showing we can indeed define a dynamic risk measure by an f-expectation under certain conditions. Then, by the Inverse Theorem, arguably one of the most important theorems in the entire thesis, it shows that we can represent an f-expectation by a BSDE with jumps under rather general conditions. We will continue to generalise the connection between BSDEs with jumps and dynamic risk measures in the time-changed setting in Chapter 3 and 4.

1.1 Static and Dynamic Risk Measures

This section recalls definitions and properties of the static and dynamic risk measures. It is easy to observe that dynamic risk measures are a generalisation of static ones, and there is need to impose extra conditions on dynamic risk measures to deal with filtration-related issues.

Static risk measures

A risk measure is a functional that determines the riskiness of financial positions within a certain time window. We express a financial position as a mapping $X : \Omega \to \mathbb{R}$, where Ω is the scenario space, and $X(\omega)$ represents the discounted net worth of the financial position at the end of the trading period. We recall Definition 4.1 from [FS04] as the definition of a *static risk measure*.

Definition 1.1.1 (Static Risk Measures)**.** A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a *monetary measure of risk* if it satisfies the following conditions for all $X, Y \in \mathcal{X}$, where $\mathcal{X}$ is a given class of financial positions:

(i) *Monotonicity:* If $X \leq Y$, then $\rho(X) \geq \rho(Y)$;

(ii) *Translation Invariance:* if $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$.

Remark 1.1.2. Note that translation invariance implies:

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$

While for $m \in \mathbb{R}$,

$$\rho(m) = \rho(0) - m.$$

Basic properties of a risk measure justify the intuition that a bigger financial position faces bigger risk. One can neutralise the riskiness of a position X by adding the amount $\rho(X)$ to the position.

If we impose further conditions, we can obtain *convex* and *coherent risk measures.*

Definition 1.1.3. A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a *convex measure of risk*, if it satisfies:

(iii) *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

A convex risk measure is called a *coherent risk measure*, if it satisfies:

(iv) *Positive Homogenity:* If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

Remark 1.1.4. Under the assumption of positive homogenity, given a coherent risk measure ρ and $\lambda = 0$, then

$$\rho(0) = 0.$$

This property is referred by [FS04] as *Normalisation.* More importantly, if we combine positive homogenity and convexity, then we can obtain a stronger property:

(v) *Subadditivity:* $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

The axiom of convexity and its stronger version, subadditivity, have an important financial interpretation about diversification. Subadditivity implies that a diversified portfolio $(X+Y)$ faces smaller risk than two single positions combined. The convexity is a more general version where the portfolio is constructed with weighted splits. The property of positive homogeneity, in addition, expresses the idea that the riskiness of a position grows in proportion with its size.

In the meantime, a risk measure $\rho : \mathcal{X} \to \mathbb{R}$ induces an *acceptance set* $\mathcal{A}_\rho$, defined as:

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \mid \rho(X) \le 0\}.$$

We can also characterise a risk measure via a given acceptance $\mathcal{A}$ set by defining:

$$\rho_{\mathcal{A}} := \inf \{m \in \mathbb{R} \mid m + X \in \mathcal{A}\}.$$

By remark 1.1.2, we see that, given a risk measure ρ and a position X, $X + \rho(X) \in \mathcal{A}_\rho$. This justifies the use of risk measures as a tool to determine the riskiness of X, in the sense that $\rho(X)$ can neutralise its risk and thus make X "acceptable". We recall from [FS02] an important connection between a convex risk measure and its corresponding acceptance set.

Proposition 1.1.5. *Suppose $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure with its associated acceptance set $\mathcal{A}_\rho$. Then*

$$\rho_{\mathcal{A}_\rho} = \rho.$$

Moreover, $\mathcal{A} := \mathcal{A}_\rho$ has the following properties:

(i) $\mathcal{A}$ is convex and non-empty;

(ii) If $X \in \mathcal{A}$ and $Y \in \mathcal{X}$ satisfies $Y \ge X$, then $Y \in \mathcal{A}$;

(iii) If $X \in \mathcal{A}$ and $Y \in \mathcal{X}$, then

$$\{\lambda \in [0, 1] \mid \lambda X + (1 - \lambda)Y \in \mathcal{A}\}$$

is closed in [0,1].

Proof. First we see $\mathcal{A} := \mathcal{A}_\rho$ by definition. Property (i) and (ii) are straightforward from the properties of convex risk measures and the definition of acceptance set. For property (iii), we note that the mapping $\lambda \to \rho(\lambda X + (1 - \lambda)Y)$ is continuous, as it is bounded and convex. The result follows accordingly. For details of the proof, we refer to Proposition 2 in [FS02]. □

Now we recall the representation of convex and coherent risk measures from [Ros06] and [FR02]. We follow [Ros06] in restricting the general class of positions $\mathcal{X}$ to $L^2(\Omega, \mathcal{F}, P)$. We denote by $\mathcal{X}'$ the dual space of $\mathcal{X}$.

Remark 1.1.6. Researchers in the field usually study the class of positions $\mathcal{X}$ in a general space of $L^p(\Omega, \mathcal{F}, P)$, where $p \in [1, \infty]$. Here we have chosen $L^2(\Omega, \mathcal{F}, P)$ in order to facilitate our discussions in connection with BSDEs in the following chapters.

Theorem 1.1.7. *A functional $\rho : \mathcal{X}(L^2(\Omega, \mathcal{F}, P)) \to \mathbb{R}$ is a coherent risk measure if and only if there exits a non-empty closed convex set $\mathcal{P}$ of P-absolutely continuous probability measures such that*

$$\rho(X) = \sup_{Q \in \mathcal{P}} \mathrm{E}_Q[-X], \quad \textit{for all } X \in \mathcal{X}.$$

The representation of convex risk measures is of a more general form.

Theorem 1.1.8. *A functional $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure if and only if there exits a convex functional $F : \mathcal{X}' \to \mathbb{R} \cup \{\infty\}$ satisfying $\inf_{x' \in \mathcal{X}'} F(x') = 0$ such that*

$$\rho(X) = \sup_{Q \in \mathcal{P}} \Big\{ \mathrm{E}_Q[-X] - F\Big(\frac{\mathrm{d}Q}{\mathrm{d}P}\Big) \Big\}, \quad \text{for all } X \in \mathcal{X},$$

where $\mathcal{P} := \{Q \ll P : \frac{\mathrm{d}Q}{\mathrm{d}P} \in \mathcal{X}'$ and $F(\frac{\mathrm{d}Q}{\mathrm{d}P}) < \infty\}$ is a non-empty convex set.

Proof. This is a classical result in the literature for risk measures and is obtained by applying convex analysis theories. We refer to Corollary 7 in [FR02] for details of this proof. Following this idea, we give a proof of this result under the time-changed BSDE setting in Theorem 3.2.4. □

Remark 1.1.9. Note that Theorem 1.1.7 is a special case of Theorem 1.1.8, where $F \equiv 0$. This is a direct consequence of the coherent risk measure's property of subadditivity. We refer to Corollary 7 *vi)* in [FR02] for details of the proof in the general setting. We apply the same argument to prove this result under the time-changed BSDE setting in Corollary 3.2.5.

Remark 1.1.10. We note that authors in [FS04] have obtained similar results independently. To be more specific, authors in [FS04] have expressed the penalty term F in Theorem 1.1.8 through the acceptance set $\mathcal{A}_\rho$:

$$F\Big(\frac{\mathrm{d}Q}{\mathrm{d}P}\Big) \Longleftrightarrow \alpha_{min}(Q) := \sup_{X \in \mathcal{A}_\rho} \mathrm{E}_Q[-X] \quad \text{for } Q \in \mathcal{P}.$$

Here the α_{min} is the minimal penalty term for the risk measure ρ. In other words, $\alpha \geq \alpha_{min}$ for any penalty function α that can represent ρ in Theorem 1.1.8. It is easy to show that the two expressions are equivalent. We refer to Theorem 4.12 [FS04], Theorem 6 in [FR02] and Theorem 4 in [Ros06] for details.

By Theorem 1.1.7, a coherent risk measure can be interpreted as the supremum of the expected loss over a set of "generalised scenarios" $\mathcal{P}$. We interpret F in Theorem 1.1.8 as a penalty term that depends on the generalised scenarios. In this way, a convex risk measure is the supremum of the "corrected expected loss" over the scenarios.

Dynamic Risk Measures

We have so far discussed risk measures in a static setting. Now we are ready to generalise them in a dynamic setting. We first expand the setting into a filtered probability space $L^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$, where a fixed time horizon $T \in [0, \infty]$, $(\mathcal{F}_t)_{t \in [0,T]}$ is a sequence of filtrations such that $\mathcal{F} = \mathcal{F}_T$, if $T < \infty$, and $\mathcal{F} = \sigma\big(\bigcup_{t \geq 0} \mathcal{F}_t\big)$, if $T = \infty$. We define $\mathcal{X} := L^2(\Omega, \mathcal{F}, P)$, while $\mathcal{X}_t := L^2(\Omega, \mathcal{F}_t, P)$. We recall the follwoing definitions from [AP11].

Remark 1.1.11. As mentioned earlier in Remark 1.1.6, here we have chosen $\mathcal{X} := L^2(\Omega, \mathcal{F}, P)$ with the purpose of studying the connection between risk measures and BSDEs.

Definition 1.1.12 (Conditional Convex Risk Measures)**.** For $t \in [0, T]$, where $T \in [0, \infty]$, a mapping $\rho_t : \mathcal{X} \to \mathcal{X}_t$ is called a *conditional convex risk measure*, if it satisfies the following properties for $X, Y \in \mathcal{X}$:

(i) *Monotonicity:* If $X \leq Y$, then $\rho_t(X) \geq \rho_t(Y)$;

(ii) *Conditional Invariance:* For all $m_t \in \mathcal{X}_t$,

$$\rho_t(X + m_t) = \rho(X) - m_t;$$

(iii) *Conditional Convexity:* For all $\lambda \in \mathcal{X}_t$ and $0 \leq \lambda \leq 1$,

$$\rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y).$$

A conditional convex risk measure is called *normalised*, if it enjoys in addition the following property:

(iv) *Normalisation:* $\rho_t(0) = 0$.

A normalised conditional convex risk measure is called a *coherent risk measure* if it satisfies the following extra property:

(v) *Conditional Positive Homogenity:* For all $\lambda \in \mathcal{X}_t$ and $\lambda \geq 0$,

$$\rho_t(\lambda X) = \lambda \rho_t(X).$$

As explain in Remark 1.1.4, Conditional Positive Homogenity implies normalisation. Therefore, all conditional coherent risk measures are normalised.

Remark 1.1.13. It is easy to observe that dynamic risk measures are a generalisation of static ones. We want to point out that the conditional risk measures are parametrised with time, and depend on the filtrations, and the constants used in the properties of static measures need to be replaced with a random variable with respect to the filtration in the dynamic setting.

Remark 1.1.14. We note that for a position X that is $\mathcal{F}_t$-measurable, it is also $\mathcal{F}_T$-measurable.

As with static risk measures, a conditional convex risk measure ρ_t also has an associated acceptance set:

$$\mathcal{A}_t := \{X \in L^2(\Omega, \mathcal{F}, P) \mid \rho_t(X) \leq 0\}.$$

We recall from [AP11] the following properties for such an acceptance set.

Proposition 1.1.15. *The acceptance set $\mathcal{A}_t$ of a normalised conditional convex risk measure ρ is*

(i) Conditionally Convex:

$$\alpha X + (1 - \alpha Y) \in \mathcal{A}_t,$$

for all $X, Y \in \mathcal{A}_t$ and $\mathcal{F}_t$-measurable for $\alpha \in [0, 1]$;

(ii) Solid: If $Y \geq X$ for some $X \in \mathcal{A}_t$, $Y \in \mathcal{A}_t$;

(iii) Such that $0 \in \mathcal{A}_t$ and $\operatorname{ess\,inf}\{X \in L^2(\Omega, \mathcal{F}_t, P)) \mid X \in \mathcal{A}_t\} = 0$.

Moreover, ρ_t is uniquely determined through its acceptance set, since

$$\rho_t(X) = \operatorname{ess\,inf}\big\{Y \in L^2(\Omega, \mathcal{F}_t, P) \;\big|\; X + Y \in \mathcal{A}_t\big\}.$$

Conversely, if some set $\mathcal{A}_t \subseteq L^2$ satisfies the above conditions, then the functional $\rho_t : L^2(\Omega, \mathcal{F}, P)) \to L^2(\Omega, \mathcal{F}_t, P))$ defined in the above definition is a normalised conditional convex risk measure.

Proof. It is very easy to see the correspondence between a static acceptance set and a conditional one. The above three properties follow naturally from the definition of a normalised conditional convex risk measure. We refer to Proposition 1.2 in [AP11] for the rest of the proof. □

The random variable ρ_t represents the riskiness of position X at time t, conditioning on all the information up to time t. We can then look at a process of conditional convex risk measures ρ_t, keeping control of the risk from the beginning to the terminal time T, and thus obtain a definition of a *dynamic convex risk measure.*

Definition 1.1.16 (Dynamic Convex Risk Measures)**.** A sequence $(\rho_t)_{t\in[0,T]}$ is called a *dynamic convex risk measure*, if ρ_t is a conditional convex risk measure for all $t \in [0, T]$.

One important feature of the dynamic setting is the conditional convex risk measures' dependence upon the filtrations. This motives an extra property of dynamic risk measures. We recall the following definitions from [AP11].

Definition 1.1.17. Assume that $(\rho_t)_{t\in[0,T]}$ is a normalised dynamic convex risk measure and let $\mathcal{Y}_t$ be a subset of L^2 such that $0 \in \mathcal{Y}_t$ and $\mathcal{Y}_t + \mathbb{R} = \mathcal{Y}$ for each $t \in [0, T]$. Then $(\rho_t)_{t\in[0,T]}$ is called *acceptance (resp. rejection) consistent* with respect to $(\mathcal{Y}_t)_{t\in[0,T]}$ if for all $t \in [0, T)$ and for any $X \in L^2$ and $Y \in \mathcal{Y}_{t+1}$, the following condition holds:

$$\rho_{t+1}(X) \le \rho_{t+1}(Y) \quad (\text{respectively} \ge) \Longrightarrow \rho_t(X) \le \rho_t(Y) \quad (\text{respectively} \ge). \tag{1.1}$$

Depending on different properties of $\mathcal{Y}_t$, the normalised dynamic convex risk measure $(\rho_t)_{t\in[0,T]}$ has different properties.

Definition 1.1.18. We call a *normalised dynamic convex risk measure* $(\rho_t)_{t\in[0,T]}$:

(i) *Strongly Time-consistent:* If $(\rho_t)_{t\in[0,T]}$ is either acceptance consistent or rejection consistent with respect to $\mathcal{Y}_t = L^2$ for all t in the sense of Definition 1.1.17;

(ii) *Middle Acceptance-consistent (respectively, Middle Rejection-consistent):* If for all t, we have $\mathcal{Y}_t = L^2(\Omega, \mathcal{F}_t, P))$ in Definition 1.1.17;

(iii) *Weakly Acceptance-consistent (respectively, Weakly Rejection-consistent)*: If for all t, we have $\mathcal{Y}_t = \mathbb{R}$ in Definition 1.1.17.

In the following proposition, we provide a unifying summary of time-consistency property for normalised dynamic risk measures from different sources in the literature, such as [Ros06] and [AP11].

Proposition 1.1.19 (Time-consistency Property)**.** *For a normalised dynamic risk measure $(\rho_t)_{t\in[0,T]}$, the following properties are equivalent with (1.1) in Definition 1.1.17:*

(i) Time-consistency: If for all $t \in [0,T]$, $X \in \mathcal{X}$ and $A \in \mathcal{F}_t$,

$$\rho_0(X\mathbb{1}_A) = \rho_0(-\rho_t(X)\mathbb{1}_A); \tag{1.2}$$

(ii) Recursiveness: For all $t, s \geq 0$ such that $t, t+s \in [0,T]$,

$$\rho_t(X) = \rho_t(-\rho_{t+s}(X)). \tag{1.3}$$

❇ *Proof.* It is easy to see 1.2 and 1.3 are equivalent. Given that $A \in \mathcal{F}_t$, by normalisation of ρ_t, we have

$$-\rho_t(X\mathbb{1}_A) = -\rho_t(X)\mathbb{1}_A.$$

Hence the claim follows.

To show 1.1 and 1.3 are equivalent, we recall the proof from Proposition 1.16 in [AP11]. Here 1.1 implies that for all $t \in [0,T)$ and for all $X, Y \in L^2$,

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \quad P\text{-a.s} \implies \rho_t(X) = \rho_t(Y) \quad P\text{-a.s} \,. \tag{1.4}$$

By conditional invariance, defined in Definition 1.1.12, we know that $\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X)$. Thus by 1.4, we obtain one-step recursiveness for $s = 1$. Now we assume the claim holds for each t and all $k \leq s$ for some $s \geq 1$. Then we have:

$$\begin{aligned}\rho_t(\rho_{t+s+1}(X)) &= \rho_t(-\rho_{t+s}(-\rho_{t+s+1}(X)))\\ &= \rho_t(-\rho_{t+s}(X))\\ &= \rho_t(X).\end{aligned}$$

We thus obtain recursiveness by 1.1 through induction. Finally, we see that 1.3 implies 1.1 by conditional monotonicity. This concludes the proof. □

Before we end this section, we recall two families of examples of dynamic coherent and convex risk measures from [Ros06] and [FR04].

Example 1.1.20. Let $\mathcal{P}$ be a convex set of P-absolutely continuous probability measures defined on $(\Omega, \mathcal{F})$, and set

$$\rho_t \coloneqq \operatorname*{ess\,sup}_{Q\in\mathcal{P}} \mathrm{E}_Q[-X \mid \mathcal{F}_t], \quad \text{for all } X \in \mathcal{X},\ t \in [0,T].$$

Then $(\rho_t)_{t\in[0,T]}$ is a dynamic coherent risk measure. ■

Example 1.1.21. Let $\mathcal{P}$ be a convex set of P-absolutely continuous probability measures defined on $(\Omega, \mathcal{F})$, and for any $t \in [0,T]$, let $\mathcal{F}_t : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ be a convex functional such that $\inf_{Q\in\mathcal{P}} F_t(Q) = 0$. Then $(\rho_t)_{t\in[0,T]}$, defined by

$$\rho_t \coloneqq \operatorname*{ess\,sup}_{Q\in\mathcal{P}} \{\mathrm{E}_Q[-X \mid \mathcal{F}_t] - F_t(Q)\}, \quad \text{for all } X \in \mathcal{X},\ t \in [0,T],$$

is a dynamic convex measure. ■

It is clear that Example 1.1.20 is a special case of Example 1.1.21. Same as in the static setting, the dynamic coherent risk measure has such representation because of the stronger conditional subadditivity condition, a combination of conditional convexity and positive homogeneity.

We note that not all dynamic risk measures in this family have the time-consistency property. In order to guarantee the time-consistency property, We usually need to require extra properties on the probability set $\mathcal{P}$ and the penalty term $F_t(\cdot)$. We do not go into further discussions about this issue, and refer to [FR04] and [Ros06] for details.

1.2 BSDEs with Jumps

Since its introduction by Bismut to solve optimal control problems by maximum principle, backwards stochastic differential equations (BSDEs) have developed into a powerful tool in finance. The study of BSDEs started with the linear form, which Pardoux and Peng later extended into a non-linear framework. The continuous setting, where the BSDEs are driven by Brownian motions, has been well studied in the literature.

In this section, we investigate the discontinuous framework where we associate the natural filtration with both a Brownian motion and a Poisson random measure. By summarising works including [QS13], [Pha09], [CFS08],[Del13], [Ros06], [TL94], [Roy06] and [DS14], we recall various important properties of BSDEs with jumps and their associated f-expectations.

We focus especially on the Comparison Theorem which is an instrumental tool to the study of optimisation problems associated with BSDEs. Royer is one of the first to prove a comparison theorem for BSDEs with jumps in [Roy06], while a few years later, authors in [QS13] presented comparison theorems under a even weaker condition. We also recall one of the optimisation principles from [QS13] to characterise *minima* of BSDEs under the discontinuous framework.

In this section, we follow mainly the work that has been done in [QS13], and present the Comparison Theorem as well as the Optimisation Principle, which are useful in the next section.

Existence and Uniqueness of Solution

Given the set-up in Section 0.1 for BSDEs with jumps, we recall from [QS13] and [TL94] an important theorem regarding the existence and uniqueness of solution for BSDE with jumps. We note here in our setting, the Brownian motion W is assumed to be one-dimensional.

Theorem 1.2.1 (Existence and Uniqueness of Solution in One-dimensional Brownian Motion Setting)**.** *Let $T > 0$. For every Lipschitz driver f defined in Definition 0.1.2, and every terminal condition $\xi \in L^2(\mathcal{F}_T)$, there exists a unique solution $(Y, \pi, l) \in S^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$ of the BSDE with jumps in Definition 0.1.3.*

Proof. We give a brief account of the proof for this fundamental result. With a series of lemmata, authors in [TL94] make use of properties including the quasi-left-continuous property of the right-continuous natural filtration associated with a Brownian motion and a Poisson random measure, and obtain an orthogonal decomposition of adapted, square-integrable martingales that starts at 0. By this decomposition, authors in [TL94] succeed in proving a Martingale Representation Theorem in Lemma 2.3 in the paper.

With the help of the Martingale Representation Theorem, authors in [TL94] manage to construct a contracting map, which eventually leads to the proof of the result in Lemma 2.4 in the paper.

We refer to Lemma 2.3, Lemma 2.4 and lemmata in Appendix in [TL94] for details of the proof. □

The results we present in this thesis are in the one-dimensional setting. To extend to a multi-dimensional ($d \in \mathbb{N}\backslash\{0\}$) Brownian motion setting, we need to impose some extra conditions on the generator f to guarantee the existence of a unique solution, as stated in the following assumption. We refer to [Roy06] for a detailed explanation of this assumption. Yet despite these extra conditions, our results in this thesis can be generalised into the multi-dimensional situation with no problem.

Assumption 1.2.2 (Multi-dimensional Brownian Motion Setting)**.** We need the following conditions on the driver f defined in Definition 0.1.3 to guarantee a unique solution for the related BSDE with a multi-dimensional ($d \in \mathbb{N}\backslash\{0\}$) Brownian motion:

(i) f is Lipschitz with respect to π, ℓ:

$$\exists K \geq 0 \text{ such that } \forall t \in [0,T], \forall y \in \mathbb{R}, \forall \pi, \pi' \in \mathbb{R}^d, \forall \ell, \ell' \in L^2_\nu$$
$$|f(t,y,\pi,\ell) - f(t,y,\pi',\ell')| \leq K\|\pi - \pi'\| + K\|\ell - \ell'\|_{2,\nu}$$

where $\|\pi - \pi'\|$ is the Euclidean norm in $\mathbb{R}^d$.

(ii) f is continuous with respect to y, and there exits an $\mathbb{R}^*$-valued process $\varphi_t \in \mathcal{H}^2_{\mathcal{F}}(0,T)$ for $0 \leq t \leq T$, and

$$|f(t,y,\pi,\ell) \leq \varphi_t + K(|y| + \|\pi\| +)| + \|\ell\|_{2,\nu}$$

(iii) f is monotonic with respect to y:

$$\exists \alpha \in \mathbb{R} \text{ such that } \forall t \geq 0, \forall y, y' \in \mathbb{R}, \forall \pi \in \mathbb{R}^d, \forall \ell, \ell' \in L^2_\nu$$
$$(y - y')(f(t,y,\pi,\ell) - f(t,y',\pi,\ell)) \leq \alpha|y - y'|^2 \ P\text{-a.s.}$$

We refer to Assumption ($\mathbf{H}_{ex}$) in [Roy06].

Exponential Local Martingales

We first take a look at linear BSDEs with jumps, before we present the Comparison Theorem. The result and properties of linear BSDEs can give us an insight into the general comparison theorems.

We follow the notation of [QS13] and first recall the definition of exponential local martingale from Definition 15.1.1 in [Coh15].

Definition 1.2.3 (Exponential Local Martingales)**.** For X a semimartingale, we define the *stochastic exponential* (also known as the Doléans-Dade exponential) to be the process $\mathcal{E}(X)$ by

$$\mathcal{E}(X) = \exp\left(X_t - \frac{1}{2}\langle X^c, X^c\rangle_t\right)\prod_{s\leq t}(1 + \Delta X_s)e^{-\Delta X_s}$$

where X^c is the continuous martingale part of X, and $\langle X^c, X^c\rangle$ is the predictable quadratic variation of X^c and for process X_t, $\Delta X_t := X_t - X_{t^-}$.

Let (β_t) be a $\mathbb{R}$-valued predictable process, a.s. integrable with respect to $\mathrm{d}W_t$. Let $(\gamma_t(u))$be an $\mathbb{R}$-valued *predictable* process defined on $[0,T] \times \Omega \times \mathbb{R}^*$, in other words, $\Sigma_p \otimes \mathcal{B}(\mathbb{R}^*)$-measurable, and a.s. integrable with respect to $\widetilde{N}(\mathrm{d}s, \mathrm{d}u)$.

Let $M = (M_t)_{0\le t\le T}$ be a local martingale given by

$$M_t := \int_0^t \beta_s \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^*} \gamma_s(u)\widetilde{N}(\mathrm{d}s, \mathrm{d}u). \tag{1.5}$$

Let $Z = (Z_t)_{0\le t\le T}$ be the solution of $\mathrm{d}Z_s = Z_{s^-}\mathrm{d}M_s$ where $Z_0 = 1$. By Definition 1.2.3, and by using Itô's formula, we can denote the process Z by $\mathcal{E}(M)$ and show that it is the exponential local martingale associated with the local martingale M:

$$\begin{aligned}\mathcal{E}(M)_s = \exp\Big\{\int_s^0 \beta_u \mathrm{d}W_u - \frac{1}{2}\int_s^0 \beta_u^2 \mathrm{d}u - \int_0^s \int_{\mathbb{R}^*} \gamma_r(u)\nu(\mathrm{d}u)\mathrm{d}r\Big\}\\ \times \prod_{0<r\le s}(1+\gamma_r(\Delta Y_r))\end{aligned} \tag{1.6}$$

where $Y_t := \int_{\mathbb{R}^*} uN([0,t], \mathrm{d}u)$.

Remark 1.2.4. In Definition 1.2.3, we see that if the process $\Delta X \ge -1$, the process $\mathcal{E}(X)$ is then non-negative. In addition, if X is a local martingale, then $\mathcal{E}(X)$ is also a local martingale.

Similarly in 1.6, if $\gamma_t(\Delta Y_t)) \ge -1$ for $0 \le t \le T$ a.s., then we have $\mathcal{E}(M)_t \ge 0$ for $0 \le t \le T$ a.s.

We recall two important properties of exponential local martingales from [QS13].

Proposition 1.2.5. *Let (β_t) and $(\gamma_t(u))$ be predictable $\mathbb{R}$-valued processes and let M be the local martingale defined in (1.5). The following assertions are equivalent:*

1. *For each $n \in \mathbb{N}$, $\gamma_{T_n}(\Delta Y_{T_n} \ge -1)$ P-a.s., where $(T_n)_{n\in\mathbb{N}}$ is the increasing sequence of stopping times corresponding to the jump times of Y;*

2. *$\gamma_t(u) \ge -1$ $\mathrm{d}P \otimes \mathrm{d}t \otimes \mathrm{d}\nu(u)$-a.s.*

If one of these conditions is satisfied, we have $\mathcal{E}(M)_t \ge 0$ for $0 \le t \le T$ a.s. Moreover, if $\gamma_t(u) > -1$ $\mathrm{d}P \otimes \mathrm{d}t \otimes \mathrm{d}\nu(u)$-a.s., then for each t, $\mathcal{E}(M)_t > 0$ a.s.

Proof. We see for each $s > 0$,

$$\prod_{0<r\le s}(1+\gamma_r(\Delta Y_r)) = \prod_{n\in\mathbb{N},\ 0<T_n\le s}(1+\gamma_{T_n}(\Delta Y_{T_n})).$$

By equation 1.6, condition 1 implies for each s, $\mathcal{E}(M)_s \ge 0$ a.s.

We then show that the two conditions are equivalent. Since $\nu(\mathrm{d}u)\mathrm{d}t$ is the predictable compensator of the Poisson random measure $N(\mathrm{d}u, \mathrm{d}t)$, we have:

$$\begin{aligned}\mathrm{E}\Big[\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\gamma_{T_n}(\Delta Y_{T_n})<-1\}}\Big] &= \mathrm{E}\Big[\int_{\mathbb{R}^*\times\mathbb{R}_+} \mathbb{1}_{\{\gamma_r(u)<-1\}} N(\mathrm{d}u, \mathrm{d}r)\Big]\\ &= \mathrm{E}\Big[\int_{\mathbb{R}^*\times\mathbb{R}_+} \mathbb{1}_{\{\gamma_r(u)<-1\}} \nu(\mathrm{d}u)\mathrm{d}r\Big].\end{aligned}$$

The result follows. □

Proposition 1.2.6. *Let* (β_t) *and* $(\gamma_r(u))$ *be predictable* $\mathbb{R}$*-valued processes and let* M *be the local martingale defined in (1.5). Suppose that*

$$\int_0^T \beta_s^2 \mathrm{d}s + \int_0^T \|\gamma_s\|_\nu^2 \,\mathrm{d}s \tag{1.7}$$

is bounded. Then we have $\mathrm{E}[\mathcal{E}(M)_T^2] < \infty$.

Proof. The result follows easily from an application of the product formula or by (1.6). We refer to Proposition 3.2 in [QS13] for details of the proof. □

Remark 1.2.7. Note in the previous proposition, we can obtain $(\mathcal{E}(M)_s)_{0\le t\le T} \in S^2_{\mathcal{F}}(0,T)$ by an application of martingale inequalities. To be more precise, by Doob's L^p Inequality in Theorem 5.1.3 in [Coh15], we can obtain:

$$\|\mathcal{E}(M)\|^2_{S^2_{\mathcal{F}}(0,T)} \coloneqq \mathrm{E}\left[\sup_{0\le t\le T} |\mathcal{E}(M)_t|^2\right] \le 2 \sup_{0\le t\le T} \mathrm{E}\left[|\mathcal{E}(M)_t|^2\right] < \infty.$$

Remark 1.2.8. In the above proposition, we see that if the processes β_t and $\|\gamma_t\|_\nu$ are bounded, the random variable by (1.7) is then bounded, and by Remark 1.2.7, we have $\mathcal{E}(M)_t \in L^2$.

The condition is also satisfied if there exits $\psi \in L^2_\nu$ such that $|\gamma_t(u)| \le \psi(u)$, $\mathrm{d}t \otimes \mathrm{d}P \otimes \mathrm{d}\nu(u)$-a.s. This property will used in the proof of the Comparison Theorem.

Linear BSDEs with jumps

We first recall a fundamental result of forward SDE solution.

Proposition 1.2.9. *Let* (δ_t) *and* (β_t) *be* $\mathbb{R}$*-valued predictable processes, integrable with respect to* $\mathrm{d}t$ *and* $\mathrm{d}W_t$ *a.s. Let* $(\gamma_t(u))$ *be a predictable* $\mathbb{R}$*-valued process defined on* $[0,T] \times \Omega \times \mathbb{R}^*$*, integrable with respect to* $\widetilde{N}(\mathrm{d}s, \mathrm{d}u)$.

For each $t \in [0,T]$*, let* $(\Gamma_{t,s})_{s\in[t,T]}$ *be the unique solution of the following forward SDE:*

$$\mathrm{d}\Gamma_{t,s} = \Gamma_{t,s^-}\left[\delta_s \mathrm{d}s + \beta_s \mathrm{d}W_s + \int_{\mathbb{R}^*} \gamma_s \widetilde{N}(\mathrm{d}s, \mathrm{d}u)\right], \quad \textit{where} \quad \Gamma_{t,t} = 1.$$

The process $\Gamma_{t,s}$ *can then be written as* $\Gamma_{t,s} = e^{\int_t^s \delta_u \mathrm{d}u} Z_{t,s}$*, where* $(Z_{t,s})_{s\in[t,T]}$ *is the solution of the following SDE*

$$\mathrm{d}Z_{t,s} = Z_{t,s^-}\left[\beta_s \mathrm{d}W_s + \int_{\mathbb{R}^*} \gamma_s(u) \widetilde{N}(\mathrm{d}s, \mathrm{d}u)\right], \quad \textit{where} \quad Z_{t,t} = 1.$$

Proof. The result follows easily with an application of Itô's formula to $\ln \Gamma_{t,s}$. □

We recall a result from [QS13], showing that the solution of a linear BSDE with jumps can be represented as a conditional expectation through an exponential semimartingale.

Theorem 1.2.10. *Let (δ, β, γ) be a real-valued, bounded predictable process. Let Γ be the process defined in Proposition 1.2.9. Suppose that $\Gamma \in S^2_{\mathcal{F}}(0,T)$ and process $\varphi \in \mathcal{H}^2_{\mathcal{F}}(0,T)$.*

Let (Y_t, π_t, l_t) be the solution in $S^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$ of the linear BSDE

$$\begin{aligned} -\mathrm{d}Y_t &= (\varphi_t + \delta_t Y_t + \beta_t \pi_t + \langle \gamma_t, l_t \rangle_\nu)\mathrm{d}t - \pi_t \mathrm{d}W_t - \int_{\mathbb{R}^*} l_t(u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u); \\ Y_T &= \xi. \end{aligned} \tag{1.8}$$

The process (Y_t) satisfies

$$Y_t = \mathrm{E}\left[\Gamma_{t,T}\xi + \int_t^T \Gamma_{t,s}\varphi(s)\mathrm{d}s \;\Big|\; \mathcal{F}_t\right], \quad 0 \le t \le T, \ a.s. \tag{1.9}$$

Proof. This is an important result and we recall the proof from Theorem 3.4 in [QS13]. Fix $t \in [0,T]$. To simplify notation, we denote $\Gamma_{t,s}$ by Γ_s for $s \in [t,T]$. By Itô product formula, we have:

$$\begin{aligned} -\mathrm{d}(Y_s\Gamma_s) &= -Y_{s^-}\mathrm{d}\Gamma_s - \Gamma_{s^-}\mathrm{d}Y_s - d[X,\Gamma]_s \\ &= -Y_s\Gamma_s\delta_s\mathrm{d}s + \Gamma_s[\varphi_s + \delta_s Y_s + \beta_s\pi_s + \langle \gamma_s, l_s\rangle_\nu]\mathrm{d}s \\ &\quad - \beta_s\pi_s\Gamma_s\mathrm{d}s - \Gamma_s\langle\gamma_s, l_s\rangle_\nu\mathrm{d}s - \Gamma_s(Y_s\beta_s + \pi_s)\mathrm{d}W_s \\ &\quad - \Gamma_{s^-}\int_{\mathbb{R}} l_s(u)(1+\gamma_s(u))\widetilde{N}(\mathrm{d}u, \mathrm{d}s) \\ &= \Gamma_s\varphi_s\mathrm{d}s - \mathrm{d}M_s, \end{aligned}$$

with

$$\mathrm{d}M_s = -\Gamma_s(Y_s\beta_s + \pi_s)\mathrm{d}W_s - \Gamma_{s^-}\int_{\mathbb{R}} l_s(u)(1+\gamma_s(u))\widetilde{N}(\mathrm{d}u, \mathrm{d}s). \tag{1.10}$$

Integrate from t to T, we get

$$Y_t - \xi\Gamma_{t,T} = \int_t^T \Gamma_{t,s}\varphi_s\mathrm{d}s - M_T + M_t \quad \text{a.s.} \tag{1.11}$$

Recall that $\Gamma_{t,\cdot} \in S^2_{\mathcal{F}}(0,T)$ and that $Y \in S^2_{\mathcal{F}}(0,T)$, $\pi \in \mathcal{H}^2_{\mathcal{F}}(0,T)$ and $l \in \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$. Moreover, the processes δ, β and γ are bounded. It follows that the local martingale M is a martingale.

To see that M is indeed a martingale, we recall the classical result from stochastic analysis that, if a local martingale M satisfies $\mathrm{E}\big[[M]_t^{\frac{1}{2}}\big] < \infty$ where $[\,\cdot\,]$ denotes the quadratic variation, for all $t \in [0,T]$, then it is a martingale.

By assumption, (δ, β, γ) are real-valued, predictable process bounded by constants. In view of predictability, the compensated Poisson term in (1.10) has zero mean, according to classical stochastic analysis theories. In the meantime, because of boundedness, the Brownian motion term in (1.10) is finite. We can therefore conclude that M in (1.10) is indeed a martingale.

Note that given our time-changed framework in Section 0.2 and properties of λ defined there, the same argument also applies in the time-changed situation.

Hence, by taking the conditional expectation in (**??**) , we can obtain (1.9). □

We recall one more result before presenting the Comparison Theorem.

Corollary 1.2.11. *Suppose the assumptions of Theorem 1.2.10 are satisfied.*

- *Suppose that the inequality* $\gamma_t(u) \geq -1$ *holds* $\mathrm{d}P \otimes \mathrm{d}t \otimes \mathrm{d}\nu(u)$*-a.s.* *If* $\varphi_t \geq 0,\ t \in [0,T],\ \mathrm{d}P \otimes \mathrm{d}t$ *a.s. and* $\xi \geq 0$ *a.s., then* $Y_t \geq 0$ *a.s. for all* $t \in [0,T]$.
- *Suppose that the inequality* $\gamma_t(u) > -1$ *holds* $\mathrm{d}P \otimes \mathrm{d}t \otimes \mathrm{d}\nu(u)$*-a.s.* *If* $\varphi_t \geq 0,\ t \in [0,T],\ \mathrm{d}P \otimes \mathrm{d}t$ *a.s. and* $\xi \geq 0$ *a.s., and if* $Y_t = 0$ *a.s. for some* $t_0 \in [0,T]$, *then* $\varphi_t = 0$ $\mathrm{d}P \otimes \mathrm{d}t$ *a.s. on* $[0,T]$, *and*$\xi = 0$ *a.s.on* A, $A \in \mathcal{F}_{t_0}$.

Proof. We first use Proposition 1.2.5, where we discussed the non-negativity of exponential local martingales, to establish the non-negativity of $\Gamma_{t,T}$. Then the result follows naturally from Theorem 1.2.10 where we expressed the solution with a conditional expectation. For details we refer to Corollary 3.5 in [QS13]. □

The Comparison Theorem

We first present a comparison theorem for linear BSDEs with jumps. Based on this preliminary result, we generalise the proof and obtain a general comparison theorem.

Lemma 1.2.12 (Comparison Theorem for Linear BSDEs with Jumps)**.** *Let* (δ, β, γ) *be a bounded, real-valued predictable process and for each* t, *let* Γ_t *be the exponential semimartingale solution in Proposition 1.2.9. Suppose that*

$$\Gamma_{t,T} \in S^2_{\mathcal{F}}(0,T)\ \forall t \quad \textit{and} \quad \gamma_t(u) \geq -1 \quad \mathrm{d}P \otimes \mathrm{d}t \otimes \nu(\mathrm{d}u)\text{-a.s.}$$

Let $\xi \in L^2(\mathcal{F}_T)$ *and* h *be a driver (not necessarily Lipschitz). Let* (Y_t, π_t, l_t) *be a solution in* $S^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T) \times \mathcal{H}^p_{\mathcal{F}}(0,T,\nu)$ *of the BSDE*

$$\begin{aligned} -\mathrm{d}Y_t &= h(t, Y_{t^-}, \pi_t, l_t(.))\mathrm{d}t - \pi_t \mathrm{d}W_t - \int_{\mathbb{R}^*} l_t(u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ Y_T &= \xi \end{aligned}$$

Let $\varphi \in \mathcal{H}^2_{\mathcal{F}}(0,T)$. *Suppose that*

$$h(t, Y_{t^-}, \pi_t, l_t(.)) \geq \varphi_t + \delta_t Y_t + \beta \pi_t + \langle \gamma_t, l_t \rangle_\nu, \quad 0 \leq t \leq T,\ \mathrm{d}P \otimes \mathrm{d}t\text{-a.s.} \tag{1.12}$$

Then we have

$$Y_T \geq \mathrm{E}\left[\Gamma_{t,T}\xi + \int_t^T \Gamma_{t,s}\varphi(s)\mathrm{d}s \;\middle|\; \mathcal{F}_t\right], \quad 0 \leq t \leq T,\ a.s. \tag{1.13}$$

Proof. We give a brief account of the proof. Applying similar computation as in Theorem 1.2.10, we apply Itô product formular to $-\mathrm{d}(Y_s\Gamma_s)$ and then obtain:

$$-\mathrm{d}(Y_s\Gamma_s) \geq \Gamma_s\varphi_s\mathrm{d}s - \mathrm{d}M_s,$$

where M is a martingale. The result follows with integration. For further details of the proof, we refer to Lemma 4.1 in [QS13]. □

With all the preparations, now we are ready to present the main result of this section—a general comparison theorem for BSDEs with jumps.

Theorem 1.2.13 (Comparison Theorem for BSDEs with Jumps). *Let ξ_1 amd $\xi_2 \in L^2(\mathcal{F}_T)$. Let f_1 be a Lipschitz driver. Let f_2 be a driver. For $i = 1, 2$, let (Y_t^i, π_t^i, l_t^i) be a solution in $S^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T) \times \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$ of the BSDE*

$$
\begin{aligned}
-\mathrm{d}Y_t^i &= f_i(t, Y_t^i, \pi_t^i, l_t^i)\mathrm{d}t - \pi_t^i \mathrm{d}W_t - \int_{\mathbb{R}^*} l_t^i(u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u); \\
Y_T^i &= \xi_i
\end{aligned}
$$

Assume there exits a bounded predictable process (γ_t) such that $\mathrm{d}t \otimes \mathrm{d}P \otimes \nu(\mathrm{d}u)$-a.s.,

$$\gamma_t(u) \geq -1 \quad \text{and} \quad |\gamma_t(u) \leq \psi(u)|, \tag{1.14}$$

where $\psi(u) \in L^2_\nu$, and such that

$$f_1(t, Y_t^2, \pi_t^2, l_t^1) - f_1(t, Y_t^2, \pi_t^2, l_t^2) \geq \langle \gamma_t, l_t^1 - l_t^2 \rangle_\nu, \quad t \in [0,T],\ \mathrm{d}t \otimes \mathrm{d}P \ a.s. \tag{1.15}$$

Assume that

$$
\begin{aligned}
\xi_1 \geq \xi_2 \quad a.s. \\
f_1(t, Y_t^2, \pi_t^2, l_t^2) \geq f_2(t, Y_t^2, \pi_t^2, l_t^2), \quad t \in [0,T],\ \mathrm{d}t \otimes \mathrm{d}P \ a.s.
\end{aligned}
\tag{1.16}
$$

Then we have

$$Y_t^1 \geq Y_t^2 \quad a.s. \text{ for all } t \in [0,T]. \tag{1.17}$$

Moreover, if inequality 1.16 is satisfied for (Y_t^1, π_t^1, l_t^1) instead of (Y_t^2, π_t^2, l_t^2) and if f_2 (instead of f_1) is Lipschitz and satisfies (1.15), then inequality (1.17) still holds.

Proof. The crucial part of the proof is the linearisation of f_1. We give a brief account of the key elements.

We first rewrite the BSDE in the following way. Put $\overline{Y}_t = Y_t^1 - Y_t^2$, $\overline{\pi}_t = \pi_t^1 - \pi_t^2$, and $\bar{l}_t(u) = l_t^1(u) - l_t^2(u)$. Then the BSDE is rewritten as

$$-\mathrm{d}\overline{Y}_t = h_t \mathrm{d}t - \overline{\pi}\mathrm{d}W_t - \int_{\mathbb{R}^*} \bar{l}_t(u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u); \quad \overline{Y}_T = \xi_1 - \xi_2,$$

where $h_t \coloneqq f_1(t, Y_t^1, \pi_t^1, l_t^1) - f_2(t, Y_t^2, \pi_t^2, l_t^2)$. We can rewrite h_t further:

$$
\begin{aligned}
h_t &= f_1(t, Y_t^1, \pi_t^1, l_t^1) - f_1(t, Y_t^2, \pi_t^1, l_t^1) \\
&\quad + f_1(t, Y_t^2, \pi_t^1, l_t^1) - f_1(t, Y_t^2, \pi_t^2, l_t^1) \\
&\quad + f_1(t, Y_t^2, \pi_t^2, l_t^1) - f_1(t, Y_t^2, \pi_t^2, l_t^2) \\
&\quad + f_1(t, Y_t^2, \pi_t^2, l_t^2) - f_2(t, Y_t^2, \pi_t^2, l_t^2) \\
\varphi_t &\coloneqq f_1(t, Y_{t^-}^2, \pi_t^2, l_t^2) - f_2(t, Y_{t^-}^2, \pi_t^2, l_t^2) \\
\delta_t &\coloneqq \frac{f_1(t, Y_{t^-}^1, \pi_t^1, l_t^1) - f_1(t, Y_{t^-}^2, \pi_t^1, l_t^1)}{\overline{Y}_t} \mathbb{1}_{\{\overline{Y}_t \neq 0\}} \\
\beta_t &\coloneqq \frac{f_1(t, Y_{t^-}^2, \pi_t^1, l_t^1) - f_1(t, Y_{t^-}^2, \pi_t^2, l_t^1)}{\overline{\pi}_t} \mathbb{1}_{\{\overline{\pi}_t \neq 0\}}
\end{aligned}
$$

By assumption (1.15) about f_1, we can show that there exist δ, β such that h_t satisfies inequality (1.12). Now conditions in Lemma 1.2.12 are satisfied and by the positivity of $\Gamma_{t,\cdot}$, the result follows.

We can show the second assertion by linearising f^2 instead. We refer to Theorem 4.2 in [QS13] for details of the proof. □

Theorem 1.2.14 (Strict Comparison Theorem). *Suppose that the assumptions of Theorem 1.2.13 hold and that the inequality $\gamma_t(u) > -1$ holds $\mathrm{d}t \otimes \mathrm{d}P \otimes \mathrm{d}\nu(u)$-a.s.*

If $Y^1_{t_0} = Y^2_{t_0}$ a.s. on A for some $t_0 \in [0,T]$ and $A \in \mathcal{F}_{t_0}$, then $Y^1 = Y^2$ a.s. on $[t_0,T] \times A$, $\xi_1 = \xi_2$ a.s. on A and 1.16 holds as an equality in $[t_0,T] \times A$.

Proof. This result follows closely from the previous one. We refer to Theorem 4.4 in [QS13] for details of the proof . □

Remark 1.2.15. We draw attention to the fact that Theorem 1.2.14 is stronger than Theorem 1.2.13, as the former requires an additional condition.

Based on the comparison theorems, we recall from [QS13] an optimisation principle, which will be useful for determining *minima* of BSDEs in the next section.

Theorem 1.2.16 (Optimisation Principle). *Let ξ in $L^2(\mathcal{F}_T)$ and let $(f, f^\alpha; \alpha \in \mathcal{A}_T)$ be a family of Lipschitz drivers. Let (Y, π, l) (resp. $(Y^\alpha, \pi^\alpha, l^\alpha)$) be the solution of the BDE associated with terminal condition ξ and driver f (resp. f^α).*

Suppose that

$$f(t, Y_t, \pi_t, l_t) = \operatorname*{ess\,inf}_\alpha f^\alpha(t, Y_t, \pi_t, l_t) = f^{\bar\alpha}(t, Y_t, \pi_t, l_t),$$
$$t \in [0,T],\ \mathrm{d}P \otimes \mathrm{d}t\text{-a.s. for some parameter } \bar\alpha \in \mathcal{A}_T$$

and that for each $\alpha \in \mathcal{A}$, there exits a predictable process γ^α satisfying 1.14 and

$$f^\alpha(t, Y_t, \pi_t, l^\alpha_t) - f^\alpha(t, Y_t, \pi_t, l_t) \geq \langle \gamma^\alpha_t, l^\alpha_t - l_t \rangle_\nu, \quad t \in [0,T],\ \mathrm{d}t \otimes \mathrm{d}P \text{ a.s.}$$

Then

$$Y_t = \operatorname*{ess\,inf}_\alpha Y^\alpha_t = Y^{\bar\alpha}_t, \quad t \in [0,T] \text{ a.s.}$$

Proof. This result follows from the Comparison Theorem, see Theorem 1.2.13. We will need to use this lemma to show the representation of convex risk measures. We refer to Theorem 4.6 in [QS13] for details of proof. □

For the above theorem, a detailed proof in the time-changed setting can be found in Lemma 3.2.1.

1.3 Risk Measures and f-expectations

In this section, we establish in the classical setting the link between dynamic risk measures and BSDEs with jumps via non-linear expectations. We first recall the definition of non-linear expectations, and then the definition of "f-expectation", a non-linear expectation associated with the initial value of the solution of BSDEs with jumps. In this way, we show that we can indeed characterise a non-linear expectation via the associated BSDE.

On the other hand, we would like to be able to construct BSDEs with jumps from a given non-linear expectation, and this is what we call the *inverse problem*. So far as we know, authors in [Pen99] and [Coq+02] are the first to prove this result under the Brownian motion framework, and a few years later, Royer in [Roy06] generalised this result in the Lévy process setting. We recall the result from [Roy06] in this chapter, and will give a detailed proof of this result in the time-changed setting in Chapter 4.

Then by exploiting the natural connection between a non-linear expectation and dynamic risk measures, we show that we have established the link between dynamic risk measures and BSDEs with jumps.

We end this section by presenting a representation of convex and coherent dynamic risk measures. A corresponding result established under the time-changed framework will be presented in Section 3.2 with more details.

Non-linear Expectation

We first recall the definition of a non-linear expectation from Definition 3.1 in [Roy06].

Definition 1.3.1. We say that an operator $\mathcal{E} : L^2(\mathcal{F}_T) \mapsto \mathbb{R}$ is a non-linear expectation if

- $\mathcal{E}[c] = c$, for all $c \in \mathbb{R}$;
- if $\eta_1 \leq \eta_2$ $P-$a.s., then $\mathcal{E}[\eta_1] \leq \mathcal{E}[\eta_2]$. If moreover $\mathcal{E}[\eta_1] = \mathcal{E}[\eta_2]$, then $\eta_1 = \eta_2$ $P-$a.s.

Obviously, any traditional expectations is also a non-linear expectation.

In the following proposition, we show that we can indeed characterise a non-linear expectation via the associated BSDE with jumps. Note also that, in order to use non-linear expectations to express dynamic risk measures, we need to impose a stronger condition on the generator so as to guarantee the monotonicity property of risk measures. This is why the following proposition requires the generator f satisfying assumptions made in the Strict Comparison Theorem, see Theorem 1.2.14.

Proposition 1.3.2 (f-expectation)**.** *Consider a BSDE drived by a generator f, defined in Definition 0.1.3, such that*

(i) $f(t, x, 0, 0) = 0$ *for all* $x \in \mathbb{R}$*;*

(ii) f *is Lipschitz in* x, π*;*

(iii) f *satisfies assumptions in the Strict Comparison Theorem, see Theorem 1.2.14.*

Then for any fixed ξ in $L^2(\mathcal{F}_T)$, we denote the unique solution of the related BSDE with terminal condition ξ by (Y^ξ, π^ξ, l^ξ). We set $\mathcal{E}_f[\xi] = Y_0^\xi$, the initial value of the solution. Then $\mathcal{E}_f$ is a non-linear expectation called f-expectation.

❇ *Proof.* By checking the two properties listed in the definition of a non-linear expectation, see Definition 1.3.1, it is easy to see that the $\mathcal{E}_f$ defined above exists and it is indeed a non-linear expectation.

We first show that for $\xi = c$ where $c \in \mathbb{R}$, $\mathcal{E}_f[\xi] = c$. We convert the BSDE defined in Definition 0.1.3 into the forward form:

$$Y_t = Y_0 - \int_0^t f(s, Y_s, \pi_s, l_s(u))\mathrm{d}s + \int_0^t \pi_s \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^*} l_s(u)\widetilde{N}(\mathrm{d}s, \mathrm{d}u).$$

Then by assumption,

$$\begin{aligned} Y_T &= Y_0 - \int_0^T f(s, Y_s, \pi_s, l_s(u))\mathrm{d}s + \int_0^T \pi_s \mathrm{d}W_s + \int_0^T \int_{\mathbb{R}^*} l_s(u)\widetilde{N}(\mathrm{d}s, \mathrm{d}u) \\ &= c \end{aligned}$$

This implies, for $s \in [0, T]$,

$$\pi_s \equiv 0, \quad l_s(u) \equiv 0.$$

By our assumptions on f, it implies the trivial situation where $f \equiv 0$, and then $Y_0 = c$, and the result follows.

The monotony property is guaranteed by the assumptions in the Strict Comparison Theorem. Given $\xi_1 \geq \xi_2$ a.s. and $f_1 = f_2 = f$, then by the comparison theorem we have $Y(\xi_1) \geq Y(\xi_2)$ a.s., where $Y(\cdot)$ denotes the solution associated with respective terminal conditions. We need to use the Strict Comparison Theorem, see Theorem 1.2.14 to obtain the equality. We can therefore conclude that $\mathcal{E}_f$ is indeed a non-linear expectation. □

Remark 1.3.3. The f-expectations are generally not linear, except in situations where the generator f has a linear form.

Remark 1.3.4. As explained in the previous proof, we need assumptions in the Strict Comparison Theorem in Condition (iii) in Proposition 1.3.2 to guarantee the monotonicity property for the associated non-linear expectation and, by extension, the associated risk measures. We will impose similar conditions when we define f-expectation in the time-changed setting in Proposition 3.1.1.

We now recall some properties of non-linear expectations and see under what conditions the f-expectation satisfies them. We emphasise here we use the natural filtration $\mathbb{F}$ defined in Definition 0.1.1. Corresponding properties in Chapter 3, under the time-changed framework, replaces $\mathbb{F}$ with filtration $\mathbb{G}$ defined in Section 0.2.

Definition 1.3.5 (Filtration-consistency)**.** A non-linear expectation $\mathcal{E}$ is said to be *filtration-consistent* if, for all $A \in \mathcal{F}_t$,

$$\forall \xi \in L^2(\mathcal{F}_T),\ \forall t \in [0, T],\ \exists \eta_t \in L^2(\mathcal{F}_t) \text{ such that } \mathcal{E}[\xi \mathbb{1}_A] = \mathcal{E}[\eta_t \mathbb{1}_A]. \quad (1.18)$$

In this case, we denote $\mathcal{E}[\xi \mid \mathcal{F}_t] = \eta_t$, which is called the *non-linear conditional expectation* of ξ with respect to $\mathcal{F}_t$.

Given that $\xi \in L^2(\mathcal{F}_T)$ and the fact that $f(t,x,0,0)=0$ for all $x \in \mathbb{R}$, any f-expectation is filtration-consistent. In the meantime, we have

$$\mathrm{E}\Big[\sup_{0\le t\le T} |\mathcal{E}_f[\xi \mid \mathcal{F}_t]|^2\Big] < \infty.$$

To see that any f-expectation is filtration-consistent, we note that (1.18) can be written as

$$\mathcal{E}_f[\xi \mathbb{1}_A] = \mathcal{E}_f\Big[\mathcal{E}_f[\xi \mid \mathcal{F}_t]\mathbb{1}_A\Big]. \tag{1.19}$$

Given solution (Y^ξ, π^ξ, l^ξ) in Proposition 1.3.2, by the uniqueness of solution, it is easy to see that, for $A \in \mathcal{F}_t$, $Y_t^\xi \mathbb{1}_A$ and $\xi \mathbb{1}_A$ coincide on the interval $[0,t]$. Then we conclude:

$$\mathcal{E}_f[\xi \mid \mathcal{F}_t] = Y_t^\xi.$$

We will apply the same argument to obtain the corresponding result in the time-changed setting in (3.1).

Definition 1.3.6 (Additivity)**.** Let $\mathcal{E}$ be a filtration-consistent non-linear expectation. We call *additivity* the following property:

$$\forall \xi \in L^2(\mathcal{F}_T),\ \forall t \in [0,T],\ \forall \eta \in L^2(\mathcal{F}_t)\ \mathcal{E}[\eta+\xi \mid \mathcal{F}_t] = \eta + \mathcal{E}[\xi \mid \mathcal{F}_t]. \tag{1.20}$$

We note that any f-expectation is additive if the generator f is independent of y. This is a consequence of the uniqueness of solution for the associated BSDE, as shown in the following argument.

By definition, for $t \in [0,T]$,

$$\begin{aligned}\mathcal{E}[\eta+\xi \mid \mathcal{F}_t] = Y_t^{\eta+\xi} \\ &= \eta + \xi + \int_0^t f(s, \pi_s^{\eta+\xi}, l_s^{\eta+\xi}(u))\mathrm{d}s \\ &\quad - \int_0^t \pi_s^{\eta+\xi}\mathrm{d}W_s - \int_0^t \int_{\mathbb{R}^*} l_s^{\eta+\xi}(u)\widetilde{N}(\mathrm{d}s,\mathrm{d}u).\end{aligned}$$

We thus obtain

$$\begin{aligned}Y_t^{\eta+\xi} - \eta &= \xi + \int_0^t f(s, \pi_s^{\eta+\xi}, l_s^{\eta+\xi}(u))\mathrm{d}s \\ &\quad - \int_0^t \pi_s^{\eta+\xi}\mathrm{d}W_s - \int_0^t \int_{\mathbb{R}^*} l_s^{\eta+\xi}(u)\widetilde{N}(\mathrm{d}s,\mathrm{d}u).\end{aligned} \tag{1.21}$$

On the other hand, we have

$$\begin{aligned}Y_t^{\xi} &= \xi + \int_0^t f(s, \pi_s^{\eta+\xi}, l_s^{\eta+\xi}(u))\mathrm{d}s \\ &\quad - \int_0^t \pi_s^{\xi}\mathrm{d}W_s - \int_0^t \int_{\mathbb{R}^*} l_s^{\xi}(u)\widetilde{N}(\mathrm{d}s,\mathrm{d}u).\end{aligned} \tag{1.22}$$

We see by (1.21), $(Y^{\eta+\xi} - \eta, \pi^{\eta+\xi}, l^{\eta+\xi})$ is a solution of (1.22), since f is independent of y. By the uniqueness of solution of (1.22), we have

$$Y^{\eta+\xi} - \eta = Y^\xi.$$

And we have thus shown (1.20).

In fact, classical results in the literature tell us that we have the additivity property *if and only if the driver f is independent of y.* For details of this fact, we refer to Lemma 4.2 and 4.3 in [Bri+00], which proved this result in the Brownian motion setting. The proof carries over to the jump setting in [Roy06]. We will show the same property for time-changed f-expectation in Section 3.1.

Definition 1.3.7 ($\mathcal{E}^{C,C_1}$-domination)**.** Let $\mathcal{E}$ be a filtration-consistent non-linear expectation. We say that it is $\mathcal{E}^{C,C_1}$-dominated if there exists $C > 0$ and $-1 < C_1 \leq 0$ such that

$$\forall \xi, \xi' \in L^2(\mathrm{F}_T), \quad \mathcal{E}[\xi + \xi'] - \mathcal{E}[\xi] \leq \mathcal{E}^{C,C_1}[\xi']$$

where $\mathcal{E}^{C,C_1}[\xi']$ is the non-linear f-expectation associated with the driver

$$f_{C,C_1}(t, \pi, \ell) \coloneqq C\,|\pi| + C\int_{\mathbb{R}^*}(1 \wedge |u|)\ell^+(u)\nu(\mathrm{d}u) - C_1\int_{\mathbb{R}^*}(1 \wedge |u|)\ell^-(u)\nu(\mathrm{d}u).$$

Here $t \in [0, T]$ and in the current classical setting $\pi \in \mathcal{H}^2_{\mathcal{F}}(0, T)$ and $\ell \in \mathcal{H}^2_{\mathcal{F}}(0, T, \nu)$.

Note the filtration in this context is the natural filtration $\mathbb{F}$ defined in Definition 0.1.1. We recall from discussion following Definition 3.7 and 3.8 in [Roy06] that any f-expectation with generator f satisfying assumptions in Proposition 1.3.2 and being independent of y is both additive and $\mathcal{E}^{C,C_1}$-dominated. Additivity follows from our earlier discussion in Definition 1.3.6. It is easy to see thatf_{C,C_1} is independent of y and $\mathcal{E}^{C,C_1}$-domination follows from the Lipschitz condition imposed on f, since f satisfies assumptions in Proposition 1.3.2. We use the same argument to show a corresponding result in the time-changed situation, see Remark 3.1.3.

With the above properties in place, we can recall the important Inverse Theorem, see Theorem 4.6 in [Roy06], that allows us to represent a large class of non-linear expectations by BSDEs with jumps under rather general conditions.

Theorem 1.3.8 (Inverse Theorem)**.** *Let $\mathcal{E}$ be a filtration-consistent expectation which satisfies both properties of $\mathcal{E}^{C,C_1}$-domination and additivity. Then these exits a function $f : [0, T] \times \Omega \times \mathbb{R}^2 \times L^2_\nu \mapsto \mathbb{R}$ such that $\mathcal{E} = \mathcal{E}_f$.*

Proof. This is a fundamental result that is much more difficult to establish than it appears. In fact, in order to prove this result, it requires the establishment of the Doob-Meyer Decomposition for non-linear expectation martingales, which is very difficult since the classical argument for the decomposition relies heavily on the linearity of conventional expectation. The proof in Theorem 4.6 in [Roy06] starts with establishing the Doob-Meyer Decomposition for f-expectation martingales, before it generalises the result for general non-linear expectation martingales without a given driver f. For details, see Theorem 4.1, 4.5 and 4.6 in [Roy06]. □

So far as we know, Royer in [Roy06] is one of the first to generalise this result to the jump setting. This thesis devotes the whole Chapter 4 to establishing a corresponding result in the time-changed setting, see Theorem 4.0.1.

Dynamic risk measures as f-expectations

With f-expectations defined in Proposition 1.3.2, we can use them to represent dynamic risk measures. The following proposition shows us the natural connection between non-linear expectations and dynamic risk measures.

Proposition 1.3.9. *Let $T' > 0$ be a time horizon and $\mathcal{E}_f$ be an f-expectation generated in Proposition 1.3.2. Suppose $\mathcal{E}_f$ satisfies properties of additivity, filtration-consistence and $\mathcal{E}^{C,C_1}$-domination, all with respect to the natural filtration $\mathbb{F}$ defined in Defition 0.1.1, then we can define a normalised, time-consistent dynamic risk measure in the following way: for each $T \in [0, T']$ and $\xi \in L^2(\mathcal{F}_T)$,*

$$\rho_t^f(\xi, T) := -\mathcal{E}_f[\xi \mid \mathcal{F}_t] = -Y_t(\xi, T), \quad t \in [0, T]. \tag{1.23}$$

Moreover, if the generator f is also concave *with respect to (x, π, l), then the dynamic risk measures thus represented is* convex.

❇ *Proof.* We show that the $\rho_t^f(\xi, T)$ defined in (1.23) satisfies the properties listed in Definition 1.1.12 and Proposition 1.1.19.

First we note that the additivity of the f-expectation is equivalent to the conditional invariance for the dynamic risk measure defined in Definition 1.1.12. Thus the additivity of the f-expectation $\mathcal{E}_f$ implies the conditional invariance for the conditional risk measure ρ_t^f defined in (1.23).

The filtration-consistency property of the f-expectation is equivalent to the normalised conditional time-consistency property (1.2) in Proposition 1.1.19, as we have observed in (1.19). In this way, the filtration consistency of the f-expectation $\mathcal{E}_f$ implies time-consistency for the conditional risk measure ρ_t^f.

Here we note that, since we have the condition that $f(t, y, (0, 0)) = 0$, the associated conditional dynamic risk measure ρ_t^f defined in (1.23) has the normalisation property defined as (iv) in Definition 1.1.12.

When it comes to monotonicity property, Proposition 1.3.2 and Remark 1.3.4 tell us that the conditions on the driver f guarantees monotonicity for the f-expectation $\mathcal{E}_f$, and this implies monotonicity for the conditional risk measure ρ_t^f. Behind the curtain, this results rests on the strict Comparison Theorem, see Theorem 1.2.14.

Finally, the convexity follows by an application of the Comparison Theorem, see Theorem 1.2.13. To be more specifically, consider $\xi_1, \xi_2 \in L^2(\mathcal{F}_T)$, $\eta \in [0, 1]$. We note that, by assumptions, f is additive and is therefore independent of y, then by the concavity of f in terms of (π, l), we have

$$\begin{aligned} f(\eta\ (\pi(\xi_1), l(\xi_1)) + (1-\eta)\ (\pi(\xi_2), l(\xi_2))) \geq \eta\ & f(\pi(\xi_1), l(\xi_1)) \\ & + (1-\eta)\ f(\pi(\xi_2), l(\xi_2)) \end{aligned}$$

Then by the Comparison Theorem, the resulting risk measure is convex, and the result is established. □

Corollary 1.3.10. *In addition to assumptions in Proposition 1.3.9, if the driver f is also positively homogeneous, see (v) in Definition 1.1.12, then the resulting dynamic risk measures defined in (1.23) is* coherent.

❇ *Proof.* We note that by assumptions, the driver f is concave with respect to (π, l), and the resulting risk measure ρ_t is convex. If the driver f is, in addition, positively homogeneous, it follows that the associated risk measure ρ_t is also positively homogeneous, namely for $C \in \mathbb{R}$ and $C \geq 0$, we have $\rho_t(C\xi) = C\rho_t(\xi)$.

For the rest, we need to make use properties of the associated BSDE. Details of this proof can be found in the proof of Corollary 3.1.5, where we show a corresponding result in the time-changed setting, and we skip them here. □

Representation of Convex Dynamic Risk Measures

We now present a representation of normalised convex dynamic risk measures generated by concave BSDEs with jumps. As mentioned in [QS13], this result is established through a set of probability measures that are *absolutely continuous* with respect to P.

Following [QS13], we define a function F of f with respect to $(\pi, \ell\,)$, for each (α^1, α^2) in $\mathbb{R} \times L^2_\nu$ by

$$F(\omega, t, \alpha^1, \alpha^2) := \sup_{(\pi,\ell)\in\mathbb{R}\times L^2_\nu} [f(\omega, t, \pi, \ell) - \alpha^1\pi - \langle \alpha^2, \ell\rangle_\nu]. \tag{1.24}$$

Now we recall Theorem 5.2 in [QS13].

Theorem 1.3.11 (Representation of Normalised Convex Dynamic Risk Measures)**.** *Suppose that the Hilbert space L^2_ν is separable. Let f be a Lipschitz driver with Lipschitz constant C, which does not depend on x. Suppose also that f satisfies assumptions in the Strict Comparison Theorem, see Theorem 1.2.14, and is concave with respect to (π, l).*

Let $T' > 0$, and $T \in [0, T']$. Let $\mathcal{A}_T$ be the set of predictable processes $\alpha = (\alpha^1, \alpha^2)$ such that $F(t, \alpha^1_t, \alpha^2_t(u)) \in \mathcal{H}^2_{\mathcal{F}}(0, T)$, where F is defined by 1.24. For each $\alpha \in \mathcal{A}_T$, let Q^α be the probability absolutely continuous with respect to P which admits Z^α_T as density with respect to P on $\mathcal{F}_T$, where Z^α is the solution of

$$\mathrm{d}Z^\alpha_t = Z^\alpha_{t^-}\left(\alpha^1_t \mathrm{d}W_t + \int_{\mathbb{R}^*} \alpha^2_t(u)\mathrm{d}\widetilde{N}(\mathrm{d}t, \mathrm{d}u)\right); \quad Z^\alpha_0 = 1. \tag{1.25}$$

The convex dynamic risk measure $\rho(\xi, T)$ has the following representation: for each $\xi \in L^2(\mathcal{F}_T)$,

$$\rho_0(\xi, T) = \sup_{\alpha\in\mathcal{A}_T} [\mathrm{E}_{Q^\alpha}[-\xi] - \zeta(\alpha, T)], \tag{1.26}$$

where the function ζ, the penalty term, is defined for each T and $\alpha \in \mathcal{A}_T$ by

$$\zeta(\alpha, T) := \mathrm{E}_{Q^\alpha}\left[\int_0^T F(s, \alpha^1_s, \alpha^2_s)\mathrm{d}s\right].$$

Moreover, for each $\xi \in L^2(\mathcal{F}_T)$, there exits $\bar{\alpha} = (\bar{a}^1, \bar{\alpha}^2) \in \mathcal{A}_T$ such that

$$F(t, \bar{a}^1, \bar{\alpha}^2) = f(t, \pi_t, l_t) - \bar{a}^1\pi_t - \langle \bar{\alpha}^2, l_t\rangle_\nu, \quad t \in [0, T],\ \mathrm{d}P \otimes \mathrm{d}t\text{-a.s.},$$

where (Y, π, l) is the solution to the associated BSDE with driver f, terminal time T and terminal condition ξ. Also the process $\bar{\alpha}$ is optimal for (1.26).

Remark 1.3.12. As pointed out in [QS13], in the case of filtrations generated by only a Brownian motion, the probability measures Q^α are *equivalent* to P.

Under the discontinuous framework, the process α is valued in the Hilbert space $\mathbb{R}\times L^2_\nu$. The separability assumption will be used to fix some measurability problems in the proof.

To prove the previous theorem, we need to have the following lemmata.

Lemma 1.3.13. *For each $(t,\omega), D(\omega)$ is defined as the non-empty set of $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}\times L^2_\nu$ such that $F(\omega,t,\alpha_1,\alpha_2) > -\infty$, where F is defined in (1.24). Then for each $(t,\omega), D(\omega) \subset U$ where U is the closed subset of the Hilbert space $\mathbb{R}\times L^2_\nu$ of the elements $\alpha = (\alpha_1,\alpha_2)$ such that α_1 is bounded by C and $\nu(\mathrm{d}u)$−a.s.,*

$$\alpha_2 \geq -1 \quad \text{and} \quad |\alpha_2| \leq \psi(u)\wedge C,$$

where C is the Lipschitz constant of f.

Proof. Proof of this lemma rests on the fact that f satisfies assumptions in the Strict Comparison Theorem, see Theorem 1.2.14. Then we are able to construct a contradiction with F. For details we refer to Lemma 5.4 in [QS13]. □

We follow the same idea and give a detailed proof of a corresponding lemma in the time-changed setting in Lemma 3.2.2.

Lemma 1.3.14. *These exists a process $\bar{\alpha} = (\bar{\alpha}^1,\bar{\alpha}^2(u)) \in \mathcal{A}_T$ such that*

$$f(t,\pi,\ell_t) = \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_T}\{f^\alpha(t,\pi_t,\ell_t)\} = f^{\bar{\alpha}}(t,\pi_t,\ell_t), \quad t\in[0,T],\ \mathrm{d}P\otimes\mathrm{d}t\text{-}a.s.$$

Proof. We need to recall results from convex analysis to obtain this result. For details we refer to Lemma 5.5 in [QS13]. □

Now we can present a brief proof for the previous Theorem.

Proof of Theorem 1.3.11 . We first need to fix the measurability issue with F. Given that the space $\mathbb{R}\times L^\nu_2$ is separable, there is a dense countable subset I, and since f is concave and continuous with respect to (π,ℓ) by assumption, the superemum in (1.24) coincides with the supremum over I. And thus we can establish the measurability of F. By results from convex analysis, F is also convex.

Since f is concave and continuous, we have the conjugacy relation between f and F:

$$f(\omega,t,\pi,\ell) = \inf_{\alpha\in D_t(\omega)}\{F(\omega,t,\alpha_1,\alpha_2) + \alpha_1\pi + \langle\alpha_2,\ell\rangle_\nu\},$$

where $D_t(\omega)$ is defined in Lemma 1.3.13.

Thus with Lemma 1.3.13 established, for each process $\alpha_t = (\alpha^1_t,\alpha^2_t) \in \mathcal{A}_T$, let f^α be the associated linear driver defined by

$$f^\alpha(\omega,t,\pi,\ell) := F(\omega,t,\alpha^1_t(\omega),\alpha^2_t(\omega)) + \alpha^1_t(\omega)\pi + \langle\alpha^2_t(\omega),\ell\rangle_\nu.$$

Here we can also see that by the infimum, for each $\alpha\in\mathcal{A}_T, f^\alpha \geq f$.

Let $T\in[0,T']$ and $\xi\in L^2(\mathcal{F}_T)$. Let $(Y(\xi,T),\pi(\xi,T),l(\xi,T))$ be the solution in $S^2_{\mathcal{F}}(0,T)\times\mathcal{H}^2_{\mathcal{F}}(0,T)\times\mathcal{H}^p_{\mathcal{F}}(0,T,\nu)$ of the BSDE associated with driver f, terminal time T and terminal condition ξ.

By Lemma 1.3.14 and the Optimisation Principle in Theorem 1.2.16, we can derive that

$$Y_0(\xi, T) = \inf_{\alpha \in \mathcal{A}_T} Y_0^{\alpha}(\xi, T) = Y_0^{\bar{\alpha}}(\xi, T)$$

where for each $\alpha \in \mathcal{A}_T, Y^{\alpha}(\xi, T)$ is the solution of the linear BSDE associated with driver f^{α}, terminal time T and terminal condition ξ.

By Lemma 1.3.13, the process Z^{α} defined by (1.25) belongs to $S^2_{\mathcal{F}}(0, T)$ by Proposition 1.2.6. Thus by the representation of linear BSDEs as conditional expectations in Theorem 1.2.10, we have:

$$Y_0^{\alpha}(\xi, T) = \mathrm{E}\bigg[Z_T^{\alpha}\xi + \int_0^T Z_s^{\alpha} F(s, \alpha_s^1, \alpha_s^2)\mathrm{d}s\bigg].$$

Then by Lemma 1.3.13 where $\alpha_t^2 \geq -1$ $\mathrm{d}t \otimes \mathrm{d}P \otimes \mathrm{d}\nu$-a.s., we can establish the non-negativity of $(Z_t^{\alpha})_{0 \leq t \leq T}$ and it is a martingale. Thus as a density for Q^{α} with respect to P on $\mathcal{F}_T$, Z_T^{α} is well-defined. And the result follows. For other details, we refer to Theorem 5.2 in [QS13]. □

We apply a similar argument and prove a corresponding result in the time-changed setting. For more details, see Theorem 3.2.4.

Corollary 1.3.15. *Given assumptions in Theorem 1.3.11, we note that if the driver f is also positively homogeneous, then $F = 0$.*

❇ *Proof.* If the driver f is, in addition, positively homogeneous, it follows that the associated risk measure is also positively homogeneous, namely for $C \in \mathbb{R}$ and $C \geq 0$, we have $\rho_t(C\xi) = C\rho_t(\xi)$. It implies that the resulting risk measure is coherent by Corollary 1.3.10. For the rest of the proof, we apply the argument used in Corollary 3.2.5, where we show a corresponding result in the time-changed setting. We skip the details here. □

Chapter 2

Change of Time Methods

The Change of Time Methods (CTMs) has been widely studied in the literature of mathematical finance. The central idea of CTMs is to perturb the time line of a "complicated" process with another stochastic process and then obtain a well-known process with comparatively "simpler" structure. The change of time process is usually interpreted as "operational time" or "business time".

Section 2.1 recalls the basic definitions and theory of CTMs. Focus in this section is on the mathematical properties of both the original process, which are usually referred to as *base* processes, and the resulting time-changed processes.

Section 2.2 goes deeper in CTMs by considering Lévy processes in the semi-martingale setting, and there one needs to take into consideration issues related to filtrations.

Section 2.2 also reviews two classes of change of time processes that have been extensively studied in the literature, namely, *subordinators* and *absolutely continuous time changes*. Subordinated Lévy processes have arguably "nicer" properties, since they stay Lévy processes and results in the literature tell us that we can figure out the characteristic triplet for the subordinated Lévy processes based on those of the base process.

On the other hand, absolutely continuous time changes causes more problems, since time-changed Lévy processes in this case may no longer stay Lévy processes. In the meantime, they can be used to construct processes with conditional independent stationary increments, as classical results in the literature show. This section addresses these problems.

In section 2.3, we present a framework for BSDEs with jumps driven by noises associated with absolutely continuous time changes, based on results from [DS14]. This is a novel framework to generalise the discussion of BSDEs with jumps, and it has several "nice" properties, thanks to the absolutely continuous time change. We will give a detailed discussion about this framework in this section.

This chapter tries to approach the CTMs in a more conceptual way, in the sense that it presents more examples to offer intuitions about CTMs instead of detailed proofs. One of the reasons for this is that, CTMs-related results usually involve large amount of computations, something that is highly technical, yet sheds little lights on the main topic of this thesis. After establishing the time-changed framework in Section 2.3, we continue the discussion of time-changed BSDEs in this setting in the next chapter.

2.1 Change of Time Methods

We first recall some basic definitions of CTMs, before presenting some important properties. Results for this section are based on works done in [BS10] and [Swi16].

Basic Definitions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, with filtration $(\mathcal{F}_t)_{t\geq 0}$ being right-continuous. We first define the random change of time.

Definition 2.1.1 (Random Change of Time)**.** A family of random variables $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$ is said to be a *random change of time*, if

(i) $(\widehat{T}(\theta))_{\theta\geq 0}$ is a non-decreasing, right-continuous family of $[0,\infty]$-valued random variables $\widehat{T}(\theta), \theta \geq 0$;

(ii) for all $\theta \geq 0$ the random variables $\widehat{T}(\theta)$ are stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, meaning

$$\{\widehat{T}(\theta) \leq t\} \in \mathcal{F}_t, \quad \theta \geq 0,\ t \geq 0.$$

Definition 2.1.2. The random variable

$$\widehat{\zeta} = \inf\{\theta : \widehat{T}(\theta) = \infty\}$$

is called the *life time* of the process $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$. The change of time is said to be finite if $\widehat{T}(\theta) < \infty$ $P-$a.s., for all $\theta \in [0,\infty)$, or equivalently, $P(\widehat{\zeta} = \infty) = 1$.

Definition 2.1.3 (Subordinator)**.** The change of time $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$ is called a *subordinator*, if this randome process $\widehat{T}$ on the interval $[0,\widehat{\zeta}]$ is a Lévy process. If $P(\widehat{\zeta} = \infty) = 1$, then the change of time $\widehat{T}$ is said to be a *subordinator in the strong sense.*

General Idea and Construction

With the basic definitions in place, we now provide a general construction of change of time processes.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the above-mentioned probability space, and let $A = (A_t)_{t\geq 0}$ be an increasing, right-continuous random process, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, with $A_0 = 0$.

We also assume the stochastic processes we consider in this section to be *progressively measurable*, see Definition A.0.1, with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

We define

$$\widehat{T}(\theta) = \inf\{t : A_t > 0\}, \quad \theta > 0, \tag{2.1}$$

where, following the convention, $\inf \varnothing = \infty$.

We show that such a process constitutes a random change of time.

Lemma 2.1.4. *The family of random variables $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$ as defined in (2.1) constitutes a random change of time.*

Proof. Property (i) in Definition 2.1.1 follows easily from construction in (2.1), given that

$$\{t : A_t > \theta\} = \bigcup_{\varepsilon>0} \{t : A_t > \theta + \varepsilon\},$$

we have $\widehat{T}(\theta) = \lim_{\varepsilon\downarrow 0} \widehat{T}(\theta + \varepsilon)$, namely, the process $\widehat{T}$ is right-continuous for each $\theta \geq 0$.

To verify Property (ii) in Definition 2.1.1, we first note that since the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. Given this assumption, we see $\{\widehat{T}(\theta) \leq t\} \in \mathcal{F}_t$ is equivalent to $\{\widehat{T}(\theta) < t\} \in \mathcal{F}_t$, as we can see if we start with $\{\widehat{T}(\theta) \leq t\} \in \mathcal{F}_t$,

$$\{\widehat{T}(\theta) < t\} = \bigcup_{\varepsilon>0} \{\widehat{T}(\theta) \leq t - \varepsilon\} \in \mathcal{F}_{t-\varepsilon} \subseteq \mathcal{F}_t.$$

Conversely, if we have $\{\widehat{T}(\theta) < t\} \in \mathcal{F}_t$, then by the right-continuity of the filtration,

$$\{\widehat{T}(\theta) \leq t\} \in \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t.$$

Hence the claim for equivalence is established.

Then by the definition of (2.1), we have

$$\{\widehat{T}(\theta) < t\} = \bigcup_{s<t, s\in\mathbb{Q}} \{A_s > \theta\} \in \mathcal{F}_t,$$

where $\mathbb{Q}$ is the set of the rational number on $[0, \infty)$. So the second property also holds. □

With the change of time, we will be able to construct another filtration $\widehat{\mathcal{F}_\theta}$, where the process we used in (2.1) becomes a stopping time for each t.

Lemma 2.1.5. *If $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$ is defined as in (2.1), then*

$$A_t = \inf\{\theta : \widehat{T}(\theta) > t\},$$

and, for all $t \geq 0$, the random variables A_t are $(\widehat{\mathcal{F}_\theta})_{\theta\geq 0}$-stopping times, where

$$\widehat{\mathcal{F}_\theta} = \mathcal{F}_{\widehat{T}(\theta)}.$$

Proof. We prove here the equation for A_t in the lemma, for the rest of the proof, we refer to Lemma 1.2 in [BS10]. We note that for given $\theta \in [0, \infty)$, we have $\widehat{T}(\theta) > t$, then by (2.1), we have $A_t \leq \inf\{\theta : \widehat{T}(\theta) > t\}$.

Conversely, since $\widehat{T}(A_t) \geq t$ for each $t \in [0, \infty)$, so $\widehat{T}(A_{t+\varepsilon} \geq t + \varepsilon > t)$. Therefore, $A_{t+\varepsilon} \geq \inf\{\theta : \widehat{T}(\theta) > t\}$, and given that A_t is right-continuous, we have the reversed inequality. Hence we have proved the equality for A_t. □

The above lemma provides us with a "dual" structure for change of time processes. It is easy to observe that A is the inverse of $\widehat{T}$. To emphasise this duality, we define a process $T := (T(t))_{t\geq 0}$ *coinciding* with $A = (A_t)_{t\geq 0}$ such that $T(t) = A_t$ for $t \geq 0$. And from now on, we use T to emphasise the original process.

This is why T and $\widehat{T}$ are referred to as *mutually inverse* in the literature, as we can see $\widehat{T}$ can be constructed from T by $\widehat{T} = \inf\{t : T(t) > \theta\}$, and T can be retrieved as $T(t) = \inf\{\theta : \widehat{T}(\theta) > t\}$.

Now we take a look at a simple example where the change of time process is deterministic.

Example 2.1.6. Let $\widehat{B} = (\widehat{B}_\theta)_{\theta\geq 0}$ be a Brownian motion and $T = (T(t))_{t\geq 0}$ be a deterministic non-decreasing and right-continuous function with $T(0) = 0$. We consider the process defined as

$$X_t = F(f(t) + g(t)\widehat{B}_T(t)),$$

where $f(t), g(t)$ and $F(x)$ are continuous functions.

To be more specific, let X be a process satisfying the SDE of the Ornstein-Uhlenbeck type:

$$\mathrm{d}X_t = (\alpha(t) - \beta(t)X_t)\mathrm{d}t + \gamma(t)\mathrm{d}W_t,$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion.

Given the conditions that $\int_0^t \left|\frac{\alpha(s)}{g(s)}\right| \mathrm{d}s < \infty$ and $\int_0^t \left|\frac{\gamma(s)}{g(s)}\right|^2 \mathrm{d}s < \infty$ for $t \geq 0$, we can find a unique strong solution for X, namely,

$$X_t = g(t)\Big[X_0 + \int_0^t \frac{\alpha(s)}{g(s)}\mathrm{d}s + \int_0^t \frac{\gamma(s)}{g(s)}\mathrm{d}W_s\Big],$$

where

$$g(t) = \exp\Big\{-\int_0^t \beta(s)\mathrm{d}s\Big\}.$$

Now we put

$$T(t) = \int_0^t \Big(\frac{\gamma(s)}{g(s)}\Big)^2 \mathrm{d}s,$$

and we suppose that $T(t) \uparrow \infty$ as $t \to \infty$. Then we can defined a "new" Brownian motion by

$$\widehat{B}_{T(t)} := \int_0^t \frac{\gamma(s)}{g(s)}\mathrm{d}W_s.$$

In this way, we can represent the process X as

$$X_t = f(t) + g(t)\widehat{B}_{T(t)}$$

where

$$f(t) = g(t)\Big[X_0 + \int_0^t \frac{\alpha(s)}{g(s)}\mathrm{d}s\Big].$$

The way we constructed the Brownian motion $\widehat{B}$ is as follows:

$$\widehat{B}_\theta = \int_0^{\widehat{T}(\theta)} \frac{\gamma(s)}{g(s)}\mathrm{d}W_s,$$

where

$$\widehat{T}(\theta) = \inf\{t : T(t) > \theta\}.$$

■

Since we assume that the processes we consider in this section are progressively measurable, we see that with the change of time $\widehat{T}$, the "compound" process

$$\widehat{X}_\theta = X_{\widehat{T}(\theta)}, \quad \theta \geq 0,$$

is $(\mathcal{F}_\theta)$-adapted, that is to say, $\widehat{X}_\theta$ is $\mathcal{F}_\theta$-measurable for each $\theta \geq 0$. We refer to Section 1.1.6 in [BS10] for a more detailed explanation on this.

The following lemma describes some mathematical properties of the associated time-changed filtration, which in turn depend on the properties of the process $(A_t)_{t\geq 0}$ in (2.1).

Lemma 2.1.7 (Properties of Change of Time)**.** *Given the construction of a change of time process $\widehat{T}$ in (2.1), we have the following properties:*

(i) If the process $(A_t)_{t\geq 0}$ is continuous *and τ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, then we have:*

$$\mathcal{F}_\tau \subseteq \widehat{\mathcal{F}}_{A_\tau} = \widehat{\mathcal{F}}_{T(\tau)} = \mathcal{F}_{\widehat{T}(T(\tau))};$$

(ii) If the process $(A_t)_{t\geq 0}$ is continuous *and* strictly increasing*, then*

$$\begin{aligned} \widehat{T}(T(t)) &= t, \\ T(\widehat{T}(\theta)) &= \theta, \\ \widehat{T}(\theta) &= T^{-1}(\theta), \\ T(t) &= \widehat{T}^{-1}(t), \end{aligned}$$

and if τ is a stopping time, then

$$\mathcal{F}_\tau = \widehat{\mathcal{F}}_{A_\tau} = \widehat{\mathcal{F}}_{T(\tau)} = \mathcal{F}_{\widehat{T}(T(\tau))};$$

(iii) If the process $(A_t)_{t\geq 0}$ is continuous*,* strictly increasing *and $A_\infty = \infty$ P-a.s., then the associated change of time process $\widehat{T}$ is also continuous, and strictly increasing and its life time $\widehat{\zeta} = \infty$ P-a.s.*

Proof. These properties are very easy to check, given the construction and assumptions in (2.1) (especially the right-cotinuity of process A). We refer to Section 1.2.1 in [BS10] for a detailed explanation. □

Representations

As stated before, the central idea of CTMs is to represent a process X with a complicated structure with a comparatively "simpler" one via perturbing its time line with another stochastic process. One natural question is: given process X, how can one construct a simpler process $\widehat{X}$ via an also "simple" time-change process T, such that the representation $X = \widehat{X}_T$ holds, no matter in the strong (indistinguishable), semi-strong (P-a.s.), or the weak (distribution) sense.

We end this section by presenting two important and classical results for change of time representations, namely, for *continuous* local martingales and for local martingales obtained by compensation of counting processes.

Let $M = (M_t)_{t\geq 0}$ be a *continuous* local martingale, with $M_0 = 0$. Denote by $\langle M\rangle$ its quadratic variation as defined in Definition A.0.6 and its existence is guaranteed by Theorem A.0.5 the Doob-Meyer Decomposition for local submartingales. We have the following classical results for the *strong* representation of M as a Brownian motion via a change of time process.

Theorem 2.1.8 (Dambis, Dubins and Schwartz)**.** *Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$. Then there exits a Brownian motion $\widehat{B} = (\widehat{B}_\theta)_{\theta\geq 0}$ such that for the change of time $T(t) = \langle M\rangle_t,\ t \geq 0$, we have the strong representation $M = \widehat{B}_T$.*

Proof. Proof of this theorem is slightly technical, and we will only give a general sketch. In accordance with construction of a change of time as in (2.1), we make $A_t = \langle M\rangle_t,\ t \geq 0$. Given the assumption that $\langle M\rangle_\infty = \infty$, we conclude that $\widehat{T}(\theta)$ is finite for all $\theta \geq 0$. Then we have

$$\widehat{B}_\theta = M_{\widehat{T}(\theta)} \quad \text{and} \quad \widehat{\mathcal{F}}_\theta = \mathcal{F}_{\widehat{T}(\theta)}.$$

For the rest of the proof, we need to verify that $\widehat{B}$ is indeed a Brownian motion. We need to make use of the nice properties of $\langle M\rangle$ and refer to Theorem 1.1 in [BS10] for details of the rest of the proof. □

Remark 2.1.9. In the above theorem, we have made the assumption that $\langle M\rangle_\infty = \infty$, something that guarantees the time $\widehat{T}(\theta)$ is finite for all $\theta \geq 0$. We will need to extend the current probability space and construct a more complicated filtration if we relax this assumption by having $\langle M\rangle_\infty < \infty$. But the theorem remains true. For details of this respect, we refer to Section 1.4 in [BS10].

Remark 2.1.10. For extension of the previous theorem into a multi-dimensional time-changed Brownian motion setting, we refer to Remark 1.4 and Remark 1.5 in [BS10].

A direct consequence of the above theorem is that it can be applied to construct models of stochastic volatility, such as in the following corollary.

Corollary 2.1.11. *Let M_t be a continuous local martingale defined in our probability space $(\Omega, \mathcal{F}, P)$ equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ as*

$$M_t = \int_0^t \sigma_s \mathrm{d}B_s, \quad t \geq 0,$$

where B_t is a Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$ and σ is a positive process with $\int_0^t \sigma_s^2 \mathrm{d}s < \infty$ and $\int_0^\infty \sigma_s^2 \mathrm{d}s = \infty$. If we define

$$\widehat{T}(\theta) = \inf\Big\{t : \int_0^t \sigma_s^2 \mathrm{d}s \geq \theta\Big\},$$

Then the process $\widehat{B}_\theta = M_{\widehat{T}(\theta)}$ is a Brownian motion with respect to the filtration $(\widehat{\mathcal{F}}_\theta)_{\theta\geq 0}$, and thus

$$M_t = \widehat{B}_{T(t)},$$

where

$$T(t) = \int_0^t \sigma_s^2 \mathrm{d}s.$$

In the above theorem, the Brownian motion $\widehat{B}$ clearly played the role of the comparatively "simpler" process in the representation. A natural generalisation of this is to consider processes which are not as "nice" as Brownian motions, processes with discontinuities. Counting processes are a natural candidate, and among them, the Poisson process stands out for its "simplicity".

Theorem 2.1.12. *Let $N = (N_t)_{t\geq 0}$ be a counting process with continuous compensator (as defined in Definition A.0.7) $A = (A_t)_{t\geq 0}, N_0 = 0, A_0 = 0$. Let $M = (M_t)_{t\geq 0}$ where $M_t = N_t - A_t$. If $A_\infty = \infty$, then there exists a standard Poisson process $\widehat{N} = (\widehat{N}_\theta)_{\theta\geq 0}$, with intensity $\lambda = 1$, such that $N = \widehat{N}_T, T = A$, and $M = \widehat{M_T}$, where $\widehat{M_t} = \widehat{N}_t - t$ is a "Poisson martingale".*

Proof. Proof of the above theorem starts with the fact that by the Doob-Meyer Decomposition for counting proccesses, we have $N = A + M$, where A is the compensator process and M is a local martingale.

For the rest of the proof, we refer to Theorem 1.2 in [BS10], where it was shown that $\widehat{N}$ is indeed Poissonian by verifying its characteristic function . □

We end this section with a remark about the representation of the general Lévy process with an *independent* Brownian motion $\widehat{B}$.

Remark 2.1.13. We assume that in our probability space $(\Omega, \mathcal{F}, P)$, there exist two processes: 1) a Lévy process X, and 2) a Brownian motion $\widehat{B}$ that *does not depend* on X.

It turns out that we could construct a non-decreasing change of time process $(T(t))_{t\geq 0}$ with $T(0) = 0$ such that at least for each $t > 0$, we have the equality with probability one

$$X_t = \widehat{B}_{T(t)}.$$

Here it was shown that the process $(T(t))_{t\geq 0}$ is also a Lévy process, and because it is non-decreasing, it is a *subordinator*. For a more detailed explanation of this result, we refer to Section 1.4.6 in [BS10].

2.2 Time-changed Lévy Processes

We have introduced the general idea and construction of random change of time in the previous section, and in this section, we will look at models in a semimartingale setting. Some new aspects, such as filtration, will be also considered here.

As mentioned earlier, two classes of change of time have been widely studied in the literature, namely, subordinators and continuous and differentiable random change of time processes. This section recalls more detailed results for the former, since classical results in the literature showed that a subordinated Lévy process remains a Lévy process. In this way, it has “nicer” properties and we stay in the Lévy process framework. This also means that, if we want to study time-changed BSDEs in the setting of subordinated Lévy processes, with the time-changed noises satisfying sufficient properties (such as martingale property), results from the classical setting carry over with no problem.

Time-changed Lévy processes with continuous and differentiable change of time are more complicated, since under such a time change a Lévy process may not longer be a Lévy process. In this section, we use them to construct the framework of processes with conditional stationary independent increments, guaranteed by classical results in the literature. We review such a framework from [DS14] in the next section to study time-changed BSDEs with jumps.

Most of the results presented in this section are recalled from Chapter 8 in [BS10].

Brownian Motions

As in the previous section, let $(\Omega, \mathcal{F}, P)$ be our filtered probability space and now we have a semimartingale $X = (X_t)_{t\geq 0}$ defined in this space. See Definition A.0.8 for the definition of semimartingales.

In addition to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, we also consider the filtration $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\geq 0}$ where

$$\mathcal{F}^X_t = \bigcap_{\varepsilon>0} \sigma(X_s; s \leq t + \varepsilon),$$

so $\mathbb{F}^X$ is a right-continuous version of the so-called *natural filtration* generated by process X.

In the meantime, we will also consider another filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{F}^X_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t, \quad \text{for } t \geq 0.$$

Given the set-up, we recall the following two propositions from Chapter 8 in [BS10] that ensures the semimartingale will remain a semimartingale with respect to different filtrations.

Proposition 2.2.1. *A semimartingale $X = (X_t)_{t\geq 0}$ considered with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a semimartingale relative both to its natural filtration $\mathbb{F}^X = (\mathcal{F}^X_t)_{t\geq 0}$ and any filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ such that $\mathcal{F}^X_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t,\ t \geq 0$.*

Proposition 2.2.2. *If $X = (X_t)_{t\geq 0}$ is a semimartingale with respect to a filtration $\mathbb{F}$, then the time-changed process $\widehat{X} = (\widehat{X}_\theta)_{\theta\geq 0}$, where $\widehat{X}_\theta = X_{\widehat{T}(\theta)}$, is a semimartingale with repect to the time-changed filtration $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_\theta)_{\theta\geq 0}$.*

Proof. For detailed explanation for the above two theorems, we refer to Theorem 8.1 and Theorem 8.2 in [BS10]. □

Now we recall a classical result named "Monroe theorem", saying that under assumptions, any semimartingale can be represented as a time-changed Brownian motion in distribution. For a detailed explanation, we refer to Section 8.1 in [BS10].

Theorem 2.2.3 (Monroe Theorem). *If $X = (X_t)_{t\geq 0}$ is a semimartingale with respect to the natural filtration $\mathbb{F}^X$, then there exists a filtered probability space with a Brownian motion $\widehat{B} = (\widehat{B}_\theta)_{\theta\geq 0}$ and a change of time $T = (T(t))_{t\geq 0}$ defined on it such that*

$$X = \widehat{B}_T \quad \textit{in distribution.}$$

We note that the representation in the above theorem is generally not unique, and we need to consider the filtered probability space to study the connection between $\widehat{B}$ and T.

Yet by the Dambis-Dubins-Schwartz theorem, see Theorem 2.1.8, we can establish the following result.

Proposition 2.2.4. *A semimartingale X can be represented by $X = \widehat{B}_T$ in distribution, with a continuous process T if the provess X is a* continuous *local martingale.*

If we put in an additional condition that $\widehat{B}$ and T are independent processes, and the process T is continuous, then we have come to the following theorem by D. L. Ocone. We recall the result from Section 8.2 in [BS10].

Theorem 2.2.5. *Given above-mentioned assumptions where we have a semimartingale X, the following two conditions are equivalent:*

(i) $X = \widehat{B}_T$ in distribution;

(ii) X is a continuous local martingale such that

$$\left(\int_0^t H_s \mathrm{d}X_s;\ t \geq 0 \right) = (X_t;\ t \geq 0) \quad \textit{in distribution,}$$

for any $\mathbb{F}^X$-predictable process (Definition A.0.12) $H = (H_t)_{t\geq 0}$ such that $|H| = 1$.

Here $\widehat{B}$ and T are independent, and T is continuous.

We look at an example of subordinated Brownian motion.

Example 2.2.6 (Cauchy Process). Let $T = \inf\{\theta \geq 0 : W_\theta > t\},\ t \geq 0$ where W is a standard Brownian motion independent of $\widehat{B}$.

For any $\lambda \geq 0$ and $t \geq 0$ the process

$$\left(\exp\left\{ \lambda W_{T(t)\wedge\theta} - \frac{\lambda^2}{2}(T(t) \wedge \theta) \right\} \right)_{\theta\geq 0}$$

is a bounded martingale. Since $T(t)$ is finite and by Doob Optional Sampling Theorem, see Theorem A.0.16, we have for $\lambda \geq 0$

$$\mathrm{E}\Big[\lambda W_{T(t)} - \frac{\lambda^2}{2} T(t)\Big] = 1.$$

We can thus obtain

$$\mathrm{E}\Big[e^{-\frac{\lambda^2}{2}T(t)}\Big] = e^{-\lambda t},$$

and then for any $u \geq 0$

$$\mathrm{E}\Big[e^{-uT(t)}\Big] = e^{-\sqrt{2u}t}.$$

By Lévy-Khinchin formula for stable processes, see A.0.15, we can conclude that T is a $\frac{1}{2}$-stable process with triplet $(0, 0, \rho)$ and the Lévy measure

$$\rho(\mathrm{d}x) = (2\pi)^{-\frac{1}{2}} x^{-\frac{3}{2}} \mathrm{d}x.$$

Given that $\widehat{B}$ and T are independent, we have for any $\lambda \in \mathbb{R}$

$$\mathrm{E}\Big[e^{i\lambda \widehat{B}_{T(t)}}\Big] = \mathrm{E}\Big[e^{-\lambda^2 \frac{T(t)}{2}}\Big] = e^{-|\lambda| t}$$

and this means $X := \widehat{B}_T$ is a 1-stable symmeric process, which is called the *standard Cauchy process.* ■

Another question that arises from this set-up is: if we start with a Brownian motion $\widehat{B}$ and a subordinator T, what is the resulting process $X := \widehat{B}_T$? Theorem 2.2.12 in the following subsection answers this question, as the Brownian motion is a special case of Lévy processes.

On the other hand, we can also ask the inverse question, namely, for a Lévy process X to be represented in distribution as $X := \widehat{B}_T$, what will be the conditions on X itself as well as on the change of time process T? Theorem 2.2.11 gives an answer to this question for subordinated Lévy processes. We refer to Theorem 8.6 in [BS10] for a more general discussion on this question.

Subordinated Lévy Processes

Before we start discussing about processes being represented by subordinated Brownian motions, we recall some important facts about Lévy processes.

By the famous Lévy-Khinchine formula, see Theorem A.0.15, we recall Corollary 2.4.20 from [App09].

Theorem 2.2.7. *If X is a Lévy process for each $u \in \mathbb{R}^d, t \geq 0$, then*

$$\mathrm{E}[e^{i(u,X(t))}] = \exp(t\eta(u))$$
$$\eta(u) = \Big\{ i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^*} [e^{i(u,y)} - 1 - i(u,y)\mathbb{1}_{\hat{B}(y)}]\mu(\mathrm{d}y) \Big\} \tag{2.2}$$

where $\hat{B}$ denotes the ball of radius 1 around 0, $b \in \mathbb{R}^d$, A is a positive definite symmetric $d \times d$ matrix and μ a Lévy measure on $\mathbb{R}^*$.

Remark 2.2.8. In this way, every Lévy process can be represented with a triplet of characteristics (b, A, μ). Yet since a subordinator is a Lévy process with non-decreasing trajectories, its triplet must be of the form $(b, 0, \mu)$, with $b \geq 0$ and μ satisfying the additional requirements:

$$\mu(-\infty, 0) = 0 \quad \text{and} \quad \int_0^\infty (y \wedge 1)\mu(\mathrm{d}y) < \infty.$$

And we usually represent a subordinator T in the following way:

$$\mathrm{E}[e^{-uT(t)}] = e^{-t\mathcal{L}(u)},$$

where $t \geq 0$ and $u \geq 0$, where

$$\mathcal{L}(u) = ub + \int_0^\infty (1 - e^{-ux})\mu(\mathrm{d}x). \tag{2.3}$$

In literature, $\mathcal{L}(u)$ in (2.3) is often referred to as the *"Laplace transform"* of the subordinator. For details of this result, we refer to Theorem 1.3.15 in [App09].

Example 2.2.9 (Poisson Subordinators)**.** One of the simplest subordinators is Poisson processes. More generally, a compound Poisson process is a subordinator if and only if all its jump sizes are non-negative. ■

Example 2.2.10 (Gamma Subordinators)**.** Let $(T(t))_{t\geq 0}$ be a gamma process with parametres $a, b > 0$, and the density for $x \geq 0$ will be

$$f_{T(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}.$$

We see that for each $u > 0$

$$\int_0^\infty e^{-ux} f_{T(t)}(x)\mathrm{d}x = \Big(1 + \frac{u}{b}\Big)^{-at} = \exp\Big[- ta\log\Big(1 + \frac{u}{b}\Big)\Big].$$

With some manipulation, we can get

$$\int_0^\infty e^{-ux} f_{T(t)}(x)\mathrm{d}x = \exp\Big[- t\int_0^\infty (1 - e^{-ux}) ax^{-1} e^{-bx}\mathrm{d}x\Big].$$

By this we see that for the subordinator triplet, $b = 0$ and $\mu(\mathrm{d}x) = ax^{-1}e^{-bx}\mathrm{d}x$. ■

Now we can answer the question about what kind of Lévy processes can be represented by a subordinated Brownian motion. We focus on one-dimension situation in our discussion.

Theorem 2.2.11. *If a Lévy process X with the characteristic triplet $(0, a, \mu)$ admits the representation $X = \widehat{B}_T$ in distribution, then*

(i) the measure μ is symmetric, absolutely continuous relative to the Lebesgue measure with density $q(z) = \frac{\mu(\mathrm{d}z)}{\mathrm{d}z}$, having the property that the function $q(\sqrt{z}), z > 0$ is completely monotonous;

(ii) there exits a unique positive measure ν on $(0, \infty)$ such that

$$q(\sqrt{z}) = \int_0^\infty e^{-zy}\nu(\mathrm{d}y), \quad z > 0;$$

(iii) the triplet $(\beta, 0, \rho)$ of the subordinator T in the representation is such that

$$\beta = a$$

and

$$\rho(\mathrm{d}x) = \sqrt{2\pi x}\Big(\nu\Big(\frac{1}{2}x\Big)^{-1}\Big)(\mathrm{d}x).$$

Proof. We refer to Theorem 8.6 in [BS10] for details of the proof. □

In the above discussion, we used Brownian motion as the building block for the time-changed process. In fact, if we let general Lévy processes play the similar role, with the change of time process being a subordinator, and the resulting process will still be a Lévy processes. This is a "classical" result proved by Sato in [Sat99].

Theorem 2.2.12. *Let $L = (L_t)_{t \geq 0}$ be a Lévy processes, and T a subordinator independent of L, then the subordinated process $X = L_T$ remains a Lévy process under P with respect to the filtration $\mathbb{F}$.*

Proof. We refer to pages 197-198 in [Sat99] for details of the proof. Note that Brownian motion is just a special case of Lévy processes and is therefore included this theorem. □

Given a suordinated Lévy process $X = L_T$, we would want to figure out its triplet from those of the original Lévy process L and the subordinator T. The following proposition enables us to do that.

Proposition 2.2.13 (Triplets for Subordinated Lévy Processes)**.** *Given a Lévy Process $(L_t)_{t \geq 0}$ with triplets (b, A, μ) and its characteristic exponent $\eta(u)$ as defined in (2.2), and a subordinator $(T(t))_{t \geq 0}$, with its triplet $(\beta, 0, \nu)$ and its Laplace transform defined in (2.3). We assume L and T are independent of each other. The triplet (b_x, A_x, μ_x) for the subordinated Lévy process $X = L_T$ can then be expressed in the following way:*

$$\begin{aligned} b_x &= \beta b + \int_0^\infty \Big(\int_{|L|<1} l f_{L_s}(\mathrm{d}l)\Big)\nu(\mathrm{d}s) \\ A_x &= \beta A \\ \mu_x &= \beta\mu(\mathrm{d}x) + \int_0^\infty f_{L_s}(\mathrm{d}l)\nu(\mathrm{d}s) \end{aligned}$$

where $f_{L_s}(\mathrm{d}l)$ is the probability distribution of the Lévy Process $(L_t)_{t \geq 0}$.

Proof. To see the result, we need to compute the characteristic function for the subordinated process X, given the assumption that L and T are independent:

$$\mathrm{E}[e^{iuX}] = \mathrm{E}[e^{iuL_{T(t)}}] = \mathrm{E}[\mathrm{E}[e^{iuL_\xi} \mid T(t) = \xi]] = \mathrm{E}[e^{-T(t)\eta(u)}] = \mathcal{L}(\eta(u))$$

The result follows by expanding the equation and direct computations. □

Remark 2.2.14. We note that the time-changed process X has two sources of randomness: the original Lévy process L and the change of time process T. If we have independence between L and T, then by the proof of the previous proposition, we see that the characteristic function of the time-changed process X can be represented by the Laplace transform of the change of time process, provided that indeed such a closed form exists. We will give a more general discussion in the following where we study absolutely continuous change of time processes.

Lévy Processes with Absolutely Continuous Change of Time

We will end this section with a discussion of time-changed Lévy process with a continuous and differentiable change of time process T. Unliked subordinated Lévy processes, time-changed Lévy process with a continuous change of time may no longer remain a Lévy process and we will need a more complicated framework for our study.

The following is yet another classical result in the literature that studies processes with conditional independent increments. And this is the general framework we will use to study BSDEs driven with time-changed Lévy noises.

We first need to have a definition of processes with conditional stationary increments.

Definition 2.2.15 (Process with Conditional Stationary Independent Increments)**.** Let $(T_t)_{t\geq 0}$ be a non-negative, real-valued stochastic process with sample paths that are non-decreasing, right-continuous, and $T_0 = 0$ a.s. Let $\mathcal{A} = \sigma(T_t, t \geq 0)$, which is the σ-algebra generated by $(T_t)_{t\geq 0}$. Let X be a measurable, real-valued process that satisfies the following conditions:

(i) For any $s_1 < t_1 < \cdots < s_n < t_n \in \mathbb{R}$ and $x_1 \cdots , x_n \in \mathbb{R}$, we have

$$P[X_{t_1} - X_{s_1} \leq x_1, \cdots , X_{t_n} - X_{s_n} \leq x_n \mid \mathcal{A}] = \prod_{k=1}^{n} P[X_{t_i} - X_{s_i} \leq x_i \mid \mathcal{A}] \text{ a.s.;}$$

(ii) for any $0 \leq s \leq t$ and $\zeta \in \mathbb{R}$, we have

$$\mathrm{E}[\exp(i\zeta(X_t - X_s)) \mid \mathcal{A}] = \phi(\zeta)^{T_t - T_s} \quad \text{a.s.,}$$

where ϕ is an infinitely divisible characteristic function (Definition A.0.14).

Then we call X a (continuous parameter) *process with conditional stationary independent increments* with respect to T.

In the above definition, we can see that the first condition is about conditional independent increments, and the second condition means that the distribution of $T_t - T_s$ determines the way the distribution of $X_t - X_s$ depends on time. In this way, the behaviour of X was completely determined by the process T and the characteristic function ϕ. We can thus present two examples of processes with conditional stationary independent increments by changing the characteristic function ϕ.

Example 2.2.16 (Conditional Gaussian Processes)**.** We call the process $(X_t)_{t\geq 0}$ a *conditional Gaussian Process with variance process* $(T_t)_{t\geq 0}$ if $\phi(\zeta) = e^{-\frac{\zeta^2}{2}}$. ■

Example 2.2.17 (Conditional Poisson Processes)**.** We call a non-negative, integer-valued process $(X_t)_{t\geq 0}$ defined on the same probability space a *conditional Poisson Process with mean value process* $(T_t)_{t\geq 0}$ if $\phi(\zeta) = \exp(e^{i\zeta} - 1)$.

This is to say $(X_t)_{t\geq 0}$ is conditionally a non-homogeneous Poisson process with mean value function $(T_t)_{t\geq 0}$ given the σ-algebra $\mathcal{A}$. We can also express the conditional probability distribution for $X_t - X_s$ for any $0 \leq s \leq t$ and $x \in \mathbb{N} \cup \{0\}$ in the following way:

$$P\Big(X_t - X_s = x \mid \mathcal{A}\Big) = \frac{(T_t - T_s)^x \exp(-(T_t - T_s))}{x!} \quad \text{a.s.}$$

In the literature, there are sometimes referred to as *doubly stochastic Poisson processes (or Cox processes).* ■

Proposition 2.2.18. *Let X and Y be two random variables that are conditionally independent with respect to a σ-algebra $\mathcal{A}$, namely, for $x, y \in \mathbb{R}$,*

$$P(X \leq x, Y \leq y \mid \mathcal{A}) = P(X \leq x \mid \mathcal{A})\ P(Y \leq y \mid \mathcal{A})$$

Then we have

$$\mathrm{E}[f(X) \mid \mathcal{A} \vee \mathcal{B}] = \mathrm{E}[f(X) \mid \mathcal{A}].$$

Here $\mathcal{B} := \sigma(Y)$, namely, the σ-algebra generated by Y, and f is a measurable function such that $f(X) \in L^1(\Omega, \mathcal{F}, P)$.

❇ *Proof.* We denote by $\mathcal{G} := \sigma(X)$, the σ-algebra generated by X. We denote by $G \in \mathcal{G}$ an event G in the the σ-algebra $\mathcal{G}$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, events in σ-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\mathbb{1}_A$ and $\mathbb{1}_B$ be characteristic functions for any events $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

In the following, we show that the equality holds for the indicator function $\mathbb{1}_G$ for all $G \in \mathcal{G}$. Since f is measurable and bounded, the result follows by approximation through monotone convergence.

For events $G := \{\omega : X(\omega) \leq x\} \in \mathcal{G}$ and $B := \{\omega : Y(\omega) \leq y\} \in \mathcal{B}$ for $x, y \in \mathbb{R}$, we can rewrite the assumption of conditional independence in terms of conditional expectations with the help of characteristic functions:

$$\mathrm{E}[\mathbb{1}_{G\cap B} \mid \mathcal{A}] = P(G \cap B \mid \mathcal{A}) = P(G \mid \mathcal{A})P(B \mid \mathcal{A}) = \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathrm{E}[\mathbb{1}_B \mid \mathcal{A}]$$

From the above result, we can show that events G and $A \cup B$ for $A \in \mathcal{A}$ are also conditionally independent by using the indicator function $\mathbb{1}_{A\cup B}$:

$$\begin{aligned}
\mathrm{E}[\mathbb{1}_{G\cap(A\cup B)} \mid \mathcal{A}] &= \mathrm{E}[\mathbb{1}_G \mathbb{1}_{A\cup B} \mid \mathcal{A}] \\
&= \mathrm{E}[\mathbb{1}_G(\mathbb{1}_A + \mathbb{1}_B) \mid \mathcal{A}] \\
&= \mathrm{E}[\mathbb{1}_G \mathbb{1}_A \mid \mathcal{A}] + \mathrm{E}[\mathbb{1}_G \mathbb{1}_B \mid \mathcal{A}] \\
&= \mathbb{1}_A \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}] + \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathrm{E}[\mathbb{1}_B \mid \mathcal{A}] \qquad (2.4) \\
&= \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}](\mathbb{1}_A + \mathrm{E}[\mathbb{1}_B \mid \mathcal{A}]) \\
&= \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathrm{E}[\mathbb{1}_A + \mathbb{1}_B \mid \mathcal{A}] \\
&= \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathrm{E}[\mathbb{1}_{A\cup B} \mid \mathcal{A}]
\end{aligned}$$

Now we are ready to show the main results.

We want to show that

$$\mathrm{E}[\mathbb{1}_G \mid \mathcal{A} \vee \mathcal{B}] = \mathrm{E}[\mathbb{1}_G \mid \mathcal{A}],$$

which by the definition of conditional expectation means

$$\mathrm{E}[\mathbb{1}_G \mathbb{1}_{A\cup B}] = \mathrm{E}[\mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathbb{1}_{A\cup B}]$$

for all $G \in \mathcal{G}, A \in \mathcal{A}$ and $B \in \mathcal{B}$. We start from the right hand side:

$$\begin{aligned}
\mathrm{E}[\mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\mathbb{1}_{A\cup B}] &= \mathrm{E}[\mathrm{E}[\underbrace{\mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]}_{\mathcal{A}\text{-measurable}} \mathbb{1}_{A\cup B} \mid \mathcal{A}]] \\
&= \mathrm{E}[\mathrm{E}[\mathbb{1}_G \mid \mathcal{A}]\ \mathrm{E}[\mathbb{1}_{A\cup B} \mid \mathcal{A}]] \\
&= \mathrm{E}[\underbrace{\mathrm{E}[[\mathbb{1}_{G\cap(A\cup B)} \mid \mathcal{A}]}_{\text{Conditional independence by (2.4)}}] \\
&= \mathrm{E}[\mathbb{1}_{G\cap(A\cup B)}] \\
&= \mathrm{E}[\mathbb{1}_G \mathbb{1}_{A\cup B}].
\end{aligned}$$

And we have thus arrived at the left-hand side.

Since the result holds for the indicator function $\mathbb{1}_G$, by the linearity of conditional expectations it also holds for simple functions for $1 \leq k \leq n,\ s(X) := \sum_{k=1}^{n} a_k \mathbb{1}_{G_k}$, where $G_k := \{\omega : X(\omega) = a_k\} \in \mathcal{G}$.

If $f(X) \geq 0$, we can find a non-decreasing sequence of such simple functions $(s_m(X))_{m=1}^{\infty}$ that converges to point-wise to $f(X)$. The result follows by applying the Monotone Convergence Theorem.

For a general measurable function function $f(X)$, since it is integrable we can decompose it into the positive and negative parts, and the result follows in the same way. □

As the following theorem shows, if we start with processes with stationary independent increments, make a time change with continuous change of time process, then the resulting process will be one with *conditional stationary independent increments*. This allows us to conduct our investigation of Lévy processes with absolutely continuous change of time under the framework of processes with conditional stationary independent increments.

Theorem 2.2.19. *Let T be defined as in Definition 2.2.15. Let L be a measurable, real-valued process with stationary increments, independent of T, such that L has an infinitely divisible characteristic function for each t. Then the resulting process*

$$X_t = L_{T_t}, \quad \text{for } t \geq 0,$$

is a process with conditional stationary independent increments with respect to T_t. Conversely, every process with conditional stationary independent increments is equal in distribution to a process of the above form.

Proof. The proof can be carried out through direct probability calculations. We refer to Theorem 3.1 in [Ser72] for details of the proof. □

We will make use of this result when we discuss BSDEs driven by time-changed Lévy noises in the next section.

A widely used class of absolutely continuous change of time is locally deterministic ones represented in the following form:

$$T(t) = \int_0^t v(s)\mathrm{d}s$$

Here we characterise the random change of time process T by the *instantaneous activity rate* $v(t)$ which is a predictable càdlàg process. We note that $v(t)$ needs to be non-negative to ensure that T does not decrease. We can intuitively regard t as the calender time whereas T is the business time at calender time t.

Given the activity rate $v(t)$, we can obtain its Laplace transform in closed form,

$$\mathcal{L}_{T(t)}(u) = \mathrm{E}\Big[\exp\Big(-u\int_0^t v(s)\mathrm{d}s\Big)\Big].$$

As was shown in the proof of Proposition 2.2.13 and pointed out in Remark 2.2.14, if we have independence between L and T, with T being a subordinator, finding the characteristic function of the time-changed process $X = L_T$ can be reduced to a problem of finding the closed form of Laplace transform of the change of time process, provided that it exists.

The following theorem from [CW02] showed that, generally, we could indeed reduce the problem of characterising the time-changed process $X = L_T$ to finding the Laplace transform of T. But in case L and T do not satisfy the assumption of independence, we need to make a change of measure by a class of complex-valued measures.

There is clear intuition behind the change of measure, indeed, if there exists "leverage effect", as was described by [CW02], between the original Lévy process L and the random change of time T, we need to use the measure change to absorb it so that we can represent the time-changed process X in a leverage-neutral world. But if there exists no "leverage effect" between the original Lévy process L and the random change of time T, in other words, they are independent, then the real world is also leverage-neutral.

This thesis does not intend to go into too much details about CTMs, so we will try to substantiate the insight from [CW02] with some examples. For detailed proof of the following theorem, we refer to Theorem 1 in [CW02].

Theorem 2.2.20. *The problem of finding the characteristic function of the time-changed Lévy process $X = L_T$ under measure P reduces to the problem of finding the Laplace transform of the change of time process T under the complex-valued measure $Q(u)$, evaluated at the characteristic exponenet $\eta(u)$ of L,*

$$\mathrm{E}[e^{iuX_t}] = \mathrm{E}^u[e^{-T(t)\eta(u)}] = \mathcal{L}^u_{T(t)}(\eta(u)),$$

where $\mathrm{E}[\cdot]$ and $\mathrm{E}^u[\cdot]$ denote expectations under measures P and $Q(u)$ respectively. The new class of complex-valued measure $Q(u)$ is absolutely continuous with respect to P and is defnied by

$$\frac{\mathrm{d}Q(u)}{\mathrm{d}P} = M_t(u),$$

where

$$M_t(u) = \exp\{iuX_t + T(t)\eta(u)\}, \quad u \in \mathcal{D}$$

with $\mathcal{D} \subseteq \mathbb{C}^d$, the set of complex values for which the expectation $\mathrm{E}[e^{iuX_t}]$ is well-defined.

By the above theorem, we have somewhat simplified the question of representing the time-changed Lévy process, and thus we can focus on deriving the Laplace transform of the change of time process.

Given our current framework of instantaneous activity rate $v(t)$, we can adopt existing models for $v(t)$. In the following we present the class of affine activity rate models as an example. For a more detailed discussion of this class of models, we refer to Section 4.2.1 in [CW02].

Example 2.2.21 (Affine Activity Rate Models)**.** Let Z be a k-dimensional Markov process that starts at z_0 and satisfies the following SDE:

$$\mathrm{d}Z = \mu(Z_t)\mathrm{d}t + \sigma(Z_t)\mathrm{d}W_t + q\mathrm{d}J(\gamma(Z_t)).$$

Here we denote by W a k-dimensional Brownian motion, and J a Poisson jump component with intensity $\gamma(Z_t)$ and random jump size q, characterised by its two-sided Laplace transform $\mathcal{L}_q(\cdot)$. We also require $\mu(Z_t)$ and $\sigma(Z_t)$ to satisfy the technical conditions such that the SDE has a strong solution. Then the instantaneous rate of activity $v(t)$ is assumed to be a function of the Markov process Z_t.

Proposition 1 in [CW02] states that if the instantaneous rate of activity $v(t)$, the drift vector $\mu(Z_t)$ the diffusion covariance matrix $\sigma(Z)\sigma(Z)^\top$, and the arrival rate $\gamma(Z)$ of the Markov process are all affine in Z, then the Laplace transform $\mathcal{L}_{T(t)}(u)$ is exponential-affine in z_0.

This means that with the affine activity models, given that all the conditions satisfied as above, we have

$$\mathcal{L}_{T(t)}(u) = \mathrm{E}[e^{-uT(t)}] = \exp(-\mathbf{b}(t)^\top z_0 - c(t)), \tag{2.5}$$

where $\mathbf{b}(t) \in \mathbb{R}^k$ and $c(t)$ is a scalar.

In particular, we can let

$$\begin{aligned}
v(t) &= \mathbf{b}_v^\top Z_t + c_v, \quad \mathbf{b}_v \in \mathbb{R}^k,\ c_v \in \mathbb{R},\\
\mu(Z_t) &= a - \kappa Z_t, \quad \kappa \in \mathbb{R}^{k\times k},\ a \in \mathbb{R}^k,\\
[\sigma(Z_t)\sigma(Z_t)^\top]_{ii} &= \alpha_i + \beta_i^\top Z_t, \quad \alpha_i \in \mathbb{R},\ \beta_i \in \mathbb{R}^k,\\
[\sigma(Z_t)\sigma(Z_t)^\top]_{ij} &= 0, \quad i \neq j,\\
\gamma(Z_t) &= \alpha_\gamma + \mathbf{b}_\gamma^\top Z_t, \quad \alpha_\gamma \in \mathbb{R},\ \mathbf{b}_\gamma \in \mathbb{R}^k.
\end{aligned}$$

Then the coefficients $(\mathbf{b}(t), c(t))$ in (2.5) can be determined by the following ordinary differential equations:

$$\begin{aligned}
\mathbf{b}'(t) &= u\mathbf{b}_v - \kappa^\top \mathbf{b}(t) - \beta\mathbf{b}(t)^2/2 - \mathbf{b}_\gamma(\mathcal{L}_q(\mathbf{b}(t)) - 1),\\
c'(t) &= uc_v + \mathbf{b}(t)^\top a - \mathbf{b}(t)^\top \alpha\mathbf{b}(t)/2 - \alpha_\gamma(\mathcal{L}_q(\mathbf{b}(t)) - 1),
\end{aligned}$$

with boundary conditions $\mathbf{b}(0) = 0$ and $c(0) = 0$. Here α denotes a diagonal matrix with the i-th diagonal element given by α_i, β denotes a $k \times k$ matrix with the i-th column given by β_i, and $\mathbf{b}(t)^2$ denotes a $k \times 1$ vector with the i-th element given by $b(t)_i^2$.

As pointed by authors in [CW02], under more specific conditions, we would be able to find closed form solutions for the coefficients, such as the square-root model of Cox–Ingersoll–Ross interest model and Heston model for stochastic volatility. ■

2.3 BSDEs Driven by Time-changed Lévy Noises

In this section we study the BSDEs driven with time-changed Lévy noises. As mentioned earlier, this section focuses on Lévy noises with a continuous and differentiable change of time process.

Unlike subordinated Lévy processes, which stay Lévy processes after the time change, time-changed Lévy processes with a continuous change of time may no longer stay within the Lévy process framework. This is why we need to conduct our study under a more general framework, namely, that of processes with conditional stationary independent increments.

In doing so, we would like to point out in the very beginning the problem of dealing with two filtrations in our probability space. The bigger filtration $\mathbb{G}$ is generated by a mixture of a conditional Brownian measure and a centred doubly stochastic Poisson measure, which we denote by μ, together with the entire history of the change of time processes for the conditional Brownian measure and stochastic Poisson measure. A smaller filtration $\widetilde{\mathbb{F}}$, the smallest right-continuous filtration to which μ is adapted.

This is a novel framework proposed in [DS14] for problems related to time-changed processes and is the first to study BSDEs driven by time-changed Lévy processes in the general form. This section is based on the results from [DS14] in proving the existence and uniqueness of solution for BSDEs with jumps under this framework, and establishing a comparison theorem.

More General Framework

We first recall the framework in Section 0.2, proposed in [DS14].

The random measures B and H defined in Definition 0.2.1 are related to time-change Brownian motion and pure jump Lévy process in a specific way. We can thus define, for $t \in [0, T]$:

$$\begin{aligned} B_t &= B([0,t] \times \{0\}); \\ \Lambda_t^B &= \int_0^t \lambda_s^B \mathrm{d}s; \\ \eta_t &= \int_0^t \int_{\mathbb{R}^*} z\widetilde{H}(\mathrm{d}s, \mathrm{d}z); \\ \widehat{\Lambda}_t^H &= \int_0^t \lambda_s^H \mathrm{d}s. \end{aligned}$$

We can formulate the explicit connection by applying Theorem 2.2.19 in the previous section. We recall Theorem 2.3 from [DS14], due to Theorem 3.1 by Serfozo in [Ser72], which gives a more suitable expression in our situation.

Theorem 2.3.1. *Let $(W_t)_{t\in[0,T]}$ be a Brownian motion and $(N_t)_{t\in[0,T]}$ be a centred pure jump Lévy process with Lévy measure ν. Assume that both W and N are independent of Λ. Then B satisfies conditions* (B1) *and* (B2) *if and only if, for any $t \geq 0$,*

$$B_t = W_{\Lambda_t^B} \quad \textit{in distribution},$$

and η satisfies (H1) *and* (H2) *if and only if, for any $t \geq 0$,*

$$\eta_t = N_{\widehat{\Lambda}_t^H} \quad \textit{in distribution}.$$

Now we can construct the two filtrations mentioned earlier in Section 0.2.

We define $\mathbb{F}^\mu = (\mathcal{F}_t^\mu)_{t\in[0,T]}$ as the filtration generated by $\mu(\Delta),\ \Delta \subset [0,t]\times\mathbb{R}$. By conditions (B1) and (H1) in Definition 0.2.1, we have for any $t \in [0,T]$:

$$\mathcal{F}_t^\mu = \mathcal{F}_t^B \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda,$$

where $\mathcal{F}_t^B$ is generated by $B(\Delta \cap [0,T] \times \{0\})$, $\mathcal{F}_t^H$ by $B(\Delta \cap [0,T] \times \mathbb{R}^*$ and $\mathcal{F}_t^\Lambda$ by $\Lambda(\Delta),\ \Delta \in [0,t] \times \mathbb{R}$.

We set $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, where

$$\widetilde{\mathcal{F}}_t = \bigcap_{r>t} \mathcal{F}_r^\mu.$$

Finally, we set $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ where $\mathcal{G}_t = \mathcal{F}_t^\mu \vee \mathcal{F}^\Lambda$. This implies that $\mathcal{G}_T = \widetilde{\mathcal{F}}_T$ and $\mathcal{E}_0 = \mathcal{F}^\Lambda$, whereas $\mathcal{F}_0^\mu$ is trivial. We denote $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_T$.

Lemma 2.3.2. *The filtration $\mathbb{G}$ is right-continuous.*

Proof. We can prove this lemma by exploiting the structure of filtration $\mathbb{G}$ and apply some classical argument, given conditional independence, such as Theorem 2.1.10 in [App09]. We refer to Lemma 2.4 in [DS14]. □

By direct calculation and applying conditions (B2) and (H2) defined in Definition 0.2.1, we see that μ defined in (0.2) has the martingale property with respect to the filtration $\mathbb{G}$, as stated in the following proposition.

Proposition 2.3.3. *The signed measure μ defined in (0.2) has the following properties with respect to $\mathbb{G}$:*

(i) μ has σ-finite variance measure,

$$m(\Delta) := \mathrm{E}[\mu(\Delta)^2] = \mathrm{E}[\Lambda(\Delta)]; \tag{2.6}$$

(ii) μ is $\mathbb{G}$-adapted;

(iii) μ has conditionally orthogonal values, for $\Delta_1, \Delta_2 \subset (t,T] \times \mathbb{R}$ such that $\Delta_1 \cap \Delta_2 = \varnothing$. To be more precise, we have:

$$\mathrm{E}[\mu(\Delta_1)\mu(\Delta_2) \mid \mathcal{G}_t] = \mathrm{E}[\mu(\Delta_1) \mid \mathcal{F}^\Lambda]\ \mathrm{E}[\mu(\Delta_2) \mid \mathcal{F}^\Lambda] = 0. \tag{2.7}$$

This implies that it has the martingale property with respect to the filtration $\mathbb{G}$.

Proof. We note that adaptedness is obvious and the rest of the proof can be carried out by direct computation and applying properties of B and H defined in Definition 0.2.1.

To see that μ defined in (0.2) has the martingale property with respect to $\mathbb{G}$, we recall that Proposition 2.2.18 guarantees the conditional independence between a random variable and a σ-algebra. Then for $\Delta \subset (t,T] \times \mathbb{R}$ it follows from Proposition 2.2.18:

$$\mathrm{E}[\mu(\Delta) \mid \mathcal{G}_t] = \mathrm{E}[\mu(\Delta) \mid \mathcal{F}_t^\mu \vee \mathcal{F}^\Lambda] = \mathrm{E}[\mu(\Delta) \mid \mathcal{F}^\Lambda] = 0. \tag{2.8}$$

Here we obtain the result in the final step of (2.8), because we have used the corresponding properties of B and H in the following way:

$$\begin{aligned} &\mathrm{E}[B(\Delta) \mid \mathcal{F}^\Lambda] = 0 && \text{by (B1)} \\ &\mathrm{E}[H(\Delta) \mid \mathcal{F}^\Lambda] = \Lambda^H(\Delta) && \text{by (H1)} \\ &\mathrm{E}[\widetilde{H}(\Delta) \mid \mathcal{F}^\Lambda] = 0. \end{aligned}$$

We can also calculate the second moment in the same way:

$$\begin{aligned} \mathrm{E}[B(\Delta)^2 \mid \mathcal{F}^\Lambda] &= \Lambda^B(\Delta) \\ \mathrm{E}[\widetilde{H}(\Delta)^2 \mid \mathcal{F}^\Lambda] &= \Lambda^H(\Delta). \end{aligned}$$

By properties (B2) and (H2), it follows:

$$\mathrm{E}[\mu(\Delta)^2 \mid \mathcal{F}^\Lambda] = \Lambda(\Delta).$$

Take expectation on both sides, and then we obtain (2.6).

In the mean time, Property (BH) provides us with the following result:

$$\mathrm{E}[\mu(\Delta_1)\mu(\Delta_2) \mid \mathcal{F}^\Lambda] = \mathrm{E}[\mu(\Delta_1) \mid \mathcal{F}^\Lambda]\ \mathrm{E}[\mu(\Delta_2) \mid \mathcal{F}^\Lambda] = 0. \tag{2.9}$$

Finally, we can obtain (2.7) by combining (2.9) and (2.8), and the proof is complete. □

BSDEs driven by Time-changed Lévy noises

For the sake of completeness, we give an explanation of time-changed BSDEs and its connection with non-time-changed ones, even though we have presented its definition in Definition 0.2.7.

We recall the definition of BSDEs with jumps, with all the assumptions, from Definition 0.1.3:

$$\begin{aligned} -\mathrm{d}Y_t &= f(t, Y_{t^-}, \pi_t, l(u)_t(u))\mathrm{d}t - \pi_t \mathrm{d}W_t - \int_{\mathbb{R}^*} l(u)_t(u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ Y_T &= \xi \end{aligned}$$

where $Y \in S^2_{\mathcal{F}}(0,T)$ is a càdlàg optional process and $\pi \in \mathcal{H}^2_{\mathcal{F}}(0,T)$ (respectively $l \in \mathcal{H}^2_{\mathcal{F}}(0,T,\nu)$) is a $\mathbb{R}$-valued $\mathbb{F}$-predictable process defined on $\Omega \times [0,T]$ (respectively $\Omega \times [0,T] \times \mathbb{R}^*$) such that the stochastic integral with respect to W (respectively $\widetilde{N}$) is well defined.

With the time-changed process $(B, \widetilde{H})$, defined respectively in Definition 0.2.1 and (0.1), we need to rewrite the above BSDE in the following form:

$$\begin{aligned} -\mathrm{d}Y_t &= f(t, \lambda_t, Y_{t^-}, \phi(0)_t, \phi_t(u))\mathrm{d}t - \phi(0)_t \mathrm{d}B_t - \int_{\mathbb{R}^*} \phi(u)_t \widetilde{H}(\mathrm{d}t, \mathrm{d}u) \\ Y_T &= \xi \end{aligned}$$

Here we note that the solution pair processes (Y, ϕ) are now *adapted with respect to filtration* $\mathbb{G}$. The driver f now accepts one additional parameter, the process $\lambda = (\lambda^B, \lambda^H)$.

In the classical setting, we have $Y \in S^2_{\mathcal{F}}(0,T)$ in Definition 0.1.3. But with time change, we need Y to be adapted to filtration $\mathbb{G}$, and we denote by $S^2_{\mathcal{G}}(0,T)$ the space consisting of such Y's for simplicity, see Definition 0.2.7.

Time-changed Comparison Theorem

In establishing the Time-changed Comparison Theorem, we following the same agenda as in the classical setting. We present first a theorem relating to the existence and uniqueness of solution for time-changed BSDEs and then provide a comparison theorem for linear time-changed BSDEs.

We recall first the martingale representation theorem from [DS14], which will be used to prove the existence of unique solution for time-changed BSDEs.

Theorem 2.3.4 (Martingale Representation Theorem). *Assume* $(M_t)_{t\in[0,T]}$ *is a* $\mathbb{G}$*-martingale. Then there exists a unique* $\phi \in \mathcal{I}$ *such that*

$$M_t = \mathrm{E}[M_T \mid \mathcal{F}^\Lambda] + \int_0^t \int_{\mathbb{R}} \phi_s(z)\mu(\mathrm{d}s, \mathrm{d}z), \quad t \in [0,T].$$

Proof. Note here that, since μ has the martingale property with respect to filtration $\mathbb{G}$, shown in Proposition 2.3.3, it is a martingale random field with respect to filtration $\mathbb{G}$ in the sense of [DE10]. We refer to Thereom 3.5 in [DS14] and Theorem 2.2 in [Di 07] for details of the proof. □

Theorem 2.3.5 (Existence of Unique Solution for Time-changed BSDEs). *Let* $T > 0$ *and* (f, ξ) *standard parametres, defined in Definition 0.2.8, then there exits a unique solution* $(Y, \phi) := (Y, (\phi(0), \phi(u))) \in S^2_{\mathcal{G}}(0,T) \times \mathcal{I}$ *for BSDEs driven with time-changed noise as defined in (0.3).*

Proof. We give a brief sketch of this proof. Similar with BSDEs with jumps, authors in [DS14] used the previous martingale representation theorem for $\mathbb{G}$-martingales to define a mapping

$$\Theta : S^2_{\mathcal{G}}(0,T) \times \mathcal{I} \mapsto S^2_{\mathcal{G}}(0,T) \times \mathcal{I}, \quad \Theta(U, \psi) := (Y, \phi) \tag{2.10}$$

where Y is defined as

$$Y_t = \mathrm{E}\Big[\xi + \int_t^T f_s(\lambda_s, U_s, \phi_s)\mathrm{d}s \mid \mathcal{G}_t\Big], \quad t \in [0,T].$$

For a $\mathbb{G}$-martingale in the form

$$M_t = \mathrm{E}\Big[\xi + \int_0^T f_s(\lambda_s, U_s, \phi_s)\mathrm{d}s \mid \mathcal{G}_t\Big], \quad t \in [0,T],$$

we can find the unique element in $\mathcal{I}$, guaranteed by the previous martingale representation theorem, which we denote by ϕ in (2.10), as in the following representation:

$$\begin{aligned} M_t &= M_0 + \int_0^t \int_{\mathbb{R}^*} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u) \\ &= M_0 + \int_0^t \phi(0)_s \mathrm{d}B_s + \int_0^t \int_{\mathbb{R}^*} l_s(u)\widetilde{H}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

By a sequence of lemmata, authors in [DS14] showed that the mapping Θ in (2.10) is a contraction and the result is thus established. We refer to Theorem 4.5 in [DS14] for details of the proof. □

Similar to the previous chapter, we first take a look at linear BSDE driven by time-changed noises before we start working on the comparison theorem.

Following is what we call *time-changed linear BSDEs*, where the BSDE is driven by time-changed noises with a linear driver:

$$\begin{aligned}-\mathrm{d}Y_t &= \Big(\varphi_t + \delta_t Y_t + \beta_t \phi(0)_t \sqrt{\lambda_t^B} + \int_{\mathbb{R}^*} \gamma_t(u)\phi(u)_t(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H}\Big)\mathrm{d}t \\ &\quad - \phi(0)_t \mathrm{d}B_t - \int_{\mathbb{R}^*} \phi(u)_t(u)\widetilde{H}(\mathrm{d}t,\mathrm{d}u); \\ Y_T &= \xi,\end{aligned} \tag{2.11}$$

where the coefficients satisfy:

(i) $\varphi \in \mathcal{H}^{2,T}$ and $\mathbb{G}$-adapted;

(ii) δ is bounded by a constant C_δ for all $t \in [0,T]$, P-a.s.;

(iii) $(\beta,\gamma) \in \mathcal{I}$;

(iv) $\gamma_t(u) \geq 0$ is bounded by a constant C_{γ^u} for $u \in \mathbb{R}^*$ $\mathrm{d}t \times \mathrm{d}P$-a.s., and β is bounded by a constant C_β for all $t \in [0,T]$, P-a.s.

We define $(\Gamma_{t,s})_{s\in[t,T]}$ as the solution of the following SDE:

$$\begin{aligned}\mathrm{d}\Gamma_{t,s} &= \Gamma_{t,s^-}\left[\delta_s \mathrm{d}s + \beta_s \frac{\mathbb{1}_{\{\lambda_s^B \neq 0\}}}{\sqrt{\lambda_s^B}}\mathrm{d}B_s + \int_{\mathbb{R}^*} \gamma_s \frac{\mathbb{1}_{\{\lambda_s^H \neq 0\}}}{\sqrt{\lambda_s^H}}\widetilde{H}(\mathrm{d}s,\mathrm{d}u)\right], \\ \Gamma_{t,t} &= 1.\end{aligned} \tag{2.12}$$

Same as in Proposition 1.2.9 in Section 1.2, we can write $\Gamma_{t,s} = e^{\int_t^s \delta_u \mathrm{d}u} Z_{t,s}$, where $(Z_{t,s})_{s\in[t,T]}$ is the solution of the following SDE:

$$\begin{aligned}\mathrm{d}Z_{t,s} &= Z_{t,s^-}\left[\beta_s \frac{\mathbb{1}_{\{\lambda_s^B \neq 0\}}}{\sqrt{\lambda_s^B}}\mathrm{d}B_s + \int_{\mathbb{R}^*} \gamma_s \frac{\mathbb{1}_{\{\lambda_s^H \neq 0\}}}{\sqrt{\lambda_s^H}}\widetilde{H}(\mathrm{d}s,\mathrm{d}u)\right], \\ Z_{t,t} &= 1.\end{aligned}$$

We note $(Z_{t,s})_{s\in[t,T]}$ here as the solution to the forward SDE is well-defined, a fact that can be shown by applying Itô's formula in the same way as in Proposition 1.2.9.

Given our assumptions on the coefficients in (2.11) that $(\beta,\gamma) \in \mathcal{I}$, and given that $\lambda \in \mathcal{L}$, where $\mathcal{L}$ is defined in Section 0.2, by applying a similar argument in Proposition 1.2.6 in Section 1.2 under the time-changed framework, we conclude that $Z_{t,s} \in S^{2,T}$ and by our assumptions it is adapted to filtration $\mathbb{G}$. Thus we conclude that $\Gamma_{t,s} \in S^{2,T}$ and is adapted to $\mathbb{G}$ as well.

Then, by applying a similar argument used in Theorem 1.2.10 in our time-changed setting, we can represent the solution to the linear BSDE as a conditional expectation.

Theorem 2.3.6 (Solution for Time-changed Linear BSDEs)**.** *Given a linear BSDE defined as in (2.11), we can find a unique solution (Y,ϕ) in $S_{\mathcal{G}}^2(0,T)\times\mathcal{I}$, and Y has the following representation:*

$$Y_t = \mathrm{E}\left[\Gamma_{t,T}\xi + \int_t^T \Gamma_{t,s}\varphi(s)\mathrm{d}s \;\Big|\; \mathcal{G}_t\right], \quad 0 \leq t \leq T, \ a.s. \tag{2.13}$$

Proof. It is easy to observe the correspondence between this theorem and its classical counterpart Theorem 1.2.10. Once established in our new time-changed framework, the result follows by applying a similar argument. □

Now we are ready to present the most important result of this section, the comparison theorem for BSDEs driven by time-changed noises. We recall this result from Theorem 5.2 in [DS14].

Theorem 2.3.7 (Comparison Theorem for Time-changed BSDEs)**.** *Let* (f_1, ξ_1) *and* (f_2, ξ_2) *be two sets of standard parametres for the BSDEs with solutions* (Y_t^1, ϕ_t^1) *and* $(Y_t^2, \phi_t^2) \in S^2_{\mathcal{G}}(0,T) \times \mathcal{I}$*. Assume that*

$$f_2(t, \lambda, y, \phi) = f_2\Big(t, y, \phi(0)\beta_t\sqrt{\lambda^B}, \int_{\mathbb{R}^*} \phi(u)\gamma_t(u)\nu(\mathrm{d}u)\sqrt{\lambda^H}\Big)$$

where processes $(\beta_t, \gamma_t(u)) \in \mathcal{I}$ *are defined in condition (iv) in (2.11), and* f_2 *is a Lipschitz driver as defined in Definition 0.1.2.*

If $\xi_1 \le \xi_2$*, P-a.s., and* $f_1(t, \lambda_t, Y_t^1, \phi_t^1) \le f_2(t, \lambda_t, Y_t^1, \phi_t^1)$*,* $\mathrm{d}t \times \mathrm{d}P$*-a.s., then*

$$Y_t^1 \le Y_t^2 \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.}$$

Proof. The proof proceeds in the same way as in Theorem 1.2.13. Define the following processes and coefficients:

$$\begin{aligned}
\overline{\xi} &:= \xi_2 - \xi_1,\\
\overline{Y}_t &:= Y_t^2 - Y_t^1,\\
\overline{\phi}_t(0) &:= \phi(0)_t^2 - \phi(0)_t^1,\\
\overline{\phi}_t(u) &:= \phi(u)_t^2(u) - \phi(u)_t^1(u),\\
\varphi_t &:= f_2(t, \lambda_t, Y_{t-}^2, \phi(0)_t^2, \phi(u)_t^2) - f_1(t, \lambda_t, Y_{t-}^2, \phi(0)_t^2, \phi(u)_t^2),\\
\delta_t &:= \frac{f_2(t, \lambda_t, Y_{t-}^2, \phi(0)_t^1, \phi(u)_t^1) - f_2(t, \lambda_t, Y_{t-}^1, \phi(0)_t^1, \phi(u)_t^1)}{\overline{Y}_t}\mathbb{1}_{\{\overline{Y}_t \neq 0\}},\\
\beta_t &:= \frac{f_2(t, Y_{t-}^2, \phi(0)_t^2, \phi(u)_t^1) - f_2(t, Y_{t-}^2, \phi(0)_t^1, \phi(u)_t^1)}{\overline{\phi}_t(0)C_\beta\sqrt{\lambda^B}}\mathbb{1}_{\{\overline{\phi}_t(0)C_\beta\sqrt{\lambda^B} \neq 0\}},\\
\gamma_t &:= \frac{f_2(t, Y_{t-}^2, \phi(0)_t^1, \phi(u)_t^2) - f_2(t, Y_{t-}^2, \phi(0)_t^1, \phi(u)_t^1)}{\int_{\mathbb{R}^*}\overline{\phi}_t(u)C_{\gamma^u}\nu(\mathrm{d}u)\sqrt{\lambda^H}}\mathbb{1}_{\{\int_{\mathbb{R}^*}\overline{\phi}_t(u)C_{\gamma^u}\nu(\mathrm{d}u)\sqrt{\lambda^H} \neq 0\}}.
\end{aligned}$$

Then we can rewrite as a linear BSDE as defined in (2.11)

$$\begin{aligned}
-\mathrm{d}\overline{Y}_t = &\Big(\varphi_t + \delta_t\overline{Y}_t + C_\beta\beta_t\overline{\phi}_t(0)\sqrt{\lambda_t^B} + \int_{\mathbb{R}^*} C_{\gamma^u}\gamma_t(u)\overline{\phi}_t(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H}\Big)\mathrm{d}t\\
&- \overline{\phi}_t(0)\mathrm{d}B_t - \int_{\mathbb{R}^*}\overline{\phi}_t(u)\widetilde{H}(\mathrm{d}t, \mathrm{d}u);\\
\overline{Y}_T = &\ \overline{\xi}.
\end{aligned}$$

We see that the process δ, β and γ are bounded as f is a Lipschitz driver. $\varphi \in \mathcal{H}^{2,T}$ and $\mathbb{G}$-adapted, since it is a difference of such two functions. Moreover, $C_\beta\beta$ and $C_{\gamma^u}\gamma_t(u)\mathbb{1}_{\mathbb{R}^*}(u)$ satisfy condition (iv) in (2.11).

We can therefore apply Theorem 2.3.6, and obtain a solution

$$\overline{Y}_t = \mathrm{E}\Big[\Gamma_{t,T}\overline{\xi} + \int_t^T \Gamma_{t,s}\varphi(s)\mathrm{d}s \;\Big|\; \mathcal{G}_t\Big], \quad 0 \le t \le T, \text{ a.s.}$$

which is positive, given that $\bar{\xi}, \Gamma$ and φ are all positive a.s., and we have thus accomplished the proof. □

Remark 2.3.8. We note that in Theorem 2.3.7, we have the $\gamma_t(u) \geq 0$ in condition (iv) in (2.11). This is a stronger condition then $\gamma_t(u) > -1$ in the strict comparison theorem without change of time, see Theorem 1.2.14.

By adapting a similar argument used in Proposition 1.3.2 and Remark 1.3.4 to our current time-change setting, the stronger condition we imposed on the driver f_2 in the Time-changed Comparison Theorem, see Theorem 2.3.7, can guarantee the the monotonicity of the associated risk measures.

Chapter 3

Dynamic Risk Measures Generated by Time-changed BSDEs

This chapter and the next form the core of the present thesis. Topics we have discussed in the previous two chapters, namely, dynamic risk measures, BSDEs with jumps and Change of Time Methods, all come together in this chapter.

We have ended the previous chapter by introducing a comparison theorem for BSDEs driven by time-changed Lévy noises. Following a natural development, Section 3.1 presents the link between a dynamic risk measure and its associated BSDE, which is the non-linear expectation we denote by "f-expectation". Only this time our BSDE is driven by time-changed noises in comparison with the classical setting in Section 1.3. We also discuss the mathematical properties of the resulting dynamic risk measure, given the conditions we have imposed on the driver f.

In Section 3.2, we present the representation of convex and coherent risk measures generated by a concave driver f. Most of these results have been presented in Section 1.3 in the classical setting, but now we need to establish their counterparts in our new, time-changed setting. Since we are dealing with an absolutely continuous time change, it does not cause too much trouble to carry out the proof in our current setting.

Yet when we look at the question from the opposite direction, a challenge arises. To be more precise, we need to answer the question that, given a so-called f-expectation, can we construct a time-changed BSDE with a corresponding driver f? If we impose some conditions on the f-expectation, the answer is affirmative. One key element in the proof of this inverse problem is the Doob-Meyer Decomposition in our time-changed setting. The difficulty lies in the fact that the classical proof of the Doob-Meyer Decomposition relies on the linearity of classical expectations, something that no longer applies in our current non-linear expectation situation. To give a proper presentation of this problem is the task for the next chapter.

3.1 Dynamic Risk Measures and Time-changed BSDEs

We follow the same agenda as in Section 1.3, namely, we will first derive a non-linear expectation generated by the driver f of the associated time-changed BSDE, and then define a dynamic risk measure with it. We note that all the results we obtain in this chapter and the next are adapted to the filtration $\mathbb{G}$, and the same applies to the dynamic risk measures we derive in this section.

Time-changed f-expectations

We recall the definition of non-linear expectation in Definition 1.3.1, and we claim that we can derive a time-changed non-linear expectation from the associated BSDE by the following proposition.

Proposition 3.1.1 (Time-changed f-expectation)**.** *Consider a time-changed BSDE drived by a generator f defined in (0.3) such that*

(i) $f(t, \lambda, y, (0, 0)) = 0$ for all $y \in \mathbb{R}$;

(ii) f satisfies conditions for f_2 in the Time-changed Comparison Theorem, Theorem 2.3.7.

Then for any fixed ξ in $L^2(\mathcal{G}_T)$, we denote the unique solution of the related time-changed BSDE with terminal condition ξ by (Y^ξ, ϕ^ξ). We set $\mathcal{E}_f[\xi] = Y_0^\xi$, the initial value of the solution. Then $\mathcal{E}_f$ is a non-linear expectation called time-changed f-expectation.

Proof. We note this proposition corresponds to Propostion 1.3.2 in the classical setting. We can indeed adapt the argument used in Propostion 1.3.2 to our current time-changed framework, and prove this proposition by applying the Time-changed Comparison Theorem, see Theorem 2.3.7. □

Remark 3.1.2. In view of Remark 2.3.8, we note that the Time-changed Comparison Theorem, see Theorem 2.3.7, imposes a stronger condition on f than assumptions made in the Strict Comparison Theorem in the classical setting, see Theorem 1.2.14. Using the same argument from Proposition 1.3.2 and Remark 1.3.4 under our time-changed framework, we can conclude that the Time-changed Comparison Theorem guarantees the monotony property of the associated risk measures.

Now we can investigate the properties we listed in Section 1.3 in the time-changed setting and check under what conditions the f-expectation defined above satisfies them.

(i) Filtration consistency.
This property is defined in Definition 1.3.5 with respect to filtration $\mathbb{F}$, defined in Definition 0.1.1 and in this chapter and the next, we consider filtration $\mathbb{G}$, defined in Section 0.2. In fact, we see that this property can also be written as, for $A \in \mathcal{G}_t$,

$$\mathcal{E}_f[\xi \mathbb{1}_A] = \mathcal{E}_f\Big[\mathcal{E}_f[\xi \mid \mathcal{G}_t] \mathbb{1}_A\Big]. \tag{3.1}$$

We see that all time-changed f-expectations defined in Proposition 3.1.1 are filtration-consistent. Given solution (Y^ξ, ϕ^ξ) in Proposition 3.1.1, by the uniqueness of solution, it is easy to see that, for $A \in \mathcal{G}_t$, $Y_t^\xi \mathbb{1}_A$ and $\xi \mathbb{1}_A$ coincide on the interval $[0,t]$. We can thus express the conditional expectation, in the similar way as in Definition 1.3.5:

$$\mathcal{E}_f[\xi \mid \mathcal{G}_t] = Y_t^\xi.$$

(ii) Additivity.
Additivity is defined in Definition 1.3.6 with respect to filtration $\mathbb{F}$. Now we consider the filtration $\mathbb{G}$. By adapting discussion following Definition 1.3.6 to our time-changed setting, it is easy to see that we can obtain additivity if the driver f is independent of y under our time-changed framework.

(iii) $\mathcal{E}^{C,C_1}$-domination.
This property can be directly applied to our time-changed situation. We only need to replace the filtration with filtration $\mathbb{G}$ and change the driver f_{C,C_1} in Definition 1.3.7 to the following: for $t \in [0,T]$ and $\phi \in \mathcal{I}$, where $\mathcal{I}$ is defined in (i) in Definition 0.2.6,

$$\begin{aligned} f_{C,C_1}(t,\phi) &:= C\left|\phi(0)\lambda_t^B\right| + C\int_{\mathbb{R}^*}(1\wedge|u|)\phi^+(u)\nu(\mathrm{d}u)\lambda_t^H \\ &\quad - C_1\int_{\mathbb{R}^*}(1\wedge|u|)\phi^+(u)\nu(\mathrm{d}u)\lambda_t^H. \end{aligned}$$

Remark 3.1.3. We note that independence between the driver f and y and assumptions in Proposition 3.1.1 ensure both additivity and $\mathcal{E}^{C,C_1}$-domination. We can obtain the same result as in the classical setting by applying the argument in our time-changed framework, see our discussion about this following Definition 1.3.7.

Dynamic Risk Measures as Time-changed f-expectations

Now we can use time-changed f-expectation to represent a dynamic risk measure, and thus establish a link from non-linear expectation to risk measures.

Proposition 3.1.4. *Let $T' > 0$, and $\mathcal{E}_f$ be a time-changed f-expectation generated in Proposition 3.1.1. Suppose $\mathcal{E}_f$ satisfies properties of additivity, filtration-consistence and $\mathcal{E}^{C,C_1}$-domination, all with respect to filtration $\mathbb{G}$ defined in Section 0.2, then we can define a normalised, time-consistent dynamic risk measure in the following way: for each $T \in [0,T']$ and $\xi \in L^2(\mathcal{G}_T)$,*

$$\rho_t^f(\xi,T) := -\mathcal{E}_f[\xi \mid \mathcal{G}_t] = -Y_t(\xi,T), \quad t \in [0,T]. \tag{3.2}$$

Moreover, if the generator f is also concave *with respect to (y,ϕ), then the dynamic risk measures thus represented is* convex.

❇ *Proof.* We have showed similar results in Proposition 1.3.9, and here we carry out the proof by adapting the argument used in the classical setting to our time-changed framework.

In the following, we check that the conditional risk measure $\rho_t^f(\xi, T)$ defined in (3.2) satisfies the properties listed in Definition 1.1.12 and Proposition 1.1.19.

First we note that the additivity of the f-expectation is equivalent to the conditional invariance for the dynamic risk measure defined in Definition 1.1.12. Thus the additivity of the time-changed f-expectation $\mathcal{E}_f$ implies the conditional invariance for the conditional risk measure ρ_t^f defined in (3.2).

The filtration-consistency property of the f-expectation is equivalent to the normalised conditional time-consistency property (1.2) in Proposition 1.1.19, as we have observed in (3.1). In this way, the filtration consistency of the time-changed f-expectation $\mathcal{E}_f$ implies time-consistency for the conditional risk measure ρ_t^f.

Here we note that, since we have the condition that $f(t, \lambda, y, (0,0)) = 0$, the associated conditional dynamic risk measure ρ_t^f defined in (3.2) has the normalisation property defined as (iv) in Definition 1.1.12.

When it comes to monotonicity property, Remark 3.1.2 tells us that the conditions on the driver f guarantees monotonicity for the f-expectation $\mathcal{E}_f$, and this implies monotonicity for the conditional risk measure ρ_t^f. Behind the curtain, this results rests on the Time-changed Comparison Theorem, see Theorem 2.3.7.

Finally, the convexity follows by an application of the Time-changed Comparison Theorem, Theorem 2.3.7. To be more specifically, consider $\xi_1, \xi_2 \in L^2(\mathcal{G}_T)$, $\eta \in [0,1]$. We note that, by assumptions, f is additive and is therefore independent of y, then by the concavity of f in terms of ϕ, we have

$$f(\eta\ \phi(\xi_1) + (1-\eta)\ \phi(\xi_2)) \geq \eta\ f(\phi(\xi_1)) + (1-\eta)\ f(\phi(\xi_2)) \qquad (3.3)$$

Then by the Time-changed Comparison Theorem, the resulting risk measure is convex, and the result is established. □

Corollary 3.1.5. *In addition to assumptions in Proposition 3.1.4, if the driver f is also positively homogeneous, see (v) in Definition 1.1.12, then the resulting dynamic risk measures defined in (3.2) is* coherent.

❇ *Proof.* We note that by assumptions, the driver f is concave with respect to ϕ, and the resulting risk measure ρ_t is convex. If the driver f is, in addition, positively homogeneous, it follows that the associated risk measure ρ_t is also positively homogeneous, namely for $C \in \mathbb{R}$ and $C \geq 0$, we have $\rho_t(C\xi) = C\rho_t(\xi)$.

This is because,

$$\begin{aligned}\mathcal{E}_f[C\ \xi \mid\ \mathcal{G}_t] &= Y_t \\ &= C\ \xi + \int_t^T f(s, \lambda_s, \phi)\mathrm{d}s - \int_t^T \int_{\mathbb{R}^*} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u)\end{aligned}$$

Note that $f(s, \lambda_s, \phi)$ by our assumptions is independent of Y and positively homogeneous, it follows:

$$\begin{aligned}Y_t &= C\ \Big[\xi + \int_t^T \frac{f(s, \lambda_s, \phi)}{C}\mathrm{d}s - \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u)\Big] \\ &= C\ \Big[\xi + \int_t^T f(s, \lambda_s, \frac{\phi}{C})\mathrm{d}s - \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u)\Big],\end{aligned}$$

Now we denote

$$\overline{Y}_t = \xi + \int_t^T f(s, \lambda_s, \frac{\phi}{C})\mathrm{d}s - \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u),$$

and we note this BSDE has unique solution, given the conditions we imposed on f. Thus we conclude, by the definition of $\mathcal{E}_f[\xi \mid \mathcal{G}_t]$ coincides with $\overline{Y}$:

$$\mathcal{E}_f[C\ \xi \mid \mathcal{G}_t] = Y_t = C\ \overline{Y}_t = C\ \mathcal{E}_f[\xi \mid \mathcal{G}_t].$$

This means that the associated risk measure has subadditivity property, namely, for two financial positions X, Y, we choose $\eta = \frac{1}{2}$ in (3.3), then we obtain:

$$\rho_t(X + Y) \leq \rho_t(X) + \rho_t(Y)$$

By definition 1.1.3, the resulting risk measure is coherent. □

3.2 Representation of Dynamic Risk Measures

In this section, we present a representation of dynamic risk measures generated by time-changed BSDE with jumps. We use a similar approach to proving this theorem under our time-changed framework as Theorem 1.3.11 in the classical setting. Before we present the main result, we need the following lemma which corresponds to Optimisation Principle in the classical setting, see Theorem 1.2.16.

Lemma 3.2.1 (Time-changed Optimisation Principle). *Let ξ in $L^2(\mathcal{G}_T)$ and let (f, f^α) be a family of Lipschitz drivers parametrised by predictable processes α, and $\mathcal{A}_T \in \mathcal{G}_T$ is the set of α. Let (Y, ϕ) (resp. (Y^α, ϕ^α)) be the solution of the BSDE associated with terminal condition ξ and driver f (resp. f^α).*

Suppose that

$$f(t, \lambda_t, Y_t, \phi_t) = \operatorname*{ess\,inf}_{\alpha} f^\alpha(t, \lambda_t, Y_t, \phi_t) = f^{\bar{\alpha}}(t, \lambda_t, Y_t, \phi_t),$$
$$t \in [0, T],\ \mathrm{d}P \otimes \mathrm{d}t\text{-a.s. for some parameter } \bar{\alpha} \in \mathcal{A}_T$$

and that for each $\alpha \in \mathcal{A}_T$, $f^\alpha(t, \lambda_t, Y_t, \phi_t)$ satisfies conditions for f_2 in the Time-changed Comparison Theorem, Theorem 2.3.7. Then

$$[Y_t = \operatorname*{ess\,inf}_{\alpha} Y_t^\alpha = Y_t^{\bar{\alpha}}, \quad t \in [0, T] \text{ a.s.} \tag{3.4}$$

Proof. This result is straightforward with an application of the Time-changed Comparison Theorem, see Theorem 2.3.7.

Given that for each α, $f(t, \lambda_t, Y_t, \phi_t) \le f^\alpha(t, \lambda_t, Y_t, \phi_t)$, by the Time-changed Comparison Theorem, we obtain $Y_t \le Y_t^\alpha$ for all $t \in [0, T]$ almost surely. It follows that, for all $t \in [0, T]$,

$$Y_t \le \operatorname*{ess\,inf}_{\alpha} Y_t^\alpha.$$

On the other hand, Y_t is the solution of the BSDE associated with $f^\alpha(t, \lambda_t, Y_t, \phi_t)$ by assumption. By the uniqueness of solution, we conclude that $Y_t = \operatorname{ess\,inf}_\alpha Y_t^\alpha$, for all $t \in [0, T]$ almost surely. This gives us the equality (3.4). □

Similar as in (1.24), now we define a function F of the driver $f(t, \lambda_t, Y_t, \phi_t)$ with respect to ϕ_t, for each $\alpha := (\alpha^1, \alpha^2) \in \Phi$, where Φ is defined in (ii) in Definition 0.2.6, and $u \in \mathbb{R}^*$, as the following:

$$\begin{aligned} F(\omega, t, \lambda, \alpha) &:= \sup_{\phi \in \Phi} [f(\omega, t, \lambda, \phi) - \alpha \phi \lambda] \\ &= \sup_{\phi \in \Phi} \Big[f(\omega, t, \lambda, \phi) - \alpha^1 \phi(0) \sqrt{\lambda^B} \\ &- \int_{\mathbb{R}^*} \alpha^2 \phi(u) \sqrt{\lambda^H} \nu(\mathrm{d}u) \Big]. \end{aligned} \tag{3.5}$$

Here f is a Lipschitz driver with Lipschitz constant C, which does not depend on y. Suppose also that f satisfies conditions for f_2 in the Time-changed Comparison Theorem, Theorem 2.3.7 and is concave with respect to ϕ. We also need the following two lemmata to prove the main theorem.

Lemma 3.2.2. *For each $(t,\omega), D(\omega)$ is defined as the non-empty set of $\alpha = (\alpha_1, \alpha_2) \in \Phi$, where Φ is defined in (ii) in Definition 0.2.6, such that $F(\omega, t, \lambda_t, \alpha_1, \alpha_2) < \infty$, where $F(\omega, t, \lambda_t, \alpha_1, \alpha_2)$ and the associated driver $f(\omega, t, \lambda, \phi)$ is defined as in (3.5). Then for each $(t,\omega), D(\omega) \subset U$, where U is the closed subset of the Hilbert space Φ of the elements $\alpha = (\alpha_1, \alpha_2)$ such that α_1 is bounded by C and $\nu(\mathrm{d}u)-a.s.$,*

$$\alpha_2 \geq -1 \quad and \quad |\alpha_2| \leq \psi(u) \wedge C,$$

where C is the Lipschitz constant of f, and $\psi(u) \in \Phi$.

❇ *Proof.* We follow the proof of Lemma 5.4 in [QS13] in proving the lemma in the time-changed setting. First we show that $a_2 \leq -1$. We assume for now that

$$\nu(\{u \in \mathbb{R}^* : \alpha_2(u) < -1\}) > 0.$$

We note that from the our assumptions on the driver $f(\omega, t, \lambda, \phi)$ and independent of Y. For the sake of simplicity, we assume the Lipschitz coefficient $C = 1$ for now. Then for $\phi^1(u)$ and $\phi^2(u)$ we have, similarly to expression used in the Time-changed Comparison Theorem, Theorem 2.3.7, namely, for $i = \{1, 2\}$,

$$f(t, \omega, \lambda_t, \phi^i) = f\Big(t, 0, \int_{\mathbb{R}^*} \phi^i(u)\gamma_t(u)\nu(\mathrm{d}u)\sqrt{\lambda^H}\Big)$$

Then by assumptions imposed on f, we can find a process $\gamma_t^{1,2}(u)$, such that

$$f(t, \omega, \lambda_t, \phi^1) - f(t, \omega, \lambda_t, \phi^2) \leq \int_{\mathbb{R}^*} (\phi^1(u) - \phi^2(u))\gamma_t^{1,2}(u)\nu(\mathrm{d}u)\sqrt{\lambda^H},$$

and

$$f(t, \omega, \lambda_t, \phi^2) - f(t, \omega, \lambda_t, \phi^1) \geq \int_{\mathbb{R}^*} (\phi^2(u) - \phi^1(u))\gamma_t^{1,2}(u)\nu(\mathrm{d}u)\sqrt{\lambda^H}.$$

Here we note $\gamma_t^{1,2}(u) \geq 0$ and is bounded by a constant C_{γ^u} that depends on u, as defined in condition (iv) in (2.11), for any $\phi(u) \in \mathcal{I}$. Now we make $\phi^1(u) = 0$, then by (3.5), it follows:

$$\begin{aligned}
F(\omega, t, \lambda, \alpha) &\geq f(t, \omega, \lambda_t, \phi^2) - \int_{\mathbb{R}^*} \alpha^2\phi^2(u)\sqrt{\lambda^H}\nu(\mathrm{d}u) \\
&\geq f(t, \omega, \lambda_t, 0) + \int_{\mathbb{R}^*} \phi^2(u)\gamma_t^{1,2}(u)\nu(\mathrm{d}u)\sqrt{\lambda^H} \\
&\quad - \int_{\mathbb{R}^*} \alpha^2\phi^2(u)\sqrt{\lambda^H}\nu(\mathrm{d}u) \\
&= f(t, \omega, \lambda_t, 0) + \int_{\mathbb{R}^*} (\gamma_t^{1,2}(u) - \alpha^2)\phi^2(u)\sqrt{\lambda^H}\nu(\mathrm{d}u)
\end{aligned}$$

Then we make $\phi^2(u) = n\mathbb{1}_{\{\alpha_2 < -1\}}$, we end up with

$$F(\omega, t, \lambda, \alpha) \geq f(t, \omega, \lambda_t, 0) + n\int_{\{\alpha_2 < -1\}} (\gamma_t^{1,2}(u) - \alpha^2)\sqrt{\lambda^H}\nu(\mathrm{d}u)$$

and this shows $F(\omega, t, \lambda, \alpha) \to \infty$ as $n \to \infty$, since $\gamma_t^{1,2}(u) \geq 0$ and $\alpha_2 < -1$. This is a contradiction since $(\alpha_1, \alpha_2) \in D_t(\omega)$. And the same argument can be applied to prove the other results. □

Lemma 3.2.3. *Given assumptions in (3.5) and Lemma 3.2.2, and we assume that the Hilbert space Φ is separable. Then there exists a process $\bar{\alpha} = (\bar{\alpha}^1, \bar{\alpha}^2(u)) \in \mathcal{A}_T$ such that*

$$f(t,\pi,\ell_t) = \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_T}\{f^\alpha(t,\pi_t,\ell_t)\} = f^{\bar{\alpha}}(t,\pi_t,\ell_t), \quad t\in[0,T],\ \mathrm{d}P\otimes\mathrm{d}t\text{-}a.s.$$

Proof. We refer to the proof of Lemma 5.5 in [QS13]. This is a result based on classical arguments of convex analysis. The proof is rather technical and it carries over to our time-changed setting with no problem. □

Theorem 3.2.4 (Representation of Time-changed Normalised Convex Dynamic Risk Measures)**.** *Suppose that the Hilbert space Φ is separable. Let f be a Lipschitz driver with Lipschitz constant C, which does not depend on y. Suppose also that f satisfies conditions for f_2 in the Time-changed Comparison Theorem, Theorem 2.3.7 and is concave with respect to ϕ.*

Let $T'>0$ be a time horizon and $T\in[0,T']$. Let $\mathcal{A}_T$ be the set of predictable processes $\alpha=(\alpha^1,\alpha^2)$ such that $F(t,\alpha^1_t,\alpha^2_t(u))\in\mathcal{H}^2_G(0,T)$, where F is defined by (3.5). For each $\alpha\in\mathcal{A}_T$, let Q^α be the probability absolutely continuous with respect to P which admits Z^α_T as density with respect to P on $\mathcal{F}_T$, where Z^α is the solution of

$$\mathrm{d}Z^\alpha_t = Z^\alpha_{t-}\left(\alpha^1_t\mathrm{d}B_t + \int_{\mathbb{R}^*}\alpha^2_t(u)\widetilde{H}(\mathrm{d}t,\mathrm{d}u)\right); \quad Z^\alpha_0 = 1. \tag{3.6}$$

The convex dynamic risk measure $\rho(\xi,T)$ has the following representation: for each $\xi\in L^2(\mathcal{F}_T)$,

$$\rho_0(\xi,T) = \sup_{\alpha\in\mathcal{A}_T}[\mathrm{E}_{Q^\alpha}[-\xi] - \zeta(\alpha,\lambda,T)], \tag{3.7}$$

where the function ζ, the penalty term, is defined for each T and $\alpha\in\mathcal{A}_T$ by

$$\zeta(\alpha,\lambda,T) := \mathrm{E}_{Q^\alpha}\left[\int_0^T F(s,\lambda,\alpha^1_s,\alpha^2_s)\mathrm{d}s\right].$$

Moreover, for each $\xi\in L^2(\mathcal{F}_T)$, there exits $\bar{\alpha}=(\bar{a}^1,\bar{\alpha}^2)\in\mathcal{A}_T$ such that for $t\in[0,T]$, $\mathrm{d}P\otimes\mathrm{d}t$-a.s.:

$$F(\omega,t,\lambda,\bar{\alpha}) = f(\omega,t,\lambda,\phi) - \bar{\alpha}^1\phi(0)\sqrt{\lambda^B} - \int_{\mathbb{R}^*}\bar{\alpha}^2\phi(u)\sqrt{\lambda^H}\nu(\mathrm{d}u),$$

where (Y,ϕ) is the solution to the BSDE with driver f, terminal time T and terminal condition ξ. Also the process $\bar{\alpha}$ is optimal for (3.7).

Proof. We follow the argument used in Theorem 5.3 in [QS13] in the classical setting but now try to establish the result in our time-changed setting. Here we need the assumption of the Φ space being separable so as to solve the measurability issues for F, defined in (3.5).

Given the separability assumption, Φ is separable, then it admits a dense countable subset I. Since our driver f is Lipschits with respect to ϕ, it is continuous with respect to ϕ, then by the definition in (3.5), the supremum in (3.5) coincides with the supremum over I. This shows that F is measurable.

Then classical convex analysis arguments provide us with the result that F is also lower semi-continuous with respect to α. We refer to Theorem 5.2 in [QS13] for details concerning this part.

Since f is concave and continuous, we have

$$f(\omega, t, \lambda, \phi) = \inf_{\alpha \in D_t(\omega)} \{F(\omega, t, \lambda, \phi) + \alpha\phi\lambda\}.$$

Here $D_t(\omega)$ is defined as in Lemma 3.2.2, and the Lemma holds.

For each $\alpha = (\alpha^1, \alpha^2) \in \mathcal{A}_T$, we define:

$$f^\alpha(\omega, t, \lambda, \phi) := F(\omega, t, \lambda, \phi) + \alpha\phi\lambda.$$

Then we note here $f^\alpha \geq f$ for each $\alpha \in \mathcal{A}_T$, because of the construction of f.

Now we consider $T \in [0, T']$ and $\xi \in L^2(\mathcal{G}_T)$. Let (Y, ϕ) be the solution of the BSDE associated with the driver f, terminal time T and terminal condition ξ. Then we apply Lemma 3.2.3.

Then by the optimisation principle for Time-changed BSDEs with jumps, Lemma 3.2.1, it follows that

$$Y_0(\xi, T) = \inf_{\alpha \in \mathcal{A}_T} Y_0^\alpha(\xi, T) = Y_0^{\bar{\alpha}}(\xi, T).$$

Here for each $\alpha \in \mathcal{A}_T$, $Y^\alpha(\xi, T)$ is the solution of the BSDE associated with driver f^α.

Now we let $\alpha = (\alpha^1, \alpha^2) \in \mathcal{A}_T$. By Lemma 3.2.2, we have $|\alpha^2| \leq \psi(u) \wedge C$. Then by Proposition 1.2.6 in Section 1.2, we know $Z^\alpha \in S^2_{\mathcal{G}}(0, T)$ for Z^α defined in (3.6), and by our assumptions it is adapted to filtration $\mathbb{G}$. As a result, we can represent the solution of linear BSDE in the following way, proved in Theorem 2.3.6:

$$Y_0^{\alpha(\xi,T)} = \mathrm{E}\left[Z_T^\alpha \xi + \int_0^T Z_s^\alpha F(s, \lambda, \alpha_s^1, \alpha_s^2) \mathrm{d}s\right], \quad 0 \leq t \leq T, \ a.s.$$

On the other hand, also by Lemma 3.2.2, we have that $\alpha^2 > -1$. Hence $(Z_t^\alpha)_{0 \leq t \leq T}$ is a non-negative martingale and the probability Q^α that admits Z_T^α as density with respect to probability P on $\mathcal{G}_T$ is well-defined. We have thus obtained

$$Y_0^{\alpha(\xi,T)} = \mathrm{E}_{Q^\alpha}\left[\xi + \int_0^T F(s, \lambda, \alpha_s^1, \alpha_s^2) \mathrm{d}s\right],$$

and the result is established. □

Finally, we show that the representation for coherent dynamic risk measures is just a special case of convex risk measures with $F = 0$.

Corollary 3.2.5. *Given assumptions in Theorem 3.2.4, we note that if the driver f is also positively homogeneous, then $F = 0$.*

❇ *Proof.* If the driver f is, in addition, positively homogeneous, it follows that the associated risk measure is also positively homogeneous, namely for $C \in \mathbb{R}$ and $C \geq 0$, we have $\rho_t(C\xi) = C\rho_t(\xi)$. It implies that the resulting risk measure is coherent. We have shown this result in Corollary 3.1.5.

Now we only need to show that given the above assumptions, the $F = 0$ and we can thus obtain the representation for coherent risk measures. By positive homogeneity, we have in (3.7), $\rho_0(0,T) = 0$. This implies that, for $\xi = 0$:

$$\zeta(\alpha, \lambda, T) = \sup_{\alpha \in \mathcal{A}_T} [\mathrm{E}_{Q^\alpha}[-\xi] - \rho_0(\xi, T)] \geq 0.$$

Now we assume that for $\bar{\xi} \in L^2(\mathcal{G}_T)$, we have $\mathrm{E}_{Q^\alpha}[-\bar{\xi}] - \rho_0(\bar{\xi}, T) > 0$, then by construction, it follows that, for a constant $C > 0$,

$$\begin{aligned}
\zeta(\alpha, \lambda, T) &:= \sup_{\alpha \in \mathcal{A}_T} [\mathrm{E}_{Q^\alpha}[-\xi] - \rho_0(\xi, T)] \\
&\geq \sup_{C>0} [\mathrm{E}_{Q^\alpha}[-C\bar{\xi}] - \rho_0(C\bar{\xi}, T)] \\
&= \sup_{C>0} C[\mathrm{E}_{Q^\alpha}[-\bar{\xi}] - \rho_0(\bar{\xi}, T)] \\
&\to \infty.
\end{aligned}$$

We can thus conclude that, for all $\xi \in L^2(\mathcal{G}_T)$

$$\mathrm{E}_{Q^\alpha}[-\xi] - \rho_0(\xi, T) \leq 0,$$

which implies

$$\zeta(\alpha, \lambda, T) = \sup_{\alpha \in \mathcal{A}_T} [\mathrm{E}_{Q^\alpha}[-\xi] - \rho_0(\xi, T)] \leq 0.$$

We have thus shown $\zeta(\alpha, \lambda, T) = 0$ and therefore $F = 0$, and the result is established. $\square$

Chapter 4

The Inverse Theorem

This chapter addresses the Inverse Theorem. It is arguably the most difficult result in the entire thesis. Establishing the Inverse Theorem enables us to represent, under rather general assumptions, a given non-linear expectation, and by extension, a given dynamic risk measure, by a time-changed BSDE.

Thanks to the work done by authors in [Pen99], [Coq+02], where the proof is established in the Brownian motion setting, and in [Roy06], where the author extends the proof in a more general setting of BSDEs with jumps, a result has been proved in the classical setting in Theorem 1.3.8. This section gives a detailed proof of the corresponding theorem in the time-changed setting with absolutely continuous time change, the framework constructed in Section 2.3 and used in the previous chapter.

Proof of the Inverse Theorem is more complicated than it appears. The result depends in fact on establishing the Doob-Meyer decomposition in our time-changed setting. Given the time change $\Lambda_t = \int_0^t \lambda_s \mathrm{d}s, t \geq 0$ and the filtration $\mathbb{G}$, both defined in Section 0.2, we have the following theorem:

Theorem 4.0.1 (Time-changed Inverse Theorem)**.** *Let $\mathcal{E}$ be a filtration-consistent time-changed non-linear expectation which satisfies both properties of $\mathcal{E}^{C,C_1}$-domination and additivity, all with respect to filtration $\mathbb{G}$, both defined in Section 0.2. Then these exits a function $f : [0,T] \times \Omega \times \Phi \mapsto \mathbb{R}$ such that $\mathcal{E} = \mathcal{E}_f$.*

In Section 4.1, we give first the definition of supermartingales (or submartingales) with respect to non-linear expectations and establish some of their properties, which are used in the proofs in the following sections.

Section 4.2 proves the important results of the Doob-Meyer Decomposition for $\mathcal{E}_f$-supermartingales. Proof of this theorem follows the idea from [Pen99], where the result is established in a Brownian motion setting. To carry over this result in our time-changed setting is not easy, as now we have to take into consideration the extra term, namely, the time-changed jumps of the BSDEs. We adapt to our time-changed framework some of the arguments used in [Roy06] to generalise this result from the Brownian motion setting to the one of BSDEs with jumps in the classical setting.

Section 4.3 further generalises the result from Section 4.2 by establishing the Doob-Meyer Decomposition for a general non-linear expectation $\mathcal{E}$-supermartingales with a given driver f. This is an important step towards the proof of the main theorem. And the key here is to express the results in

Section 4.2 as conditional (non-linear) expectations. In order to do so, it requires proving several important lemmata, including one comparison theorem for conditional (non-linear) expectations, before establishing the decomposition for non-linear expectation martingales. One good thing is that, most of the hard work has already been done in Section 4.2, so the proof of decomposition theorem in Section 4.3 can follow a correspondingly similar argument. In this section, we base our proof on works done mainly in [Coq+02] and [Roy06].

We end this chapter and the main part of this thesis by presenting the proof of the Inverse Theorem, see Theorem 4.0.1. The result follows from the theorems we have established in the previous sections. What is important about the Inverse Theorem is that it enables us to convert a question about non-linear expectations into a study of time-changed BSDEs. In this way, we establish a full connection between BSDEs and risk measures via non-linear expectations, fulfilling the goal of this thesis.

4.1 Properties of $\mathcal{E}$-supermartingales

Definition 4.1.1 ($\mathcal{E}$-martingale)**.** Let $(X_t)_{0\le t\le T} \in L^2([0,T]\times\Omega, \mathcal{B}_{[0,T]}\times\widetilde{\mathcal{F}}, m\times P)$ be a $\mathbb{G}$-adapted, càdlàg process, and $\mathcal{E}$ be a time-changed non-linear expectation. $(X_t)_{0\le t\le T}$ is called an $\mathcal{E}$-martingale ($\mathcal{E}$-supermartingale, $\mathcal{E}$-submartingale, respectively) if for all $0 \le s \le t \le T$:

$$X_s = \mathcal{E}[X_t \mid \mathcal{G}_s] \qquad (\ge,\ \le,\ \text{respectively}).$$

We recall from Definition 1.3.7 the definition of the $\mathcal{E}^{C,C_1}$-domination property in our time-changed setting.

Definition 4.1.2 (Time-changed $\mathcal{E}^{C,C_1}$-domination property)**.** Let $\mathcal{E}$ be a filtration-consistent non-linear expectation with respect to filtration $\mathbb{G}$. We say that it is $\mathcal{E}^{C,C_1}$-dominated if there exists $C \in \mathbb{R}$ and $-1 < C_1 \le 0$ such that, for all $\xi, \xi' \in L^2(\mathcal{G}_T)$:

$$\mathcal{E}[\xi+\xi'] - \mathcal{E}[\xi] \le \mathcal{E}^{C,C_1}[\xi']$$

where $\mathcal{E}^{C,C_1} := \mathcal{E}_{f_{C,C_1}}$ is the non-linear f-expectation associated with the following driver: for $t \in [0,T]$ and $\phi \in \mathcal{I}$, where $\mathcal{I}$ is defined in (i) in Definition 0.2.6,

$$\begin{aligned} f_{C,C_1}(t,\phi) := {} & C\left|\phi(0)\sqrt{\lambda_t^B}\right| + |C|\int_{\mathbb{R}^*}(1\wedge|u|)\phi^+(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H} \\ & - C_1\int_{\mathbb{R}^*}(1\wedge|u|)\phi^-(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H} \end{aligned} \tag{4.1}$$

We denote another non-linear f-expectation as $\overline{\mathcal{E}}^{C,C_1} := \mathcal{E}_{\overline{f}_{C,C_1}}$ associated with the driver $\overline{f}_{C,C_1}$ defined as:

$$\begin{aligned} \overline{f}_{C,C_1}(t,\phi) := {} & -C\left|\phi(0)\sqrt{\lambda_t^B}\right| - |C|\int_{\mathbb{R}^*}(1\wedge|u|)\phi^+(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H} \\ & + C_1\int_{\mathbb{R}^*}(1\wedge|u|)\phi^-(u)\nu(\mathrm{d}u)\sqrt{\lambda_t^H} \end{aligned}$$

Remark 4.1.3. As mentioned in Remark 3.1.3, it is easy to see that any f-expectation with generator f independent of y is both additive and $\mathcal{E}^{C,C_1}$-dominated.

Given the definition of $\mathcal{E}^{C,C_1}$, we can deduce the following property.

Lemma 4.1.4. *For all constant* $C > 0,\ \xi \in L^2(\mathcal{G}_T)$,

$$\mathcal{E}^{C,C_1}[C\ \xi \mid\ \mathcal{G}_t] = C\ \mathcal{E}^{C,C_1}[\xi \mid\ \mathcal{G}_t].$$

For all constant $C < 0$,

$$\mathcal{E}^{C,C_1}[C\ \xi \mid\ \mathcal{G}_t] = -C\ \mathcal{E}^{C,C_1}[-\xi \mid\ \mathcal{G}_t].$$

❇ *Proof.* We have proved a similar result in Corollary 3.1.5. We prove this result for $C > 0$, the same argument can be applied to the situation where $C < 0$. By the definition of $\mathcal{E}^{C,C_1}[C\ \xi \mid\ \mathcal{G}_t]$, it is associated with the following BSDE:

$$\begin{aligned}\mathcal{E}^{C,C_1}[C\ \xi \mid\ \mathcal{G}_t] &= Y_t \\ &= C\ \xi + \int_t^T f_{C,C_1}(s, \lambda_s, \phi)\mathrm{d}s + \int_t^T \int_{\mathbb{R}^*} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u)\end{aligned}$$

Note that by $f_{C,C_1}(s, \lambda_s, \phi)$ by definition is independent of Y and linear in ϕ, it follows:

$$\begin{aligned}Y_t &= C\ \Big[\xi + \int_t^T \frac{f_{C,C_1}(s, \lambda_s, \phi)}{C}\mathrm{d}s + \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u)\Big] \\ &= C\ \Big[\xi + \int_t^T f_{C,C_1}(s, \lambda_s, \frac{\phi}{C})\mathrm{d}s + \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u)\Big],\end{aligned}$$

Now we denote

$$\overline{Y}_t = \xi + \int_t^T f_{C,C_1}(s, \lambda_s, \frac{\phi}{C})\mathrm{d}s + \int_t^T \int_{\mathbb{R}^*} \frac{\phi_s(u)}{C}\mu(\mathrm{d}s, \mathrm{d}u),$$

and we note this BSDE has unique solution, thanks to the construction of f_{C,C_1}. Thus we conclude, by the definition of $\mathcal{E}^{C,C_1}[\xi \mid\ \mathcal{G}_t]$ coincides with $\overline{Y}$:

$$\mathcal{E}^{C,C_1}[C\ \xi \mid\ \mathcal{G}_t] = Y_t = C\ \overline{Y}_t = C\ \mathcal{E}^{C,C_1}[\xi \mid\ \mathcal{G}_t].$$

And the result is established. □

We can also express the $\mathcal{E}^{C,C_1}$-domination property in terms of conditional non-linear expectations, as stated in the following proposition.

Proposition 4.1.5. *If $\mathcal{E}$ is a filtration-consistent and $\mathcal{E}^{C,C_1}$-dominated and additive non-linear expectation, then for all $\xi, \xi' \in L^2(\mathcal{G}_T)$:*

$$\overline{\mathcal{E}}^{C,C_1}[\xi' \mid \mathcal{G}_t] \le \mathcal{E}[\xi + \xi' \mid \mathcal{G}_t] - \mathcal{E}[\xi \mid \mathcal{G}_t] \le \mathcal{E}^{C,C_1}[\xi' \mid \mathcal{G}_t].$$

Proof. By the construction of drivers f_{C,C_1} and $\overline{f}_{C,C_1}$, we note that:

$$\overline{\mathcal{E}}^{C,C_1}[\xi' \mid \mathcal{G}_t] = -\mathcal{E}^{C,C_1}[-\xi' \mid \mathcal{G}_t].$$

The result follows from Lemma 4.3, 4.4 and 4.5 in [Coq+02], and we skip the proof here. □

We give an estimation that is very useful in proving the following theorems.

Lemma 4.1.6. *For $\xi \in L^2(\mathcal{G}_T)$, we have*

$$\mathrm{E}\big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\big] \leq e^{(2(|C|+1)^2)(T-t)}\ \mathrm{E}[\xi^2].$$

❇ *Proof.* We write out the expression for $\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]$:

$$\mathcal{E}^\mu[\xi \mid \mathcal{G}_t] = \xi + \int_t^T f_{C,C_1}(t,\phi)\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s,\mathrm{d}u),$$

where f_{C,C_1} is defined in (4.1). Then we apply Itô's formula, see Theorem A.0.17, to $\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2$ and obtain

$$\begin{aligned}\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2 = \xi^2 &+ \int_t^T 2\mathcal{E}^\mu[\xi \mid \mathcal{G}_t] f_{C,C_1}\mathrm{d}s - \int_t^T 2\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\phi_s(0)\mathrm{d}B_s \\ &- \int_t^T \int_{\mathbb{R}^*} \Big(2\mathcal{E}^\mu[\xi \mid \mathcal{G}_{t-}]\phi_s(u) + \phi_s^2(u)\Big)\widetilde{H}(\mathrm{d}s,\mathrm{d}u) - \int_t^T \phi_s^2(0)\lambda_s^B \mathrm{d}s \\ &- \int_t^T \int_{\mathbb{R}^*} \phi_s^2(u)\nu(\mathrm{d}u)\lambda_s^H \mathrm{d}s.\end{aligned}$$

Then we take expectation on both sides,

$$\begin{aligned}\mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big] &= \mathrm{E}[\xi^2] + \int_t^T 2\mathrm{E}[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ f_{C,C_1}]\mathrm{d}s - \int_t^T \mathrm{E}[\phi_s^2(0)\lambda_s^B]\mathrm{d}s \\ &\quad - \int_t^T \int_{\mathbb{R}^*} \mathrm{E}[\phi_s^2(u)\lambda_s^H]\nu(\mathrm{d}u)\mathrm{d}s \\ &= \mathrm{E}[\xi^2] + \int_t^T 2\mathrm{E}\left[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ C\left|\phi_s(0)\sqrt{\lambda_t^B}\right|\right]\mathrm{d}s \\ &\quad + \int_t^T \int_{\mathbb{R}^*} 2\mathrm{E}\Bigg[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ (1 \wedge |u|)\Big(\ |C|\,\phi^+(u) \\ &\quad - C_1\phi^-(u)\Big)\nu(\mathrm{d}u)\sqrt{\lambda_t^H}\Bigg]\mathrm{d}s - \int_t^T \mathrm{E}[\phi_s^2(0)\lambda_s^B]\mathrm{d}s \\ &\quad - \int_t^T \int_{\mathbb{R}^*} \mathrm{E}\Big[\phi_s^2(u)\lambda_s^H\Big]\nu(\mathrm{d}u)\mathrm{d}s.\end{aligned} \tag{4.2}$$

Here we note by the inequality $2ab \leq a^2 + b^2$, and by the linearity of ordinary expectation,

$$\begin{aligned}\int_t^T 2\mathrm{E}\left[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ C\left|\phi(0)\sqrt{\lambda_t^B}\right|\right]\mathrm{d}s &\leq C^2\int_t^T \mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big]\mathrm{d}s \\ &\quad + \int_t^T \mathrm{E}\Big[\phi_s^2(0)\lambda_t^B\Big]\mathrm{d}s,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^*} 2\mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ (1 \wedge |u|)\Big(\,|C|\,\phi^+(u) - C_1\phi^-(u)\Big)\nu(\mathrm{d}u)\sqrt{\lambda_t^H}\Big]\mathrm{d}s \\
&\le \int_t^T \int_{\mathbb{R}^*} 2\mathrm{E}\Big[(1 \wedge |u|)\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]\ \Big(|C|+1\Big)\,|\phi(u)|\,\nu(\mathrm{d}u)\sqrt{\lambda_t^H}\Big]\mathrm{d}s \qquad (4.4) \\
&\le \Big(|C|+1\Big)^2 \int_t^T \mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big]\mathrm{d}s + \int_t^T \int_{\mathbb{R}^*} \mathrm{E}\Big[\phi_s^2(u)\lambda_s^H\Big]\nu(\mathrm{d}u)\mathrm{d}s
\end{aligned}$$

Now we insert (4.3) and (4.4) into (4.2), cancel the last four terms, and we end up with:

$$\begin{aligned}
\mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big] &= \mathrm{E}[\xi^2] + C^2 \int_t^T \mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big]\mathrm{d}s \\
&\quad + \Big(|C|+1\Big)^2 \int_t^T \mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big]\mathrm{d}s \\
&\le \mathrm{E}[\xi^2] + 2\Big(|C|+1\Big)^2 \int_t^T \mathrm{E}\Big[\mathcal{E}^\mu[\xi \mid \mathcal{G}_t]^2\Big]\mathrm{d}s \\
&\le e^{(2(|C|+1)^2)(T-t)}\ \mathrm{E}[\xi^2].
\end{aligned}$$

In the final step we used Grönwall inequality, and we have thus obtained the result for general non-linear expectation $\mathcal{E}$. □

Combining Proposition 4.1.5 and the above lemma, we can obtain the following result.

Corollary 4.1.7. *Let $\mathcal{E}$ be a filtration-consistent and $\mathcal{E}^{C,C_1}$-dominated and additive non-linear expectation, and $\xi, \xi' \in L^2(\mathcal{G}_T)$. Then we have*

$$\mathrm{E}\big[\big(\mathcal{E}[\xi \mid \mathcal{G}_t] - \mathcal{E}[\xi' \mid \mathcal{G}_t]\big)^2\big] \le e^{(2(|C|+1)^2)(T-t)}\ \mathrm{E}\Big[\,|\xi - \xi'|^2\,\Big].$$

Proof. From Proposition 4.1.5, we have:

$$\begin{aligned}
|\mathcal{E}[\xi \mid \mathcal{G}_t] - \mathcal{E}[\xi' \mid \mathcal{G}_t]| &\le \big|\mathcal{E}^{C,C_1}[\xi - \xi' \mid \mathcal{G}_t]\big| \vee \Big|\overline{\mathcal{E}}^{C,C_1}[\xi - \xi' \mid \mathcal{G}_t]\Big| \\
&\le \mathcal{E}^{C,C_1}[|\xi - \xi'| \mid \mathcal{G}_t].
\end{aligned}$$

This is because

$$\overline{\mathcal{E}}^{C,C_1}[\xi' \mid \mathcal{G}_t] = -\mathcal{E}^{C,C_1}[-\xi' \mid \mathcal{G}_t].$$

Then we apply Lemma 4.1.6 and obtain the result as we want. □

Note the above corollary provides us with a nice bound for convergence. This is useful in the following theorem, where we see that a general $\mathcal{E}$-supermartingale, under mild conditions, admits a càdlàg modification, although we will skip details of the proof. By the following theorem, we consider $\mathcal{E}$-supermartingales as their càdlàg modifications for the rest of this section.

Theorem 4.1.8. *Let $(M_t)_{t\in[0,T]}$ be an $\mathcal{E}$-supermartingale such that*

$$\mathrm{E}\Big[\sup_{0\le t\le T} |M_t|^2\Big] < \infty,$$

then it admits a càdlàg modification.

Proof. We adapt to our current, time-changed setting the argument used in Theorem 3.12 in [Roy06] in the classical setting. Note here we need to use Corollary 4.1.7 when there is need to show convergence of conditional non-linear expectations in the form $\mathcal{E}[\xi \mid \mathcal{G}_t] \to \mathcal{E}[\xi' \mid \mathcal{G}_t]$ in $L^2([0,T] \times \Omega, \mathcal{B}_{[0,T]} \times \widetilde{\mathcal{F}}, m \times P)$. □

We will also need the following property for filtration-consistent non-linear expectations in our proof of the main theorem.

Lemma 4.1.9. *Let $\mathcal{E}$ be a filtration-consistent non-linear expectation, and $\xi \in L^2(\mathcal{G}_T)$. Then we have almost surely for all $A \in \mathcal{G}_T$,*

$$\mathcal{E}[\xi \ \mathbb{1}_A \mid \mathcal{G}_t] = \mathcal{E}[\xi \mid \mathcal{G}_t]\mathbb{1}_A.$$

Proof. The result is rather straightforward. Consider for all $B \in \mathcal{G}_T$, by the filtration-consistent property, it follows:

$$\begin{aligned}
\mathcal{E}\Big[\mathcal{E}[\xi \ \mathbb{1}_A \mid \mathcal{G}_t]\mathbb{1}_B\Big] &= \mathcal{E}[\xi\mathbb{1}_A\mathbb{1}_B] \\
&= \mathcal{E}\Big[\mathcal{E}[\xi \mid \mathcal{G}_t]\mathbb{1}_{A\cap B}\Big] \\
&= \mathcal{E}\Big[[\mathcal{E}[\xi \mid \mathcal{G}_t]\mathbb{1}_A]\mathbb{1}_B\Big].
\end{aligned}$$

□

4.2 Decomposition for $\mathcal{E}_f$-supermartingales

This sections establishes the Doob-Meyer Decomposition for $\mathcal{E}_f$-supermartingales. This is the first step towards proving the Inverse Theorem. Proof of the decomposition under this framework is difficult, since we can make use of the linearity of classical expectations. To our best knowlegde, Peng in [Pen99] is the first to prove this result under the Brownian motion framework, and his argument gets further generalised in the discontinuous setting by Royer in [Roy06].

Proof of this result under our new, time-changed setting requires adapting the classical arguments from [Pen99] and [Roy06] to our current setting, and we manage to do so, thanks to the "nice" properties of our time-changed setting.

To establish the decomposition, we first need the following lemma to control jumps of càdlàg processes by choosing a sequence of stopping times.

Lemma 4.2.1. *Let $(A_t)_{t\in[0,T]}$ be an increasing predictable càdlàg process defined on $[0,T]$ with $A_0 = 0$ and $\mathcal{E}[A_T^2] < \infty$. Then for any δ, ϵ there exits a finite number of pairs of stopping times $\{\sigma_k, \tau_k\}$, $k = 0, 1, 2, \dots, N$ with $0 < \sigma_k \le \tau_k \le T$ such that*

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \varnothing$ *if* $j \neq k$;

(ii) $\mathrm{E}[\sum_{k=0}^N [\tau_k - \sigma_k](\omega)] \ge T - \epsilon$;

(iii) $\sum_{k=0}^N \mathrm{E}\Big[\sum_{\sigma_k<t\le\tau_k} (\Delta A_t)^2\Big] \le \delta$.

Here ΔA_t denotes the jump $A_t - A_{t^-}$.

Proof. Peng has proved a similar Lemma 2.3 in [Pen99] in the Brownian motion setting. In Peng's setting, an arbitrary stopping time is also predictable. We do not enjoy such convenience in our current setting, but given that A is a predictable process, we can apply Peng's arguments in our current setting and make use of the announcing sequence of stopping time to complete the proof in our setting.

We first construct a sequence f non-decreasing, predictable stopping times $\{\sigma_k\}_{k=0}^{N+1}$ with $\sigma_0 = 0$ and $\sigma_{N+1} = T$ such that $\sigma_k < \sigma_{k+1}$ and that

$$\sum_{k=0}^{N} \mathrm{E}\Big[\sum_{\sigma_k<t<\sigma_{k+1}} (\Delta A_t)^2\Big] \le \delta. \tag{4.5}$$

Intuitively, this corresponds to the fact that a major part of jumps by a predictable càdlàg process take place over a finite number of (random) time points. To see this result, we follow Peng's notation and denote for jump size $\nu > 0$,

$$A_t(\nu) = A_t - \sum_{s\le t} \Delta A_s \mathbb{1}_{\{\Delta A_s > \nu\}}.$$

We have thus removed all the jumps with size larger than ν. In this way, we can choose a $\nu > 0$ that is small enough such that

$$\mathrm{E}\Big[\sum_{s\le T} (\Delta A_s(\nu))^2\Big] \le \frac{\epsilon}{2}.$$

Now we fix a sequence of predictable stopping times $\{\tau_k\}_{k=1}^{\infty}$ where jumps of A have size bigger than ν, and there exists an N such that

$$\mathrm{E}\Big[\sum_{\tau_N<t<T}(\Delta A_s)^2\Big] \leq \frac{\epsilon}{2}$$

Then we denote $\sigma_k = \tau_k \wedge T$ for $k \leq N$ and $\sigma_0 = 0,\ \sigma_{N+1} = T$, and such a sequence of predictable stopping time $\{\sigma_k\}_{k=0}^{N+1}$ satisfies (4.5).

Note that we have thus obtained a sequence of open intervals in the form of (σ_k, σ_{k+1}), but this is not so convenient. To obtain the sequence of stopping times we wanted, we need to cut away a small portion of the open interval and thus keep the remaining part as $(\sigma_k, \tau_k]$. Peng in [Pen99] achieved this under the Brownian motion framework, and here we can achieve the same result, thanks to the fact that our process A_t is *predictable.* We can thus approach a predictable stopping time σ with an announcing sequence $(\sigma_i)_{i=1}$ such that $0 < \sigma_i < \sigma$ for all i and that $\sigma_i \uparrow \sigma$.

Given the sequence of stopping times $(\sigma_k)_{k=0}^{N+1}$ we have obtained in the first step, for each $0 \geq k \geq N$, we can find a stopping time τ'_k in the announcing sequance for σ_{k+1} such that

$$\mathrm{E}\Big[\sum_{k=0}^{N}(\sigma_{k+1} - \tau'_k)\Big] \leq \epsilon.$$

We set

$$\tau_0 = \tau'_0,\ \tau_1 = \sigma_1 \vee \tau'_1,\ \ldots,\ \tau_N = \sigma_N \vee \tau'_N$$

By this construction, we make sure that $\tau_k \in [\sigma_k, \sigma_{k+1}) \cap [\tau'_{k+1}, \sigma_{k+1}]$ and that $\tau_k < \sigma_{k+1}$ if $\sigma_k < T$.

Then we have

$$\mathrm{E}\Big[\sum_{k=0}^{N}(\sigma_{k+1} - \tau_k)\Big] \leq \epsilon.$$

By construction, $\sigma_{N+1} = T$, thus the above equation can be converted to

$$\mathrm{E}\Big[\sum_{k=0}^{N}(\tau_k - \sigma_k)\Big] \geq T - \epsilon,$$

and

$$\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k<t\leq\tau_k}(\Delta A_t)^2\Big] \leq \sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k<t<\sigma_{k+1}}(\Delta A_t)^2\Big] \leq \delta,$$

and the proof is accomplished. □

Now we are ready to prove the main result of this section.

Theorem 4.2.2 (Doob-Meyer Decomposition for $\mathcal{E}_f$-supermartingales)**.** *Suppose we have a driver f that satisfies assumptions in Proposition 3.1.1. Let $(Y_t)_{0\leq t\leq T} \in S^2_{\mathcal{G}}(0,T)$. If $(Y_t)_{0\leq t\leq T}$ is an $\mathcal{E}_f$-supermartingale, then there exist a process $(\phi_t)_{0\leq t\leq T} \in \mathcal{I}$ where $\phi_t := (\phi_t(0), \phi_t(u))$ for $u \in \mathbb{R}^*$ and an increasing* *càdlàg process $(A_t)_{0\leq t\leq T}$, predictable with respect to filtration $\mathbb{G}$, with $A_0 = 0$, and $\mathrm{E}[A_T^2] < \infty$ such that*

$$\begin{aligned}
Y_t &= Y_T + \int_t^T f(\lambda_s, Y_s, \phi_s)\mathrm{d}s + (A_T - A_t) - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u) \\
&= Y_T + \int_t^T f(\lambda_s, Y_s, \phi_s)\mathrm{d}s + (A_T - A_t) - \int_t^T \phi_s(0)\mathrm{d}B_s \\
&\quad - \int_t^T \int_{\mathbb{R}^*} \phi_s(u)\widetilde{H}(\mathrm{d}s, \mathrm{d}u).
\end{aligned}$$

Moreover, processes ϕ and A are unique in their respective spaces.

❇ *Proof.* This theorem corresponds to Theorem 4.1 in [Roy06] which is established in the classical setting, but here we need to prove the result under our time-changed framework.

We take three steps to prove the result. In the first step, we construct the so-called "penalised sequence" mentioned in [Pen99], in the following form:

$$Y_t^n = Y_T + \int_t^T f(\lambda_s, Y_s^n, \phi_s^n)\mathrm{d}s + n\int_t^T |Y_t - Y_t^n|\,\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u). \tag{4.6}$$

We denote $A_t^n := n\int_t^T |Y_t - Y_t^n|\,\mathrm{d}s$.

We then show that the sequence in (4.6) would converge to our $\mathcal{E}_f$-supermartingale process Y. In the meatime, the limit must be of the following form:

$$Y_t = Y_T + \int_t^T g(\lambda_s, Y_s, \phi_s)\mathrm{d}s + (A_T - A_t) - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u). \tag{4.7}$$

Here $g(\lambda_s, Y_s, \phi_s)$ is the weak limit for $(f(\lambda_s, Y_s^n, \phi_s^n))_{n\in\mathbb{N}}$ in $\mathcal{H}^2_{\mathcal{G}}(0,T)$, A_t the weak limit for A_t^n in $\mathcal{H}^2_{\mathcal{G}}(0,T)$ and ϕ the weak limit for $(\phi^n)_{n\in\mathbb{N}}$ in $\mathcal{I}$.

Step two is the key part, where we show that $(\phi^n)_{n\in\mathbb{N}}$ in (4.6) converges to ϕ in (4.7) in the strong sense in $L^p([0,T]\times\mathbb{R}\times\Omega, \mathcal{B}_X\times P, \Lambda\times P)$ for all $p\in[1,2)$. We obtain this result by applying the Itô's formula, see Theorem A.0.17, to achieve a convergence in measure, and given that ϕ_n are bounded in $\mathcal{I}$, we can have a strong convergence in $L^p([0,T]\times\mathbb{R}\times\Omega, \mathcal{B}_X\times P, \Lambda\times P)$ for all $p\in[1,2)$.

By the strong convergence in step two, we show that $(f(\lambda_s, Y_s^n, \phi_s^n))_{n\in\mathbb{N}}$ in (4.6) converge strongly to $f(\lambda_s, Y_s, \phi_s)$. By the uniqueness of weak limits, we can show that this $f(\lambda_s, Y_s, \phi_s)$ coincides with our earlier weak limit $(g(\lambda_s, Y_s, \phi_s))$ in (4.7). And this is our final step and the result is proven.

Step 1. In order to prove the theorem, we first consider the following family of BSDEs parametrised by $n = 1, 2, \ldots$:

$$Y_t^n = Y_T + \int_t^T f(\lambda_s, Y_s^n, \phi_s^n)\mathrm{d}s + n\int_t^T (Y_t - Y_t^n)^+\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

Here Y is the $\mathcal{E}_f$-supermartingale, with Y_T being its terminal condition, and f the driver given in the theorem. Thus there exit unique solutions for this sequence of BSDEs, which we denote by $(Y_t^n, \phi_t^n)_{0\leq t\leq T}$.

Note that we can write the driver for Y^n as $f_n(\lambda_s, Y_s, \phi_s) = f(\lambda_s, Y_s, \phi_s) + n(Y_t - Y)^+$. In this way, we can apply the Time-changed Comparison Theorem, see Theorem 2.3.7, and conclude that $(Y_t^n)_{n\in\mathbb{N}}$ is a non-decreasing sequence.

Given that Y_t is an $\mathcal{E}_f$-supermartingale, we can observe by this construction of driver $f_n(\lambda_s, Y_s, \phi_s) = f(\lambda_s, Y_s, \phi_s) + n(Y_t - Y)^+$ that, for all $n \in \mathbb{N}$ and for all $t \in [0, T]$, by the time-changed comparison theorem again, we have:

$$\mathcal{E}_f[Y_T \mid \mathcal{G}_t] \leq Y_t^n \leq Y_t. \tag{4.8}$$

In this way, we see that $(Y_t^n)_{n\in\mathbb{N}}$ and $(\phi_t^n)_{n\in\mathbb{N}}$ are bounded in respective spaces.

By (4.8), we can denote $A_t^n = n\int_0^t |Y_s - Y_s^n|^+ \,\mathrm{d}s = n\int_0^t |Y_s - Y_s^n| \,\mathrm{d}s = n\int_0^t (Y_s - Y_s^n)\mathrm{d}s$. We have thus constructed a “penalised sequence” expressed in (4.6).

We note for a fixed $n \in \mathbb{N}$, A_t^n is a non-decreasing process with respect to t. We denote by C the common bound for all sequences, which changes it values and potentially depends on T, and we obtain:

$$\begin{aligned}
\mathrm{E}\Big[\,|A_T^n|^2\,\Big] &= n^2\mathrm{E}\Big[\Big(\int_0^T |Y_s - Y_s^n| \,\mathrm{d}s\Big)^2\Big] \\
&\leq n^2\mathrm{E}\Big[\Big(\int_0^T |Y_s - Y_s^n|^2 \,\mathrm{d}s\Big)\Big] \\
&\leq C.
\end{aligned}$$

This shows that $(Y_t^n)_{n\in\mathbb{N}}$ converges to Y_t almost surely as $n \to \infty$. We can also derive the convergence in $S^2_{\mathcal{G}}(0, T)$ by applying dominated convergence theorem. In addition, we can obtain that $\mathrm{E}[\int_0^T |f(\lambda_s, Y_s^n, \phi_s^n)|^2 \,\mathrm{d}s] \leq C$, and here C is the (new) common bound.

Note here that we have obtained the boundedness of $(f(\lambda_s, Y_s^n, \phi_s^n))_{n\in\mathbb{N}}$, $(Y_t^n)_{n\in\mathbb{N}}$ and $(\phi_t^n)_{n\in\mathbb{N}}$ in respective spaces based on the following assumptions on f:

$$\begin{aligned}
&|f(\lambda_s, Y_s^n, \phi_s^n)| \leq K_1\|\phi_s^n(0)\| + K_2\Big(\int_{\mathbb{R}^*} |\phi_s^n(u)|^2 \lambda_s^H \nu(\mathrm{d}s)\Big)^{\frac{1}{2}} \\
&f(\lambda_s, Y_s^n, (0,0)) = 0
\end{aligned} \tag{4.9}$$

Here it is sufficient that constants K_1 and K_2 do not depend on n. We will make use of these assumptions when we prove Theorem 4.3.3.

We can thus find subsequences of $(\phi_t)_{n\in\mathbb{N}}$ and $f_s^n \coloneqq (f(\lambda_s, Y_s^n, \phi_s^n))_{n\in\mathbb{N}}$ that are weakly convergent to ϕ_t and g_t in respective spaces. Then for each stopping time $\tau \in [0, T]$, the weak convergence holds in their respective spaces:

$$\int_0^\tau f_s^n \mathrm{d}s \to \int_0^\tau g_s \mathrm{d}s, \quad \int_0^\tau \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u) \to \int_0^\tau \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

We can rewrite the BSDE into a forward SDE:

$$A_\tau^n = Y_0^n - Y_\tau - \int_0^\tau f_s^n \mathrm{d}s + \int_0^\tau \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u),$$

and we denote the weak limit for A^n as A. Since A is equal to its predictable projection, it is predictable. Thus we have:

$$A^n_\tau \to A_\tau := Y_0 - Y_\tau - \int_0^\tau g_s \mathrm{d}s + \int_0^\tau \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u),$$

and since A and its predictable projection coincide in any stopping time, they are indistinguishable. And we have thus established (4.7).

Step 2. Our focus now is to prove $(\phi^n_t)_{n\in\mathbb{N}}$ converges to ϕ_t in the strong sense, and by this result, we would be able to show that f^n_s converges strongly to f_t in $\mathcal{H}^p_{\mathcal{G}}(0,T))$ for all $p \in [1,2)$. This convergence is particularly important, given that we are now dealing with non-linear expectations.

We apply Itô's formula, see Theorem A.0.17, to $(Y^n_t - Y_t)^2$ on a given subinterval $(\sigma, \tau]$, where $0 \le \sigma \le \tau \le T$ and $\sigma,\ \tau$ are two stopping times.

$$\begin{aligned}
&\int_\sigma^\tau |\phi^n_s(0) - \phi_s(0)|^2 \lambda^B_s \mathrm{d}s + \int_\sigma^\tau \int_{\mathbb{R}^*} |\phi^n_s(u) - \phi_s(u)|^2 \lambda^H_s \nu(\mathrm{d}u)\mathrm{d}s = \\
&\Big(Y^n_\tau - Y_\tau\Big)^2 - \Big(Y^n_\sigma - Y_\sigma\Big)^2 + 2\int_\sigma^\tau (f^n_s - g_s)(Y^n_s - Y_s)\mathrm{d}s + 2\int_\sigma^\tau (Y^n_s - Y_s)\mathrm{d}A^n_s \\
&\quad -2\int_\sigma^\tau (Y^n_s - Y_s)\mathrm{d}A_s - 2\int_\sigma^\tau (Y^n_s - Y_s)\mathrm{d}B_s \\
&\quad - \int_\sigma^\tau \int_{\mathbb{R}^*} \Big[|\phi^n_s(u) - \phi_s(u)|^2 + 2(Y^n_{s^-} - Y_{s^-})(\phi^n_s(u) - \phi_s(u))\Big] \widetilde{H}(\mathrm{d}s, \mathrm{d}u)
\end{aligned}$$

We take expectation on both sides, and since $Y^n_t - Y_t \le 0$, we can thus derive the following inequality:

$$\begin{aligned}
&\mathrm{E}\Big[\int_\sigma^\tau |\phi^n_s(0) - \phi_s(0)|^2 \lambda^B_s \mathrm{d}s + \int_\sigma^\tau \int_{\mathbb{R}^*} |\phi^n_s(u) - \phi_s(u)|^2 \lambda^H_s \nu(\mathrm{d}u)\mathrm{d}s\Big] \\
&\le \mathrm{E}\Big[|Y^n_\tau - Y_\tau|^2\Big] + 2\mathrm{E}\Big[\int_\sigma^\tau |f^n_s - g_s|\, |Y^n_s - Y_s|\,\mathrm{d}s\Big] \\
&\quad + 2\mathrm{E}\Big[\int_\sigma^\tau (Y^n_s - Y_s)\mathrm{d}A_s\Big] \\
&= \mathrm{E}\Big[|Y^n_\tau - Y_\tau|^2\Big] + 2\mathrm{E}\Big[\int_\sigma^\tau |f^n_s - g_s|\, |Y^n_s - Y_s|\,\mathrm{d}s\Big] \\
&\quad + 2\mathrm{E}\Big[\int_\sigma^\tau \Delta(Y^n_s - Y_s)\mathrm{d}A_s\Big] \\
&\quad + 2\mathrm{E}\Big[\int_\sigma^\tau |Y^n_{s^-} - Y_{s^-}|\,\mathrm{d}A_s\Big]
\end{aligned} \tag{4.10}$$

Given that A terms have jumps, we obtained the last two terms in (4.10) in the following way:

$$\begin{aligned}
\mathrm{E}\Big[\int_\sigma^\tau (Y^n_s - Y_s)\mathrm{d}A_s\Big] &= \mathrm{E}\Big[\int_\sigma^\tau [(Y^n_s - Y_s) - (Y^n_{s^-} - Y_{s^-}) + (Y^n_{s^-} - Y_{s^-})]\mathrm{d}A_s\Big] \\
&= \mathrm{E}\Big[\int_\sigma^\tau \Delta(Y^n_s - Y_s)\mathrm{d}A_s\Big] + \mathrm{E}\Big[\int_\sigma^\tau (Y^n_{s^-} - Y_{s^-})\mathrm{d}A_s\Big],
\end{aligned}$$

The first term on the right hand side of (4.10) goes to 0, and so does the second term. To see the latter, we apply Hölder's inequality and obtain:

$$\mathrm{E}\Big[\int_\sigma^\tau |f_s^n - g_s|\,|Y_s^n - Y_s|\,\mathrm{d}s\Big] \le C\,\mathrm{E}\Big[\Big(\int_\sigma^\tau (Y_s^n - Y_s)^2 \mathrm{d}s\Big)^{\frac{1}{2}}\Big] \to 0,$$

since we have that f_t^n and g_t are bounded by C.

The last term on the right hand side of (4.10) also goes to 0 almost surely. We note that since $(Y_t^n)_{n\in\mathbb{N}}$ is non-decreasing, for all $t \in [0,T]$,

$$\big|Y_{t^-}^1 - Y_{t^-}\big| \ge |Y_{t^-}^n - Y_{t^-}| \to 0.$$

Then by

$$\mathrm{E}\Big[\int_0^T \big|Y_{s^-}^1 - Y_{s^-}\big|\,\mathrm{d}A_s\Big] \le \Big[\mathrm{E}\Big[\sup_s (Y_{s^-}^1 - Y_{s^-})^2\Big]\Big]^{\frac{1}{2}}\Big[\mathrm{E}[A_T]^2\Big]^{\frac{1}{2}} < \infty,$$

we can apply the dominated convergence theorem and conclude that

$$\mathrm{E}\Big[\int_{(0,T]} |Y_{s^-}^n - Y_{s^-}|\,\mathrm{d}A_s\Big] \to 0.$$

The only remaining term on the right hand side of (4.10) is the jump term. But this is tricky as in our setting we have jumps both from the càdlàg process A as well as the (conditional) Poisson integral.

Now we recall a classical result. For an increasing predictable process A, we can decompose it as a sum of continuous and a purely discontinuous process: $A_t = A_t^c + A_t^d$. Given a càdlàg martingale N that is bounded in $L^2(\Omega, \mathcal{G}_T, P)$, then for any stopping time $\tau \in [0,T]$:

$$\mathrm{E}\Big[\int_0^\tau \Delta N_s \mathrm{d}A_s^c = 0\Big]. \tag{4.11}$$

And for any predictable stopping time $\tau \in [0,T]$:

$$\mathrm{E}\Big[\int_0^\tau \Delta N_s \mathrm{d}A_s^d\Big] = \mathrm{E}\Big[\sum_{0\le s\le\tau} \Delta N_s \Delta A_s^d\Big]. \tag{4.12}$$

Recall in Lemma 4.2.1, we have constructed a sequence of predictable stopping times $\{\sigma_k, \tau_k\}$, $k = 0,1,2,\ldots,N$ with $0 < \sigma_k \le \tau_k \le T$ such that

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \varnothing$ if $j \ne k$;

(ii) $\mathrm{E}[\sum_{k=0}^N [\tau_k - \sigma_k](\omega)] \ge T - \epsilon$;

(iii) $\sum_{k=0}^N \mathrm{E}\Big[\sum_{\sigma_k < t \le \tau_k} (\Delta A_t)^2\Big] = \sum_{k=0}^N \mathrm{E}\Big[\sum_{\sigma_k < t \le \tau_k} (\Delta A_t^d)^2\Big] \le \delta.$

Now we denote $N_t := \int_0^t \int_{\mathbb{R}^*} |\phi_s^n(u) - \phi_s(u)|\,\widetilde{H}(\mathrm{d}s, \mathrm{d}u)$ and as mentioned earlier, N_t is bounded in $\mathcal{H}_{\mathcal{E}}^2(0,T,\nu)$. Since $\phi_t^n(u)$ and $\phi_t(u)$ are bounded in $\mathcal{I}$ by C, by applying the inequality $(a+b)^2 \le 2(a^2+b^2)$, we can obtain the following bound:

$$\mathrm{E}\Big[\sup_{0\le t\le T} |N_t|^2\Big] \le 4C. \tag{4.13}$$

Another observation we can make is that, jumps in Y_t come from both the A_t and $\int_{\mathbb{R}^*}\phi_t(u)\widetilde{H}(\mathrm{d}t,\mathrm{d}u)$ terms, whereas in Y_t^n, A_t^n is continuous and jumps are caused by the $\int_{\mathbb{R}^*}\phi_t^n(u)\widetilde{H}(\mathrm{d}t,\mathrm{d}u)$ term only. It follows:

$$\Delta(Y_t^n - Y_t) = \Delta A_t + \Delta N_t$$

Thus, by a convenient choice of stopping times, $\{\sigma_k,\tau_k\}$, $k=0,1,2,\ldots,N$ with $0<\sigma_k\le\tau_k\le T$, which is guaranteed by Lemma 4.2.1, we can obtain:

$$\begin{aligned}
2\mathrm{E}\Big[\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}\Delta(Y_s^n-Y_s)\mathrm{d}A_s\Big] &= 2\mathrm{E}\Big[\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}(\Delta A_s+\Delta N_s)\mathrm{d}A_s\Big]\\
&= 2\mathrm{E}\Big[\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}(\Delta A_s+\Delta N_s)\mathrm{d}A_s^d\Big] \quad (\text{by } (4.11))\\
&= 2\sum_{k=0}^{N}\mathrm{E}\Big[\int_{\sigma_k}^{\tau_k}\Delta A_s\mathrm{d}A_s^d\Big]+2\sum_{k=0}^{N}\mathrm{E}\Big[\int_{\sigma_k}^{\tau_k}\Delta N_s\mathrm{d}A_s^d\Big]\\
&= 2\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k\le s\le\tau_k}\Delta A_s\Delta A_s^d\Big]\\
&\quad+2\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k\le s\le\tau_k}\Delta N_s\Delta A_s^d\Big] \quad (\text{by } (4.12))\\
&\le 2\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k\le s\le\tau_k}(\Delta A_s^d)^2\Big]\\
&\quad+2\Bigg[\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k\le s\le\tau_k}(\Delta A_s^d)^2\Big]\Bigg]^{\frac{1}{2}}\\
&\quad\Bigg[\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k\le s\le\tau_k}(\Delta N_s)^2\Big]\Bigg]^{\frac{1}{2}}.
\end{aligned}$$

Here we applied Hölder's inequality to obtain the last inequality.

Now we apply property (iii) of these stopping times as the following, for $\epsilon,\delta>0$:

$$\sum_{k=0}^{N}\mathrm{E}\Big[\sum_{\sigma_k<t\le\tau_k}(\Delta A_t^d)^2\Big]\le\frac{\epsilon^2\delta^2}{64(C+1)}<1.$$

Combining with (4.13), it follows:

$$\begin{aligned}
2\mathrm{E}\Big[\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}\Delta(Y_s^n-Y_s)\mathrm{d}A_s\Big] &\le 2\Big[\frac{\epsilon^2\delta^2}{64(C+1)}+\Big(\frac{\epsilon^2\delta^2}{64(C+1)}\Big)^{\frac{1}{2}}2C^{\frac{1}{2}}\Big]\\
&\le 2\Big[\frac{\epsilon\delta}{4}+\frac{\epsilon\delta}{4}\Big]=\epsilon\delta,
\end{aligned}$$

and we have thus shown that

$$\mathrm{E}\Big[\int_{\sigma}^{\tau}|\phi_s^n(0)-\phi_s(0)|^2\,\lambda_s^B\mathrm{d}s+\int_{\sigma}^{\tau}\int_{\mathbb{R}^*}|\phi_s^n(u)-\phi_s(u)|^2\,\lambda_s^H\nu(\mathrm{d}u)\mathrm{d}s\Big]\le\epsilon\delta.$$

Note this result will provide us with a strong convergence in measure. To be more specific, we denote by m the Lebesgue measure on $[0,T]$, then we have

$$m \times P\Big\{(s,\omega) \in \bigcup_{k=0}^{N} (\sigma_k(\omega), \tau_k(\omega)] \times \Omega : |\phi_s^n(0) - \phi_s(0)|^2 \geq \delta\Big\} \leq \epsilon,$$
$$m \times P\Big\{(s,\omega,u) \in \bigcup_{k=0}^{N} (\sigma_k(\omega), \tau_k(\omega)] \times \Omega \times \mathbb{R}^* : \int_{\mathbb{R}^*} |\phi_s^n(u) - \phi_s(u)|^2 \lambda_s^H \nu(\mathrm{d}u) \geq \delta\Big\} \leq \epsilon$$

The fact that $\mathrm{E}[\sum_{k=0}^{N}[\tau_k - \sigma_k](\omega)] \geq T - \epsilon$ implies that for n big enough,

$$m \times P\Big\{(s,\omega) \in [0,T] \times \Omega : |\phi_s^n(0) - \phi_s(0)|^2 \geq \delta\Big\} \leq 2\epsilon,$$
$$m \times P\Big\{(s,\omega,u) \in [0,T] \times \Omega \times \mathbb{R}^* : \int_{\mathbb{R}^*} |\phi_s^n(u) - \phi_s(u)|^2 \lambda_s^H \nu(\mathrm{d}u) \geq \delta\Big\} \leq 2\epsilon$$

It follows that for $\delta > 0$,

$$\lim_{n\to\infty} m \times P\Big\{(s,\omega) \in [0,T] \times \Omega : |\phi_s^n(0,\omega) - \phi_s(0,\omega)|^2 \geq \delta\Big\} = 0,$$
$$\lim_{n\to\infty} m \times P\Big\{(s,\omega,u) \in [0,T] \times \Omega \times \mathbb{R}^* : \int_{\mathbb{R}^*} |\phi_s^n(u) - \phi_s(u)|^2 \lambda_s^H \nu(\mathrm{d}u) \geq \delta\Big\} = 0$$

We have thus established strong convergence in $L^p([0,T]\times\mathbb{R}\times\Omega, \mathcal{B}_X \times P, \Lambda \times P)$ for $p \in [1,2)$ where sequences ϕ_t^n are uniformly integrable.

Step 3. With the strong convergence established, we can find a strong convergence of $(f(\lambda_s, Y_s^n, \phi_s^n))_{n\in\mathbb{N}}$ to $f(\lambda_s, Y_s, \phi_s)$ in $\mathcal{H}_{\mathcal{G}}^p(0,T)$ for $p \in [1,2)$, by the Lipschitz conditions and Minkowski's inequality:

$$\begin{aligned}\|f(\lambda_s, Y_s^n, \phi_s^n) - f(\lambda_s, Y_s, \phi_s)\|_p &\leq C\|(Y_s^n - Y_s) + (\phi_s^n - \phi_s)\|_p \\ &\leq C\|Y_s^n - Y_s\|_p + \|\phi_s^n - \phi_s\|_p \\ &\to 0.\end{aligned}$$

By the uniqueness of weak limit, we have

$$\int_0^t g_s \mathrm{d}s = \int_0^t f_s \mathrm{d}s.$$

We can thus conclude that

$$Y_t = Y_T + \int_t^T f(\lambda_s, Y_s, \phi_s)\mathrm{d}s + A_T - A_t - \int_0^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u),$$

and the result is established. □

By applying the same proof, we can obtain a similar result for $\mathcal{E}_f$-submartingales, which is stated in the following theorem.

Theorem 4.2.3 (Doob-Meyer Decomposition for $\mathcal{E}_f$-submartingales)**.** *Suppose we have a driver f that satisfies assumptions in Proposition 3.1.1. Let $(Y_t)_{0\le t\le T} \in S^2_{\mathcal{G}}(0,T)$. If $(Y_t)_{0\le t\le T}$ is an $\mathcal{E}_f$-submartingale, then there exist a process $(\phi_t)_{0\le t\le T} \in \mathcal{I}$ where $\phi_t := (\phi_t(0), \phi_t(u))$ for $u \in \mathbb{R}^*$ and an increasing càdlàg process $(A_t)_{0\le t\le T}$, predictable with respect to filtration $\mathbb{G}$, with $A_0 = 0$, and $\mathrm{E}[A_T^2] < \infty$ such that*

$$\begin{aligned} Y_t &= Y_T + \int_t^T f(\lambda_s, Y_s, \phi_s)\mathrm{d}s + (A_t - A_T) - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u) \\ &= Y_T + \int_t^T f(\lambda_s, Y_s, \phi_s)\mathrm{d}s + (A_t - A_T) - \int_t^T \pi_s \mathrm{d}B_s - \int_t^T \int_{\mathbb{R}^*} l_s(u)\widetilde{H}(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

Moreover, processes ϕ and A are unique in their respective spaces.

Now we derive a corollary from the above two theorems, such that we can express a general $\mathcal{E}$-supermartingale (or $\mathcal{E}$-submartingale) as an $\overline{\mathcal{E}}^{C,C_1}$-martingale (or $\mathcal{E}^{C,C_1}$-martingale). We follow Corollary 4.3 from [Roy06].

Corollary 4.2.4. *Let $\mathcal{E}$ be a $\mathcal{E}^{C,C_1}$-dominated, additive and filtration-consistent non-linear expectation. Given process $(Y_t)_{t\in[0,T]} \in S^2_{\mathcal{G}}(0,T)$, and if it is a $\mathcal{E}$-supermartingale (or $\mathcal{E}$-submartingale), then there exits an increasing predictable process A (or A') such that $Y + A$ (or $Y - A'$) is an $\overline{\mathcal{E}}^{C,C_1}$-martingale (or $\mathcal{E}^{C,C_1}$-martingale).*

❇ *Proof.* We give a brief proof for the $\mathcal{E}$-supermartingale case. Same argument can be applied to the $\mathcal{E}$-submartingale situation.

Given $(Y_t)_{t\in[0,T]} \in S^2_{\mathcal{G}}(0,T)$ such that it is a $\mathcal{E}$-supermartingale. In the meantime by the assumptions, we have $\mathcal{E}$ is $\mathcal{E}^{C,C_1}$-dominated. By Proposition 4.1.5, we have

$$\overline{\mathcal{E}}^{C,C_1}[\xi \mid \mathcal{G}_t] \le \mathcal{E}[\xi \mid \mathcal{G}_t] \le Y_t.$$

This shows that Y_t is an $\overline{\mathcal{E}}^{C,C_1}$-supermartingale. Then we can apply Theorem 4.2.2 and obtain

$$Y_t = Y_T + \int_t^T \overline{f}_{C,C_1}(\lambda_s, \phi_s)\mathrm{d}s + (A_T - A_t) - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u),$$

then we see

$$Y_t + A_t = (Y_T + A_T) + \int_t^T \overline{f}_{C,C_1}(\lambda_s, \phi_s)\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

Since $\overline{f}_{C,C_1}$ is independent of Y, and by the uniqueness of solution, we see that $Y_t + A_t$ is an $\overline{\mathcal{E}}^{C,C_1}$-martingale, and the proof is complete. □

4.3 Decomposition for $\mathcal{E}$-supermartingales

In the previous section, we establish decomposition of $\mathcal{E}_f$-martingales with a given driver f. In order to prove the Inverse Theorem, we need to look at martingales with respect to a general non-linear expectation $\mathcal{E}$ without a given driver f.

This result is first established by authors in [Coq+02] in the Brownian motion setting, three years after Peng proved the result in the previous section in [Pen99], and later extended to the setting of BSDEs with jumps in [Roy06]. Similar with the previous section, we need to adapt the classical arguments in our time-changed setting.

As mentioned earlier, one of the most important arguments for this result is to express the non-linear expectation martingales as conditional non-linear expectations. This gives rise to new challenges, since it requires showing that such representation is unique, and there also needs to be a comparison theorem in the corresponding, conditional non-linear expectation form.

The first proposition, based on results from the previous section, enables us to construct BSDEs for general $\mathcal{E}$-martingales. It corresponds to Proposition 4.4 from [Roy06] in the classical setting.

Proposition 4.3.1. *Let $(Y_t)_{t\in[0,T]}, (\widehat{Y}_t)_{t\in[0,T]} \in S^2_{\mathcal{G}}(0,T)$ such that they are $\mathcal{E}$-martingales. Then there exits functions $g, \widehat{g}$ and processes $\phi, \widehat{\phi} \in \mathcal{I}$ such that*

$$Y_t = Y_T + \int_t^T g_s \mathrm{d}s - \int_t^T \phi_s(u)\mu(\mathrm{d}s,\mathrm{d}u),$$
$$\widehat{Y}_t = \widehat{Y}_T + \int_t^T \widehat{g}_s \mathrm{d}s - \int_t^T \widehat{\phi}_s(u)\mu(\mathrm{d}s,\mathrm{d}u).$$

Moreover,

$$\overline{f}_{C,C_1}(\lambda_s, \phi_s - \widehat{\phi}_s) \le g - \widehat{g} \le f_{C,C_1}(\lambda_s, \phi_s - \widehat{\phi}_s).$$

Proof. Note here Y and $\widehat{Y}$ are their càdlàg modifications, as mentioned earlier. We apply Corollary 4.2.4 to Y, and obtain processes A and A'.

$$Y_t = Y_T + \int_t^T \overline{f}_{C,C_1}(\lambda_s,\phi_s)\mathrm{d}s + (A_T - A_t) - \int_t^T \int_{\mathbb{R}} \phi_s(u)\mu(\mathrm{d}s,\mathrm{d}u)$$
$$Y_t = Y_T + \int_t^T f_{C,C_1}(\lambda_s,\phi'_s)\mathrm{d}s + (A'_t - A'_T) - \int_t^T \int_{\mathbb{R}} \phi'_s(u)\mu(\mathrm{d}s,\mathrm{d}u)$$

By this result, we have:

$$\phi'_t = \phi_t,$$
$$f_{C,C_1}(\lambda_t,\phi'_t)\mathrm{d}t - \mathrm{d}A'_t = \overline{f}_{C,C_1}(\lambda_t,\phi_t)\mathrm{d}t + \mathrm{d}A_t$$

This shows us

$$\begin{aligned}\mathrm{d}A'_t + \mathrm{d}A_t &= (f_{C,C_1}(\lambda_t,\phi_t) - \overline{f}_{C,C_1}(\lambda_t,\phi_t))\mathrm{d}t \\ &= \Big[2C\left|\phi(0)\sqrt{\lambda^B_t}\right| + (|C| - C_1)\int_{\mathbb{R}^*}(1\wedge|u|)\phi^+(u)\nu(\mathrm{d}u)\sqrt{\lambda^H_t} \\ &+ (|C| - C_1)\int_{\mathbb{R}^*}(1\wedge|u|)\phi^-(u)\nu(\mathrm{d}u)\sqrt{\lambda^H_t}\Big]\mathrm{d}t\end{aligned}$$

By this we see A and A' are both càdlàg, and we write that $\mathrm{d}A_t = a_t \mathrm{d}t$ and $\mathrm{d}A'_t = a'_t \mathrm{d}t$. Given that

$$\int_{\mathbb{R}^*} (1 \wedge |u|)\phi_s(u)\nu(du)\sqrt{\lambda^H} = \int_{\mathbb{R}^*} (1 \wedge |u|)\phi_s^+(u)\nu(du)\sqrt{\lambda^H} - \int_{\mathbb{R}^*} (1 \wedge |u|)\phi_s^-(u)\nu(du)\sqrt{\lambda^H}$$

we can construct the driver g in the following way:

$$\begin{aligned} g_s &= \frac{|C| + C_1}{2} \int_{\mathbb{R}^*} (1 \wedge |u|)\phi_s(u)\nu(du)\sqrt{\lambda^H} + \frac{1}{2}(a_s + a'_s) - a'_s \\ &= \frac{|C| + C_1}{2} \int_{\mathbb{R}^*} (1 \wedge |u|)\phi_s(u)\nu(du)\sqrt{\lambda^H} + \frac{1}{2}(a_s - a'_s). \end{aligned}$$

Clearly this satisfies our requirements in the proposition. We apply the same construction to $\widehat{Y}$, and the result follows. □

One direct consequence of the previous proposition is that we can obtain càdlàg modifications for the general non-linear conditional expectation.

Corollary 4.3.2. *Let $\mathcal{E}$ be a $\mathcal{E}^{C,C_1}$-dominated, additive and filtration-consistent non-linear expectation. Then for any $\xi \in L^2(\mathcal{G}_T)$ and $h \in \mathcal{H}^2_{\mathcal{G}}(0,T)$, the process $\mathcal{E}\Big[\xi + \int_t^T h(s)\mathrm{d}s \mid \mathcal{G}_t\Big]$ admits a càdlàg modification.*

Proof. By additivity we obtain

$$\begin{aligned} \mathcal{E}\Big[\xi + \int_t^T h(s)\mathrm{d}s \mid \mathcal{G}_t\Big] &= \mathcal{E}\Big[\xi + \int_0^T h(s)\mathrm{d}s - \int_0^t h(s)\mathrm{d}s \mid \mathcal{G}_t\Big] \\ &= \mathcal{E}\Big[\xi + \int_0^T h(s)\mathrm{d}s \mid \mathcal{G}_t\Big] - \int_0^t h(s)\mathrm{d}s. \end{aligned}$$

Then we apply Proposition 4.3.1, and the proof is complete. □

The following theorem is the generalised version of Theorem 4.2.2 in the sense that we are now dealing with general $\mathcal{E}$-supermartingales with a given driver f, whereas in Theorem 4.2.2 the driver f is given.

Theorem 4.3.3 (Doob-Meyer Decomposition for $\mathcal{E}$-supermartingale)**.** *Let $\mathcal{E}$ be a $\mathcal{E}^{C,C_1}$-dominated, additive and filtration-consistent non-linear expectation, and $(Y_t)_{t\in[0,T]} \in S^2_{\mathcal{G}}(0,T)$ such that it is an $\mathcal{E}$-supermartingale. Then there exits an increasing cádlág predictable process $(A_t)_{t\in[0,T]}$ such that $A_0 = 0$ and $\mathrm{E}[A_T^2] \leq \infty$ and that $(Y_t + A_t)_{t\in[0,T]}$ is an $\mathcal{E}_f$-martingale, namely, $Y_t + A_t = \mathcal{E}_f[Y_T + A_T \mid \mathcal{G}_t]$.*

Proof of Theorem 4.3.3 requires a similar argument used in the proof of Theorem 4.2.2. But before we start working on the proof, we need the following two lemmata to fix some technical issues. They correspond to Lemma 6.1 and 6.2 from [Coq+02] in the classical setting and we adapt the classical arguments to our new, time-changed setting.

Lemma 4.3.4 (Existence of unique solution)**.** *Given a function $h : (\Omega \times [0,T] \times \mathbb{R}) \mapsto \mathbb{R}$ such that for a constant $K > 0$*

(i) $h(t,\omega,\lambda,y) \in \mathcal{H}^2_{\mathcal{G}}(0,T)$, *for all* $y \in \mathbb{R}$;

(ii) $|h(t,\lambda_t,y_1) - h(t,\lambda_t,y_2)| \leq K\,|y_1 - y_2|$ *for all* $y_1, y_2 \in \mathbb{R}$.

Then given a terminal condition $\xi \in L^2(\mathcal{G}_T)$ *the following type of equation*

$$\overline{Y_t} = \mathcal{E}\Big[\xi + \int_t^T h(s,\lambda_s,\overline{Y_s})\mathrm{d}s \mid \mathcal{G}_t\Big] \tag{4.14}$$

has a unique process $\{\overline{Y_t}\}_{t\in[0,T]}$ *solution in* $S^2_{\mathcal{G}}(0,T)$, *and it admits a càdlàg modification.*

❇ *Proof.* We prove this lemma by first showing the following $\mathbb{G}$-adapted mapping $\Phi(y(\omega))(t) : L^2([0,T]\times\Omega, \mathcal{B}_{[0,T]}\times\widetilde{\mathcal{F}}, m\times P) \mapsto L^2([0,T]\times\Omega,\mathcal{B}_{[0,T]}\times\widetilde{\mathcal{F}}, m\times P)$ is a contraction:

$$\Phi(y(\omega))(t) = \mathcal{E}\Big[\xi + \int_t^T h(s,\lambda_s,y_s)\mathrm{d}s \mid \mathcal{G}_t\Big].$$

We denote $\overline{Y}_1(t) \coloneqq \Phi(y_1(\omega))(t)$ and $\overline{Y}_2(t) \coloneqq \Phi(y_2(\omega))(t)$. Then by the properties of $\mathcal{E}^{C,C_1}$-domination and additivity, and basic property of BSDEs, we obtain:

$$\begin{aligned}
\big|\overline{Y}_1(t) - \overline{Y}_2(t)\big| &\leq \big|\mathcal{E}^{C,C_1}[\overline{Y}_1(t) - \overline{Y}_2(t) \mid \mathcal{G}_t]\big| \vee \Big|\overline{\mathcal{E}}^{C,C_1}[\overline{Y}_1(t) - \overline{Y}_2(t) \mid \mathcal{G}_t]\Big| \\
&\leq \mathcal{E}^{C,C_1}[\big|\overline{Y}_1(t) - \overline{Y}_2(t)\big| \mid \mathcal{G}_t] \\
&= \mathcal{E}^{C,C_1}\left[\left|\int_t^T h(s,\lambda_s,y_1(s)) - h(s,\lambda_s,y_2(s))\mathrm{d}s\right| \mid \mathcal{G}_t\right] \\
&\leq \mathcal{E}^{C,C_1}\left[\int_t^T |h(s,\lambda_s,y_1(s)) - h(s,\lambda_s,y_2(s))|\,\mathrm{d}s \mid \mathcal{G}_t\right] \\
&\leq \mathcal{E}^{C,C_1}\left[\int_t^T K\,|y_1 - y_2|\,\mathrm{d}s \mid \mathcal{G}_t\right] \\
&= K\mathcal{E}^{C,C_1}\left[\int_t^T |y_1 - y_2|\,\mathrm{d}s \mid \mathcal{G}_t\right]
\end{aligned}$$

Equality in the final step is a result of Lemma 4.1.4, where we used the properties of the BSDE associated with $\mathcal{E}^{C,C_1}$, since f_{C,C_1} is independent of y. Now we recall Corollary 4.1.7 and obtain

$$\begin{aligned}
\mathrm{E}\Big[\big|\overline{Y}_1(t) - \overline{Y}_2(t)\big|^2\Big] &\leq K^2\mathrm{E}\left[\mathcal{E}^{C,C_1}\Big[\int_t^T |y_1 - y_2|\,\mathrm{d}s \mid \mathcal{G}_t\Big]^2\right] \\
&\leq K^2 e^{(2(|C|+1)^2)(T-t)}\,\mathrm{E}\left[\Big(\int_t^T |y_1 - y_2|\,\mathrm{d}s\Big)^2\right] \qquad (4.15) \\
&\leq K^2 e^{(2(|C|+1)^2)T}(T-t)\,\mathrm{E}\left[\int_t^T |y_1 - y_2|^2\,\mathrm{d}s\right].
\end{aligned}$$

We used Hölder's inequality in the last step. This implies

$$\mathrm{E}\Big[\int_t^T \big|\overline{Y}_1(t)-\overline{Y}_2(t)\big|^2\,\mathrm{d}s\Big] \le K^2 e^{(2(|C|+1)^2)T}(T-t)^2\,\mathrm{E}\Big[\int_t^T |y_1-y_2|^2\,\mathrm{d}s\Big].$$

Now we pick a time interval $\eta > 0$ such that $K^2 e^{(2(|C|+1)^2)T}\eta^2 < 1$, then Φ on this interval $[T-\eta, T]$ is a contraction, and therefore there exits a fixed point such that it solves (4.14).

Once this is established, we just need to repeat this procedure over the whole $[0, T]$ time interval and find a solution. For $t \le T-\eta$, we can define the mapping Φ in the same way, and $\xi' := \xi + \int_{T-\eta}^T h(s, y_s)\mathrm{d}s$, then we have:

$$\begin{aligned}\Phi(y(\omega))(t) &= \mathcal{E}\Big[\Big(\xi + \int_{T-\eta}^T h(s,\lambda_s,y_s)\mathrm{d}s\Big) + \int_t^{T-\eta} h(s,\lambda_s,y_s)\mathrm{d}s \mid \mathcal{G}_t\Big]\\ &= \mathcal{E}\Big[\xi' + \int_t^{T-\eta} h(s,\lambda_s,y_s)\mathrm{d}s \mid \mathcal{G}_t\Big]\end{aligned}$$

By the same computation as before, we obtain:

$$\begin{aligned}\mathrm{E}\Big[\int_t^T \big|\overline{Y}_1(t)-\overline{Y}_2(t)\big|^2\,\mathrm{d}s\Big] \le & K^2 e^{(2(|C|+1)^2)T}(T-t-\eta)^2\\ & \mathrm{E}\Big[\int_t^{T-\eta} |y_1-y_2|^2\,\mathrm{d}s\Big].\end{aligned}$$

In this way, we can find a solution by iterating over the whole $[0, T]$ interval. Suppose Y_1 and Y_2 are two solutions we find in this way, then by (4.15), we have:

$$\mathrm{E}\Big[\,|Y_1(t)-Y_2(t)|^2\,\Big] \le K^2 e^{(2(|C|+1)^2)T}T\,\mathrm{E}\Big[\int_t^T |y_1-y_2|^2\,\mathrm{d}s\Big].$$

and this implies $Y_1 = Y_2$ and we have thus shown uniqueness of solution.

Finally, by applying Corollary 4.3.2, we conclude that the solution of (4.14) admits a càdlàg modification and the proof is complete. □

Now we establish a comparison theorem for the conditional $\mathcal{E}$-expectation expressed as in (4.14).

Lemma 4.3.5 (Comparison Thoerem)**.** *Let $\overline{Y}$ be the solution of (4.14) and let $\overline{Y}'$ be the solution of*

$$\overline{Y}'_t = \mathcal{E}\Big[\xi' + \int_t^T [h(s,\lambda_s,\overline{Y}'_s) + \zeta_s]\mathrm{d}s \mid \mathcal{G}_t\Big].$$

Here $\xi' \in L^2(\mathcal{G}_T)$ and $\zeta \in H^2_{\mathcal{G}}(0,T)$. If

$$\xi' \ge \xi, \quad \zeta_t \ge 0 \quad \mathrm{d}P \times \mathrm{d}t\text{-}a.s. \tag{4.16}$$

then we have

$$\overline{Y}'_t \ge \overline{Y}_t, \quad \mathrm{d}P \times \mathrm{d}t\text{-}a.s. \tag{4.17}$$

Moreover, (4.17) becomes equality if and only if (4.16) takes equalities.

❇ *Proof.* Proving this lemma requires adapting to the time-changed setting the classical argument used in Theorem 6.2 in [Coq+02]. First we assume $\zeta_t \equiv 0$, then we define, for each $\delta > 0$, the following stopping time τ_1^δ and its corresponding set A^δ:

$$\tau_1^\delta := \inf\{t \geq 0;\ \overline{Y}'_t \leq \overline{Y}_t - \delta\} \wedge T,$$
$$A^\delta := \{\tau_1^\delta < T\} \in \mathcal{G}_{\tau_1^\delta}$$

Then we see if for $\delta > 0$, $\tau_1^\delta = T$, then we can conclude that (4.17) holds. Now we can assume rather for some $\delta > 0$, $P(A^\delta) > 0$. Then we can define another stopping time τ_2,

$$\tau_2 := \inf\{t \geq \tau_1^\delta;\ \overline{Y}'_t \geq \overline{Y}\}.$$

Since $\overline{Y}'_T = \xi' \geq \xi = \overline{Y}_T$, we can conclude that $\tau_2 \leq T$, and that $\mathbb{1}_{A^\delta}\overline{Y}'_{\tau_2} = \mathbb{1}_{A^\delta}\overline{Y}_{\tau_2}$. Then for $t \in [\tau_1^\delta, \tau_2]$, by definition

$$\mathbb{1}_{A^\delta}\overline{Y}'_t = \mathcal{E}\Big[\mathbb{1}_{A^\delta}\overline{Y}_{\tau_2} + \int_t^{\tau_2} \mathbb{1}_{A^\delta} h(s, \lambda_s, \mathbb{1}_{A^\delta}\overline{Y}'_s)\mathrm{d}s \mid \mathcal{G}_t\Big]$$
$$\mathbb{1}_{A^\delta}\overline{Y}_t = \mathcal{E}\Big[\mathbb{1}_{A^\delta}\overline{Y}_{\tau_2} + \int_t^{\tau_2} \mathbb{1}_{A^\delta} h(s, \lambda_s, \mathbb{1}_{A^\delta}\overline{Y}_s)\mathrm{d}s \mid \mathcal{G}_t\Big]$$

By Lemma 4.3.4, solutions of the above two equations coincide. This means $\mathbb{1}_{A^\delta}\overline{Y}'_{\tau_1^\delta} = \mathbb{1}_{A^\delta}\overline{Y}_{\tau_1^\delta}$ a.s., and this is a contradiction to $P(A^\delta) > 0$.

Now we consider generally $\zeta_t \geq 0$. To do this, we first construct a sequence of $\overline{Y}^n_t$ for $n = \{1, 2, 3, \ldots\}$, $i = \{1, 2, 3, \ldots, n-1\}$ and $t \in \Big[\frac{iT}{n}, \frac{(i+1)T}{n}\Big)$, as solutions of

$$\overline{Y}^n_t = \mathcal{E}\left[\Big[\xi' + \int_{\frac{iT}{n}}^T \zeta_s \mathrm{d}s\Big] + \int_t^T h(s, \lambda_s, \overline{Y}^n_t)\mathrm{d}s \mid \mathcal{G}_t\right]$$
$$\overline{Y}^n_T = \xi'.$$

Then by the additivity, we can rewrite the above equation into

$$\overline{Y}^n_t = \mathcal{E}\left[\Big[\overline{Y}_{\frac{(i+1)T}{n}} + \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \zeta_s \mathrm{d}s\Big] + \int_t^{\frac{(i+1)T}{n}} h(s, \lambda_s, \overline{Y}^n_t)\mathrm{d}s \mid \mathcal{G}_t\right],$$

for $t \in \Big[\frac{iT}{n}, \frac{(i+1)T}{n}\Big)$.

Now we consider a small time interval $t \in [\frac{(n-1)T}{n}, T)$. Here $\overline{Y}^n_t$ can be written as (4.14) with the same function h but a different terminal condition:

$$\xi'' := \xi' + \int_{\frac{(n-1)T}{n}}^T \zeta_s \mathrm{d}s \geq \xi' \geq \xi.$$

Then by the proof in the first part, we can conclude that $\overline{Y}^n_t \geq \overline{Y}_t$ on the interval $t \in [\frac{(n-1)T}{n}, T)$. We note in particular, $\overline{Y}^n_{\frac{(n-1)T}{n}} \geq \overline{Y}_{\frac{(n-1)T}{n}}$ on the left point of the interval.

This result allows us to iterate over all $i = \{1, 2, 3, \dots, n-1\}$ as $t \in \left[\frac{iT}{n}, \frac{(i+1)T}{n}\right)$, and conclude that $\overline{Y}^n_t \geq \overline{Y}_t$ for all $t \in [0, T]$.

What remains to show is the fact that $\overline{Y}^n_t$ converges to $\overline{Y}'$. Once again, we look at an interval $t \in \left[\frac{iT}{n}, \frac{(i+1)T}{n}\right)$. With correspondingly big enough constants K_1 and the Lipschitz constant K from Lemma 4.3.4, we can follow the same procedure as in Lemma 4.3.4 and obtain the following result:

$$\begin{aligned}\mathrm{E}\left[\left|\overline{Y}^n_t - \overline{Y}'_t\right|^2\right] &\leq K_1 \,\mathrm{E}\left[\Big(\int_{\frac{iT}{n}}^t |\zeta_s|\,\mathrm{d}s + K\int_t^T |y_1 - y_2|\,\mathrm{d}s\Big)^2\right] \\ &\leq K_1 \,\mathrm{E}\left[2\Big(\int_{\frac{iT}{n}}^t |\zeta_s|\,\mathrm{d}s\Big)^2 + 2K^2(T-t)\int_t^T |y_1 - y_2|^2\,\mathrm{d}s\right]\end{aligned}$$

Now we apply Schwarz's inequality, and for $t \in [0, T)$ we have:

$$\mathrm{E}\left[\left|\overline{Y}^n_t - \overline{Y}'_t\right|^2\right] \leq 2K_1\frac{T}{n}\,\mathrm{E}\Big[\int_0^T |\zeta_s|^2\,\mathrm{d}s\Big] + 2K^2K_1T\,\mathrm{E}\Big[\int_t^T |y_1 - y_2|^2\,\mathrm{d}s\Big]$$

Then we can apply Grönwall's inequality and obtain the convergence. We can thus conclude that $\overline{Y}'_t \geq \overline{Y}_t$.

Finally, we need to check when (4.17) becomes equality, (4.16) take equalities. We can no longer follow the proof in [Coq+02] to prove this point, as we do not have continuity for $\overline{Y}$ and $\overline{Y}'$. We assume $\xi' \geq \xi$, $\zeta_t \geq 0$ and $\overline{Y}_0 = \overline{Y}'_0$. Then we can see:

$$\xi + \int_0^T h(s, \lambda_s, \overline{Y}_s)\mathrm{d}s \leq \xi' + \int_0^T [h(s, \lambda_s, \overline{Y}'_s) + \zeta_s]\mathrm{d}s$$

Yet in the meantime, given $\overline{Y}_0 = \overline{Y}'_0$, by definition we have

$$\mathcal{E}\Big[\xi + \int_0^T h(s, \lambda_s, \overline{Y}_s)\mathrm{d}s \mid \mathcal{G}_t\Big] = \overline{Y}_0 = \overline{Y}'_0 = \mathcal{E}\Big[\xi' + \int_0^T [h(s, \lambda_s, \overline{Y}'_s) + \zeta_s]\mathrm{d}s \mid \mathcal{G}_t\Big].$$

By the strict monotonicity of non-linear expectations, we conclude

$$\xi + \int_0^T h(s, \lambda_s, \overline{Y}_s)\mathrm{d}s = \xi' + \int_0^T [h(s, \lambda_s, \overline{Y}'_s) + \zeta_s]\mathrm{d}s.$$

We have thus shown that $\xi = \xi'$ and $\zeta_t = 0$ d$P \times$ dt-a.s., and the proof is complete. □

Now we are ready to prove Theorem 4.3.3.

❄ *Proof of Theorem 4.3.3.* Much of the hard work has been done in the proof of Theorem 4.2.2, and in this proof we will adopt a very similar approach, with the only difference being that we now express the processes as non-linear conditional expectations. We now give a sketch of proof by adapting to our time-changed setting the classical argument used in Theorem 4.5 in [Roy06].

Similar as in the proof of Theorem 4.2.2, for a given $\mathcal{E}$-supermartingale Y, we construct

$$Y_t^n := \mathcal{E}\Big[Y_T + n\int_t^T (Y_s - Y_s^n)^+ \mathrm{d}s \mid \mathcal{G}_t\Big],$$

and define $A_t^n := n\int_0^t (Y_s - Y_s^n)^+ \mathrm{d}s$. By the comparison theorem established in Lemma 4.3.5, we conclude in the similar way as in the proof of Theorem 4.2.2 that $(Y_t^n)_n$ is an increasing sequence and $Y_t^n \le Y_t$ for all $n \in \mathbb{N}$ and all $t \in [0,T]$. This implies that $(Y_t^n)_n$ converges almost surely to a certain limit, and what remains to show is that this limit can be expressed as a non-linear conditional expectation, and that this limit indeed coincides with our $\mathcal{E}$-supermartingale Y.

Because $Y_t^n \le Y_t$ for all $n \in \mathbb{N}$ and all $t \in [0,T]$, we can rewrite

$$A_t^n = n\int_0^t (Y_s - Y_s^n)^+ \mathrm{d}s = n\int_0^t (Y_s - Y_s^n)\mathrm{d}s = n\int_0^t |Y_s - Y_s^n|\,\mathrm{d}s.$$

For $0 \le t \le r \le T$, we apply Lemma 4.3.4 and find the unique solution for $Y_t^n := \mathcal{E}\Big[Y_r^n + n\int_t^r |Y_s - Y_s^n|\,\mathrm{d}s \mid \mathcal{G}_t\Big]$. By additivity of the non-linear expectation, we have

$$Y_t^n + A_t^n = \mathcal{E}\Big[Y_r^n + A_r^n \mid \mathcal{G}_t\Big],$$

and this satisfies the definition of an $\mathcal{E}$-martingale.

Then we apply Proposition 4.3.1 to $Y_t^n + A_t^n$ and construct a corresponding BSDE as the following

$$Y_t^n + A_t^n = Y_T + A_T^n + \int_t^T g_s^n(\lambda_s)\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

Here we note $\overline{f}_{C,C_1}(s, \lambda_s, \phi_s^n) \le g_s^n(\lambda_s) \le f_{C,C_1}(s, \lambda_s, \phi_s^n)$.

Now we manipulate some terms on both sides and obtain the following

$$Y_t^n = Y_T + \int_t^T \Big(g_s^n(\lambda_s) + n\,|Y_s - Y_s^n|\Big)\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \phi_s^n(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

The only difference here from the proof of Theorem 4.2.2 is that now the driver g_t^n also depends on n. But given that $\overline{f}_{C,C_1}(s, \lambda_s, \phi_s^n) \le g_s^n(\lambda_s) \le f_{C,C_1}(s, \lambda_s, \phi_s^n)$, this does not pose any problem, as we have discussed in (4.9) in the proof of Theorem 4.2.2. For a detailed discussion of this condition, we refer to theorems 4.1 and 4.5 in [Roy06].

We can thus proceed as in the proof of Theorem 4.2.2, and then conclude that Y^n converges almost surely to Y, and obtain an increasing, predictable càdlàg process A, such that A^n converges almost surely to A.

Finally, we need to show that the limit of Y^n can indeed be expressed as a non-linear conditional expectation. We note

$$\Big|Y_t^n - \mathcal{E}\Big[Y_T + A_T - A_t \mid \mathcal{G}_t\Big]\Big| = \Big|\mathcal{E}\Big[Y_T + A_T^n - A_t^n \mid \mathcal{G}_t\Big] - \mathcal{E}\Big[Y_T + A_T - A_t \mid \mathcal{G}_t\Big]\Big|$$

Then we apply Corollary 4.1.7 and note

$$\begin{aligned} &\mathrm{E}\bigg[\Big|\mathcal{E}\Big[Y_T + A_T^n - A_t^n \mid \mathcal{G}_t\Big] - \mathcal{E}\Big[Y_T + A_T - A_t \mid \mathcal{G}_t\Big]\Big|^2\bigg] \\ &\qquad \le e^{(2(|C|+1)^2)(T-t)}\,\mathrm{E}\Big[\,|(A_T^n - A_t^n) - (A_T - A_t)|^2\,\Big]. \end{aligned}$$

Therefore, by this convergence, there exists a subsequence of $(Y_t^n)_n$ we denote also by $(Y_t^n)_n$ that converges almost sure to $\mathcal{E}\Big[Y_T + A_T - A_t \mid \mathcal{G}_t\Big]$. Since Y^n converges almost surely to Y, by the uniqueness of limit, we conclude that Y_t coincides with this process and thus obtain that

$$Y_t = \mathcal{E}\Big[Y_T + A_T - A_t \mid \mathcal{G}_t\Big].$$

And the proof is complete. □

4.4 Proof of the Time-changed Inverse Theorem

With all the above results in place, we are ready to prove the Time-changed Inverse Theorem, see Theorem 4.0.1.

❇ *Proof of Theorem 4.0.1 .* We take three steps to prove the Time-changed Inverse Theorem. In the first step, we construct a function f from an $\mathcal{E}$-martingale, given that the non-linear expectation $\mathcal{E}$ is $\mathcal{E}^{C,C_1}$-dominated and has additivity property. We show that f is well-defined and bounded between $\overline{f}_{C,C_1}$ and f_{C,C_1}.

In step two, we check that the operator $\mathcal{E}_f$ derived from our constructed f is indeed a non-linear expectation. This would follow naturally from assumptions on the driver f.

Our last step is to show that $\mathcal{E}_f$ coincides with $\mathcal{E}$, and this completes our proof.

Step 1. Given that $\mathcal{E}$ is $\mathcal{E}^{C,C_1}$-dominated, we consider a deterministic $\phi_0 \in \Phi$, where Φ is defined in (ii) in Definition 0.2.6, and we consider a process in the following form:

$$Y_t^{\phi_0} = -tf_{C,C_1}(\lambda_t, \phi_0) + \int_0^t \int_{\mathbb{R}} \phi_0(u)\mu(\mathrm{d}s, \mathrm{d}u). \tag{4.18}$$

Here we note that $\phi_0 \in \mathcal{I}$, where $\mathcal{I}$ is defined in (i) in Definition 0.2.6. Then we conclude that, for a BSDE associated with $(f_{C,C_1}, Y_T^{\phi_0})$, there exists a unique solution. By the uniquess of solution for BSDEs, we have that

$$(Y_t, \phi_t) = (Y_t^{\phi_0}, \phi_0)$$

We note Y is an $\mathcal{E}^{C,C_1}$-martingale, and by the $\mathcal{E}^{C,C_1}$-domination property, it is a $\mathcal{E}$-supermartingale in $S^2_{\mathcal{G}}(0,T)$. Apply Theorem 4.3.3 to Y and then we can find an increasing predictable càdlàg process $(A_t)_{t\in[0,T]}$ such that $A_0 = 0, \mathrm{E}[A_T^2] < \infty$ and $(Y_t + A_t)_{t\in[0,T]}$ is an $\mathcal{E}$-martingale.

In the meantime, Proposition 4.3.1 allows us to construct from $\mathcal{E}$-martingale $(Y_t + A_t)_{t\in[0,T]}$ a function $f(\lambda_s, \phi_0)$ and process $\widehat{\phi}$ such that

$$Y_t + A_t = Y_T^{\phi_0} + A_T + \int_t^T f_s(\lambda_s, \phi_0)\mathrm{d}s - \int_t^T \int_{\mathbb{R}} \widehat{\phi}(u)\mu(\mathrm{d}s, \mathrm{d}u).$$

Here,

$$\overline{f}_{C,C_1}(\lambda_s, \widehat{\phi}_s) \leq f_s(\lambda_s, \phi_0) \leq f_{C,C_1}(\lambda_s, \widehat{\phi}_s),$$

and given (4.18), we conclude:

$$A_t = \int_0^t f_s(\lambda_s, \phi_0)\mathrm{d}s - tf_{C,C_1},$$
$$\widehat{\phi} = \phi_0,$$

and f is well-defined.

Step 2. We check that $\mathcal{E}_f$ is indeed a non-linear expectation, by Proposition 3.1.1. We conclude it is so, because:

(i) $f(t, \lambda, (0,0)) = 0$ as $\overline{f}_{C,C_1}(\lambda_s, \phi_0 \leq f(\lambda_s, \phi_0) \leq f_{C,C_1}(\lambda_s, \phi_0)$;

(ii) f satisfies all other conditions, thanks to Proposition 4.3.1.

We have thus shown that $\mathcal{E}_f$ is indeed a non-linear expectation and it is $\mathcal{E}^{C,C_1}$-dominated and additive, since f is independent of Y.

Step 3. Now we show that $\mathcal{E}_f$ coincides with $\mathcal{E}$. To see this result, we first claim that for processes $\widetilde{\phi} \in \mathcal{I}$, for all $r \leq t \in [0,T]$, we have:

$$\mathcal{E}\Big[- \int_r^t f_s(\lambda_s, \widetilde{\phi}_s) + \int_r^t \int_{\mathbb{R}} \widetilde{\phi}_s(u) \mu(\mathrm{d}s, \mathrm{d}u) \mid \mathcal{G}_r \Big] = 0. \tag{4.19}$$

From the previous argument and the construction of f, for all $r \leq t \in [0,T]$, we have the martingale property and can thus obtain:

$$\begin{aligned} \mathcal{E}\Big[- \int_r^t f_s(\lambda_s, \phi_{0,s}) + \int_r^t \int_{\mathbb{R}} \phi_{0,s} \mu(\mathrm{d}s, \mathrm{d}u) \mid \mathcal{G}_r \Big] &= \\ \mathcal{E}\Big[(Y_t^{\phi_0} + A_t) - (Y_r + A_r) \mid \mathcal{G}_r \Big] &= 0. \end{aligned} \tag{4.20}$$

Now let $\{A_i\}_{i=1}^N$ be a $\mathcal{G}_r$-measurable partition of Ω, then we consider $\{\phi_i\}_{i=1}^N$ with ϕ_i defined in the same way as ϕ_0, by Lemma 4.1.9 and the fact that $f_s(\lambda_s, (0,0)) = 0$, it follows:

$$\begin{aligned} &\mathcal{E}\Big[- \int_r^t f_s(\lambda_s, \sum_{i=1}^N \phi_{i,s} \mathbb{1}_{A_i}) + \int_r^t \int_{\mathbb{R}} \sum_{i=1}^N \phi_{i,s} \mathbb{1}_{A_i} \mu(\mathrm{d}s, \mathrm{d}u) \mid \mathcal{G}_r \Big] = \\ &\mathcal{E}\Big[\sum_{i=1}^N \mathbb{1}_{A_i} \Big(- \int_r^t f_s(\lambda_s, \phi_{i,s}) + \int_r^t \int_{\mathbb{R}} \sum_{i=1}^N \phi_{i,s} \mu(\mathrm{d}s, \mathrm{d}u) \Big) \mid \mathcal{G}_r \Big] = \\ &\sum_{i=1}^N \mathbb{1}_{A_i} \mathcal{E}\Big[\Big(- \int_r^t f_s(\lambda_s, \phi_{i,s}) + \int_r^t \int_{\mathbb{R}} \sum_{i=1}^N \phi_{i,s} \mu(\mathrm{d}s, \mathrm{d}u) \Big) \mid \mathcal{G}_r \Big] = 0. \end{aligned}$$

We obtain the final equality by (4.20). In this way, we have shown that (4.19) holds for any simple function in $\mathcal{I}$. Now we apply Corollary 4.1.7, and the fact that f is Lipschitz with respect to $\phi \in \mathcal{I}$ gives us dominant convergence which enables us to establish (4.19).

Finally, we prove that $\mathcal{E}_f[\xi]$ coincides with $\mathcal{E}[\xi]$ for all $\xi \in L^2(\mathcal{G}_T)$. Now we consider the following BSDE:

$$\begin{aligned} -\mathrm{d}Y_t &= f_t(\lambda_t, \phi_t)\mathrm{d}t + \int_{\mathbb{R}^*} \phi_t(u) \mu(\mathrm{d}t, \mathrm{d}u) \\ Y_T &= \xi \end{aligned}$$

Here $\xi \in L^2(\mathcal{G}_T)$. By the definition of f-expectation, $\mathcal{E}_f[\xi] = Y_0$. On the other hand, by applying (4.19), we can obtain:

$$\begin{aligned} \mathcal{E}[\xi] &= \mathcal{E}\Big[Y_0 - \int_0^T f_s(\lambda_s, \phi_s)\mathrm{d}s + \int_0^T \int_{\mathbb{R}^*} \phi_s(u) \mu(\mathrm{d}s\mathrm{d}u) \Big] \\ &= Y_0 + \mathcal{E}\Big[- \int_0^T f_s(\lambda_s, \phi_s)\mathrm{d}s + \int_0^T \int_{\mathbb{R}^*} \phi_s(u) \mu(\mathrm{d}s\mathrm{d}u) \Big] \\ &= Y_0 = \mathcal{E}_f[\xi]. \end{aligned}$$

The proof is thus complete. □

Conclusion

This thesis sets out to establish the connection between dynamic risk measures and BSDEs with jumps driven by time-changed Lévy noises. As mentioned in Introduction, the natural link between them is the non-linear expectation.

In Chapter 1, this thesis reviews definitions and mathematical properties of both static and dynamic risk measures. Here we also recall basic definitions and theories of BSDEs with jumps. Several important results, such as the Existence and Uniqueness of solution for BSDE, see Theorem 1.2.1, and Comparison Theorem for BSDEs with Jumps, see Theorem 1.2.13, are also included in this chapter.

Chapter 1 establishes the link between dynamic risk measures and BSDEs with jumps under the classical framework. By Proposition 1.3.2, we are able to characterise a non-linear expectation via the associated BSDE. On the other hand, Proposition 1.3.9 shows that we can define a dynamic risk measure through a non-linear expectation. Theorem 1.3.11 gives a specific representation of dynamic risk measures associated with a BSDE that satisfies our assumptions. By this line of argument, given a BSDE, we are able to generate a corresponding dynamic risk measure.

To represent a given dynamic risk measure, under reasonable conditions, by a BSDE with jumps is considerably more difficult. In Chapter 1, we recall the Inverse Theorem from [Roy06], see Theorem 1.3.8, without giving a detailed proof. This is because this result is much more difficult than it appears to be. In order to prove this theorem, it requires establishing the Doob-Meyer Decomposition for non-linear expectation martingales. The classical argument for the decomposition is no longer relevant here, since it relies heavily on the linearity of ordinary expectations.

In this way, Chapter 1 provides us with all the key elements in this thesis that we can develop further in our time-changed setting in Chapter 3 and 4.

Chapter 2 starts by recalling basic theories of the Change of Time Method (CTMs) as well as two of the most widely studied time change processes, namely, subordinators and absolutely continuous time changes. According to a well-established result in the literature, see Theorem 2.2.12, a subordinated Lévy process remains a Lévy process. This makes our job of investigating properties of BSDEs driven by subordinated Lévy processes conceptually easier, because now we are back in the classical framework discussed in Chapter 1. In the meantime, given the original process, Proposition 2.2.13 enables us to compute the characteristic triplets for the subordinated Lévy process, under the condition that the original process and the subordinator are mutually independent.

Things are trickier with absolutely continuous time changes, as time-changed

Lévy processes in such cases may no longer stay Lévy processes. Theorem 2.2.19 shows that, given all the necessary assumptions, we can use absolutely continuous time change processes to construct processes with conditional stationary independent increments. Similar with subordinators, we would like to be able to figure out the characteristic triplets for the time-changed process based on those of the original process. Theorem 2.2.20 tells us that we can reduce this problem into finding the closed form of the Laplace transform of the time change process, provided that it exits.

In the Section 2.3, we review the time-changed framework proposed by authors in [DS14], where we define BSDEs with jumps driven by time-changed Lévy noises. Given the conditional stationary independent increments and absolute continuity, this time-changed framework has several "nice" properties. For example, Proposition 2.3.3 tells us that the signed measure μ constructed under this framework has the martingale property with respect to the filtration $\mathbb{G}$. Without the martingale property of μ, many of the results we try to establish in later chapters are not possible.

In general, Chapter 2 tries to approach CTMs in a more conceptual way, in the sense that there are more examples in this chapter rather than detailed proofs. The main reason for this is that many of the proofs for this chapter are quite technical and get quickly bogged down in large amount of detailed computations, something that sheds little light on the main topic of this thesis. More examples, on the other hand, could help us develop an intuition of the usefulness of CTMs, and gain a better understanding of the framework proposed in [DS14].

Chapter 3 and 4 constitute the core of this thesis, and it is also here we have put in most effort. In Chapter 3, we try to further develop the subject presented in Chapter 1, under the time-changed framework established in Section 2.3. Corresponding to their non-time-changed counterparts, Proposition 3.1.1 enables us to characterise a non-linear expectation via the associated time-changed BSDE, whereas Proposition 3.1.4 shows that we can indeed produce a dynamic risk measure by a non-linear expectation generated by a time-changed BSDE. In the meantime, Theorem 3.2.4 gives us a specific representation of dynamic risk measures. In this way, we can define a dynamic risk measure in accordance with the associated time-changed BSDE.

But Chapter 3 only tells half of the story. To establish a mutual connection, we need to be able to represent a given dynamic risk measure by time-changed BSDEs with jumps, under general enough conditions. We devote the entire Chapter 4 to proving this important Inverse Theorem, see Theorem 4.0.1.

This is an important result as much as it is a difficult one. We have to rely on ideas from [Pen99], [Coq+02] which proved this result in the Brownian motion setting, and [Roy06], which generalised it in the setting of BSDEs with jumps.

To prove this theorem, it requires first establishing the Doob-Meyer Decomposition for f-expectation martingales. This is difficult, because, as mentioned earlier, the classical argument is based on the linearity of ordinary expectation, something we no longer have with our non-linear f-expectation. To solve this problem, we adapt to the time-changed setting the method used in [Pen99], namely, to construct a so-called "penalised" sequence and push it up so hard that it finally converges to a supermartingale with respect to the f-expectation, see Theorem 4.2.2 for details.

The second step is to obtain the decomposition for a general non-linear expectation, and here we no longer have a given driver f. The key element in this step is to express the "penalised" sequence in the form of conditional non-linear expectations, and in order to show convergence, there needs to be a comparison theorem in the corresponding form, see Lemma 4.3.5 for details.

Results from the previous steps enables us to prove the time-changed Inverse Theorem in Section 4.4, and we have thus achieved a full connection between time-changed BSDEs and dynamic risk measures.

One thing in particular that we would like to point out is that, all the results we obtain in Chapter 3 and 4 are adapted to the filtration $\mathbb{G}$. This is a big, technical filtration that includes "anticipating-information", which is the entire history of the time-changed noises that we use to generate the BSDEs. In applications, we can still solve an optimal control problem with a classical performance functional, and this is achieved by projecting the results we obtain in filtration $\mathbb{G}$ onto filtration $\widetilde{\mathbb{F}}$, the smallest right-continuous filtration to which our random signed measure μ is adapted. For a detailed implementation of this idea, we refer to Section 6 in [DS14]. This can well serve as a motivation for further developments of this thesis in terms of applications.

Another direction for further studies is to investigate other types of time change processes and see if our results still hold. We have benefited immensely from the "nice" properties of our current framework, such as continuity of the time change process and martingale property, as mentioned earlier.

These are all ambitious and promising projects for future studies, but at the moment they are, regrettably, beyond the scope of the present thesis.

Appendix A

Elements of Stochastic Processes and Calculus

Definition A.0.1 (Progressively measurable processes)**.** Suppose $(\mathcal{F}_t)_{t\in[0,\infty)}$ is a filtration on the probability space $(\Omega, \mathcal{F})$, where X is a stochastic process with values in $(E, \mathcal{E})$. Then X is said to be *progressively measurable* if for every $t \in [0, \infty)$, the map $(s, \omega) \mapsto X_s(\omega)$ of $[0, t] \times \Omega$ into $(E, \mathcal{E})$ is measurable with respect to the product $\sigma-$algebra $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

Definition A.0.2 (Class (D))**.** A right-continuous uniformly integrable supermartingale X is said to be of class (D) if the set of random variables $(X_T)_{T\in\mathcal{T}}$ is uniformly integrable (where $\mathcal{T}$ is the set of all stopping times).

Definition A.0.3 (Potential)**.** A non-negative, right-continuous supermartingale X is called a *potential* if $\lim_{t\to\infty} \mathrm{E}[X_t] = 0$.

Definition A.0.4 ($\widetilde{\mathcal{H}}^p$*spaces*)**.** For M a martingale and $p \in [1, \infty)$, write

$$\|M\|_{\widetilde{\mathcal{H}}^p} := \|M^*_\infty\|_p = \mathrm{E}[\sup_t |M_t|^p]^{1/p}.$$

Here $\|\cdot\|_p$ denotes the norm in L^p. Then $\widetilde{\mathcal{H}}^p$ is the space of martingales such that

$$\|M\|_{\widetilde{\mathcal{H}}^p} < \infty.$$

Theorem A.0.5 (Doob-Meyer Decomposition: Class (D))**.** *Suppose Z is a potential , defined in Definition A.0.3, of class (D), defined in Definition A.0.2. Then there is a unique predictable integrable increasing càdlàg process $A \in \mathcal{A}_0^+$ such that Z is the potential generated by A. That is, upto indistinguishability,*

$$Z_t = \mathrm{E}[A_\infty \mid \mathcal{F}_t] - A_t.$$

Here $\mathcal{A}_0^+$ denotes the set of adapted (with respect to the relevant filtration), integrable and increasing processes that starts at 0 at $t = 0$.

Definition A.0.6 (Predictable quadratic variation)**.** For $M \in \widetilde{\mathcal{H}}^2$ where $\widetilde{\mathcal{H}}^2$ is defined in Definition A.0.4, we denote by $\langle M\rangle$ the *unique* predictable *increasing* *process* in $\mathcal{A}_0^+$ given by Theorem A.0.5 the Doob-Meyer decomposition of the class (D) potential X defined by

$$X_t = \mathrm{E}[M_\infty^2 \mid \mathcal{F}_t] - M_t^2.$$

The process $\langle M \rangle$ is called the *predictable quadratic variation* of M.

Definition A.0.7 (Compensated Poisson Process)**.** The process $\widetilde{N}$ defined by $\widetilde{N}_t = N_t - \lambda t$ is a martingale, and is called the *compensated Poisson Process.*

We call λ the *parameter* of the Poisson Process and λt the *compensator* of the increasing process N.

Definition A.0.8 (Semimartingale)**.** A process $X = \{X_t\}_{t\geq 0}$ is a *semimartingale* if it has a decomposition of the form

$$X = X_0 + M + A,$$

where M is a local martingale and A is a càdlàg adapted process of almost surely finite variation, and $M_0 = A_0 = 0$. Clearly, semimartingales are càdlàg and adapted.

Definition A.0.9 (Evanescent Sets)**.** Suppose A is a subset of $[0,\infty) \times \Omega$ and that $\mathbb{1}_A(t,\omega) = \mathbb{1}_A$ is the indicator function of A. Then A is said to be *evanescent* if $\mathbb{1}_A$ is indistinguishable from the zero process.

Definition A.0.10 (Stochastic Intervals)**.** Suppose S and T are maps $\Omega \mapsto [0,\infty]$ and $S \leq T$ a.s. The *(half open) stochastic interval* denoted by $[\![S,T[\![$ is the set

$$\{(t,\omega) \in [0,\infty) \times \Omega : S(\omega) \leq t < T(\omega)\}.$$

The stochastic intervals $[\![S,T]\!],]\!]S,T]\!],]\!]S,T[\![$ are defined similarly. The stochastic interval

$$[\![T,T]\!] = \{(t,\omega) \in [0,\infty) \times \Omega : T(\omega) = t\}$$

is denoted by $[\![T]\!]$, and is called the *graph* of T.

Definition A.0.11 (Optional and Predictable σ-algebras)**.** The *optional (respectively predictable)* σ-algebra Σ_o (respectively Σ_p) on $[0,\infty] \times \Omega$ is the σ-algebra generated by the evanescent sets, defined in Definition A.0.9, and all stochastic intervals, defined in Definition A.0.10, of the form $[\![T,\infty[\![$ for T an arbitrary (respectively predictable) stopping time.

Definition A.0.12 (Optional and Predictable Processes)**.** A stochastic process $\{X_t\}_{t\in[0,\infty)}$ defined on $(\Omega,\mathcal{F})$, with values in the measurable space $(E,\mathcal{E})$, is said to be *optional (respectively, predictable)* if the map $X : [0,\infty) \times \Omega \mapsto E$ is measurable with respect to the optional (respectively, predictable) σ-algebra, defined in Definition A.0.11.

Definition A.0.13 (Random measure)**.** Suppose we are working on a probability space $(\Omega,\mathcal{F},P)$ which has a complete, right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. We also have an auxiliary Blackwell Space $(\mathcal{Z},\mathfrak{Z})$. However, we do not require such generality as the applications we have in mind are when $\mathcal{Z} \subset \mathbb{R}^n$.

A *non-negative random measure* μ is an $\mathcal{F}$-measurable family $\{\mu(\omega,t)\}_{\omega\in\Omega}$ of σ-finite measures on $([0,\infty)\times\mathcal{Z}, \mathcal{B}([0,\infty)\otimes\mathfrak{Z})$. A function that can be written as the difference of two non-negative random measures is called a *random measure.*

Definition A.0.14 (Infinitely divisible characteristic function)**.** Let X be a random variable taking values in $\mathbb{R}^d$ with law μ_X. We say that X is *infinitely divisible* if, for all $n \in \mathbb{N}$, there exits identically and independently distributed random variables $Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)}, \ldots, Y_n^{(n)}$ such that

$$X = Y_1^{(n)} + Y_2^{(n)} + Y_3^{(n)} + \ldots + Y_n^{(n)} \qquad \text{in distribution.}$$

Let $\phi_x(u) = \mathrm{E}[e^{i(u,X)}]$ denote the characteristic function of X, where $u \in \mathbb{R}^d$. More generally, if $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ where $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$, then

$$\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu(\mathrm{d}y).$$

If $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is *infinitely divisible* if it has a convolution n-th root $\mu^{\frac{1}{n}} \in \mathcal{M}_1(\mathbb{R}^d)$ for each $n \in \mathbb{N}$ such that, for each $x \in \mathbb{R}^d$,

$$\phi_\mu(x) = [\phi_{\mu^{\frac{1}{n}}}(x)]^n.$$

Theorem A.0.15 (Lévy-Khinchine Formula)**.** *$\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is infinitely divisible, defined in Definition A.0.14, if there exists a vector $b \in \mathbb{R}^d$, a positive definite symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d/\{0\}$ such that, for all $u \in \mathbb{R}^d$,*

$$\phi_\mu(u) = \exp\left\{ i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d/\{0\}} \left[e^{i(u,y)} - 1 - i(u,y)\chi_{\hat{B}}(y) \right] \nu(\mathrm{d}y) \right\}, \tag{A.1}$$

where $\hat{B} = B_1(0)$, namely the ball with radius 1 and centred at 0.

Conversely, any mapping of the form (A.1) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.

Theorem A.0.16 (Doob Optional Sampling Theorem)**.** *Suppose X is a uniformly integrable or non-negative right-continuous supermartingale with respect to the filtation $\{\mathcal{F}_t\}_{t\in[0,\infty]}$. If S and T are two stopping times such that $S < T$ a.s., then the random variables X_S and X_T are integrable and $X_S \geq \mathrm{E}[X_T \mid \mathcal{F}_S]$ a.s.*

Theorem A.0.17 (One-Dimensional Itô Formula for Itô-Lévy Processes)**.** *Suppose $Y(t) \in \mathbb{R}$ is an Itô-Lévy Process of the form*

$$\mathrm{d}X(t) = \alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\bar{N}(\mathrm{d}t,\mathrm{d}z),$$

where,

$$\bar{N}(\mathrm{d}t,\mathrm{d}z) = \begin{cases} N(\mathrm{d}t,\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t, & \text{if} \quad |z| < R \\ N(\mathrm{d}t,\mathrm{d}z), & \text{if} \quad |z| \geq R \end{cases}$$

for some $R \in [0,\infty]$.

Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, X(t))$. Then $G(t)$ is again an Itô-Lévy Process and

$$\begin{aligned}
\mathrm{d}Y(t) = &\frac{\partial f}{\partial t}(t, X(t))\mathrm{d}t + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t)] \\
&+ \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t, X(t))\mathrm{d}t \\
&+ \int_{|z|<R} \Big\{ f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-)) \\
&\qquad - \frac{\partial f}{\partial x}(t, X(t^-))\gamma(t,z) \Big\} \nu(\mathrm{d}z)\mathrm{d}t \\
&+ \int_{\mathbb{R}} \{ f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-)) \} \bar{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

We recall this formula from Theorem 1.14 in [ØS07]. We note that if $R = 0$, then $\bar{N} = N$ everywhere. If $R = \infty$, then $\bar{N} = \widetilde{N}$, where $\widetilde{N}$ is defined in Deinfition A.0.7, everywhere.

Bibliography

[AP11] Beatrice Acciaio and Irina Penner. “Dynamic Risk Measures”. In: *Advanced Mathematical Methods for Finance*. Ed. by Giulia Di Nunno and Bernt Øksendal. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 1–34.

[App09] David Applebaum. *Levy Processes and Stochastic Calculus*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[Art+99] Philippe Artzner et al. “Coherent Measures of Risk”. In: *Mathematical Finance* 9.3 (1999), pp. 203–228. ISSN: 0960-1627.

[Bri+00] Philippe Briand et al. “A Converse Comparison Theorem for BSDEs and Related Properties of g-Expectation”. In: *Electron. Commun. Probab.* 5 (2000), pp. 101–117.

[BS10] Ole E Barndorff-Nielsen and Albert Shiryaev. *Change of Time and Change of Measure*. 2nd ed. WORLD SCIENTIFIC, 2010.

[CFS08] R. Carbone, B. Ferrario, and M. Santacroce. “Backward Stochastic Differential Equations Driven By Càdlàg Martingales”. In: *Theory of Probability & Its Applications* 52.2 (2008), pp. 304–314.

[Coh15] Samuel N Cohen. *Stochastic Calculus and Applications*. eng. 2nd ed. 2015. Probability and Its Applications. New York, NY: Birkhäuser Basel, 2015.

[Coq+02] François Coquet et al. “Filtration-consistent nonlinear expectations and related g-expectations”. In: *Probability Theory and Related Fields* 123.1 (May 2002), pp. 1–27.

[CW02] Peter Carr and Liuren Wu. “Time-Changed Levy Processes and Option Pricing”. In: *Journal of Financial Economics* 71 (Jan. 2002).

[DE10] Giulia Di Nunno and Inga Baadshaug Eide. “Minimal-Variance Hedging in Large Financial Markets: Random Fields Approach”. In: *Stochastic Analysis and Applications* 28.1 (2010), pp. 54–85.

[Del13] Łukasz Delong. *Backward Stochastic Differential Equations with Jumps and Their Actuarial and Financial Applications: BSDEs with Jumps*. eng. EAA Series. London: Springer London, 2013.

[Di 07] Giulia Di Nunno. “On orthogonal polynomials and the Malliavin derivative for Lévy stochastic measures”. In: *SMF Seminaires et Congres* 16 (2007), pp. 55–69.

[DS14] Giulia Di Nunno and Steffen Sjursen. "BSDEs driven by time-changed Lévy noises and optimal control". In: *Stochastic Processes and their Applications* 124.4 (2014), pp. 1679–1709.

[FR02] Marco Frittelli and Emanuela Rosazza Gianin. "Putting order in risk measures". In: *Journal of Banking & Finance* 26.7 (2002), pp. 1473–1486.

[FR04] Marco Frittelli and Emanuela Rosazza Gianin. "Dynamic Convex Risk Measures". In: *New Risk Measures in the 21st Century* (2004), pp. 227–248.

[FS02] Hans Föllmer and Alexander Schied. "Convex measures of risk and trading constraints". In: *Finance and Stochastics* 6.4 (Oct. 2002), pp. 429–447.

[FS04] Hans Föllmer and Alexander Schied. *Stochastic Finance : An Introduction in Discrete Time*. eng. 2nd ed. De Gruyter Studies in Mathematics. Berlin: De Gruyter, 2004.

[Gri75] Bronius Grigelionis. "Characterization of stochastic processes with conditionally independent increments". In: *Lithuanian Mathematical Journal* 15(4) (1975), pp. 562–567.

[ØS07] Bernt Øksendal and Agnès Sulem. *Applied stochastic control of jump diffusions*. 2nd ed. Universitext. Berlin: Springer, 2007.

[Pen99] Shige Peng. "Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyers type". In: *Probability Theory and Related Fields* 113.4 (Mar. 1999), pp. 473–499.

[Pha09] Huyên Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. eng. Vol. 61. Stochastic Modelling and Applied Probability. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[QS13] Marie-Claire Quenez and Agnès Sulem. "BSDEs with jumps, optimization and applications to dynamic risk measures". In: *Stochastic Processes and their Applications* 123.8 (2013), pp. 3328–3357. ISSN: 0304-4149.

[Ros06] Emanuela Rosazza Gianin. "Risk measures via g-expectations". In: *Insurance: Mathematics and Economics* 39.1 (2006), pp. 19–34.

[Roy06] Manuela Royer. "Backward stochastic differential equations with jumps and related non-linear expectations". In: *Stochastic Processes and their Applications* 116.10 (2006), pp. 1358–1376. ISSN: 0304-4149.

[Sat99] Ken-iti Sato. *Lévy processes and infinitely divisible distributions*. Vol. 68. Cambridge studies in advanced mathematics. Cambridge: Cambridge University Press, 1999.

[Ser72] Richard F. Serfozo. "Processes with Conditional Stationary Independent Increments". In: *Journal of Applied Probability* 9.2 (1972), pp. 303–315.

[Swi16] Anatoliy Swishchuk. *Change of Time Methods in Quantitative Finance*. eng. SpringerBriefs in Mathematics. Cham: Springer International Publishing, 2016.

[TL94] Shanjian Tang and Xunjing Li. "Necessary Conditions for Optimal Control of Stochastic Systems with Random Jumps". eng. In: *SIAM Journal on Control and Optimization* 32.5 (1994), pp. 1447–1475.