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The LIBOR Forward Rate in a HJM-Lévy Framework

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

ABSTRACT

An extended LIBOR forward rate model is derived through what we call the HJM-Lévy framework. The resulting model is a geometric Itô-Lévy process, of which the well known geometric Brownian motion with deterministic volatility is one of many special cases. One specific case of the LIBOR forward rate in the HJM-Lévy framework is a geometric Brownian motion with stochastic volatility. This special case is analyzed and implemented. Two caplet valuation formulas expressed by power series are derived for the model. One for the general geometric Itô-Lévy process, and one for the specific case of a geometric Brownian motion with stochastic volatility.

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CONTENTS

Chapter 1

INTRODUCTION

1.1 The thesis in brief

In the field of mathematical finance predictive models are derived by use of mathematical theory. The models are predicting financial assets, and from these models prices of financial contracts are derived. It is an inexact field in the way that it is impossible to predict the future. Researchers are continuously developing new models, which they hope can do better predictions than any other model derived before. However, better predictions are not always best. In applications we are often dependent on efficiency, which means that models should be able to provide instantaneous information. This is a good reason for the fact that a model as "simple" as the Black-Scholes is extensively used.

Different people and situations require different attributes from a predictive model. The main goal of this thesis was to develop a model that is as general as possible, such that it can be customized for different uses. The focus in this thesis is set on the interest rate market, more specifically, on the LIBOR forward rates. The LIBOR forward rates are generally modeled by a geometric Brownian motion with deterministic volatility, and are constructed such that option prices on them easily can be computed. The general model which is derived for the LIBOR forward rates in this thesis has the classical geometric Brownian motion as a special case, and is therefore referred to as the extended LIBOR forward rate. The extended LIBOR forward rate is derived in what we call the HJM-Lévy framework, because the derivation is based on the HJM framework, where the instantaneous forward rate is an Itô-Lévy process instead of a simple Itô process or Lévy process. This leads to a model for the LIBOR forward rate which is a geometric Itô-Lévy process.

One special case of the geometric Itô-Lévy process is pursued and analyzed in this thesis. This special case is a geometric Brownian motion with an exponential subordinator stochastic volatility. By specifying that the subordinator is a non-Gaussian OU subordinator driven by a compound Poisson process with exponential jumps, we are able to derive the distribution of the stochastic volatility. In this specific case two non-calibrated versions of the model are also implemented, and compared to actual LIBOR forward rate data.

An important part of predictive models in mathematical finance is that it is possible to derive prices on financial contracts from them. As mentioned above, the original LIBOR forward rate model is constructed such that options on them easily can be computed. In this thesis we will focus on the special case of caplets, and two different approaches are applied to derive a caplet valuation formula. The first caplet valuation formula is applicable to general geometric Itô-Lévy processes, while the other is applicable to the special case of a geometric Brownian motion with a general stochastic volatility. We emphasize that both the derived caplet valuation formulas are expressed with power series, and thus has to be computed as approximations in applications.

1.2 Chapter overview: approach and contributions

Before the description each chapter, we emphasize that all theory which are borrowed is marked with a reference. If there are things that are inspired by external sources this is also mentioned.

- **Chapter 2:** Stochastic theory beyond an introductory course is presented in this chapter. Originally my supervisor Fred Espen proposed to study an exponential negative subordinator stochastic volatility model, driven by a compound Poisson process. However, inspired by the derivation of LIBOR forward rates in [Fil09], where the derivation is based on an Itô process, I soon decided to derive an extended model based on an Itô-Lévy process (as [ØS07] calls them). This chapter is a result of that decision. It is an attempt to do a thorough but efficient introduction of Itô-Lévy processes. All material in this chapter is achieved from external sources, but the structure and discussions around all mathematical statements are worked out by me. As my knowledge of Lévy processes from before was almost equal to zero, this chapter has been developed over several months. It has been challenging to understand every aspect of the presented theory, and I had to use a lot of sources to be able to give an introduction as complete as I wanted.
- **Chapter 3:** As this thesis focuses on LIBOR forward rates, an introduction to the interest rate market was needed. I have tried to give an introduction which gives both a mathematical and financial/economical aspect of the topic. This was done to be sure that I was able to use the necessary mathematical approaches, and at the same time understand the application of it. All theory in this chapter is inspired by external sources.
- **Chapter 4:** The derivation of the extended LIBOR forward rate model in this chapter is highly inspired by the derivation in [Fil09]. That is, the approach is similar in the way that I define an instantaneous forward rate model, derives a zero-coupon bond price model from it, and then find an expression for its discounted model. From that model, and its characteristics, the LIBOR forward rate model and the LIBOR market model are derived. The difference between my derivation and the one done in [Fil09], is that the instantaneous forward rate model in [Fil09] is driven by an Itô process with deterministic coefficients, while in this thesis the instantaneous forward rate model is driven by

an Itô-Lévy process. This adds a lot of extra work in more complicated theory and heavier expressions. Because the model is derived in this way, I have chosen to call it the LIBOR forward rate in the HJM-Lévy framework. In the end I achieve a flexible model, and the model derived in [Fil09] is a special case of it. All calculations and discussions in this chapter are performed by myself.

- **Chapter 5:** If the extended LIBOR forward rate model derived in Ch. 4 is considered with zero jump part, it is reduced to a geometric Brownian motion with stochastic volatility. As mentioned above, my supervisor initially proposed to analyze a specific stochastic volatility model in this thesis. Based on these facts I decided to analyze the extended LIBOR forward rate model further, with zero jump part, and the proposed stochastic volatility. The volatility model he proposed is on skeleton form an exponential negative subordinator process. I deduce characteristics of this stochastic volatility, and of the LIBOR forward rate driven by it. It was cool to realize (even if it is quite obvious now) that all the characteristics are functions of the exponential moments and joint exponential moments of the subordinator. One possible choice for the subordinator is a non-Gaussian OU subordinator (nGOUS). The chapter is introduced with calculations of the exponential moments and joint exponential moments of this subordinator, to set the stage. Except from the calculations of the characteristic function of the nGOUS, which is inspired by calculations done in a course I attended at UiO, all other calculations is done exclusively by me.

- **Chapter 6:** I wanted to get an impression of how a special case of the LIBOR forward rate in the HJM-Lévy framework could behave. It was natural to continue the analysis on the geometric Brownian motion model with stochastic volatility, which was considered in Ch. 5. To be able to analyze the model further, a specific Poisson random measure had to be chosen for the nGOUS. Earlier I have worked with compound Poisson processes (CPPs) with exponential jumps in a course which I attended at UiO. Because of this I chose to consider this specific Poisson random measure with exponential jumps. The derivation of the characteristic function of the CPP nGOUS with exponential jumps was is inspired by the lecture notes from the above mentioned course. Other than that calculations are done independent of any source of inspiration.

- **Chapter 7:** The LIBOR market model was derived such that options could be prices on them in a simple way. That is, when the LIBOR forward rate is log-normally distributed and a martingale, Black's formula (Prop. 3.3.1) can be used to find the fair price. That is not the case for the LIBOR forward rate in the HJM-Lévy framework, except for the special case when it is a geometric Brownian motion with deterministic stochastic volatility. Based on this I wanted to explore if it was possible to derive a general caplet valuation formula for the LIBOR forward rate in the HJM-Lévy framework. In this chapter a Fourier transformation method is used. The method is derived in

[EGP10], and the two first sections in this chapter are very much inspired by that publication.

- **Chapter 8:** I though that there had to be possible to use another approach than Fourier transformations to derive a caplet valuation formula for the special case of the LIBOR forward rate in the HJM-Lévy framework, that was considered in Ch. 5. That is, Black's formula is based on the Black-Scholes approach for log-normally distributed random variables. In my special case from Ch. 5, the LIBOR forward rates are not log-normally distributed, but at least they are driven by a Brownian motion with stochastic volatility. Inspired by the classical Black-Scholes derivation in [Ben04] I therefore decided to derive a similar formula for models with stochastic volatilities. Further work in that chapter is all done by me.

Chapter 2

THEORY AND NOTATION: ITÔ-LÉVY PROCESSES

In this chapter we are going to introduce theory relevant for this thesis, as well as notation and assumptions used throughout. The presented theory is mainly obtained from and inspired by [ØS07] and [App04]. We assume that the reader is familiar with basic theory in mathematical and stochastic analysis. If nothing else mentioned we will always work with function values in $\mathbb{R}$, $t \in [0, \mathcal{T}]$ and $\omega \in \Omega$. To set the framework, we define a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \leq \mathcal{T}}, P)$ where Ω is an appropriate sample space, $\mathcal{F}$ is a σ-algebra of subsets of Ω, $\{\mathcal{F}_t\}_{t \leq \mathcal{T}}$ is a filtration on the measurable space $(\Omega, \mathcal{F})$, and P is the market probability measure on $\mathcal{F}$. This probability space will be used throughout the thesis, with a variation in which probability measure we work with respect to.

Let $\{X(t)\}_{t \leq \mathcal{T}}$ be a stochastic process in the sense that $X(t, \omega) : [0, \mathcal{T}] \times \Omega \to \mathbb{R}$ is a $\mathcal{B} \otimes \mathcal{F}$-measurable function, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}^+$. For simplicity, we introduce the following notation.

Notation 2.0.1. Stochastic processes $\{X(t)\}_{t \leq \mathcal{T}}$ are denoted by $X(t)$. It will be clear from the context if we discuss the process or the measurable function.

Processes called Itô-Lévy processes are extensively used in this thesis. We are going to introduce such processes through the theory of Lévy processes. The introduction will be fairly thorough, but efficient, such that the reader gets an impression of what conditions we have to use in forthcoming derivations and analyzes.

Definition 2.0.1 (Lévy processes, [App04]). Let $X(t)$ be a stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \leq \mathcal{T}}, P)$. $X(t)$ is called a Lévy process if the following conditions are satisfied:

1. $X(0) = 0$ a.s.;
2. $X(t)$ has independent and stationary increments;
3. $X(t)$ is continuous in probability, i.e. for all $\epsilon > 0$ and $t \geq 0$

$$\lim_{s \to t} P\left(\left|X(s) - X(t)\right| > \epsilon\right) = 0.$$

Notation 2.0.2. Lévy processes are denoted by $\mathcal{L}(t)$.

There is a variety of different stochastic processes that are Lévy processes. In the following section we will introduce Brownian motion processes. They are important examples of Lévy processes, and we will see that they are key in representing general Lévy processes. The same applies to the Poisson random measure, which will be introduced later.

2.1 Brownian motion processes

Brownian motion processes are the most widely used Lévy processes. In this section they are introduced briefly, and we will consider some of their most important characteristics.

Definition 2.1.1 (Brownian motions, [App04]). A Lévy process $\mathcal{L}(t)$ is a Brownian motion if

1. $\mathcal{L}(t)$ is normally distributed for each $t \geq 0$, with mean 0 and variance given by the process volatility and the applied time interval;
2. $\mathcal{L}(t)$ has continuous sample paths.

Notation 2.1.1. Brownian motions are denoted by $W(t)$.

Integrals with respect to Brownian motions are called Itô integrals. For an Itô integral to exist, the integrand has to satisfy certain conditions. Functions satisfying these conditions are contained in a function space which we will call $\mathcal{V}$.

Definition 2.1.2 (The function space $\mathcal{V}$, [Øks10]). Let $\mathcal{V} = \mathcal{V}([0,\mathcal{T}] \times [0,\mathcal{T}])$ be the class of functions $f(t,T)$ such that

1. $f(t,T,\omega) : [0,\mathcal{T}] \times [0,\mathcal{T}] \times \Omega \to \mathbb{R}$ is $\mathcal{B}^2 \otimes \mathcal{F}$-measurable;
2. $(T,\omega) \to f(t,T,\omega)$ is $\mathcal{B}([0,\mathcal{T}]) \otimes \mathcal{F}_t$-measurable for each $t \leq \mathcal{T}$;
3. $E\left[\int_0^{\mathcal{T}} f(t,T)^2 dt\right] < \infty$.

We write $\mathcal{V}([0,\mathcal{T}] \times [0,\mathcal{T}]) := \mathcal{V}([0,\mathcal{T}]_2)$ for simplicity. If the function is not given by any parameter T, then the space is reduced to $\mathcal{V}([0,\mathcal{T}])$.

Geometric Brownian motions are SDE's on the form

$$\frac{dX(t)}{X(t)} = \alpha(t,T)dt + \sigma(t,T)dW(t),$$

where $\alpha(t,T)$ is integrable and $\sigma(t,T) \in \mathcal{V}[(0,\mathcal{T})_2]$. Its solution is given by (see App. A.2)

$$X(t) = X(0)\exp\left(\int_0^t \sigma(s,T)dW(s) + \int_0^t \left(\alpha(s,T) - \frac{1}{2}\sigma^2(s,T)\right) ds\right).$$

In no-arbitrage frameworks we work with processes on the form

$$X(t) = X(0)\exp\left(\int_0^t \sigma(s,T)dW(s) - \frac{1}{2}\int_0^t \sigma^2(s,T)ds\right), \tag{2.1}$$

because, under appropriate conditions, they are martingales with respect to the measure under which $W(t)$ is a Brownian motion. We will introduce two notations which are useful when we work with such processes. The first notation is a simplification of the semimartingale dynamics inside the exponential function in Eq. (2.1).

Notation 2.1.2 (Stochastic integral I, [Fil09]). We define the dynamics

$$(f \circ W) := f(t,T)dW(t) - \frac{1}{2}f^2(t,T)dt,$$

where $f(t,T) \in \mathcal{V}([0,\mathcal{T}]_2)$ and $W(t)$ is a Brownian motion process.

Next we introduce a notation which we call the stochastic exponential, also called the Doléans-Dade exponential, which represents the solution of a geometric Brownian motion. In Sect. 2.3 we will see that the stochastic exponential represents the solution of a geometric Itô-Lévy process as well, and that the geometric Brownian motion is a special case of the geometric Itô-Lévy process.

Notation 2.1.3 (Stochastic exponential, [Fil09]). Let $dX(t)$ be a real valued stochastic process dynamics. Then the stochastic exponential is defined as

$$\mathcal{E}_t(X) = \exp\left(\int_0^t dX(s)\right).$$

Finally, we will present two important theorems called Itô isometry and Novikov's condition. Itô isometry is invaluable when it comes to analyses of Brownian motion processes, as it states the connection between the expectation of squared Itô integrals and the expectation of classical time integrals.

Theorem 2.1.1 (Itô isometry, [Øks10]). *Let $f(u,T) \in \mathcal{V}([s,t] \times [0,\mathcal{T}])$. Then*

$$E\left[\left(\int_s^t f(u,T)dW(u)\right)^2\right] = E\left[\int_s^t f^2(u,T)du\right],$$

where $s \leq t \leq \mathcal{T}$.

Novikov's condition is important because it states a sufficient condition for the stochastic exponential, with respect to the dynamics given in Nota. 2.1.2, to be a martingale. In no-arbitrage frameworks we are dependent on the fact that the model representing the underlying discounted financial asset price actually is a martingale.

Theorem 2.1.2 (Novikov's condition, [Øks10]). *A sufficient condition for*

$$\mathcal{E}_t(\lambda \circ W)$$

to be a martingale for $t \leq \mathcal{T}$ is that

$$E\left[\exp\left(\frac{1}{2}\int_0^{\mathcal{T}} \lambda^2(s)ds\right)\right] < \infty.$$

2.2 Poisson random measure

To be able to give a complete introduction of Lévy processes we have to introduce a type of measure which is called the Poisson random measure. As mentioned earlier we will see that the Poisson random measure is inevitable when it comes to expressing general Lévy processes. To define the Poisson random measure we have to consider càdlàg functions, and we therefore start this section by stating a notation needed for the definition of that concept.

Notation 2.2.1 (Left and right limits, [App04]).

- When $s < t$ the left limit of a function is denoted by $f(t-) = \lim_{s\uparrow t} f(s)$.
- When $s > t$ the right limit of a function is denoted by $f(t+) = \lim_{s\downarrow t} f(s)$.

We can now define the concept of càdlàg functions, as well as càglàd functions, which we also need in order to define integrals with respect to Lévy processes in Sect. 2.3.

Definition 2.2.1 (Càglàd and càdlàg functions, [App04]). Let $I = [a,b] \subseteq \mathbb{R}^+$. A function $f : I \to \mathbb{R}$ is said to be càglàd if, $\forall t \in (a,b]$, $f(t)$ has right limits and is left-continuous at t, i.e.

1. for all sequences (t_n, $n \in \mathbb{N}$) in I with each $t_n \geq t$ and $\lim_{n\to\infty} t_n = t$ we have that $\lim_{n\to\infty} f(t_n)$ exists. $f(t+) = f(t)$ if and only if f is continuous at t;
2. for all sequences (t_n, $n \in \mathbb{N}$) in I with each $t_n < t$ and $\lim_{n\to\infty} t_n = t$ we have that $\lim_{n\to\infty} f(t_n) = f(t)$.

A càdlàg function is defined similarly with left limits and right-continuity $\forall t \in [a,b)$.

A Poisson random measure is measuring the number of jumps of a certain size occurring over some given time interval. The main attribute of Poisson random measures is their ability to jump. Càdlàg functions have the ability to jump, and it will be clear what this means by the following definition.

Definition 2.2.2 (Stochastic jump, [BBK08]). Let $X(t)$ be a càdlàg stochastic process. The jump of $X(t)$ at time t is denoted by

$$\Delta X(t) = X(t) - X(t-).$$

As we are considering Lévy processes in this thesis, the following theorem is important to verify their jump attribute.

Theorem 2.2.1 (Existence of càdlàg Lévy process, [ØS07]). *Let $\mathcal{L}(t)$ be a Lèvy process. Then there exists a càdlàg version of $\mathcal{L}(t)$ which is also a Lèvy process.*

Assumption 2.2.1. Based on Thm. 2.2.1 we assume that Lévy processes considered in this thesis are their càdlàg versions.

We are now ready to give a formal definition of the Poisson random measure.

Definition 2.2.3 (Poisson random measure, [ØS07], [BBK08]). For each $t > 0$ and Borel subset $U \in \mathbb{R} \setminus \{0\}$ where $0 \notin \bar{U}$, define

$$N(t, U) = N(t, U, \omega) := \sum_{s=0}^{t} \mathbb{1}_{\{\Delta\mathcal{L}(s) \in U\}}.$$

Since $\mathcal{L}(t)$ is càdlàg the sum is finite. $N(t, U)$ is a counting measure and is called the Poisson random measure associated to $\mathcal{L}(t)$. Since $N(t, U)$ is a sum of independent increments $\Delta\mathcal{L}(t)$, $N(t, U)$ is also a Lévy process.

Another important quantity in the theory of Lévy processes is called the Lévy measure. The Lévy measure measures the expected number of jumps of a certain size occurring over a time interval equal to 1.

Definition 2.2.4 (Lévy measure, [BBK08], [ØS07], [App04]). For each $t > 0$ and Borel subset $U \in \mathbb{R} \setminus \{0\}$ where $0 \notin \bar{U}$, define

$$\nu(U) = E\left[N(1, U)\right],$$

such that

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, x^2)\nu(dx) < \infty.$$

Then ν is called the Lévy measure of the stochastic process $\mathcal{L}(t)$.

Remark. Because of stationarity of Lévy processes, we have that the expected number of jumps of a certain size occurring over a time interval $(0, t]$ is given by

$$E\left[N(t, dx)\right] = t\nu(dx).$$

We note that the Lévy measure is defined for $U \in \mathbb{R} \setminus \{0\}$, and make the following assumption for Lévy measures in this thesis to ease the notation.

Assumption 2.2.2. We assume that $\nu\left(\{0\}\right) = 0$, such that we are allowed to consider integrals over $\mathbb{R}$.

To ease the notation even more we introduce a notation which will make the expression of general Lévy processes neater (see Thm. 2.3.1).

Notation 2.2.2 (Poisson random measure over $\mathbb{R}^+$, [ØS07]).

$$\bar{\boldsymbol{N}}(dt, dx) = \begin{cases} \tilde{N}(dt, dx) & \text{if } |x| < R \\ N(dt, dx) & \text{if } |x| \geq R \end{cases}$$

for some $R \in [0, \infty]$, where $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$ is called the compensated Poisson random measure.

Remark. We refer to the case $|x| < R$ as the small jumps, and to $|x| \geq R$ as the big jumps.

2.3 General Lévy processes and Itô-Lévy processes

In this thesis we consider Itô-Lévy processes as driving processes. The derivation of such processes is highly complex, however, we will try to give the reader a feeling about their construction. We will also present conditions that ensure existence of such processes. Our approach to introduce Itô-Lévy processes will be to give a presentation of general Lévy processes, and then give a short but fairly thorough introduction of Lévy integrals. Then we state the definition of Itô-Lévy processes, and we will see that Lévy integrals are special cases of such processes.

General Lévy processes

We start this section by introducing maybe the most important result in the theory of Lévy processes. That is the Itô-Lévy decomposition, which gives a way of representing general Lévy processes.

Theorem 2.3.1 (Itô-Lévy decomposition, [ØS07]). *Let $\mathcal{L}(t)$ be a Lévy process. Then $\mathcal{L}(t)$ has the decomposition*

$$\mathcal{L}(t) = \alpha t + \sigma W(t) + \int_{\mathbb{R}} x \bar{\boldsymbol{N}}(t, dx),$$

for some constants $\alpha, \sigma \in \mathbb{R}$. $W(t)$ is a Brownian motion such that $W(t) \perp \tilde{N}(t, U)$, where the compensated Poisson random measure $\tilde{N}(t, U)$ is a martingale as long as $N(t, U)$ is defined as in Def. 2.2.3.

Finiteness of moments and exponential moments is very important when it comes to applications of Lévy processes. That is, it is important to be sure that characteristics such as mean, variance and moment generating functions exist. Fortunately, there are simple conditions that ensure these properties for Lévy processes.

Theorem 2.3.2 (Finiteness of moments end exponential moments of Lévy processes, [Ebe14]). *Let $p \in \mathbb{R}$. Then a Lévy process $\mathcal{L}(t)$ has*

- *finite absolute p-th moment if and only if*

$$\int_{|x| \geq R} |x|^p \, \nu(dx) < \infty;$$

- *finite exponential p-th moment if and only if*

$$\int_{|x|\geq R} e^{px}\nu(dx) < \infty.$$

It is also worth mentioning the condition for finite variation of almost all paths of Lévy processes, because it is a necessary condition in the derivations of Ch. 4.

Theorem 2.3.3 (Finite variation of a.a. paths of Lévy processes, [Ebe14]). *A Lévy process $\mathcal{L}(t)$ has finite variation for a.a. paths if $\sigma = 0$ and*

$$\int_{|x|<R} |x|\, \nu(dx) < \infty. \tag{2.2}$$

A.a. paths of $\mathcal{L}(t)$ have infinite variation if $\sigma \neq 0$ or if Eq. 2.2 does not hold.

Remark ([Ebe14]). If the condition in Eq. 2.2 is satisfied, the small jumps in the Itô-Lévy decomposition converges, and we can split the small jump integral as

$$\int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|<R\}} \tilde{N}(dt,dx) = \int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|<R\}} N(dt,dx) - t\int_{\mathbb{R}} x\mathbb{1}_{\{|x|<R\}}\nu(dx).$$

As mentioned earlier we are dependent on that processes modeling the underlying discounted financial assets prices are martingales in no-arbitrage frameworks. The next theorem gives us an opportunity to exploit the martingale property of $\tilde{N}(t,U)$. That is, the following theorem gives a way of considering Lévy processes where the jump part only consists of $\tilde{N}(t,U)$, and thus it is straight forward to pinpoint when the Lévy process is a martingale.

Theorem 2.3.4 (Itô-Lévy decomposition when $R = \infty$, [ØS07]). *If $E\left[\left|\mathcal{L}(1)\right|\right] < \infty$ we have that*

$$\int_{|x|\geq R} |x|\, \nu(dx) < \infty,$$

and we may then choose $R = \infty$ such that

$$\mathcal{L}(t) = \alpha t + \sigma W(t) + \int_{\mathbb{R}} x\tilde{N}(t,dx)$$

for α, σ, $W(t)$ and $\tilde{N}(t,x)$ as in Thm. 2.3.1.

Remark. If $\alpha = 0$, then $\mathcal{L}(t)$ is called a Lévy martingale.

Another essential theorem in the theory of Lévy processes is the Lévy-Khintchine formula, which gives us an easy way to find the characteristic function of any given Lévy process. We will use Lévy-Khintchine formula in the proof of Thm. 2.4.3, which states a formula for the characteristic function of exponential Lévy integrals.

Theorem 2.3.5 (Lévy-Khintchine formula, [ØS07], [Ebe14]). *Let $\mathcal{L}(t)$ be a Lévy process with Lévy measure ν. Then for $\theta \in \mathbb{R}$*

$$E\left[e^{i\theta\mathcal{L}(t)}\right] = e^{t\psi(\theta)},$$

where

$$\psi(\theta) = i\alpha\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - \mathbb{1}_{\{|x|<R\}} i\theta x\right) \nu(dx) \tag{2.3}$$

is called the Lévy symbol of $\mathcal{L}(t)$ when $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$. Conversely, given constants α, σ^2 and a measure ν such that $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$, Eq. (2.3) is the Lévy symbol of some Lévy process $\mathcal{L}(t)$. (α, σ^2, ν) is called the triplet characterizing the Lévy process.

Examples of Lévy processes

We have already introduced the Brownian motion, which is an important special case of Lévy processes, and now we are going to introduce further two important examples.

Example 2.3.1 (Poisson processes, [App04],[ØS07]). A Lévy process $\mathcal{L}(t) = N(t)$ is called a Poisson process of intensity $\lambda > 0$ if it takes values in $\mathbb{N} \cup \{0\}$ such that

$$P\left(N(t) = n\right) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

That is, $N(t)$ is a Poisson random variable with mean λt. Notice that the Poisson process is such that

$$N(t) = N(t, U = 1, \omega),$$

which means that $N(t)$ is a Poisson random measure with intensity $\lambda = \nu(U = 1)$.

In Ch. 5 we consider a specific stochastic volatility process, and in Ch. 6 we will analyze this stochastic volatility explicitly. The Lévy process in the following example (when it has exponential jump sizes) will be the driver of the stochastic volatility in that explicit case.

Example 2.3.2 (Compound Poisson processes (CPP), [ØS07]). Let $Y(n)$, for $n \in \mathbb{N}$, be a sequence of i.i.d. random variables with values in $\mathbb{R}$ and common law μ_Y. Also, let $N(t)$ be a Poisson process with intensity λ, such that it is independent of each $Y(n)$. A Lévy process $\mathcal{L}(t) = Z(t)$ is called a CPP if it has the form

$$Z(t) = \sum_{n=1}^{N(t)} Y(n)$$

for each $t \geq 0$. An increment of the process can be expressed as

$$Z(s) - Z(t) = \sum_{n=N(t)+1}^{N(s)} Y(n)$$

for $s > t$, and its Lévy measure ν is given by

$$\nu(U) = E\left[N(1,U)\right] = \lambda \mu_Y.$$

Remark. Notice that $Z(t) = N(t)$ if $Y(n) := 1\ \forall n$.

Subordinators

In Ch. 5 we introduce a stochastic volatility driven by what is called a subordinator. We define such processes here, as well as a theorem giving their characteristic function.

Definition 2.3.1 (Subordinator, [BBK08]). A monotonically increasing Lévy processes is called a subordinator.

The following corollary, which states the formula for the characteristic function of subordinators, is the the Lévy-Khintchine formula with a specific Lévy symbol. This specific Lévy symbol is needed when we want to find the characteristic function of a subordinator which is defined in Ch. 5.

Corollary 2.3.1 (Characteristic function of subordinators, [App04]). *Let $\mathcal{L}(t)$ be a subordinator with Lévy measure ν. Then the Lévy symbol takes the form*

$$\psi(\theta) = i\alpha\theta + \int_0^\infty \left(e^{i\theta x} - 1\right)\nu(dx), \tag{2.4}$$

where $\theta \in \mathbb{R}$, $\alpha \geq 0$, $\nu(-\infty, 0] = 0$ and $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$. Conversely, given a constant $\alpha \geq 0$ and a measure ν such that $\nu(-\infty, 0] = 0$ and $\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty$, Eq. (2.4) is the Lévy symbol of some subordinator $\mathcal{L}(t)$.

Integrals with respect to Lévy processes

A short introduction of Itô integrals was presented in the preceding. The integral with respect to general Lèvy processes is also an important tool in stochastics, and will be introduced next. As mentioned earlier Lévy integrals are special cases of Itô-Lévy processes, and we will use this special case to model instantaneous forward rates in Ch. 4. We will keep the introduction to the Lévy integrals short, but thorough enough for the reader to understand what kind of processes that are Lévy integrable.

First we define the mode of convergence required on the space of Lévy integrable processes.

Definition 2.3.2 (Processes that uniformly converges on compacts in probability, [Low09]). A sequence of jointly measurable stochastic processes $X_n(t)$ are said to uniformly converge on compacts in probability (ucp) to the limit $X(t)$ if

$$P\left(\sup_{s \leq t} \left|X_n(s) - X(s)\right| > \epsilon\right) \to 0$$

as $n \to \infty$, $\forall t$ and $\epsilon > 0$.

Next we define two function spaces. In Thm. 2.3.7 we will see that Lévy integrable processes are contained in one of these spaces, and that the Lévy integrals are contained in the second.

Definition 2.3.3 (The function spaces L_{ucp} and D_{ucp}, [ØS07]).

- Define L_{ucp} as the space of cáglád adapted processes which are ucp.
- Define D_{ucp} as the space of cádlág adapted processes which are ucp.

By Assum. 2.2.1 all Lévy processes in this thesis are their càdlàg version. The next theorem then ensures that the considered Lévy processes are semimartingales as well.

Theorem 2.3.6 (Lévy process as a semimartingale, [Low10]). *Every càdlàg Lévy process is a semimartingale.*

Now we have what we need to state the theorem which gives the necessary conditions to define an integral with respect to general Lévy processes.

Theorem 2.3.7 (Integral with respect to general Lévy processes, [ØS07]). *If the stochastic process $H(t,T)$ is such that $H(t,T) \in L_{ucp}$ and the Lévy process $\mathcal{L}(t)$ is a semimartingale we can define the stochastic integral*

$$X(t) = \int_0^t H(s,T)d\mathcal{L}(s), \tag{2.5}$$

where $X(t)$ is a continuous linear map

$$X(t) : L_{ucp} \to D_{ucp}.$$

Itô-Lévy processes

Following [ØS07], we consider Thm. 2.3.7, and observe that we can split the integral in Eq. (2.5) into three terms. That is, we can split it into three terms which are integrals with respect to dt, $dW(t)$ and $\bar{\boldsymbol{N}}(dt,dx)$. By this observation, and by having the Itô-Lévy decomposition (Thm. 2.3.1) in mind, it is natural to think that more general SDE's of the form

$$d\hat{\mathcal{L}}(t) = \alpha(t,T,\omega)dt + \sigma(t,T,\omega)dW(t) + \int_{\mathbb{R}} \gamma(t,T,x,\omega)\bar{\boldsymbol{N}}(dt,dx) \tag{2.6}$$

are possible to define. As long as the coefficient processes $\alpha(t,T)$, $\sigma(t,T)$ and $\gamma(t,T,x)$ satisfy certain conditions such that the integrals exist, they are indeed possible to define. Processes of the form as in Eq. (2.6) are called Itô-Lévy processes, and they are used as driving processes in this thesis. We note that general Lévy processes and Lévy integrals are special cases of Itô-Lévy processes.

Notation 2.3.1. Itô-Lévy processes are denoted by $\hat{\mathcal{L}}(t)$.

Because we use Itô-Lévy processes as driving processes in this thesis, the next definition will be important to ensure that those processes are well defined.

Definition 2.3.4 (The function space $\mathcal{U}$). Given an Itô-Lévy process $\hat{\mathcal{L}}(t)$, let $\mathcal{U}^* = \mathcal{U}^*([0,\mathcal{T}]^3 \times [0,\mathcal{T}]^3 \times U)$ be the class of triplets $(\alpha(t,T), \sigma(t,T), \gamma(t,T,x))$ such that

1. $\alpha(t,T,\omega), \sigma(t,T,\omega) : [0,\mathcal{T}]^2 \times \Omega \to \mathbb{R}$ are $\mathcal{B}^2 \otimes \mathcal{F}$-measurable and $\gamma(t,T,x,\omega) : [0,\mathcal{T}]^2 \times U \times \Omega$ is $\mathcal{B}^3 \otimes \mathcal{F}$-measurable;

2. $E\left[\int_0^{\mathcal{T}} \left|\alpha(t,T)\right| dt\right] < \infty$;

3. $E\left[\int_0^{\mathcal{T}} \sigma(t,T)^2 dt\right] < \infty$;

4. $E\left[\int_0^{\mathcal{T}} \int_{\mathbb{R}} \gamma(t,T,x)^2 \nu(dx)dt\right] < \infty$.

Then $\mathcal{U} = \mathcal{U}([0,\mathcal{T}]^3 \times [0,\mathcal{T}]^3 \times U)$ is the class of triplets $(\alpha(t,T), \sigma(t,T), \gamma(t,T,x)) \in L_{ucp} \cap \mathcal{U}^*([0,\mathcal{T}]^3 \times [0,\mathcal{T}]^3 \times U)$. We write $\mathcal{U}([0,\mathcal{T}]^3 \times [0,\mathcal{T}]^3 \times U) := \mathcal{U}([0,\mathcal{T}]_2^3 \times U)$ for simplicity. Also, if $\alpha(t,T)$ or $\sigma(t,T)$ or both are zero in a triplet we will omit writing them. That is, we write $(f(t,T), \gamma(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U)$ if one of the functions is zero, and simply $\gamma(t,T,x) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U)$ if both are zero. If the triplet is not given by a parameter T, then the space is reduced to $\mathcal{U}([0,\mathcal{T}]^3 \times U)$.

Remark ([ØS07]). Define $M(t) := \int_0^t \int_{\mathbb{R}} \gamma(s,T,x)\tilde{N}(dx,ds)$, and let $T \leq \mathcal{T}$. Then

- $M(t)$ is a local martingale on $t \leq \mathcal{T}$ if
$$\int_0^t \int_{\mathbb{R}} \gamma^2(s,T,x)\nu(dx)ds < \infty;$$

- $M(t)$ is a martingale on $t \leq \mathcal{T}$ if
$$E\left[\int_0^t \int_{\mathbb{R}} \gamma^2(s,T,x)\nu(dx)ds\right] < \infty.$$

Itô formula for Itô-Lèvy processes is an inevitable theorem when we want to interchange between a Itô-Lévy process and its dynamics. We will use this theorem several times throughout the thesis.

Theorem 2.3.8 (The one-dimensional Itô formula, [ØS07]). *Suppose that $X(t)$ is an Itô-Lévy process as defined in Eq. (2.6). Let $f \in C^2(\mathbb{R}^2)$ and define*

$Y(t) = f(t, X(t))$. *Then $Y(t)$ is again an Itô-Levy process and*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))\big(\alpha(t)dt + \beta(t)dW(t)\big) \\ &+ \frac{1}{2}\beta^2(t)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt \\ &+ \int_{|x|<R} \Big(f(t, X(t-) + \gamma(t,x)) - f(t, X(t-)) \\ &\qquad\qquad - \frac{\partial f}{\partial x}(t, X(t-))\gamma(t,x)\Big)\nu(dx)dt \\ &+ \int_{\mathbb{R}} \big(f(t, X(t-) + \gamma(t,x)) - f(t, X(t-))\big)\bar{\boldsymbol{N}}(dt, dx). \end{aligned}$$

Remark. If $R = 0$ then $\bar{\boldsymbol{N}} = N$ everywhere, and if $R = \infty$ then $\bar{\boldsymbol{N}} = \tilde{N}$, as long as the sufficient conditions are satisfied.

In the section where we introduced Brownian motions, we mentioned that geometric Brownian motions are special cases of geometric Itô-Lévy processes. Geometric Itô-Lévy processes are SDE's on the form $dX(t) = X(t-)d\hat{\mathcal{L}}(t)$. By use of Itô formula (Thm. 2.3.8) it is straight forward to show that the solution of the geometric Brownian motion is $\mathcal{E}_t(f \circ W)$, where we have used Nota. 2.1.2 and 2.1.3. The solution of a geometric Itô-Lévy process is calculated by use of Itô formula (Thm. 2.3.8) in App. A.2, and we see that the stochastic exponential contain two extra terms in that case, compared to the geometric Brownian motion case. To ease the notation when we work with geometric Itô-Lévy processes, we present another notation which will give a neat representation of the stochastic exponential in that case as well.

Notation 2.3.2 (Stochastic integral II). We define the dynamics

$$\begin{aligned} (f \circ \bar{\boldsymbol{N}}) := \int_{|x|<R} \Big(\log\big(1 + f(t,x)\big) - f(t,x)\Big)\nu(dx)dt \\ + \int_{\mathbb{R}} \log\big(1 + f(t,x)\big)\bar{\boldsymbol{N}}(dx, dt), \end{aligned}$$

where $\bar{\boldsymbol{N}}(t, U)$ is the Poisson random measure, ν is the Lévy measure, $f(t,x) \geq -1$ and $f(t,x), \log\big(1 + f(t,x)\big) \in \mathcal{U}([0, \mathcal{T}]^3 \times U)$.

Then, by App. A.2 and Nota. 2.1.2, 2.1.3 and 2.3.2, we see that the SDE $dX(t) = X(t-)d\hat{\mathcal{L}}$ has a solution of the form $X(t) = X(0)\mathcal{E}_t\Big(f_1 \circ W + f_2 \circ \bar{\boldsymbol{N}}\Big)$.

Another theorem which is very important for the derivations in this thesis is Girsanov's theorem. Girsanov's theorem makes it possible to do measure changes, and consider stochastic processes under the given probability measure. This is a powerful tool in stochastic analysis, as some situations become considerably simplified under certain probability measures. We define predictable processes before we introduce Girsanov's theorem and Girsanov's theorem for Itô-Lévy processes.

Definition 2.3.5 (Predictable processes and the predictable σ-algebra, [App04], [Pro95]). Let $X(t, a, \omega) : [0, \mathcal{T}] \times A \times \Omega \to \mathbb{R}$ be a function satisfying

1. $(a, \omega) \to X(t, a, \omega)$ is $\mathcal{B}(A) \otimes \mathcal{F}_t$-measurable for each $t \leq \mathcal{T}$;
2. $t \to X(t, a, \omega)$ is left-continuous with right limits (càglàd) for each $a \in A$, $\omega \in \Omega$.

Let $\mathcal{P}$ denote the smallest σ-algebra of $\mathcal{P}$-measurable mappings $X(t, a)$. We then call $\mathcal{P}$ the predictable σ-algebra, and $\mathcal{P}$-measurable mappings $X(t, a)$ are said to be predictable.

Remark. We see from Def. 2.3.3 and 2.3.4 that all functions in triplets which belong to $\mathcal{U}([0, \mathcal{T}]^3_2 \times U)$ are predictable.

Girsanov's theorem gives us a connection between two Brownian motion processes under two different probability measures, and a connection between two compensated Poisson random measures under two different probability measures.

Theorem 2.3.9 (Girsanov's theorem, [ØS07]). *Let $h(t)$ and $\theta(t, x) \leq 1$ be predictable processes such that the process*

$$Z(t) := \mathcal{E}_t\left(h \circ W + (-\theta) \circ \tilde{N}\right)$$

exists for $t \leq \mathcal{T}$ and satisfies $E[Z(\mathcal{T})] = 1$. Define the probability measure Q on $\mathcal{F}_{\mathcal{T}}$ by $dQ = Z(\mathcal{T})dP$. Then the process

$$dW^Q(t) := dW(t) - h(t)dt$$

is a Brownian motion with respect to Q, and the random measure

$$\tilde{N}^Q(dt, dx) := \theta(t, x)\nu(dx)dt + \tilde{N}(dt, dx)$$

is the Q-compensated Poisson random measure of $N(\cdot, \cdot)$, in the sense that the process

$$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(s, x)\tilde{N}^Q(ds, dx)$$

is a local Q-martingale for all predictable processes $\gamma(t, x)$ where

$$\int_0^{\mathcal{T}} \int_{\mathbb{R}} \gamma^2(s, x)\theta^2(s, x)\nu(dx)ds < \infty \quad a.s.$$

2.4 Exponential Itô-Lévy processes

In Ch. 3 (Def. 3.3.6) we see that the LIBOR forward rate is modeled by a geometric Brownian motion. We will derive an extended model for LIBOR forward rates in Ch. 4, where they turn out to be given by geometric Itô-Lévy processes. As mentioned previously, the solution of a geometric Itô-Lévy process is given by a stochastic exponential. In Ch. 5 we will also encounter the exponential of a Lévy integral, when we consider a stochastic volatility given by an exponential non-Gaussian OU subordinator (nGOUS). To sum up, we state the exponential processes which we will work with in this thesis. We have the very general exponential Itô-Lévy process which has the form

$$X(t) = X(0)e^{\hat{\mathcal{L}}(t)},$$

and we have the special case of the an exponential Itô-Lévy process, which is the exponential Lévy integral process

$$X(t) = X(0)e^{\int_0^t f(s)d\mathcal{L}(s)}.$$

Since we derive a model for the LIBOR forward rate which is an exponential Itô-Lévy process it is important to know when such processes are martingales, such that we can use the model in no-arbitrage frameworks. The following theorem gives a necessary condition.

Theorem 2.4.1 (Exponential martingale, [App04]). *Let $\hat{\mathcal{L}}(t)$ be an Itô-Lévy process such that $Y(t) = Y(0)e^{\hat{\mathcal{L}}(t)}$ is a local martingale. Then $Y(t)$ is a martingale if and only if $E\left[Y(t)\right] = Y(0)$, $\forall t$.*

In Ch. 5 we want to find the characteristic function of the nGOUS which might drive the stochastic volatility in interest, because we want to analyze the distributional properties of that specific stochastic volatility. We will now state and prove a theorem which we can use to find the characteristic function of a general Lévy integral. To be able to prove the theorem, we need a result called dominated convergence in measure.

Theorem 2.4.2 (Dominated convergence in measure, [MW13]). *Let $(\Sigma, \mathcal{S}, \mu)$ be a measure space. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of complex-valued $\mathcal{S}$-measurable functions that converges in measure to the $\mathcal{S}$-measurable function f. Further suppose that there is a non-negative Lebesgue integrable function g such that $|f_n| \leq g$ μ-a.e. for each $n \in \mathbb{N}$. Then*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$$

for each $E \in \mathcal{S}$.

Now we are ready to state and prove the theorem which is essential for some of the calculations in Ch. 5. This is a known result, and is found in [Ebe14]) without proof. It is proved here due to notational uncertainty from the reference, and at the same time because it is a "cool" proof. Similar calculations were done in a lecture of a previous course at UiO.

Theorem 2.4.3 (Characteristic function of the exponential Lévy integral). *Suppose that $f : \mathbb{R}_+ \to \mathbb{C}$ is a continuous function such that the triplet $(\alpha f(t), \sigma f(t), x f(t)) \in \mathcal{U}([0, \mathcal{T}]^3, U)$. Also, suppose that Re $\left(f(t)\right) \leq M$, $\forall t \leq \mathcal{T}$, where M is such that the p-th exponential moment exists for $p \in [-M, M]$. Then*

$$E\left[\exp\left(i\int_0^t f(s)d\mathcal{L}(s)\right)\right] = \exp\left(\int_0^t \psi(f(s))ds\right),$$

where ψ is the Lévy symbol with triplet (α, σ^2, ν), such that

$$\int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty.$$

Proof. Since $\mathcal{L}(t)$ is a semimartingale and $f(t) \in L_{ucp}$, we have by Thm. 2.3.7 that $\int_0^t f(s)d\mathcal{L}(s)$ is a Lévy integral. Then, by the definition of stochastic integrals in [ØS07] we have that

$$\begin{aligned} E\left[\exp\left(i\int_0^t f(s)d\mathcal{L}(s)\right)\right] &= E\left[\exp\left(\lim_{j\to\infty} i\sum_j f(s_j)\left(\mathcal{L}(s_{j+1}) - \mathcal{L}(s_j)\right)\right)\right] \\ &= E\left[\exp\left(\lim_{j\to\infty} i\sum_j f(s_j)\Delta\mathcal{L}(s_j)\right)\right]. \end{aligned}$$

Clearly

$$i\sum_j f(s_j)\Delta\mathcal{L}(s_j)$$

is $\mathcal{F}_t$-measurable $\forall s_j \leq t$. Thus, since $\mathrm{Re}\left(f(t)\right) \leq M$ $\forall t \leq \mathcal{T}$ and the exponential function is continuous, we can use Thm. 2.4.2 to get

$$\begin{aligned} E\left[\exp\left(i\int_0^t f(s)d\mathcal{L}(s)\right)\right] &= \lim_{j\to\infty} E\left[\exp\left(i\sum_j f(s_j)\Delta\mathcal{L}(s_j)\right)\right] \\ &= \lim_{j\to\infty} E\left[\prod_j \exp\left(if(s_j)\Delta\mathcal{L}(s_j)\right)\right]. \end{aligned}$$

Next, as Lévy processes have stationary increments we have that

$$\begin{aligned} E\left[\exp\left(i\int_0^t f(s)d\mathcal{L}(s)\right)\right] &= \lim_{j\to\infty} \prod_j E\left[\exp\left(if(s_j)\Delta\mathcal{L}(s_j)\right)\right] \\ &= \lim_{j\to\infty} \prod_j e^{\psi(f(s_j))\Delta s_j} \\ &= \exp\left(\int_0^t \psi(f(s_j))ds\right), \end{aligned}$$

where we used the Lévy-Khintchine formula (Thm. 2.3.5) in the second equality. ■

2.5 Assumptions

This section is not meant to be read and understood out of context, but rather as a reference list for the reader throughout the thesis. In this way the reader can easily find the assumptions which are referred to in the text.

Chapter 2

Assumption 2.2.1. We assume that Lévy processes considered in this thesis are their càdlàg version.

Assumption 2.2.2. We assume that Lévy measures considered in this thesis are such that $\nu\left(\{0\}\right) = 0$.

Chapter 3

Assumption 3.1.1. We assume that

- there exist a frictionless market for T-bonds $\forall T > 0$;
- $P(t,T) > 0$;
- $P(T,T) = 1 \quad \forall T$;
- $P(t,T)$ is differentiable in T.

Chapter 4

Assumption 4.1.1. We assume that

$$(\alpha(t,T), \sigma(t,T), \gamma(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Assumption 4.1.2. For all $T \leq \mathcal{T}$ we assume that

- $\int_0^T \int_0^T \big|\alpha(t,s)\big|\, dtds < \infty$;
- $\sup_{s,t \leq T} \big|\sigma(t,s)\big| < \infty$;
- $\sup_{s,t \leq T} \big|\gamma(t,s,x)\big| < \infty$, $\forall x \in U$;
- Condition 4. in Thm. 4.1.2 holds for the entire triplet.

Assumption 4.1.3. We assume that

- $\int_0^T f(0,s)ds < \infty$;
- $\big((r(s) + b(t,T)), v(t,T), \Delta(t,T,x)\big) \in \mathcal{U}\left([0,\mathcal{T}]_2^3 \times U\right)$.

Assumption 4.1.4. We assume that

$$\Big(v^2(t,T+\delta), (e^{\Delta_\delta(t,T,x)} - 1)\Big) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Assumption 4.2.1. For given measurable functions $v_\delta(t,T)$ and $\gamma_2(t,T,x)$, we assume that the predictable processes $h(t)$ and $\theta(t,x) \leq 1$ satisfy

$$\big(h(t)v_\delta(t,T), \theta(t,x)\gamma_2(t,T,x)\big) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Assumption 4.2.2. We assume that the four listed conditions in Lemma 4.2.2 hold for the stochastic exponential

$$\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right).$$

Assumption 4.2.3. We assume that the stochastic exponential

$$Y(t) = Y(0)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

is such that

$$E\left[Y(t)\right] = Y(0).$$

Assumption 4.2.4. We assume that

$$\int_{\mathbb{R}} \gamma_\delta(t,T,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du}\left(1-\theta(t,x)\right) - \mathbb{1}_{\{|x|<R\}}\right)\nu(dx) < \infty.$$

Assumption 4.3.1. We assume that $\lambda(t,T)$ and $\xi(t,T,x)$ are such that the four listed conditions in Lemma 4.2.2 hold.

Assumption 4.3.2. We assume that the stochastic exponential

$$L(t,T) = L(0,T)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

is such that

$$E\left[L(t,T)\right] = L(0,T).$$

Chapter 5

Assumption 5.1.1. We assume that the stochastic volatility $\lambda(t)$ is independent of the Poisson random measure $N(t,U)$.

Assumption 5.1.2. We assume that the stochastic volatility $\lambda(t)$ satisfies Novikov's condition (Thm. 2.1.2).

Assumption 5.2.1. $Z(t)$ is such that $\int_0^\infty \nu(dx) < \infty$ holds.

Chapter 7

Assumption 7.2.1. We assume that $H \geq 1$.

Appendix A

Assumption A.4.1. We assume that

1. $g \in L^1_{bc}(\mathbb{R})$;
2. $M_{X_T}(H)$ exists;
3. $\hat{g} \in L^1(\mathbb{R})$

Chapter 3

PRELIMINARIES ON THE INTEREST RATE MARKET

Except for the concept of money, the concept of interest rates is the most important element in the world of finance. Interest rates is a powerful tool when the value of money is to be expressed explicitly at different times. The term 'interest rates' in itself is highly general, because there exist several definitions, each intended for different uses. We are going to consider three of these definitions in this thesis. Interest rates are directly linked to what we call zero-coupon bond prices, and we will start out by introducing this concept. The information in Sect. 3.1 is from [Fil09] and [Inv], Sect. 3.2 is based on [Fil09] and [BM07], and Sect. 3.3 is inspired by [Fil09], [BM07] and [IBA].

3.1 Zero-coupon bonds

Inflation is a rate measure expressing the average price level increase of goods and services in an economy. Two important reasons for this effect is increased money supply to the economy due to increased money printing by the monetary authorities, and stronger growth in demand than in supply of goods and services, such that the demand/supply ratio increases.[Inv] The consequence of such events is that the value of money decreases over time.

Definition 3.1.1 (Zero-coupon bond price, [Fil09]). A zero-coupon bond price $P(t,T)$ with maturity T is defined as the time t value of 1 dollar[1] at the future time $T \geq t$.

Notation 3.1.1. Zero-coupon bond prices $P(t,T)$ will be referred to as T-bond prices.

For the purpose of deriving mathematical expressions in the interest rate market we will do some assumptions on the zero-coupon bond price that not necessarily is correct in real markets.

Assumption 3.1.1. We assume that

1. there exists a frictionless market for T-bonds $\forall T > 0$;
2. $P(t,T) > 0 \quad \forall T$;
3. $P(T,T) = 1 \quad \forall T$;
4. $P(t,T)$ is differentiable in T.

[1] Or 1 unit of another currency. We will use dollars in this thesis.

Due to the fact that deflation, which is the opposite of inflation, is also a possible outcome (although more rarely), the event $P(t,T) > 1$ is allowed. Assum. 3.1.1.2 is somewhat logical in the sense that you do not want to pay someone to give them money in the future, or give them money in the future for free. Still, negative interest rates do occur in the real world due to risk assessments.

The mathematical expression of zero-coupon bond prices is obtained through no-arbitrage theory. Before we state that result, even if we assume that the reader is known with no-arbitrage theory, it is worth recalling its core theorem.

Theorem 3.1.1 (Fundamental theorem of asset pricing, [Fil09]). *If the market i arbitrage-free there exist at least one equivalent (local) martingale measure (ELMM) $Q \sim P$, under which the relevant discounted price process is a (local) martingale.*

The general mathematical no-arbitrage expression of the zero-coupon bond price is not used directly in this thesis, other than when we prove Black's formula (Proof I of Prop. 3.3.1). Even so, we state and prove the formula to give the reader a feeling about its mathematical appearance. In the proof we will use an important theorem called Bayes theorem, which is also referred to later in this chapter.

Theorem 3.1.2 (Bayes theorem, [Øks10]). *Let Q^1 and Q^2 be two probability measures on $(\Omega, \mathcal{F})$ with $\frac{dQ^1}{dQ^2} = g$, and let X be a random variable on $(\Omega, \mathcal{F})$, such that $g(\omega) \in L^1(\Omega)$ and $\left|X(\omega)\right| g(\omega) \in L^1(\Omega)$. Let $\mathcal{G}$ be a σ-algebra $\mathcal{G} \subset \mathcal{F}$. Then*

$$E_{Q^1}\left[X|\mathcal{G}\right] = \frac{E_{Q^2}\left[gX|\mathcal{G}\right]}{E_{Q^2}\left[g|\mathcal{G}\right]} \quad a.s.$$

Now we are ready to state and prove the no-arbitrage zero-coupon bond price.

Lemma 3.1.1 (General expression of zero-coupon bond prices, [Fil09]). *The general mathematical expression of zero-coupon bond prices is*

$$P(t,T) = E_Q\left[D(t,T) \quad \Bigg| \quad \mathcal{F}_t\right] = E\left[D(t,T)\frac{g(T)}{g(t)} \quad \Bigg| \quad \mathcal{F}_t\right]$$

where $D(t,T)$ is the discount factor between the two times t and T, and $g(t) = \frac{dQ}{dP}\big|_{\mathcal{F}_t}$.

Proof. From no-arbitrage theory, Assum. 3.1.1.3 and Thm. 3.1.2 it is easy to realize that

$$P(t,T) = E_Q\left[D(t,T)P(T,T) \;\middle|\; \mathcal{F}_t\right] = E_Q\left[D(t,T) \;\middle|\; \mathcal{F}_t\right]$$
$$= \frac{E\left[D(t,T)g(T) \;\middle|\; \mathcal{F}_t\right]}{E\left[g(T) \;\middle|\; \mathcal{F}_t\right]} = E\left[D(t,T)\frac{g(T)}{g(t)} \;\middle|\; \mathcal{F}_t\right].$$

■

3.2 Interest rates

Interest rates are alternative measures on the time-value of money. They are expressed by zero-coupon bond prices, and as already mentioned, there exists a variety of them. In this thesis we will focus on simply compounded forward rates, instantaneous forward rates and short rates. In the following definitions we consider the time instants $t \leq S < T$.

Simply compounded forward rates describe the time average rate of change of the time t value of money between two time instants S and T. The LIBOR forward rate, which is the main subject in this thesis, is a simply compounded forward rate.

Definition 3.2.1 (Simply compounded forward rates, [Fil09]). The simply compounded forward rate applicable in the period $[S,T]$ prevailing at time t is given by

$$F(t;S,T) = \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right),$$

where $P(t,T)$ is the time t value of 1 dollar at the future time T.

The instantaneous rate of change of the time t value of money at some future time T is called the instantaneous forward rate.

Definition 3.2.2 (Instantaneous forward rates, [Fil09]). The instantaneous forward rate with maturity T prevailing at time t is given by

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T},$$

where $P(t,T)$ is the time t value of 1 dollar at the future time T.

And, finally, the short rate is the immediate rate of change of the time t value of money.

Definition 3.2.3 (Short rates, [Fil09]). The short rate at time t is given by

$$r(t) = f(t,t),$$

where $f(t,T)$ is the instantaneous forward rate.

Money-market account and short-rate models

If you deposit 1 dollar to your bank account today, the value of your deposit will change over time according to the banks given short rate. The change of the deposit is a payment from the bank to you, as the deposit works as a loan from you to the bank. The t value of a deposit done at time $t_0 = 0$ is denoted by $B(t)$, and is called the money-market account.

Definition 3.2.4 (Money-market account). The dynamics of a money-market account is given by

$$dB(t) = r(t)B(t)dt,$$

where $r(t)$ is the short rate. Thus, with the normalized initial value $B(0) = 1$ the money-market account is given by

$$B(t) = e^{\int_0^t r(s)ds}.$$

The money-market account is called a risk-free asset, and $r(t)$ the risk-free rate of return, as the future value of $B(t)$ is known up to time $t + dt$ at time t. An important application of the money-market account is its use as a discount factor. The ratio

$$\frac{B(t)}{B(T)} = e^{-\int_t^T r(s)ds} \tag{3.1}$$

is the number of dollars you have to have in your money-market account at time $t \leq T$ to be sure to have 1 dollar at time T. Short rates $r(t)$ are not known before time t, but the importance of being able to say something about the future short rate reveals itself in no-arbitrage theory, where we need the discount factor in Eq. (3.1) to say something about prices in the derivatives market. Just remember from Lemma 3.1.1 that the T-bond prices are given by the discount factor, and thus

$$P(t,T) = E_Q\left[e^{-\int_t^T r(s)ds} \quad \middle| \quad \mathcal{F}_t\right]. \tag{3.2}$$

The assumption of a constant short rate $r(t) = r$ is frequently used in calculations, but that is generally not true. Many stochastic models are therefore developed to be able to say something about the future short rates, where the Vasiček model is an important example. In the Vasiček model the short rate is modeled by the SDE

$$dr(t) = (b + \beta r(t))dt + \sigma dW^Q(t),$$

where $b, \beta, \sigma \in \mathbb{R}$, and W^Q is a Q-Brownian motion. From this the zero-coupon bond price can be derived, and thus other types of interest rates.

HJM methodology for instantaneous forward rates

Short-rate models provide flexibility in the way that you are relatively free to choose drift and volatility without having to think about restrictions. On the other hand, problems do occur in some cases, for example when you are to calibrate models to the observed initial term-structure, or complexity of the derived forward rates.

Heath, Jarrow and Morton (HJM) proposed a new method in the 1980s, namely to model the entire forward curve directly. The forward curve was proposed to be given by the Itô dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s), \quad \forall t \leq T,$$

for each T, where $T \to f(0,T)$ is a given integrable initial forward curve. This model is more analytically tractable, and virtually any interest rate model might be derived from it. In Ch. 4 we will use an extended version of the HJM-framework, which we will call the HJM-Lévy framework, to derive an extended version of the LIBOR market model. The commonly used LIBOR market model is introduced in the following section.

3.3 The LIBOR market model

The London Interbank Offered Rate (LIBOR), currently also called the Intercontinental Exchange LIBOR (ICE LIBOR), is presented at around 11:55 am every London business day by the ICE Benchmark Administration Limited (IBA). There are 35 different LIBORs, each differing in underlying currency and loan maturity. IBA determines the daily LIBOR by computing a weighted average of everyday submissions from between 11 to 16 panel banks, each answering the question "At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am?"[2][IBA]. This means that the LIBOR is based on the rate at which each of the panel banks are willing to lend a short-term loan to other banks. The LIBOR is a simply compounded forward rate with maturities ranging from 1 day to 12 months.

Definition 3.3.1 (LIBOR forward rate, [Fil09]). The LIBOR forward rate applicable in the period $[T, T+\delta]$ prevailing at time t is given by

$$L(t,T) := F(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right),$$

where $P(t,T)$ is the time t-value of 1 dollar at a future time T and $\delta \in \mathbb{R}^+$ is a constant[3].

All over the world the LIBOR is used as a benchmark in all types of markets where rates play a central role, and the LIBOR 3-month maturity

[2]IBA is currently in a transition phase where they are introducing a new framework (the Waterfall Methodology) for how the panel banks are to set their submissions.

[3]For convenience.

dollar rate is the most commonly used. Even if $L(t,T)$ is simply compounded, and thus a fixed rate in the interval $[T, T+\delta]$, there are floating rate systems based on it. A typical example of how LIBOR is used as a floating rate benchmark could be a mortgage issued by a bank to a private individual with floating rate 3-month LIBOR+2.5%. That is, the 3-month LIBOR is updated every time IBA submits a new rate.

Derivatives on the LIBOR are traded as well. One of the most traded derivatives in the interest rate market are caps, which are structures of caplets. Caplets are derivatives similar to call options in the stock market, and ensures the holder a maximum rate of K percent over a period of time. A formal definition of caplets and caps follows, but first we have to introduce a LIBOR term structure.

Definition 3.3.2 (LIBOR term structure). Define some future dates $T_0 < T_1 < ... < T_n$ with $T_m - T_{m-1} = \delta$. The LIBOR term structure is a system of n rates $L(T_{m-1}, T_{m-1})$ reset at each time $T_{m-1} + \delta$, for $m = 1, ..., n$. The term structure is illustrated in Fig. 3.3.

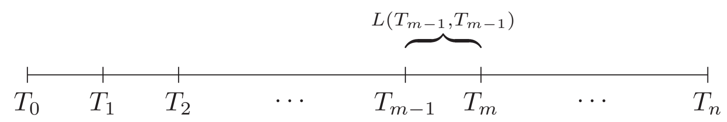

Figure 3.1: Term structure for LIBOR rates. The applicable interval for a LIBOR forward rate prevailing at time T_{m-1}, with expiry T_{m-1} and maturity T_m, is marked.

Now we are ready to formally define caplets and caps.

Definition 3.3.3 (Caplets and caps, [Fil09]). Let $F(T;T,T+\delta)$ be a simply compounded forward rate with maturity $T+\delta$ prevailing at time T, and let N be the nominal value. A caplet pays the holder $N(F(T;T,T+\delta) - K)$ dollars if $F(T;T,T+\delta)$ exceeds the caplet rate K for the time interval $[T, T+\delta]$. Since $L(T,T) = F(T;T,T+\delta)$ the holder of a caplet on a LIBOR gets the payoff

$$c = \delta N \left(L(T,T) - K \right)^+$$

at time $T + \delta$. Now consider the term structure defined in Def. 3.3.2 and a cap rate K. The holder of a cap on a LIBOR gets a payoff

$$c_i = \delta N \left(L(T_{i-1}, T_{i-1}) - K \right)^+$$

every maturity $T_{i-1} + \delta$, for $i = 1, ..., n$. This leads to a system of n payments, where the last payment is executed at time T_n.

We categorize caplets into three different states, which are dependent on the value of the LIBOR forward rate the day you enter the contract.

Definition 3.3.4 (ATM, ITM and OTM, [BM07]). Consider a caplet with payoff time $T+\delta$, such that $L(t,T)$ is the initial value of the LIBOR model. The caplet is said to be

- at-the-money (ATM) if the strike price is such that $K = L(t,T)$;
- in-the-money (ITM) if the strike price is such that $K < L(t,T)$;
- out-of-the-money (OTM) if the strike price is such that $K > L(t,T)$.

This categorization can be generalized to caps as well, see [Fil09].

From Def. 3.3.3 it is easy to realize that the price of a cap equals the sum of the n underlying caplet prices. That is, if

$$\text{Cpl}\left(t; T_{i-1}, T_i\right), \quad i = 1, ..., n$$

denotes the i-th caplet price at time $t \leq T_0$, then

$$\text{Cp}(t) = \sum_{i=1}^{n} \text{Cpl}\left(t; T_{i-1}, T_i\right) \tag{3.3}$$

denotes the price of the cap at time t. It is common market practice to price caplets, and thus caps, with Black's formula. The original proof of the formula was performed with inconsistencies as we will see in Proof I, but it is possible to provide a rigorous proof with a change of probability measure, as we will see in Proof II. As the generalization to pricing caps from caplets is straight forward, we will state Black's formula in a caplet framework.

Proposition 3.3.1 (Black's formula, [BM07]). *Let $t \leq T$, and suppose that the simply compounded forward rate prevailing at time t follows an analytical formula $F(t; T, T+\delta, r(t)) := F(t; T, T+\delta)$, where $r(t)$ is a given short rate model. We assume that the dynamics of $F(t; T, T+\delta)$ follows a geometric Brownian motion of the form*

$$dF(t; T, T+\delta) = \sigma F(t; T, T+\delta) dW^Q(t), \tag{3.4}$$

where σ is a constant[4] volatility and $W^Q(t)$ is a standard Brownian motion process under the risk-free probability measure Q. Let K be the caplet rate and N the nominal value. Then the caplet price at time t is given by

$$\begin{aligned} &\textit{Cpl}\left(t; T, T+\delta, K, N\right) \\ &\quad = P(t, T+\delta)\delta N \left(F(t; T, T+\delta)\Phi\left(d_1\right) - K\Phi(d_2)\right), \end{aligned}$$

where $P(t, T+\delta)$ is the time t-value of 1 dollar at the future time $T+\delta$, $\Phi(\cdot)$ is the standard normal cdf, and

$$d_{1,2} = \frac{1}{\sigma\sqrt{T-t}} \left(\ln\left(\frac{F(t; T, T+\delta)}{K}\right) \pm \frac{\sigma^2(T-t)}{2} \right).$$

[4] For simplicity.

Proof I (Black's formula). From no-arbitrage theory we know that the fair price of a caplet is the risk-free expectation of the discounted payoff, conditioned on a filtration in which the filtration generated by the stochastic process is contained. That is,

$$\begin{aligned}
&\mathrm{Cpl}\,(t;T,T+\delta,K,N)\\
&\quad= E_Q\left[\exp\left(-\int_t^{T+\delta} r(s)ds\right)\delta N\left(F(T;T,T+\delta)-K\right)^+ \;\middle|\; \mathcal{F}_t\right].
\end{aligned}$$

We assume that the discount factor $\exp\left(-\int_t^{T+\delta} r(s)ds\right)$ is deterministic, and identifies it with $P(t,T+\delta)$ as stated in Eq. (3.2). Then

$$\begin{aligned}
&\mathrm{Cpl}\,(t;T,T+\delta,K,N)\\
&\quad= P(t,T+\delta)\delta N E_Q\left[\left(F(T;T,T+\delta)-K\right)^+ \;\middle|\; \mathcal{F}_t\right].
\end{aligned} \tag{3.5}$$

Now we go back to the assumption that the short-rate $r(t)$ is a stochastic process, such that the forward rate follows the dynamics in Eq. (3.4). Then we recognize $E_Q\left[\left(F(T;T,T+\delta)-K\right)^+ \mid \mathcal{F}_t\right]$ as the T-maturity call option price with strike K in a market with zero risk-free rate, whose price is given by the Black-Scholes formula. We assume that the reader is known with this formula. If not, it can be proved by following the steps in the proof of Prop. 8.1.1 in Ch. **??**. Black's formula follows. ■

In Prop. 3.3.1 the forward rate is based on a stochastic short-rate model, but in the proof we assume the short-rate dependent discount factor to be deterministic. In addition we make an assumption that seems rather arbitrary; that the short-rate driven forward rate dynamics are lognormally distributed. These facts make the original version of Black's formula an apparent approximation. Indeed it is possible to give Black's formula a rigorous proof by use of what is called forward measures. We will perform this proof in Proof II, but first we have to define forward measures, and present a needed lemma.

Definition 3.3.5 (Forward measures, [Fil09]). Assume that there exists an equivalent martingale measure Q for the bond market such that all B-discounted T-bond price processes are Q-martingales. For a fixed maturity T we define the T-forward measure $Q^T \sim Q$ on $\mathcal{F}_T$ as

$$\frac{dQ^T}{dQ} = \frac{B(0)P(T,T)}{B(T)P(0,T)} = \frac{1}{B(T)P(0,T)},$$

which is a valid probability measure because $B(T)P(0,T) > 0$, and

$$E_Q\left[\frac{B(0)P(T,T)}{B(T)P(0,T)}\right] = E_Q\left[\frac{1}{B(T)P(0,T)}\right] = 1$$

by Lemma 3.1.1. For $t \le T$ we have that

$$\left.\frac{dQ^T}{dQ}\right|_{\mathcal{F}_t} = E_Q\left[\frac{dQ^T}{dQ} \;\middle|\; \mathcal{F}_t\right] = \frac{P(t,T)}{B(t)P(0,T)}.$$

As we see, the forward measure is associated to the zero-coupon bond price discounted by the money-market account. In the following lemma we will see that the forward measure also is associated to zero-coupon bond prices discounted by other zero-coupon bond prices. In addition, we will see that these fractions are martingales, a fact which adds rigour to Proof II. The following lemma is also inevitable in the derivations of the next chapter.

Lemma 3.3.1 (Martingale property for discounted bond prices, [Fil09]).

a) *For any $S > 0$ and $0 \leq t \leq \min(S, T)$, the T-bond discounted S-bond price process*

$$\frac{P(t,S)}{P(t,T)}$$

is a Q^T-martingale.

b) *The S- and T-forward measures are related by*

$$\left.\frac{dQ^S}{dQ^T}\right|_{\mathcal{F}_t} = \frac{P(t,S)P(0,T)}{P(t,T)P(0,S)}.$$

We are now ready to state Proof II of Black's formula.

Proof II (Black's formula). As in Proof I we have given the caplet price from no-arbitrage theory. Rewriting the discount factor in terms of the money market account an defining $X := \big(F(T;T,T+\delta) - K\big)^+$ we have that

$$\begin{aligned}
&\mathrm{Cpl}\,(t;T,T+\delta,K,N)\\
&\quad = \delta N E_Q\left[\frac{B(t)}{B(T+\delta)}X \;\middle|\; \mathcal{F}_t\right]\\
&\quad = P(0,T+\delta)B(t)\delta N E_Q\left[\frac{1}{B(T+\delta)P(0,T+\delta)}X \;\middle|\; \mathcal{F}_t\right],
\end{aligned}$$

because the money market account is $\mathcal{F}_t$-measurable at time t. Further, using the fact that $B(0) = P(T+\delta, T+\delta) = 1$, then Def. 3.3.5, Thm. 3.1.2 and then Def. 3.3.5 again, we find that

$$\begin{aligned}
&\mathrm{Cpl}\,(t;T,T+\delta,K,N)\\
&\quad = P(0,T+\delta)B(t)\delta N E_Q\left[\frac{B(0)P(T+\delta,T+\delta)}{B(T+\delta)P(0,T+\delta)}X \;\middle|\; \mathcal{F}_t\right]\\
&\quad = P(0,T+\delta)B(t)\delta N E_Q\left[\frac{dQ^{T+\delta}}{dQ}X \;\middle|\; \mathcal{F}_t\right]\\
&\quad = P(0,T+\delta)B(t)\delta N E_Q\left[\frac{dQ^{T+\delta}}{dQ} \;\middle|\; \mathcal{F}_t\right] E_{Q^{T+\delta}}\left[X \mid \mathcal{F}_t\right]\\
&\quad = P(0,T+\delta)B(t)\delta N \frac{P(t,T+\delta)}{P(0,T+\delta)B(t)} E_{Q^{T+\delta}}\left[X \mid \mathcal{F}_t\right]\\
&\quad = P(t,T+\delta)\delta N E_{Q^{T+\delta}}\left[X \mid \mathcal{F}_t\right].
\end{aligned}$$

Inserting $X = \big(F(T;T,T+\delta) - K\big)^+$ gives us the desired result. ■

In the previous proof we do not at any time assume that something stochastic is deterministic. Also, the assumption that the forward rate dynamics are lognormally distributed is not just a random assumption anymore, as the forward rate is defined as a $P(T,T+\delta)$-discounted zero-coupon bond price process, and thus is a $Q^{T+\delta}$-martingale according to Lemma 3.3.1a).

The next issue is that no known short-rate model leads to Black's formula, as the resulting forward rate fails to be lognormally distributed. There are calibrating methods that make short-rate based models reproduce results well, but the models tend to involve complicated functions, making them hard to work with.[BM07] This is where the LIBOR market model comes to rescue.

Definition 3.3.6 (LIBOR market model, [Fil09]). Assume the LIBOR term structure as defined in Def. 3.3.2. Set $T_0 = 0$ such that $T_m = m\delta$ for $m = 0, \dots, M-1$, where we then have that $T_M = T_{n+1}$ from Def. 3.3.2. The dynamics of $L(t,T_m)$ under the forward measure $Q^{T_{m+1}}$ is then given by

$$dL(t,T_m) = L(t,T_m)\lambda(t,T_m)dW^{T_{m+1}}(t), \quad t \in [0,T_m],$$

which gives the log-normally distributed processes

$$L(t,T_m) = L(0,T_m)\mathcal{E}_t\left(\lambda \circ W^{T_{m+1}}\right).$$

Proof. The fact that the given dynamics are equivalent to that $L(t,T_m)$ is driven by a stochastic exponential is achieved by the calculations in App. A.2. A proof that the dynamics actually apply to the LIBOR term structure is given for a more complicated LIBOR model in Sect. 4.3. ■

It is possible to derive the LIBOR market model from the HJM framework. As stated earlier we will introduce the HJM-Lévy framework in the next chapter, and from it derive an extended version of the LIBOR market model. This extended model will give users a greater deal of freedom in modeling, and the LIBOR market model in Def. 3.3.6 is actually a special case of that model.

Chapter 4

THE LIBOR FORWARD RATE DRIVEN BY GEOMETRIC ITÔ-LÈVY PROCESSES

We are going to derive an extended version of the LIBOR market model presented in Def. 3.3.6. We saw that the LIBOR forward rate is driven by a geometric Brownian motion, and thus is distributed as a log-normal random variable, such that it coincides with Black's pricing formula for caplets which is stated in Prop. 3.3.1. One possible derivation of this LIBOR market model is through the HJM-framework, as done in [Fil09]. Inspired by [Fil09] we are going to derive an extended LIBOR market model through an extended HJM-framework, which we call the HJM-Lévy framework in this thesis. That is, we start with the instantaneous forward rate modeled directly in the market as an Itô-Lévy process, and from this we derive what we call the extended LIBOR market model. We will see that the LIBOR forward rate is driven by a geometric Itô-Lévy process in this framework.

4.1 The HJM-Lévy framework

The instantaneous forward rate

As we saw in Ch. 3, the HJM framework models instantaneous forward rates directly in the market as Itô processes, in stead of deriving them through short-rate models. We will in stead model instantaneous forward rates directly in the market as Itô-Lévy processes, and we will refer to these as instantaneous forward rates in the HJM-Lévy framework. A derivation through this framework results in a LIBOR forward rate model with an increased grade of flexibility, compared to the LIBOR forward rate model in Def. 3.3.6. By increased flexibility we mean that several kinds of models can be achieved from it, either one wants a clean Brownian motion model, a clean jump model, or a combination of both, either with deterministic or stochastic coefficients.

Let $\mathcal{L}(t)$ be a Lévy process and $\gamma(t,T)$ a measurable function, and define the triplet

$$(\alpha\gamma(t,T), \sigma\gamma(t,T), x\gamma(t,T)) := (\alpha(t,T), \sigma(t,T), \gamma(t,T,x)),$$

for $\alpha, \sigma \in \mathbb{R}$, to which we add the following assumption.

Assumption 4.1.1. We assume that

$$(\alpha(t,T), \sigma(t,T), \gamma(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Then, in the HJM-Lévy framework, the instantaneous forward rate is given by the Itô-Lévy process

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \gamma(s,T)d\mathcal{L}(s) \qquad (4.1) \\ &= f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,T,x)\bar{\mathbf{N}}(ds,dx) \end{aligned}$$

for each T, where $t \leq T \leq \mathcal{T}$, and $T \to f(0,T)$ is a given initial instantaneous forward curve. By Assum. 4.1.1 this process is well defined.

Zero-coupon bond prices

Our goal is to find an extended model for the LIBOR forward rate. By Def. 3.3.1 the LIBOR forward rate is expressed by the $(T+\delta)$-bond discounted T-bond price process. This is why we have to find an expression for the zero-coupon bond price process in the HJM-Lévy framework to derive the extended LIBOR market model. According to Def. 3.2.2 the zero-coupon bond prices are associated to instantaneous forward rates through the relationship

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \qquad (4.2)$$

which we can use to find $P(t,T)$ associated with $f(t,T)$ given in Eq. (4.1). This version of a zero-coupon bond price is given in the next proposition, but first we have to state Fubini's theorem for stochastic integrals and an assumption, to be able to prove it. First we introduce a well known result called Fubini's theorem, which also will be used in this thesis.

Theorem 4.1.1 (Fubini's theorem, [MW13]). *Suppose that $(\Sigma, \mathcal{S}, \mu)$ and $(\Pi, \mathcal{P}, \nu)$ are σ-finite measure spaces. Let f be a complex-valued $\mathcal{S} \times \mathcal{P}$-measurable function on $\Sigma \times \Pi$ such that at least one of the integrals*

- $\int_{\Sigma\times\Pi} \big| f(x,y) \big| \, d(\mu \times \nu)(x,y)$,
- $\int_{\Sigma} \int_{\Pi} \big| f(x,y) \big| \, d\nu(y) d\mu(x)$,
- $\int_{\Pi} \int_{\Sigma} \big| f(x,y) \big| \, d\mu(x) d\nu(y)$

is finite. Then it holds that

$$\begin{aligned} \int_{\Sigma\times\Pi} \big| f(x,y) \big| \, d(\mu \times \nu)(x,y) &= \int_{\Sigma} \int_{\Pi} \big| f(x,y) \big| \, d\nu(y) d\mu(x) \\ &= \int_{\Pi} \int_{\Sigma} \big| f(x,y) \big| \, d\mu(x) d\nu(y). \end{aligned}$$

In Fubini's theorem for stochastic integrals one of the integrals in Thm. 4.1.1 is a stochastic integral (semimartingale in this case). There are more conditions to consider when we work with stochastic integrals, and as a result the next theorem is somewhat more complicated.

Theorem 4.1.2 (Fubini's theorem for stochastic integrals, [Pro95], [Fil09]). *Let $X(t)$ be a semimartingale, and $\phi(t,s)$ a stochastic process satisfying*

1. $\phi(t,s,\omega) : [0,\mathcal{T}]^2 \times \Omega \to \mathbb{R}$ *is* $\mathcal{B}^2 \otimes \mathcal{F}$*-measurable;*
2. $\phi(t,s,\omega)$ *is* $\mathcal{B} \otimes \mathcal{P}$*-measurable;*
3. $\sup_{t,s}\left|\phi(t,s)\right| < \infty$*;*
4. $Z(s) = \int_0^T \phi(t,s,\omega)dX(t)$ *is* $\mathcal{B}([0,T])^2 \otimes \mathcal{F}_T$*-measurable such that for each* s*,* $Z(s)$ *is a càdlàg version of* $\int \phi(t,s,\omega)dX(t)$*.*

Then $\int_0^T \phi(t,s)ds \in L^1$*, and* $\int_0^T \left|Z(s)\right| ds < \infty \; \forall t$ *a.s. Moreover,*

$$\int_0^T \left(\int_0^T \phi(t,s)dX(t) \right) ds = \int_0^T \left(\int_0^T \phi(t,s)ds \right) dX(t).$$

To be able to use Fubini's theorem for stochastic integrals (Thm. 4.1.2) in the proof of the following proposition, we add an assumption to the triplet $(\alpha(t,T), \sigma(t,T), \gamma(t,T,x))$.

Assumption 4.1.2. For all $T \leq \mathcal{T}$ we assume that

- $\int_0^T \int_0^T \left|\alpha(t,s)\right| dtds < \infty$;
- $\sup_{s,t\leq T}\left|\sigma(t,s)\right| < \infty$;
- $\sup_{s,t\leq T}\left|\gamma(t,s,x)\right| < \infty$, $\forall x \in U$;
- Condition 4. in Thm. 4.1.2 holds for the entire triplet.

We also add an assumption which ensures the process in the following proposition to be well defined.

Assumption 4.1.3. We assume that

- $\int_0^T f(0,s)ds < \infty$;
- $\big((r(s) + b(t,T)), v(t,T), \Delta(t,T,x)\big) \in \mathcal{U}\left([0,\mathcal{T}]_2^3 \times U\right)$.

Now we have what we need to state and prove the proposition which gives an expression for the zero-coupon bond price process in the HJM-Lévy framework.

Proposition 4.1.1 (Zero-coupon bond price process in the HJM-Lévy framework). *Let the instantaneous forward rate $f(t,T)$ be as given in Eq. (4.1). The associated zero-coupon bond price process is given by*

$$\begin{aligned} P(t,T) = & P(0,T) \exp\Bigg(\int_0^t \big(r(s) + b(s,T)\big) ds + \int_0^t v(s,T)dW(s) \qquad \textbf{(4.3)} \\ & + \int_0^t \int_{\mathbb{R}} \Delta(s,T,x)\bar{\mathbf{N}}(ds,dx) \Bigg), \end{aligned}$$

where

$$P(0,T) = e^{-\int_0^T f(0,u)du}, \quad b(s,T) = -\int_s^T \alpha(s,u)du,$$

$$v(s,T) = -\int_s^T \sigma(s,u)du, \quad \textit{and} \quad \Delta(s,T,x) = -\int_s^T \gamma(s,u,x)du.$$

Finally, $r(s)$ *is the short-rate associated to* $f(t,T)$ *in the sense that*

$$\begin{aligned} r(s) =& f(s,s) \\ =& f(0,s) + \int_0^s \alpha(u,s)du + \int_0^s \sigma(u,s)dW(u) + \int_0^s \int_{\mathbb{R}} \gamma(u,s,x)\bar{\boldsymbol{N}}(du,dx). \end{aligned}$$

Proof. By Eq. (4.2) and Eq. (4.1) we see that

$$\begin{aligned} \log P(t,T) = &- \int_t^T f(0,u)du - \int_t^T \int_0^t \alpha(s,u)dsdu - \int_t^T \int_0^t \sigma(s,u)dW(s)du \\ &- \int_t^T \int_0^t \int_{\mathbb{R}} \gamma(s,T,x)\bar{\boldsymbol{N}}(ds,dx)du. \end{aligned}$$

By Def. 3.2.3, the short rate associated to $f(t,T)$ is given by $r(t) = f(t,t)$. Thus, working through the derivation in [Fil09] with an extra stochastic term and some slight modifications, we achieve

$$\begin{aligned} \log P(t,T) =& \log P(0,T) + \int_0^t \big(r(s) + b(s,T)\big)\, ds \\ &+ \int_0^t v(s,T)dW(s) + \int_0^t \int_{\mathbb{R}} \Delta(s,T,x)\bar{\boldsymbol{N}}(ds,dx). \end{aligned}$$

The calculations of this last result is found in App. A.1. The expression of $P(t,T)$ in Eq. (4.3) follows. ■

Now that the zero-coupon bond price associated to the instantaneous forward rate in the HJM-Lévy framework is derived, we are interested in finding the $(T+\delta)$-bond discounted T-bond price process, which is needed to derive the LIBOR forward rate according to Def. 3.3.1. For notational tractability we first introduce the following notation.

Notation 4.1.1. For simplicity we write

$$Y(t) := \frac{P(t,T)}{P(t,T+\delta)}.$$

We always choose δ such that $T + \delta \leq \mathcal{T}$. Also, to be sure that the SDE in the following proposition is well defined, we have to add another assumption.

Assumption 4.1.4. We assume that

$$\Big(v^2(t,T+\delta), (e^{\Delta_\delta(t,T,x)} - 1)\Big) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Remark. We have that

$$v_\delta(t,T))^2 = v^2(t,T) - 2v(t,T)v(t,T+\delta) + v^2(t,T+\delta),$$

and when $v(t,T+\delta)$ is in $\mathcal{U}$-space, $v(t,T)$ is obviously in $\mathcal{U}$-space as well.

With these assumptions in play, we can state a well defined SDE which represents the $(T+\delta)$-bond discounted T-bond price process in the following proposition.

Proposition 4.1.2 (The $(T+\delta)$-bond discounted T-bond price process in the HJM-Lévy framework). *Let the zero-coupon bond price $P(t,T)$ be as stated in Prop. 4.1.1. Then the $(T+\delta)$-bond discounted T-bond price process is given by the stochastic exponential*

$$\begin{aligned} &Y(t) \\ &= Y(0)\exp\left(\int_0^t b_\delta(s,T)ds + \int_0^t v_\delta(s,T)dW(s) + \int_0^t\int_{\mathbb{R}} \Delta_\delta(s,T,x)\bar{\boldsymbol{N}}(ds,dx)\right), \end{aligned}$$

where $z_\delta(\cdot,T) := z(\cdot,T) - z(\cdot,T+\delta)$. Its dynamics is given by

$$\begin{aligned} \frac{dY(t)}{Y(t-)} &= \left(b_\delta(t,T) + \frac{1}{2}\left(v_\delta(t,T)\right)^2\right)dt + v_\delta(t,T)dW(t) \\ &\quad + \int_{|x|<R}\left(e^{\Delta_\delta(t,T,x)} - 1 - \Delta_\delta(t,T,x)\right)\nu(dx)dt \qquad \textbf{(4.4)} \\ &\qquad\qquad + \int_{\mathbb{R}}\left(e^{\Delta_\delta(t,T,x)} - 1\right)\bar{\boldsymbol{N}}(dt,dx). \end{aligned}$$

Proof. From Eq. (4.3) we easily see that

$$\begin{aligned} &Y(t) \\ &= Y(0)\exp\Bigg(\int_0^t \big(b(s,T) - b(s,T+\delta)\big)\,ds + \int_0^t \big(v(s,T) - v(s,T+\delta)\big)\,dW(s) \\ &\qquad + \int_0^t\int_{\mathbb{R}} \big(\Delta(s,T,x) - \Delta(s,T+\delta,x)\big)\,\bar{\boldsymbol{N}}(ds,dx)\Bigg), \end{aligned}$$

which equals the desired stochastic exponential when we define $z_\delta(\cdot,T) = z(\cdot,T) - z(\cdot,T+\delta)$. Since T is a parameter, we achieve the stated dynamics by use of the Itô formula calculations done in App. A.2, by using $\alpha(t) = b_\delta(t,T)$, $\sigma(t) = v_\delta(t,T)$ and $\gamma(t,x) = \Delta_\delta(t,T,x)$. ∎

4.2 The HJM-Lévy drift condition

At this point we have everything we need to derive an extended LIBOR forward rate model based on the instantaneous forward rate in the HJM-Lévy framework. But we have to be careful. As we know from Lemma 3.3.1a), the $(T+\delta)$-bond discounted T-bond price process $Y(t)$ is a $Q^{T+\delta}$-martingale.

Because the instantaneous forward rate $f(t,T)$ is the starting point of the derivation of the discounted zero-coupon bond price processes, this fact has an implication on the modeling of $f(t,T)$. That is, for $Y(t)$ to be a martingale under $Q^{T+\delta}$, we have to add a restriction to $f(t,T)$. In the final theorem of this section will see that the restriction is on the instantaneous forward rate drift, and we call it the HJM-Lévy drift condition.

We will derive the HJM-Lévy drift condition from the SDE version of the $(T+\delta)$-bond discounted T-bond price process. This SDE is given in Eq. (4.4) in Prop. 4.1.2. The derivation will be done through several steps. First we state and prove a corollary which represents the SDE in Eq. (4.4) in a more convenient way.

Corollary 4.2.1 (Rewritten dynamics of the $(T+\delta)$-bond discounted T-bond price process in the HJM-Lévy framework). *Let the dynamics of the $(T+\delta)$-bond discounted T-bond price process, $dY(t)$, be as given in Prop. 4.1.2. Then it can be rewritten as*

$$\begin{aligned}\frac{dY(t)}{Y(t-)} = &\left(\beta(t,T) + \int_{\mathbb{R}} \gamma_1(t,T,x)\nu(dx)\right) dt \\ &+ v_\delta(t,T)dW(t) + \int_{\mathbb{R}} \gamma_2(t,T,x)\tilde{N}(dt,dx),\end{aligned}$$

where we have defined

$$\beta(t,T) = b_\delta(t,T) + \frac{1}{2}\left(v_\delta(t,T)\right)^2,$$

$$\gamma_1(t,T,x) = e^{\Delta_\delta(t,T,x)} - 1 - \mathbb{1}_{\{|x|<R\}}\Delta_\delta(t,T,x)$$

and

$$\gamma_2(t,T,x) = e^{\Delta_\delta(t,T,x)} - 1.$$

Proof. From Prop. 4.1.2 we know that

$$\begin{aligned}\frac{dY(t)}{Y(t-)} = &\left(b_\delta(t,T) + \frac{1}{2}\left(v_\delta(t,T)\right)^2\right) dt + v_\delta(t,T)dW(t) \\ &+ \int_{|x|<R} \left(e^{\Delta_\delta(t,T,x)} - 1 - \Delta_\delta(t,T,x)\right)\nu(dx)dt \\ &+ \int_{\mathbb{R}} \left(e^{\Delta_\delta(t,T,x)} - 1\right)\bar{\boldsymbol{N}}(dt,dx).\end{aligned} \tag{4.5}$$

According to Notat. 2.2.2 we can rewrite the two last terms in $dY(t)/Y(t-)$

as

$$\begin{aligned}
&\int_{|x|<R} \left(e^{\Delta_\delta(t,T,x)} - 1 - \Delta_\delta(t,T,x)\right) \nu(dx)dt \\
&\qquad + \int_{\mathbb{R}} \left(\mathbb{1}_{\{|x|<R\}} \left(e^{\Delta_\delta(t,T,x)} - 1\right) \tilde{N}(dx,dt) \right. \\
&\qquad\qquad \left. + \mathbb{1}_{\{|x|\geq R\}} \left(e^{\Delta_\delta(t,T,x)} - 1\right) \left(\tilde{N}(dx,dt) + \nu(dx)dt\right) \right) \\
&= \int_{|x|<R} \left(e^{\Delta_\delta(t,T,x)} - 1 - \Delta_\delta(t,T,x)\right) \nu(dx)dt \\
&\qquad + \int_{|x|\geq R} \left(e^{\Delta_\delta(t,T,x)} - 1\right) \nu(dx)dt + \int_{\mathbb{R}} \left(e^{\Delta_\delta(t,T,x)} - 1\right) \tilde{N}(dt,dx) \\
&= \int_{\mathbb{R}} \left(e^{\Delta_\delta(t,T,x)} - 1 - \mathbb{1}_{\{|x|<R\}} \Delta_\delta(t,T,x)\right) \nu(dx)dt \\
&\qquad\qquad + \int_{\mathbb{R}} \left(e^{\Delta_\delta(t,T,x)} - 1\right) \tilde{N}(dt,dx).
\end{aligned}$$

If we insert this representation into Eq. (4.5), we reach the desired expression when $\beta(t,T) := b_\delta(t,T) + \frac{1}{2}\left(v_\delta(t,T)\right)^2$, $\gamma_1(t,T,x) := e^{\Delta_\delta(t,T,x)} - 1 - \mathbb{1}_{\{|x|<R\}}\Delta_\delta(t,T,x)$ and $\gamma_2(t,T,x) = e^{\Delta_\delta(t,T,x)} - 1$. ■

Further, we want to express the $(T+\delta)$-bond discounted T-bond price process as a function of a $Q^{T+\delta}$-Brownian motion and a $Q^{T+\delta}$-compensated Poisson random measure. Then we can utilize the martingale properties of the Brownian motion and compensated Poisson random measure under the probability measure $Q^{T+\delta}$ to find the HJM-Lévy drift condition. To ensure that the $(T+\delta)$-bond discounted T-bond price process is well defined under $Q^{T+\delta}$, we add the following assumption.

Assumption 4.2.1. For given measurable functions $v_\delta(t,T)$ and $\gamma_2(t,T,x)$, we assume that the predictable processes $h(t)$ and $\theta(t,x) \leq 1$ satisfy

$$\left(h(t)v_\delta(t,T), \theta(t,x)\gamma_2(t,T,x)\right) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U).$$

Lemma 4.2.1 (The $(T+\delta)$-bond discounted T-bond price process under the forward measure $Q^{T+\delta}$). *Let $Q^{T+\delta}$ be a forward measure $dQ^{T+\delta} := Z(T+\delta)dP$, where $Z(t) := \mathcal{E}_t\left(h \circ W + (-\theta) \circ \tilde{N}\right)$ for some predictable processes $h(t)$ and $\theta(t,x) \leq 1$ such that $E\left[Z(T+\delta)\right] = 1$. Then the $(T+\delta)$-bond discounted T-bond price process SDE in Cor. 4.2.1 can be rewritten as*

$$\begin{aligned}
\frac{dY(t)}{Y(t-)} &= \left(\beta(t,T) - h(t)v_\delta(t,T) + \int_{\mathbb{R}} \left(\gamma_1(t,T,x) - \theta(t,x)\gamma_2(t,T,x)\right)\nu(dx)\right) dt \\
&\qquad + v_\delta(t,T)dW^{T+\delta}(t) + \int_{\mathbb{R}} \gamma_2(t,T,x)\tilde{N}^{T+\delta}(dt,dx), \qquad (4.6)
\end{aligned}$$

where $W^{T+\delta}(t)$ is a $Q^{T+\delta}$-Brownian motion and $\tilde{N}^{T+\delta}(dt,dx)$ is a $Q^{T+\delta}$-compensated Poisson random measure.

Proof. Define the probability $Q^{T+\delta}$ on $\mathcal{F}_{T+\delta}$ by

$$dQ^{T+\delta} = Z(T+\delta)dP,$$

where for two predictable processes $h(t)$ and $\theta(t,x) \leq 1$, the process

$$Z(t) := \mathcal{E}_t\left(h \circ W + (-\theta) \circ \tilde{N}\right)$$

exists for $t \leq T+\delta$ and satisfies $E\left[Z(T+\delta)\right] = 1$. Then, by Girsanov's theorem (Thm. 2.3.9)

$$dW^{T+\delta}(t) = h(t)dt + dW(t)$$

is a $Q^{T+\delta}$-Brownian motion, and

$$\tilde{N}^{T+\delta}(dt,dx) = \theta(t,x)\nu(dx)dt + \tilde{N}(dt,dx)$$

is a $Q^{T+\delta}$-compensated Poisson random measure. Then, we can rewrite the $T+\delta$-bond discounted T-bond price process in Cor. 4.2.1 as

$$\frac{dY(t)}{Y(t-)} = \left(\beta(t,T) + \int_{\mathbb{R}} \gamma_1(t,T,x)\nu(dx)\right)dt + v_\delta(t,T)\left(dW^{T+\delta}(t) - h(t)dt\right) + \int_{\mathbb{R}} \gamma_2(t,T,x)\left(\tilde{N}^{T+\delta}(dt,dx) - \theta(t,x)\nu(dx)dt\right).$$

By rearranging the terms we find the required expression. ∎

Almost everything we need to state and prove the HJM-Lévy drift condition is in place. That is, by Lemma 3.3.1a) $Y(t)$ is a $Q^{T+\delta}$-martingale, and by Lemma 4.2.1 there exists a version of $Y(t)$ where it is expressed by integrals over $W^{T+\delta}(t)$ and $\tilde{N}^{T+\delta}(t,U)$. We know that these integrals are $Q^{T+\delta}$-martingales under appropriate conditions, and thus we can utilize this fact to find the HJM-Lévy drift condition that ensures $Y(t)$ to be a $Q^{T+\delta}$-martingale, such as Lemma 3.3.1a) states. First we state a lemma which gives the appropriate conditions for the stochastic exponential, expressed by integrals over $W^{T+\delta}(t)$ and $\tilde{N}^{T+\delta}(t,U)$, to be a $Q^{T+\delta}$-martingale.

Lemma 4.2.2 (Exponential martingale). *Let $f_1(t,T)$ and $f_2(t,T,x)$ be stochastic processes, such that*

1. $(f_1(t,T), f_2(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U)$*;*
2. $f_2(t,T,x) \geq -1$*;*
3. $\log\left(1 + f_2(t,T,x)\right) \in \mathcal{U}([0,\mathcal{T}]_2^3 \times U)$*;*
4. $\mathcal{E}_t\left(f_1 \circ W + f_2 \circ \tilde{N}\right) \in L^1(\Omega)$.

Consider the SDE

$$\frac{dX(t)}{X(t-)} = f_1(t,T)dW(t) + \int_{\mathbb{R}} f_2(t,T,x)\tilde{N}(dx,dt),$$

with solution

$$X(t) = X(0)\mathcal{E}_t\left(f_1 \circ W + f_2 \circ \tilde{N}\right).$$

Then $X(t)$ is a P-martingale if and only if $E[X(t)] = X(0)$ for all $t \leq \mathcal{T}$.

Proof. According to App. A.2, the solution of the SDE is easily seen to be the stochastic exponential as stated. Also, according to Notat. 2.1.2, 2.1.3 and 2.3.2, Eq. 2.6, Thm. 2.3.4, and conditions 1., 2. and 3. in the lemma, the process

$$\begin{aligned}
&f_1 \circ W + f_2 \circ \tilde{N} \\
&= f_1(t)dW(t) + \left(\int_{|x|<R} \Big(\log\big(1 + f_2(t,x)\big) - f_2(t,x) \Big) \nu(dx) - \frac{1}{2} f_1^2(t) \right) dt \\
&\quad + \int_{\mathbb{R}} \log\big(1 + f_2(t,x)\big) \tilde{N}(dx, dt)
\end{aligned}$$

is a well defined Itô-Lévy process. We want to use Thm. 2.4.1 to prove the lemma, and thus have to show that $X(t)$ is a local martingale. It is well known from stochastic analysis that a process $X(t)$ is a local martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \leq \mathcal{T}}$ if there exists an increasing sequence of $\mathcal{F}_t$-stopping times τ_k, such that $\tau_k \to \infty$ a.s. as $k \to \infty$, and such that $Z(t \wedge \tau_k)$ is a $\mathcal{F}_t$-martingale for all t.[Øks10] The following local martingale proof is a further developed version of a proof presented by the author in an oral exam at UiO, where it was proved that the geometric Brownian motion is a local martingale. Define the increasing sequence of stopping times

$$\tau_k = \left\{ t > 0 : \left(\int_0^t X^2(s-) f_1^2(s,T) ds + \int_0^t X^2(s-) \int_{\mathbb{R}} f_2^2(s,T,x) \nu(dx) ds \right) \geq k \right\}.$$

Then we have that

$$\begin{aligned}
X(t \wedge \tau_k) =& \mathcal{E}_{t \wedge \tau_k} \left(f_1 \circ W + f_2 \tilde{N} \right) \\
=& \mathcal{E}_t \left(\mathbb{1}_{[s,\infty]}(\tau_k) f_1 \circ W \right) \\
& \exp\left(\int_0^t \mathbb{1}_{[s,\infty]}(\tau_k) \Bigg(\int_{|x|<R} \Big(\log\big(1 + f_2(s,T,x)\big) - f_2(s,T,x) \Big) \nu(dx) \right. \\
& \left. + \int_{\mathbb{R}} \log\big(1 + f_2(s,T,x)\big) \tilde{N}(dx, ds) \Bigg) ds \right),
\end{aligned}$$

and by use of App. A.2 we find after some calculations that this is equivalent to (when $u := t \wedge \tau_k$)

$$\begin{aligned}
\frac{dX(u)}{X(u-)} =& \mathbb{1}_{\{[u,\infty]\}}(\tau_k) f_1(u,T) dW(u) - \mathbb{1}_{\{[u,\infty]\}}(\tau_k) \int_{|x|<R} f_2(u,T,x) \nu(dx) du \\
& + \int_{|x|<R} \left(\big(1 + f_2(u,T,x)\big)^{\mathbb{1}_{\{[u,\infty]\}}(\tau_k)} - 1 \right) \nu(dx) du \\
& + \int_{\mathbb{R}} \left(\big(1 + f_2(u,T,x)\big)^{\mathbb{1}_{\{[u,\infty]\}}(\tau_k)} - 1 \right) \tilde{N}(dx, du).
\end{aligned} \tag{4.7}$$

We have to look at the two cases $\mathbb{1}_{\{[u,\infty]\}}(\tau_k) = 0$ and $\mathbb{1}_{\{[u,\infty]\}}(\tau_k) = 1$ separately. When $\mathbb{1}_{\{[u,\infty]\}}(\tau_k) = 0$ we see that

$$\frac{dX(u)}{X(u-)} = \int_{|x|<R} (1 - 1)\, \nu(dx) du + \int_{\mathbb{R}} (1 - 1)\, \tilde{N}(dx, du) = 0,$$

and when $\mathbb{1}_{\{[u,\infty]\}}(\tau_k) = 1$ we see that

$$\begin{aligned}\frac{dX(u)}{X(u-)} =& f_1(u,T)dW(u) - \int_{|x|<R} f_2(u,T,x)\nu(dx)du \\ &+ \int_{|x|<R} \big(1 + f_2(u,T,x) - 1\big)\,\nu(dx)du \\ &+ \int_{\mathbb{R}} \big(1 + f_2(u,T,x) - 1\big)\,\tilde{N}(dx,du) \\ =& f_1(u,T)dW(u) + \int_{\mathbb{R}} f_2(u,T,x)\tilde{N}(dx,du)\end{aligned}$$

This means that the expression in Eq. (4.7) is equivalent to writing

$$\frac{dX(u)}{X(u-)} = \mathbb{1}_{\{[u,\infty]\}}(\tau_k) f_1(u,T)dW(u) + \int_{\mathbb{R}} \mathbb{1}_{\{[u,\infty]\}}(\tau_k) f_2(u,T,x)\tilde{N}(dx,du),$$

which gives us

$$\begin{aligned}X(u) =& 1 + \int_0^t \mathbb{1}_{\{[s,\infty]\}}(\tau_k) X(s-) f_1(s,T)dW(s) \\ &+ \int_0^t \mathbb{1}_{\{[s,\infty]\}}(\tau_k) X(s-) \int_{\mathbb{R}} f_2(s,T,x)\tilde{N}(dx,ds) \\ =& 1 + \int_0^u X(s-) f_1(s,T)dW(s) + \int_0^u X(s-) \int_{\mathbb{R}} f_2(s,T,x)\tilde{N}(dx,ds).\end{aligned}$$

By definition of the sequence of stopping times we have that

$$\big(X(t-)f_1(t,T), X(t-)f_2(t,T,x)\big) \in \mathcal{U}\left([0,\mathcal{T}]_2^3 \times U\right),$$

which means that $X(t \wedge \tau_k)$ is a $\mathcal{F}_t$-martingale. The stopped process is a martingale for every k, and thus $X(t)$ is a local martingale. Finally, according to condition 4. in the lemma, the expectation of the stochastic exponential is finite and thus well defined. By these facts we can conclude that $X(t)$ is a martingale if and only if $E\big[X(t)\big] = 1$ for all $t \leq \mathcal{T}$, according to Thm. 2.4.1. ∎

In Lemma 4.2.2 we state

$$\mathcal{E}_t\left(f_1 \circ W + f_2 \circ \tilde{N}\right) \in L^1(\Omega) \tag{4.8}$$

as one of the conditions. There exist results which ensure that this condition generally holds for stochastic processes $f_1(t,T)$ and $f_2(t,T,x)$, but we will not dig into these results here. By the theory introduced in this thesis we can ensure the condition in Eq. (4.8) to hold for several special cases of $f_1(t,T)$ and $f_2(t,T,x)$ (as long as the other conditions in Lemma 4.2.2 hold as well). That is, e.g. if

- $f_1(t,T)$ is a stochastic process and the jump-part is zero, then Eq. (4.8) holds by Novikov's condition (Thm. 2.1.2);
- $f_2(t,T,x)$ is a constant and $f_1(t,T) = 0$, then Eq. (4.8) holds by the finite exponential moment condition in Thm. 2.3.2;

- $f_1(t,T)$ is a stochastic process which is independent of $\tilde{N}(t,U)$ and $f_2(t,T,x)$ is a constant, then Eq. (4.8) holds by Novikov's condition and the finite exponential moment condition together, because of independence between $W(t)$ and $\tilde{N}(t,U)$ (see Thm. 2.3.1).

With Lemma 4.2.2 in place we are ready to state and prove the HJM-Lévy drift condition. We also note that this lemma is inevitable to derive the extended LIBOR market model in the next section, because we then can utilize the martingale-property of $Y(t)$ to define the LIBOR forward rates.

Three more assumptions are needed in order to state a well defined HJM-Lévy drift condition. First, the SDE in Lemma 4.2.1 has the solution

$$Y(t) = Y(0)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

by App. A.2, and is well defined by the following assumption.

Assumption 4.2.2. We assume that the four listed conditions in Lemma 4.2.2 hold for the stochastic exponential

$$\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right).$$

Secondly, for the $(T+\delta)$-bond discounted T-bond price process to be a martingale, as required, we need the following assumption.

Assumption 4.2.3. We assume that the stochastic exponential

$$Y(t) = Y(0)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

is such that

$$E\left[Y(t)\right] = Y(0).$$

Thirdly, we need this last assumption for the HJM-Lévy drift condition to make sense.

Assumption 4.2.4. We assume that

$$\int_{\mathbb{R}} \gamma_\delta(t,T,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du}\left(1-\theta(t,x)\right) - \mathbb{1}_{\{|x|<R\}}\right)\nu(dx) < \infty.$$

Theorem 4.2.1 (The HJM-Lévy drift condition). *Let $Q^{T+\delta}$ be the forward measure as described in Lemma 4.2.1. Then the HJM-Lévy drift condition on the instantaneous forward rate model $f(t,T)$ represented in Eq. (4.1) is given by*

$$\begin{aligned}\alpha_\delta(t,T) =& \sigma_\delta(t,T)\left(h(t) - \int_T^{T+\delta}\sigma(t,u)du\right)\\ &- \int_{\mathbb{R}}\gamma_\delta(t,T,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du}\left(1-\theta(t,x)\right) - \mathbb{1}_{\{|x|<R\}}\right)\nu(dx),\end{aligned}$$

where $z_\delta(\cdot,T) := z(\cdot,T) - z(\cdot,T+\delta)$.

Proof. By the remark in Thm. 2.3.8, the SDE stated in Eq. (4.6) is seen to have the solution given in App. A.2. That is,

$$\begin{aligned} Y(t) =& Y(0)e^{\int_0^t \left(\beta(t,T)-h(t)v_\delta(t,T)+\int_{\mathbb{R}}\left(\gamma_1(t,T,x)-\theta(t,x)\gamma_2(t,T,x)\right)\nu(dx)\right)dt} \\ & \mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right). \end{aligned} \tag{4.9}$$

By Assum. 4.2.2 and 4.2.3 we know from Lemma 4.2.2 that the stochastic exponential

$$Y(0)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

is a $Q^{T+\delta}$-martingale. Therefore, it is easy to realize that the condition

$$\beta(t,T) - h(t)v_\delta(t,T) = \int_{\mathbb{R}} \left(\theta(t,x)\gamma_2(t,T,x) - \gamma_1(t,T,x)\right)\nu(dx). \tag{4.10}$$

ensures $Y(t)$ to be a martingale. The only remaining task is thus to rewrite the condition in Eq. (4.10), such that it is expressed as a condition on the drift of the instantaneous forward rate $f(t,T)$ (defined in Eq. (4.1)). From Prop. 4.1.1 and 4.1.2, and Cor. 4.2.1, we know that the left hand side of Eq. (4.10) is given by

$$\begin{aligned} &\beta(t,T) - h(t)v_\delta(t,T) \\ &\quad = \int_T^{T+\delta} \alpha(t,u)du + \frac{1}{2}\left(\int_T^{T+\delta} \sigma(t,u)du\right)^2 - h(t)\int_T^{T+\delta}\sigma(t,u)du, \end{aligned}$$

and that the right hand side is given by

$$\begin{aligned} &\int_{\mathbb{R}} \left(\theta(t,x)\gamma_2(t,T,x) - \gamma_1(t,T,x)\right)\nu(dx) \\ &= \int_{\mathbb{R}} \Bigg(\theta(t,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du} - 1\right) \\ &\qquad - \left(e^{\int_T^{T+\delta}\gamma(t,u,x)du} - 1 - \mathbb{1}_{\{|x|<R\}}\int_T^{T+\delta}\gamma(t,u,x)du\right)\Bigg)\nu(dx), \\ &= \int_{\mathbb{R}} \left((\theta - 1)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du} - 1\right) + \mathbb{1}_{\{|x|<R\}}\int_T^{T+\delta}\gamma(t,u,x)du\right)\nu(dx). \end{aligned}$$

Differentiating both sides with respect to T we find that

$$\begin{aligned} &\frac{\partial}{\partial T}\left(\beta(t,T) - h(t)v_\delta(t,T)\right) \\ &\qquad = \alpha(t,T+\delta) - \alpha(t,T) + \int_T^{T+\delta}\sigma(t,u)du\left(\sigma(t,T+\delta) - \sigma(t,T)\right) \\ &\qquad\quad - h(t)\left(\sigma(t,T+\delta) - \sigma(t,T)\right) \\ &\qquad = \sigma_\delta(t,T)\left(h(t) - \int_T^{T+\delta}\sigma(t,u)du\right) - \alpha_\delta(t,T), \end{aligned}$$

and that

$$\frac{\partial}{\partial T}\int_{\mathbb{R}}\big(\theta(t,x)\gamma_2(t,T,x-\gamma_1(t,T,x))\big)\,\nu(dx)$$
$$=\int_{\mathbb{R}}\Bigg(\big(\theta(t,x)-1\big)\,e^{\int_T^{T+\delta}\gamma(t,u,x)du}\,\big(\gamma(t,T+\delta,x)-\gamma(t,T,x)\big)$$
$$+\mathbb{1}_{\{|x|<R\}}\,\big(\gamma(t,T+\delta,x)-\gamma(t,T,x)\big)\Bigg)\nu(dx)$$
$$=\int_{\mathbb{R}}\gamma_\delta(t,T,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du}\,\big(1-\theta(t,x)\big)-\mathbb{1}_{\{|x|<R\}}\right)\nu(dx).$$

So, by the last calculations, we can rewrite Eq. (4.10) as

$$\sigma_\delta(t,T)\left(h(t)-\int_T^{T+\delta}\sigma(t,u)du\right)-\alpha_\delta(t,T)$$
$$=\int_{\mathbb{R}}\gamma_\delta(t,T,x)\left(e^{\int_T^{T+\delta}\gamma(t,u,x)du}\,\big(1-\theta(t,x)\big)-\mathbb{1}_{\{|x|<R\}}\right)\nu(dx),$$

which gives us the HJM-Lévy drift condition as stated. ■

We can conclude this section by noting that with the HJM-Lévy drift condition on the instantaneous forward rate $f(t,T)$, and by all assumptions so far, we have a SDE for its associated $(T+\delta)$-bond discounted T-bond price process of the form

$$\frac{dY(t)}{Y(t-)}=v_\delta(t,T)dW^{T+\delta}(t)+\int_{\mathbb{R}}\gamma_2(t,T,x)\tilde{N}^{T+\delta}(dt,dx). \tag{4.11}$$

The solution of this SDE is

$$Y(t)=Y(0)\mathcal{E}_t\left(v_\delta\circ W^{T+\delta}+\gamma_2\circ\tilde{N}^{T+\delta}\right),$$

and it fulfills the martingale property for discounted zero-coupon bond prices which is stated in Lemma 3.3.1a) in the preliminaries, because it is a $Q^{T+\delta}$-martingale.

4.3 The Extended LIBOR forward rate

In this section we will use the theory from the last section to derive an extended model for the LIBOR forward rate. That is, we derive a LIBOR forward rate which is driven by a geometric Itô-Lévy process, for the benefit of a geometric Brownian motion with deterministic volatility, which is the model derived in [Fil09].

From Def. 3.3.1 we know that the LIBOR forward rate is defined as

$$L(t,T)=\frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)}-1\right).$$

According to Lemma 3.3.1a) $\frac{P(t,T)}{P(t,T+\delta)}$ is a $Q^{T+\delta}$-martingale, and thus, $L(t,T)$ have to be a $Q^{T+\delta}$-martingale as well. We define the extended LIBOR forward rate directly under the probability measure $Q^{T+\delta}$, as this eases the notation considerably. Combining Def. 3.3.1 with Eq. (4.11) gives the dynamics

$$\begin{aligned} dL(t,T) &= \frac{1}{\delta} d\left(\frac{P(t,T)}{P(t,T+\delta)}\right) \\ &= \frac{P(t-,T)}{\delta P(t-,T+\delta)}\left(v_\delta(t,T)dW^{T+\delta}(t) + \int_{\mathbb{R}} \gamma_2(t,T,x)\tilde{N}^{T+\delta}(dx,dt)\right) \end{aligned}$$

for the LIBOR forward rate. From Def. 3.3.1 it is also easy to deduce that $\frac{P(t,T)}{P(t,T+\delta)} = \big(\delta L(t,T)+1\big)$, which allows us to rewrite the above LIBOR forward rate dynamics as

$$\begin{aligned} dL(t,T) = \frac{1}{\delta}\Bigg(&\big(\delta L(t,T)+1\big)\, v_\delta(t,T)dW^{T+\delta}(t) \\ &+ \big(\delta L(t-,T)+1\big)\int_{\mathbb{R}} \gamma_2(t,T,x)\tilde{N}^{T+\delta}(dx,dt)\Bigg). \end{aligned}$$

To be able to rewrite the dynamics in a neat way, we define the stochastic processes

$$\lambda(t,T) := \frac{\delta L(t,T)+1}{\delta L(t,T)} v_\delta(t,T) \quad \text{and} \quad \xi(t,T,x) := \frac{\delta L(t-,T)+1}{\delta L(t-,T)}\gamma_2(t,T,x),$$

and add the following assumption to them.

Assumption 4.3.1. We assume that $\lambda(t,T)$ and $\xi(t,T,x)$ are such that the four listed conditions in Lemma 4.2.2 hold.

By Assum. 4.3.1 there exists a well defined LIBOR forward rate dynamics on the form

$$dL(t,T) = L(t,T)\lambda(t,T)dW^{T+\delta}(t) + L(t-,T)\int_{\mathbb{R}} \xi(t,T,x)\tilde{N}^{T+\delta}(dx,dt). \tag{4.12}$$

We see that this version of the LIBOR forward rate dynamics has exactly the same form as the $(T+\delta)$-bond discounted T-bond price process dynamics derived in Eq. (4.11). Dynamics of this type are known as geometric Itô-Lévy processes. The following theorem gives a formal statement of the extended LIBOR forward rate model, which we will call the LIBOR forward rate in the HJM-Lévy framework. First we state an assumption which together with Assum. 4.3.1 ensure the martingale property of the LIBOR forward rate in the HJM-Lévy framework.

Assumption 4.3.2. We assume that the stochastic exponential

$$L(t,T) = L(0,T)\mathcal{E}_t\left(v_\delta \circ W^{T+\delta} + \gamma_2 \circ \tilde{N}^{T+\delta}\right)$$

is such that

$$E\left[L(t,T)\right] = L(0,T).$$

Theorem 4.3.1 (The LIBOR forward rate in the HJM-Lévy framework). *In the HJM-Lévy framework the LIBOR forward rate model is given by*

$$L(t,T) = L(0,T)\mathcal{E}_t\left(\lambda \circ W^{T+\delta} + \xi \circ \tilde{N}^{T+\delta}\right).$$

This is called a geometric Itô-Lévy processes, and it is a martingale by Assum. 4.3.1 and 4.3.2. Its dynamics is given in Eq. (4.12).

Proof. By looking at the dynamics in Eq. (4.12), the solution of $L(t,T)$ is easily obtained from App. A.2. By Assum. 4.3.1 the model is well defined, and by Assum. 4.3.2 and Thm. 4.2.2 it is a martingale. ■

The extended LIBOR market model

In the previous we have derived an extended LIBOR forward rate model, which expresses the LIBOR forward rate $L(t,T)$ as a $Q^{T+\delta}$-martingale. That is, we have derived a model for the forward rate $L(t,T)$ prevailing at time t, applicable to the time interval $[T, T+\delta]$. In the interest rate markets we are often not only interested in modeling a forward rate with one given set of expiry T and maturity $T+\delta$, but rather in a set of multiple forward rates applicable at different time intervals. An example of when you need this set of forward rates is when you want to price caps. Such sets of forward rates are modeled with what we call term structure models. In the following we will derive a term structure model for LIBOR forward rates in the HJM-Lévy framework (Thm. 4.3.1), and we will call it the extended LIBOR market model.

Consider the term structure defined in Def. 3.3.2. Set $T_0 = 0$ such that $T_m = m\delta$ for $m = 0, \ldots, M-1$, where we then have that $T_M = M\delta = T_{n+1}$ from Def. 3.3.2. We assume that $T_M \leq \mathcal{T}$. Introduce the complete filtered probability space

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T_M]}, Q^{T_M}\right),$$

where the filtration generated by the Itô-Lévy process

$$\lambda_{T_{M-1}} \circ W^{T_M}(t) + \xi_{T_{M-1}} \circ \tilde{N}^{T_M}(t), \quad t \in [0, T_M],$$

for $\lambda_{T_{M-1}} := \lambda(t, T_{M-1})$ and $\xi_{T_{M-1}} := \xi(t, T_{M-1})$, is contained in $\{\mathcal{F}_t\}_{t\in[0,T_M]}$. The same holds for all similar complete filtered probability spaces

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T_{m+1}]}, Q^{T_{m+1}}\right)$$

and Itô-Lévy processes

$$\lambda_{T_m} \circ W^{T_{m+1}}(t) + \xi_{T_m} \circ \tilde{N}^{T_{m+1}}(t), \quad t \in [0, T_{m+1}].$$

We also require that Assum. 4.3.1 and 4.3.2 hold for all m.

Since the LIBOR forward rate is a martingale by definition, and by Thm. 4.3.1, it is fair to postulate that

$$\begin{aligned} dL(t,T_{M-1}) &= L(t,T_{M-1})\lambda(t,T_{M-1})dW^{T_M}(t) \\ &\quad + L(t-,T_{M-1})\int_{\mathbb{R}} \xi(t,T_{M-1},x)\tilde{N}^{T_M}(dx,dt), \quad t\in[0,T_{M-1}], \end{aligned}$$

which by Thm. 4.3.1 is equivalent to the martingale process

$$L(t,T_{M-1}) = L(0,T_{M-1})\mathcal{E}_t\left(\lambda_{T_{M-1}}\circ W^{T_M} + \xi_{T_{M-1}}\circ \tilde{N}^{T_M}\right).$$

Further, we define the probability measure $Q^{T_{M-1}}$ by

$$dQ^{T_{M-1}} = \mathcal{E}_{T_{M-1}}\left(\lambda_{T_{M-1}}\circ W^{T_M} + \xi_{T_{M-1}}\circ \tilde{N}^{T_M}\right)dQ^{T_M},$$

and use Girsanov's theorem to construct the $Q^{T_{M-1}}$-Brownian motion

$$dW^{T_{M-1}}(t) = dW^{T_M}(t) - \lambda(t,T_{M-1})dt$$

and the $Q^{T_{M-1}}$-compensated Poisson random measure of $N^{T_M}(dx,dt)$,

$$\tilde{N}^{T_{M-1}}(dx,dt) = \xi(t,T_{M-1},x)\nu(dx)dt + \tilde{N}^{T_M}(dx,dt).$$

We have already postulated a LIBOR forward rate model for the maturity T_M, which is equivalent to the extended LIBOR forward rate model presented in Thm. 4.3.1. Through this model we defined a new probability measure, and a pair of new processes $dW^{T_{M-1}}(t)$ and $\tilde{N}^{T_{M-1}}(dx,dt)$. A LIBOR forward rate with maturity T_{M-1} is a $Q^{T_{M-1}}$-martingale by definition, and thus it is reasonable to postulate the new model

$$\begin{aligned} dL(t,T_{M-2}) &= L(t,T_{M-2})\lambda(t,T_{M-2})dW^{T_{M-1}}(t) \\ &\quad + L(t-,T_{M-2})\int_{\mathbb{R}} \xi(t,T_{M-2},x)\tilde{N}^{T_{M-1}}(dx,dt), \quad t\in[0,T_{M-2}], \end{aligned}$$

with

$$L(t,T_{M-2}) = L(0,T_{M-2})\mathcal{E}_t\left(\lambda_{M-2}\circ W^{T_{M-1}} + \xi_{M-2}\circ \tilde{N}^{T_{M-1}}\right).$$

Further we can define a new probability measure $Q^{T_{M-2}}$ by

$$dQ^{T_{M-2}} = \mathcal{E}_{T_{M-2}}\left(\lambda_{M-2}\circ W^{T_{M-1}} + \xi_{M-2}\circ \tilde{N}^{T_{M-1}}\right)dQ^{T_{M-1}},$$

and again use Girsanov's theorem to define a $Q^{T_{M-2}}$-Brownian motion and a $Q^{T_{M-2}}$-compensated Poisson random measure of $N^{T_{M-1}}(dx,dt)$, and postulate a new equivalent LIBOR forward rate model for the expiry T_{M-3}. From here we can repeat the procedure for all T_{M-i}, where $i=3,\dots,M$. This leads to a family of M LIBOR forward rate processes $\{L(t,T_m)\}_{t\in[0,T_m]}$, each modeled as a geometric Itô-Lévy process.

Chapter 5

THE LIBOR FORWARD RATE WITH STOCHASTIC VOLATILITY

In the previous chapter we derived a very general model for the LIBOR forward rate, and thus a very general LIBOR market model. In the rest of this thesis we will only focus on the model for the LIBOR forward rate, because the calculations and analyses are easily extended to the full LIBOR market model. To be able to analyze the LIBOR forward rate in the HJM-Lévy framework further, we have to do some more specifications on the model. These could be that the model only is driven by Brownian motions, only driven by jump processes, or a combination of both. The triplet in $\mathcal{U}\left([0,\mathcal{T}]_2^3 \times U\right)$ which characterizes the model could be constants, functions, stochastic processes, or a combination. In this chapter we will consider one specific choice for the model. That is, we will consider the LIBOR forward rate in the HJM-Lévy framework driven by a Brownian motion only, when the volatility is given by an exponential negative subordinator.

5.1 The Brownian motion driven model with an exponential negative subordinator volatility

Consider the extended LIBOR forward rate model derived in Ch. 4. That is, the SDE

$$\frac{dL(t,T)}{L(t-,T)} = \lambda(t,T)dW^{T+\delta}(t) + \int_{\mathbb{R}} \xi(t,T,x)\tilde{N}^{T+\delta}(dx,dt),$$

with solution

$$L(t,T) = L(0,T)\mathcal{E}_t\left(\lambda \circ W^{T+\delta} + \xi \circ \tilde{N}^{T+\delta}\right).$$

In the following we will analyze this model with a triplet in $\mathcal{U}\left([0,\mathcal{T}]^3 \times U\right)$ such that the Brownian motion coefficient is a stochastic volatility $\lambda(t,T) = \lambda(t)$, and the jump part $\xi(t,T,x)$ is zero. Since the Brownian motion coefficient is the only non-zero coefficient in the triplet, the extended LIBOR forward rate model is reduced to a geometric Brownian motion model with stochastic volatility. For this reason we will from now on refer to $\lambda(t)$ as a $\mathcal{B}\otimes\mathcal{F}$-measurable stochastic volatility function $\lambda(t) : [0,\mathcal{T}] \times \Omega \to \mathbb{R}$ by which the LIBOR forward rate model is well defined when $\lambda(t) \in \mathcal{V}([0,\mathcal{T}])$. From Ch. 4 we know that the martingale property of $L(t,T)$ holds by the conditions in Lemma 4.2.2. However, since we are considering a model driven by a Brownian motion only, we can add a more specific assumption on the process to ensure that $L(t,T)$ is a $Q^{T+\delta}$-martingale. That is, if $\lambda(t)$ satisfies Novikov's condition (Thm. 2.1.2) we are sure that $L(t,T)$ is a

$Q^{T+\delta}$-martingale. Due to heavy calculations in the following, we will ease the notation by writing $W(t)$ instead of $W^{T+\delta}(t)$ in the rest of this chapter.

The stochastic volatility model which will be analyzed and applied in the LIBOR forward rate model in this thesis has the form

$$\lambda(t) = a + be^{-Z(t)}, \tag{5.1}$$

for $a, b \in \mathbb{R}$. We require that $\lambda(t)$ is $\mathcal{B} \otimes \mathcal{F}$-measurable and $\mathcal{F}_t$-adapted. As long as $Z(t)$ is a subordinator the range of $e^{-Z(t)}$ is the interval $(0, 1]$, and thus, since a and b are constants, we then clearly have that $\lambda(t) \in \mathcal{V}([0, \mathcal{T}])$ and $\sup_t |\lambda(t)| < \infty$. These properties will be used extensively in this chapter. We also add an assumption needed for the calculations in this chapter.

Assumption 5.1.1. We assume that the stochastic volatility $\lambda(t)$ is independent of the Poisson random measure $N(t, U)$.

Further, to be sure that $L(t, T)$ is a martingale we add the following assumption.

Assumption 5.1.2. We assume that the stochastic volatility $\lambda(t)$ satisfies Novikov's condition (Thm. 2.1.2).

One possible choice of $Z(t)$ is the non-Gaussian OU subordinator (nGOUS) whose dynamics is given by

$$dZ(t) = -\gamma Z(t)dt + \int_0^\infty xN(dx, dt), \tag{5.2}$$

for $\gamma \in \mathbb{R}$. In App. A.3 we have computed the solution of this SDE, and it is given by

$$Z(t) = Z(0)e^{-\gamma t} + \int_0^t \int_0^\infty e^{-\gamma(t-u)} xN(dx, du). \tag{5.3}$$

This leads to a stochastic volatility model of the form

$$\lambda(t) = a + b \exp\left(-Z(0)e^{-\gamma t} - \int_0^t \int_0^\infty e^{-\gamma(t-u)} xN(dx, du) \right), \tag{5.4}$$

To sum up, we want to analyze a LIBOR forward rate model of the form

$$L(t, T) = L(0, T)\mathcal{E}_t (\lambda \circ W), \tag{5.5}$$

where $\lambda(t)$ a stochastic volatility given by an exponential negative subordinator. One example of a subordinator is given in Eq. (5.3), and we are going to use this subordinator to analyze a specific model in the next chapter. By the discussion above we can conclude that $L(t, T)$ is both well defined, and a $Q^{T+\delta}$-martingale.

5.2 Characteristics of the nGOUS and the stochastic volatility

We are going to derive analytical expressions for some of the statistical characteristics of the stochastic volatility $\lambda(t)$. First we derive the characteristic function of $Z(t)$ and the joint characteristic function of $Z(t)$ at two different times. This is done because the characteristics of the stochastic volatility are dependent on the first and second exponential moment of $Z(t)$, as we will see in the following. That is, they are dependent on $\varphi_{Z(t)}(i)$ and $\varphi_{Z(t)}(2i)$. For the nGOUS $Z(t)$ to have finite exponential moments it has to satisfy the second conditions in Thm. 2.3.2, however, we will see that the following assumption holds in our case.

Assumption 5.2.1. $Z(t)$ is such that $\int_0^\infty \nu(dx) < \infty$ holds.

In the next section we will derive analytical expressions for some of the statistical characteristics of the logarithmic LIBOR forward rate as well. It is the logarithmic LIBOR forward rate which is interesting, because we are able to find exact analytical expressions of its characteristics, unlike what we are able to do for $L(t,T)$ directly. We will see that some of the characteristics of $\log L(t,T)$ are dependent on four joint exponential moments of the nGOUS $Z(t)$ at two different times, that is, on $\Phi_{Z(t)Z(s)}(\theta,\vartheta)$ for all possible combinations of $\theta,\vartheta = \{i, 2i\}$. We will see that these four exponential moments are well defined by Assum. 5.2.1 as well.

In the two following propositions we state and prove analytical expressions of the characteristic function of $Z(t)$, and of the joint characteristic function of $Z(t)$ and $Z(s)$ when $s \leq t$. Since the forthcoming calculations become somewhat messy, we will write the nGOUS as Z_t instead of $Z(t)$ in the rest of this chapter.

Proposition 5.2.1 (Characteristic function of the nGOUS). *Let Z_t be as given in Eq. (5.3). Then the characteristic function of Z_t is given by*

$$\varphi_{Z_t}(\theta) = \exp\left(i\theta Z_0 e^{-\gamma t} + \int_0^t \psi\left(\theta e^{-\gamma(t-u)}\right) du\right),$$

where

$$\psi(z) = \int_0^\infty \left(e^{ixz} - 1\right)\nu(dx)$$

is the Lévy symbol for subordinators.

Proof. By definition of characteristic functions we have that

$$\begin{aligned} E\left[e^{i\theta Z_t}\right] &= E\left[\exp\left(i\theta\left(Z_0 e^{-\gamma t} + \int_0^t\int_0^\infty e^{-\gamma(t-u)} x N(dx,du)\right)\right)\right] \\ &= \exp\left(i\theta Z_0 e^{-\gamma t}\right) E\left[\exp\left(i\int_0^t \theta e^{-\gamma(t-u)} d\mathcal{L}(u)\right)\right], \end{aligned}$$

where we rewrote the subordinator as a general Lévy process $\mathcal{L}(t)$, and used the fact that $\exp\left(i\theta Z_0 e^{-\gamma t}\right)$ is a deterministic function. Then we have by Thm. 2.4.3 and Cor. 2.3.1 that

$$E\left[e^{i\theta Z_t}\right] = \exp\left(i\theta Z_0 e^{-\gamma t}\right)\exp\left(\int_0^t \psi\left(\theta e^{-\gamma(t-u)}\right) du\right)$$

where $\psi(z) = \int_0^\infty \left(e^{ixz} - 1\right)\nu(dx)$ is the Lévy symbol for subordinators. ■

By inserting the Lévy symbol into the characteristic function such that $\theta = \{i, 2i\}$, we see that

$$\varphi_{Z_t}(ai) = \exp\left(-aZ_0 e^{-\gamma t} + \int_0^t \int_0^\infty \left(e^{-axe^{-\gamma(t-u)}} - 1\right)\nu(dx)du\right), \quad (5.6)$$

where $a = \{1, 2\}$. The domain of $e^{-axe^{-\gamma(t-u)}}$ is $[0, \infty)$, which mans that its range is $(0, 1]$, and thus the two first exponential moments (and all other positive exponential moments) are well defined by Assum. 5.2.1.

We see that the characteristic function of Z_t was straight forward to derive by use of Thm. 2.4.3 and Cor. 2.3.1. The same applies to the joint characteristic function of Z_t at two different times, just with some more calculations.

Proposition 5.2.2 (Joint characteristic function the nGOUS at two different times). *Let $s \leq t$, and let Z_t and Z_s be as given in Eq. (5.3). Then the joint characteristic function of Z_t and Z_s is*

$$\Phi_{Z_t Z_s}(\theta, \vartheta) = \exp\left(i\left(\theta e^{-\gamma t} + \vartheta e^{-\gamma s}\right) Z_0 + \int_0^t \psi\left(\phi_{\theta,\vartheta}(t, s, u)\right) du\right),$$

where

$$\psi(z) = \int_0^\infty \left(e^{ixz} - 1\right)\nu(dx)$$

is the Lévy symbol for subordinators and

$$\phi_{\theta,\vartheta}(t, s, u) := \theta e^{-\gamma(t-u)} + \mathbb{1}_{\{u \leq s\}}\vartheta e^{-\gamma(s-u)}.$$

Proof. By definition of joint characteristic functions we have that

$$
\begin{aligned}
&E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right]\\
&= E\Bigg[\exp\Bigg(i\theta Z_0 e^{-\gamma t}+\int_0^t\int_0^\infty i\theta e^{-\gamma(t-u)}xN\left(dx,du\right)\\
&\qquad\qquad + i\vartheta Z_0 e^{-\gamma s}+\int_0^s\int_0^\infty i\vartheta e^{-\gamma(s-u)}xN\left(dx,du\right)\Bigg)\Bigg]\\
&= \exp\Big(\left(i\theta e^{-\gamma t}+i\vartheta e^{-\gamma s}\right)Z_0\Big)\,E\Bigg[\exp\Bigg(\int_0^t\int_0^\infty i\theta e^{-\gamma(t-u)}xN\left(dx,du\right)\\
&\qquad\qquad + \int_0^s\int_0^\infty i\vartheta e^{-\gamma(s-u)}xN\left(dx,du\right)\Bigg)\Bigg],
\end{aligned}
$$

where we used the fact that $g(s,t) := \exp\Big(\left(i\theta e^{-\gamma t}+i\vartheta e^{-\gamma s}\right)Z_0\Big)$ is a deterministic function. Further, since $s \le t$, we have

$$
\begin{aligned}
&E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right]\\
&= g(s,t)E\left[\exp\left(\int_0^t\int_0^\infty i\left(\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)xN(dx,du)\right)\right]\\
&= g(s,t)E\left[\exp\left(i\int_0^t \phi_{\theta,\vartheta}(t,s,u)d\mathcal{L}(u)\right)\right],
\end{aligned}
$$

where we rewrote the subordinator as a general Lévy process $\mathcal{L}(t)$, and defined $\phi_{\theta,\vartheta}(t,s,u) = \theta e^{-\gamma(t-u)} + \mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}$. Then we find by Thm. 2.4.3 and Cor. 2.3.1 that

$$
E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right] = g(s,t)\exp\left(\int_0^t \psi\left(\phi_{\theta,\vartheta}(t,s,u)\right)du\right),
$$

where $\psi(z) = \int_0^\infty \left(e^{ixz}-1\right)\nu(dx)$ is the Lévy symbol for subordinators. ■

Inserting the Lévy symbol into the joint characteristic function such that $\theta,\vartheta = \{i,2i\}$, we see that

$$
\begin{aligned}
\Phi_{Z_tZ_s}(ai,bi) = \exp\Bigg(&-\left(ae^{-\gamma t}+be^{-\gamma s}\right)Z_0\\
&+\int_0^t\int_0^\infty\left(e^{-x\left(ae^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}be^{-\gamma(s-u)}\right)}-1\right)\nu(dx)du\Bigg),
\end{aligned}
$$

for $a,b=\{1,2\}$. By the same arguments as for Eq. (5.6) we can conclude that the four joint exponential moments which we are interested in are well defined by Assum. 5.2.1 (as well as all the joint exponential moments when

θ and ϑ are positive).

Further, we state and prove some of the statistical characteristics of the stochastic volatility $\lambda(t)$ in Eq. 5.1. The two first moments and the variance of $\lambda(t)$ are straight forward to derive. As the proofs are short and simple we state them together in the next proposition.

Proposition 5.2.3 (Some statistical characteristics of the stochastic volatility). *Let the stochastic volatility $\lambda(t)$ be as given in Eq. (5.1). Then its first two moments and its variance are given by*

$$E\left[\lambda(t)\right] = a + b\varphi_{Z_t}(i), \quad E\left[\lambda^2(t)\right] = a^2 + 2ab\varphi_{Z_t}(i) + b^2\varphi_{Z_t}(2i)$$

and

$$\mathit{Var}\left(\lambda(t)\right) = b^2\left(\varphi_{Z_t}(2i) - \varphi^2_{Z_t}(i)\right).$$

Proof. In the following calculations we will recognize some terms as the two first exponential moments of a process Z_t. Since the characteristic function with complex parameter equals the moment generating function, as long as it is well defined, we can state the two first exponential moments of Z_t by applying $\theta = i$ and $\theta = 2i$ to $\varphi_{Z_t}(\theta)$. The first moment of $\lambda(t)$ is given by

$$E\left[\lambda(t)\right] = E\left[a + be^{-Z_t}\right] = a + bE\left[e^{-Z_t}\right] = a + b\varphi_{Z_t}(i),$$

and the second moment is given by

$$\begin{aligned} E\left[\lambda^2(t)\right] &= E\left[\left(a + be^{-Z_t}\right)^2\right] = E\left[a^2 + 2abe^{-Z_t} + b^2e^{-2Z_t}\right] \\ &= a^2 + 2abE\left[e^{-Z_t}\right] + b^2E\left[e^{-2Z_t}\right] \\ &= a^2 + 2ab\varphi_{Z_t}(i) + b^2\varphi_{Z_t}(2i). \end{aligned}$$

Using the definition of variance we easily find that

$$\begin{aligned} \mathrm{Var}\left(\lambda(t)\right) &= E\left[\lambda^2(t)\right] - E\left[\lambda(t)\right]^2 \\ &= b^2\varphi_{Z_t}(2i) - b^2\varphi^2_{Z_t}(i). \end{aligned}$$

■

The characteristics of $\lambda(t)$ which are represented in Prop. 5.2.3 holds for any stochastic volatility of the form $\lambda(t) = a + be^{-Z_t}$, as long as Z_t is a subordinator and its two first exponential moments are well defined. We have derived explicit formulas for the characteristic function of the nGOUS Z_t in Prop. 5.2.1, and showed that its two first exponential moments are well defined by Assum. 5.2.1. Thus, we are able to state explicit expressions of the characteristics in Prop. 5.2.3 for a nGOUS-driven stochastic volatility $\lambda(t)$ (as stated in Eq. (5.4)).

It would be interesting to find a general expression of the characteristic function of the stochastic volatility as well. It turns out that the expression

$$\varphi_{\lambda(t)}(\theta) = E\left[e^{i\theta\lambda(t)}\right] = E\left[e^{i\theta(a+be^{-Z_t})}\right]$$

is not easy to manipulate further without specifying Z_t or doing an approximation. We will derive the pdf of $\lambda(t)$ with a specific nGOUS in the next chapter, and from it the interested reader can derive the characteristic function for that special case by applying a Fourier transformation.

5.3 Characteristics of the logarithmic LIBOR forward rate

As mentioned in the last section, we are interested in deriving characteristics of $\log L(t,T)$ instead of $L(t,T)$ directly, because it is possible to derive analytical formulas in the logarithmic case. These formulas are generally not very nice, and we will therefore state them with matrix notation for a neater representation. In the following we introduce the matrices which are used in the representation of the formulas. We will call them the coefficient matrix $\boldsymbol{C}$, the characteristic function matrix $\boldsymbol{\varphi_{t,s}}$ and the joint characteristic function matrix $\boldsymbol{\Phi_{t,s}}$.

Definition 5.3.1 (Coefficient-, characteristic function- and joint characteristic function matrices). Define the coefficient matrix

$$\boldsymbol{C} = \begin{bmatrix} 4a^2 & 2ab \\ 2ab & b^2 \end{bmatrix}$$

where $a, b \in \mathbb{R}$, the characteristic function matrix

$$\boldsymbol{\varphi}_{t,s} = \begin{bmatrix} \varphi_{Z_t}(i)\varphi_{Z_s}(i) & \varphi_{Z_t}(i)\varphi_{Z_s}(2i) \\ \varphi_{Z_t}(2i)\varphi_{Z_s}(i) & \varphi_{Z_t}(2i)\varphi_{Z_s}(2i) \end{bmatrix}$$

where $\varphi_{Z_t}(\theta)$ is the characteristic function of Z_t with parameter θ, and the joint characteristic function matrix

$$\boldsymbol{\Phi}_{t,s} = \begin{bmatrix} \Phi_{Z_tZ_s}(i,i) & \Phi_{Z_tZ_s}(i,2i) \\ \Phi_{Z_tZ_s}(2i,i) & \Phi_{Z_tZ_s}(2i,2i) \end{bmatrix}$$

where $\Phi_{Z_tZ_s}(\theta,\vartheta)$ is the joint characteristic function of Z_t and Z_s with parameters θ and ϑ.

As a reminder, the stochastic volatility is as given in Eq. (5.1). Then the characteristics of $\log L(t,T)$, and the characteristics of the log-returns of $L(t,T)$ which will be derived in the next section, are dependent on $\varphi_{Z_t}(\theta)$ and $\Phi_{Z_tZ_s}(\theta,\vartheta)$. By this fact we know that the formulas we find for their characteristics hold for any subordinator Z_t, as long as the two first exponential moments of Z_t and the four first joint exponential moments of Z_t and Z_s are well defined. By Assum. 5.2.1 this holds for the nGOUS in Eq. (5.3), and thus we know that the formulas are well defined in that special case.

First we state the expectation of the logarithmic LIBOR forward rate. It is easy to derive, and is a function of the expectation of the squared stochastic volatility, which we derived in the last section.

Proposition 5.3.1 (Expectation of the logarithmic LIBOR forward rate). *Let $L(t,T)$ be as given in Eq. (5.5). Then the expectation of $\log L(t,T)$ is given by*

$$E\left[\log L(t,T)\right] = \log L(0,T) - \frac{1}{2}\int_0^t E\left[\lambda^2(u)\right] du,$$

where $E\left[\lambda^2(t)\right]$ is as given in Thm. 5.2.3.

Proof. From the expression of $L(t,T)$ in Eq. (5.5), and by Notat. 2.1.2, we see that

$$E\left[\log L(t,T)\right] = \log L(0,T) + E\left[\int_0^t \lambda(u)dW(u)\right] + E\left[-\frac{1}{2}\int_0^t \lambda^2(u)du\right].$$

By Assum. 5.1.1 we know that $\lambda(t) \in \mathcal{V}([0,\mathcal{T}])$ is independent of $W(t)$, and hence the first expectation term disappears as the expectation of Itô integrals are zero. Applying Fubini's theorem (Thm. 4.1.1) on the second expectation term gives us

$$E\left[-\frac{1}{2}\int_0^t \lambda^2(u)du\right] = -\frac{1}{2}\int_0^t E\left[\lambda^2(u)\right] du, \tag{5.7}$$

and the proof is done. ■

Further we would like to derive the variance of $\log L(t,T)$. The derivation is more demanding than the derivation of its expectation, and we therefore state two lemmas to ease the proof. The first lemma represents the difference between two expectation expressions of the squared stochastic volatility $\lambda^2(t)$ at two different time instants.

Lemma 5.3.1 (Difference of expectation expressions of the squared stochastic volatility). *Let $\lambda(t)$ be as given in Eq. 5.1. Then*

$$E\left[\lambda^2(t)\lambda^2(s)\right] - E\left[\lambda^2(t)\right]E\left[\lambda^2(s)\right] = b^2 Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi_{t,s}} - \boldsymbol{\varphi_{t,s}}\right)\right),$$

where the matrices $\boldsymbol{C}$, $\boldsymbol{\Phi_{t,s}}$ and $\boldsymbol{\varphi_{t,s}}$ are as given in Def. 5.3.1.

Proof. By straight forward calculations, and by applying $\theta,\vartheta = \{i, 2i\}$ to the characteristic function of Z_t and the joint characteristic function of Z_t

at two different times, we find that

$$
\begin{aligned}
E&\left[\lambda^2(t)\lambda^2(s)\right] \\
&= E\left[\left(a+be^{-Z_t}\right)^2\left(a+be^{-Z_s}\right)^2\right] \qquad (5.8)\\
&= a^4 + 2a^3bE\left[e^{-Z_t}+e^{-Z_s}\right] + a^2b^2E\left[4e^{-Z_t-Z_s}+e^{-2Z_t}+e^{-2Z_s}\right] \\
&\quad + 2ab^3E\left[e^{-Z_t-2Z_s}+e^{-2Z_t-Z_s}\right] + b^4E\left[e^{-2Z_t-2Z_s}\right] \\
&= a^4 + 2a^3b\left(\varphi_{Z_t}(i)+\varphi_{Z_s}(i)\right) + a^2b^2\left(4\Phi_{Z_tZ_s}(i,i)+\varphi_{Z_t}(2i)+\varphi_{Z_s}(2i)\right) \\
&\quad + 2ab^3\left(\Phi_{Z_tZ_s}(i,2i)+\Phi_{Z_tZ_s}(2i,i)\right) + b^4\Phi_{Z_tZ_s}(2i,2i).
\end{aligned}
$$

Further, by use of Prop. 5.2.3 we multiply the expectation of $\lambda^2(t)$ at two different times with each other, and find that

$$
\begin{aligned}
E&\left[\lambda^2(t)\right]E\left[\lambda^2(s)\right] \\
&= a^4 + 2a^3b\left(\varphi_{Z_t}(i)+\varphi_{Z_s}(i)\right) + 2ab^3\left(\varphi_{Z_t}(i)\varphi_{Z_s}(2i)+\varphi_{Z_t}(2i)\varphi_{Z_s}(i)\right) \\
&\quad + a^2b^2\left(\varphi_{Z_t}(2i)+4\varphi_{Z_t}(i)\varphi_{Z_s}(i)+\varphi_{Z_s}(2i)\right) + b^4\varphi_{Z_t}(2i)\varphi_{Z_s}(2i).
\end{aligned}
$$

Performing the difference gives the result

$$
\begin{aligned}
E&\left[\lambda^2(t)\lambda^2(s)\right] - E\left[\lambda^2(t)\right]E\left[\lambda^2(s)\right] \\
&= 4a^2b^2\left(\Phi_{Z_tZ_s}(i,i)-\varphi_{Z_t}(i)\varphi_{Z_s}(i)\right) \\
&\quad + 2ab^3\left(\Phi_{Z_tZ_s}(i,2i)+\Phi_{Z_tZ_s}(2i,i)-\varphi_{Z_t}(i)\varphi_{Z_s}(2i)-\varphi_{Z_t}(2i)\varphi_{Z_s}(i)\right) \\
&\quad + b^4\left(\Phi_{Z_tZ_s}(2i,2i)-\varphi_{Z_t}(2i)\varphi_{Z_s}(2i)\right),
\end{aligned}
$$

where it is easy to verify that this equals the sum of the diagonal elements of the matrix product $\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{t,s}-\boldsymbol{\varphi}_{t,s}\right)$ when $\boldsymbol{C}$, $\boldsymbol{\Phi}_{t,s}$ and $\boldsymbol{\varphi}_{t,s}$ are as given in Def. 5.3.1. ■

The second lemma gives a nice expression of squared time-integrals, which are also encountered in the derivation of the variance of $\log L(t,T)$.

Lemma 5.3.2 (Squared time-integral). *Let $f(t)$ be an integrable function and $t_0 \le t$. Then*

$$
\left(\int_{t_0}^t f(s)ds\right)^2 = 2\int_{t_0}^t\int_{t_0}^t \mathbf{1}_{\{u\le s\}}f(u)f(s)duds.
$$

Proof. By the fundamental theorem of calculus we have that

$$
\frac{d}{dt}\left(\int_{t_0}^t f(s)ds\right)^2 = 2f(t)\int_{t_0}^t f(s)ds.
$$

Integrating the previous expression we find that

$$
\left(\int_{t_0}^t f(s)ds\right)^2 = 2\int_{t_0}^t f(s)\int_{t_0}^s f(u)duds = 2\int_{t_0}^t\int_{t_0}^t \mathbf{1}_{\{u\le s\}}f(u)f(s)duds,
$$

as stated. ■

To derive the formula of the variance of $\log L(t,T)$ a new filtration also has to be introduced. That is, we have to introduce the filtration generated by the stochastic volatility process $\lambda(t)$ which is defined in Eq. (5.1), such that we we can utilize the measurability-property of $\lambda(t)$ with respect to that filtration.

Definition 5.3.2 (Filtration generated by the stochastic volatility process). We define $\{\mathcal{F}_t^\lambda\}_{t\leq\mathcal{T}}$ as the filtration generated by the stochastic volatility process $\lambda(t)$ in Eq. (5.1). Notice that $\{\mathcal{F}_t^\lambda\}_{t\leq\mathcal{T}}$ is a filtration on the measurable space $(\Omega,\mathcal{F})$, and that $\mathcal{F}_t^\lambda \subset \mathcal{F}_t \ \forall t \leq \mathcal{T}$.

We finally have what we need to state and prove the variance formula of $\log L(t,T)$.

Proposition 5.3.2 (Variance of the logarithmic LIBOR forward rate). *Let the LIBOR forward rate $L(t,T)$ be as given in Eq. (5.5). Then the variance of $\log L(t,T)$ is given by*

$$\textit{Var}\left(\log L(t,T)\right) = \int_0^t \left(E\left[\lambda^2(s)\right] + \frac{b^2}{2} C(s) \right) ds,$$

where $E\left[\lambda^2(t)\right]$ is given in Prop. 5.2.3 and

$$C(s) = \int_0^s Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{u,s} - \boldsymbol{\varphi}_{u,s}\right)\right) du$$

for the matrices $\boldsymbol{C}$, $\boldsymbol{\Phi}_{t,s}$ and $\boldsymbol{\varphi}_{t,s}$ which are given in Def. 5.3.1.

Proof. From the definition of $L(t,T)$ in Eq. (5.5) and Notat. 2.1.2, we have from the addition rule of variance that

$$\begin{aligned}\text{Var}\left(\log L(t,T)\right) = \text{Var}\left(\int_0^t \lambda(s)dW(s)\right) + \frac{1}{4}\text{Var}\left(\int_0^t \lambda^2(s)ds\right) \\ - \text{Cov}\left(\int_0^t \lambda(s)dW(s), \int_0^t \lambda^2(s)ds\right).\end{aligned}$$

For simplicity we consider the three terms separately.

T16.1 By definition of variance we have that

$$\text{Var}\left(\int_0^t \lambda(s)dW(s)\right) = E\left[\left(\int_0^t \lambda(s)dW(s)\right)^2\right] - \left(E\left[\int_0^t \lambda(s)dW(s)\right]\right)^2$$

The second term disappears by Assum. 5.1.1, because the expectation of $dW(t)$ is zero for all t. Further, by Itô isometry (Thm. 2.1.1) and Fubuni's theorem (Thm. 4.1.1) the first term gives

$$\text{Var}\left(\int_0^t \lambda(s)dW(s)\right) = E\left[\int_0^t \lambda^2(s)ds\right] = \int_0^t E\left[\lambda^2(s)\right] ds.$$

T16.2 By definition of variance we have that

$$\frac{1}{4}\text{Var}\left(\int_0^t \lambda^2(s)ds\right) = \frac{1}{4}E\left[\left(\int_0^t \lambda^2(s)ds\right)^2\right] - \frac{1}{4}\left(E\left[\int_0^t \lambda^2(s)ds\right]\right)^2.$$

Applying Lemma 5.3.2 and Fubuni's theorem (Thm. 4.1.1) to the first term gives us

$$\begin{aligned}\frac{1}{4}E\left[\left(\int_0^t \lambda^2(s)ds\right)^2\right] &= \frac{1}{2}E\left[\int_0^t\int_0^t \mathbb{1}_{\{u\leq s\}}\lambda^2(u)\lambda^2(s)duds\right]\\ &= \frac{1}{2}\int_0^t\int_0^t \mathbb{1}_{\{u\leq s\}}E\left[\lambda^2(u)\lambda^2(s)\right]duds. \quad (5.9)\end{aligned}$$

Further, by first applying Fubuni's theorem (Thm. 4.1.1) to the second term, and then Lemma 5.3.2, we find that

$$\begin{aligned}-\frac{1}{4}\left(E\left[\int_0^t \lambda^2(s)ds\right]\right)^2 &= -\frac{1}{4}\left(\int_0^t E\left[\lambda^2(s)\right]ds\right)^2\\ &= -\frac{1}{2}\int_0^t\int_0^t \mathbb{1}_{\{u\leq s\}}E\left[\lambda^2(u)\right]E\left[\lambda^2(s)\right]duds.\end{aligned}$$

Finally, by adding the two resulting terms above and then use Lemma 5.3.1, we find that

$$\begin{aligned}&\frac{1}{4}\text{Var}\left(\int_0^t \lambda^2(s)ds\right)\\ &= \frac{1}{2}\int_0^t\int_0^t\left(\mathbb{1}_{\{u\leq s\}}\left(E\left[\lambda^2(u)\lambda^2(s)\right] - E\left[\lambda^2(u)\right]E\left[\lambda^2(s)\right]\right)\right)duds\\ &= \frac{b^2}{2}\int_0^t\int_0^t\left(\mathbb{1}_{\{u\leq s\}}Tr\left(\boldsymbol{C}\cdot(\boldsymbol{\Phi_{u,s}} - \boldsymbol{\varphi_{u,s}})\right)\right)duds\end{aligned}$$

T16.3 By definition of covariance we see that

$$\begin{aligned}c(t) &= -\text{Cov}\left(\int_0^t \lambda(s)dW(s), \int_0^t \lambda^2(s)ds\right)\\ &= -E\left[\int_0^t \lambda(s)dW(s)\int_0^t \lambda^2(s)ds\right] + E\left[\int_0^t \lambda(s)dW(s)\right]E\left[\int_0^t \lambda^2(s)ds\right]\\ &= -E\left[\int_0^t \lambda(s)dW(s)\int_0^t \lambda^2(s)ds\right],\end{aligned}$$

where we used Assum. 5.1.1 to find that the expectation of the Itô integral is zero. Remember the filtration $\{\mathcal{F}_t^\lambda\}_{t\leq\mathcal{T}}$, which is defined in

Def. (5.3.2). We apply the tower rule of expectations on $c(t)$ to obtain

$$\begin{aligned} c(t) &= -E\left[E\left[\int_0^t \lambda(s)dW(s)\int_0^t \lambda^2(s)ds \quad \Big| \quad \mathcal{F}_t^\lambda\right]\right] \\ &= -E\left[\int_0^t \lambda^2(s)dsE\left[\int_0^t \lambda(s)dW(s) \quad \Big| \quad \mathcal{F}_t^\lambda\right]\right], \end{aligned}$$

where we utilized the fact that $\lambda(t)$ is $\mathcal{F}_t^\lambda$-measurable. Since the Brownian motion process $W(t)$ is independent of the volatility process $\lambda(t)$ by Assum. 5.1.1, the conditional expectation of the Itô integral equals zero. This leaves us with $c(t) = 0$.

Adding the resulting terms in T16.1, T16.2 and T16.3 we obtain the desired expression. ■

5.4 Characteristics of the LIBOR forward rate log-returns

The introduction to this section is inspired by [Ben04] and [Qua]. Let $S(t_i)$ represent a stock price observed at time t_i, for $i = 0, \dots, n$. We will assume in this section that $\Delta t := t_i - t_{i-1}$ is equal to 1 (London business) day. Then the return of a stock at time t_i from the investment of the given stock at time t_{i-1} is given by

$$y(t_i) = \frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}.$$

That is, the return is measured as the growth rate, such that it is easy to compare the return of a stock with the return of other stocks. Even so, there are several benefits for using what we call log-returns over raw returns, that is

$$z(t_i) = \log\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \log S(t_i) - \log S(t_{t-1}).$$

One benefit is that stock prices often are thought to be log-normally distributed, making the log-returns more analytically tractable. It is also worth mentioning that small changes between $S(t_{t-1})$ and $S(t_i)$ gives log-returns that are approximately equal to the returns. Other benefits are listed in [Qua]. By Def. 3.3.6 we know that the LIBOR forward rate also is thought to be log-normally distributed. However, in the extended case which we are considering now, that is generally not the case. The extended model is still an exponential process, so the log-returns might be more analytically tractable in this case as well. Also, due to the other benefits listed, we stick to the tradition of considering log-returns.

Define the process $X(t) = \log L(t,T) - \log L(t-1,T)$, which is modeling LIBOR forward rate log-returns. We still assume the time measure to be 1 (London business) day. The explicit formula of $X(t)$ is easily obtained by

Eq. (5.5), that is

$$X(t) = \int_{t-1}^{t} \lambda(s)dW(s) - \frac{1}{2}\int_{t-1}^{t} \lambda^2(s)ds. \tag{5.10}$$

Since we are going to derive an autocorrelation formula for the LIBOR forward rate log-returns in this chapter, we also state that $X(t+k)$ is the LIBOR forward rate log-return with lag $k \in \mathbb{N}$. We need both the variance of $X(t)$ and the covariance between $X(t)$ and $X(t+k)$ to obtain a general autocorrelation formula. The LIBOR forward rate log-return variance follows.

Corollary 5.4.1 (Variance of LIBOR forward rate log-returns). *Let the LIBOR forward rate $L(t,T)$ be as given in Eq. (5.5), and let $X(t)$ be its log-return. Then the variance of $X(t)$ is given by*

$$\mathit{Var}\left(X(t)\right) = \int_{t-1}^{t} \left(E\left[\lambda^2(s)\right] + \frac{b^2}{2}C(s)\right) ds,$$

where $E\left[\lambda^2(t)\right]$ is given in Prop. 5.2.3 and

$$C(s) = \int_{t-1}^{s} Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{u,s} - \boldsymbol{\varphi}_{u,s}^2\right)\right) du$$

for $t-1 \le s \le t$. The matrices $\boldsymbol{C}$, $\boldsymbol{\Phi}_{t,s}$ and $\boldsymbol{\varphi}_{t,s}$ are given in Def. 5.3.1.

Proof. From Eq. (5.10) it is clear that Var$(X(t))$ is given by Prop. 5.3.2 with initial value $t_0 = t-1$ instead of zero. That is,

$$\begin{aligned}&\text{Var}\left(X(t)\right) \\ &\quad = \text{Var}\left(\log L(t,T) - \log L(t-1,T)\right) = \text{Var}\left(\log L(t,T)_{t_0=t-1}\right).\end{aligned} \tag{5.11}$$

So, using Eq. (5.11) and Prop. 5.3.2 we find the desired result. ■

Now we want to find the covariance between the LIBOR forward rate log-return at time t and the lagged time $t+k$. The derivation requires computations on several terms, so in an attempt to ease the readability of the proof, we split up the calculations such that we look at one term at a time.

Proposition 5.4.1 (Covariance between LIBOR log-return and lagged LIBOR log-return). *Let the LIBOR forward rate $L(t,T)$ be as given in Eq. (5.5), and let $k \in \mathbb{N}$ be a time-lag such that $X(t)$ is the LIBOR forward rate log-return and $X(t+k)$ is the lagged LIBOR forward rate log-return. Then the covariance between $X(t)$ and $X(t+k)$ is given by*

$$\begin{aligned}&\mathit{Cov}\left(X(t), X(t+k)\right) \\ &\quad = \frac{b^2}{4}\int_{t-1}^{t}\int_{t+k-1}^{t+k} Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{u,s} - \boldsymbol{\varphi}_{u,s}\right)\right) duds,\end{aligned}$$

where the matrices $\boldsymbol{C}$, $\boldsymbol{\Phi}_{t,s}$ and $\boldsymbol{\varphi}_{t,s}$ are given in Def. 5.3.1.

Proof. By definition of covariance we have that

$$\mathrm{Cov}\left(X(t), X(t+k)\right) = E\left[X(t)X(t+k)\right] - E\left[X(t)\right] E\left[X(t+k)\right].$$

By the expression of LIBOR forward rate log-returns in Eq. (5.10) we see that the first term in the covariance expression above becomes

$$\begin{aligned}
&E\left[X(t)X(t+k)\right] \\
&= E\Bigg[\left(\int_{t-1}^{t} \lambda(s)dW(s) - \frac{1}{2}\int_{t-1}^{t} \lambda^2(s)ds\right) \\
&\qquad\qquad \left(\int_{t+k-1}^{t+k} \lambda(s)dW(s) - \frac{1}{2}\int_{t+k-1}^{t+k} \lambda^2(s)ds\right)\Bigg] \\
&= E\left[\int_{t-1}^{t} \lambda(s)dW(s) \int_{t+k-1}^{t+k} \lambda(s)dW(s)\right] - \frac{1}{2}E\left[\int_{t-1}^{t} \lambda^2(s)ds \int_{t+k-1}^{t+k} \lambda(s)dW(s)\right] \\
&\quad - \frac{1}{2}E\left[\int_{t-1}^{t} \lambda(s)dW(s) \int_{t+k-1}^{t+k} \lambda^2(s)ds\right] + \frac{1}{4}E\left[\int_{t-1}^{t} \lambda^2(s)ds \int_{t+k-1}^{t+k} \lambda^2(s)ds\right].
\end{aligned}$$

That is, to find $\mathrm{Cov}(X(t), X(t+k))$, we have to compute five terms. We will consider these five terms separately.

T17.1 Using Itô isometry on the first term we find that

$$\begin{aligned}
&E\left[\int_{t-1}^{t} \lambda(s)dW(s) \int_{t+k-1}^{t+k} \lambda(s)dW(s)\right] \\
&\qquad = E\left[\int_{t-1}^{t+k} \mathbf{1}_{\{s\le t\}}\lambda(s)dW(s) \int_{t-1}^{t+k} \mathbf{1}_{\{s\ge t+k-1\}}\lambda(s)dW(s)\right] \\
&\qquad = E\left[\int_{t-1}^{t+k} \mathbf{1}_{\{s\le t\}}\mathbf{1}_{\{s\ge t+k-1\}}\lambda^2(s)ds\right] = 0,
\end{aligned}$$

because $\mathbf{1}_{\{s\le t\}}\mathbf{1}_{\{s\ge t+k-1\}}$ equals zero as the interval $[t+k-1, t]$ is empty.

T17.2 We want to calculate the second term in $\mathrm{Cov}(X(t), X(t+k))$. In the following calculations we first use the tower rule of expectations and then $\mathcal{F}^{\lambda}_{t+k}$-measurability.

$$\begin{aligned}
&-\frac{1}{2}E\left[\int_{t-1}^{t} \lambda^2(s)ds \int_{t+k-1}^{t+k} \lambda(s)dW(s)\right] \\
&\qquad = -\frac{1}{2}E\left[E\left[\int_{t-1}^{t} \lambda^2(s)ds \int_{t+k-1}^{t+k} \lambda(s)dW(s) \;\middle|\; \mathcal{F}^{\lambda}_{t+k}\right]\right] \\
&\qquad = -\frac{1}{2}E\left[\int_{t-1}^{t} \lambda^2(s)ds E\left[\int_{t+k-1}^{t+k} \lambda(s)dW(s) \;\middle|\; \mathcal{F}^{\lambda}_{t+k}\right]\right] = 0,
\end{aligned}$$

where we in the last step used Assum. 5.1.1.

T17.3 For the third term in $\mathrm{Cov}(X(t), X(t+k))$ we use the exact same methods and arguments as in term T17.2 to obtain

$$-\frac{1}{2}E\left[\int_{t-1}^{t}\lambda(s)dW(s)\int_{t+k-1}^{t+k}\lambda^2(s)ds\right] = 0.$$

T17.4 In the calculations for the fourth term in $\mathrm{Cov}(X(t), X(t+k))$ we combine the two integrals, and then use Fubini's theorem (Thm. 4.1.1).

$$\begin{aligned}\frac{1}{4}E\left[\int_{t-1}^{t}\lambda^2(s)ds\int_{t+k-1}^{t+k}\lambda^2(u)du\right] &= \frac{1}{4}E\left[\int_{t-1}^{t}\int_{t+k-1}^{t+k}\lambda^2(s)\lambda^2(u)duds\right]\\ &= \frac{1}{4}\int_{t-1}^{t}\int_{t+k-1}^{t+k}E\left[\lambda^2(s)\lambda^2(u)\right]duds.\end{aligned}$$

T17.5 Finally, in the last term, we easily obtain by Fubini's theorem (Thm. 4.1.1) that

$$\begin{aligned}E\left[X(t)\right]E\left[X(t+k)\right] &= E\left[-\frac{1}{2}\int_{t-1}^{t}\lambda^2(s)ds\right]E\left[-\frac{1}{2}\int_{t+k-1}^{t+k}\lambda^2(u)du\right]\\ &= \frac{1}{4}\int_{t-1}^{t}E\left[\lambda^2(s)\right]ds\int_{t+k-1}^{t+k}E\left[\lambda^2(u)\right]du\\ &= \frac{1}{4}\int_{t-1}^{t}\int_{t+k-1}^{t+k}E\left[\lambda^2(s)\right]E\left[\lambda^2(u)\right]duds.\end{aligned}$$

Now, collecting all the terms T17.1-T17.5, we find that

$$\begin{aligned}&\mathrm{Cov}\left(X(t), X(t+k)\right)\\ &\quad= \frac{1}{4}\int_{t-1}^{t}\int_{t+k-1}^{t+k}E\left[\lambda^2(s)\lambda^2(u)\right]-E\left[\lambda^2(s)\right]E\left[\lambda^2(u)\right]duds.\end{aligned}$$

By the expression in Lemma 5.3.1 we obtain the stated covariance expression. ■

Since the autocorrelation is a function of variance and covariance we now have what we need to present it as a result. The autocorrelation is an important characteristic, because it gives us a way to study the impact earlier log-returns have on current log-returns. This means that we can use the autocorrelation structure as a helping tool to predict future LIBOR forward rates.[Ben04]

Corollary 5.4.2 (Autocorrelation of LIBOR forward rate log-returns). *Let the LIBOR forward rate $L(t,T)$ be as given in Eq. (5.5), and let $k \in \mathbb{N}$ be a time-lag such that $X(t)$ is the LIBOR forward rate log-return and $X(t+k)$ is the lagged LIBOR forward rate log-return. Then the autocorrelation between*

$X(t)$ and $X(t+k)$ is given by

$$\textit{Corr}\left(X(t), X(t+k)\right)$$
$$= \frac{\frac{b^2}{4}\int_{t-1}^{t}\int_{t+k-1}^{t+k} Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{s,u}-\boldsymbol{\varphi}_{s,u}\right)\right)duds}{\left(\int_{t-1}^{t}\int_{t+k-1}^{t+k}\left(E\left[\lambda^2(s)\right]+\frac{b^2}{2}C(s)\right)\left(E\left[\lambda^2(v)\right]+\frac{b^2}{2}C(v)\right)dvds\right)^{\frac{1}{2}}},$$

where $E\left[\lambda^2(t)\right]$ is given in Prop. 5.2.3,

$$C(s) = \int_{t-1}^{s} Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{u,s}-\boldsymbol{\varphi}_{u,s}\right)\right)du$$

for $t-1 \leq s \leq t$, and

$$C(v) = \int_{t+k-1}^{v} Tr\left(\boldsymbol{C}\cdot\left(\boldsymbol{\Phi}_{w,v}-\boldsymbol{\varphi}_{w,v}\right)\right)dw$$

for $t+k-1 \leq v \leq t+k$. The matrices $\boldsymbol{C}$, $\boldsymbol{\Phi}_{t,s}$ and $\boldsymbol{\varphi}_{t,s}$ are given in Def. 5.3.1.

Proof. By definition of autocorrelation we have that

$$\text{Corr}\left(X(t), X(t+k)\right) = \frac{\text{Cov}\left(X(t), X(t+k)\right)}{\sqrt{\text{Var}\left(X(t)\right)\text{Var}\left(X(t+k)\right)}}.$$

The desired result is achieved by inserting the expressions in Cor. 5.4.1 and Prop. 5.4.1. ■

It would also be interesting to derive an expression for the autocorrelation of squared log-returns (or absolute log-returns), to consider the actual magnitude of dependency. An attempt has been made to do so, but it turns out to involve messy calculations, and thus is very time consuming to do. Firstly there are a lot of terms to consider, and secondly we have to compute time integrals over the squared stochastic volatility with powers as high as 5. It is possible to do, but we are not going to spend time on that in this thesis.

Chapter 6

MODEL ANALYSIS: A COMPOUND POISSON PROCESS WITH EXPONENTIAL JUMPS

In the last chapter we defined a specific type of the LIBOR forward rate in the HJM-Lévy framework, which was derived in Ch. 4. That is, we specified the model to be a geometric Brownian motion, $L(t,T) = L(0,T)\mathcal{E}_t\left(\lambda \circ W^{T+\delta}\right)$, where $\lambda(t)$ is a stochastic volatility of the form $\lambda(t) = a + be^{-Z(t)}$, for a subordinator $Z(t)$. We proposed the nGOUS process as a possible driver of the stochastic volatility, and derived general formulas for the nGOUS characteristic function, $\phi_{Z(t)}(\lambda)$, and its joint characteristic function at two different times, $\Phi_{Z(t)Z(s)}(\theta,\vartheta)$. Statistical characteristics of $\lambda(t)$, $\log L(t,T)$ and the LIBOR forward rate log-returns were calculated as well, and we saw that these characteristics were functions of $\phi_{Z(t)}(\theta)$ for $\theta = \{i, 2i\}$, and $\Phi_{Z(t)Z(s)}(\theta,\vartheta)$ for $\theta,\vartheta = \{i, 2i\}$. As a result these characteristics are possible to calculate for all types of subordinators $Z(t)$, as long as $Z(t)$'s two first exponential moments and four first joint exponential moments are finite. In this chapter we are going to explore the distribution of the the stochastic volatility $\lambda(t)$ when it is driven by a nGOUS, for the specific case when the nGOUS is driven by a compound Poisson process (CPP) with exponential jumps. The derivations of this chapter leaves results which makes this specific LIBOR forward rate model ready for thorough statistical analysis and calibration.

6.1 Limiting distribution of the stochastic volatility driven by a CPP nGOUS with exponential jumps

Our strategy to obtain a distribution of the nGOUS-driven stochastic volatility in Eq. (5.4), is to derive it through the probability distribution of the nGOUS $Z(t)$. In the last chapter we derived the characteristic function of $Z(t)$, and we know that the characteristic function of a real random variable uniquely defines its probability distribution. The goal is to find a distribution which is independent of time, and we will see that this is possible if we study the limiting ($t \to \infty$) characteristic function of $Z(t)$. To find an explicit limiting distribution of $Z(t)$ we have to specify a Poisson random measure for the model. In this thesis we choose to analyze the nGOUS driven by a CPP with exponential jumps. By use of the general characteristic function derived for the nGOUS in Ch. 5 we derive an explicit expression for the characteristic function in this special case. Then the limiting characteristic function is easily obtained, and we will recognize it as the characteristic function of a known probability distribution.

Proposition 6.1.1 (Characteristic function of the CPP nGOUS with exponential jumps). *Let $Z(t)$ be as given in Eq. (5.3). When the Poisson random measure is a CPP with exponential jumps, the characteristic function of $Z(t)$ is given by*

$$E\left[e^{i\theta Z(t)}\right] = g_\theta(t)\left(\frac{\mu - i\theta e^{-\gamma t}}{\mu - i\theta}\right)^{\frac{\lambda}{\gamma}},$$

where $g_\theta(t) = \exp\left(i\theta Z_0 e^{-\gamma t}\right)$. In this case the limiting characteristic function of $Z(t)$ is given by

$$\lim_{t\to\infty} E\left[e^{i\theta Z(t)}\right] = \left(\frac{\mu}{\mu - i\theta}\right)^{\frac{\lambda}{\gamma}}.$$

Proof. We know from Prop. 5.2.1 that

$$E\left[e^{i\theta Z(t)}\right] = g_\theta(t)\exp\left(\int_0^t\int_0^\infty \left(e^{ix\theta e^{-\gamma(t-u)}} - 1\right)\nu(dx)du\right),$$

where $g_\theta(t) = \exp\left(i\theta Z_0 e^{-\gamma t}\right)$. From Ex. 2.3.2 we see that the Lévy measure for a CPP is given by $\nu(U) = \lambda\mu_X$, where λ is the Poisson process intensity and μ_X is the law of the jumps. In this case the law of the jumps is the pdf of the exponential distribution given in Def. B.3.2. That is, the characteristic function takes the form

$$E\left[e^{i\theta Z(t)}\right] = g_\theta(t)\exp\left(\int_0^t\int_0^\infty \left(e^{ix\theta e^{-\gamma(t-u)}} - 1\right)\lambda\mu e^{-\mu x}dxdu\right), \tag{6.1}$$

where μ is the parameter of the exponential distribution. Now, just focusing on the integral with respect to x, we find by straight forward calculations that

$$\begin{aligned}
&\int_0^\infty \left(e^{ix\theta e^{-\gamma(t-u)}} - 1\right)\lambda\mu e^{-\mu x}dx \\
&\quad = \int_0^\infty \left(e^{x\left(i\theta e^{-\gamma(t-u)} - \mu\right)} - e^{-\mu x}\right)\lambda\mu dx \\
&\quad = \lambda\mu\left[\frac{1}{i\theta e^{-\gamma(t-u)} - \mu}e^{x\left(i\theta e^{-\gamma(t-u)} - \mu\right)} + \frac{1}{\mu}e^{-\mu x}\right]_0^\infty \\
&\quad = -\lambda\mu\left(\frac{1}{i\theta e^{-\gamma(t-u)} - \mu} + \frac{1}{\mu}\right) \\
&\quad = \frac{i\lambda\theta e^{-\gamma(t-u)}}{\mu - i\theta e^{-\gamma(t-u)}},
\end{aligned}$$

as $e^{ix\theta e^{-\gamma(t-u)}}$ is always bounded. We insert this expression into the characteristic function expressed in Eq. (6.1), and find that

$$E\left[e^{i\theta Z(t)}\right] = g_\theta(t)\exp\left(\int_0^t \frac{i\lambda\theta e^{-\gamma(t-u)}}{\mu - i\theta e^{-\gamma(t-u)}}du\right). \tag{6.2}$$

The following trick is inspired by lecture notes from a course at UiO. Consider the derivative

$$\frac{d}{du}\log\left(\mu - i\theta e^{-\gamma(t-u)}\right) = -\gamma\frac{i\theta e^{-\gamma(t-u)}}{\mu - i\theta e^{-\gamma(t-u)}}, \tag{6.3}$$

and recognize it as the integrand of the expression in Eq. (6.2) times the constant $-\frac{\lambda}{\gamma}$. Thus, by the fundamental theorem of calculus, we have that

$$\begin{aligned} E\left[e^{i\theta Z(t)}\right] &= g_\theta(t)\exp\left(-\frac{\lambda}{\gamma}\left(\log\left(\mu - i\theta\right) - \log\left(\mu - i\theta e^{-\gamma t}\right)\right)\right) \\ &= g_\theta(t)\left(\frac{\mu - i\theta e^{-\gamma t}}{\mu - i\theta}\right)^{\frac{\lambda}{\gamma}}. \end{aligned}$$

Finally, if we let $t \to \infty$, we trivially achieve the limiting characteristic function. ∎

In the following corollary we will see that the limiting characteristic function in Prop. 6.1.1 can be identified as the characteristic function of the gamma distribution. That is, in the long run the nGOUS driven by a CPP with exponential jumps will be distributed according to the gamma distribution in Def. B.3.3.

Corollary 6.1.1 (Limiting distribution of the CPP nGOUS with exponential jumps). *Let the limiting characteristic function of the nGOUS $Z(t)$ be as given in Prop. 6.1.1. Then Z is gamma-distributed as*

$$Z \sim \textit{Gamma}\left(\frac{\lambda}{\gamma}, \mu\right),$$

where $\frac{\lambda}{\gamma}$ and μ is the shape and rate of the distribution, respectively.

Proof. According to Def. B.3.3 the characteristic function of the gamma distribution is given by

$$\left(1 - \frac{i\theta}{\kappa}\right)^{-k},$$

where $\theta \in \mathbb{R}$, and k and κ is the shape and rate of the gamma distribution, respectively. From Prop. 6.1.1 the limiting characteristic function of $Z(t)$ is given by

$$\lim_{t\to\infty} E\left[e^{i\theta Z(t)}\right] = \left(\frac{\mu}{\mu - i\theta}\right)^{\frac{\lambda}{\gamma}} = \left(\frac{\mu - i\theta}{\mu}\right)^{-\frac{\lambda}{\gamma}} = \left(1 - \frac{i\theta}{\mu}\right)^{-\frac{\lambda}{\gamma}},$$

which proves that $Z(t)$ has the claimed limiting distribution. ∎

Further, we want to derive the limiting distribution of the stochastic volatility $\lambda(t)$ which is given in Eq. (5.4), when $Z(t)$ is the nGOUS driven by a CPP with exponential jumps. As $\lambda(t)$ is given by the exponential of $Z(t)$, it is possible to derive its limiting distribution by use of the limiting distribution of $Z(t)$, which is given in Cor. 6.1.1.

Theorem 6.1.1 (Limiting distribution of the exponential CPP nGOUS stochastic volatility with exponential jumps). *Let $Z \sim$ Gamma $\left(\frac{\lambda}{\gamma}, \mu\right)$ as stated in Cor. 6.1.1. Then the limiting distribution of $\lambda(t) = a + be^{-Z(t)}$ is the unit-gamma distribution. That is,*

$$\lambda \sim UG\left(\frac{\lambda}{\gamma}, \mu\right).$$

Then the cdf is given by

$$F(x) = \frac{\gamma\left(\frac{\lambda}{\gamma}, -\mu \log\left(\frac{x-a}{b}\right)\right)}{\Gamma\left(\frac{\lambda}{\gamma}\right)},$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function and $\Gamma(\cdot)$ is the gamma function (Def. B.1.1), and the pdf is given by

$$p(x) = \frac{1}{\Gamma(k)}\left|\frac{-\mu}{b}\right|\left(\frac{x-a}{b}\right)^{\mu-1}\left(-\mu\log\left(\frac{x-a}{b}\right)\right)^{k-1}.$$

Proof. Define $k = \frac{\lambda}{\gamma}$. Then we know from Cor. 6.1.1 that $Z \sim$ Gamma (k, μ). Since $\lambda = a + be^{-Z}$, the cdf of λ is equivalent to

$$F(x) = P(\lambda \le x) = P\left(a + be^{-Z} \le x\right) = P\left(Z \le -\log\left(\frac{x-a}{b}\right)\right).$$

Define $z(x) = -\log\left(\frac{x-a}{b}\right)$. Then, given the pdf of the gamma distribution in Def. B.3.3, we find that

$$\begin{aligned}
F(x) &= \int_0^{z(x)} \frac{\mu^k}{\Gamma(k)} y^{k-1} e^{-\mu y} dy \\
&= \frac{\mu^k}{\Gamma(k)} \int_0^{\mu z(x)} \left(\frac{u}{\mu}\right)^{k-1} e^{-u} \frac{du}{\mu} \\
&= \frac{1}{\Gamma(k)} \int_0^{\mu z(x)} u^{k-1} e^{-u} du \qquad (6.4)\\
&= \frac{\gamma\left(k, \mu z(x)\right)}{\Gamma(k)},
\end{aligned}$$

where we did the substitution $u = \mu y$, and where we in the last step used the definition of the incomplete gamma function which is stated in Def. B.1.1. By inserting the values for k and $z(x)$, we reach the cdf which is stated in the theorem. Now we have to show that $F(x)$ actually is the cdf of a unit-gamma random variable, as claimed. From Def. B.3.4 we know the expression of the pdf of the unit-gamma distribution, and we want to show that the pdf of λ is equivalent. By the general definition of a pdf we know that $p(x) = \frac{d}{dx}F(x)$, so by Eq. (6.4) and Leibniz rule we have that

$$\begin{aligned} p(x) &= \frac{d}{dx}\left(\frac{1}{\Gamma(k)}\int_0^{\mu z(x)} u^{k-1}e^{-u}du\right) \\ &= \frac{e^{-\mu z(x)}}{\Gamma(k)}\left(\mu z(x)\right)^{k-1}\frac{d}{dx}\left(\mu z(x)\right). \end{aligned}$$

Inserting for $z(x)$ we find that

$$\begin{aligned} p(x) &= \frac{\left(\frac{x-a}{b}\right)^{\mu}}{\Gamma(k)}\left(-\mu\log\left(\frac{x-a}{b}\right)\right)^{k-1}\left(-\frac{\mu}{b\left(\frac{x-a}{b}\right)}\right) \\ &= \frac{1}{\Gamma(k)}\left(\frac{-\mu}{b}\right)\left(\frac{x-a}{b}\right)^{\mu-1}\left(-\mu\log\left(\frac{x-a}{b}\right)\right)^{k-1}. \end{aligned}$$

We recognize this as the pdf of the unit-gamma distribution. ■

Remark. We note that the factor $\frac{-\mu}{b}$ is stated with an absolute value in the theorem, but not in the proof. We state this factor with an absolute value in the theorem, because it is stated with an absoulte value in [Cro]. This probably has something to do with the sign of the normalizing factor of the pdf. We will not dig deeper into this question here, but just assume that the absolute value of the factor is needed in order to obtain a well defined pdf.

We would of course be interested in obtaining a probability distribution for the specific LIBOR forward rate model in this chapter as well. That is, to find the probability distribution of $L(t,T) = L(0,T)\mathcal{E}_t\left(\lambda \circ W\right)$ when $\lambda(t)$ is given by the CPP nGOUS with exponential jumps. That is not an easy task, and the author is not convinced that it is possible to prove that $L(t,T)$ is distributed as any known probability distribution. At least we know some of the distributional properties of $\log L(t,T)$ from the previous chapter. We will not analyze these properties further in this thesis, other than having a look at the expected logarithmic LIBOR forward rate in the next section. Even so, we add a proposition stating the joint characteristic function for our specific nGOUS at two different times, such that the interested reader can do further statistical analyses.

Lemma 6.1.1 (Joint characteristic function of the CPP nGOUS with exponential jumps at two different times). *Let $Z(t)$ and $Z(s)$ be the nGOUS given in Eq. (5.3) at two different times, where $s \leq t$. When the Poisson random measure is a CPP with exponential jumps, the joint characteristic function of $Z(t)$ and $Z(s)$ is given by*

$$E\left[e^{i(\theta Z_t + \vartheta Z_s)}\right] = g_\theta(t)g_\vartheta(s)\left(\frac{\left(\mu - i\left(\theta e^{-\gamma t} + \vartheta e^{-\gamma s}\right)\right)\left(\mu - i\theta e^{-\gamma(t-s)}\right)}{\left(\mu - i\left(\theta e^{-\gamma(t-s)} + \vartheta\right)\right)\left(\mu - i\theta\right)}\right)^{\frac{\lambda}{\gamma}}$$

where $g_\theta(t) = \exp\left(i\theta Z_0 e^{-\gamma t}\right)$*, and equivalent for* $g_\vartheta(s)$*. In this case the limiting joint characteristic function of* $Z(t)$ *and* $Z(s)$ *is given by*

$$\lim_{s\to\infty} E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right] = \left(\frac{\mu\left(\mu - i\theta e^{-\gamma\varrho}\right)}{\left(\mu - i\left(\theta e^{-\gamma\varrho}+\vartheta\right)\right)\left(\mu - i\theta\right)}\right)^{\frac{\lambda}{\gamma}},$$

where ϱ *is a constant time difference between* t *and* s*.*

Proof. We know from Ex. 2.3.2 that the Lévy measure for a CPP is given by $\nu(U) = \lambda\mu_X$, where λ is the Poisson process intensity and μ_X is the law of the jumps. In this case the law is the pdf of the exponential distribution Def. B.3.2. Thus, by Prop. 5.2.2, the joint characteristic function takes the form

$$E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right] = \tag{6.5}$$
$$g_\theta(t)g_\vartheta(s)\exp\left(\int_0^t\int_0^\infty\left(e^{ix\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}-1\right)\lambda\mu e^{-\mu x}dxdu\right),$$

where μ is the parameter from the exponential distribution. Focusing just on the integral with respect to x, and doing the exact same calculations as in the proof of Prop. 6.1.1, we find that

$$\begin{aligned}\int_0^\infty\left(e^{ix\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}-1\right)\lambda\mu e^{-\mu x}dx \\ = \frac{i\lambda\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}{\mu - i\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}.\end{aligned}\tag{6.6}$$

Further, inserting Eq. (6.6) into the joint characteristic function expressed in Eq. (6.5), we find

$$\begin{aligned}E&\left[e^{i(\theta Z_t+\vartheta Z_s)}\right] \\ &= g_\theta(t)g_\vartheta(s)\exp\left(\int_0^t \frac{i\lambda\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}{\mu - i\left(\mathbb{1}_{\{u\le t\}}\theta e^{-\gamma(t-u)}+\mathbb{1}_{\{u\le s\}}\vartheta e^{-\gamma(s-u)}\right)}\right) \\ &= g_\theta(t)g_\vartheta(s)\exp\left(\int_0^s \frac{i\lambda\left(\theta e^{-\gamma(t-u)}+\vartheta e^{-\gamma(s-u)}\right)}{\mu - i\left(\theta e^{-\gamma(t-u)}+\vartheta e^{-\gamma(s-u)}\right)}du\right. &&(6.7)\\ &\qquad\left.+\int_s^t \frac{i\lambda\theta e^{-\gamma(t-u)}}{\mu - i\theta e^{-\gamma(t-u)}}du\right) &&(6.8)\end{aligned}$$

We recognize the derivative

$$\begin{aligned}\frac{d}{du}\log&\left(\mu - i\left(\theta e^{-\gamma(t-u)}+\vartheta e^{-\gamma(s-u)}\right)\right) \\ &= -\gamma\frac{i\left(\theta e^{-\gamma(t-u)}+\vartheta e^{-\gamma(s-u)}\right)}{\mu - i\left(\theta e^{-\gamma(t-u)}+\vartheta e^{-\gamma(s-u)}\right)}\end{aligned}$$

as the integrand of Eq. (6.7) times the constant $-\frac{\lambda}{\gamma}$, and the derivative in Eq. (6.3) as the integrand of Eq. (6.8) times the constant $-\frac{\lambda}{\gamma}$. Thus, by the fundamental theorem of calculus we have that

$$\begin{aligned}
&E\left[e^{i(\theta Z_t+\vartheta Z_s)}\right]= \\
&g_\theta(t)g_\vartheta(s)\exp\left(\frac{\lambda}{\gamma}\Bigg(-\log\left(\mu-i\left(\theta e^{-\gamma(t-s)}+\vartheta\right)\right)+\log\left(\mu-i\left(\theta e^{-\gamma t}+\vartheta e^{-\gamma s}\right)\right)\right. \\
&\qquad\left.-\log\left(\mu-i\theta\right)+\log\left(\mu-i\theta e^{-\gamma(t-s)}\right)\Bigg)\right) \\
&\qquad = g_\theta(t)g_\vartheta(s)\left(\frac{\left(\mu-i\left(\theta e^{-\gamma t}+\vartheta e^{-\gamma s}\right)\right)\left(\mu-i\theta e^{-\gamma(t-s)}\right)}{\left(\mu-i\left(\theta e^{-\gamma(t-s)}+\vartheta\right)\right)\left(\mu-i\theta\right)}\right)^{\frac{\lambda}{\gamma}}.
\end{aligned}$$

The limiting distribution when $\lim_{s\to\infty}$ is straight forward to derive. ■

6.2 Numerical analysis of the CPP nGOUS stochastic volatility with exponential jumps and the LIBOR forward rate

In this section we will do a restricted numerical analysis, just to get a feeling about the LIBOR forward rate model with exponential nGOUS stochastic volatility which is driven by a CPP with exponential jumps. Further stochastic analysis and calibration of the model is left to the interested reader. The non-calibrated LIBOR forward rate model will be implemented, and its paths will be compared to the actual LIBOR 3-month (3M) forward rate path during the year of 2018. That is, the parameters of the model is set by guessing, such that the paths of the model looks as similar to the path of the actual LIBOR 3M forward rate as possible, and such that relative error values of $E\left[\lambda(t)\right]$ and $E\left[\log L(t,T)\right]$ are acceptable. The paths of two sets of parameters will be considered, and we will also have a look at the pdf of the stochastic volatility in these specific cases.

It is not straight forward to implement the nGOUS $Z(t)$ numerically. Since this section can be viewed as a detour in this thesis, we will do the implementation as simple as possible, and thus use the following approximation to the nGOUS. From the SDE expression of $Z(t)$ in Eq. (5.2) we know from the calculations in App. A.3 that

$$d\left(Z(t)e^{\gamma t}\right)=\int_0^\infty e^{\gamma t}xN(dx,dt),$$

which we can write as

$$
\begin{aligned}
Z(t) &= Z(t)e^{\gamma t}e^{-\gamma(t+dt)} + \int_t^{t+dt}\int_0^\infty e^{\gamma s}e^{-\gamma(t+dt)}xN(dx,ds) \\
&\approx Z(t)e^{-\gamma dt} + e^{-\gamma dt}\int_t^{t+dt}\int_0^\infty xN(dx,ds) \\
&\approx Z(t)e^{-\gamma dt} + e^{-\gamma dt}\big(N(t+dt)-N(t)\big) \\
&= Z(t)e^{-\gamma dt} + e^{-\gamma dt}\sum_{k=\pi(t)+1}^{\pi(t+dt)} J_k,
\end{aligned}
$$

where we used Ex. 2.3.2 to state the Poisson random measure difference as a sum. Here $\pi(\cdot)$ is a Poisson random variable with intensity λ, and J_k is an exponential distributed random variable with rate μ. This expression is possible to implement, and the rest of the implementation is straight forward to do. The code is found in App. C.1 and C.2.

To do a simulation which is as realistic as possible we start with a simulation which produces 253 LIBOR 3M forward rate data points, one rate for each London business day in 2018. The resulting paths are graphed in Fig. 6.1, together with the actual LIBOR 3M forward rates of that year. In Tab. 6.1 the parameters for the simulation are listed, as well as the computed expectation of the simulated values of the stochastic volatility and the logarithmic LIBOR 3M forward rate, such that we can compare them to the theoretical limiting equivalents, and to the LIBOR 3M data set. It turns out that the relative error values are not too bad considering the fact that the model is non-calibrated. They are 25.7% between the limiting and simulated $E\left[\lambda(t)\right]$, and 4.3% between the limiting and simulated $E\left[\log L(t,T)\right]$. To double check the calculations done earlier we also compute the mean value of $\lambda(t)$ according to the the mean value formula of the unit-gamma distribution. From Def. B.3.4 we know that the mean is given by $a + b\left(\mu/(\mu+1)\right)^{\frac{\lambda}{\gamma}}$, and this gives $E[\lambda(t)] = 0.008352$ for the given parameter values. That is spot on the limiting mean value which is computed by the derived formula in Ch. 5 (Prop. 5.2.3), as we see in Tab. 6.1. The relative error of the expected value of $\log L(t,T)$ between the simulation values and the 2018-data is also promising, as it is as low as 4.3%.

The simulation of the daily LIBOR 3M forward rates is somewhat unstable. The relative error of the expected stochastic volatility is registered as high as about 50%, and as low as about 0.5%. This causes paths which sometimes are flat, with little variation, and paths which sometimes have big and sudden jumps. Even so, the mean relative error of $E\left[\log L(t,T)\right]$ between the theoretical limiting value and the simulated value seems to be stable, and always below 5%. The mean relative error of $E\left[\log L(t,T)\right]$ between the 2018-data and the simulated values is also quite stable, but alternates between 0.5%, and 20%.

The numbers above seem somewhat promising, but still the paths of the simulation are not able to recreate the “long” curvy movements of the

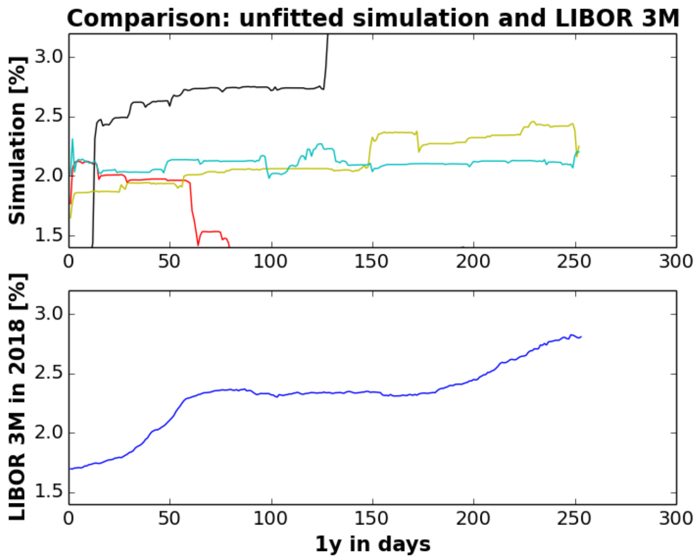

Figure 6.1: Execution No. 1. Comparison of simulated daily LIBOR 3M rates and LIBOR 3M data from 2018. The LIBOR 3M data from 2018 are obtained from [IBA].

real LIBOR 3M forward rate path. Also, most of the jumps that occur in the simulation are more drastic than the changes in the real values. Such jumps do occur in real data as well, but it is more common in graphs over a larger time interval than one year.

In Tab. 6.1 we see that both executions are done with $a = 0$ and $b = 1$, and by Def. B.3.4 we know that the distribution of $\lambda(t)$ supports values in the interval $[0, 1]$ for that specific case. In Fig. 6.2 the pdf of the stochastic volatility is graphed. It is apparent that values in the far lower end of the interval are most probable to occur. When looking at the pdf in Fig. 6.2 it might seem odd that the mean of the stochastic volatility is as high as 0.008. However, considering that the graph is continuing all the way to $\lambda(t) = 1$, where every value has a positive probability to occur, it is a reasonable value.

As we know, the LIBOR forward rate is submitted every London business day. Consider that the rate is submitted every hour, every minute, or even every second, instead of once a day. In these cases we would have a much bigger data set during one year. In the next example we increase the time resolution during a year, and make a simulation which is equivalent to submitting a new LIBOR forward rate about every 6-th minute (~ 5 minutes and 42 seconds), as long as it is a London business day. In Fig. 6.3 the resulting paths are compared with the daily LIBOR 3M forward rates in 2018.

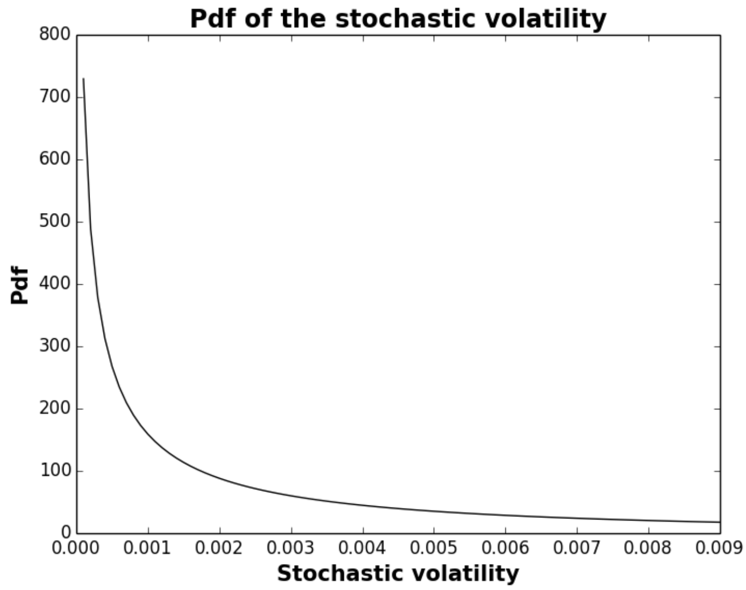

Figure 6.2: The pdf of the stochastic volatility $\lambda(t)$ when the simulation is done with daily rate submittions.

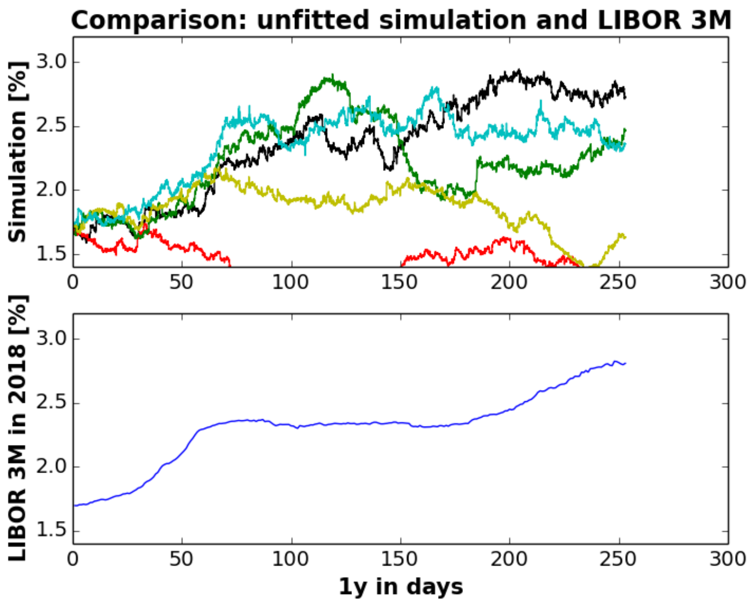

Figure 6.3: Execution No. 2. Comparison of simulated LIBOR 3M rates submitted about every 6-th minute and LIBOR 3M data from 2018. The LIBOR 3M data from 2018 are obtained from [IBA].

From Tab. 6.1 we see that the relative error values for the expectations of the stochastic volatility and the logarithmic LIBOR 3M forward rate are at least as good as the relative errors in the simulations for daily submittions. They are 1.5% between simulated and limiting values for $E\left[\lambda(t)\right]$, and 6.5% between the limiting and simulated values of $E\left[\log L(t,T)\right]$. The simulations seem stable, and the relative error values just mentioned are almost never registered to exceed 6% in executions done so far (even if the relative error value for this specific execution did). The relative error value between the simulation and the 2018-data for the expected value of $\log L(t,T)$ is as low as 0.5% from this specific execution. That is a lucky shot, because this relative error value is more fluctuating from one execution to another than the other relative error values. Relative errors touching 15% are registered.

The relative error values stated for execution no. 2 are more promising than the values for execution no. 1. Also, comparing the paths from the simulation with the path of the actual LIBOR 3M forward rate path in Fig. 6.3, we see that the high time resolution model is able to recreate the "long" curvy movements of the LIBOR 3M forward rate path, which the daily model is not able to recreate. However, one problem with the high time resolution model is that the variation of the paths are much larger than the variation of the real path. One interesting method to lower the variation of the paths of the high time resolution model is to use its moving average to model the paths, in stead of the model itself. We will not go further with this idea in this thesis, but the interested reader could give it a try.

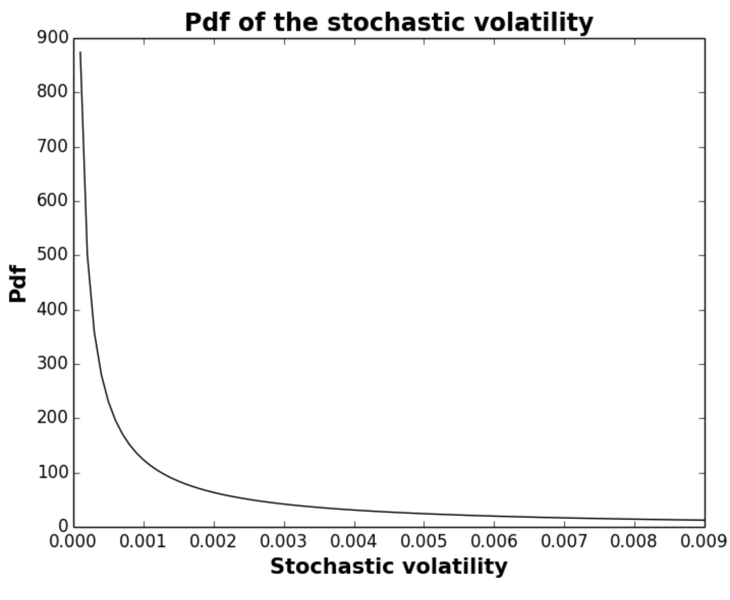

Figure 6.4: The pdf of the stochastic volatility $\lambda(t)$ when the simulation is done with rate submittions about every 6-th minute.

In Fig. 6.4 the pdf of the distribution of $\lambda(t)$ is graphed for execution no. 2. It is not easy to see the difference between the pdf of the two executions with the naked eye. The mean value formula from the unit-gamma distribution gives $E\left[\lambda(t)\right] = 0.009487$, which is slightly higher than the corresponding value for execution no. 1. This means that the pdf of execution no. 2 has slightly heavier tail. This is not a realistic result, because we would expect to have a lower volatility value between every 6-th minute, than between every day. This is probably also why the paths of the simulation in Fig. 6.3 have a great extent of variation.

Execution no.	1	2
Resolution	1 day	$1/253$ day
Parameters		
a	0.0	0.0
b	1.0	1.0
λ	3.0	1.0
μ	1.20	0.65
γ	0.38	0.20
LIBOR 3M data		
$E\left[\log L(t,T)\right]$	-3.77696	-3.77696
Limiting distr.		
$E\left[\lambda(t)\right]$	0.008352	0.009487
$E\left[\log L(t,T)\right]$	-4.13077	-4.18822
Simulation		
$E\left[\lambda(t)\right]$	0.006203	0.009628
$E\left[\log L(t,T)\right]$	-3.85968	-3.75839
Rel. error: Data and sim.		
$E\left[\log L(t,T)\right]$	0.02190	0.004915
Rel. error: Lim. distr. and sim.		
$E\left[\lambda(t)\right]$	0.2573	0.01480
$E\left[\log L(t,T)\right]$	0.04326	0.06479

Table 6.1: Data from two different executions, where execution no. 1 is done with 253 points over 253 days, and execution no. 2 is done with $63,757$ points over 253 days.

Chapter 7

CAPLET VALUATION WITH A FOURIER TRANSFORM APPROACH

In this chapter we want to explore the possibility of deriving an analytical formula for the caplet price when the LIBOR forward rate is given by the geometric Itô-Lévy process derived in Ch. 4 (Eq. (4.11)). The theory in this chapter is based on [EGP10]. First we will derive a general valuation formula by use of a given Fourier transform valuation method, and then look at the possibility of deriving an explicit analytical valuation formula for the stochastic volatility model considered in Ch. 5, by use of the general formula. We will see that it is possible to state the general caplet price for the geometric Itô-Lévy process analytically, as an integral over the process characteristic function times the Fourier transformed payoff function. However, to solve this integral for our specific stochastic volatility model, we will have to use power series. Notice that only caplet prices are considered in this chapter. The results are easily extended to the case of caps by use of Def. 3.3.2 and Eq. (3.3) in the preliminaries, and also to floorlets and floors.

7.1 Caplet valuation with a Fourier transform valuation approach on a geometric Itô-Lévy process

It is common to price options by use of Fourier transform valuation methods. An impressive Fourier transform valuation formula for general frameworks is derived in [EGP10]. The formula can be applied on models dependent on the path of the underlying financial asset, and the payoff function is allowed to be discontinuous. In our case the model only depends on the value of the interest rate process at time T when expressed in the payoff function, and the payoff function is continuous. Therefore, a version of this Fourier transform valuation formula which holds for arbitrary continuous payoff functions only, is stated in App. A.4. In the next corollary we will show that caplets can be priced with the Fourier transform valuation formula in Thm. A.4.1 within our framework. First we introduce the following notation for simplicity.

Notation 7.1.1. Let $X(t)$ be the stochastic process

$$X(t) := \int_0^t \left(\lambda \circ W^{T+\delta} + \xi \circ \tilde{N}^{T+\delta} \right),$$

where $\lambda \circ W^{T+\delta}$ and $\xi \circ \tilde{N}^{T+\delta}$ is given in Nota. 2.1.2 and 2.3.2, respectively.

Since the valuation formula considered in this chapter is based on a Fourier transform, we state its definition next.

Definition 7.1.1 (Fourier transform, [BBK08]). Let $f(x) \in L^1(\mathbb{R})$. Then the Fourier transform of $f(x)$ is defined by

$$\hat{f}(u) = \int_{\mathbb{R}} e^{iuy} f(y) dy$$

for $u \in \mathbb{R}$. The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \hat{f}(u) du.$$

A damped version of the payoff function is needed in order to ensure finiteness in the derivation of the formula in Thm. A.4.1. A damped function means that if we denote the payoff function as $f(x)$, then the damped payoff function is given by

$$g(x) = e^{-Hx} f(x), \tag{7.1}$$

where $H \in \mathbb{R}$. To be sure that the Fourier transform valuation formula exists, conditions on the damped payoff function and the moment generating function of the Itô-Lévy process $X(t)$ have to be satisfied. Let the moment generating function of $X(t)$ be denoted by M_{X_t}, and its characteristic function by φ_{X_t}. We denote by L^1_{bc} the space of functions which are bounded and continuous in L^1. Then the conditions for the Fourier transform valuation formula to exist are

1. $g \in L^1_{bc}(\mathbb{R})$;
2. $M_{X_T}(K)$ exists;
3. $\hat{g} \in L^1(\mathbb{R})$.

We are finally ready to state and prove the aforementioned corollary.

Corollary 7.1.1 (Fourier transform valuation formula for a geometric Itô-Lévy process). *Let the LIBOR forward rate be modeled as a geometric Itô-Lévy process of the form*

$$L(t,T) = L(0,T)\mathcal{E}_t \left(\lambda \circ W^{T+\delta} + \xi \circ \tilde{N}^{T+\delta} \right) = L(0,T)e^{X(t)},$$

such that Assum. 4.3.1 and 4.3.2 hold. As long as $g \in L^1_{bc}(\mathbb{R})$ and $\hat{g} \in L^1(\mathbb{R})$, the caplet price at time $t_0 = 0$ is given by

$$C(X;l_0) = \delta N P(0,T+\delta) \frac{e^{-Hl_0}}{2\pi} \int_{\mathbb{R}} e^{-iul_0} \varphi_{X_T}(u-iH) \hat{f}(iH-u) du,$$

where $l_0 := -\log L(0,T)$ and $\hat{f}(x)$ is the Fourier transformed payoff function $f(x) = (e^x - K)^+$.

Remark. The characteristic function $\varphi_{X_T}(\cdot)$ is computed with respect to the probability (forward) measure $Q^{T+\delta}$.

Proof. By Assum. 4.3.1 and 4.3.2, and by the definition of semimartingales (see e.g. [BBK08]), the Itô-Lévy process $X(t)$ is a (semi)martingale with respect to $Q^{T+\delta}$. Thus, by App. A.4 the process

$$L(t,T) = L(0,T)e^{X(t)}$$

is a (semi)martingale. By condition 4. in Lemma 4.2.2 it is also satisfied that M_{X_T} exists.

Further, by use of Proof II of Black's formula in Ch. 3, we know that the fair price of a caplet at time $t_0 = 0$ is given by

$$\delta NP(0,T+\delta)E_{Q^{T+\delta}}\left[(L(T,T)-K)^+\right]$$

in our framework. Rewriting $(L(T,T)-K)^+$ by defining $f(x) := (e^x - K)^+$ and $l_0 := -\log L(0,T)$, we find that

$$\begin{aligned}(L(T,T)-K)^+ &= \left(L(0,T)e^{X(T)} - K\right)^+\\ &= \left(e^{\left(X(T)-l_0\right)} - K\right)^+\\ &= f(X(T)-l_0),\end{aligned}$$

where $X(T)$ is as defined in Nota. 7.1.1. Thus, the fair price of a caplet at time $t_0 = 0$ can be stated as

$$\delta NP(0,T+\delta)E_{Q^{T+\delta}}\left[f(X(T)-l_0)\right],$$

where the payoff function has the same form as in Eq. (A.2), and the factor $\delta NP(0,T+\delta)$ can be viewed as a constant discount factor. Assume that the damped payoff function satisfies $g \in L^1_{bc}(\mathbb{R})$ and $\hat{g} \in L^1(\mathbb{R})$. From the proof of Thm. A.4.1 in [EGP10] we can thus conclude that the caplet price takes the form as stated, where φ_{X_T} is the characteristic function of $X(T)$ with respect to $Q^{T+\delta}$. ■

We notice that $C(X;l_0)$ is dependent of the characteristic function of the Itô-Lévy process in Nota. 7.1.1 at time T. This means that we are able to find an explicit formula for the caplet price as long as we are able to derive an analytical expression of $\varphi_{X_T}(u-iH)$, such that the integrand of $C(X;l_0)$ in Cor. 7.1.1 is integrable. If $X(T)$ was a Lévy process this would be attainable in most cases because of the Lévy-Khintchine formula (Thm. 2.3.5). This is not the case in our framework. There are almost no special cases of the process $X(T)$ that is a Lévy process. The most obvious example is when $\log(1+\xi(t,T,x)) = 0$ and $\lambda(t,T)$ is constant, that is when the model is an Itô process with constant volatility.

7.2 Caplet valuation with a Fourier transform valuation approach for a geometric Brownian motion with stochastic volatility

In the last section we proved the fact that the general Fourier transform valuation formula in Thm. A.4.1 is applicable to the LIBOR forward rate in the HJM-Lévy framework when caplets are to be priced. In Ch. 5 we specified a stochastic volatility model, and studied the LIBOR forward rate model with that stochastic volatility and zero jump-part. We want to derive an explicit caplet valuation formula for that model. That is, in this section we derive expressions which can be used to calculate the caplet price from the model

$$L(t,T) = L(0,T)\mathcal{E}_t\left(\lambda \circ W^{T+\delta}\right),$$

for any stochastic volatility satisfying the required conditions. In the next section we will derive the explicit caplet valuation formula from this model with $\lambda(t)$ as defined in Eq. (5.1), and discuss the nGOUS case.

Considering the caplet valuation formula in Cor. 7.1.1, we see that we have to find an expression for the characteristic function of

$$X(T) = \int_0^T \lambda \circ W^{T+\delta},$$

and an expression for the Fourier transform of the caplet payoff function. We state these expressions in two lemmas.

Lemma 7.2.1 (Characteristic function of $X(T)$). *Let $\lambda(t,T)$ be a stochastic volatility process. Then the characteristic function of*

$$X(T) = \int_0^T \lambda \circ W^{T+\delta}$$

is given by

$$\varphi_{X_T}(\eta) = E_{Q^{T+\delta}}\left[e^{-g(\eta)\sigma_T^2}\right],$$

where $\eta \in \mathbb{C}$,

$$g(\eta) = \frac{i\eta + \eta^2}{2} \quad \textit{and} \quad \sigma_T^2 = \int_0^T \lambda^2(t,T)dt.$$

Proof. Let $\eta \in \mathbb{C}$ and $X(T) = \int_0^T \lambda \circ W^{T+\delta}$, for some stochastic volatility process $\lambda(t,T)$. By the tower rule of expectations we find that

$$\begin{aligned}
\varphi_{X_T}(\eta) &= E_{Q^{T+\delta}}\left[e^{i\eta X_T}\right] \\
&= E_{Q^{T+\delta}}\left[\exp\left(i\eta\left(\int_0^T \lambda(t,T)dW^{T+\delta}(t) - \frac{1}{2}\int_0^T \lambda^2(t,T)dt\right)\right)\right] \\
&:= E_{Q^{T+\delta}}\left[e^{\Lambda(T)}\exp\left(i\eta\left(\int_0^T \lambda(t,T)dW^{T+\delta}(t)\right)\right)\right] \\
&= E_{Q^{T+\delta}}\left[E_{Q^{T+\delta}}\left[e^{\Lambda(T)}\exp\left(i\eta\left(\int_0^T \lambda(t,T)dW^{T+\delta}(t)\right)\right) \;\middle|\; \mathcal{F}_T^{\lambda}\right]\right],
\end{aligned}$$

where $\mathcal{F}_T^{\lambda}$ is the filtration defined in Def. 5.3.2. Now we can utilize the fact that $\lambda(t,T)$ is deterministic with respect to $\mathcal{F}_T^{\lambda}$. First, we use the measurability property of $e^{\Lambda(T)}$, such that we can move it outside the conditional expectation. Next, we recognize the expectation of the exponential Itô integral as the characteristic function of the Itô integral. It is well known from stochastic analysis that

$$\int_0^T \lambda(t,T)dW^{T+\delta}(t) \sim N\left(0, \int_0^T \lambda^2(t,T)dt\right),$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2. Thus, we find that

$$\begin{aligned}
\varphi_{X_T}(\eta) &= E_{Q^{T+\delta}}\left[e^{\Lambda(T)} \exp\left(-\frac{\eta^2}{2}\int_0^T \lambda^2(t,T)dt\right)\right] \\
&= E_{Q^{T+\delta}}\left[\exp\left(-\frac{i\eta}{2}\int_0^T \lambda^2(t,T)dt - \frac{\eta^2}{2}\int_0^T \lambda^2(t,T)dt\right)\right] \\
&:= E_{Q^{T+\delta}}\left[e^{-g(\eta)\sigma_T^2}\right],
\end{aligned}$$

where we defined

$$g(\eta) = \frac{i\eta + \eta^2}{2} \quad \text{and} \quad \sigma_T^2 = \int_0^T \lambda^2(t,T)dt.$$

■

As stated in Cor. 7.1.1, the payoff function for a caplet in our framework can be expressed as $f(x) = (e^x - K)^+$. In the following lemma we compute the Fourier transform of this payoff function.

Lemma 7.2.2 (Fourier transform of the caplet payoff function). *Let* $Re(i\tilde{u} + 1) \leq 0$. *Then the Fourier transform of the caplet payoff function* $f(x) = (e^x - K)^+$ *is given by*

$$\hat{f}(\tilde{u}) = \frac{K^{i\tilde{u}+1}}{i\tilde{u}(i\tilde{u}+1)}.$$

Proof. Let the caplet payoff function be given by $f(x) = (e^x - K)^+$. Then, by Def. 7.1.1 we have that its Fourier transform is given by

$$\begin{aligned}
\hat{f}(\tilde{u}) &= \int_{\mathbb{R}} e^{i\tilde{u}x} f(x)dx = \int_{\mathbb{R}} e^{i\tilde{u}x}\left(e^x - K\right)^+ dx \\
&= \int_{\mathbb{R}} e^{i\tilde{u}x}\left(e^x - K\right)\mathbb{1}_{\{x > \log K\}}dx,
\end{aligned}$$

where we used that $e^x - K > 0 \Longleftrightarrow x > \log K$. Further, by straight forward calculations we find that

$$\begin{aligned}\hat{f}(\tilde{u}) &= \int_{\log K}^{\infty} e^{(i\tilde{u}+1)x} - Ke^{i\tilde{u}x}dx = \left[\frac{1}{i\tilde{u}+1}e^{(i\tilde{u}+1)x} - \frac{K}{i\tilde{u}}e^{i\tilde{u}x}\right]_{\log K}^{\infty} \\ &= -\frac{1}{i\tilde{u}+1}e^{(i\tilde{u}+1)\log K} + \frac{K}{i\tilde{u}}e^{i\tilde{u}\log K} = \frac{K^{i\tilde{u}+1}}{i\tilde{u}} - \frac{K^{i\tilde{u}+1}}{i\tilde{u}+1} \\ &= K^{i\tilde{u}+1}\left(\frac{1}{i\tilde{u}\,(i\tilde{u}+1)}\right),\end{aligned}$$

where we in the third equality assumed that $Re\,(i\tilde{u}+1) \leq 0$ (which also implies that $Re\,(i\tilde{u}) < 0$). ∎

In the proof of Lemma 7.2.2 we used the fact that $Re(i\tilde{u}+1) \leq 0$. This fact has an implication on the choice of H in the damped payoff function $g(x)$, because the formula in Cor. 7.1.1 is given by the Fourier transformed payoff function when $\tilde{u} = iH - u$. That is, for the Fourier transformed payoff function to be finite we have to require that

$$Re\,\big(i(iH-u)+1\big) \leq 0 \Longleftrightarrow Re\,(1-H-iu) \leq 0 \Longleftrightarrow H \geq 1.$$

Therefore, to make sure that the caplet valuation formula is well defined, we add the following assumption.

Assumption 7.2.1. We assume that $H \geq 1$.

We also have to make sure that the conditions $g \in L^1_{bc}(\mathbb{R})$ and $\hat{g} \in L^1(\mathbb{R})$ hold in our case. By introducing a result which is provided by [EGP10], we can easily check if the condition $\hat{g} \in L^1(\mathbb{R})$ holds. Then we first have to introduce a specific Sobolev space.

Definition 7.2.1 (Sobolev space 1, [EGP10]). Define the Sobolev space $H^{1,2}(\mathbb{R})$ as the space of functions

$$H^{1,2}(\mathbb{R}) = \left\{g \in L^2(\mathbb{R}) \;\middle|\; \partial g \text{ exists and } \partial g \in L^2(\mathbb{R})\right\},$$

where ∂g denotes the weak derivative of a function (a reference to this concept is given in [EGP10]).

Then we can use the following lemma to show that $\hat{g} \in L^1(\mathbb{R})$.

Lemma 7.2.3 ([EGP10]). *Let $g \in H^{1,2}(\mathbb{R})$, then $\hat{g} \in L^1(\mathbb{R})$.*

We start by proving that the condition $g \in L^1_{bc}(\mathbb{R})$ holds, and that g is square integrable over $\mathbb{R}$ as well. From Eq. (7.1) and Cor. 7.1.1 we know that

$$g(x) = e^{-Hx}(e^x - K)^+ = e^{-Hx}(e^x - K)\mathbb{1}_{\{x > \log K\}}.$$

When $x \leq \log K$ we have that $g(x) = 0$, and it is obviously in L^1_{bc} and L^2 in that case. When $x > \log K$ we have

$$\left|g(x)\right| = \left|e^{-Hx}(e^x - K)\right| < \infty,$$

where we used the fact that the exponential function is bounded from above when $H \in [1, \infty)$ and $x > \log K$, with one maximum point. This can be verified with derivative methods. Also, the value of $g(x)$ goes to zero as x approaches $\log K$ and infinity. It is also well known that exponential functions are continuous. By these facts we know that $g(x) \in L^1_{bc}$. Now it is straight forward to show that $g(x) \in L^2$ as well, because all the facts about $\left|g(x)\right|$ holds for $\left|g(x)\right|^2$ as well. Thus, we have that $g \in L^1_{bc}(\mathbb{R}) \cap L^2(\mathbb{R})$. An example of $g(x)$ is presented in Fig. 7.1, where the properties stated above are visible.

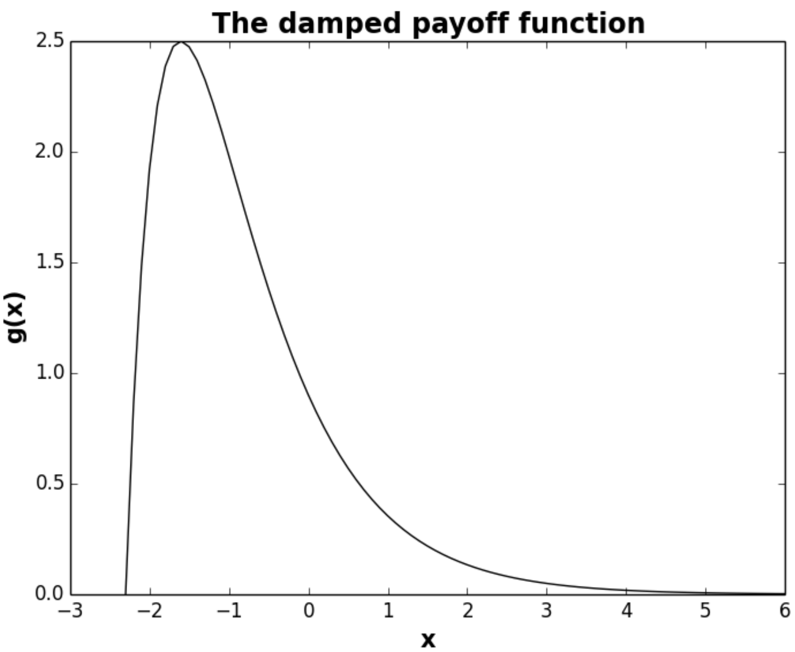

Figure 7.1: Example of a damped caplet payoff function.

Next we use Lemma 7.2.3 to prove that $\hat{g} \in L^1(\mathbb{R})$. The calculatons to find the weak derivative will not be executed here. We just state the weak derivative of g, which is given in [EGP10].

$$\partial g(x) \begin{cases} 0, & \text{if } x < \log K \\ e^{-Hx}\left(e^x - He^x + HK\right), & \text{if } x > \log K. \end{cases}$$

We obviously have that $\partial g(x) \in L^2(\mathbb{R})$ when $x < \log K$. When $x > \log K$ we can prove with derivative methods that ∂g has one minimum point, and is thus bounded by it from below. Also, when $x \to \infty$ the function goes to zero by Assum. 7.2.1, and when $x \to \log K$ it is bounded by the value of

K. Again, these arguments holds for the square of $|\partial g|$ as well, and hence $\partial g \in L^2(\mathbb{R})$. Thus, by Lemma 7.2.3 we have that $\hat{g} \in L^1(\mathbb{R})$. An example of the weak derivative of the damped caplet payoff function is presented in Fig. 7.2.

By Cor. 7.1.1, Lemma 7.2.1 and 7.2.2, and by the fact that $g \in L^1_{bc}(\mathbb{R})$ and $\hat{g} \in L^1(\mathbb{R})$ hold, we are able to derive a caplet valuation formula for the Brownian motion driven LIBOR forward rate with stochastic volatility. We will see that it is not possible to achieve an explicit analytical formula expressed without a power series, and thus we will have to use an approximation of caplet prices -in applications. We will also see that the power series contains integrals which not necessarily are easy to solve analytically, and thus may have to be computed numerically.

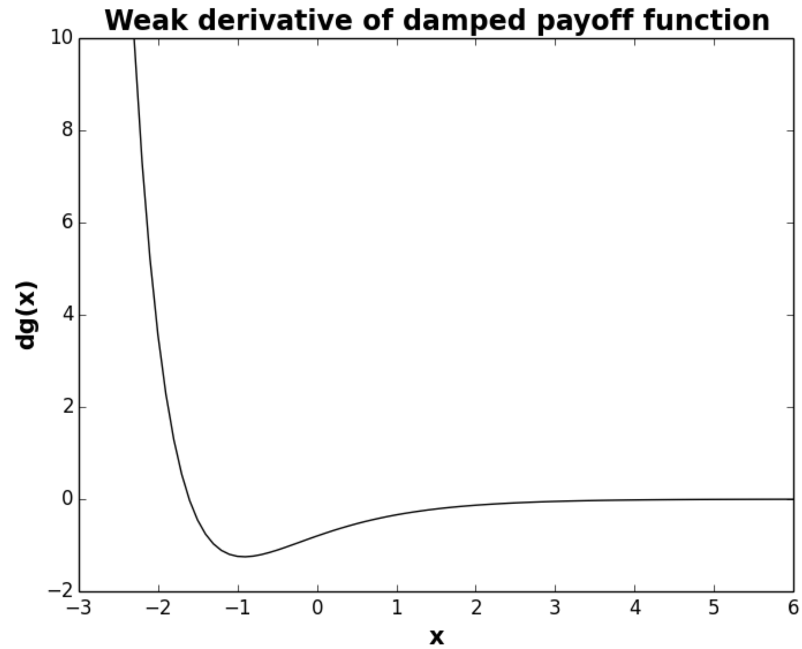

Figure 7.2: Example of the weak derivative of the damped caplet payoff function.

Theorem 7.2.1 (Caplet valuation formula for a Brownian motion driven LIBOR forward rate with stochastic volatility). *Let the caplet payoff function be given as*

$$f(X(T) - l_0) = \left(e^{\left(X(T)-l_0\right)} - K\right)^+,$$

for the measurable function

$$X(T) = \int_0^T \lambda \circ W^{T+\delta}$$

with stochastic volatility $\lambda(t,T)$. Then the caplet price formula is given by

$$C(X;l_0)$$
$$= -\delta NP(0,T+\delta)\frac{K}{2\pi}\int_{\mathbb{R}} e^{i\tilde{u}l_0}K^{i\tilde{u}}\sum_{k=0}^{\infty}\left(\frac{1}{2^k k!}\left(i\tilde{u}-\tilde{u}^2\right)^{k-1}E_{Q^{T+\delta}}\left[\sigma_T^{2k}\right]\right)d\tilde{u},$$

where $\tilde{u} = iH - u$ for $H \geq 1$.

Proof. From Cor. 7.1.1 we know that the caplet valuation formula can be expressed as

$$\begin{aligned}
C(X;l_0) &= \delta NP(0,T+\delta)\frac{e^{-Hl_0}}{2\pi}\int_{\mathbb{R}} e^{-iul_0}\varphi_{X_T}(-\tilde{u})\hat{f}(\tilde{u})du \\
&= -\delta NP(0,T+\delta)\frac{e^{-Hl_0}}{2\pi}\int_{\mathbb{R}} e^{(H+i\tilde{u})l_0}\varphi_{X_T}(-\tilde{u})\hat{f}(\tilde{u})d\tilde{u} \\
&= -\delta NP(0,T+\delta)\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\tilde{u}l_0}\varphi_{X_T}(-\tilde{u})\hat{f}(\tilde{u})d\tilde{u}, \qquad (7.2)
\end{aligned}$$

where we did the substitution $\tilde{u} = iH - u$. In our case the payoff function is given by $f(x) = (e^x - K)^+$, and the stochastic process is given by $X(T) = \int_0^T \lambda \circ W^{T+\delta}$ for a stochastic volatility $\lambda(t,T)$. That is, by use of Lemma 7.2.1 and 7.2.2 we have the expressions for the characteristic function of $X(T)$ and the Fourier transform of the caplet payoff function, which are needed to compute Eq. (7.2). This gives us a caplet valuation formula of the form

$$C(X;l_0) = -\delta NP(0,T+\delta)\frac{K}{2\pi}\int_{\mathbb{R}} e^{i\tilde{u}l_0}E_{Q^{T+\delta}}\left[e^{-g(-\tilde{u})\sigma_T^2}\right]\left(\frac{K^{i\tilde{u}}}{i\tilde{u}(i\tilde{u}+1)}\right)d\tilde{u}. \qquad (7.3)$$

It is not straight forward to compute the expectation

$$E_{Q^{T+\delta}}\left[e^{-g(-\tilde{u})\sigma_T^2}\right],$$

and it might even be impossible to state it as an explicit analytical expression. We will use the power series of e^x, which is given in Def. B.2.1, to be able to analyze the expression of $C(X;l_0)$ further. That is, considering only the integral in Eq. (7.3), we find that

$$\begin{aligned}
&\int_{\mathbb{R}} e^{i\tilde{u}l_0}E_{Q^{T+\delta}}\left[e^{-g(-\tilde{u})\sigma_T^2}\right]\left(\frac{K^{i\tilde{u}}}{i\tilde{u}(i\tilde{u}+1)}\right)d\tilde{u} \\
&\quad = \int_{\mathbb{R}} e^{i\tilde{u}l_0}E_{Q^{T+\delta}}\left[\sum_{k=0}^{\infty}\frac{1}{k!}\left(-g(-\tilde{u})\right)^k\sigma_T^{2k}\right]\left(\frac{K^{i\tilde{u}}}{i\tilde{u}(i\tilde{u}+1)}\right)d\tilde{u} \\
&\quad = \int_{\mathbb{R}} e^{i\tilde{u}l_0}\sum_{k=0}^{\infty}\left(\frac{1}{k!}\left(\frac{i\tilde{u}-\tilde{u}^2}{2}\right)^k E_{Q^{T+\delta}}\left[\sigma_T^{2k}\right]\right)\left(\frac{K^{i\tilde{u}}}{i\tilde{u}-\tilde{u}^2}\right)d\tilde{u} \\
&\quad = \int_{\mathbb{R}} e^{i\tilde{u}l_0}K^{i\tilde{u}}\sum_{k=0}^{\infty}\left(\frac{1}{2^k k!}\left(i\tilde{u}-\tilde{u}^2\right)^{k-1}E_{Q^{T+\delta}}\left[\sigma_T^{2k}\right]\right)d\tilde{u},
\end{aligned}$$

where we used the fact that $g(-\tilde{u}) = \frac{\tilde{u}^2 - i\tilde{u}}{2}$. Inserting this expression into Eq. (7.3) we find the desired formula. ∎

7.3 Caplet valuation formula for a geometric Brownian motion with an exponential nGOUS stochastic volatility

As mentioned earlier, we have to use an approximation of the caplet valuation formula in Thm. 7.2.1 to state a caplet price, because the formula is expressed as a power series. If we approximate the caplet price with the three first terms of the power series, we find that the caplet price is given by

$$\begin{aligned}
C(X; l_0) &\simeq -\delta N P(0, T+\delta) \frac{K}{2\pi} \\
&\quad \int_{\mathbb{R}} e^{i\tilde{u}l_0} K^{i\tilde{u}} \left(\frac{1}{i\tilde{u} - \tilde{u}^2} + \frac{E_{Q^{T+\delta}}\left[\sigma_T^2\right]}{2} + \frac{E_{Q^{T+\delta}}\left[\sigma_T^4\right]}{8} (i\tilde{u} - \tilde{u}^2) \right) d\tilde{u} \\
&= -\delta N P(0, T+\delta) \frac{K}{2\pi} \\
&\quad \left(\int_{\mathbb{R}} \frac{e^{i\tilde{u}l_0} K^{i\tilde{u}}}{i\tilde{u} - \tilde{u}^2} d\tilde{u} + \frac{E_{Q^{T+\delta}}\left[\sigma_T^2\right]}{2} \int_{\mathbb{R}} e^{i\tilde{u}l_0} K^{i\tilde{u}} d\tilde{u} \right. \\
&\quad \left. + \frac{E_{Q^{T+\delta}}\left[\sigma_T^4\right]}{8} \int_{\mathbb{R}} e^{i\tilde{u}l_0} K^{i\tilde{u}} (i\tilde{u} - \tilde{u}^2) d\tilde{u} \right).
\end{aligned}$$

Because $\sigma_T^2 = \int_0^T \lambda^2(t, T) dt$, we have to find the two first moments of the time integral over the squared stochastic volatility. We are interested in the special case when $\lambda(t) = a + be^{-Z(t)}$, as defined in Eq. (5.1), for a nGOUS $Z(t)$. Remember from Ch. 5 that we computed formulas for the characteristic function of $Z(t)$, $\varphi_{Z_t}(\theta)$, in Prop. 5.2.1, and for the joint characteristic function of $Z(t)$ at two different times, $\Phi_{Z_t Z_s}(\theta, \vartheta)$, in Prop. 5.2.2. These formulas can be used to express the first and second moments of σ_T^2. That is, by Eq. (5.7) and Prop. 5.2.3 in Ch. 5, it is easy to see that the first moment is given by

$$\begin{aligned}
E_{Q^{T+\delta}}\left[\int_0^T \lambda^2(t) dt\right] &= \int_0^T E_{Q^{T+\delta}}\left[\lambda^2(t)\right] dt \\
&= \int_0^T \left(a^2 + 2ab\varphi_{Z_t}(i) + b^2 \varphi_{Z_t}(2i)\right) dt.
\end{aligned}$$

Also, by Eq. (5.9) and Eq. (5.8) in Ch. 5, the second moment of the time integral of the squared stochastic volatility is given by

$$E_{Q^{T+\delta}}\left[\left(\int_0^T \lambda^2(t) dt\right)^2\right] = \int_0^T \int_0^T \mathbb{1}_{\{s \le t\}} E_{Q^{T+\delta}}\left[\lambda^2(s)\lambda^2(t)\right] ds dt,$$

where

$$\begin{aligned} &E_{Q^{T+\delta}}\left[\lambda^2(s)\lambda^2(t)\right] \\ &\quad = a^4 + 2a^3b\left(\varphi_{Z_t}(i) + \varphi_{Z_s}(i)\right) + a^2b^2\left(4\Phi_{Z_tZ_s}(i,i) + \varphi_{Z_t}(2i) + \varphi_{Z_s}(2i)\right) \\ &\qquad + 2ab^3\left(\Phi_{Z_tZ_s}(i,2i) + \Phi_{Z_tZ_s}(2i,i)\right) + b^4\Phi_{Z_tZ_s}(2i,2i). \end{aligned}$$

If we would want an explicit analytical formula for the approximated caplet price with more terms, e.g. m terms, we would have to compute the $(m-1)$-th moment of the time integral of the squared stochastic volatility. As mentioned in the end of Ch. 5, that is possible to do, but requires a lot of time due to the exploding number of terms as the exponent m grows. In addition to that the number of terms grows fast, each term is also time consuming to compute. That is, each term in $C(X;l_0)$ consists of several integrals which have to be solved, and as m grows, the number of integrals in each term that has to be solved grows as well. By this we conclude that, in theory, it is possible to compute an explicit analytical approximation of $C(X;l_0)$ with an arbitrarily small error. However, the time consumption doing this would be so large that it is recommended to use numerical methods to compute $C(X;l_0)$ if the desired number of terms is bigger than three.

In the next chapter we will use another approach to derive a caplet valuation formula, which can be used to approximate the ATM caplet price for a geometric Brownian motion with exponential nGOUS stochastic volatility.

Chapter 8

CAPLET VALUATION WITH A BLACK-SCHOLES APPROACH

In the last chapter we found a caplet valuation formula for the LIBOR forward rate modeled by a general geometric Itô-Lévy process, by use of Fourier transformation. Then we applied that formula on the special case when the LIBOR forward rate is driven by a geometric Brownian motion with an exponential nGOUS stochastic volatility, as introduced in Ch. 5. We know that there is possible to derive caplet valuation formulas for geometric Brownian motions with deterministic volatility, by use of a Black-Scholes approach. Inspired by that we are going to derive a caplet valuation formula for a geometric Brownian motion with stochastic volatility. It turns out that it is not possible to state the expectations in this general formula explicitly, and thus we precede the calculations by rewriting functions as power series. As a warning, the calculations turn out to be quite messy.

8.1 Caplet valuation formula by a Black-Scholes approach

In this section we will use a Black-Scholes approach to derive a general caplet valuation formula for a geometric Brownian motion with stochastic volatility. In [Fil09] an equivalent formula is stated for volatilities $\lambda(t,T)$ which are deterministic functions, without proof. Our version of this formula is stated in Prop. 8.1.1, with a proof which is inspired by the proof of the easier case in [Ben04], when the volatility is a constant.

So, we want to derive a caplet valuation formula from a LIBOR forward rate which is given by

$$L(T,T) = L(t,T)\exp\left(\lambda \circ W^{T+\delta}\right), \tag{8.1}$$

where $t \leq T \leq \mathcal{T}$, $\lambda(t) \in \mathcal{V}([0,\mathcal{T}]_2)$ and $W^{T+\delta}$ is a $Q^{T+\delta}$-Brownian motion. That is, a LIBOR forward rate prevailing at time T, applicable to the interval $[T, T+\delta]$. The caplet valuation formula derived in this section is applicable to all stochastic volatilities as long as they are in the function space $\mathcal{V}([0,\mathcal{T}]_2)$ and satisfies Novikov's conditon (Thm. 2.1.2).

Proposition 8.1.1 (Caplet valuation formula for a geometric Brownian motion with stochastic volatility). *Let $L(T,T)$ be as given in Eq. (8.1) with stochastic volatility $\lambda(t,T) \in \mathcal{V}([0,\mathcal{T}]_2)$. Then the general caplet price is* *given by*

$$\begin{aligned}
&\mathbf{\mathit{Cpl}}(t;T,T+\delta,K,N) \\
&\qquad = \delta N P(t,T+\delta) E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right)^+ \;\middle|\; \mathcal{F}_t \right] \\
&\qquad = \delta N P(t,T+\delta) E_{Q^{T+\delta}}\left[L(t,T)\Phi(d_1) - K\Phi(d_2) \right], \qquad (8.2)
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cdf, $\{\mathcal{F}_t\}_{t\leq\mathcal{T}}$ is containing both $\{\mathcal{F}_t^\lambda\}_{t\leq\mathcal{T}}$ (see Def. 5.3.2) and the natural filtration of the $Q^{T+\delta}$-Brownian motion, and

$$d_{1,2} = \frac{\ln\left(\frac{L(t,T)}{K}\right) \pm \frac{1}{2}\int_t^T \lambda^2(s,T)ds}{\left(\int_t^T \lambda^2(s,T)ds\right)^{\frac{1}{2}}}.$$

Remark. Note that $\Phi(\cdot)$ is stochastic in Eq. (8.2), as $\lambda(t,T)$ is a stochastic process.

Proof ([Ben04]). In Proof II of Prop. 3.3.1 we proved that the no-arbitrage price of a caplet at time t is given by

$$\begin{aligned}
&\mathrm{Cpl}(t;T,T+\delta,K,N) = \\
&\qquad \delta N P(t,T+\delta) E_{Q^{T+\delta}}\left[\left(F(T;T,T+\delta) - K \right)^+ \;\middle|\; \mathcal{F}_t \right]
\end{aligned}$$

when the dynamics of $F(t;T,T+\delta)$ follows a geometric Brownian motion with constant volatility. In the current case we have a stochastic volatility, and therefore extends the information flow to ensure adaptedness of the model. That is, define a filtration $\{\mathcal{F}_t\}_{t\leq\mathcal{T}}$ such that $\mathcal{F}_t^N \cup \mathcal{F}_t^\lambda \subseteq \mathcal{F}_t$ for all t, where $\{\mathcal{F}_t^N\}_{t\leq\mathcal{T}}$ is the filtration generated by the Brownian motion and $\{\mathcal{F}_t^\lambda\}_{t\leq\mathcal{T}}$ is the filtration generated by the stochastic volatility, both under $Q^{T+\delta}$. The LIBOR is a simply compound forward rate according to Def. 3.3.1 and 3.2.1, and we can define $L(T,T) := F(T;T,T+\delta)$. Combining these facts we have that

$$\begin{aligned}
&\mathrm{Cpl}(t;T,T+\delta,K,N) \qquad (8.3) \\
&\qquad = \delta N P(t,T+\delta) E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right)^+ \;\middle|\; \mathcal{F}_t \right]
\end{aligned}$$

holds. Now we only have to focus on the expectation given above. First we see that

$$\begin{aligned}
&E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right)^+ \;\middle|\; \mathcal{F}_t \right] \\
&\qquad = E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right)^+ \right] \\
&\qquad = E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right) \mathbb{1}_{\{L(T,T)>K\}} \right] \qquad (8.4) \\
&\qquad = E_{Q^{T+\delta}}\left[E_{Q^{T+\delta}}\left[\left(L(T,T) - K \right) \mathbb{1}_{\{L(T,T)>K\}} \;\middle|\; \mathcal{F}_T^\lambda \right] \right]
\end{aligned}$$

where we used $\mathcal{F}_t$-independence in the first step and the tower rule of expectations in the last step. Further we can rewrite the indicator function according to the model of $L(T,T)$ given in Eq. (8.1). Define $\sigma_T^2 := \int_t^T \lambda^2(s,T)ds$.

Then we have that

$$\begin{aligned} & L(T,T) > K \\ \Rightarrow L(t,T)\exp & \left(\int_t^T \lambda(s,T)dW^{T+\delta} - \frac{1}{2}\sigma^2 \right) > K \\ & \Rightarrow \int_t^T \lambda(s,T)dW^{T+\delta} > \ln\left(\frac{K}{L(t,T)}\right) + \frac{1}{2}\sigma^2. \end{aligned}$$

It is a known fact from stochastic analysis that Itô integrals are normally distributed, and we have that

$$\int_t^T \lambda(s,T)dW^{T+\delta} \sim N\left(0, \int_t^T \lambda^2(s,T)ds\right),$$

where $N(\mu,\sigma^2)$ denotes the normal distribution with mean μ and variance σ^2. As normally distributed variables can be written as $X = \mu + \sigma Z$, where $Z \sim N(0,1)$, the inequality above becomes

$$\begin{aligned} Z\sigma &> \ln\left(\frac{K}{L(t,T)}\right) + \frac{1}{2}\sigma^2 \\ \Rightarrow Z &> \frac{\ln\left(\frac{K}{L(t,T)}\right) + \frac{1}{2}\sigma^2}{\sigma} \\ \Rightarrow Z &> -\frac{\ln\left(\frac{L(t,T)}{K}\right) - \frac{1}{2}\sigma^2}{\sigma} := -d_2, \end{aligned}$$

where we defined d_2 for convenience. This gives us, from Eq. (8.4), that

$$\begin{aligned} & E_{Q^{T+\delta}}\left[\left(L(T,T)-K\right)^+ \;\middle|\; \mathcal{F}_t\right] \\ &\quad = E_{Q^{T+\delta}}\left[E_{Q^{T+\delta}}\left[\left(L(T,T)-K\right)\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right]\right] \\ &\quad = E_{Q^{T+\delta}}\Big[E_{Q^{T+\delta}}\left[L(T,T)\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right] \\ &\qquad\qquad - E_{Q^{T+\delta}}\left[K\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right]\Big]. \end{aligned} \tag{8.5}$$

For the second expectation above we then reach

$$\begin{aligned} -E_{Q^{T+\delta}}\left[K\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right] &= -K\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= -KP(Z > -d_2) \\ &= -KP(Z < d_2), \end{aligned}$$

where we in the last equality used the fact that the pdf of the standard normal distribution is symmetric about zero. By conventional notation we then have that $-E_{Q^{T+\delta}}\left[K\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right] = -K\Phi(d_2)$. For the first expectation in Eq. (8.5), inserting the model of $L(T,T)$ in Eq. (8.1) and using the introduced notation, we get

$$
\begin{aligned}
E_{Q^{T+\delta}}\left[L(T,T)\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right] &= \int_{-d_2}^{\infty} L(t,T)e^{z\sigma-\frac{\sigma^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= \int_{-d_2}^{\infty} \frac{L(t,T)}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(z^2-z\sigma+\sigma^2\right)}dz \\
&= \int_{-d_2}^{\infty} \frac{L(t,T)}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma)^2}dz \\
&= \int_{-d_2-\sigma}^{\infty} \frac{L(t,T)}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}dz,
\end{aligned}
$$

where we in the last step used the substitution $u = z - \sigma$. Now we consider the lower limit in the last integral above.

$$
\begin{aligned}
-d_2-\sigma &= -\frac{\ln\left(\frac{L(t,T)}{K}\right)-\frac{1}{2}\sigma^2}{\sigma}-\sigma \\
&= -\frac{\ln\left(\frac{L(t,T)}{K}\right)+\frac{1}{2}\sigma^2}{\sigma} := -d_1
\end{aligned}
$$

This leaves us with

$$
\begin{aligned}
E_{Q^{T+\delta}}\left[L(T,T)\mathbb{1}_{\{Z>-d_2\}} \;\middle|\; \mathcal{F}_T^\lambda\right] &= L(t,T)\int_{-d_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}dz \\
&= L(t,T)P(U>-d_1) \\
&= L(t,T)\Phi(d_1).
\end{aligned}
$$

Combining the results for the expectations in Eq. (8.5) with the caplet pricing formula in Eq. (8.3) gives us the desired result. ■

8.2 The Black-Scholes type caplet valuation formula as power series

When the volatility is constant the caplet formula in Prop. 8.1.1 is a standard Black-Scholes formula, and when the volatility is a deterministic function it is straight forward to find the value of $d_{1,2}$ as long as the time integral over the squared volatility is possible to solve. In our case the volatility is stochastic, and the caplet valuation formula in Prop. 8.1.1 is therefore dependent on the expectation of standard normal cdf's given by the time integral over a squared stochastic volatility. This expected value is most likely impossible to derive analytically in general, and thus we want to derive a formula which is not dependent on the expectation of stochastic standard normal cdf's.

To obtain a caplet valuation formula independent of the expectation of the stochastic standard normal cdf's, an intuitive approach is to express the cdf's as a power series. Then we can apply the expectation $E_{Q^{T+\delta}}[\cdot]$ to each term of the series, expectations which should be easier to compute analytically than expectations of the form $E_{Q^{T+\delta}}[\Phi(d_{1,2})]$. From Prop. B.1.1 we know that the standard normal cdf is given by

$$\Phi(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right), \tag{8.6}$$

which means that we can obtain a power series for $\Phi(d_{1,2})$ through the power series of the error function. By Def. B.1.2 we have that the error function is given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Using the power series of e^x with $x = -t^2$ and then integrate each term in that series, one can prove that the power series around zero of the error function is given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)} \tag{8.7}$$

(see Def. B.2.2). We notice that the power series in Eq. (8.7) consists of odd terms only, which means that we will face a problem with the square root of the stochastic volatility integral in $d_{1,2}$, that is with the expression

$$\left(\int_t^T \lambda^2(s,T)ds\right)^{\frac{1}{2}}.$$

We will see that this problem results in that the square root has to be expanded as a power series as well. This fact will cause the final approximation of the caplet price to be less accurate.

Through the power series of the error function we can derive a power series of the standard normal cdf.

Proposition 8.2.1 (Power series of the standard normal cdf). *The standard normal cdf can be expressed as the following power series.*

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} C_k x^{2k+1},$$

where $C_{1,k} = \frac{(-1)^k}{2^k k!(2k+1)}$.

Proof. We will use that fact that

$$\Phi(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right),$$

which is given in Eq. (8.6).

From Eq. (8.7) we know that the error function is given by the power series

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} y^{2k+1}.$$

Inserting $y = \frac{x}{\sqrt{2}}$ we easily see that

$$\begin{aligned}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{x}{\sqrt{2}}\right)^{2k+1} \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{k=0}^{\infty} C_k x^{2k+1},\end{aligned}$$

where we defined $C_k = \frac{(-1)^k}{2^k k!(2k+1)}$. We find the desired formula by inserting this expression into Eq. (8.6). ■

In the following corollary we will state the caplet valuation formula in Prop. 8.1.1 when the standard normal cdf is written as the power series in Prop. 8.2.1.

Corollary 8.2.1 (Power series of the caplet valuation formula). *Let the caplet valuation formula be as given in Prop. 8.1.1. Then the caplet valuation formula can be expressed as the following power series.*

$$\begin{aligned}\boldsymbol{Cpl}(t,T,T+\delta,K,N) = \delta N P(t,T+\delta)\Bigg(&\frac{1}{2}\left(L(t,T)-K\right) \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} C_k \left(L(t,T) E_{Q^{T+\delta}}\left[d_1^{2k+1}\right] - K E_{Q^{T+\delta}}\left[d_2^{2k+1}\right]\right)\Bigg)\end{aligned}$$

where $C_k = \frac{(-1)^k}{2^k k!(2k+1)}$.

Proof. From Prop. 8.1.1 we know that

$$\operatorname{Cpl}(t,T,T+\delta,K,N) = \delta N P(t,T+\delta) E_{Q^{T+\delta}}\left[L(t,T)\Phi(d_1) - K\Phi(d_2)\right]. \tag{8.8}$$

In this proof we write $\delta N P(t,T+\delta) := 1$ for simplicity. From Prop. 8.2.1 we know that

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} C_k x^{2k+1}$$

for $C_k = \frac{(-1)^k}{2^k k!(2k+1)}$. By inserting this expression of the standard normal cdf into Eq. (8.8), with d_1 and d_2 in Prop. 8.1.1 as input, we see that

$$
\begin{aligned}
&\mathrm{Cpl}(t,T,T+\delta,K,N)\\
&= E_{Q^{T+\delta}}\left[L(t,T)\left(\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}C_k d_1^{2k+1}\right)\right.\\
&\qquad\qquad\qquad\qquad\left.-K\left(\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}C_k d_2^{2k+1}\right)\right]\\
&= \frac{1}{2}\left(L(t,T)-K\right)\\
&\quad+\frac{1}{\sqrt{2\pi}}\left(L(t,T)\sum_{k=0}^{\infty}C_k E_{Q^{T+\delta}}\left[d_1^{2k+1}\right]-K\sum_{k=0}^{\infty}C_k E_{Q^{T+\delta}}\left[d_2^{2k+1}\right]\right)\\
&= \frac{1}{2}\left(L(t,T)-K\right)\\
&\quad+\frac{1}{\sqrt{2\pi}}\sum_{k=0}^{\infty}C_k\left(L(t,T)E_{Q^{T+\delta}}\left[d_1^{2k+1}\right]-KE_{Q^{T+\delta}}\left[d_2^{2k+1}\right]\right)
\end{aligned}
$$

In the second equality we used the fact that $d_{1,2}$ are the only stochastic functions in the expression. ■

By the caplet valuation formula in Cor. 8.2.1 we are one step closer to an explicit analytical formula. We have reduced the problem from being to solve expectations of stochastic standard normal cdf's, to solve expectations of the stochastic function $d_{1,2}$. It turns out to be hard to compute these expectations analytically as well. In the next section we will find an explicit analytical formula for the caplet price, but just for the special case ATM (Def. 3.3.4).

8.3 An explicit ATM caplet valuation formula

In Cor. 8.2.1 we have stated a caplet valuation formula which is dependent on the expectation of the odd powers of the two stochastic expressions

$$
d_{1,2} = \frac{\log\left(\frac{L(t,T)}{K}\right) \pm \frac{1}{2}\int_t^T \lambda^2(s,T)ds}{\left(\int_t^T \lambda^2(s,T)ds\right)^{\frac{1}{2}}} = \frac{\log\left(\frac{L(t,T)}{K}\right) \pm \frac{1}{2}\sigma_T^2}{\sigma_T}.
$$

That is, the formula involves expectations of the form

$$
E_{Q^{T+\delta}}\left[d_{1,2}^{2k+1}\right],
$$

so our challenge is to derive explicit analytical expressions of these terms. Consider the general expression

$$E_{Q^{T+\delta}}\left[\left(\frac{\log\left(\frac{L(t,T)}{K}\right)\pm\frac{1}{2}\sigma_T^2}{\sigma_T}\right)^{k+1}\right],$$

which can be rewritten as

$$E_{Q^{T+\delta}}\left[\left(\frac{1}{\sigma_T}\right)^{k+1}\left(\log\left(\frac{L(t,T)}{K}\right)\pm\frac{1}{2}\sigma_T^2\right)^{k+1}\right].$$

By this last expression we see that the caplet valuation formula in Cor. 8.2.1 always will contain terms which involves expectations applied to an inverse stochastic function. That is, we will have to consider expectations of the form $E_{Q^{T+\delta}}\left[\sigma_T^{-m}\right]$, where $m := k+1$, which is not trivial to solve analytically. It is possible to derive a lower limit for the caplet price by use of Jensen's inequality.

Theorem 8.3.1 (Jensen's inequality, [Øks10]). *If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $E\left[\left|\phi(X)\right|\right] < \infty$ we have that*

$$\phi\left(E\left[X|\mathcal{F}\right]\right) \leq E\left[\phi(X)\middle|\mathcal{F}\right].$$

Considering the graph of a function x^{-1}, it is easy to realize that it is a convex function as long as $x > 0$. Since σ_T is a volatility, the function σ_T^{-m} will always be positive, and we can therefore apply Jensen's inequality (Thm. 8.3.1) in our case. A lower limit, which we would have obtained in that case, might be interesting to consider in applications, but we will focus on deriving an explicit analytical formula with equality in this thesis.

Another approach would be to rewrite σ_T^{-m} as an integral, and then use Fubini's theorem (Thm. 4.1.1) such that

$$\begin{aligned}
E\left[\sigma_T^{-m}\right] = E\left[X^{-1}\right] &= \int_0^\infty\left(\int_0^\infty e^{-yx}dy\right)p(x)dx \\
&= \int_0^\infty\left(\int_0^\infty e^{-yx}p(x)dx\right)dy \\
&= \int_0^\infty M_{\sigma_T^m}(-y)dy,
\end{aligned}$$

where $p(x)$ is the pdf of σ_T^m, and $M_{\sigma_T^m}(-y)$ is its moment generating function with parameter $-y$. This calculation is inspired by [Hal]. We notice that the expectation integral goes from 0 to ∞ because of positivity of the volatility function, and also that this approach requires the condition $M_{\sigma_T^m}(-y) \in L^1(\mathbb{R}^+)$. By this approach we would have to derive the moment generating functions of random variables which are given by the m-th power of σ_T, where the maximum value of m depend on the number of terms one decides to use in the approximation of the caplet valuation formula. In this thesis we have not derived the moment generation function of neither the general stochastic volatility in Eq. (5.1), nor any of its special cases. If this moment generation function is possible to derive, it is certainly not easy to do, and it is even harder to derive the moment generating function of its powers. We will not go further with this approach here. However, there might be stochastic volatility functions for which this approach is fairly doable.

In this thesis we will go further with the approach of expanding functions as power series. As mentioned above, we have a problem with the inverse stochastic volatility term. However, by considering the special case with an ATM caplet only, we will be able to derive an explicit analytical caplet valuation formula. This is because we know that $L(t,T) = K$ for an ATM caplet (see Def. 3.3.4), and we are thus left with a caplet valuation formula (from Cor. 8.2.1) involving terms of the form

$$E_{Q^{T+\delta}}\left[d_{1,2}\right] = E_{Q^{T+\delta}}\left[\left(\frac{\pm\frac{1}{2}\sigma_T^2}{\sigma_T}\right)^m\right] = \left(\pm\frac{1}{2}\right)^m E_{Q^{T+\delta}}\left[\sigma_T^m\right]. \tag{8.9}$$

That is, with the special case of ATM caplets we do not have the problem with an inverse stochastic function. However, there is another problem, as mentioned in the last section. There is not possible to derive a general analytical formula of the expectation of random variables with fractional powers, without using approximation methods. Since the function σ_T is the square root of a random variable, and the caplet valuation formula in Cor. 8.2.1 consists of odd terms only, we have terms involving the expectation of functions on the form

$$\sigma_T^m = \left(\int_t^T \lambda^2(s,T)ds\right)^{\frac{2k+1}{2}}.$$

It is possible to derive an explicit analytical ATM caplet valuation formula by use of power series. The formula is stated and proved in the following theorem.

Theorem 8.3.2 (Explicit ATM caplet valuation formula). *Let the caplet valuation formula be as given in Cor. 8.2.1. Then the explicit ATM caplet valuation formula is given by*

$$\boldsymbol{Cpl}(t,T,T+\delta,K,N) = \frac{\delta N P(t,T+\delta)L(t,T)}{\sqrt{2\pi}}$$
$$\sum_{k=0}^{\infty}\frac{C_k}{4^k}E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right)^{2k+1}\right],$$

where $C_k = \frac{(-1)^k}{2^k k!(2k+1)}$.

Proof. Consider the caplet valuation formula in Cor. 8.2.1. Since we are considering an ATM caplet, we know that $L(t,T) = K$ according to Def.

3.3.4, and thus we are left with the formula

$$\frac{\delta N P(t,T+\delta)L(t,T)}{\sqrt{2\pi}} \sum_{k=0}^{\infty} C_k \left(E_{Q^{T+\delta}}\left[d_1^{2k+1}\right] - E_{Q^{T+\delta}}\left[d_2^{2k+1}\right]\right). \tag{8.10}$$

Now $E_{Q^{T+\delta}}\left[d_{1,2}^{2k+1}\right]$ has to be computed explicitly for an ATM caplet, and according to Eq. (8.9) that is equivalent to computing

$$\left(\pm\frac{1}{2}\right)^{2k+1} E_{Q^{T+\delta}}\left[\sigma_T^{2k+1}\right] = \pm\frac{1}{2^{2k+1}} E_{Q^{T+\delta}}\left[\sigma_T^{2k+1}\right], \tag{8.11}$$

where the signs of $d_{1,2}$ are retained since $2k+1$ is an odd number. In the following we focus on the expectation only. Since σ_T is the square root of the integral of the squared stochastic volatility, we have to write it as a power series, such that we are able to apply the expectation on a function without fractional exponent. First we rewrite the expectation as

$$E_{Q^{T+\delta}}\left[\sigma_T^{2k+1}\right] = E_{Q^{T+\delta}}\left[\left(\left(1+(\sigma_T^2-1)\right)^{\frac{1}{2}}\right)^{2k+1}\right].$$

Now we can use Def. B.2.3 twice to rewrite the expectation as

$$\begin{aligned}
E_{Q^{T+\delta}}\left[\sigma_T^{2k+1}\right] &= E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\left(\sigma_T^2-1\right)^n\right)^{2k+1}\right] \\
&= E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n\left(1-\sigma_T^2\right)^n\right)^{2k+1}\right] \\
&= E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n\sum_{j=0}^{\infty}\binom{n}{j}(-\sigma_T^2)^j\right)^{2k+1}\right],
\end{aligned}$$

and thus Eq. (8.11) is equivalent to

$$\pm\frac{1}{2^{2k+1}} E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right)^{2k+1}\right].$$

By computing

$$E_{Q^{T+\delta}}\left[d_1^{2k+1}\right] - E_{Q^{T+\delta}}\left[d_2^{2k+1}\right]$$

and insert it into Eq. (8.10), we easily see that we reach the desired formula. ∎

As for the caplet valuation formula derived by Fourier transformation in the previous chapter, the caplet valuation formula in Thm. 8.3.2 has to be computed by approximation. The caplet valuation formula derived by Fourier transformation was approximated by three terms in Sect. 7.3, and we discussed the calculations which has to be done to compute this approximation for the special case when the stochastic volatility is driven by an exponential nGOUS. The conclusion was that we can compute this three-terms approximation by use of analytical expressions computed previously in this thesis. If we need better approximations with more terms it is still possible to compute the price approximation analytically, but it is way too time consuming.

To give an idea of how the caplet valuation formula in Thm. 8.3.2 is to compute analytically, we will look at an approximation with $k = 0, 1$, $n = 0, 1$ and $j = 0, 1$. As a tool to keep track of the sums, we add a table where the first and second terms are stated for each of the sums.

$k = 0:$	$C_0 E_{Q^{T+\delta}}\left[\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right]$
$k = 1:$	$\frac{C_1}{4}E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right)^3\right]$
$n = 0:$	$\binom{1/2}{0}\sum_{j=0}^{\infty}\binom{0}{j}(-1)^{j}\sigma_T^{2j}$
$n = 1:$	$\binom{1/2}{1}\sum_{j=0}^{\infty}\binom{1}{j}(-1)^{1+j}\sigma_T^{2j}$
$j = 0:$	$\binom{n}{0}(-1)^{n}$
$j = 1:$	$\binom{n}{1}(-1)^{n+1}\sigma_T^{2}$

Table 8.1: The first and second terms of each of the three sums in the caplet valuation formula in Thm. 8.3.2.

The expressions in Table 8.1 lead to the following approximation of the caplet price.

$$
\begin{aligned}
&\frac{\sqrt{2\pi}\mathbf{Cpl}(t,T,T+\delta,K,N)}{\delta N P(t,T+\delta)L(t,T)} \\
&\simeq C_0 E_{Q^{T+\delta}}\left[\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right] \\
&\qquad + \frac{C_1}{4}E_{Q^{T+\delta}}\left[\left(\sum_{n=0}^{\infty}\binom{1/2}{n}\sum_{j=0}^{\infty}\binom{n}{j}(-1)^{n+j}\sigma_T^{2j}\right)^3\right] \\
&= C_0 E_{Q^{T+\delta}}\left[\binom{1/2}{0}\sum_{j=0}^{\infty}\binom{0}{j}(-1)^{j}\sigma_T^{2j} + \binom{1/2}{1}\sum_{j=0}^{\infty}\binom{1}{j}(-1)^{1+j}\sigma_T^{2j}\right] \\
&\quad + \frac{C_1}{4}E_{Q^{T+\delta}}\left[\left(\binom{1/2}{0}\sum_{j=0}^{\infty}\binom{0}{j}(-1)^{j}\sigma_T^{2j} + \binom{1/2}{1}\sum_{j=0}^{\infty}\binom{1}{j}(-1)^{1+j}\sigma_T^{2j}\right)^3\right]
\end{aligned}
$$

Inserting for $C_k = \frac{(-1)^k}{2^k k!(2k+1)}$, and for the last sum when $j = 0, 1$, we find that

$$
\begin{aligned}
&\frac{\sqrt{2\pi}\mathrm{Cpl}(t,T,T+\delta,K,N)}{\delta NP(t,T+\delta)L(t,T)} \\
&\simeq E_{Q^{T+\delta}}\Bigg[\binom{1/2}{0}\left(\binom{0}{0}+\binom{0}{1}(-1)\sigma_T^2\right) \\
&\qquad\qquad + \binom{1/2}{1}\left(\binom{1}{0}(-1)+\binom{1}{1}(-1)^2\sigma_T^2\right)\Bigg] \\
&\quad - \frac{1}{24}E_{Q^{T+\delta}}\Bigg[\Bigg(\binom{1/2}{0}\left(\binom{0}{0}+\binom{0}{1}(-1)\sigma_T^2\right) \\
&\qquad\qquad + \binom{1/2}{1}\left(\binom{1}{0}(-1)+\binom{1}{1}(-1)^2\sigma_T^2\right)\Bigg)^3\Bigg] \\
&= E_{Q^{T+\delta}}\left[1+\frac{1}{2}\left(\sigma_T^2-1\right)\right] - \frac{1}{24}E_{Q^{T+\delta}}\left[\left(1+\frac{1}{2}\left(\sigma_T^2-1\right)\right)^3\right] \\
&= \frac{1}{2}E_{Q^{T+\delta}}\left[1+\sigma_T^2\right] - \frac{1}{192}E_{Q^{T+\delta}}\left[\left(1+\sigma_T^2\right)^3\right].
\end{aligned}
$$

Just the work to get to the approximated caplet valuation formula

$$
\begin{aligned}
&\mathrm{Cpl}(t,T,T+\delta,K,N) \qquad (8.12)\\
&\quad\simeq \frac{\delta NP(t,T+\delta)L(t,T)}{\sqrt{2\pi}}\left(\frac{1}{2}E_{Q^{T+\delta}}\left[1+\sigma_T^2\right] - \frac{1}{192}E_{Q^{T+\delta}}\left[\left(1+\sigma_T^2\right)^3\right]\right)
\end{aligned}
$$

is quite hard work. However, the resulting formula in itself is nice. The only thing we have to find is the expectation of σ_T^2, σ_T^4 and σ_T^6. We see that the power of the stochastic volatility grows fast, and as discussed in Sect. 7.3, even to calculate the analytical expression of $E_{Q^{T+\delta}}\left[\sigma_T^4\right]$ is time consuming. In this thesis we have only computed $E_{Q^{T+\delta}}\left[\sigma_T^2\right]$ and $E_{Q^{T+\delta}}\left[\sigma_T^4\right]$ analytically, which means that we are not able to state the explicit analytical formula for the special case when the stochastic volatility is driven by an exponential nGOUS here. The interested reader could compute the analytical expression of $E_{Q^{T+\delta}}\left[\sigma_T^6\right]$, and thus test the accuracy of the approximation in Eq. (8.12).

Even if it is messy just to find an approximation to the caplet valuation formula when each of the three sums are stated with two terms, it might be worth the work. Just as for the caplet valuation formula derived from Fourier transforms in the last chapter, the formula holds for every stochastic volatility σ_T^2 with finite moments, and there might be special cases where the moments are easy to calculate. Also, by considering the approximation in Eq. 8.12, the higher terms seem to approach zero quite fast, and hence it might not be necessary to add a lot of terms to achieve a good approximation. According to the analysis in Ch. 6, the volatility values typically has values which are less that 0.01. This means that the higher powers of σ^2 also will tend to zero quite fast.

The caplet valuation formula derived in this section is only for the special case ATM, while the formula in Ch. 7 holds for all three cases; ATM, OTM and ITM. If we consider only the ATM case, without doing any further analysis of the two caplet valuation formulas, the author of this thesis has more faith in the formula derived in this chapter. In both approximations one will have to compute the moments of the integral of the squared stochastic volatilities

$$\sigma_T = \left(\int_t^T \lambda^2(s,T)ds \right)^{\frac{1}{2}},$$

which one can choose to do numerically or analytically. Each term of the caplet valuation formula derived in Ch. 7 are much easier to state, because the formula only consists of one sum. Even so, each term of this formula also contain an integral, and these integrals are not necessarily easy to solve analytically. Also, the terms does not seem to approach zero as fast as the terms of the caplet valuation formula derived in this chapter (even if they might because of the integrals which is not solved). As already mentioned, this discussion is only based on the knowledge above. The two formulas should be analyzed further, and compared in a ATM case.

Chapter 9

COMMENTS AND SUGGESTIONS TO FURTHER WORK

A elegant and very general formula is derived for the LIBOR forward rate in in Ch. 4. It is called the LIBOR forward rate in the HJM-Lévy framework, and is a geometric Itô-Lévy process. The interested reader is encouraged to develop some specific model from it, a model which trigger his or hers interest. The possibilities are many and diversified.

In this thesis the geometric Brownian motion with stochastic volatility was chosen to be analyzed further. An exponential negative CPP nGOUS with exponential jumps was chosen as the stochastic volatility, and its distribution was derived. Also, two non-calibrated versions of this specific LIBOR forward rate model was implemented and compared to real LIBOR forward rate data. Based on the fact that the model was non-calibrated, the model did surprisingly well. Calibration and further analysis of this specific model is a very exciting task!

The two last chapters were devoted to derivations of caplet valuation formulas. One caplet valuation formula for the general geometric Itô-Lévy process derived by Fourier transformations, and one for the special case of a geometric Brownian motion with stochastic volatility derived with a Black-Scholes approach. In full generality both formulas had to be expressed as power series, which means that only approximated caplet prices can be used in applications. Also, by the Black-Scholes approach we were only able to state an exact caplet valuation formula for the special case when the caplet is ATM. The performance of these two formulas should be analyzed further, and be compared to each other. It would be interesting to see if it is possible to derive analytical formulas without power series for some other special cases of the geometric Itô-Lévy process. At the same time, it would have been extraordinary impressive if someone manages to derive an analytical caplet valuation formula without power series for the general geometric Itô-Lévy process, or even for the general geometric Brownian motion with stochastic volatility.

Thank you for reading. It has been exiting and delightful to write this thesis, and I hope you have enjoyed reading it. ★

Appendix A

THEORY

A.1 Rewrite the logarithmic zero-coupon bond price

These calculations are similar to, but more complicated than, the calculations in [Fil09].

By Assum. 4.1.1 we can use Fubini's theorem (Thm. 4.1.1), and by Assum. 4.1.2 we can use Fubini's theorem for stochastic integrals (Thm. 4.1.2). Then we achieve

$$
\begin{aligned}
\log P(t,T) = &-\int_t^T f(0,u)du - \int_t^T \int_0^t \alpha(s,u)dsdu - \int_t^T \int_0^t \sigma(s,u)dW(s)du \\
&- \int_t^T \int_0^t \int_{\mathbb{R}} \gamma(s,u,x)\bar{\boldsymbol{N}}(ds,dx)du \\
= &-\int_t^T f(0,u)du - \int_0^t \int_t^T \alpha(s,u)duds - \int_0^t \int_t^T \sigma(s,u)dudW(s) \\
&- \int_0^t \int_{\mathbb{R}} \int_t^T \gamma(s,u,x)du\bar{\boldsymbol{N}}(ds,dx) \\
= &-\int_0^T f(0,u)du - \int_0^t \int_s^T \alpha(s,u)duds - \int_0^t \int_s^T \sigma(s,u)dudW(s) \\
&- \int_0^t \int_{\mathbb{R}} \int_s^T \gamma(s,u,x)du\bar{\boldsymbol{N}}(ds,dx) \\
&+ \int_0^t f(0,u)du + \int_0^t \int_s^t \alpha(s,u)duds + \int_0^t \int_s^t \sigma(s,u)dudW(s) \\
&+ \int_0^t \int_{\mathbb{R}} \int_s^t \gamma(s,u,x)du\bar{\boldsymbol{N}}(ds,dx)
\end{aligned}
$$

Further, using indicator functions, Thm. 4.1.1 and Thm. 4.1.2 once more, and defining

$$
b(s,T) = -\int_s^T \alpha(s,u)du, \quad v(s,T) = -\int_s^T \sigma(s,u)du,
$$

and

$$
\Delta(s,T,x) = -\int_s^T \gamma(s,u,x)du,
$$

we find that

$$\begin{aligned}\log P(t,T) =& \log P(0,T) + \int_0^t b(s,T)ds + \int_0^t v(s,T)dW(s)\\ &+ \int_0^t \int_{\mathbb{R}} \Delta(s,T,x)\bar{\boldsymbol{N}}(ds,dx) + \int_0^t \bigg(f(0,u) + \int_0^u \alpha(s,u)ds\\ &+ \int_0^u \sigma(s,u)ds + \int_0^u \int_{\mathbb{R}} \gamma(s,u,x)\bar{\boldsymbol{N}}(ds,dx)\bigg)du.\end{aligned}$$

We recognize the integrand inside the integral with respect to u as the short-rate $r(u)$ associated to the instantaneous forward rate in Eq. 4.1, and thus we find that

$$\begin{aligned}\log P(t,T) =& \log P(0,T) + \int_0^t \big(r(s) + b(s,T)\big)\, ds\\ &+ \int_0^t v(s,T)dW(s) + \int_0^t \int_{\mathbb{R}} \Delta(s,T,x)\bar{\boldsymbol{N}}(ds,dx)\end{aligned}$$

A.2 Geometric Itô-Lévy process SDE and solution

From SDE to stochastic exponential

Consider the SDE

$$\frac{dX(t)}{X(t-)} = \alpha(t,T)dt + \sigma(t,T)dW(t) + \int_{\mathbb{R}} \gamma(t,T,x)\bar{\boldsymbol{N}}(dx,dt),$$

where the triplet $(\alpha(t,T), \sigma(t,T), \gamma(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3, U)$, and satisfies the three first conditions in Lemma 4.2.2. Define the process $f(t,X(t)) = \log X(t)$. Then $f(t,X(t)) \in C^2([0,\mathcal{T}] \times \mathbb{R})$ and

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial X} = \frac{1}{X(t)} \quad \text{and} \quad \frac{\partial^2 f}{\partial X^2} = -\frac{1}{X^2(t)}.$$

By Itô formula (Thm. 2.3.8) we then have that

$$\begin{aligned}d\log X(t) =& \frac{1}{X(t)}\big(\alpha(t,T)X(t)dt + \sigma(t,T)X(t)dW(t)\big)\\ &- \frac{1}{X^2(t)}\left(\frac{1}{2}\sigma^2(t,T)X^2(t)dt\right)\\ &+ \int_{|x|<R} \bigg(\log\big(X(t-) + X(t-)\gamma(t,T,x)\big) - \log\big(X(t-)\big)\\ &\qquad - \frac{1}{X(t-)}\big(X(t-)\gamma(t,T,x)\big)\bigg)\nu(dx)dt\\ &+ \int_{\mathbb{R}} \Big(\log\big(X(t-) + X(t-)\gamma(t,T,x)\big) - \log\big(X(t-)\big)\Big)\,\bar{\boldsymbol{N}}(dx,dt)\\ =& \left(\alpha(t,T) - \frac{1}{2}\sigma^2(t,T)\right)dt + \sigma(t,T)dW(t)\\ &+ \int_{|x|<R} \Big(\log\big(1+\gamma(t,T,x)\big) - \gamma(t,T,x)\Big)\,\nu(dx)dt\\ &+ \int_{\mathbb{R}} \log\big(1+\gamma(t,T,x)\big)\,\bar{\boldsymbol{N}}(dx,dt).\end{aligned}$$

Integrate from 0 to t and apply the exponential function to find

$$\begin{aligned} X(t) =& X(0)\exp\Bigg(\int_0^t \left(\alpha(s,T)-\frac{1}{2}\sigma^2(s,T)\right)ds + \int_0^t \sigma(s,T)dW(s) \\ &+ \int_0^t\int_{|x|<R}\Big(\log\big(1+\gamma(s,T,x)\big)-\gamma(s,T,x)\Big)\nu(dx)ds \\ &+ \int_0^t\int_{\mathbb{R}}\log\big(1+\gamma(s,T,x)\big)\bar{\boldsymbol{N}}(dx,ds)\Bigg) \\ =& X(0)e^{\int_0^t \alpha(s,T)ds}\mathcal{E}_t\left(\sigma\circ W+\gamma\circ\bar{\boldsymbol{N}}\right), \end{aligned}$$

where we used Notat. 2.3.2.

From stochastic exponential to SDE

Consider the stochastic exponential

$$Z(t) = Z(0)\exp\left(\int_0^t \alpha(s,T)ds\int_0^t \sigma(s,T)dW(s) + \int_0^t\int_{\mathbb{R}}\gamma(s,T,x)\bar{\boldsymbol{N}}(dx,ds)\right),$$

where we have that

$$(\alpha(t,T),\sigma(t,T),\gamma(t,T,x)) \in \mathcal{U}([0,\mathcal{T}]_2^3,U),$$

and

$$(\sigma^2(t,T),(e^{\gamma(t,T,x)}-1)) \in \mathcal{U}([0,\mathcal{T}]_2^3,U).$$

Define the process $f(t,X(t)) = Z(0)\exp\big(X(t)\big) = Z(t)$. Then $f(t,X(t)) \in C^2([0,\mathcal{T}]\times\mathbb{R})$ and

$$\frac{\partial f}{\partial t} = 0 \quad \text{and} \quad \frac{\partial f}{\partial X} = \frac{\partial^2 f}{\partial X^2} = \exp(X(t)) = Z(t).$$

By Itô formula (Thm. 2.3.8) we then have that

$$\begin{aligned} dZ(t) =& Z(t)\left(\alpha(t,T)dt+\sigma(t,T)dW(t)+\frac{1}{2}\sigma^2(t,T)dt\right) \\ &+ \int_{|x|<R}\Big(Z(t-)e^{\gamma(t,T,x)}-Z(t-)-Z(t-)\gamma(t,T,x)\Big)\nu(dx)dt \\ &+ \int_{\mathbb{R}}\Big(Z(t-)e^{\gamma(t,T,x)}-Z(t-)\Big)\bar{\boldsymbol{N}}(dx,dt) \\ =& Z(t)\left(\left(\alpha(t,T)+\frac{1}{2}\sigma(t,T)^2\right)dt+\sigma(t,T)dW(t)\right) \\ &+ Z(t-)\int_{|x|<R}\left(e^{\gamma(t,T,x)}-1-\gamma(t,T,x)\right)\nu(dx)dt \\ &+ Z(t-)\int_{\mathbb{R}}\left(e^{\gamma(t,T,x)}-1\right)\bar{\boldsymbol{N}}(dx,dt). \end{aligned}$$

A.3 nGOUS SDE and solution

From SDE to jump diffusion

Consider the SDE

$$dX(t) = -\gamma X(t)dt + \int_0^\infty xN(dx, dt),$$

Define the process $f(t, X(t)) = X(t)e^{\gamma t}$. Then $f(t, X(t)) \in C^2([0, \mathcal{T}] \times \mathbb{R})$ and

$$\frac{\partial f}{\partial t} = \gamma X(t)e^{\gamma t}, \quad \frac{\partial f}{\partial X} = e^{\gamma t} \quad \text{and} \quad \frac{\partial^2 f}{\partial X^2} = 0.$$

By Itô formula (Thm. 2.3.8) we then have that

$$\begin{aligned} d(X(t)e^{\gamma t}) &= \gamma X(t)e^{\gamma t}dt + e^{\gamma t}\left(-\gamma X(t)dt\right) \\ &\quad + \int_0^\infty \big(X(t) + x\big)\, e^{\gamma t} - X(t)e^{\gamma t}N(dx, dt) \\ &= \int_0^\infty e^{\gamma t}xN(dx, dt). \end{aligned}$$

Integrate from 0 to t and multiply by $e^{-\gamma t}$ to obtain

$$X(t) = X(0)e^{-\gamma t} + \int_0^t \int_0^\infty e^{-\gamma(t-s)}xN(dx, ds).$$

A.4 Fourier-based valuation formula

This section is inspired by and is closely following [EGP10].

A process of the form

$$S(t) = S(0)e^{X(t)}, \quad t \leq \mathcal{T}, \tag{A.1}$$

is an exponential semimartingale as long as $X(t)$ is a semimartingale with $X(0) = 0$. We also assume the semimartingale to have finite exponential moment of first order, and that there exists a valid martingale condition (see [EGP10]), such that S is a martingale under the measure Q. Then, by no-arbitrage theory the price of a derivative at time $t_0 = 0$ is given by

$$E_Q\left[f(x)\right],$$

when $f(x)$ is a payoff function (we assume discount factor equal to 1 for simplicity). In view of the following theorem we need a damped payoff function to ensure boundedness. That is, let

$$g(x) = e^{-Hx}f(x)$$

be the damped payoff function for some $H \in \mathbb{R}$. Denote by M_{X_T} and φ_{X_T} the moment generating function and the characteristic function of the random variable X_T, respectively. Also, denote by $\hat{f}$ and $\hat{g}$ the Fourier transform of the function g and f, respectively. For the following theorem to be valid, we need some assumption.

Assumption A.4.1. We assume that

1. $g \in L^1_{bc}(\mathbb{R})$;
2. $M_{X_T}(H)$ exists;
3. $\hat{g} \in L^1(\mathbb{R})$

Theorem A.4.1 ([EGP10]). *Let a financial asset be modeled as an exponential semimartingale, such as in Eq. (A.1). Also, let an arbitrary continuous payoff function be given as*

$$f(X_T - s), \tag{A.2}$$

where X is the underlying semimartingale process and $s = -\log S(0)$. If the conditions in Assum. A.4.1 are satisfied the fair price of a derivative at time $t_0 = 0$ is given as

$$V_f(X; s) = \frac{e^{-Hs}}{2\pi} \int_{\mathbb{R}} e^{-ius} \varphi_{X_T}(u - iH) \hat{f}(iH - u) du.$$

Proof. See [EGP10]. ■

Remark. The discounting factor is assumed to be 1, and is located in front of the integral.

Appendix B

SPECIAL FUNCTIONS, POWER SERIES DISTRIBUTIONS

B.1 Special functions

Definition B.1.1 (Gamma function and incomplete gamma functions, [Boa06], [Misc]). For any $p, x \in \mathbb{R}^+ \setminus \{0\}$

- the gamma function is given by

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx; \tag{B.1}$$

- the upper and lower incomplete gamma function is given by

$$\Gamma(p, y) = \int_y^\infty x^{p-1} e^{-x} dx \quad \text{and} \quad \gamma(p, y) = \int_0^y x^{p-1} e^{-x} dx,$$

  respectively.

Definition B.1.2 (Error function, [Boa06]). The error function is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Proposition B.1.1 (Connection between the standard normal cdf and the error function, [Boa06]). *The connection between the standard normal cdf and the error function is given by*

$$\Phi(x) = \frac{1}{2}\left(1 + \textit{erf}\left(\frac{x}{\sqrt{2}}\right)\right).$$

B.2 Power series

Definition B.2.1 (Power series (Maclaurin series) of the exponential function, [Boa06]). The power series of the exponential function is given by

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!}.$$

for all x.

Definition B.2.2 (Power series (Maclaurin series) of the error function, [Misa]). The power series of the error function is given by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{k!(2k+1)}$$

for all x.

Definition B.2.3 (Power series of $(1+x)^p$**, [Boa06]).** The power series of $(1+x)^p$ is given by

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n,$$

and converges for $|x| < 1$.

Remark. If $p = \frac{1}{2}$ we can write the power series as

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3!!)}{(2n)!!} x^n.$$

B.3 Distributions

Definition B.3.1 (Poisson distribution, [Ros14]). A random variable $X = 0, 1, 2, \ldots$ is Poisson distributed with parameter $\lambda > 0$ if its pmf is given by

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda}.$$

Definition B.3.2 (The exponential distribution, [Ros14]). A continuous random variable X is exponentially distributed with parameter $\mu > 0$ if its pdf is given by

$$p(x) = \begin{cases} \mu e^{-\mu x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Definition B.3.3 (The gamma distribution, [Misb]). The pdf of a gamma-distributed variable $x \sim \Gamma(k, \kappa)$ is given by

$$p(x) = \frac{\kappa^k}{\Gamma(k)} x^{k-1} e^{-\kappa x}$$

where k and κ is the shape and rate of the distribution, respectively. The characteristic function is given by

$$\left(1 - \frac{i\theta}{\kappa}\right)^{-k}$$

in this case.

Definition B.3.4 (The unit gamma distribution, [Cro]). The pdf of a unit-gamma distributed variable $x \sim \Gamma(a, b, \alpha, \beta)$ is given by

$$p(x) = \frac{1}{\Gamma(\alpha)} \left| \frac{\beta}{b} \right| \left(\frac{x-a}{b} \right)^{\beta - 1} \left(-\beta \log \left(\frac{x-a}{b} \right) \right)^{\alpha - 1}$$

for $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+ \setminus \{0\}$. The support of the distribution is

- $[a, a+b]$, $b > 0$, $\beta > 0$;

- $[a+b, a]$, $b < 0$, $\beta > 0$;
- $[a+b, \infty]$, $b > 0$, $\beta < 0$;
- $[-\infty, a+b]$, $b < 0$, $\beta < 0$,

and its mean is given by

$$a + b\left(\frac{\mu}{\mu+1}\right)^{\frac{\lambda}{\gamma}}.$$

Appendix C

PYTHON CODE

C.1 Code

Chapter 6

```
import numpy as np
from math import exp, sqrt, log
import matplotlib.pyplot as plt
from scipy.special import gamma, gammainc
import cmath as cm
import random
from datetime import datetime
from matplotlib.pyplot import gca


"""
Initiating the distribution parameters and cf's and joint cf's
of the non-Gaussian OU-process
"""
k = 1.0 # >0 (!!) (poisson distr. parameter)
m = 0.65 # >0 (!!) (exponential distr. parameter)
g = 0.2

#k = 3.0 # >0 (!!) (poisson distr. parameter)
#m = 1.2 # >0 (!!) (exponential distr. parameter)
#g = 0.38

print "Poisson distribution parameter (lambda) = ", k
print "Exponential distribution parameter (mu) = ", m
print "nGOUS parameter (gamma) =                 ", g

psi_1 = limiting_cf_ou_CCP_exp_jumps(k, m, g, 1.0j)
psi_2 = limiting_cf_ou_CCP_exp_jumps(k, m, g, 2.0j)
ups_11 = limiting_jointcf_ou_CCP_exp_jumps(k, m, g, 1.0j, 1.0j, 1.0)
ups_12 = limiting_jointcf_ou_CCP_exp_jumps(k, m, g, 1.0j, 2.0j, 1.0)
ups_21 = limiting_jointcf_ou_CCP_exp_jumps(k, m, g, 2.0j, 1.0j, 1.0)
ups_22 = limiting_jointcf_ou_CCP_exp_jumps(k, m, g, 2.0j, 2.0j, 1.0)


"""
We have to make sure that the cf's and joint cf's are real valued,
and therefore perform a test:
"""
cf_vec_test = [psi_1,psi_2,ups_11,ups_12,ups_21,ups_22]
cf_vec = []

for i in range(0, len(cf_vec_test)):
    if cf_vec_test[i].imag != 0j:
        print("Immaginary cf or joint cf values:")
        print(cf_vec[i])
        break
    cf_vec.append(cf_vec_test[i].real)
"""
The cf's and joint cf's are contained in an
array [psi_1,psi_2,ups_11,ups_12,ups_21,ups_22]
"""


"""
Initiating the coefficients a and b, and the
coefficient matrix:
"""
x_max = 1.0 #maximum value of the stochastic volatility
```

```
a = 0.0
b = 1.0
coeff_matrix = [[4*(a**2),2*a*b],[2*a*b,b**2]]


"""
Compute two first moments and variance of the
stochastic volatility:
"""
E_lambda = expectation_lambda(a, b, cf_vec)
E_lambda_sq = expectation_lambda_sq(a, b, cf_vec)


"""
Set the time frame:
"""
t_0=1.0
t_1 = 253.0
#t_steps = 1.0
t_steps = 1.0/t_1


"""
Import the LIBOR 3M data for every business day
of the year 2018, and initiate the model with the
LIBOR 3M rate from the first day of that year:
"""
filename = "libor_3M_18.txt"
libor_3M = np.array(np.loadtxt(filename,delimiter=";"))
days = np.array(np.arange(1,254,1))

logL0 = log(libor_3M[0]/100.0)
L0 = libor_3M[0]/100.0


"""
Compute the expectation of the logarithm
of the LIBOR (3M)
"""
E_logL = expectation_logL(logL0, t_steps, t_0, t_1, E_lambda_sq)


"""
Simulation of the stochastic volatility model, and of the
LIBOR (3M) driven by a brownian motion with that stochastic
volatility:
"""
Z0 = 2.0 #Guessing
mean_error_logL = []
mean_error_vol = []

plt.subplot(2, 1, 1)
ax = gca()
ax.xaxis.set_tick_params(labelsize=14)
ax.yaxis.set_tick_params(labelsize=14)
plt.ylim(top=3.2,bottom=1.4)
plt.title('Comparison: unfitted simulation and LIBOR '
          '3M',fontweight='bold',fontsize='17')
plt.ylabel('Simulation [%]',fontweight='bold',fontsize='15')
colors = ['k-','r-','g-','y-','c-']
for j in range(0,5):
    Zt = Z_t(Z0, t_0, t_1, t_steps, g, m, k)
    Z_vec = np.array(Zt[1])
    lambda_vec = (a+b*(np.exp(-1*Z_vec)))
    libor = libor_stoch_vol(Z_vec, t_0, t_1, t_steps, L0)

    mean_error_logL.append(np.mean(np.log(libor[1])))
    mean_error_vol.append(np.mean(lambda_vec))

    plt.plot(libor[0],np.array(libor[1])*100.0,colors[j])

plt.subplot(2, 1, 2)
ax = gca()
ax.xaxis.set_tick_params(labelsize=14)
```

```
ax.yaxis.set_tick_params(labelsize=14)
plt.ylim(top=3.2,bottom=1.4)
plt.xlabel('1y in days',fontweight='bold', fontsize='15')
plt.ylabel('LIBOR 3M in 2018 [%]',fontweight='bold',fontsize='15')
plt.plot(days,libor_3M)
plt.show()


"""
Comparison between simulated mean and
theoretical mean
"""
print
print "E[lambda] limit =                "\
      ,E_lambda
print "E[lambda] simulation =           "\
      ,np.mean(mean_error_vol)
print "Abs. error =                     "\
      ,abs((E_lambda - np.mean(mean_error_vol))/E_lambda)
print

print "E[log(L)] limit =                "\
      ,E_logL[1][-1]
print "E[log(L)] simulatio              "\
      ,np.mean(np.log(libor[1]))
print "Abs. error =                     "\
      ,abs((E_logL[1][-1]-np.mean(mean_error_logL))/E_logL[1][-1])
print

"""
Mean from unit-gamma distribution
"""
l = (m/(m+1))**(k/g)
print "UG mean = ",l


"""
Rel. error data and simulated mean of log(L(t,T))
"""
datamean_logL_3M = np.mean(np.log(libor_3M/100.0))
print "Data logL mean = ",datamean_logL_3M
print "Rel. error data/simulation logL = ",\
abs((datamean_logL_3M - np.mean(np.log(libor[1])))/datamean_logL_3M)
print


"""
Plot the pdf of the distribution of the stochastic
voltility (the unit-gamma distribution):
"""

p_lambda = UG_probability_density_func(k,m,g,a,b,0.0001,x_max)
stoch_lambda = p_lambda[0]
density_lambda = p_lambda[1]
#m = len(p_lambda[1])

n = 90

plt.plot(stoch_lambda[0:n],density_lambda[0:n],'k-')
plt.title('Pdf of the stochastic volatility',fontweight='bold',fontsize='17')
plt.ylabel('Pdf',fontweight='bold',fontsize='15')
plt.xlabel('Stochastic volatility',fontweight='bold', fontsize='15')
plt.show()
```

Chapter 7

```
import numpy as np
import matplotlib.pyplot as plt
from math import log
```

```
H = 2.0  #H>=1.0
K = 0.1
x = np.arange(log(K),6.0,0.1)
g = np.exp(-H*x)*(np.exp(x)-K)
g2 = np.exp(-H*x)*(np.exp(x)-H*np.exp(x)+H*K)

plt.plot(x,g2,'k-')
plt.title('Weak derivative of damped payoff function',fontweight='bold',\
          fontsize='17')
plt.ylabel('dg(x)',fontweight='bold',fontsize='15')
plt.xlabel('x',fontweight='bold', fontsize='15')
plt.show()
```

C.2 Functions with documentation

Chapter 6

```
def limiting_cf_ou_CCP_exp_jumps(kappa, mu, gamma, theta):
    """
    Return(cf) (float): The limiting (t to infinity) characteristic
    function of a nGOUS driven by a compound Poisson process
    with exponential jumps.

    Variables (float if nothing else mentioned):
    kappa (integer) = parameter Poisson distribution
    mu = parameter exponential distribution
    gamma = parameter nGOUS
    theta = cf parameter
    """
    k1 = kappa/gamma
    cf = (mu/(mu-1j*theta))**k1
    return(cf)
```

```
def limiting_jointcf_ou_CCP_exp_jumps(kappa, mu, gamma, theta, vartheta, delta):
    """
    Return(j_cf) (float): The limiting (t to infinity) joint characteristic
    function of a nGOUS, driven by a compound Poisson process with
    exponential jumps, at two different points in time t and s.

    Variables (float if nothing else mentioned):
    kappa (integer) = parameter Poisson distribution
    mu = parameter exponential distribution
    gamma = parameter OU-process
    theta = cf parameter 1
    vartheta = cf parameter 2
    delta = constant time diff. t-s
    """
    k1 = kappa/gamma
    k2 = gamma*delta
    j_cf = ((mu*(mu - 1j*theta*exp(-k2)))/((mu-1j*(theta*exp(-k2)+vartheta))\
           *(mu-1j*theta)))**k1
    return(j_cf)
```

```
def expectation_lambda(a, b, vec):
    """
    Return(e) (float): The limiting (t to infinity) expectation of stochastic
    volatility driven by a subordinator.

    Variables:
    a,b = (floats) coefficients
    vec = (vector) containing exp. moment and joint exp. moments of
          the subordinator: [psi_1,psi_2,ups_11,ups_12,ups_21,ups_22]
    """
    e = a + b*vec[0]
    return(e)
```

```
def expectation_lambda_sq(a, b, vec):
    """
    Return(e) (float): The limiting (t to infinity) expectation of the squared
    stochastic volatility driven by a subordinator.

    Variables:
    a,d = (floats) coefficients
    vec = (vector) containing exp. moment and joint exp. moments of
          the subordinator: [psi_1,psi_2,ups_11,ups_12,ups_21,ups_22]
    """
    e = a**2 + 2*a*b*vec[0] + (b**2)*vec[1]
    return(e)
```

```
def expectation_logL(logL_0, dt, t0, t_stop, E_lambda_sq):
    """
    Return(t) (vector): Time points
    Return(e) (vector): The expectation of the logarithm of the LIBOR
    forward rate when modeled by a geometric Brownian motion with
    stochastic volatility lambda = a + b*exp(-Z(t)). The espectation
    is computed with the limiting (t to infinity) verison of lambda.

    Variables (all floats):
    logL_0 = initial value of the logarithmic LIBOR forward rate
    dt = time increment giving the time resolution
    t0 = initial time
    t_stop = last time point
    E_lambda_sq = expectation_lambda_sq(a, b, vec)
    """
    t = np.arange(t0,t_stop,dt)
    e = logL_0 - 0.5*E_lambda_sq*(t-t0)
    return(t,e)
```

```
def UG_probability_density_func(kappa, mu, gam, eps, x_max):
    """
    Return(x) (vector): stochastic volatility values
    Return(p_y) (vector): pdf-values from the unit-gamma
    distribution with a=0 and b=1.0

    Variables (float if nothing else mentioned):
    kappa (integer) = parameter Poisson distribution
    mu = parameter exponential distribution
    gam = parameter OU-process
    eps = value to keep x away from zero
    x_max = maximum value of stochastic volatility
    """
    k1 = kappa/gam
    x = np.arange(eps,x_max,0.00001)
    p_y = (abs(mu)/gamma(k1))*(x)**(mu-1)*((-mu)*np.log(x))**(k1-1)
    return(x,p_y)
```

```
def Z_t(Z_0, t0, T, dt0, gam, mu, kappa):
    """
    Return(t) (vector): Time points
    Return(Z) (vector): Values from the nGOUS driven by a compound
    Poisson process with exponential jumps

    Variables (float if nothing else mentioned):
    Z_0 = initial value of the nGOUS
    t0 = initial time
    T = last time point
    dt = time increment giving the time resolution
    kappa (integer) = parameter Poisson distribution
    mu = parameter exponential distribution
    gam = parameter OU-process
    """
    t = np.arange(t0,T,dt0)
    Z = [exp(-gam*dt0)*Z_0]
    for k in range(1,len(t)):
        random.seed(datetime.now())
        N = np.random.poisson(kappa)
        jump = np.random.exponential(mu,N)
```

```
        delta_N = np.cumsum(jump)
        if N == 0:
            Z.append(exp(-gam*dt0)*Z[k-1])
        else:
            Z.append(exp(-gam*dt0)*Z[k-1] + exp(-gam)*delta_N[N-1])
    return(t,Z)
```

```
def libor_stoch_vol(Z_t, t0, T, dt0, L0):
    """
    Return(t) (vector): Time points
    Return(L) (vector): LIBOR forward rate values from a geometric
    Brownian motion model with stochastic volatility lambda = a+b*exp(Z(t))
    (a=0 and b=1.0 in this case)

    Variables (float if nothing else mentioned):
    Z_t = (vector) A subordinator process
    t0 = initial time
    T = last time point
    dt = time increment giving the time resolution
    L0 = initial LIBOR forward rate value
    """
    t = np.arange(t0,T,dt0)
    #volatility
    vol = np.exp(-1*Z_t)
    #mean
    mean = []
    for i in range(0,len(t)):
        mean.append(vol[i]*dt0)
    #LIBOR forward rate
    dB = 0
    L = []
    for i in range(0,len(t)):
        dB = dB + np.random.normal(0, vol[i]*sqrt(dt0))
        L.append(L0*exp(dB-0.5*mean[i]))
    return(t,L)
```

ABBREVIATIONS AND SYMBOLS

Abbreviations

3M	3-month
ATM	At-the-money
cdf	Cumulative density function
CPP	Compound Piosson process
HJM	Heath, Jarrow and Morton
IBA	ICE Benchmark Administration Limited
ICE	Intercontinental Exchange
i.i.d.	Independent and identically distributed
ITM	In-the-money
LIBOR	The London Interbank Offered Rate
nGOUS	non-Gaussian OU subordinator
OTM	Out-of-the-money
OU	Ornstein-Uhlenbeck
pdf	Probability density function
pmf	Probability mass function
SDE	Stochastic differential equation
ucp	Uniformly converges on compacts in probability

Symbols

$\mathcal{B}$	Borel σ-algebra
D_{ucp}	The space of cádlág adapted processes which are ucp
$E[\cdot]$	Expectation with respect to P
$E_Q[\cdot]$	Expectation with respect to Q
erf(x)	The error function
$H^{1,2}(\cdot)$	The Sobolev space (1st derivative, $L^2(\cdot)$-space)
$\mathcal{L}(t)$	Lévy process
$\hat{\mathcal{L}}(t)$	Itô-Lévy process
$L^p(\cdot)$	The L^p-space of functions
$L^p_{bc}(\cdot)$	The L^p-space of functions on $\mathbb{R}$, which are also bounded and continuous

L_{ucp}	The space of cáglád adapted processes which are ucp
$M_X(\theta)$	Moment generating function of a random variable X
$N(t, U)$	Poisson random measure under P
$\tilde{N}(t, U)$	Compensated Poisson random measure under P
$\tilde{N}^{T+\delta}(t, U)$	Compensated Poisson random measure under $Q^{T+\delta}$
P	Market probability measure
$\Phi_{X,Z}(\theta, \vartheta)$	Joint characteristic function of two random variables X and Z
Q	Equivalent (local) martingale measure
Q^T	Forward measure for some time $T \geq t$
$\mathbb{R}$	1-dim space of real numbers (the real line)
$\mathbb{R}^+$	1-dim space of positive real numbers including 0
Tr $(\cdot)$	Trace of a $n \times n$-matrix
$\mathcal{U}([0, \mathcal{T}]_2^3 \times U)$	Space of triplets $(\alpha(t, T, \omega), \sigma(t, T, \omega), \gamma(t, T, x, \omega))$ which make Itô-Lévy processes well defined
ν	Lévy measure
$\mathcal{V}([0, \mathcal{T}])$	Space of functions $f(t, \omega)$ which are Itô integrable
$\varphi_X(\theta)$	Characteristic function of a random variable X
$W(t)$	Brownian motion process under P
$W^{T+\delta}(t)$	Brownian motion process under $Q^{T+\delta}$
$Y(t)$	$(T + \delta)$-bond discounted T-bond price process
$z_\delta(\cdot, T)$	The difference $z(\cdot, T) - z(\cdot, T + \delta)$

BIBLIOGRAPHY

[App04] Applebaum, D. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, 2004.

[BBK08] Benth, F. E., Benth, J. S., and Koekebakker, S. *Stochastic Modelling of Electricity and Related Markets*. World Scientific Publishing Co. Pte. Ltd., 2008.

[Ben04] Benth, F. E. *Option Theory with Stochastic Analysis. An Introduction to Mathematical Finance*. Springer-Verlag Berlin Heidelberg, 2004.

[BM07] Brigo, D. and Mercurio, F. *Interest Rate Models - Theory and Practice*. Springer, 2007.

[Boa06] Boas, M. L. *Mathematical Methods in the Physical Sciences*. Kaye Pace (John Wiley & Sons, Inc.), 2006.

[Cro] Crooks, G. E. *Field Guide to Continuous Probability Distributions*. URL: http://threeplusone.com/fieldguide. (Accessed: 29.01.2019).

[Ebe14] Eberlein, E. "Fourier-Based Valuation Methods in Mathematical Finance". In: *Quantitative Energy Finance - Modeling, Pricing, and Hedging in Energy and Commodity Markets*. Ed. by Benth, F. E., Kholodnyi, V. A., and Laurence, P. Springer, 2014. Chap. 3, pp. 85–114.

[EGP10] Eberlein, E., Glau, K., and Papapantoleon, A. "Analysis of Fourier Transform Valuation Formulas and Applications". In: *Applied Mathematical Finance* 17.3 (2010), pp. 211–240. (Accessed: 09-04-2019).

[Fil09] Filipovic, D. *Term-structure Models*. Springer, 2009.

[Hal] Halvorsen, Kjetil B. *Expectation of resiprocal of a variable*. URL: https://stats.stackexchange.com/questions/80874/expectation-of-reciprocal-of-a-variable. (Accessed: 13.05.2019).

[IBA] IBA. *LIBOR*. URL: https://www.theice.com/iba/libor. (Accessed: 17.01.2019).

[Inv] Investopedia. *Inflation*. URL: https://www.investopedia.com/terms/i/inflation.asp. (Accessed: 26.01.2019).

[Low09] Lowther, G. *U.C.P. and Semimartingale Convergence*. 2009. URL: https://almostsure.wordpress.com/2009/12/22/u-c-p-convergence/. (Accessed: 23.10.2018).

[Low10] Lowther, G. *Lévy Processes*. 2010. URL: https://almostsure.wordpress.com/2010/11/23/levy-processes/#more-1012. (Accessed: 06.11.2018).

[Misa] Miscellaneous authors. *Error function*. URL: https://en.wikipedia.org/wiki/Error_function. (Accessed: 24.04.2019).

[Misb] Miscellaneous authors. *Gamma distribution*. URL: https://en.wikipedia.org/wiki/Gamma_distribution. (Accessed: 29.01.2019).

[Misc] Miscellaneous authors. *Incomplete gamma function*. URL: https://en.wikipedia.org/wiki/Incomplete_gamma_function. (Accessed: 29.01.2019).

[MW13] McDonald, J. N. and Weiss, N. A. *A Course in Real Analysis*. Elsevier, 2013.

[Øks10] Øksendal, B. *Stochastic Differential Equations - An introduction with Applications*. Springer, 2010.

[ØS07] Øksendal, B. and Sulem, A. *Applied Stochastic Control of Jump Diffusions*. Springer, 2007.

[Pro95] Protter, P. *Stochastic Integration and Differential Equations. A New Approach*. Springer-Verlag, 1995.

[Qua] Quantivity. *Why Log Returns*. URL: https://quantivity.wordpress.com/2011/02/21/why-log-returns/. (Accessed: 07.04.2019).

[Ros14] Ross, S. M. *Introduction to Probability Models*. Elsevier, 2014.