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Convergence of penalty methods for American options. Variational solutions and the compactness method.

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This master's thesis is submitted under the master's programme *Stochastic Modelling, Statistics and Risk Analysis*, with programme option *Finance, Insurance and Risk*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

Pricing American options is, in comparison with European options, much more complicated. This is due to the holders of American options having the ability to choose any day for exercising options (before or at a given maturity time). Therefore, the problem attracts the work of many mathematicians. To date, there are three methodologies to solve the American option pricing problems, namely the *free boundary*, the *variational method* and the *semilinear Black and Scholes equation*. This thesis present and study the American options pricing problem with variational method.

A significant part in this thesis aims to prove that there exists exactly one solution to the American option pricing problem, namely the value of an American option. In particular, we analyze the existence and uniqueness of a solution for the classical penalised problem and its improvement-the k-power penalty problem. Later, we apply the operator splitting method for solving the power penalty problem by showing that the solutions generated by this technique will converge to the one of the k-power penalty problem. The analysis requires using compactness theory as well as functional analysis and mostly deal with put options. Finally, we wish to give a numerical testing for the results.

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CHAPTER 1

Introduction

In a financial market, people make profit by investing money in some underlying assets, such as stocks and they have to pay out K value in T years by entering a buying contract. Such contract is named by call option which guarantees the buyer can have the stock with the fixed price K value in T years despite of the fact that the value of this stock in T years in the market might be higher than K. But if the market price of the stock is less than K, then the buyer is not obligated to buy it from the seller. He can choose to buy the stock from another seller for a cheaper price. So he can earn a profit $S - K$ or 0, respectively. Similarly, a put option gives holder the right (but not obligation) to sell it at the date of expiration (*maturity date*). The profit in this case is $K - S$ or 0. Mathematically, for a call option and a put option we can write profits, respectively

$$\max(S - K, 0) = (S - K)^+, \qquad \max(K - S, 0) = (K - S)^+.$$

where $\max(S - K, 0)$ is maximum of x and 0, K is strike price, S is the market price of the stock and $T > 0$ is the maturity date.

The valuation of options is one of the most interesting matters in modern finance. European options and American options are two major types of options. While European options give the buyer the right to exercise only at the date of expiration and have an explicit formula, American options can conversely be exercised at any time up to that maturity date and have no explicit formula for their values. The main reason is that the American values depend on each strategy of the holders for exercising the options. Therefore, pricing American options has inspired the work of many mathematicians. Roughly speaking, the American option problem is to determine the value at which the option can be traded in a security market with arbitrary free. In this thesis, we will concern the American option pricing problem.

In the literature there are three main partial differential equation related approaches for determining the value of an American option, namely *the free boundary problem*, *the quasi- variational inequalities* and *the semilinear Black and Scholes equation.*

In *free boundary problem*, one splits the domain into two regions, namely the stopping region and the continuation region. One solves the price function such that it coincides with the payoff function in the stopping region, while it is the solution to the partial differential equation in the other region. In other words, one of these two conditions is not satisfied and there is strict complementary in this solving (see [1, Chapter 6.3]). The earliest analysis American option pricing problem in connection with a free boundary problem (or *Stefan*) was first studied by McKean [11]. Lately, McKean wrote the problem in an explicit form (*the optimal stopping boundary*), which was taken further by van Moerbeke [12], who studied properties of the optimal stopping boundary.

Besides the free boundary method, there is another technique that finds the value of an American option, namely a *quasi-variational inequalities*, which is the main interest of our work in this thesis. This second methodology was developed by Jaillet, Lamberton and Lapeyre [10], who got inspired from the work on variational inquality field of Bensoussan and Lions [2]. In this approach, one do not need to calculate the free boundary. Moreover, an advantage of studying variational method is that it gives stability. We will present theory on well-posedness of variational solutions later.

Recently, Benth, Karlsen and Reikvam introduced the *semilinear Black and Scholes partial differential equation* for studying the American option pricing problem [3, 4]. The main results imply that there exists exactly one solution of the semilinear Black and Scholes equation, namely the American option value, suggested by viscosity solutions. Motivation for studying this methodology is that it allows designing and analyzing "easy to implement" numerical algorithms for computing the value of an American option. Indeed, a simple numerical algorithms of the semilinear Black and Scholes equation were constructed in [7] and it has been shown that the approximate solution converges to the American option value as the discretization parameters tend to zero by this method.

In this thesis, we consider the American option pricing problem with variational method. We are interested here in proving that there exists exactly one unique solution to the problem of pricing American option, namely the value of an American option by using compactness method. We analyze in particular the classical penalty method, the k-power penalty method to extract a convergent subsequence that converges to a limit function. Passing to the limit, then we obtain the existence of a solution of the original problem. We apply operator splitting method for solving the k-power penalty problem and show that the solution generated by this technique converges to the one of k-power penalty problem. Our analysis will use compactness method, functional analysis as well as some of stochastic analysis to give results. We focus on analyzing American put options.

The thesis begins with an introduction for the American option pricing problem. We discuss briefly the problem in three main approaches in Chapter 6.42. Mathematical preliminaries of the weighted Sobolev space and its dual space as well as the Black and Scholes equation are also introduced in Chapter 3. Our analysis begins with Chapter 4 where we study the *classical penalised* equation. The *k-power penalty* problem, an improvement of the classical penalty method, is studied in Chapter 5. Finally, we apply the *operator splitting* method to analyze the well-posedness for the American option pricing problem in Chapter 7. We attempt to test a numerical scheme for the k-power penalty method in the Chapter 8. In closure, we introduce some significant notations as well as theory which we have used in the thesis.

CHAPTER 2

The American option pricing problem

The model is built as follows. Let $T > 0$ be fixed and a time variable $t < T$. Suppose that the price dynamics of a stock X_s follows a geometric Brownian motion (under the unique equivalent martingale measure $\mathbb{Q}$):

$$dX_s = (r - d)X_s ds + \sigma X_s dW_s, \quad s \in (t, T]. \tag{2.1}$$

where $d \geq 0$ is the constant dividend yield for the stock, $r \geq 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, and $W_s\big|_s \in [0, T]$ is a standard Brownian motion (A.0.6). Starting at time t with initial condition $X_t = x$, the (arbitrary-free) value of an American option with expiration at time T is given by

$$V(t, x) = \sup_{t \leq \tau \leq T} \mathbb{E}^{t,x} \left[e^{\int_t^\tau -r(s)ds} g(X\tau) \right]. \tag{2.2}$$

where the supremum is taken over all F_t stopping times $\tau \in [t, T]$. $\mathbb{E}^{t,x}$ denotes expectation under the equivalent martingale measure conditioned on $X_t = x$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is the payoff function. Typical examples of g come from call and put options, i.e, options with payoff functions,

$$g(x) = \begin{cases} (x - K)^+ & \text{, for a call option} \\ (K - x)^+ & \text{, for a put option} \end{cases} \tag{2.3}$$

where $K > 0$ is the strike price.

As mentioned before, three methodologies can be used for determining the price of an American option, namely those based on the *free boundary problem* formulation, those based on the *quasi-variational inequality* formulation and recently, those based on the *semilinear Black and Scholes partial differential equation* formulation. In this thesis, we choose the second method to present and analyse the American options pricing problem as well as we focus to analyze for put options. More precisely, this method is to seek a function $V(t, x)$ which satisfies

$$\begin{cases} \max\Big(\mathcal{L}_{BS}V(t, x) - rV(t, x), g(x) - V(t, x)\Big) = 0, & (t, x) \in Q_T \\ V(T, x) = g(x), & x \in \mathbb{R}_+. \end{cases} \tag{2.4}$$

where the payoff function $g(x)$ is given by (2.3) and the linear Black and Scholes differential operator $\mathcal{L}_{BS}$ takes the following form

$$\mathcal{L}_{BS}u(t,x) = \partial_t u + (r-d)x\partial_x u + \frac{1}{2}\sigma^2 x^2 \partial^2{}_x u. \tag{2.5}$$

We briefly review the three approaches to the solution of the American options pricing problem (2.4) as follows.

2.1 The free boundary problem

Letting $x(t)$ denote the free boundary, for $t \in [0,T]$. We introduce the sets

$$C(t) = \begin{cases} (0, x(t)), & \text{call option,} \\ (x(t), \infty), & \text{put option,} \end{cases} \qquad E(t) = \begin{cases} [x(t), \infty), & \text{call option,} \\ (0, x(t)], & \text{put option.} \end{cases}$$

The free boundary problem is to determine the pricing function $V(t,x)$ and the free boundary $x(t)$ which satisfy the following form:

$$\begin{cases} \mathcal{L}_{BS}V(t,x) - rV(t,x) = 0, & t \in [0,T], \quad x \in C(s), \\ V(T,x) = g(x), & x \in \mathbb{R}_+, \\ V(t,x) = g(x), & t \in [0,T], \quad x \in E(t), \\ \partial_x V(t,x) = \pm 1, & t \in [0,T], \quad x \in E(t). \end{cases}$$

where "$\pm 1 = 1$" for call options, and "$\pm 1 = -1$" for a put options.
In particular, it has been shown that the free boundary $x(t)$ satisfies

$$\begin{cases} x(t) > \max(\frac{r}{d}K, K) \quad \text{call options,} \quad x(t) < \min(\frac{r}{d}K, K) \quad \text{put options,} \\ x \in C(t) \Longleftrightarrow V(t,x) > g(x), \quad \mathcal{L}_{BS}V(t,x) - rV(t,x) = 0, \\ x \in E(t) \Longleftrightarrow V(t,x) = g(x), \quad \mathcal{L}_{BS}V(t,x) - rV(t,x) < 0. \end{cases}$$

Observe that if $d = 0$,i.e., there is no dividend, American call options are equal to European call options with the same strike price. In case of $r = 0$, American put options are equal to European put options with the same price. In both cases, we do not need to calculate the free boundary and it is not optimal to exercise before the maturity date T. Moreover, as we mentioned in Chapter 1, the continuation region and the exercise region are equivalent to $C(t)$ and $E(t)$, respectively.

2.2 The quasi-variational inequality

The method does not need to determine the free boundary and the problem is considered in the whole domain $[0,T] \times \mathbb{R}_+$. The method is to seek functions $V(t,x)$ that satisfy the following setting

$$\begin{cases} \mathcal{L}_{BS}V(s,x) - rV(s,x) \le 0, \\ g(x) - V(s,x) \le 0, \\ \Big(\mathcal{L}_{BS}V(s,x) - rV(s,x)\Big)\Big(g(x) - V(s,x)\Big) = 0, \end{cases} \tag{2.6}$$

with terminal data

$$V(T,x) = g(x), \quad x \in \mathbb{R}_+.$$

The continuity of the pricing function $V(t,x)$ has been proved in [10, Proposition 2.2]. Moreover, it has been known that studying this variational method gives an advantage that is its stability. We will study and present theory for well-posedness of variational solutions to (2.6) as we will deal with American put options through this thesis later.

2.3 The semilinear Black and Scholes partial differential equation

Recalling that for every locally bounded function $f : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 1$, its upper and lower semicontinuous envelopes, denoted by f^* and f_* respectively, are defined as

$$f^*(x) = \limsup_{y\to x} f(y), \qquad f_*(x) = \liminf_{y\to x} f(y).$$

Introducing the Heaviside function H as

$$H(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi \geq 0, \end{cases}$$

where

$$H^*(\xi) = \begin{cases} 0, & \xi < 0, \\ 1, & \xi \geq 0, \end{cases} \qquad H_*(\xi) = \begin{cases} 0, & \xi \leq 0, \\ 1, & \xi > 0. \end{cases}$$

The *cash flow* function c is defined as

$$c(x) = \begin{cases} rK - dx, & \text{call option} \\ dx - rK, & \text{put option.} \end{cases} \tag{2.7}$$

and the nonlinear *reaction* $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ takes the form

$$q(x,V) = c(x)H\big(g(x) - V\big). \tag{2.8}$$

We observe that since H is upper semi-continuous, $H^* \equiv H$, so is q, and $q^* \equiv q$, where

$$\begin{aligned} q^*\big(x, V(t,x)\big) &= c(x)H^*\big(g(x) - V(t,x)\big), \\ q_*\big(x, V(t,x)\big) &= c(x)H_*\big(g(x) - V(t,x)\big). \end{aligned} \tag{2.9}$$

The semilinear Black and Scholes partial differential equation for valuing American options is to seek functions $V(t,x)$ such that

$$\begin{cases} \mathcal{L}_{BS}V(t,x) - rV(t,x) = -q(x, V(t,x)), & \text{a.e in } Q_T \\ V(T,x) = g(x), \quad x \in \mathbb{R}_+. \end{cases} \tag{2.10}$$

where $\mathcal{L}_{BS}$ is given in (2.5).

Starting point from an assumption that $V \in C^{1,2}(Q_T) \cap C(\overline{Q_T})$ is an solution of the optimal stopping problem and $V \geq g$ in Q_T (which is the so-called *early* *exercise constraint*), the semilinear Black and Scholes partial differential equation (2.10) was formulated by using guidance of the dynamic programming principle (A.0.8) and definition of classical sub- and supersolutions (A.0.10). We refer [3] for the derivation of (2.10). In particular, *Dynamic programming principle* suggests that

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0, \qquad \text{in } Q_T.$$

In case of τ_0 being an optimal stopping time, then

$$e^{-r(t\wedge\tau_0 - t)}v(t\wedge\tau_0, X(t\wedge\tau_0))$$

is a martingale, and we obtained that

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) = 0$$

(Proposition 3.2.1 and Theorem 3.2.2). Thus, we formulate the following setting

$$\begin{cases} V(t,x) \geq g(x), \\ \mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0, \\ \Big(\mathcal{L}_{BS}V(t,x) - rV(t,x)\Big)\big(g(x) - V(t,x)\big) = 0. \end{cases}$$

Clearly, the above formulation is equivalent to (2.4). Since

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0,$$

for almost everywhere, it suggests that

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) = 0, \qquad \textit{in the continuation region.}$$

Therefore,

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0, \qquad \textit{in the exercise region.}$$

In addition, a lower bound of $\mathcal{L}_{BS}V(t,x) - rV(t,x)$ can be derived in the exercise region. Since $V(t,x) \geq g(x)$ almost everywhere and $V(t,x) = g(x)$ in the exercise region, we use argument of the maximum principle of $g(x) - V(t,x)$. We say that $V(t,x)$ *touches* $g(x)$ from above at a local maximizer (t,x) and obtain

$$\begin{cases} -(rK - dx)^+ \leq \mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0, & \text{for a put option} \\ -(dx - rK)^+ \leq \mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0, & \text{for a call option.} \end{cases}$$

when $V(t,x) = g(x)$. Using the notation of the cash flow function $c(x)$ given by (2.7), it is equivalent to

$$-\,c(x) \leq \mathcal{L}_{BS}V(t,x) - rV(t,x) \leq 0. \tag{2.11}$$

Remark 2.3.1. Let us discuss the inequality (2.11). If we use knowledge of the free boundary problem, in the exercise region where it holds that $V(t,x) = g(x)$, inserting this into the Black and Scholes equation, we get

$$\mathcal{L}_{BS}V(t,x) - rV(t,x) = -c(x).$$

So (2.11) is indeed an equality in the exercise region. However, it does not mean that this equation holds for every point in this region. Instead of claiming this, we have to use (2.11). Furthermore, the semilinear Black and Scholes equation (2.10) allows for some points in the exercise region to have the possibility that $\mathcal{L}_{BS}V - rV = 0$. This implies that we can use the seminlinear Black and Scholes equation without any priori knowledge of the free boundary. It is flexible to carry out results on well-posedness of solution.

On the other hand, when using the definition of classical sub- and supersolution and notations q^*, q_* given by (2.9), it leads to

$$\begin{cases} \mathcal{L}_{BS}V(t,x) - rV(t,x) \leq -q^*(x, V(t,x)), & \text{for classical subsolution} \\ \mathcal{L}_{BS}V(t,x) - rV(t,x) \geq -q_*(x, V(t,x)), & \text{for classical supersolution.} \end{cases} \tag{2.12}$$

almost everywhere and we get the semilinear Black and Scholes equation (2.10) as desired.

Remark 2.3.2. The reaction term q is discontinuous. The question is how one can interpret the semilinear Black and Scholes equation (2.10). It suggests that we should use inequalities (2.12) for interpretation [3]. In addition, the monotonicity of q is an important property for proving well-posedness of solution to the American options pricing problem.

CHAPTER 3

Preliminaries

3.1 The weighted Sobolev space

We introduce a weighted Sobolev space M which is used in our thesis. Let us denote by $L^2(\mathbb{R}_+)$ the Hilbert space of square integral functions on $\mathbb{R}_+$. The norm and the inner product in $L^2(\mathbb{R}_+)$ are defined

$$\|v\|_{L^2(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} v(x)^2 dx\right)^{\frac{1}{2}}, \qquad (v, w) = \int_{\mathbb{R}_+} v(x)w(x)dx,$$

respectively. We define the weighted Sobolev space M as follows

$$M = \left\{v \in L^2(\mathbb{R}_+) : \quad x\frac{dv}{dx} \in L^2(\mathbb{R}_+)\right\}, \tag{3.1}$$

where the inner product and the norm are given

$$(v, w)_M = (v, w) + \left(x\frac{dv}{dx}, x\frac{dw}{dx}\right), \qquad \|v\|_M = \sqrt{(v, v)_M},$$

respectively. We denote by $(\cdot, \cdot)$ the inner product in $L^2(\mathbb{R}_+)$. Moreover, we denote by M' the dual space of M, where the norm is defined such that

$$\|w\|_{M'} = \sup_{\|v\|_M \leq 1} \frac{\langle w, v\rangle_{M',M}}{\|v\|_M}, \quad \forall v \in M.$$

Lemma 3.1.1 (Poincáre's inequality)**.** *[1, p. 30] If $v \in M$, then*

$$\|v\|_{L^2(\mathbb{R}_+)} \leq 2\left\|x\frac{dv}{dx}\right\|_{L^2(\mathbb{R}_+)}. \tag{3.2}$$

Proof. We have

$$2\int_{\mathbb{R}_+} xv\frac{dv}{dx}dx = -\int_{\mathbb{R}_+} v^2 dx.$$

Using Cauchy-Schwartz inequality (A.1), we deduce that

$$\|v^2\|_{L^2(\mathbb{R}_+)} \leq 2\|v\|_{L^2(\mathbb{R}_+)}\|x\frac{dv}{dx}\|_{L^2(\mathbb{R}_+)}.$$

Hence we obtain (3.2). ■

3.2 The Black and Scholes equation

The Black-Scholes equation is well-known because the equation is solved for the values of an European option. Due to American options and European options having similar characteristics (the difference being the date of exercise), the consideration of this equation will provide a fundamental view for studying the American option pricing problem. We follow [1, Chapter 2] for briefly reviewing this equation as well as some important properties.

Proposition 3.2.1. *Assume that the functions $\sigma(t)$ and $r(t)$ are continuous nonnegative and bounded on $[0,T]$. Then, for any function $v:(t,x)\mapsto v(t,x)$ continuous in $\mathbb{R}_+\times[0,T]$, C^1 - regular with respect to t and C^2 - regular with respect to x in $\mathbb{R}_+\times[0,T)$, and such that $\left|x\frac{\partial v}{\partial x}\right|\leq C(1+x)$ with C indepent of t, the process*

$$M_t = e^{-\int_0^t r(s)ds}v(t,x) - \int_0^t e^{-\int_0^\tau r(\nu)d\nu}\Big(\mathcal{L}_{BS}v(t,x) - r(\tau)v(t,x)\Big)d\tau$$

is a martingale under F_t, where $\mathcal{L}_{BS}$ is given in (2.5).

Proof. Consider the *Itô* process

$$dX(s) = (r-d)X(s)ds + \sigma X(s)dW(s), \text{ for } s\in[0,t]$$

Assume that h, $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial x}$, $\frac{\partial^2 h}{\partial x^2}$ are bounded. Applying *Itô*' s formula (A.0.9) for the function $h(t,x) = e^{-\int_0^t r(s)ds}v(t,x)$, we get

$$\begin{aligned} e^{-\int_0^t r(s)ds}v(t,x) = &\int_0^t \left(-rv + \underbrace{\frac{\partial v}{\partial t} + (r-d)x\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2x^2\frac{\partial^2 v}{\partial x^2}}_{=\mathcal{L}_{BS}v}\right)e^{-\int_0^\tau r(\nu)d\nu}d\tau \\ &+ \int_0^t \sigma x\frac{\partial v}{\partial x}e^{-\int_0^\tau r(\nu)d\nu}dW_\tau. \end{aligned}$$

Using the notation of $\mathcal{L}_{BS}$, we rewrite the equation and obtain

$$\begin{aligned} e^{-\int_0^t r(s)ds}v(t,x) = &\int_0^t e^{-\int_0^\tau r(\nu)d\nu}\left(\mathcal{L}_{BS}v(t,x) - r(\tau)v(t,x)\right)d\tau \\ &+ \underbrace{\int_0^t \sigma x\frac{\partial v}{\partial x}(t,x)e^{-\int_0^\tau r(\nu)d\nu}dW_\tau}_{=:\,M_t}. \end{aligned}$$

Because we know that the condition $\left|x\frac{\partial v}{\partial x}\right|$ is bounded,

$$\mathbb{E}\left[\int_0^t \left|\sigma x\frac{\partial v}{\partial x}\right|^2 ds\right] < \infty$$

and by the definition of martingale, we have M_t is a martingale such that

$$M_t = e^{-\int_0^t r(s)ds}v(t,x) - \int_0^t e^{-\int_0^\tau r(\nu)d\nu}\left(\mathcal{L}_{BS}v(t,x) - r(\tau)v(t,x)\right)d\tau. \quad (3.3)$$

■

Using Proposition 3.2.1, the Black and Scholes equation is presented in the following Theorem.

Theorem 3.2.2. *Assume that the functions $\sigma(t)$ and $r(t)$ are continuous non-negative and bounded $[0,T]$. Consider a function $v : \mathbb{R}_+ \times [0,T] \mapsto \mathbb{R}$, that is continuous in $\mathbb{R}_+ \times [0,T]$ and C^1 - regular with respect to t and C^2 - regular with respect to x in $\mathbb{R}_+ \times [0,T)$, and such that $\left|x\frac{\partial P}{\partial x}\right| \leq C(1+x)$ with C independent of t. Assume that v satisfies*

$$\mathcal{L}_{BS}v(t,x) - r(t)v(t,x) = 0 \tag{3.4}$$

and

$$v(T,x) = g(x), \quad x \in \mathbb{R}_+. \tag{3.5}$$

Then we have

$$v(t,x) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} g(x)|F_t\right], \quad t \leq s \leq T.$$

Remark 3.2.3. (3.4)-(3.5) are called the *backward-in-time* parabolic boundary value problem. By changing the time variable t by the time to maturity $T-t$, one can define a new formulation, namely the *forward-in-time* parabolic problem, such that

$$\begin{cases} \partial_t v - \frac{1}{2}\sigma^2 x^2 \partial^2{}_x v - (r-d)x\partial_x v + rv = 0, & \text{in } Q_T, \\ v(0,x) = g(x), & \text{in } \in \mathbb{R}_+. \end{cases} \tag{3.6}$$

Proof. Given (3.4) and the formulation of M_t (3.3), we get

$$M_t = e^{-\int_0^t r(s)ds} v(t,x)$$

and

$$\begin{aligned} \mathbb{E}\left[e^{-\int_0^T r(s)ds} g(x)|F_t\right] &= \mathbb{E}\left[e^{-\int_0^T r(s)ds} v(T,x)|F_t\right] \\ &= \mathbb{E}\left[M_T|F_t\right]. \end{aligned}$$

We know above that M_t is a martingale. It follows from the properties of a martingale which states that the best estimate for a value of a random variable is equal to its current value and independent of all information from previous events, that

$$\mathbb{E}\left[e^{-\int_0^T r(s)ds} g(x)|F_t\right] = e^{-\int_0^t r(s)ds} v(t,x)$$

Multiplying by $e^{\int_0^t r(s)ds}$ gives

$$v(t,x) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} g(x)|F_t\right], \quad \text{for } t \leq s \leq T.$$

■

The Weak Formulation

We follow [1, p. 32] for introducing the weak formulation of the forward-in-time problem (3.6) as well as definition of solution. In particular, we seek a function $v \in C^0([0,T]; L^2(\mathbb{R}_+)) \cap L^2(0,T;M)$ and $\frac{\partial v}{\partial t} \in L^2(0,T;M')$, such that

$$\begin{cases} \left(\frac{\partial v}{\partial t}, w\right) + a(t; v, w) = 0, \quad \forall w \in M, \\ v(0,x) = g(x), \quad \forall x \in \mathbb{R}_+. \end{cases} \tag{3.7}$$

where the bilinear form $a(s; v, w)$ is defined as follows

$$a(t; v, w) = \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \left(-(r-d) + \sigma^2 + \sigma x \frac{\partial \sigma}{\partial x}\right) x \frac{\partial \sigma}{\partial x} w dx \\ + \int_{\mathbb{R}_+} rvwdx, \qquad \text{for } v, w \in M. \tag{3.8}$$

The bounded associated operator $A(t) : M \to M'$, such that $(A(t)v, w) = a(t; v, w)$, for any $v, w \in M$, is defined as

$$A(t)v = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} - (r-d)x\frac{\partial v}{\partial x} + rv \tag{3.9}$$

Before stating properties of the bilinear form $a(t; v, w)$, we need to impose some conditions

Assumption 3.2.4. Assume that there exist two positive constants, σ and $\bar{\sigma}$, such that for all $t \in [0,T]$ and all $x \in \mathbb{R}_+$

$$0 < \underline{\sigma} \leq \sigma(t) \leq \bar{\sigma}$$

The continuity of the bilinear form $a(t, v, w)$ is now stated by this lemma

Lemma 3.2.5. *Under Assumption 3.2.4, the bilinear form a_t is continuous on M; i.e., there exists a positive constant μ such that for all $v, w \in M$,*

$$|a_t(v,w)| \leq \mu \|v\|_M \|w\|_M. \tag{3.10}$$

Proof. [1, p. 32] Recalling the norm in M

$$\|u\|_M = \|u\|_{L^2(\mathbb{R}_+)} + \left\| x \frac{\partial u}{\partial x} \right\|_{L^2(\mathbb{R}_+)}, \quad u \in M.$$

Consider the bilinear form $a(t, u, v)$ given by (3.8). Using Hölder's inequality along with Assumption 3.2.4, we have

$$\left| \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \mathrm{d}x \right| \leq \frac{1}{2}\bar{\sigma}^2 \|u\|_M \|v\|_M.$$

Moreover, letting $R = \max_{t \in [0,T]} r(t)$, we get

$$\left| \int_{\mathbb{R}_+} \left(-(r-d) + \sigma^2\right) x \frac{\partial u}{\partial x} v \mathrm{d}x \right| \leq (R - d + \bar{\sigma}^2)\|u\|_{L^2(\mathbb{R}_+)} \|v\|_{L^2(\mathbb{R}_+)} \\ \leq 2(R - d + \bar{\sigma}^2)\|u\|_M \|v\|_M.$$

Finally, we have

$$\left| \int_{\mathbb{R}_+} ruv\mathrm{d}x \right| \leq R\|u\|_{L^2(\mathbb{R}_+)}\|v\|_{L^2(\mathbb{R}_+)}$$
$$\leq 4R\|u\|_M\|v\|_M.$$

which gives us the estimate (3.10) with $\mu = \frac{5}{2}\bar{\sigma}^2 + 6R$. ■

Besides that, the bilinear form $a(t;.,.)$ satisfies Gårding's inequality

Lemma 3.2.6 (Gårding's inequality)**.** *Under Assumption 3.2.4, there exists a nonnegative constant λ, such that*

$$a(t;v,v) \geq \frac{\underline{\sigma}^2}{4}\|v\|_M^2 - \lambda\|v\|^2_{L^2(\mathbb{R}_+)}, \quad \forall v \in M \tag{3.11}$$

Proof. [1, p. 32] Recalling the Poincáre inequality (3.2)

$$\|v\|_{L^2(\mathbb{R}_+)} \leq \left\| x\frac{\partial v}{\partial x} \right\|_{L^2(\mathbb{R}_+)}.$$

Consider the bilinear form $a(t;v,v)$ given by (3.8). Using Assumption 3.2.4, we observe that

$$\left| \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x \right| \geq \frac{1}{2}\underline{\sigma}^2\|v\|_M^2.$$

Moreover, letting $R = \max_{0\leq t\leq T} r(t)$, we have

$$\left| \int_{\mathbb{R}_+} \left(-(r-d)+\sigma^2\right) x\frac{\partial v}{\partial x} v\mathrm{d}x \right| \leq \left(R - d + \bar{\sigma}^2\right)\|v\|_M\|v\|_{L^2(\mathbb{R}_+)}$$
$$= \left(2\frac{\underline{\sigma}}{2}\|v\|_M\right)\left(\frac{R-d+\bar{\sigma}^2}{\underline{\sigma}}\right)\|v\|_{L^2(\mathbb{R}_+)}.$$

Using Cauchy's inequality (A.4), we deduce

$$\left| \int_{\mathbb{R}_+} \left(-(r-d)+\sigma^2\right) x\frac{\partial v}{\partial x} v\mathrm{d}x \right| \leq \frac{1}{4}\underline{\sigma}^2\|v\|_M^2 + \lambda\|v\|^2_{L^2(\mathbb{R}_+)}.$$

where $\lambda = \frac{\left(R-d+\bar{\sigma}^2\right)^2}{\underline{\sigma}^2}$. ■

The weak formulation (3.7) has exactly one continuous solution which is stated in the following theorem:

Theorem 3.2.7. *If $g \in L^2(\mathbb{R}_+)$, and under Assumption 3.2.4, the weak formulation (3.7) has a unique solution, and we have the estimate, for all t, $0 < t < T$,*

$$e^{-2\lambda t}\|v(t)\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{2}\underline{\sigma}^2\int_0^t e^{-2\lambda t}|v(\tau)|_V^2 d\tau \leq \|g\|^2_{L^2(\mathbb{R}_+)}. \tag{3.12}$$

Proof. We refer to [] for a proof of uniqueness. The estimate (5.34) is obtained by taking $w = v(t)e^{-2\lambda t}$ in the weak formulation (3.7), we get

$$\underbrace{\left(\frac{\partial v}{\partial t}, ve^{-2\lambda t}\right)} + a(t; v, ve^{-2\lambda t}) = 0 \tag{3.13}$$

$$\begin{aligned}\left(\frac{\partial v}{\partial t}, ve^{-2\lambda t}\right) &= \int_0^t \frac{\partial v}{\partial s} ve^{-2\lambda s} ds \\ &= \int_0^t \frac{1}{2}\left(v(s)^2 e^{-2\lambda s}\right)' ds + \int_0^t \lambda v(s)^2 e^{-2\lambda s} ds \\ &= \frac{1}{2}\|v(s)^2\|_{L^2(\mathbb{R}_+)} e^{-2\lambda s}\Big|_{s=0}^t + \int_0^t \lambda v(s)^2 e^{-2\lambda s} ds\end{aligned}$$

Therefore, we obtain from (3.13)

$$\frac{1}{2}\|v(s)^2\|_{L^2(\mathbb{R}_+)} e^{-2\lambda s}\Big|_{s=0}^t + \int_0^t \lambda v(s)^2 e^{-2\lambda s} ds + a(t; v, ve^{-2\lambda t}) = 0.$$

Applying *G*årding's inequality (3.11), (3.12) is obtained as desired. ∎

Remark 3.2.8. Theorem 3.2.7 shows that the function $(t, x) \mapsto v(t, x)$ is continuous and $v(t, x)$ is bounded by function $g(x)$ in the estimate (3.12) . Therefore, if $g \in L^2(\mathbb{R}_+)$, then $v \in C^0(0, T; L^2(\mathbb{R}_+))$.

CHAPTER 4

The quasi-variational inequality

Our motivation for studying the American option pricing problem with variational method is that it gives stability, which is one of our interests here. This section is dedicated to presenting an approach for solving the American option pricing problem through the variational method (2.6). Mathematically, the American option pricing problem is to seek the price function $V(t,x)$ such that

$$\begin{cases} -\frac{\partial V}{\partial t} + A(t)V \geq 0, & \text{in } Q_T, \\ g - V \leq 0, & \text{in } Q_T, \\ \left(-\frac{\partial V}{\partial t} + A(t)V\right)(g - V) = 0, & \text{in } Q_T, \\ V(T,x) = g(x), & x \in \mathbb{R}_+. \end{cases} \tag{4.1}$$

where $Q_T = [0,T) \times \mathbb{R}_+$ and the operator $A(t)v$ has the form

$$A(t)v = -\frac{1}{2}\sigma^2 x^2 \partial^2{}_x v - (r-d)x\partial_x v + rv. \tag{4.2}$$

For simplicity the presentation, we drop herein the dependence of V on (t,x).

As mentioned before, it has been shown that the solutions V are continuous [10, Proposition 2.2]. In addition, we do not need to determine the free boundary and we will consider (4.1) in the whole domain $Q_T = [0,T] \times \mathbb{R}_+$. Furthermore, the problem will be solved in the weighted Sobolev space M. Recalling that

$$M = \left\{ v \in L^2(\mathbb{R}_+), \quad x\frac{dv}{dx} \in L^2(\mathbb{R}_+) \right\},$$

which is equipped with the inner product $(v,w)_M$ and the norm $\|.\|_M$, respectively

$$\langle v,w\rangle_M = (v,w) + \left(x\frac{dv}{dx}, x\frac{dw}{dx}\right), \qquad \|v\|_M = \sqrt{\langle v,v\rangle_M}.$$

where the inner product in $L^2(\mathbb{R}_+)$ is denoted by $(,)$. The dual space of M is denoted by M' which endows with the norm

$$\|w\|_{M'} = \sup_{\|v\|_M \leq 1} \frac{\langle w,v\rangle_{M',M}}{\|v\|_M}, \quad \forall v \in M.$$

and the inner product in M' is denoted by $\langle \cdot,\cdot\rangle$. Solving (4.1) in the weighted Sobolev space M is one of the key conditions which ensures coercivity property needed in the well-possedness analysis later (see discussion below). Our goal here is to present an existence and uniqueness for (4.1).

In what follows, we rely on [1] for deriving the weak formulation of (4.1) as well as definition of weak solutions to (4.1).

4.1 The weak formulation

Let us first introduce the nonempty, convex set $\mathcal{K}$

$$\mathcal{K} = \big\{ w \in M : w \geq g \text{ in } \mathbb{R}_+ \big\}. \tag{4.3}$$

In fact, the convex set K depends on time t, which is one of the basic difficulties for solving the problem. In what follows, we will consider a special case where $\mathcal{K}$ does not depend on t which is significantly simpler.

Let $C_c^\infty(Q_T)$ denote the space of infinitely differentiable functions with compact support in Q_T. We usually call a function ϕ belonging to $C_c^\infty(Q_T)$ a *test function*. Assume that $V \in M$, such that $\frac{\partial V}{\partial t} \in M'$. For any test function $w \in \mathcal{K}$, we multiply the first inequality of (4.1) by w, integrate in x over $\mathbb{R}_+$ and doing integration by parts, we obtain

$$\int_{\mathbb{R}_+} \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - (r-d)x\frac{\partial V}{\partial x} + rV \right) w \mathrm{d}x \geq 0. \tag{4.4}$$

Doing integration by parts for $\int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x} w \mathrm{d}x$, it yields

$$\int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x} w \mathrm{d}x = -\int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial V}{\partial x}\frac{\partial w}{\partial x} + \sigma^2 x \frac{\partial V}{\partial x} w \mathrm{d}x.$$

Inserting this equation into (4.4) and using the notion of the inner product, then we obtain

$$-\left\langle \frac{\partial V}{\partial t}, w \right\rangle_{M',M} + a(t; V, w) \geq 0, \quad \forall w \in \mathcal{K}. \tag{4.5}$$

where the bilinear form $a(t; u, v)$ is defined by

$$\begin{aligned} a(t; u, v) = &\int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(-(r-d) + \sigma^2\right) x \frac{\partial u}{\partial x} v \mathrm{d}x \\ &+ \int_{\mathbb{R}_+} ruv \mathrm{d}x, \qquad \forall u, v \in M. \end{aligned} \tag{4.6}$$

Let $\mathcal{K}_0$ denote the cone of non-negative functions in M, the set K in (4.3) is exactly $\mathcal{K} = g + \mathcal{K}_0$. For any $w \in \mathcal{K}_{\prime}$, we have obtained from (4.5) that

$$-\left\langle \frac{\partial V}{\partial t}, w \right\rangle_{M',M} + a(t; V, w) \geq 0, \quad \forall w \in \mathcal{K}_0.$$

It is equivalent to

$$-\left\langle \frac{\partial V}{\partial t}, w - g \right\rangle_{M',M} + a(t; V, w - g) \geq 0, \quad \forall w \in \mathcal{K}_0. \tag{4.7}$$

Moreover, integrating the third equation of (4.1) in x over $\mathbb{R}_+$ and doing integration by parts. Similarly from the above, we easily obtain

$$-\left\langle \frac{\partial V}{\partial t}, V-g \right\rangle_{M',M} + a(t;V,V-g) = 0. \tag{4.8}$$

Subtracting (4.8) from (4.7) and using the linearity of $a(t;u,v)$, we obtain

$$-\left\langle \frac{\partial V}{\partial t}, w-V \right\rangle_{M',M} + a(t;V,w-V) \geq 0, \quad \forall w \in \mathcal{K}. \tag{4.9}$$

We introduce definition of weak solutions as follows

Definition 4.1.1 (Weak solutions)**.** We call V a weak solution of (4.1) if

$$V \in L^2(0,T;\mathcal{K}), \quad \frac{\partial V}{\partial t} \in L^2(0,T;M'),$$

and V solves the weak formulation (4.9). In addition,

$$V(T,x) = g(x), \quad x \in \mathbb{R}_+. \tag{4.10}$$

Remark 4.1.2. Since $V \in L^2(0,T;M)$ and $\frac{\partial V}{\partial t} \in L^2(0,T;M')$, it follows from the chain rule (A.0.15) that

$$V \in C^0\left([0,T];L^2(\mathbb{R}_+)\right).$$

In what follows, we present well-posedness result for the American option pricing problem (4.1). A general context for studying variational inequalities problems was introduced in [2] with the penalisation method for solving the problem.

4.2 Well-posedness

The bilinear form $a(t;u,v)$ in (4.6) has the same properties as the one linked to the Black and Scholes equation. In particular, recalling Assumption 3.2.4, it follows that the bilinear form $a(t;u,v)$ is bounded in the sense that

$$|a(t;u,v)| \leq \mu\|u\|_M\|v\|_M, \quad u,v \in M$$

and satisfies the following Gårding's inequality

$$a(t;v,v) \geq \alpha\|v\|_M^2 - \lambda\|v\|_{L^2(\mathbb{R}_+)}^2, \quad \forall v \in M,$$

where $\alpha \geq 0$ and $\lambda > 0$ are some constants. The first inequality also implies that $a(t;u,v)$ is continuous in M uniformly in time t. We refer to (3.10) and (3.11) in Chapter 3 for details.

At this point, we should explain why the weighted Sobolev space M is suitable for solving the problem. The key condition is the Gårding inequality which ensures the existence and uniqueness of a solution for (4.1). In the general context [2], this amounts to the coercivity property

$$a(t;v,v) \geq \alpha\|v\|_M^2 - \lambda\|v\|_{L^2(\mathbb{R}_+)}^2, \quad \forall v \in M,$$

for constants $\alpha > 0$ and $\lambda \geq 0$. We know that the stock price in practice can be zero. When $x \to 0$, using the weighted space M we obtain the Gårding inequality in (3.11) with $\alpha = \frac{1}{4}\underline{\sigma}^2$. Making use of Assumption 3.2.4, thus $\alpha > 0$. Hence, the condition still holds in case of $x \to 0$.

Regarding the well-posedness result for the American option pricing problem (4.1), we have the following theorem:

Theorem 4.2.1. *Assume that $g \in L^2(\mathbb{R}_+)$. Then there exist a unique weak solution of (4.1).*

Our goal here is to prove Theorem 4.2.1. However, inspired by [2], we use penalisation method to study (4.1). Thus, our proof of Theorem 4.2.1 will give later in the next chapter where we present and analyze the associated penalised problem of (4.1).

CHAPTER 5

The penalised problem

The *classical penalty* method was studied by Bensoussan and Lions in [2]. It is used to construct numerical schemes for computing the value of an American option [4, 8, 13]. In particular, the penalised equation seeks an approximation V_ϵ to the weak solution V of (4.1) for each $\epsilon > 0$. When $\epsilon \downarrow 0$, the approximate solutions V_ϵ converge to the unit weak solution V.

We consider the American option pricing problem (4.1). The associated penalised problem of (4.1) is to seek a function V_ϵ that satisfies

$$\begin{cases} -\frac{\partial V_\epsilon}{\partial t} + A(t)V_\epsilon - \frac{1}{\epsilon}(g - V_\epsilon)^+ = 0, & \text{in } Q_T, \\ V_\epsilon(T) = g, & \text{in } \mathbb{R}_+. \end{cases} \tag{5.1}$$

$\epsilon > 0$ is called the "penalization parameter".

Let us first motivate definition of weak solutions. We consider a mapping

$$V_\epsilon : [0, T] \mapsto M$$

defined by

$$\big[V_\epsilon(t)\big](x) := V_\epsilon(t, x), \quad \text{in } Q_T.$$

In other words, we are now considering V_ϵ not as a function of t and x together, but rather as a mapping V_ϵ of time t into the space M. This makes it easier for us to understand the following argument.

Returning to (5.1), if $V_\epsilon \in \mathcal{K}$, where $\mathcal{K}$ is given by (4.3), it gives

$$\frac{\partial V_\epsilon}{\partial t} = -\frac{1}{\epsilon}(g - V_\epsilon)^+ + A(t)V_\epsilon.$$

Inserting $A(t)$ given by (4.2), we get

$$\frac{\partial V_\epsilon}{\partial t} = -\frac{1}{\epsilon}(g - V_\epsilon)^+ - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_\epsilon}{\partial x^2} - (r - d)x\frac{\partial V_\epsilon}{\partial x} + rV_\epsilon. \tag{5.2}$$

In a general form, we have

$$\frac{\partial V_\epsilon}{\partial t} = h^0 + h, \quad \text{in } Q_T, \tag{5.3}$$

where

$$h^0 = -\frac{1}{\epsilon}(g - V_\epsilon)^+ - (r-d)x\frac{\partial V_\epsilon}{\partial x} + rV_\epsilon$$

and

$$h^1 = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_\epsilon}{\partial x^2}.$$

Consequently, (5.3) and the definition from Chapter 3 imply that the right-hand side of (5.3) lies in M'. Indeed, recalling that

$$\|\partial_t V_\epsilon\|_{M'} = \sup_{\|w\|_M \le 1} \frac{\langle \partial_t V_\epsilon, w\rangle_{M',M}}{\|w\|_M}. \tag{5.4}$$

Using $\|w\|_M \le 1$, we utilize the equation above and use (5.3) to deduce that

$$\begin{aligned}\|\partial_t V_\epsilon\|_{M'} &\le \left|\langle \partial_t V_\epsilon, w\rangle_{M',M}\right| \\ &= \left|\langle h^0 + h^1, w\rangle\right| \\ &\le \left|\langle h^0, w\rangle\right| + \left|\langle h^1, w\rangle\right|.\end{aligned} \tag{5.5}$$

To estimate $\left|\langle h^0, w\rangle\right|$ and $\left|\langle h^1, w\rangle\right|$, notice that since $V_\epsilon \in M$ and $g \in L^2(\mathbb{R}_+)$, then $(g - V_\epsilon)^+ \in L^2(\mathbb{R}_+)$. For $\|w\|_M \le 1$, using Cauchy-Schwartz inequality (A.1), the triangle inequality and the definition of the norm in M, we obtain

$$\begin{aligned}\left|\langle h^0, w\rangle\right| &\le \left|\left\langle -\frac{1}{\epsilon}(g - V_\epsilon)^+, w\right\rangle\right| + \left|\left\langle -(r-d)x\frac{\partial V_\epsilon}{\partial x}, w\right\rangle\right| + |\langle rV_\epsilon, w\rangle| \\ &\le C\left(\|(g - V_\epsilon)^+\|_{L^2(\mathbb{R}_+)} + \left\|x\frac{\partial V_\epsilon}{\partial x}\right\|_{L^2(\mathbb{R}_+)} + \|V_\epsilon\|_{L^2(\mathbb{R}_+)}\right)\|w\|_{L^2(\mathbb{R}_+)} \\ &\le C\left(\|g\|_{L^2(\mathbb{R}_+)} + \|V_\epsilon\|_M\right),\end{aligned}$$

for a constant C not depending on ϵ. Moreover, let us estimate $\left|\langle h^1, w\rangle\right|$. Assume that for any nonnegative test function $w \in C_0^\infty(\mathbb{R}_+)$, we integrate by parts, then apply Cauchy-Schwartz inequality (A.1) and using Assumption 3.2.4, it yields

$$\begin{aligned}\left|\langle h^1, w\rangle\right| &= \left|\left\langle -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_\epsilon}{\partial x^2}, w\right\rangle\right| \\ &= \left|\int_{R_+} -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_\epsilon}{\partial x^2} w \mathrm{d}x\right| \\ &= \int_{R_+}\left|-\frac{1}{2}\sigma^2 x^2 \frac{\partial V_\epsilon}{\partial x}\frac{\partial w}{\partial x} + \sigma^2 x \frac{\partial V_\epsilon}{\partial x} w\right| \mathrm{d}x \\ &\le C\left(\int_{R_+}\left|\left(x\frac{\partial V_\epsilon}{\partial x}\right)\left(x\frac{\partial w}{\partial x}\right)\right| \mathrm{d}x + \int_{R_+}\left|x\frac{\partial V_\epsilon}{\partial x} w\right| \mathrm{d}x\right) \\ &\le C\left(\left\|x\frac{\partial V_\epsilon}{\partial x}\right\|_{L^2(\mathbb{R}_+)}\left\|x\frac{\partial w}{\partial x}\right\|_{L^2(\mathbb{R}_+)} + \left\|x\frac{\partial V_\epsilon}{\partial x}\right\|_{L^2(\mathbb{R}_+)}\|w\|_{L^2(\mathbb{R}_+)}\right) \\ &\le C\|V_\epsilon\|_M\|w\|_M \\ &\le C\|V_\epsilon\|_M,\end{aligned}$$

for some positive constants C not depending on ϵ. Thus, it follows from (5.5) that

$$\begin{aligned}\|\partial_t V_\epsilon\|_{M'} &\leq \left|\langle h^0, w\rangle\right| + \left|\langle h^1, w\rangle\right| \\ &\leq C\left(\|V_\epsilon\|_M + \|g\|_{L^2(\mathbb{R}_+)}\right).\end{aligned}$$

Finally, integrating in time from 0 to T and using $(a+b)^2 \leq 2(a^2+b^2)$, we arrive at

$$\begin{aligned}\int_0^T \|\partial_t V_\epsilon\|_{M'}^2 \,\mathrm{d}t &\leq C\int_0^T \left(\|g\|_{L^2(\mathbb{R}_+)} + \|V_\epsilon\|_M\right)^2 \mathrm{d}t \\ &\leq C_1 \int_0^T \left(\|g\|_{L^2(\mathbb{R}_+)}^2 + \|V_\epsilon\|_M^2\right)\mathrm{d}t \\ &\leq C_1\left(T\|g\|_{L^2(\mathbb{R}_+)}^2 + \|V_\epsilon\|_{L^2(0,T;M)}^2\right) \\ &\leq C_T,\end{aligned}$$

where the constants C, C_1 and C_T do not depend on ϵ. The last inequality comes from the facts that $V_\epsilon \in L^2(0,T;M)$, $g \in L^2(\mathbb{R}_+)$ and T is finite. Hence,it follows that $\frac{\partial V_\epsilon}{\partial t}$ is bounded in M' for almost every time in $[0,T]$.

The weak formulation of (5.1) is obtained by multiplying (5.1) by an arbitrary function $w \in L^2(0,T;M)$. We integrate in x over $\mathbb{R}_+$ and then do integration by parts. Thus, we obtain

$$\begin{cases} -\left\langle \frac{\partial V_\epsilon}{\partial t}, w\right\rangle_{M',M} + a(t;V_\epsilon,w) - \frac{1}{\epsilon}\left((g-V_\epsilon)^+, w\right) = 0, \quad \text{in } Q_T \\ V_\epsilon(T) = g, \quad \text{in } \mathbb{R}_+, \end{cases} \tag{5.6}$$

where the bilinear form $a(t;.,.)$ takes the form (4.6). From the above, we have the following definition of weak solutions:

Definition 5.0.1 (Weak solution)**.** We call V_ϵ a weak solution to (5.1) if

$$V_\epsilon \in L^2(0,T;\mathcal{K}), \quad \frac{\partial V_\epsilon}{\partial t} \in L^2(0,T;M'). \tag{5.7}$$

and V_ϵ satisfies the weak formulation (5.6), where the set $\mathcal{K}$ is given by (4.3).

Observe that since $g \in L^2(\mathbb{R}_+)$, then $(g-V_\epsilon)^+ \in L^2(0,T;M)$. Define a nonempty set $\mathcal{N}$ such that

$$\mathcal{N} = \left\{ w \middle| \quad w \in L^2(0,T;M); \quad \frac{\partial w}{\partial t} \in L^2(0,T;M'); \quad w \geq g \text{ a.e in } Q_T \right\}. \tag{5.8}$$

Notice that since $V_0 \in \mathcal{N}$, $g - V_0 \leq 0$. Thus,

$$\begin{aligned}(g-V_\epsilon)^+ &= (g - V_0 + V_0 - V_\epsilon)^+ \\ &\leq (V_0 - V_\epsilon)^+.\end{aligned}$$

5.1 Galerkin approximations

To analyze the penalised problem (5.1), we use the Galerkin approximations [5, p. 375]. More precisely, the Galerkin's method is used to construct approximate solutions V_m and then passing to the limit by sending $m \to \infty$.

To construct the approximate solutions V_m, we assume the existence of smooth functions $e_k = e_k(x)$ for $k = 1, 2, \ldots$, such that $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis of M and $L^2(\mathbb{R}_+)$. For a fixed positive integer m, we look for a function $V_m : [0, T] \to M$ of the form

$$V_m(t) = \sum_{k=1}^{m} d_m^k(t) e_k, \quad t \in [0, T]; \quad k = 1, \ldots, m, \tag{5.9}$$

where the coefficients $d_m^k(t)$ are chosen so that, for $k = 1, \ldots, m$,

$$\begin{cases} -\left\langle \frac{\partial V_m}{\partial t}, e_k \right\rangle_{M',M} + a(t; V_m, e_k) - \frac{1}{\epsilon}\big((g - V_m)^+, e_k\big) = 0, & \text{in } Q_T, \\ d_m^k(T) = (g, e_k), & \text{in } \mathbb{R}_+. \end{cases} \tag{5.10}$$

The weak formulation (5.10) admits a unique solution.

Theorem 5.1.1 (Construction of the approximate solutions)**.** *For each integer $m = 1, 2, \ldots$ there exists a unique function V_m of the form (5.9) satisfying (5.10).*

Proof. Consider (5.10). Since V_m has the form of (5.9) and $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis of $L^2(\mathbb{R}_+)$, we have

$$\left\langle \frac{\partial V_m}{\partial t}, e_k \right\rangle_{M',M} = \partial_t d_m^k(t).$$

Moreover, substituting V_m in (5.9) into the bilinear form (3.8), we have

$$\begin{aligned} a(t; V_m, e_k) = & \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial\big(\sum_{l=1}^m d_m^l(t) e_l\big)}{\partial x} \frac{\partial e_k}{\partial x} \mathrm{d}x \\ & + \int_{\mathbb{R}_+} \big(-(r-d) + \sigma^2\big)\, x \frac{\partial\big(\sum_{l=1}^m d_m^l(t) e_l\big)}{\partial x} e_k \mathrm{d}x \\ & + \int_{\mathbb{R}_+} r \left(\sum_{l=1}^{m} d_m^l(t) e_l\right) e_k \mathrm{d}x. \end{aligned}$$

Using the linearity of $a(t; u, v)$, we move out $\sum_{l=1}^m d_m^l(t)$ before the integral and get

$$\begin{aligned} & a(t; V_m, e^k) \\ & = \left(\sum_{l=1}^{m} d_m^l(t)\right) \int_{\mathbb{R}_+} \left(\frac{1}{2}\sigma^2 x^2 \frac{\partial e_l}{\partial x}\frac{\partial e_k}{\partial x} + \big(-(r-d) + \sigma^2\big)\, x \frac{\partial e^l}{\partial x} e_k + r e^l e_k\right) \mathrm{d}x \\ & = \sum_{l=1}^{m} d_m^l(t) a(t; e_l, e_k). \end{aligned}$$

Also, we have

$$\begin{aligned} \big((g - V_m)^+, e_k\big) &= \Big(\big[\, \big(d_m^k(T), e_k\big) - \big(d_m^k(t), e_k\big) \,\big]^+, e_k\Big) \\ &= \big(d_m^k(T) - d_m^k(t)\big)^+, \end{aligned}$$

where $\left(d_m^k(T) - d_m^k(t)\right)^+ = \max\left\{d_m^k(T) - d_m^k(t), 0\right\}$. Inserting these into (5.10), thus

$$-\partial_t d_m^k(t) + \sum_{l=1}^{m} d_m^l(t) a(t; e_l, e_k) - \frac{1}{\epsilon}\left(d_m^k(T) - d_m^k(t)\right)^+ = 0, \quad \text{for } k = 1, \ldots, m.$$

We rewrite the equation above and get

$$-\partial_t d_m^k(t) + \sum_{l=1}^{m} d_m^l(t) a(t; e_l, e_k) = f^k(t), \tag{5.11}$$

where $f^k(t) = \frac{1}{\epsilon}\left[d_m^k(T) - d_m^k(t)\right]^+$ for $k = 1, \ldots, m$.

According to standard existence theory for ordinary differential equations, there exists a unique absolutely continuous function $d_m^k(t)$, for $k = 1, \ldots, m$ satisfying (5.11) almost every time in $[0, T]$ and the terminal data $d_m^k(T) = (g, e^k)$ in $\mathbb{R}_+$. Then V_m is uniquely determined by (5.9) and solves (5.10) almost every time in $[0, T]$. ■

We have already obtained the existence and uniqueness of the approximate solution V_m in the finite dimensional subspace spanned by orthogonal basis $\{e^k\}_{k=1}^{\infty}$. In the next step, we will send m to infinity and show that a subsequence of the approximation V_m converges to the weak solution V. Before doing this, we propose an estimate which is useful to analyse the well-posedness of the solution later.

5.2 Energy estimates

The following theorem gives us some useful estimates:

Theorem 5.2.1 (Energy estimates)**.** *There exists some positive constants C_1 and C_2 that do not depend on $m = 1, 2, \ldots$ such that*

$$\max_{0\le t\le T} \|V_m(t,.)\|^2_{L^2(\mathbb{R}_+)} + \|V_m\|^2_{L^2(0,T;M)} + \left\|\frac{\partial V_m}{\partial t}\right\|^2_{L^2(0,T;M')} \le e^{C_1 t}\|g\|^2_{L^2(\mathbb{R}_+)}\left(1 + C_2 t\right). \tag{5.12}$$

Proof. Multiplying the first equation of (5.10) by $d_m^k(t)$, summing over $k = 1, \ldots, m$ and using the formulation (5.9) of V_m, we obtain

$$-\left\langle \frac{\partial V_m}{\partial t}, V_m \right\rangle_{M',M} + a(t; V_m, V_m) - \frac{1}{\epsilon}\Big((g - V_m)^+, V_m\Big) = 0. \tag{5.13}$$

Observe that using the chain rule (A.0.15), we have

$$-\left\langle \frac{\partial V_m}{\partial t}, V_m \right\rangle_{M',M} = -\frac{d}{dt}\left(\frac{1}{2}\|V_m(t)\|^2_{L^2(\mathbb{R}_+)}\right).$$

Moreover, applying Cauchy-Schwarz's inequality (A.4) and Minkowski's inequality (**??**), it yields

$$\Big((g - V_m)^+, V_m\Big) \le 1/2\left(\left\|(g - V_m)^+\right\|^2_{L^2(\mathbb{R}_+)} + \|V_m\|^2_{L^2(\mathbb{R}_+)}\right)$$
$$\le C_1\left(\|V_m\|^2_{L^2(\mathbb{R}_+)} + \|g\|^2_{L^2(\mathbb{R}_+)}\right),$$

for a constant C_1 not depending on m. Finally, using Gårding inequality (3.11), (5.13) implies that

$$-\frac{d}{dt}\left(\frac{1}{2}\|V_m(t)\|^2_{L^2(\mathbb{R}_+)}\right)+\alpha\|V_m\|^2_M \le C_3\|V_m\|^2_{L^2(\mathbb{R}_+)}+C_2\|g\|^2_{L^2(\mathbb{R}_+)}. \quad (5.14)$$

It follows that

$$-\frac{d}{dt}\left(\frac{1}{2}\|V_m(t)\|^2_{L^2(\mathbb{R}_+)}\right)\le C_3\|V_m\|^2_{L^2(\mathbb{R}_+)}+C_2\|g\|^2_{L^2(\mathbb{R}_+)}, \quad (5.15)$$

almost every time in $[0,T]$ and for some positive constants C_3, C_4, independent of m. Setting $\eta(t):=\|V_m(t)\|^2_{L^2(\mathbb{R}_+)}$, rewriting (5.15)

$$-\eta'(t)\le C_3\eta(t)+C_2\|g\|^2_{L^2(\mathbb{R}_+)}.$$

Applying the Gronwall's inequality (A.10), it gives

$$\eta(t)\le e^{C_3t}\left(\eta(T)+C_2t\|g\|^2_{L^2(\mathbb{R}_+)}\right),\quad t\in[0,T].$$

Since $d^k_m(T)=(g,e_k)$, we multiply both sides of this equation by e_k and use the fact that $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis of $L^2(\mathbb{R}_+)$ so $\|e_k\|^2_{L^2(\mathbb{R}_+)}\le 1$, it yields $\eta(T)=\|V_m(T)\|^2_{L^2(\mathbb{R}_+)}\le\|g\|^2_{L^2(\mathbb{R}_+)}$. Taking maximum over $t\in[0,T]$ and substituting $\eta(t)$ by $\|V_m(t)\|^2_{L^2(\mathbb{R}_+)}$, we find that

$$\max_{0\le t\le T}\|V_m(t,.)\|^2_{L^2(\mathbb{R}_+)}\le e^{C_3t}\|g\|^2_{L^2(\mathbb{R}_+)}\left(1+C_2t\right). \quad (5.16)$$

for some positive constants C_3, C_2 not depending on m.

Returning to (5.14), we integrate in time from 0 to T and employ (5.16), it yields

$$\begin{aligned}\|V_m(t)\|^2_{L^2(0,T;M)}&=\int_0^T\|V_m(t)\|^2_M\,\mathrm{d}t\\&\le e^{C_3t}\|g\|^2_{L^2(\mathbb{R}_+)}\left(1+C_2t\right).\end{aligned} \quad (5.17)$$

It remains to prove

$$\|\partial_tV_m\|^2_{L^2(0,T;M')}\le e^{C_3t}\|g\|^2_{L^2(\mathbb{R}_+)}\left(1+C_2t\right). \quad (5.18)$$

Fix any $w\in M$, with $\|w\|_M\le 1$, and write $w=w_1+w_2$, where $w_1\in\operatorname{span}\{e_k\}_{k=1}^{\infty}$ and $(w_2,e_k)=0$, for $k=1,\dots,m$. Since $\{e_k\}_{k=1}^{\infty}$ are orthogonal in M, $\|w_1\|_M\le\|w\|_M\le 1$. From (5.10), we deduce

$$-\left(\partial_tV_m,w^1\right)+a(t;V_m,w^1)-\frac{1}{\epsilon}\left(\left(g-V_m\right)^+,w^1\right)=0,\quad\text{in } Q_T.$$

It is equivalent to,

$$\left(\partial_tV_m,w^1\right)=a(t;V_m,w^1)-\frac{1}{\epsilon}\left(\left(g-V_m\right)^+,w^1\right). \quad (5.19)$$

Using $\|w_1\|_M \leq \|w\|_M \leq 1$, we observe that

$$\begin{aligned} a(t; V_m, w^1) &\leq C\|V_m\|_M \|w^1\|_M \\ &\leq C\|V_m\|_M. \end{aligned}$$

for a constant C not depending on m. Moreover, using Cauchy's inequality (A.1) and the triangle inequality, we also have

$$\begin{aligned} \left|\left((g - V_m)^+, w^1\right)\right| &\leq \left\|(g - V_m)^+\right\|_{L^2(\mathbb{R}_+)} \|w^1\|_{L^2(\mathbb{R}_+)} \\ &\leq C\left(\|V_m\|_{L^2(\mathbb{R}_+)} + \|g\|_{L^2(\mathbb{R}_+)}\right) \\ &\leq C\left\|V_m\right\|_M, \end{aligned}$$

for a constant C not depending of m. Thus, we deduce from (5.19) that

$$\left|\left(\partial_t V_m, w^1\right)\right| \leq C\left\|V_m\right\|_M. \tag{5.20}$$

Moreover, we have $\partial_t V_m \in M'$ and recall the norm in M'

$$\|\partial_t V_m\|_{M'} = \sup_{\|w\|_M \leq 1} \frac{\langle \partial_t V_m, w\rangle_{M',M}}{\|w\|_M}.$$

Consider $\langle \partial_t V_m, w\rangle_{M',M}$. Since $w = w_1 + w_2$, where $w_1 \in \text{span}\ \{e_k\}_{k=1}^{\infty}$ and $(w_2, e_k) = 0$, for $k = 1, \ldots, m$, we have

$$\left\langle \partial_t V_m, w\right\rangle_{M',M} = (\partial_t V_m, w) = (\partial_t V_m, w_1),$$

Here we use $(.,.)$ which denotes the inner product in $L^2(\mathbb{R}_+)$. Using $\|w^1\|_M \leq \|w\|_M \leq 1$, then

$$\begin{aligned} \|\partial_t V_m\|_{M'} &\leq \left|\left\langle \partial_t V_m, w^1\right\rangle_{M',M}\right| \\ &\leq C\left\|V_m\right\|_M. \end{aligned}$$

Integrating in time from 0 to T

$$\int_0^T \left\|\partial_t V_m(t)\right\|_{M'}^2 \mathrm{d}t \leq C\int_0^T \|V_m(t)\|_M^2\,\mathrm{d}t.$$

Using (5.17), thus we obtain (5.18). ■

5.3 Well-posedness

The existence and uniqueness result for the penalised problem (5.1) is now stated in the following theorem:

Theorem 5.3.1 (Well-posedness)**.** *(5.1) admits a unique weak solution defined by (5.7).*

To prove Theorem 5.3.1, we send m to ∞, then the approximation V_m converges to the weak solution V of the associated penalised problem (5.1). Since the penalty term $(g - V_m)^+$ is nonlinear and it is not usually continuous with respect to weak convergence (see [5, p. 531]). Therefore, we need strong convergence of V_m to pass to the limit. To find out strong convergence of V_m, we first point out the connection between the weighted space M and the traditional Sobolev space H^1 ([5]). We say that a function $u \in H^1(\Omega)$ if and only if u and the first derivative $\frac{\partial u}{\partial x}$ both belong to $L^2(\Omega)$. Let us recall the norm of V_m in M and in $H^1((a,\infty))$ for any $a > 0$, respectively:

$$\begin{aligned} \|V_m\|_M &= \|V_m\|_{L^2(\mathbb{R}_+)} + \left\| x \frac{\partial V_m}{\partial x} \right\|_{L^2(\mathbb{R}_+)}, \\ \|V_m\|_{H^1((a,\infty))} &= \|V_m\|_{L^2((a,\infty))} + \left\| \frac{\partial V_m}{\partial x} \right\|_{L^2((a,\infty))}. \end{aligned} \tag{5.21}$$

Since $x \in (a,\infty)$, $x > a$, then $\frac{x}{a} > 1$. Thus

$$\begin{aligned} \int_a^\infty \left| \frac{\partial V_m}{\partial x} \right|^2 \mathrm{d}x &\le \frac{1}{a^2} \int_a^\infty \left| x \frac{\partial V_m}{\partial x} \right|^2 \mathrm{d}x \\ &\le \frac{1}{a^2} \int_0^\infty \left| x \frac{\partial V_m}{\partial x} \right|^2 \mathrm{d}x. \end{aligned}$$

In other words,

$$\left\| \frac{\partial V_m}{\partial x} \right\|_{L^2((a,\infty))} \le C_a \left\| x \frac{\partial V_m}{\partial x} \right\|_{L^2(\mathbb{R}_+)}$$

for a constant C_a depending on $a > 0$. It follows from (5.21) that

$$\|V_m\|_{H^1((a,\infty))} \le C_a \|V_m\|_M, \qquad \forall a > 0. \tag{5.22}$$

In other words,

$$M \subset H^1((a,\infty)), \qquad \forall a > 0. \tag{5.23}$$

The next result is an extension of this observation to time dependent functions

Lemma 5.3.2. *If $u \in L^2(0,T;M)$, then*

$$u \in L^2\big(0,T;H^1((a,\infty))\big), \quad \forall a > 0.$$

Furthermore, it follows from (5.23) that

$$\Big(H^1((a,\infty))\Big)' \subset M', \qquad \forall a > 0, \tag{5.24}$$

where M' and $\big(H^1((a,\infty))\big)'$ denote the dual space of M and $H^1((a,\infty))$, respectively. Since we have from [5, p. 299]

$$H^1((a,\infty)) \subset L^2((a,\infty)) \subset \Big(H^1((a,\infty))\Big)',$$

along with (5.23), (5.24), we obtain

$$M \subset H^1((a,\infty)) \subset L^2((a,\infty)) \subset \Big(H^1((a,\infty))\Big)' \subset M'.$$

Thus,

$$M \subset L^2((a,\infty)) \subset M'. \tag{5.25}$$

This shows that M is compactly embedded in $L^2((a,\infty))$. This information will be applied for the next lemma.

Lemma 5.3.3 (Aubin-Lions)**.** *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X. For $1 \leq p, q \leq +\infty$, let*

$$W = \left\{ u \in L^p(0,T;X_0), \quad \frac{\partial u}{\partial t} \in L^q(0,T;X_1) \right\}$$

If $p \in [1, \infty)$, then the embedding of W into $L^p(0,T;X)$ is compact.

Define a nonempty set W_a of all functions V_m such that

$$W_a = \left\{ V_m \in L^2\left(0,T;M\right), \quad \frac{\partial V_m}{\partial t} \in L^2\left(0,T;M'\right) \right\}.$$

For the identities $X_0 = M$, $X = L^2\left((a,\infty)\right)$ and $X_1 = M'$, we have (5.25) and apply Lemma 5.3.3, then W_a is compactly embedded in $L^2\big((0,T) \times (a,\infty)\big)$

Lemma 5.3.4. *Let M denotes the weighted Sobolev space defined by*

$$M = \left\{ u \in L^2(\mathbb{R}_+), \quad x\frac{\partial u}{\partial x} \in L^2(\mathbb{R}_+) \right\},$$

whereas M' denotes its dual space. Then, M is compactly embedded in $L^2\big((a,\infty)\big)$, for a constant $a > 0$. Furthermore, let

$$W_a = \left\{ u \in L^2(0,T;M), \quad \frac{\partial u}{\partial t} \in L^2\left(0,T;M'\right) \right\},$$

Then W_a is compactly embedded in $L^2\big(0,T;L^2\left((a,\infty)\right)\big)$, for $a > 0$.

It follows from weak compactness Theorem A.0.14 that there exists a subsequence $\{V_{m_j}\}_{j>0} \subseteq \{V_m\}$ and a function V such that

$$V_{m_j} \text{ converges strongly to } V \text{ in } L^2\left((0,T) \times (a,\infty)\right),$$

for any $a > 0$. Thus, by a diagonal argument, we conclude that

$$V_{m_j} \text{ converges strongly to } V \text{ in } L^2\left((0,T) \times (0,\infty)\right).$$

Having obtained strong convergence of approximation solutions V_m in $L^2\left((0,T) \times \mathbb{R}_+\right)$, we are now ready to prove the well-posedness theory of weak solutions to the penalised problem (5.1).

Proof of Theorem 5.3.1. We will first prove for the existence. According to the energy estimates (5.12), $\{V_m\}_{m=1}^{\infty}$ and $\{\frac{\partial V_m}{\partial t}\}_{m=1}^{\infty}$ are both bounded in $L^2(0,T;M)$ and $L^2(0,T;M')$, respectively. It follows from weak compactness Theorem A.0.14 that there exists a subsequence $\{V_{m_l}\}_{l=1}^{\infty} \subseteq \{V_m\}$ and a function $V \in L^2(0,T;M)$, such that

$$V_{m_l} \text{ converges weakly to } V \text{ in } L^2(0,T;M).$$

Similarly, there also exists a subsequence $\{\frac{\partial V_{m_l}}{\partial t}\}_{l=1}^{\infty} \subseteq \{\frac{\partial V_m}{\partial t}\}_{m=1}^{\infty}$ and a function $\frac{\partial V}{\partial t} \in L^2(0,T;M')$, such that

$$\frac{\partial V_{m_l}}{\partial t} \text{ converges weakly to } \frac{\partial V}{\partial t} \text{ in } L^2(0,T;M').$$

By definition A.0.13, weak convergence means that for any $w \in M$,

$$\left\langle \frac{\partial V_m}{\partial t}, w \right\rangle_{M',M} \longrightarrow \left\langle \frac{\partial V}{\partial t}, w \right\rangle_{M',M}.$$

Next, we fix an integer N and choose a function $q \in C^1([0,T];M)$ having the form

$$q(t) = \sum_{k=1}^{N} d^k(t) e_k, \tag{5.26}$$

where $\{e_k\}_{k=1}^N$ are given smooth functions and the function satisfies the condition $q(t=0)=0$. Choosing $m \geq N$, multiplying (5.10) by $d^k(t)$, summing for $k = 1,\dots,N$ and then integrating with respect to t, we get

$$\int_0^T -\left\langle \frac{\partial V_m}{\partial t}, q \right\rangle_{M',M} + a(t;V_m,q) - \frac{1}{\epsilon}\Big(\left(g-V_m\right)^+, q\Big)\, dt = 0, \quad \text{in } Q_T. \tag{5.27}$$

Setting $m = m_l$, we send $j \to \infty$ and obtain

$$\int_0^T -\left\langle \frac{\partial V}{\partial t}, q \right\rangle_{M',M} + a(t;V,q) - \frac{1}{\epsilon}\Big(\left(g-V\right)^+, q\Big)\, \mathrm{d}t = 0. \tag{5.28}$$

This equality holds for all functions $q \in C^1([0,T];\mathbb{R}_+)$

In order to prove $V(T) = g$, we integrate by parts (5.28) and use $q(t=0)=0$, thus

$$-\left(V(T), q(T)\right) + \int_0^T (V,q') + a(t;V,q) - \frac{1}{\epsilon}\Big(\left(g-V\right)^+, q\Big)\, \mathrm{d}t = 0. \tag{5.29}$$

for each $q \in C^1([0,T];\mathbb{R}_+)$. On the other hand, recalling (5.27), we set $m = m_l$ and integrate by parts, we arrive at

$$-\big(V_{m_j}(T), q(T)\big) + \int_0^T \big(V_{m_j}, q'\big) + a(t;V_{m_j},q) - \frac{1}{\epsilon}\Big(\big(g-V_{m_j}\big)^+, q\Big)\, \mathrm{d}t = 0.$$

Sending $j \to \infty$ and using $V_{m_l}(T) \to g$ in $L^2(\mathbb{R}_+)$, thus

$$-\left(g, q(T)\right) + \int_0^T (V,q') + a(t;V,q) - \frac{1}{\epsilon}\Big(\left(g-V\right)^+, q\Big)\, \mathrm{d}t = 0. \tag{5.30}$$

Consider (5.29) and (5.30). Since $q(T)$ is arbitrary, we conclude $V(T) = g$ in $\mathbb{R}_+$. Hence, we finished proving the existence of weak solution to the associated penalised problem (5.1) by the energy estimates and the compactness method.

The *uniqueness* of V is an immediate consequence of the monotonicity of the operator $V \to (g-V)^+$. In fact, if V^1 and V^2 are two solutions of the associated penalised problem, then V^1 and V^2 satisfy (5.6), respectively

$$-\left\langle \frac{\partial V^1}{\partial t}, u \right\rangle_{(M',M)} + a(t;V^1,u) - \frac{1}{\epsilon}\Big((g-V^1)^+, u\Big) = 0, \quad \forall u \in M.$$

and

$$-\left\langle \frac{\partial V^2}{\partial t}, u \right\rangle_{(M',M)} + a(t; V^2, u) - \frac{1}{\epsilon}\Big((g - V^2)^+, u\Big) = 0, \quad \forall u \in M.$$

In addition,

$$V^1(T) = V^2(T) = g, \quad \text{in } \mathbb{R}_+.$$

Setting $z = V^1 - V^2$, we subtract these two equations and get

$$-\left\langle \frac{\partial z}{\partial t}, u \right\rangle_{M',M} + a(t; z, u) - \frac{1}{\epsilon}\Big((g - V^1)^+ - (g - V^2)^+, u\Big) = 0, \quad \forall u \in M.$$

Choosing $u = z$, we arrive at

$$-\left\langle \frac{\partial z}{\partial t}, z \right\rangle_{M',M} + a(t; z, z) - \frac{1}{\epsilon}\Big((g - V^1)^+ - (g - V^2)^+, z\Big) = 0, \quad \forall u \in M. \tag{5.31}$$

We consider the last term of the left hand side of (5.31). Observe that If $V^1 > V^2$, then $z > 0$ and $(g - V^1)^+ < (g - V^2)^+$, we have

$$\Big((g - V^2)^+ - (g - V^1)^+\Big) z > 0.$$

On the other hand, if $V^1 \le V^2$, then $z \le 0$ and $(g - V^1)^+ \ge (g - V^2)^+$. Again,

$$\Big((g - V^2)^+ - (g - V^1)^+\Big) z \ge 0.$$

Therefore, the following inequality always holds

$$\Big((g - V^2)^+ - (g - V^1)^+\Big) z \ge 0.$$

From (5.31), the last term of the left hand side is non-negative and thus,

$$-\left\langle \frac{\partial z}{\partial t}, z \right\rangle_{M',M} + a(t; z, z) \le 0. \tag{5.32}$$

The chain rule (A.0.15) gives

$$\frac{d}{dt}\left(-\frac{1}{2}\|z(t)\|^2_{L^2(\mathbb{R}_+)}\right) + a(t; z, z) \le 0.$$

Integrating (5.32) in time from t to T and using the fact that $z(T) = 0$ together with Gårding inequality (3.11), we get

$$\frac{1}{2}\|z(t)\|^2_{L^2(\mathbb{R}_+)} + \alpha \int_t^T \|z(s)\|^2_M \, ds \le \lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)} \, \mathrm{d}s.$$

Hence,

$$\|z(t)\|^2_{L^2(\mathbb{R}_+)} \le 2\lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)} \, \mathrm{d}s.$$

Applying Gronwall's inequality (A.10) yields $z(t) = 0$ for all $t \in [0, T]$. Hence, we have obtained $V^1 = V^2$ as we desired. ■

5.4 Proof of the well-posedness result for the American option pricing problem

Since we already have obtained the existence and uniqueness of solution for the associated penalised problem (5.1), we can now prove the well-posedness result for the American option pricing problem (4.1).

Proof of Theorem 4.2.1. Regarding the existence, we first propose two supplementary estimates on V_ϵ and $\frac{\partial V_\epsilon}{\partial t}$. Continuing from (5.7), a supplementary estimate on V_ϵ is derived from the following equation

$$-\left\langle \frac{\partial V_\epsilon}{\partial t}, u \right\rangle_{M',M} + a(t; V_\epsilon, u) - \frac{1}{\epsilon}\big((g - V_\epsilon)^+, u\big) = 0, \quad \forall u \in M.$$

Putting $u = V_0 - V_\epsilon$, for $V_0 \in \mathcal{N}$ where the set $\mathcal{N}$ is given by (5.8), then

$$-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon \right\rangle_{M',M} + a(t; V_\epsilon, V_0 - V_\epsilon) - \frac{1}{\epsilon}\big((g - V_\epsilon)^+, V_0 - V_\epsilon\big) = 0.$$

Writing $V_0 - V_\epsilon = V_0 - g + g - V_\epsilon$,

$$\begin{aligned}-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon \right\rangle_{M',M} &+ a(t; V_\epsilon, V_0 - V_\epsilon)\\ &- \frac{1}{\epsilon}\big((g - V_\epsilon)^+, V_0 - g + g - V_\epsilon\big) = 0.\end{aligned}$$

Thus,

$$\begin{aligned}-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon \right\rangle_{M',M} &+ a(t; V_\epsilon, V_0 - V_\epsilon) - \frac{1}{\epsilon}\big((g - V_\epsilon)^+\big)^2\\ &- \frac{1}{\epsilon}\big((g - V_\epsilon)^+, V_0 - g\big) = 0.\end{aligned} \tag{5.33}$$

Since $V_0 \in \mathcal{N}$, $V_0 - g \geq 0$. It follows that

$$\big((g - V_\epsilon)^+, V_0 - g\big) \geq 0.$$

Thus,

$$-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon \right\rangle_{M',M} + a(t; V_\epsilon, V_0 - V_\epsilon) - \frac{1}{\epsilon}\big((g - V_\epsilon)^+\big)^2 \geq 0.$$

It is equivalent to

$$\begin{aligned}-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 \right\rangle_{M',M} &+ a(t; V_\epsilon, V_0)\\ &\geq -\left\langle \frac{\partial V_\epsilon}{\partial t}, V_\epsilon \right\rangle_{M',M} + a(t; V_\epsilon, V_\epsilon) + \frac{1}{\epsilon}\big((g - V_\epsilon)^+\big)^2.\end{aligned}$$

Applying the chain rule (A.0.15) for $-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_\epsilon \right\rangle_{M',M}$, it yields

$$\begin{aligned}-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 \right\rangle_{M',M} &+ a(t; V_\epsilon, V_0)\\ &\geq \frac{d}{dt}\left(-\frac{1}{2}\|V_\epsilon(t)\|^2_{L^2(\mathbb{R}_+)}\right) + a(t; V_\epsilon, V_\epsilon) + \frac{1}{\epsilon}\big((g - V_\epsilon)^+\big)^2.\end{aligned}$$

We continue by integrating in time from t to T and using the fact that $V_\epsilon(T) = g$, we obtain

$$\begin{aligned}\frac{1}{2}\|V_\epsilon(t)\|^2_{L^2(\mathbb{R}_+)} + \int_t^T a(s; V_\epsilon, V_\epsilon)\mathrm{d}s + \frac{1}{\epsilon}\int_t^T \left((g - V_\epsilon)^+\right)^2 \mathrm{d}s \\ \leq \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}_+)} + \int_t^T -\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 \right\rangle_{M',M} + a(s; V_\epsilon, V_0)\mathrm{d}s.\end{aligned} \tag{5.34}$$

Since $V_0 \in \mathcal{N}$, where $\mathcal{N}$ is given by (5.8), we have $\|V_0\|_{L^2(0,T;M)}$ and $\left\|\frac{\partial V_\epsilon}{\partial t}\right\|_{L^2(0,T;M')}$ are bounded by a constant C independent of ϵ. Also, Hölder inequality (A.7) for $p = q = 2$ gives

$$\begin{aligned}\int_0^T \left| -\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 \right\rangle_{M',M} \right| \mathrm{d}t &\leq \int_0^T \left\|\frac{\partial V_\epsilon}{\partial t}\right\|_{M'} \|V_0\|_M \mathrm{d}t \\ &\leq \frac{1}{2}\int_0^T \left\|\frac{\partial V_\epsilon}{\partial t}\right\|^2_{M'} \mathrm{d}t + \frac{1}{2}\int_0^T \|V_0\|^2_M \mathrm{d}t \\ &= \frac{1}{2}\left\|\frac{\partial V_\epsilon}{\partial t}\right\|^2_{L^2(0,T;M')} + \frac{1}{2}\|V_0\|^2_{L^2(0,T;M)}.\end{aligned}$$

where the second inequality is using Cauchy inequality (A.4). Consider the bilinear form $a(t; V_\epsilon, V_0)$, we have

$$a(t; V_\epsilon, V_0) \leq \mu \|V_\epsilon\|_M \|V_0\|_M$$

Since $V_\epsilon \in M$, $\|V_\epsilon\|_M \leq C$ for a constant C independent of ϵ. Then $a(t; V_\epsilon, V_0)$ is also bounded in M. Hence, from (5.34), we deduce that

$$\begin{aligned}\|V_\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} &\leq C_1, \\ \frac{1}{\sqrt{\epsilon}}\left\|(g - V_\epsilon)^+\right\|_{L^2(0,T;L^2(\mathbb{R}_+))} &\leq C_2.\end{aligned} \tag{5.35}$$

for some constants C_1, C_2 independent of ϵ.

The first inequality in (5.35) shows that V_ϵ is bounded in $L^\infty\left(0, T; L^2(\mathbb{R}_+)\right)$. Since $L^2(0,T) \subseteq L^\infty(0,T)$, V_ϵ is then bounded in $L^2\left((0,T) \times \mathbb{R}_+\right)$. By weak compactness Theorem A.0.14, there exists a subsequence $\{V_\epsilon\}_{\epsilon>0}$ (not relabelled) and a function V such that

$$V_\epsilon \text{ converges strongly to } V \text{ in } L^2\left((0,T) \times \mathbb{R}_+\right).$$

The second part of (5.35) implies that $(g - V_\epsilon)^+ \to 0$ in $L^2(Q_T)$ when $\epsilon \downarrow 0$. Since V_ϵ converges strongly to V in this space, then $(g - V)^+ = 0$ in $L^2(Q_T)$. This implies $V \geq g$ in Q_T. Hence, we have obtained the existence of weak solutions V.

A supplementary estimate for $\frac{\partial V_\epsilon}{\partial t}$ can also be derived as follows. Setting

$$w_\epsilon = \frac{\partial V_\epsilon}{\partial t}.$$

We differentiate (5.6) in time and get

$$-\left\langle \frac{\partial w_\epsilon}{\partial t}, w \right\rangle_{M',M} + a\left(t; w_\epsilon, w\right) + a'\left(t; V_\epsilon, w\right) \\ - \frac{1}{\epsilon}\left(\frac{\partial (g-w_\epsilon)^+}{\partial t}, w\right) = 0, \quad \forall w \in M, \tag{5.36}$$

where $a(t;u,v)$ and $a'(t;u,v)$ take the form

$$a(t;u,v) = \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \int_{\mathbb{R}_+} \left(-(r-d)+\sigma^2\right) x \frac{\partial u}{\partial x} v \mathrm{d}x \\ + \int_{\mathbb{R}_+} ruv\mathrm{d}x, \quad \forall u, v \in M,$$

and

$$a'(t;u,v) = \int_{\mathbb{R}_+} \frac{d}{dt}\left(\frac{1}{2}\sigma^2\right) x^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{d}{dt}\left(-(r-d)+\sigma^2\right) x \frac{\partial u}{\partial x} v \mathrm{d}x \\ + \int_{\mathbb{R}_+} \frac{dr(t)}{dt} uv\mathrm{d}x, \quad \forall u, v \in M,$$

respectively. Since $\frac{\partial g}{\partial t} = 0$, we have

$$\frac{\partial (g-V_\epsilon)^+}{\partial t} = -\frac{\partial V_\epsilon}{\partial t}\mathbb{1}_{g>V_\epsilon} = -w_\epsilon \mathbb{1}_{g>V_\epsilon} = \begin{cases} -w_\epsilon, & g > V_\epsilon, \\ 0, & g \le V_\epsilon. \end{cases} \tag{5.37}$$

Letting $w = w_\epsilon$, we substitute w and (5.37) into (5.36) to obtain

$$-\left\langle \frac{\partial w_\epsilon}{\partial t}, w_\epsilon \right\rangle_{M',M} + a\left(t; w_\epsilon, w_\epsilon\right) + a'\left(t; V_\epsilon, w_\epsilon\right) + \frac{1}{\epsilon}\left(w_\epsilon, w_\epsilon\right) = 0.$$

Using the chain rule (A.0.15), then integrating in time from t to T, we rewrite the equation

$$\int_t^T \left[\frac{d}{ds}\left(-\frac{1}{2}\|w_\epsilon(s)\|^2_{L^2(\mathbb{R}_+)}\right) + a\left(t; w_\epsilon, w_\epsilon\right) + \frac{1}{\epsilon}\|w_\epsilon\|^2_{L^2(\mathbb{R}_+)}\right]\mathrm{d}s \\ = \int_t^T -a'\left(t; V_\epsilon, w_\epsilon\right)\mathrm{d}s.$$

Employing Gårding inequality (3.11) yields

$$\frac{1}{2}\|w_\epsilon(t)\|^2_{L^2(\mathbb{R}_+)} + \alpha\int_t^T \|w_\epsilon\|^2_M \mathrm{d}s + \frac{1}{\epsilon}\int_t^T \|w_\epsilon\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}s \\ \le \frac{1}{2}\|w_\epsilon(T)\|^2_{L^2(\mathbb{R}_+)} + \int_t^T -a'\left(t; V_\epsilon, w_\epsilon\right)\mathrm{d}s. \tag{5.38}$$

Consider the right-hand side of (5.38). The bilinear form is continuous in the sense that

$$a'\left(t; V_\epsilon, w_\epsilon\right) \le \mu\|V_\epsilon\|_M\|w_\epsilon\|_M \le C,$$

for a constant C not depending on ϵ. Moreover, (5.1) gives $w_\epsilon(T) = A(T)g$, where the operator $A(T)$ is bounded in M. Therefore $w_\epsilon(T)$ is also bounded by a constant C that is independent of ϵ. Hence, it follows from (5.38) that

$$\frac{1}{2}\|w_\epsilon(t)\|^2 + \alpha\int_0^T \|w_\epsilon\|_M^2 \,\mathrm{d}s \le C$$

This implies that

$$\frac{\partial V_\epsilon}{\partial t} \in L^\infty(0,T;L^2(\mathbb{R}_+)) \cap L^2(0,T;M)$$

Therefore, we can find a subsequence of $\{\frac{\partial V_\epsilon}{\partial t}\}_{\epsilon>0}$ converging weakly to $\frac{\partial V}{\partial t}$ in $L^2(0,T;M)$ and star in $L^\infty(0,T;L^2(\mathbb{R}_+))$.

Finally, to show that V satisfies the first inequality of (4.1), returning to (5.33):

$$-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon\right\rangle_{M',M} + a(t;V_\epsilon, V_0 - V_\epsilon) - \frac{1}{\epsilon}\left((g-V_\epsilon)^+\right)^2 - \frac{1}{\epsilon}\big((g-V_\epsilon)^+, V_0 - g\big) = 0, \quad \forall V_0 \in \mathcal{N}$$

As observed before, $((g - V_\epsilon)^+, V_0 - g) \ge 0$. Hence,

$$-\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon\right\rangle_{M',M} + a(t;V_\epsilon, V_0 - V_\epsilon) \ge 0, \quad \forall V_0 \in \mathcal{N}. \tag{5.39}$$

Integrating from 0 to T and using the linearity of $a(t;u,v)$, we obtain

$$\int_0^T -\left\langle \frac{\partial V_\epsilon}{\partial t}, V_0 - V_\epsilon\right\rangle_{M',M} + a(t;V_\epsilon, V_0)\mathrm{d}t \ge \int_0^T a(t;V_\epsilon,V_\epsilon)\mathrm{d}t, \quad \forall V_0 \in \mathcal{N}. \tag{5.40}$$

Using (5.39) and (5.40), letting $\epsilon \downarrow 0$,

$$\int_0^T -\left\langle \frac{\partial V}{\partial t}, V_0 - V\right\rangle_{M',M} + a(t;V, V_0)\mathrm{d}t \ge \liminf_{\epsilon\to 0}\int_0^T a(t;V_\epsilon,V_\epsilon)\mathrm{d}t, \quad \forall V_0 \in \mathcal{N}.$$

Since $\liminf_{\epsilon\to 0}\int_0^T \left(\int_{\mathbb{R}_+} |x\frac{\partial V_\epsilon}{\partial x}|^2 \mathrm{d}x\right)\mathrm{d}t \ge \int_0^T \left(\int_{\mathbb{R}_+} |x\frac{\partial V}{\partial x}|^2 \mathrm{d}x\right)\mathrm{d}t$ [5, p. 469],

$$\int_0^T -\left\langle \frac{\partial V}{\partial t}, V_0 - V\right\rangle_{M',M} + a(t;V, V_0)\mathrm{d}t \ge \int_0^T a(t;V,V)\mathrm{d}t, \quad \forall V_0 \in \mathcal{N}.$$

In other words,

$$\int_0^T -\left\langle \frac{\partial V}{\partial t}, V_0 - V\right\rangle_{M',M} + a(t;V, V_0 - V)\mathrm{d}t \ge 0, \quad \forall V_0 \in \mathcal{N}.$$

Hence,

$$-\left\langle \frac{\partial V}{\partial t}, V_0 - V \right\rangle_{M',M} + a(t; V, V_0 - V)\mathrm{d}t \geq 0, \quad \forall V_0 \in \mathcal{N}.$$

Regarding *uniqueness*, we assume V^1 and V^2 are two weak solutions to the American option pricing problem (4.1), then V^1, V^2 solve (4.9). In particular, V^1, V^2 satisfy

$$-\left\langle \frac{\partial V^1}{\partial t}, w - V^1 \right\rangle_{M',M} + a(t; V^1, w - V^1) \geq 0, \quad \forall w \in \mathcal{N} \tag{5.41}$$

and

$$-\left\langle \frac{\partial V^2}{\partial t}, w - V^2 \right\rangle_{M',M} + a(t; V^2, w - V^2) \geq 0, \quad \forall w \in \mathcal{N}, \tag{5.42}$$

respectively. In addition,

$$V^1(T) = V^2(T) = g, \quad \text{in } \mathbb{R}_+.$$

Replacing $w = V^1 - u$ and $w = V^2 - u$, $u \in M$, in (5.41) and (5.41) respectively, we obtain that

$$\left\langle \frac{\partial V^1}{\partial t}, u \right\rangle_{M',M} - a(t; V^1, u) \geq 0$$

and

$$\left\langle \frac{\partial V^2}{\partial t}, u \right\rangle_{M',M} - a(t; V^2, u) \geq 0.$$

Setting $z = V^1 - V^2$, we subtract these above inequalities and use linearity of $a(t; u, v)$ to find that

$$\left\langle \frac{\partial z}{\partial t}, u \right\rangle_{M',M} - a(t; z, u) \geq 0, \quad \forall u \in M.$$

Multiplying inequality the above result by -1, it yields

$$-\left\langle \frac{\partial z}{\partial t}, u \right\rangle_{M',M} + a(t; z, u) \leq 0, \quad \forall u \in M.$$

Choosing $u = z$,

$$-\left\langle \frac{\partial z}{\partial t}, z \right\rangle_{M',M} + a(t; z, z) \leq 0.$$

The chain rule (A.0.15) gives

$$\frac{d}{dt}\left(-\frac{1}{2} \|z(t)\|^2_{L^2(\mathbb{R}_+)} \right) + a(t; z, z) \leq 0. \tag{5.43}$$

Using Gårding inequality (3.11), then integrating (5.43) in time from t to T and using the fact that $z(T) = 0$, we arrive at

$$\frac{1}{2} \|z(t)\|^2_{L^2(\mathbb{R}_+)} + \alpha \int_t^T \|z(t)\|^2_M \, \mathrm{d}s \leq \lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)} \, \mathrm{d}s,$$

which implies

$$\|z(t)\|^2_{L^2(\mathbb{R}_+)} \leq 2\lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)}\,\mathrm{d}s.$$

Applying Gronwall's inequality (A.10) yields $z(t) = 0$, for all $t \in [0, T]$. Hence, we have obtained $V^1 = V^2$ as desired. ■

CHAPTER 6

The power penalty problem

The *power penalty method* was proposed and analyzed by S. Wang, X. Q. Yang and K. L. Teo in [14, 16]. This approach has been considered as an improvement of the *classical penalty method* because of its accurate solutions to values of American options [14] and it overcomes computational problems due to $\epsilon \to 0$. In this chapter, we will study this method and our main focus is to present theory for well-posedness of solutions to this problem.

Recalling that the American option pricing problem takes the following form

$$\begin{cases} -\dfrac{\partial V}{\partial t} + A(t)V \geq 0, & \text{in } Q_T, \\ V \geq g, & \text{in } Q_T, \\ \left(-\dfrac{\partial V}{\partial t} + A(t)V\right)(g - V) = 0, & \text{in } Q_T. \end{cases} \tag{6.1}$$

where

$$V(T) = g, \quad \text{in } \mathbb{R}_+,$$

and the operator $A(t)$ takes the form

$$A(t)v = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} - (r - d)x\frac{\partial v}{\partial x} + rv, \quad \forall v \in M. \tag{6.2}$$

The main idea of power penalty method is to replace the nonlinear equation in classical penalty problem by the "more nonlinear" equation. Mathematically, the associated k-power penalty problem of (6.1) is to solve a nonlinear partial differential equation of the form

$$-\frac{\partial V_k}{\partial t} + A(t)V_k - \frac{1}{\epsilon}\left[(g - V_k)^+\right]^{\frac{1}{k}} = 0, \quad \text{in } Q_T. \tag{6.3}$$

where $k > 1$ is the penalty parameter. We denote by V_k a solver of the k-power penalty problem (6.3) and $V_k(T) = g$ in $\mathbb{R}_+$. The operator $A(t)$ takes the form (6.2) and $k > 0$ is an additional parameter (typically between 0 and 1).

Remark 6.0.1.

- When $k = 1$, (6.3) reduces to the classical penalty problem.
- Since $V_k \geq g$ in Q_T, $(g - V_k)^+ = 0$. In this case, the k-power penalty problem (6.3) turns into the Black-Scholes equation which is simpler to

solve. Otherwise, if $V_k < g$, (6.3) gives $(g - V_k)^+ = \epsilon^k \left(-\frac{\partial V_k}{\partial t} + A(t)V_k\right)^k$. We see that since $\epsilon \to 0$ and if $\left(-\frac{\partial V_k}{\partial t} + A(t)V_k\right)$ is bounded, then $(g - V_k)^+ \approx 0$. Therefore, when ϵ is sufficiently small, the nonlinear term $\frac{1}{\epsilon}\left[(g - V_k)^+\right]^{\frac{1}{k}}$ is used to penalize the positive part of $(g - V_k)$.

We introduce the weak formulation of (6.3) as follows

$$-\left\langle \frac{\partial V_k}{\partial t}, w \right\rangle_{M',M} + a(t; V_k, w) - \frac{1}{\epsilon}\left(\left[(g - V_k)^+\right]^{\frac{1}{k}}, w\right) = 0, \quad w \in M, \tag{6.4}$$

where the bilinear form $a(t; u, v)$ takes the form

$$\begin{aligned} a(t; u, v) = \int_{\mathbb{R}_+} \left(\frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(-(r-d) + \sigma^2\right) x \frac{\partial u}{\partial x} v\right) \mathrm{d}x \\ + \int_{\mathbb{R}_+} ruv \mathrm{d}x, \quad \text{for } u, v \in M. \end{aligned} \tag{6.5}$$

We define weak solutions of (6.3) as follows:

Definition 6.0.2 (Weak solution)**.** We call V_k a weak solution to the k-power penalty problem (6.3) if

$$V_k \in L^2(0, T; M), \quad \frac{\partial V_k}{\partial t} \in L^2(0, T; M').$$

Moreover, V_k satisfies the weak formulation (6.4) and $V_k(T) = g$.

6.1 Well-posedness

For the existence and uniqueness of weak solutions to the k-power penalty problem (6.3), we have the following theorem

Theorem 6.1.1 (Well-posedness)**.** *There exists a unique weak solution to the k-power penalty problem (6.3).*

Before proving Theorem 6.1.1, we propose some uniform estimates in the following Theorem:

Theorem 6.1.2. *Let V_k denotes solution to the k-power penalty problem (6.3). Then there exists some positive constants C and C_1, independent of k such that*

$$\begin{aligned} \max_{0 \le t \le T} \|V_k(t, .)\|^2_{L^2(\mathbb{R}_+)} + \left\|V_k(t)\right\|^2_{L^2(0,T;M)} \\ + \left\|\frac{\partial V_k}{\partial t}\right\|^2_{L^2(0,T;M')} \le C_1\big(\exp(CT) + T\big)\|g\|^2_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{6.6}$$

Proof. The proof is analogous with the one in the classical penalised problem. Substituting $w = V_k$ in (6.4), we obtain the following equation

$$-\left\langle \frac{\partial V_k}{\partial t}, V_k \right\rangle_{M',M} + a(t; V_k, V_k) - \frac{1}{\epsilon}\left(\left[(g - V_k)^+\right]^{1/k}, V_k\right) = 0. \tag{6.7}$$

Observe that since $V_k \le g$ in Q_T,

$$\left(\left[(g - V_k)^+ \right]^{1/k}, V_k \right) \le C \left\| V_k \right\|^2_{L^2(\mathbb{R}_+)}$$

Using the chain rule (A.0.15) and Gårding inequality (3.11), from (5.13) we deduce the following inequality

$$\frac{d}{dt}\left(-\frac{1}{2} \left\| V_k(t) \right\|^2_{L^2(\mathbb{R}_+)} \right) + \alpha \left\| V_k \right\|^2_M \le C \left\| V_k \right\|^2_{L^2(\mathbb{R}_+)}, \tag{6.8}$$

for some constants $\alpha > 0$ and C not depending on k. It follows that

$$\frac{d}{dt}\left(-\frac{1}{2} \left\| V_k(t) \right\|^2_{L^2(\mathbb{R}_+)} \right) \le C \left\| V_k \right\|^2_{L^2(\mathbb{R}_+)}, \tag{6.9}$$

almost every time in $[0, T]$ and for some positive constant C, independent of k. Setting $\eta(t) := \left\| V_m(t) \right\|^2_{L^2(\mathbb{R}_+)}$, then we apply Gronwall's inequality (A.10), it yields the following result

$$\eta(t) \le e^{Ct}\eta(T), \quad t \in [0, T].$$

Taking maximum over $t \in [0, T]$ and substituting $\eta(t)$ by $\left\| V_k(t) \right\|^2_{L^2(\mathbb{R}_+)}$, we find that

$$\max_{0 \le t \le T} \left\| V_k(t, .) \right\|^2_{L^2(\mathbb{R}_+)} \le \exp(CT) \|g\|^2_{L^2(\mathbb{R}_+)}. \tag{6.10}$$

for some positive constant C not depending on k.

Returning to (6.8), we integrate in time from 0 to T and employ (6.10), then we obtain that

$$\begin{aligned} \left\| V_k(t) \right\|^2_{L^2(0,T;M)} &= \int_0^T \left\| V_k(t) \right\|^2_M \,\mathrm{d}t \\ &\le \exp(CT) \|g\|^2_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{6.11}$$

We next give an estimate for the time derivative of V_k. Since (6.3) gives

$$\frac{\partial V_k}{\partial t} = -\frac{1}{2}\sigma^2 x^2 \frac{\partial V_k}{\partial x} - (r - d)x\frac{\partial V_k}{\partial x} + rV_k - \frac{1}{\epsilon}\big(\left[(g - V_k)^+ \right] \big)^{1/k}, \quad \text{in } Q_T,$$

the formulation suggests that $\frac{\partial V_k}{\partial t} \in M'$, where M' is the dual space of M. Recalling the norm in M', for any $w \in M$ such that $\|w\|_M \le 1$, we have

$$\left\| \partial_t V_k \right\|_{M'} = \sup_{\|w\|_M \le 1} \frac{\left| \langle \partial_t V_k, w \rangle_{M',M} \right|}{\|w\|_M}.$$

From (6.7), we have

$$\langle \partial_t V_k, w \rangle_{M',M} = a(t; V_k, w) - \frac{1}{\epsilon}\Big(\big(\left[(g - V_k)^+ \right] \big)^{1/k}, w \Big).$$

Observe that, using Cauchy-Schwartz inequality (A.1) and the triangle inequality, we find that

$$\begin{aligned} \Big(\big(\left[(g - V_k)^+ \right] \big)^{1/k}, w \Big) &\le C \Big\| \left[(g - V_k)^+ \right] \big)^{1/k} \Big\|_{L^2(\mathbb{R}_+)} \|w\|_{L^2(\mathbb{R}_+)} \\ &\le C\big(\|g\|_{L^2(\mathbb{R}_+)} + \|V_k\|_{L^2(\mathbb{R}_+)} \big) \|w\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Using the above result and the continuity of the bilinear form $a(t;.,.)$, for a fixed function $w \in M$, such that $\|w\|_M \leq 1$, we deduce that

$$\left|\langle \partial_t V_k, w\rangle_{M',M}\right| \leq C\left(\|V_k\|_M + \|g\|_{L^2(\mathbb{R}_+)}\right)\|w\|_{L^2(\mathbb{R}_+)},$$

for some constant C not depending on k. For details in the above estimate, we refer to (5.20) in Chapter 5. using $\|w\|_M \leq 1$, we utilize the above inequality and find that

$$\|\partial_t V_k\|_{M'} \leq C\left(\|V_k\|_M + \|g\|_{L^2(\mathbb{R}_+)}\right).$$

Integrating in time from 0 to T, employing $(a+b)^2 \leq 2(a^2+b^2)$ and (6.10), we arrive at

$$\begin{aligned}\int_0^T \left\|\partial_t V_m(t)\right\|^2_{M'} \mathrm{d}t &\leq C\int_0^T \left(\|V_m(t)\|_M + \|g\|_{L^2(\mathbb{R}_+)}\right)^2 \mathrm{d}t\\ &\leq C_1 \int_0^T \|V_m(t)\|^2_M + \|g\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}t\\ &\leq C_1\left(\exp(CT)+T\right)\|g\|_{L^2(\mathbb{R}_+)},\end{aligned}$$

for some constants C and C_1 not depending on k. We now collect the above estimate together with (6.10) and (6.11), thus we have proved (6.6) in the Theorem. ■

Having obtained a prior estimates for V_k in (6.6), we next define the following set W_k such that

$$W_{a*} = \left\{V_k \in L^2(0,T;M), \quad \frac{\partial V_k}{\partial t} \in L^2(0,T;M')\right\}.$$

Apply Lemma 5.3.4 (cf. Chapter 5), it shows that W_{a*} is compactly embedded in $L^2\big((0,T)\times(a,\infty)\big)$, for $a>0$. It then follows from compactness Theorem A.0.14 that there exists a subsequence $\{V_{k_j}\}_{j>0} \subseteq \{V_k\}$ and a function V such that

$$V_{k_j} \text{ converges strongly to } V \text{ in } L^2\big((0,T)\times(a,\infty)\big),$$

for any $a>0$. Thus, by a diagonal argument, we then conclude that

$$V_{k_j} \text{ converges strongly to } V \text{ in } L^2\big((0,T)\times(0,\infty)\big).$$

Since we have obtained strong convergence of sequences $\{V_k\}_{k>0}$ in $L^2(Q_T)$ as well as (6.6), we are now ready to prove the well-posedness theory for (6.3).

Proof of Theorem 6.1.1. For the existence, according to energy estimates (6.6), the sequences $\{V_k\}_{k>0}$ and $\{\frac{\partial V_k}{\partial t}\}_{k>0}$ are bounded in $L^2(0,T;M)$ and $L^2(0,T;M')$, respectively. Weak compactness Theorem A.0.14 shows that there exists a subsequence $\{V_{k_j}\}_{j=1}^\infty \subseteq \{V_k\}_{k>0}$ and a function $V \in L^2(0,T;M)$, such that

$$V_{k_j} \text{ converges weakly to } V \text{ in } L^2(0,T;M).$$

Similarly, there exists a subsequence $\{\frac{\partial V_{k_j}}{\partial t}\}_{j=1}^\infty \subseteq \{\frac{\partial V_k}{\partial t}\}_{k>0}$ and a function $\frac{\partial V}{\partial t} \in L^2(0,T;M')$, such that

$$\frac{\partial V_{k_j}}{\partial t} \text{ converges weakly to } \frac{\partial V}{\partial t} \text{ in } L^2(0,T;M').$$

By definition A.0.13, weak convergence means that for any $w \in M$,

$$\left\langle \frac{\partial V_k}{\partial t}, w \right\rangle_{M',M} \longrightarrow \left\langle \frac{\partial V}{\partial t}, w \right\rangle_{M',M}.$$

For any fixed function $q \in C^1([0,T];M)$ such that $q(0)=0$, the following integral holds

$$\int_0^T -\left\langle \frac{\partial V_k}{\partial t}, q \right\rangle_{M',M} + a(t;V_k,q) - \frac{1}{\epsilon}\left((g-V_k)^+, q \right) dt = 0, \quad \text{in } Q_T. \quad (6.12)$$

Setting $k=k_j$ and sending $j \to \infty$, we obtain that

$$\int_0^T -\left\langle \frac{\partial V}{\partial t}, q \right\rangle_{M',M} + a(t;V,q) - \frac{1}{\epsilon}\left((g-V)^+, q \right) \mathrm{d}t = 0. \quad (6.13)$$

In order to prove $V(T)=g$, we integrate by parts (6.13) and use $q(0)=0$, thus

$$-\left(V(T), q(T)\right) + \int_0^T (V,q') + a(t;V,q) - \frac{1}{\epsilon}\left((g-V)^+, q \right) \mathrm{d}t = 0. \quad (6.14)$$

for all $q \in C^1([0,T];M)$. On the other hand, doing integration by parts (6.12) and setting $k=k_j$, we arrive at

$$-\big(V_{k_j}(T), q(T)\big) + \int_0^T \big(V_{k_j}, q'\big) + a(t;V_{k_j},q) - \frac{1}{\epsilon}\left(\left(g-V_{k_j}\right)^+, q \right) \mathrm{d}t = 0.$$

Sending $j \to \infty$ and using the fact that $V_{k_j}(T) \to g$ in $L^2(\mathbb{R}_+)$, thus

$$-\big(g, q(T)\big) + \int_0^T (V,q') + a(t;V,q) - \frac{1}{\epsilon}\left((g-V)^+, q \right) \mathrm{d}t = 0. \quad (6.15)$$

Consider (6.14) and (6.15). Since $q(T)$ is arbitrary, we conclude $V(T)=g$ in $\mathbb{R}_+$. Hence, we finished proving the existence of a weak solution for (6.3) by using energy estimates and the compactness method.

The *uniqueness* of V_k is an immediate consequence of the monotonicity of the operator $V_k \to (g-V_k)^+$. For our convenience, we will drop dependence of V_k on k in what follows. Assume that V^1 and V^2 are two solutions to (6.3), then V^1 and V^2 satisfy the weak formulation (6.4) such that

$$-\left\langle \frac{\partial V^1}{\partial t}, u \right\rangle_{(M',M)} + a(t;V^1,u) - \frac{1}{\epsilon}\left(\left[(g-V^1)^+\right]^{1/k}, u \right) = 0, \quad \forall u \in M,$$

and

$$-\left\langle \frac{\partial V^2}{\partial t}, u \right\rangle_{(M',M)} + a(t;V^2,u) - \frac{1}{\epsilon}\left(\left[(g-V^2)^+\right]^{1/k}, u \right) = 0, \quad \forall u \in M.$$

Moreover,

$$V^1(T) = V^2(T) = g, \quad \text{in } \mathbb{R}_+.$$

Setting $z = V^1 - V^2$, we subtract the second equation from the first one and get the following equation, for all $u \in M$,

$$-\left\langle \frac{\partial z}{\partial t}, u \right\rangle_{M',M} + a(t; z, u) - \frac{1}{\epsilon}\left(\left[(g - V^1)^+\right]^{1/k} - \left[(g - V^2)^+\right]^{1/k}, u \right) = 0.$$

Choosing $u = z$, it yields

$$-\left\langle \frac{\partial z}{\partial t}, z \right\rangle_{M',M} + a(t; z, z) - \underbrace{\frac{1}{\epsilon}\left(\left[(g - V^1)^+\right]^{1/k} - \left[(g - V^2)^+\right]^{1/k}, z \right)}_{=E \leq 0} = 0. \tag{6.16}$$

Setting

$$E = \frac{1}{\epsilon}\left(\left[(g - V^1)^+\right]^{1/k} - \left[(g - V^2)^+\right]^{1/k}, z \right).$$

For all $\epsilon > 0$ and $k > 0$, observe that $E \leq 0$. Indeed, if $V^1 > V^2$, then

$$z > 0, \quad \left[(g - V^1)^+\right]^{1/k} < \left[(g - V^2)^+\right]^{1/k},$$

it follows that

$$E < 0.$$

Otherwise,

$$z \leq 0, \quad \left[(g - V^1)^+\right]^{1/k} \geq \left[(g - V^2)^+\right]^{1/k},$$

then again, we have

$$E \leq 0.$$

(6.16) implies that

$$-\left\langle \frac{\partial z}{\partial t}, z \right\rangle_{M',M} + a(t; z, z) \leq 0. \tag{6.17}$$

Applying the chain rule (A.0.15), it gives

$$\frac{d}{dt}\left(-\frac{1}{2}\|z(t)\|^2_{L^2(\mathbb{R}_+)} \right) + a(t; z, z) \leq 0.$$

Integrating in time from t to T and using the fact that $z(T) = 0$ together with Gårding inequality (3.11), we get

$$\frac{1}{2}\|z(t)\|^2_{L^2(\mathbb{R}_+)} + \alpha \int_t^T \|z(s)\|^2_M \, ds \leq \lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)} \, \mathrm{d}s.$$

Hence,

$$\|z(t)\|^2_{L^2(\mathbb{R}_+)} \leq 2\lambda \int_t^T \|z(s)\|^2_{L^2(\mathbb{R}_+)} \, \mathrm{d}s.$$

Gronwall's inequality (A.10) yields $z(t) = 0$ for all $t \in [0, T]$. Hence, we have obtained $V^1 = V^2$ as we desired. ■

6.2 The standard formulation

In what follows, we follow [14] to reformulate the original problem (6.1) into an equivalent standard form which satisfies homogeneous Dirichlet boundary conditions. Assume that $\sigma(t)$ and $r(t)$ satisfy

$$\underline{\sigma} \leq \sigma(t) \leq \overline{\sigma}, \quad \underline{r} \leq r(t) \leq \overline{r},$$

for some positive constants $\underline{\sigma}$, $\overline{\sigma}$,$\underline{r}$ and $\overline{r}$.

For $L >> K$, assume that we solve (6.1) in an finite interval $[0, L]$ with boundary conditions given by

$$V(t,0) = b(t), \quad V(t,L) = 0, \quad t \in [0,T].$$

When $x \to 0$, (6.1) reads

$$\begin{cases} -\dfrac{\partial b(t)}{\partial t} + rb(t) \geq 0, \\ b(t) - g(0) \geq 0, \\ \Big(-\dfrac{\partial b(t)}{\partial t} + rb(t)\Big)\Big(b(t) - g(0)\Big) = 0. \end{cases}$$

See that $b(t) = g(0) = V(T,0) = K$ satisfies the last equation and also satisfies the above system. Thus, the boundary condition are defined as follows

$$V(t,0) = K, \quad V(t,L) = 0 \quad \text{for } t \in [0,T]. \tag{6.18}$$

The second boundary should be read as

$$\lim_{x\to\infty} V(t,x) = 0$$

We start the transformation by letting V_0 be the linear function satisfying the boundary conditions (6.18). V_0 is defined by

$$V_0(x) = \Big(1 - \frac{x}{L}\Big) K. \tag{6.19}$$

Introduce a new variable

$$U(t,x) = e^{\beta t}\big[V(t,x) - V_0(x)\big], \quad \text{where } \beta = \sup_{0<t<T} \sigma^2(t). \tag{6.20}$$

Then,

$$V(t,x) = e^{-\beta t}U(t,x) + V_0(x). \tag{6.21}$$

To simplify the presentation, we will drop the dependence of functions on (t,x) in what follows. Substituting (6.21) into the first inequality of (6.1), it yields

$$e^{-\beta t}\Big[-\frac{\partial U}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 U}{\partial x^2} - (r-d)x\frac{\partial U}{\partial x} + (r+\beta)U\Big]$$
$$- \frac{\partial V_0}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_0}{\partial x^2} - (r-d)x\frac{\partial V_0}{\partial x} + rV_0 \geq 0.$$

We rewrite the above inequality as follows

$$-\frac{\partial U}{\partial t}-\frac{1}{2}\sigma^2x^2\frac{\partial^2 U}{\partial x^2}-(r-d)x\frac{\partial U}{\partial x}+(r+\beta)U\geq -f(t,x), \tag{6.22}$$

where

$$f(t,x)=e^{\beta t}\left[-\frac{\partial V_0}{\partial t}-\frac{1}{2}\sigma^2x^2\frac{\partial^2 V_0}{\partial x^2}-(r-d)x\frac{\partial V_0}{\partial x}+rV_0\right].$$

(6.19) gives

$$\frac{\partial V_0}{\partial t}=0,\quad \frac{\partial V_0}{\partial x}=-\frac{K}{L},\quad \frac{\partial^2 V_0}{\partial x^2}=0.$$

Thus we find that

$$\begin{aligned} f(t,x)&=e^{\beta t}\left[(r-d)x\frac{K}{L}+r\left(1-\frac{x}{L}\right)K\right]\\ &=e^{\beta t}\left(r-d\frac{x}{L}\right)K. \end{aligned} \tag{6.23}$$

We would like to show that (6.22) takes the following form

$$-\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial U}{\partial x}+b(t)xU\right]+c(t)U\geq -f(t,x).$$

From (6.22), these functions $a(t), b(t), c(t)$ must satisfy the following set

$$\begin{cases} a(t)=\frac{1}{2}\sigma^2,\\ 2a(t)+b(t)=(r-d),\\ -b(t)+c(t)=r+\beta. \end{cases} \tag{6.24}$$

Solving (6.24), it yields

$$\begin{cases} a(t)=\frac{1}{2}\sigma^2,\\ b(t)=(r-d)-\sigma^2,\\ c(t)=r+\beta+b(t)=2r+\beta-d-\sigma^2. \end{cases} \tag{6.25}$$

(6.1) is now equivalent to the following system where we seek functions U that satisfy

$$\begin{cases} -\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial U}{\partial x}+b(t)xU\right]+c(t)U\geq -f(t,x),\\ U-U^*\geq 0,\\ \left(-\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial U}{\partial x}+b(t)xU\right]+c(t)U\right)(U-U^*)=0, \end{cases} \tag{6.26}$$

where $f(t,x)$, $a(t)$, $b(t)$ and $c(t)$ are given by (6.23) and (6.25), respectively. Moreover,

$$U^*=U(T,x)=e^{\beta t}\left(g-V_0\right)$$

and the boundary conditions (6.18) becomes

$$U(t,0)=U(t,L)=0,\quad t\in[0,T),$$

where we have used (6.21) to obtain the above boundary conditions. Finally, letting $u = -U$ and multiplying the resulting system by -1, we obtain the following problem

$$\begin{cases} -\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left[a(t)x^2\dfrac{\partial u}{\partial x} + b(t)xu\right] + c(t)u \leq f(t,x), \\ u - u^* \leq 0, \\ \left(-\dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left[a(t)x^2\dfrac{\partial u}{\partial x} + b(t)xu\right] + c(t)U\right)(u - u^*) = 0, \end{cases} \tag{6.27}$$

where $u^* = -U^* = e^{\beta t}(V_0 - g)$. We define u^* as follows. Using (6.19) and $g = (K - x)^+$, we consider two cases:
If $K > x$,

$$\begin{aligned} u^* = e^{\beta t}(V_0 - g) =& e^{\beta t}\left[\left(1 - \frac{x}{L}\right)K - (K - x)\right] \\ =& e^{\beta t}\left(1 - \frac{K}{L}\right)x. \end{aligned}$$

Otherwise, if $x \in [K, L]$ then $g = 0$,

$$u^* = e^{\beta t}(V_0 - g) = e^{\beta t}\left(1 - \frac{x}{L}\right)K.$$

Hence,

$$u^* = \begin{cases} e^{\beta t}\left(1 - K/L\right)x, & x \in [0, K), \\ e^{\beta t}\left(1 - x/L\right)K, & x \in [K, L]. \end{cases} \tag{6.28}$$

We summarize the above analysis by the following lemma

Lemma 6.2.1. *The problem of pricing American options (6.1) is equivalent to the new problem (6.27) in the sense that their solutions are related by*

$$u = -e^{\beta t}\left(V - V_0\right),$$

where u and V denote solutions corresponding to the systems (6.27) and (6.1), respectively. Moreover, V_0 and the associated pay-off function u^* of (6.27) are defined by (6.19) and (6.28), respectively.

We now follows [2, Chapter 3.2] to introduce the variational inequality of (6.27). We first denote by M_1 the weighted Sobolev space as follows:

$$M_1 = \left\{w \in L^2\big([0, L]\big), \quad x\frac{w}{x} \in L^2\big([0, L]\big)\right\}.$$

Its dual space is denoted by M_1'. We next define a convex and closed set $\mathcal{K}_1$ such that

$$\mathcal{K}_1 = \big\{w \in M_1 : w \leq u^*\big\},$$

Following [2, p. 236], the variational inequality of (6.27) is to seek a function u that satisfies

$$-\left\langle \frac{\partial u}{\partial t}, w - u\right\rangle_{M_1', M_1} + b(t; u, w - u) \leq (f, w - u), \quad w \in M_1, \tag{6.29}$$

where we have introduced the bilinear form $b(t; u, w)$ such that

$$\begin{aligned} b(t; u, w) &= \left(a(t)x^2 \frac{\partial u}{\partial x} + b(t)xu, \frac{\partial w}{\partial x} \right) + \left(c(t)u, w \right) \\ &= \int_0^L \left(a(t)x^2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + b(t)xu \frac{\partial w}{\partial x} + c(t)uw \right) \mathrm{d}x, \end{aligned} \tag{6.30}$$

where the functions $a(t)$, $b(t)$ and $c(t)$ are given in (6.25). We state some key properties of the bilinear form $b(t; , .,)$ given by (6.30) in the following lemma:

Lemma 6.2.2. *Assume that $\sigma(t)$ and $r(t)$ satisfy*

$$\underline{\sigma} \leq \sigma(t) \leq \overline{\sigma}, \quad \underline{r} \leq r(t) \leq \overline{r},$$

for some positive constants $\underline{\sigma}$, $\overline{\sigma}$,$\underline{r}$ and $\overline{r}$. Then there exists some positive constants α and C such that, for any $v, w \in M_1$,

$$b(t; v, v) \geq \alpha \|v\|^2_{M_1}, \quad b(t; v, w) \leq C \|v\|_{M_1} \|w\|_{M_1}.$$

Proof. Given (6.30), we do integration by parts and find that, for all $u \in M_1$,

$$\begin{aligned} b(t; u, u) &= \int_0^L a(t)x^2 \left(\frac{\partial u}{\partial x} \right)^2 + \underbrace{b(t)x \frac{\partial u}{\partial x} u}_{IBP} + c(t)u^2 \mathrm{d}x \\ &= \int_0^L a(t)x^2 \left(\frac{\partial u}{\partial x} \right)^2 - 1/2 b(t)u^2 + c(t)u^2 \mathrm{d}x \end{aligned}$$

Inserting $a(t)$, $b(t)$ and $c(t)$ in (6.25), we obtain that

$$\begin{aligned} b(t; u, u) &= \frac{1}{2} \int_0^L \sigma^2 \left(x \frac{\partial u}{\partial x} \right)^2 \mathrm{d}x + \frac{1}{2} \int_0^L (3r + 2\beta - d - \sigma^2) u^2 \mathrm{d}x \\ &\geq \frac{1}{2} \min\{\sigma^2, 3r - d\} \left(\left\| x \frac{\partial u}{\partial x} \right\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right) \\ &\geq C \|u\|^2_{M_1}, \end{aligned} \tag{6.31}$$

where we have used the assumption $\beta = \sup_{0<t<T} \sigma^2(t)$ in (6.20) and C is a constant. Hence, the first inequality in Lemma has been proved.
The second inequality can be obtained easily since $b(t; ., .)$ has the same form of the blinear form $a(t; ., .)$ given by (4.6) linked to the Black and Scholes equation. Thus we skip the proof here. We just only comment that these inequalities establish the Gårding inequality and the continuity of $b(t; ., .)$. Moreover, these inequalities ensure well-posedness of solution to the problem, which we will use for our analysis later. ∎

The associated k-power penalty problem corresponding to (6.27) is now to seek a function u_k that satisfies the following equation

$$-\frac{\partial u_k}{\partial t} + B(t)u + \frac{1}{\epsilon} [(u_k - u^*)^+]^{1/k} = f, \quad u_k \in M_1, \tag{6.32}$$

where the operator $B(t)u$ is given by

$$B(t)u = -\frac{\partial}{\partial x} \left[a(t)x^2 \frac{\partial u}{\partial x} + b(t)xu \right] + c(t)u, \quad u \in M_1$$

$a(t)$, $b(t)$, $c(t)$ and u^* are defined by (6.25) and (6.28), respectively. Moreover, $k > 1$ is the penalty parameter as before .

We introduce the weak formulation of (6.32) as follows, for all $w \in M_1$

$$-\left\langle \frac{\partial u}{\partial t}, w \right\rangle_{M_1', M_1} + b(t; u, w) + \frac{1}{\epsilon}\Big(\big[(u - u^*)^+\big]^{1/k}, w\Big) = (f, w). \qquad (6.33)$$

6.3 Convergence analysis

Our analysis is relied on [14]. Here we would like to show that solutions u_k of the k-power penalty problem (6.33) converge to u of the standard problem (6.29) as $\epsilon \downarrow 0$ with the order $\mathcal{O}^{k/2}$. In particular, we first investigate an estimate for $(u_k - u^*)^+$ and then use the results to derive a bound for $u - u_k$. We start with the following lemma

Lemma 6.3.1. *Let u_k be the solution of the k-power penalty problem (6.33). Assume that*

$$u_k \in L^p(\Omega).$$

Then there exist a constant C that does not depend on k and ϵ such that

$$\begin{aligned}
&\|(u_k - u^*)^+\|_{L^p(\Omega)} \le C\epsilon^k,\\
&\|(u_k - u^*)^+\|_{L^\infty\left(0,T;L^2(\Omega)\right)} + \|(u_k - u^*)^+\|_{L^2(0,T;M)} \le C\epsilon^{k/2},
\end{aligned}$$

where $k > 0$ is the penalty parameter.

Proof. To simplify notations, setting $\phi = (u_k - u^*)^+$, $\phi \in M_1$ and $\partial_t u \in M_1'$. We substitute w by ϕ in the weak formulation (6.33) and get

$$-\left\langle \frac{\partial u_k}{\partial t}, \phi \right\rangle_{M_1', M_1} + b(t; u_k, \phi) + \frac{1}{\epsilon}\left(\phi^{\frac{1}{k}}, \phi\right) = (f, \phi),$$

Our purpose is to estimate ϕ, thus we express the above equation in term of ϕ. Adding to both sides of the above equation the amount

$$\left(\frac{\partial u^*}{\partial t}, \phi\right) - b(t; u^*, \phi)$$

and using the linearity of the bilinear form $b(t; ., .)$, we obtain

$$\begin{aligned}
-\left\langle \frac{\partial (u_k - u^*)}{\partial t}, \phi \right\rangle_{M_1', M_1} &+ b(t; u_k - u^*, \phi) + \frac{1}{\epsilon}\left(\phi^{\frac{1}{k}}, \phi\right)\\
&= (f, \phi) + \left(\frac{\partial u^*}{\partial t}, \phi\right) - b(t; u^*, \phi). \qquad (6.34)
\end{aligned}$$

We rewrite the above equation and get that

$$\begin{aligned}
-\left\langle \frac{\partial \phi}{\partial t}, \phi \right\rangle_{M_1', M_1} &+ b(t; \phi, \phi) + \frac{1}{\epsilon}\left(\phi^{\frac{1}{k}}, \phi\right)\\
&= (f, \phi) + \left(\frac{\partial u^*}{\partial t}, \phi\right) - b(t; u^*, \phi).
\end{aligned}$$

Using the chain rule (A.0.15) gives

$$\frac{d}{dt}\left(-\frac{1}{2}\big(\phi(t),\phi(t)\big)\right)+b(t;\phi,\phi)+\frac{1}{\epsilon}\left(\phi^{\frac{1}{k}},\phi\right)$$
$$=(f,\phi)+\left(\frac{\partial u^*}{\partial t},\phi\right)-b(t;u^*,\phi).$$

Integrating in time from t to T and using the fact that $\phi(T)=0$, together with inequality (6.31), we obtain

$$\frac{1}{2}\big(\phi(t),\phi(t)\big)+\beta\int_t^T\|\phi\|_{M_1}^2\,\mathrm{d}t+\frac{1}{\epsilon}\int_t^T\left(\phi^{\frac{1}{k}},\phi\right)\mathrm{d}t$$
$$\leq\int_t^T(f,\phi)\mathrm{d}t+\int_t^T\left(\frac{\partial u^*}{\partial t},\phi\right)\mathrm{d}t-\int_t^T b(t;u^*,\phi)\mathrm{d}t. \tag{6.35}$$

Let us consider boundedness on the right-hand side of (6.35). Since $\phi\in L^p(\Omega)$ and f given by (6.23) is smooth, we find that

$$\int_0^T(f,\phi)\mathrm{d}t\leq C\int_0^T\int_0^L\phi\mathrm{d}x\mathrm{d}t\leq C\bigg(\int_0^T\|\phi\|_{L^p(\Omega)}^p\mathrm{d}t\bigg)^{\frac{1}{p}}. \tag{6.36}$$

Furthermore, u^* defined by (6.28) gives $\frac{\partial u^*}{\partial t}=\beta\exp(\beta t)(V_0-g)$. Thus,

$$\begin{aligned}\int_0^T\left(\frac{\partial u^*}{\partial t},\phi\right)\mathrm{d}t&=\beta\int_0^T\exp(\beta t)\,(g-V_0,\phi)\,\mathrm{d}t\\&\leq C\int_0^T\int_0^L\phi\mathrm{d}x\mathrm{d}t\\&\leq C\bigg(\int_0^T\|\phi\|_{L^p(\Omega)}^p\mathrm{d}t\bigg)^{1/p},\end{aligned} \tag{6.37}$$

where we have used $|g-V_0|$ is uniformly bounded by a positive constant.
We consider boundedness on the last integral of the bilinear form $b(t;u^*,\phi)$. See that

$$b(t;u^*,\phi)=\int_0^L a(t)x^2\frac{\partial u^*}{\partial x}\frac{\partial\phi}{\partial x}+b(t)x\frac{\partial u^*}{\partial x}\phi+c(t)u^*\phi\mathrm{d}x.$$

Consider first the first integral on the . It follows from (6.28) that

$$\begin{aligned}\frac{\partial u^*}{\partial x}&=\begin{cases}\exp(\beta t)\,(1-K/L)\,, & x\in[0,K],\\-\exp(\beta t)K/L, & x\in(K,L],\end{cases}\\&=\begin{cases}\exp(\beta t)C_1, & x\in[0,K],\\\exp(\beta t)C_2, & x\in[K,L],\end{cases}\end{aligned}$$

where C_1,C_2 are some constants such that

$$\begin{cases}C_1=1-K/L>0, & x\in[0,K],\\C_2=-K/L<0, & x\in(K,L].\end{cases} \tag{6.38}$$

Splitting $[0, L]$ into two intervals $[0, K]$ and $[K, L]$, we have

$$\begin{aligned} -\int_0^L a(t)x^2 \frac{\partial u^*}{\partial x}\frac{\partial \phi}{\partial x}\mathrm{d}x = -C_1\exp(\beta t)a(t)\underbrace{\int_0^K x^2\frac{\partial \phi}{\partial x}\mathrm{d}x}_{IBP} \\ -C_2\exp(\beta t)a(t)\underbrace{\int_K^L x^2\frac{\partial \phi}{\partial x}\mathrm{d}x}_{IBP}. \end{aligned} \tag{6.39}$$

Doing integration by parts, it yields

$$\int_0^K x^2\frac{\partial \phi}{\partial x}\mathrm{d}x = K^2\phi(t,K) - 2\int_0^K x\phi \mathrm{d}x.$$

Moreover, using the fact that $\phi(t, L) = \big(u_k(t, L) - u^*\big)^+ = 0$, we find that

$$\int_K^L x^2\frac{\partial \phi}{\partial x}\mathrm{d}x = -K^2\phi(t,K) - 2\int_K^L x\phi(t,x) \mathrm{d}x.$$

Inserting these two results into (6.39), we obtain

$$\begin{aligned} &-\int_0^L a(t)x^2 \frac{\partial u^*}{\partial x}\frac{\partial \phi}{\partial x}\mathrm{d}x \\ &\quad = a(t)\left(\exp(\beta t)\underbrace{(C_2 - C_1)}_{<0}K^2\phi(t,K)\right) \\ &\quad + a(t)\left(2\underbrace{C_1}_{>0}\exp(\beta t)\int_0^K x\phi\mathrm{d}x + 2\underbrace{C_2}_{<0}\exp(\beta t)\int_K^L x\phi\mathrm{d}x\right). \end{aligned} \tag{6.40}$$

Since $a(t) = \frac{1}{2}\sigma^2 > 0$, it follows from (6.40) that

$$\begin{aligned} -\int_0^L a(t)x^2 \frac{\partial u^*}{\partial x}\frac{\partial \phi}{\partial x}\mathrm{d}x &\le 2C_1\exp(\beta t)a(t)\int_0^K x\phi\mathrm{d}x \\ &\le C\int_0^L \phi\mathrm{d}x, \end{aligned} \tag{6.41}$$

for a constant C not depending on k and ϵ.
We next consider the second integral of the bilinear form $b(t; u^*, \phi)$. Using the fact that $|V_0 - g|$ is uniformly bounded by a positive constant, we deduce the following inequality

$$-\int_0^L b(t)xu^*\frac{\partial \phi}{\partial x}\mathrm{d}x \le -C\underbrace{\int_0^L x\frac{\partial \phi}{\partial x}\mathrm{d}x}_{IBP}.$$

Doing integration by parts and keeping in mind that $\phi(t, X) = 0$, it yields

$$\begin{aligned} -\int_0^L b(t)xu^*\frac{\partial \phi}{\partial x}\mathrm{d}x &\le -C\left(x\phi\Big|_{x=0}^{L} - \int_0^L \phi\mathrm{d}x\right) \\ &\le C\int_0^L \phi\mathrm{d}x, \end{aligned} \tag{6.42}$$

for a positive constant C not depending on k and ϵ.
The last integrals of the bilinear form $b(t; u^*, \phi)$ also yields

$$\int_0^L c(t)u^*\phi \mathrm{d}x \leq C\int_0^L \phi \mathrm{d}x. \tag{6.43}$$

From (6.41)-(6.43), we deduce that

$$-\int_0^T b(t; u^*, \phi)\mathrm{d}t \leq C\int_0^T\int_0^L \phi \mathrm{d}x\mathrm{d}t \leq C\left(\int_0^T \|\phi\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{\frac{1}{p}}, \tag{6.44}$$

for a constant C not depending on k and ϵ.
Returning to the inequality (6.35), thus we have founded that

$$\begin{aligned}\frac{1}{2}\big(\phi(t), \phi(t)\big) + \beta\int_0^T \|\phi(t)\|^2_{M_1}\,\mathrm{d}t + \frac{1}{\epsilon}\int_0^T \left(\phi^{\frac{1}{k}}, \phi\right)\mathrm{d}t \\ \leq C\bigg(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\bigg)^{\frac{1}{p}}.\end{aligned} \tag{6.45}$$

This implies that

$$\frac{1}{\epsilon}\int_0^T \left(\phi(t)^{1/k}, \phi(t)\right)\mathrm{d}t \leq C\left(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{\frac{1}{p}}. \tag{6.46}$$

Since $\phi \in L^p(\Omega)$, it follows that

$$\frac{1}{\epsilon}\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t \leq C\left(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{1/p}, \tag{6.47}$$

where

$$p = 1 + \frac{1}{k}. \tag{6.48}$$

(6.47) is equivalent to

$$\left(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{1-\frac{1}{p}} \leq C\epsilon.$$

Clearly,

$$\left(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{1/p} \leq C\epsilon^{\frac{1}{p-1}} \leq C\epsilon^k, \tag{6.49}$$

where $k = \frac{1}{p-1}$ is obtained from (6.48). Thus, we have proved the first estimate in Lemma 6.3.1.

Now, from (6.45) and (6.49), we find that

$$\frac{1}{2}\big(\phi(t), \phi(t)\big) + \beta\int_0^T \|\phi(t)\|^2_{M_1}\,\mathrm{d}t \leq \left(\int_0^T \big\|\phi(t)\big\|^p_{L^p(\Omega)}\mathrm{d}t\right)^{1/p} \leq C\epsilon^k.$$

Choosing $\beta = \frac{1}{2}$, we have

$$\big(\phi(t), \phi(t)\big) + \int_0^T \left\|\phi(t)\right\|_{M_1}^2 \mathrm{d}t \le C_1 \epsilon^k.$$

Setting

$$a = \big(\phi(t), \phi(t)\big)^{\frac{1}{2}},$$
$$b = \left(\int_0^T \|\phi(t)\|_{M_1}^2 \, \mathrm{d}t\right)^{1/2}$$

and applying $(a+b)^2 \le 2(a^2+b^2)$, we arrive at

$$\left(\big(\phi(t), \phi(t)\big)^{1/2} + \left(\int_0^T \left\|\phi(t)\right\|_{M_1}^2 \mathrm{d}t\right)^{1/2}\right)^2$$
$$\le C\left(\big(\phi(t), \phi(t)\big) + \int_0^T \|\phi(t)\|_{M_1}^2 \, \mathrm{d}t\right)$$
$$\le C\epsilon^k.$$

Clearly,

$$\big(\phi(t), \phi(t)\big)^{1/2} + \left(\int_0^T \left\|\phi(t)\right\|_{M_1}^2 \mathrm{d}t\right)^{1/2} \le c\epsilon^{k/2},$$

which asserts the last estimate in Lemma 6.3.1, for a constant c not depending on ϵ and k as we desired. ■

Lemma 6.3.1 establishes a prior estimates for $(u_k - u^*)^+$. Using the results, we now introduce the main theorem of convergence of weak solutions as follows

Theorem 6.3.2. *Assume that*

$$\frac{\partial u}{\partial t} \in L^{k+1}(\Omega), \quad \Omega = [0, L],$$

where $k > 0$ is the penalty parameter. Assume also that Lemma 6.3.1 holds. Let u and u_k denote solutions to the problem (6.29) and the k-power penalty method (6.33), respectively. Then there exists a constant C, independent of k and ϵ, such that

$$\|u - u_k\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_k\|_{L^2(0,T;M_1)} \le C\epsilon^{k/2}, \quad k > 0.$$

Proof. In order to employ these estimates in Lemma 6.3.1, we decompose $u - u_k$ in term of $\phi = (u_k - u^*)^+$. See that

$$u - u_k = (u - u^*) - (u_k - u^*). \tag{6.50}$$

Let us define the negative part of $u_k - u^*$ as follows

$$(u_k - u^*)^- = -\min\{u_k - u^*, 0\}.$$

Observe that

$$(u_k - u^*)^-(u_k - u^*)^+ = (u_k - u^*)^- \phi = 0. \tag{6.51}$$

The composition (6.50) now becomes

$$\begin{aligned} u - u_k &= u - u^* - \left[(u_k - u^*)^+ - (u_k - u^*)^-\right] \\ &= \underbrace{u - u^* + (u_k - u^*)^-}_{r_k} - \underbrace{(u_k - u^*)^+}_{\phi} \\ &= r_k - \phi. \end{aligned}$$

To prove Theorem 6.3.2, it now suffices to estimate r_k since ϕ was estimated in Lemma 6.3.1. We start by recalling the variational inequality (6.29)

$$-\left\langle \frac{\partial u}{\partial t}, w - u \right\rangle_{M_1', M_1} + a(t; u, w - u) \geq (f, w - u), \quad \forall w \in M_1. \tag{6.52}$$

and the weak formulation (6.33) of the k-power penalty problem

$$-\left\langle \frac{\partial u_k}{\partial t}, w \right\rangle_{M_1', M_1} + a(t; u_k, w) + \frac{1}{\epsilon}\Big(\left[(u_k - u^*)^+\right]^{1/k}, w \Big) = (f, w), \quad \forall w \in M_1. \tag{6.53}$$

Substituting $w = u - r_k$ and $w = r_k$ into (6.52) and (6.53), respectively, we find that

$$\left\langle \frac{\partial u}{\partial t}, r_k \right\rangle_{M_1', M_1} - a(t; V, r_k) \geq -(f, r_k), \tag{6.54}$$

and

$$-\left\langle \frac{\partial u_k}{\partial t}, w \right\rangle_{M_1', M_1} + a(t; u_k, w) + \frac{1}{\epsilon}\Big(\left[(u_k - u^*)^+\right]^{1/k}, w \Big) = (f, r_k). \tag{6.55}$$

Adding (6.54) and (6.55) and using the notation of ϕ, the following inequality holds

$$-\left\langle \frac{\partial (u_k - u)}{\partial t}, r_k \right\rangle_{M_1', M_1} + a(t; u_k - u, r_k) + \frac{1}{\epsilon} \underbrace{\left(\phi^{1/k}, r_k\right)}_{\leq 0} \geq 0. \tag{6.56}$$

Observe that, since $u \leq u^*$,

$$\begin{aligned} \left(\phi^{1/k}, r_k\right) &= \left(\phi^{1/k}, u - u^* - (u_k - u^*)^-\right) \\ &= \left(\phi^{1/k}, u - u^*\right) \\ &\leq 0. \end{aligned}$$

Thus, (6.56) implies that

$$-\left\langle \frac{\partial (u_k - u)}{\partial t}, r_k \right\rangle_{M_1', M_1} + a(t; u_k - u, r_k) \geq 0.$$

In the other words, we obtain the following inequality

$$-\left\langle \frac{\partial (u - u_k)}{\partial t}, r_k \right\rangle_{M_1', M_1} + a(t; u - u_k, r_k) \leq 0.$$

We want to estimate r_k and remind that $u - u_k = r_k - \phi$. By substituting $r_k - \phi$ into the above inequality, we have

$$-\left\langle \frac{\partial(r_k - \phi)}{\partial t}, r_k \right\rangle_{M_1', M_1} + a(t; r_k - \phi, r_k) \leq 0.$$

Using the chain rule (A.0.15) and doing integration by parts, we rewrite the above inequality as

$$\begin{aligned}\frac{d}{ds}\left(-\frac{1}{2}\big(r_k(s), r_k(s)\big)\right) &+ a(s; r_k, r_k) \\ &\leq \underbrace{-\left\langle \frac{\partial \phi}{\partial s}, r_k \right\rangle_{M_1', M_1}}_{IBP} + a(s; \phi, r_k) \\ &\leq -(\phi, r_k) + \left\langle \frac{\partial r_k}{\partial s}, \phi \right\rangle_{M_1', M_1} + a(s; \phi, r_k).\end{aligned}$$

Integrating in time from t to T and keeping in mind that $\phi(T) = r_k(T) = 0$, thus

$$\begin{aligned}\frac{1}{2}\big(r_k(t), r_k(t)\big) &+ \alpha \int_t^T \|r_k(s)\|_{M_1}^2 \mathrm{d}s \\ &\leq -\big(\phi(t), r_k(t)\big)\Big|_{s=t}^{T} + \int_t^T \left(\left\langle \frac{\partial r_k}{\partial s}, \phi \right\rangle_{M_1', M_1} + a(s; \phi, r_k)\right) \mathrm{d}s \\ &\leq \big(\phi(t), r_k(t)\big) + \int_t^T \left(\left\langle \frac{\partial r_k}{\partial s}, \phi \right\rangle_{M_1', M_1} + a(s; \phi, r_k)\right) \mathrm{d}s.\end{aligned} \tag{6.57}$$

The estimate (6.57) implies that

$$\begin{aligned}1/2\|r_k(t)\|_{L^\infty(0,T;L^2(\Omega))}^2 &+ \alpha\|r_k(t)\|_{L^2(0,T;M_1)}^2 \\ &\leq \|\phi(t)\|_{L^\infty(0,T;L^2(\Omega))}\|r_k(t)\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \alpha\|\phi(t)\|_{L^2(0,T;M_1)}\|r_k(t)\|_{L^2(0,T;M_1)} \\ &\quad + \int_t^T \left\langle \frac{\partial r_k}{\partial s}, \phi \right\rangle_{M_1', M_1} \mathrm{d}s\end{aligned} \tag{6.58}$$

Let us consider boundedness on the last integral. Since

$$r_k = u - u^* + (u_k - u^*)^-,$$

Using (6.51), we find that

$$\begin{aligned}\int_0^T \left\langle \frac{\partial r_k}{\partial t}, \phi \right\rangle_{M_1', M_1} \mathrm{d}t &= \int_0^T \left\langle \frac{\partial(u - u^*)}{\partial t}, \phi \right\rangle_{M_1', M_1} \mathrm{d}t \\ &= \int_0^T \left\langle \frac{\partial u}{\partial t}, \phi \right\rangle_{M_1', M_1} \mathrm{d}t - \beta \int_0^T \phi \exp(\beta t)(V_0 - V^*)\mathrm{d}t\end{aligned} \tag{6.59}$$

Applying Holder inequality (A.7) for $\phi \in L^p(\Omega)$ and $\partial_t u \in L^q(\Omega)$, such that

$$p = 1 + 1/k, \qquad q = k + 1, \qquad 1/p + 1/q = 1.$$

Thus, we deduce from (6.59) that

$$\begin{aligned}\int_0^T \left\langle \frac{\partial r_k}{\partial t}, \phi \right\rangle_{M_1', M_1} \mathrm{d}t &\leq C\|\phi\|_{L^p(\Omega)} \left(\|\partial_t u\|_{L^q(\Omega)} + \|V_0 - g\|_{L^p(\Omega)} \right) \\ &\leq C\epsilon^k,\end{aligned}$$

where we have applied the estimate for ϕ in Lemma 6.3.1 and assume that function $|V_0 - g|$ is bounded in $L^q(\Omega)$.

Returning to (6.58) and using the above estimate, we find that

$$\begin{aligned}1/2\big\|r_k(t)\big\|^2_{L^\infty(0,T;L^2(\Omega))} &+ \alpha\big\|r_k(t)\big\|^2_{L^2(0,T;M_1)} \\ &\leq \big\|\phi(t)\big\|_{L^\infty(0,T;L^2(\Omega))}\big\|r_k(t)\big\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \alpha\big\|\phi(t)\big\|_{L^2(0,T;M_1)}\big\|r_k(t)\big\|_{L^2(0,T;M_1)} \\ &\quad + C\epsilon^k\end{aligned} \tag{6.60}$$

We simplify the notation by setting

$$a_1 = \big\|r_k(t)\big\|_{L^\infty(0,T;L^2(\Omega))}, \quad a_2 = \big\|r_k(t)\big\|_{L^2(0,T;M_1)},$$

and

$$b_1 = \big\|\phi(t)\big\|_{L^\infty(0,T;L^2(\Omega))}, \quad b_2 = \big\|\phi(t)\big\|_{L^2(0,T;M_1)}, \quad \rho = \epsilon^k$$

and remind that Lemma 6.3.1 gives $(b_1 + b_2) \leq C\rho^{1/2}$, the inequality (6.60) is equivalent to

$$\begin{aligned}1/2a_1^2 + \alpha a_2^2 &\leq C\left(a_1 b_1 + a_2 b_2 + \rho\right) \\ &\leq C\left((a_1 + a_2)\underbrace{(b_1 + b_2)}_{\leq C\rho^{1/2}} + \rho \right) \\ &\leq C\left[(a_1 + a_2)\rho^{1/2} + \rho\right]\end{aligned}$$

To estimate $a_1 + a_2$, using $(d + e)^2 \leq 2(d^2 + e^2)$, we see that there is some positive constant C such that

$$(a_1 + a_2)^2 \leq C\left(1/2a_1^2 + \alpha a_2^2\right).$$

Thus,

$$(a_1 + a_2)^2 \leq C\left[(a_1 + a_2)\rho^{1/2} + \rho\right] \tag{6.61}$$

Setting

$$y = a_1 + a_2.$$

Thus,

$$y = \|r_k(t)\|_{L^\infty(0,T;L^2(\mathbb{R}_+))} + \|r_k\|_{L^2(0,T;M)}.$$

(6.61) can be read as

$$y^2 \leq C\rho^{\frac{1}{2}} y + C\rho,$$

which is equivalent to

$$\left(y - \frac{1}{2}C\rho^{\frac{1}{2}}\right)^2 \leq \left(C + \frac{C^2}{4}\right)\rho.$$

Solving this inequality, it yields

$$y \leq C\rho^{\frac{1}{2}},$$

for a constant C not depending on ϵ and k. Replacing y and ρ, an estimate of r_k is given by

$$\|r_k\|_{L^\infty(0,T;L^2(\Omega))} + \|r_k\|_{L^2(0,T;M)} \leq C\epsilon^{k/2}. \tag{6.62}$$

To the end, recalling that $r_k = (u - u_k) + \phi$, we substitute it into the estimate (6.62)

$$\left\|(u - u_k) + \phi\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|(u - u_k) + \phi\right\|_{L^2(0,T;M_1)} \leq C\epsilon^{k/2}.$$

Using the triangle inequality, we arrive at

$$\begin{aligned}\|u - u_k\|_{L^\infty(0,T;L^2(\Omega))} &+ \|u - u_k\|_{L^2(0,T;M_1)} \\ &+ \underbrace{\left(\|\phi\|_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|_{L^2(0,T;M_1)}\right)}_{\leq C_1\epsilon^{k/2}} \leq C\epsilon^{k/2}.\end{aligned}$$

Again with the estimate of ϕ in Lemma 6.3.1, we conclude that

$$\|u - u_k\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_k\|_{L^2(0,T;M_1)} \leq C_2\epsilon^{k/2},$$

which proves Theorem 6.3.2. ■

CHAPTER 7

Operator Splitting Method

We study in this chapter the operator splitting method for solving the k-power penalty problem given by

$$\begin{cases} -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - rx\frac{\partial V}{\partial x} + rV = \frac{1}{\epsilon}\left[(g-V)^+\right]^{\frac{1}{k}}, & \text{in } Q_T, \\ V(T,x) = g(x), & \text{in } \mathbb{R}_+. \end{cases} \tag{7.1}$$

We see that this problem has the form of nonlinear degenerate parabolic equations with source term $\frac{1}{\epsilon}\left[(g-V)^+\right]^{\frac{1}{k}}$. Inspired by [9, p. 36], we rewrite the k-power penalty problem as an abstract Cauchy problem with initial data

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{A}(V) = 0, & \text{in } Q_T, \\ V(0,x) = g(x), & \text{in } \mathbb{R}_+. \end{cases} \tag{7.2}$$

where the nonlinear differential operator $\mathcal{A}(V)$ is given by

$$\mathcal{A}(V) = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - rx\frac{\partial V}{\partial x} + rV - \frac{1}{\epsilon}\left[(g-V)^+\right]^{\frac{1}{k}}$$

The idea of operator splitting is to choose a decomposition of the operator $\mathcal{A}$ such that each of the sub-operators $\mathcal{A}^l$ gives equations that are simpler to solve. For examples, if we split the operator $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2,$$

then we solve sequentially the simpler sub-problems with initial data for $l = 1, 2$

$$\begin{cases} \frac{\partial V_l}{\partial t} + \mathcal{A}_l(V_l) = 0, \\ V_l(0) = g. \end{cases} \tag{7.3}$$

For $l = 1, 2$, letting $V_l(t) = S_l(t)V_l(0) = S_l(t)g$ denotes the exact solution of (7.3), where S_l denotes the corresponding exact solution operator. An approximate solution of the ordinary equation (7.2) can be constructed as

$$V(n\triangle t, x) = \left[S_2(\triangle t) \circ S_1(\triangle t)\right]^n g(x), \quad n \in \mathbb{N},$$

where $\triangle t > 0$ is a small time step. We wish to analyze convergence of the approximate solution generated by the operator splitting method to the true solution of the equation (7.1). Of course, an error will occur by this process.

But this error can be controlled as we increase the numbers of time-steps in the construction. Finally, when we pass to the limit, we expect that the approximation will be converge to true solution of (7.1) in the sense that

$$V(n\triangle t) = \lim_{\triangle t \to 0, n \to \infty} \left[S_2(\triangle t) \circ S_1(\triangle t)\right]^n g,$$

where V is the exact solution to (7.1).

Concretely, let us demonstrate the idea above in our context in which we solve the equation (7.1). By splitting the nonlinear differential operator $\mathcal{A}$ into two elementary sub-operators, the algorithm is then to solve firstly the equation without source term, i.e., the Black-Scholes equation with initial data

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru = 0, \quad u(0) = g, \tag{7.4}$$

and secondly,

$$\frac{\partial z}{\partial t} = \frac{1}{\epsilon}\left[(g-z)^+\right]^{\frac{1}{k}}, \quad z(0) = g. \tag{7.5}$$

Let S_{CD} and S_P denote the exact solution operators corresponding to the sub-problems (7.4) and (7.5), respectively. For a small time-step $\triangle t > 0$, the approximate solution to the equation (7.1) generated by operator splitting method takes the following form

$$V(t,x) \approx V^n = \left[S_P(\triangle t) \circ S_{CD}(\triangle t)\right]^n g, \quad \text{for } t = n\triangle t, \quad n \in \mathbb{N}.$$

With the aim of studying convergence of the product formula (**??**), it is necessary to work with functions defined for all $t \in [0,T]$. We define the auxiliary function for a small time-step $V_{\triangle t} : Q_T \to \mathbb{R}_+$ as follows

$$V^{n+\frac{1}{2}}(t) = \left[S_{CD}(t - t_n)\right] V^n, \quad t \in \left(t_n, t_{n+\frac{1}{2}}\right] \tag{7.6}$$

and

$$V^{n+1}(t) = \left[S_P(t - t_{n+\frac{1}{2}})\right] V^{n+\frac{1}{2}}, \quad t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]. \tag{7.7}$$

Using the product formula, V^{n+1} obtained from V^n takes the following form of approximation

$$V^{n+1} \approx \left[S_P(\triangle t) \circ S_{CD}(\triangle t)\right] V^n. \tag{7.8}$$

Observe that,

$$\begin{aligned} V(n\triangle t, x) = V^n &\approx \left[S_P(\triangle t) \circ S_{CD}(\triangle t)\right]^n V^0(x) \\ &\approx \left[S_P(\triangle t) \circ S_{CD}(\triangle t)\right]^n g(x). \end{aligned} \tag{7.9}$$

Having constructed the approximate solutions $V_{\triangle t}$ defined by (7.8) and (7.9), we now search for properties of it. For $\left(t_n, t_{n+\frac{1}{2}}\right]$, $n \in \mathbb{N}$, we first consider the sub-problem (7.4) which has the exact solution defined by (7.6), the corresponding exact solution operator S_{CD} and initial value V^n. Using energy estimate, it gives us a prior estimates for $V_{\triangle t}$. In doing so, we first multiply (7.4) by $V_{\triangle t}$, integrate in x over $\mathbb{R}_+$, doing integration by parts and use the chain rule (A.0.15), thus we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|V_{\triangle t}(t)\|^2_{L^2(\mathbb{R}_+)}\right) + a(t; V_{\triangle t}, V_{\triangle t}) = 0,$$

where $a(t; v, w)$ is given by

$$a(t; v, w) = \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \big(-(r-d) + \sigma^2\big)x\frac{\partial v}{\partial x}w + rvw\mathrm{d}x, \qquad (7.10)$$

(cf. Chapter 4). Using Gårding inequality (3.11), the following inequality holds

$$\frac{1}{2}\frac{d}{dt}\left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)}\right) + \alpha\|V_{\triangle t}\|^2_M \leq \lambda\|V_{\triangle t}\|^2_{L^2(\mathbb{R}_+)}, \qquad (7.11)$$

for some constants $\alpha > 0$ and $\lambda \geq 0$. This inequality implies that

$$\frac{d}{dt}\left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)}\right) \leq C\|V_{\triangle t}\|^2_{L^2(\mathbb{R}_+)}, \qquad (7.12)$$

for a constant C not depending on $\triangle t$.

Setting $\theta(t) = \|V_{\triangle t}(t)\|^2_{L^2(\mathbb{R}_+)}$, where $t \in \left(t_n, t_{n+\frac{1}{2}}\right]$, (7.12) yields

$$\theta'(t) \leq C\theta(t),$$

Applying Gronwall's inequality (A.8) gives

$$\theta(t) \leq \exp\big(C(t - t_n)\big)\theta(t_n),$$

where C is a constant that does not depend on $\triangle t$. Thus, substituting $\theta(t)$, we get the following estimate, for $t \in \left(t_n, t_{n+\frac{1}{2}}\right]$,

$$\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} \leq \exp\left(C\triangle t\right)\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} \qquad (7.13)$$

where C is a constant, independent of $\triangle t$.

A similar estimate on $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$ can also be derived by applying energy estimates for the sub-equation (7.5). As above, we multiply (7.5) by an arbitrary $V_{\triangle t} \in L^2(\mathbb{R}_+)$, then integrate in $\mathbb{R}_+$ and use chain rule (A.0.15), thus

$$\frac{1}{2}\frac{d}{dt}\left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)}\right) = \frac{1}{\epsilon}\left(\left[(g - V_{\triangle t})^+\right]^{1/k}, V_{\triangle t}\right)$$

Since option prices $V_{\triangle t}$ satisfy $g \leq V_{\triangle t}$, using Cauchy-Schwartz inequality (A.1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)}\right) \leq & C\left\|[(g - V_{\triangle t})^+]^{\frac{1}{k}}\right\|_{L^2(\mathbb{R}_+)}\left\|V_{\triangle t}\right\|_{L^2(\mathbb{R}_+)} \\ \leq & C\left\|V_{\triangle t}\right\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Note that we have obtained $k = 1$ in the above estimate. Again using Gronwall's inequality (A.8) together with (7.13), we deduce that, for $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$, $n = 0, \cdots, N-1$,

$$\begin{aligned} \left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} &\le \exp\left(C\triangle t\right)\left\|V_{\triangle t}(t_{n+\frac{1}{2}})\right\|^2_{L^2(\mathbb{R}_+)} \\ &\le \exp\left(C\triangle t + C\triangle t\right)\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\ \vdots \\ &\le \exp\left(C(N+1)\triangle t\right)\|g\|^2_{L^2(\mathbb{R}_+)}. \end{aligned} \tag{7.14}$$

for a constant C not depending on $\triangle t$, an integer $N \in \mathbb{N}$ such that $N\triangle t = T$ and we have used induction to obtain the last inequality.

We next derive an estimate for $V_{\triangle t}$ in M on each time interval. For $t \in \left(t_n, t_{n+\frac{1}{2}}\right]$, we integrate the sub equation (7.11) in time from t_n to $t_{n+\frac{1}{2}}$ and obtain that

$$\int_{t_n}^{t_{n+\frac{1}{2}}} \left\|V_{\triangle t}(t)\right\|^2_M \mathrm{d}t \le C\left|\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} - \left\|V_{\triangle t}(t_{n+\frac{1}{2}})\right\|^2_{L^2(\mathbb{R}_+)}\right|.$$

Considering the right-hand side of the above inequality. It follows from (7.14) that

$$\left\|V_{\triangle t}(t_{n+\frac{1}{2}})\right\|^2_{L^2(\mathbb{R}_+)} \ge \exp(-C\triangle t)\left\|V_{\triangle t}(t_{n+1})\right\|^2_{L^2(\mathbb{R}_+)}.$$

Thus,

$$\begin{aligned} &\left|\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} - \left\|V_{\triangle t}(t_{n+\frac{1}{2}})\right\|^2_{L^2(\mathbb{R}_+)}\right| \\ &\qquad \le \left|\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} - \exp(-C\triangle t)\left\|V_{\triangle t}(t_{n+1})\right\|^2_{L^2(\mathbb{R}_+)}\right|. \end{aligned}$$

In other words,

$$\begin{aligned} \int_{t_n}^{t_{n+\frac{1}{2}}} \left\|V_{\triangle t}(t)\right\|^2_M \mathrm{d}t &\le \left|\left\|V_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} - \exp(-C\triangle t)\left\|V_{\triangle t}(t_{n+1})\right\|^2_{L^2(\mathbb{R}_+)}\right| \\ &\le \exp(CN\triangle t)\left\|V_{\triangle t}(t_0)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\quad + \exp(-C\triangle t)\exp\left(C(N+1)\triangle t\right)\left\|V_{\triangle t}(t_0)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\le \exp(C'N\triangle t))\|g\|^2_{L^2(\mathbb{R}_+)}, \end{aligned} \tag{7.15}$$

for some constants C and C' not depending on $\triangle t$ and an integer $N \in \mathbb{N}$ such that $N\triangle t = T$.

For $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$, besides the estimate for $\|V_{\triangle t}\|^2_{L^2(\mathbb{R}_+)}$ given by (7.14), we see that there is no term of $x\partial_x V_{\triangle t}$ in the second sub-equation (7.5). Therefore, it suffices to impose an assumption on this derivative to estimate

$\|V_{\triangle t}\|_M^2$. Assume that the exact solution $V_{\triangle t}$ to (7.5) is sufficiently smooth. We now differentiate (7.5) with respect to x and find that

$$\partial_x \left(\frac{\partial V_{\triangle t}}{\partial t} \right) = \frac{1}{\epsilon k} \left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}-1} \partial_x \left(g - V_{\triangle t} \right)^+ ,$$

where

$$\partial_x \left(g - V_{\triangle t} \right)^+ = \begin{cases} \frac{\partial g}{\partial x} - \frac{\partial V_{\triangle t}}{\partial x}, & g > V_{\triangle t}, \\ 0, & g \leq V_{\triangle t}. \end{cases} \tag{7.16}$$

Observe that

$$\partial_x \left(\frac{\partial V_{\triangle t}}{\partial t} \right) = \partial_t \left(\frac{\partial V_{\triangle t}}{\partial x} \right),$$

thus

$$\partial_t \left(\frac{\partial V_{\triangle t}}{\partial x} \right) = \frac{1}{\epsilon k} \left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}-1} \partial_x \left(g - V_{\triangle t} \right)^+ ,$$

where the last nonlinear term in the right-hand side is defined in (7.16). For any $x \in L^2(\mathbb{R}_+)$, multiplying the above equation by x and setting $w = x \frac{\partial V_{\triangle t}}{\partial x}$, we find that

$$\partial_t w = \frac{1}{\epsilon k} \left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}-1} x \partial_x \left(g - V_{\triangle t} \right)^+ . \tag{7.17}$$

Remark 7.0.1. Observe (7.17). Since $k > 0$, we see in particular that $k > 1$, i.e., $1/k - 1 < 0$, the nonlinear expression $\left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}-1} = \left[(g - V_{\triangle t})^+ \right]^{-k*}$, where $k* > 0$ has singularities when $V_{\triangle t} \geq g$. Otherwise, (7.17) will be hold. This is the difficult case for our analysis. Therefore we restrict here our analysis by considering the case $k = 1$ rather than $k > 1$.

To continue, we impose the following assumption:

Assumption 7.0.2. Assume that $k = 1$. Furthermore, there exists the first derivative with respect to x for the sub-solver to (7.5).

Under Assumption 7.0.2, (7.17) reduces to

$$\partial_t w = \frac{1}{\epsilon} x \partial_x \left(g - V_{\triangle t} \right)^+ = \begin{cases} \frac{1}{\epsilon} \left(x g_x - w \right)^+ , & g > V_{\triangle t}, \\ 0, & g \leq V_{\triangle t}. \end{cases} \tag{7.18}$$

We are in the goal to estimate $\|w(t)\|_{L^2(\mathbb{R}_+)}^2$. Following (7.18), we consider two cases as follows.

When $g > V_{\triangle t}$. Multiplying the above first equation of (7.18) by w and integrating with respect to x over $\mathbb{R}_+$, using the chain rule (A.0.15) gives

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|w(t)\|_{L^2(\mathbb{R}_+)}^2 \right) &= \frac{1}{\epsilon} \int_{\mathbb{R}_+} \left[\left(x \frac{\partial g}{\partial x} - w \right)^+ w \right] \mathrm{d}x \\ &\leq C \int_{\mathbb{R}_+} \left(\left| x \frac{\partial g}{\partial x} \right|^2 - w^2 \right) \mathrm{d}x \\ &\leq C \left(\left\| x \frac{\partial g}{\partial x} \right\|_{L^2(\mathbb{R}_+)}^2 + \|w(t)\|_{L^2(\mathbb{R}_+)}^2 \right) \\ &\leq C_1 \left(B + \|w(t)\|_{L^2(\mathbb{R}_+)}^2 \right), \end{aligned} \tag{7.19}$$

for some constant C, C_1 and B not depending on $\triangle t$, but depend on k and ϵ. We have used above $ab \leq 1/2(a^2+b^2)$ to obtain the first inequality. Also, note that as we work with put options, $x\frac{\partial g}{\partial x}$ is bounded in $L^2(\mathbb{R}_+)$ by a constant B. Indeed, recall that the payoff function g of a put option is given by

$$g = (K-x)^+ = \begin{cases} K-x, & \text{for } 0 \leq x \leq K, \\ 0, & \text{for } K < x < \infty, \end{cases}$$

it follows that

$$x\frac{\partial g}{\partial x} = \begin{cases} x, & \text{for } 0 \leq x \leq K, \\ 0, & \text{for } K < x < \infty. \end{cases}$$

Thus,

$$\int_{\mathbb{R}_+} \left| x\frac{\partial g}{\partial x} \right|^2 \mathrm{d}x = \int_0^K x^2 \mathrm{d}x + \int_K^\infty 0 \mathrm{d}x = \int_0^K x^2 \mathrm{d}x < B < \infty,$$

where B is a positive constant.

Returning to (7.19), we estimate for $\left\|w(t)\right\|^2_{L^2(\mathbb{R}_+)}$ by using Gronwall's inequality (A.8) for $\eta(t) = \frac{d}{dt}\left\|w(t)\right\|^2_{L^2(\mathbb{R}_+)}$, thus for $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$

$$\|w(t)\|^2_{L^2(\mathbb{R}_+)} \leq \exp(C\triangle t)\left(\left\|w(t_{n+1/2})\right\|^2_{L^2(\mathbb{R}_+)} + \triangle t B \right).$$

Substituting w by $x\partial_x V_{\triangle t}$, we find that for $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$

$$\begin{aligned} \|x\partial_x V_{\triangle t}(t)\|^2_{L^2(\mathbb{R}_+)} &\leq \exp(C\triangle t)\left(\left\|x\partial_x V_{\triangle t}(t_{n+1/2})\right\|^2_{L^2(\mathbb{R}_+)} + \triangle t B \right) \\ &\leq \exp(C\triangle t)\left(\left\|V_{\triangle t}(t_{n+1/2})\right\|^2_M + \triangle t B \right) \\ &\leq \exp(C\triangle t)\left(\exp(CN\triangle t)\|g\|^2_{L^2(\mathbb{R}_+)} + \triangle t B \right), \end{aligned}$$

where we have used (7.15) to obtain the last inequality. Using this result and the estimate for $\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)}$ given by (7.14), we deduce that

$$\begin{aligned} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left\|V_{\triangle t}(t)\right\|^2_M \mathrm{d}t &= \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} + \|x\partial_x V_{\triangle t}(t)\|^2_{L^2(\mathbb{R}_+)} \right) \mathrm{d}t \\ &\leq \exp\left(C(N+1)\triangle t\right) \|g\|^2_{L^2(\mathbb{R}_+)} + \exp(C\triangle t)\triangle t B. \end{aligned}$$

for some constants C and B not depending on $\triangle t$ and N is an integer such that $N\triangle t = T$.

On the other hand, when $V_{\triangle t} \geq g$, we obtain the following estimate

$$\begin{aligned} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left\|V_{\triangle t}(t)\right\|^2_M \mathrm{d}t &= \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left(\left\|V_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} + \underbrace{\|x\partial_x V_{\triangle t}(t)\|^2_{L^2(\mathbb{R}_+)}}_{=0} \right) \mathrm{d}t \\ &\leq \exp\left(C(N+1)\triangle t\right) \|g\|^2_{L^2(\mathbb{R}_+)} \end{aligned}$$

Remark 7.0.3. We comment again that we here do not treat the problem when $k > 1$. The above results just only hold when $k = 1$.

Regarding time derivative of $V_{\triangle t}$. The sub-equation (7.4) gives for $t \in \left(t_n, t_{n+\frac{1}{2}}\right]$,

$$\frac{\partial V_{\triangle t}}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_{\triangle t}}{\partial x^2} + (r-d)x\frac{\partial V_{\triangle t}}{\partial x} - rV_{\triangle t},$$

which suggests that $\frac{\partial V_{\triangle t}}{\partial t}$ belongs to space M'. Recalling the norm of a function in space M'

$$\left\|\partial_t V_{\triangle t}(t)\right\|_{M'} = \sup_{\|w\|_M \le 1} \frac{\left|\langle \partial_t V_{\triangle t}, w\rangle_{M',M}\right|}{\|w\|_M}. \tag{7.20}$$

For a fixed function $w \in M$, such that $\|w\|_M \le 1$, we utilize (7.20) and get

$$\begin{aligned}\left|\langle \partial_t V_{\triangle t}, w\rangle_{M',M}\right| &= \left|\left\langle \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_{\triangle t}}{\partial x^2}, w\right\rangle + \left\langle (r-d)x\frac{\partial V_{\triangle t}}{\partial x}, w\right\rangle - rV_{\triangle t}w\right| \\ &\le C\left(\left|\left\langle \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_{\triangle t}}{\partial x^2}, w\right\rangle\right| + \left|\left\langle (r-d)x\frac{\partial V_{\triangle t}}{\partial x}, w\right\rangle\right| + |rV_{\triangle t}w|\right),\end{aligned}$$

for a constant C not depending on $\triangle t$. To estimate on the right-hand side above, we refer to Chapter 4 and do similarly step by step. We get

$$\left\|\partial_t V_{\triangle t}(t)\right\|_{M'} \le C\left\|V_{\triangle t}(t)\right\|_M,$$

for a constant C not depending on $\triangle t$. We next square and integrate in time over $\left(t_n, t_{n+\frac{1}{2}}\right]$, thus

$$\begin{aligned}\int_{t_n}^{t_{n+\frac{1}{2}}} \|\partial_t V_{\triangle t}(t)\|_{M'}^2 \mathrm{d}t &\le C\int_{t_n}^{t_{n+\frac{1}{2}}} \|V_{\triangle t}(t)\|_M^2 \mathrm{d}t \\ &\le C\exp(C' N\triangle t)\|g\|_{L^2(\mathbb{R}_+)}^2.\end{aligned}$$

where the last inequality is obtained from (7.15).
For $t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]$, the formula of sub-equation (7.5) also suggests that $\frac{\partial V_{\triangle t}}{\partial t} \in M'$. Similarly, we utilize the norm of $\partial_t V_{\triangle t}$ in M' by any fixed function $w \in M$ such that $\|w\|_M \le 1$, thus

$$\begin{aligned}C\left|\langle \partial_t V_{\triangle t}, w\rangle_{M',M}\right| &= C\left|\left([(g - V_{\triangle t})^+]^{\frac{1}{k}}, w\right)\right| \\ &\le C\left\|(g - V_{\triangle t})^+]^{\frac{1}{k}}\right\|_{L^2(\mathbb{R}_+)} \|w\|_{L^2(\mathbb{R}_+)} \\ &\le C\left(\|g\|_{L^2(\mathbb{R}_+)} + \left\|V_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}\right)\end{aligned}$$

where we have used Cauchy-Schwartz (A.1) and the triangle inequality. Note that $k = 1$ in the above estimate. We next square the above inequality, integrate in time over $\left(t_{n+\frac{1}{2}}, t_{n+1}\right]$ and apply $(a+b)^2 \le 2(a^2+b^2)$, it yields

$$\begin{aligned}\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left\|\partial_t V_{\triangle t}(t)\right\|_{M'}^2 \mathrm{d}t \le& C\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left(\|g\|_{L^2(\mathbb{R}_+)} + \left\|V_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}\right)^2 \mathrm{d}t \\ \le& C_1\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left(\|g\|_{L^2(\mathbb{R}_+)}^2 + \left\|V_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}^2\right) \mathrm{d}t,\end{aligned}$$

for some constants C and C_1 not depending on $\triangle t$, but depend on k and ϵ. Employing (7.14), we arrive at

$$\int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left\|\partial_t z_{\triangle t}(t)\right\|_{M'}^2 \mathrm{d}t \leq C_1\Big(\triangle t + \exp\big(C(N+1)\triangle t\big)\Big)\|g\|_{L^2(\mathbb{R}_+)}^2,$$

for some constants C and C_1 not depending on $\triangle t$, but on k and ϵ.

Summarizing, the following lemma collects a prior estimates of the splitting approximate solutions $V_{\triangle t}$

Lemma 7.0.4. *Under Assumption 7.0.2, the splitting approximate solutions $V_{\triangle t} : \mathbb{R}_+ \times [t_n, t_{n+1}) \to \mathbb{R}_+$, where $n = 0, \cdots, N-1$, $N \in \mathbb{N}$ such that $N\triangle t = T$, satisfy the three estimates uniformly in $\triangle t$ such as*

- *Boundedness*

$$\max_{t_n \leq t \leq t_{n+1}} \left\|V_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}^2 \leq \exp(CT)\|g\|_{L^2(\mathbb{R}_+)}^2, \tag{7.21}$$

- *L^2 stability in the weighted Sobolev space M*

$$\int_{t_n}^{t_{n+1}} \left\|V_{\triangle t}(t)\right\|_M^2 \mathrm{d}t \leq \exp(CT)\|g\|_{L^2(\mathbb{R}_+)}^2, \tag{7.22}$$

- *L^2 stability in the dual space M'*

$$\int_{t_n}^{t_{n+1}} \left\|\partial_t V_{\triangle t}(t)\right\|_{M'}^2 \mathrm{d}t \leq C' \exp(CT)\|g\|_{L^2(\mathbb{R}_+)}^2, \tag{7.23}$$

where C and C' are some constants that do not depend on $\triangle t$, but depend on ϵ and k. Moreover, we have used the fact that $N\triangle t = T$.

Lemma 7.0.4 establishes three fundamental properties of the approximations $V_{\triangle t}$ generated by operator splitting method. Using these properties, we have the following theorem of convergence

Theorem 7.0.5 (Convergence)**.** *Using Assumption 7.0.2, we assume also that Lemma 7.0.4 holds. Then there exists a subsequence $\{V_{\triangle t}^m\}_{\triangle t>0} \subseteq \{V_{\triangle t}\}$ (not relabelled) that converges in $L^2_{loc}((0,T)\times\mathbb{R}_+)$ to a limit function $V \in L^\infty\big(0,T,L^2(\mathbb{R}_+)\big)$ as $\triangle t \downarrow 0$.*

Proof. In view of Lemma 7.0.4, there exist some constants c and c' such that

$$\max_{0\leq t\leq T} \left\|V_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}^2 \leq \exp(cT)\|g\|_{L^2(\mathbb{R}_+)}^2,$$

$$\int_0^T \left\|V_{\triangle t}(t)\right\|_M^2 \mathrm{d}t \leq \exp(cT)\|g\|_{L^2(\mathbb{R}_+)}^2,$$

$$\int_0^T \left\|\partial_t V_{\triangle t}(t)\right\|_{M'}^2 \mathrm{d}t \leq c' \exp(cT)\|g\|_{L^2(\mathbb{R}_+)}^2.$$

Using these estimates and invoking Lemma 5.3.4 from Chapter 5-an application of Aubin-Lions Lemma, it follows that there exists a subsequence $\{V_{\triangle t}^m\}_{\triangle t>0} \subseteq \{V_{\triangle t}\}$ (not relabelled) that converges strongly in $L^2_{loc}((0,T)\times\mathbb{R}_+)$ to a limit function V as $\triangle t \to 0$. Moreover, it follows from the first estimate that $V \in L^\infty\big(0,T,L^2(\mathbb{R}_+)\big)$. ∎

Finally, we would like to show that the limit function V is exactly the weak solution to the original equation (7.2). Recalling that V is a weak solution to the equation (7.2) if it satisfies the following weak formulation for any fixed test function $\phi \in C_0^\infty(Q_T)$,

$$\int_0^T \int_{\mathbb{R}_+} -V\partial_t\phi \mathrm{d}x\mathrm{d}t + \int_0^T a^*(t;V;\phi)\mathrm{d}t = \int_{\mathbb{R}_+} V(0,x)\phi(0,x)\mathrm{d}x, \qquad (7.24)$$

where the bilinear form $a^*(t;V,\phi)$ is given by

$$\begin{aligned} a^*(t;V;\phi) = \int_{\mathbb{R}_+} \frac{1}{2}\sigma^2 x^2 \partial_x V \partial_x \phi + \big(-(r-d)+\sigma^2\big)x\frac{\partial V}{\partial x}\phi \mathrm{d}x \\ + \int_{\mathbb{R}_+} \left(r - \frac{1}{\epsilon}\left[(g-V)^+\right]^{\frac{1}{k}}\right)\phi \mathrm{d}x. \end{aligned} \qquad (7.25)$$

Theorem 7.0.6. *Using Assumption 7.0.2. Assume also that Lemma 7.0.4 and Theorem 7.0.5 hold. Then the approximations $V_{\triangle t}$ defined by (7.8) and (7.9) converges to a weak solution to the original equation (7.2).*

Proof. We wish to show that the limit function V in Theorem 7.0.5 satisfies the weak formula (7.24). To this end, fixed a test function $\phi \in C_0^\infty$ and we introduce a new test function ψ by

$$\psi(t,x) = \phi\left(\frac{t}{2}, x\right).$$

since $V_{\triangle t}$ solves (7.4) in $\left[t_n, t_{n+1/2}\right)$ with initial data V^n, the following integral equality holds

$$\begin{aligned} \int_{t_n}^{t_{n+1/2}} \int_{\mathbb{R}_+} -\frac{1}{2}V_{\triangle t}\partial_t\phi \mathrm{d}x\mathrm{d}t + \int_{t_n}^{t_{n+1/2}} a\left(t;V_{\triangle t},\phi\right)\mathrm{d}t \\ = -\frac{1}{2}\int_{\mathbb{R}_+} V_{\triangle t}\phi \mathrm{d}x\Big|_{t=t_n}^{t=t_{n+1/2}}, \end{aligned} \qquad (7.26)$$

where $a\left(t;V_{\triangle t},\phi\right)$ takes the form (7.25). Changing the integral equation (7.26) in time interval $[0,\triangle t]$ such that the new time variable τ satisfies

$$\tau = 2\left(t - t_n\right),$$

and introducing

$$U^n(t) = S_{CD}(t)V^n, \quad t \in [0,\triangle t],$$

(7.26) becomes

$$\begin{aligned} \int_0^{\triangle t} \int_{\mathbb{R}_+} -\frac{1}{2}U^n\psi_\tau\left(\tau + 2t_{n+1/2}, x\right)\mathrm{d}x\mathrm{d}\tau + \int_{\mathbb{R}_+} a(t;U^n,\psi)\mathrm{d}\tau \\ = -\frac{1}{2}\int_{\mathbb{R}_+} V^{n+1/2}\phi\left(t_{n+1/2},x\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}_+} V^n\phi\left(t_n,x\right)\mathrm{d}x. \end{aligned} \qquad (7.27)$$

Moreover, since $V_{\triangle t}$ solves the sub-equation (7.5) in time interval $\left[t_{n+1/2}, t_{n+1}\right)$ with initial data V^n, we have

$$\int_{t_{n+1/2}}^{t_{n+1}} \int_{\mathbb{R}_+} \left(-\frac{1}{2} V_{\triangle t} \partial_t \phi - \frac{1}{\epsilon} \left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}} \phi \right) \mathrm{d}x \mathrm{d}t$$
$$= -\frac{1}{2} \int_{\mathbb{R}_+} V_{\triangle t} \phi \Big|_{t=t_{n+1/2}}^{t=t_{n+1}} \mathrm{d}x.$$

Similarly, we change time variable t to $\tau \in [0, \triangle t]$ such that

$$\tau = 2(t - t_{n+1/2}),$$

and letting

$$W^{n+1/2}(t) = S_P(t) V^{n+1/2}, \quad t \in [0, \triangle t].$$

then the above integral equality is equivalent to

$$\begin{aligned} &\int_0^{\triangle t} \int_{\mathbb{R}_+} -\frac{1}{2} W^{n+1/2} \psi_\tau \left(\tau + 2t_{n+1/2}, x\right) \mathrm{d}x \mathrm{d}\tau \\ &\qquad - \int_0^{\triangle t} \int_{\mathbb{R}_+} \left(\frac{1}{\epsilon} \left[\left(g - W^{n+1/2} \right)^+ \right]^{\frac{1}{k}} \right) \psi \left(\tau + 2t_{n+1/2}, x\right) \mathrm{d}x \mathrm{d}\tau \\ &\qquad = -\frac{1}{2} \int_{\mathbb{R}_+} V^{n+1} \phi\left(t_{n+1}, x\right) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}_+} V^{n+1/2} \phi \left(t_{n+1/2}, x\right) \mathrm{d}x. \end{aligned} \tag{7.28}$$

We now use (7.27) and (7.11) to deduce an integral equality of $V_{\triangle t}$ in $t \in [t_n, t_{n+1})$. To this end, let us denote the characteristic function by $\mathcal{X}_{\triangle t}$ and define it as follows

$$\mathcal{X}_{\triangle t} = \begin{cases} 1, & t \in \left(t_n, t_{n+\frac{1}{2}}\right], \\ 0, & t \in \left(t_{n+\frac{1}{2}}, t_{n+1}\right]. \end{cases}$$

Note that $\mathcal{X}_{\triangle t} \rightharpoonup \frac{1}{2}$ in $L^2\left([0,T] \times \mathbb{R}_+\right)$. The following equation holds

$$\begin{aligned} &\mathcal{X}_{\triangle t} \left(\int_{t_n}^{t_{n+1}} \int_{\mathbb{R}_+} -\frac{1}{2} V_{\triangle t} \partial_t \phi \mathrm{d}x \mathrm{d}t + \int_{t_n}^{t_{n+1/2}} a\left(t; V_{\triangle t}, \phi\right) \mathrm{d}t \right) \\ &\qquad - (1 - \mathcal{X}_{\triangle t}) \int_{t_{n+1/2}}^{t_{n+1}} \int_{\mathbb{R}_+} \frac{1}{\epsilon} \left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}} \phi \mathrm{d}x \mathrm{d}t \\ &\qquad\qquad = -\frac{1}{2} \int_{\mathbb{R}_+} V^{n+1} \phi\left(t_{n+1}\right) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}_+} V^n \phi\left(t_n\right) \mathrm{d}x. \end{aligned}$$

For a fixed time $T > 0$ such that $T = N \triangle t$, $N \in \mathbb{N}$, we take sum over $n = 0, \cdots, N-1$ and use the fact that $\phi(T) = 0$, we obtain that

$$\begin{aligned} &\sum_0^{N-1} \int_{t_n}^{t_{n+1}} \int_{\mathbb{R}_+} -\frac{1}{2} V_{\triangle t} \partial_t \phi \mathrm{d}x \mathrm{d}t + \mathcal{X}_{\triangle t} \int_{t_n}^{t_{n+1/2}} a\left(t; V_{\triangle t}, \phi\right) \mathrm{d}t \\ &\qquad - (1 - \mathcal{X}_{\triangle t}) \int_{t_{n+1/2}}^{t_{n+1}} \int_{\mathbb{R}_+} \frac{1}{\epsilon} \left(\left[(g - V_{\triangle t})^+ \right]^{\frac{1}{k}} \phi \right) \mathrm{d}x \mathrm{d}t \\ &\qquad\qquad = \frac{1}{2} \int_{\mathbb{R}_+} \phi\left(0, x\right) V_{\triangle t}(0) \mathrm{d}x. \end{aligned}$$

Letting $\triangle t \downarrow 0$, Theorem 7.0.5 shows that $V_{\triangle t}$ converges strongly to V as $\triangle t \downarrow 0$. Using the fact that $\mathcal{X}_{\triangle t} \rightharpoonup \frac{1}{2}$ in $L^2(Q_T)$ and multiply the resulting equation by 2, thus we arrive at

$$\int_0^T \int_{\mathbb{R}_+} -V\partial_t \phi \mathrm{d}x\mathrm{d}t + \int_0^T a(t;V,\phi)\mathrm{d}t \\ - \int_0^T \int_{\mathbb{R}_+} \frac{1}{\epsilon}\left(\left[(g-V)^+\right]^{\frac{1}{k}} \phi\right)\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}_+} \phi(0,x)\, V(0,x)\mathrm{d}x,$$

which asserts that V is the weak solution to equation (7.2) according to the weak formulation (7.24). ■

We comment that our proof above works only under Assumption 7.0.2. We have not proved yet in case of $k > 1$. In this part, we wish to show that another convergence of the weak solution to equation (7.2) can also be obtained by a new approximation. Using this new approximation, we may avoid the problem we met before when $k > 1$.

Guided by [9], we define the new time interpolant $\tilde{V}_{\triangle t}$ as follows:

$$\tilde{V}_{\triangle t}(t,x) = \left[S_{CD}(t-t_n) \circ S_P(\triangle t)\right]\tilde{V}^n, \quad t \in (t_n, t_{n+1}]\,. \tag{7.29}$$

(7.29) can be interpreted as $\tilde{V}_{\triangle t}$ solves for the sub-problem (7.4) in the time interval $(t_n, t_{n+1}]$ with new initial data $\tilde{V}^{n+}$ given by

$$\tilde{V}^{n+} = S_P(\triangle t)\tilde{V}^n. \tag{7.30}$$

Since $\tilde{V}_{\triangle t}$ solves for both sub-problems (7.4) and (7.5), $\tilde{V}_{\triangle t}$ satisfies the three basic estimates (7.21)-(7.23) which imply convergence of $\tilde{V}_{\triangle t}$ to a limit function V by Theorem 7.0.5. We wish to show that $\tilde{V}_{\triangle t}$ is sufficiently close to $V_{\triangle t}$ defined in (7.8) and (7.9) in $L^2(Q_T)$. Using energy estimates as before for the first sub-equation (7.4) with initial data V^{n+}, we have the following inequality $t \in (t_n, t_{n+1}]$

$$\left\|\tilde{V}_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} \leq \exp(C\triangle t)\left\|\tilde{V}^{n+}\right\|^2_{L^2(\mathbb{R}_+)}. \tag{7.31}$$

By the approximation (7.30), we deduce an estimate for $\tilde{V}^{n+}$ from the second sub-equation (7.5) with initial data $\tilde{V}^n$. As before, we obtain that

$$\left\|\tilde{V}^{n+}\right\|^2_{L^2(\mathbb{R}_+)} \leq \exp(C\triangle t)\left\|\tilde{V}^{n}\right\|^2_{L^2(\mathbb{R}_+)}. \tag{7.32}$$

From (7.31) and (7.32), we find that, for $t \in (t_n, t_{n+1}]$,

$$\begin{aligned} \left\|\tilde{V}_{\triangle t}(t)\right\|^2_{L^2(\mathbb{R}_+)} &\leq \exp(C\triangle t)\left\|\tilde{V}^{n+}\right\|^2_{L^2(\mathbb{R}_+)} \\ &\leq \exp(C\triangle t)\exp(C\triangle t)\left\|\tilde{V}_{\triangle t}(t_n)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\vdots \\ &\leq \exp\big(C(N+1)\triangle t\big)\|g\|^2_{L^2(\mathbb{R}_+)}, \end{aligned} \tag{7.33}$$

where we have used induction and the initial data $\tilde{V}_{\triangle t}(t_0) = V(0) = g$. Moreover, $t_n = n\triangle t$, $n = 0, \cdots, \mathbb{N}-1$ where N is an integer such that $N\triangle t = T$.

Doing similarly energy estimates for $V_{\triangle t}$ in the first sub-equation (7.5) as before and use (7.33), we also obtain that

$$\int_{t_n}^{t_{n+1}} \left\|\tilde{V}_{\triangle t}(t)\right\|_M^2 \mathrm{d}t \leq \exp\left(C(N+1)\triangle t\right) \|g\|^2_{L^2(\mathbb{R}_+)}$$

and

$$\int_{t_n}^{t_{n+1}} \left\|\partial_t\tilde{V}_{\triangle t}(t)\right\|_{M'}^2 \mathrm{d}t \leq C' \exp\left(C(N+1)\triangle t\right) \|g\|^2_{L^2(\mathbb{R}_+)},$$

for some constants C and C' not depending on $\triangle t$. These above estimates together with (7.33) show that $\tilde{V}_{\triangle t}$ satisfies (7.21)-(7.23), which implies convergence of the new interpolant $\tilde{V}_{\triangle t}$ defined by (7.29).

Remark 7.0.7. Using the new approximation (7.29), by letting the first sub-solver be defined in all of the time interval $(t_n, t_{n+1}]$, we have used energy estimates to derive easily the estimates for $\left\|\tilde{V}_{\triangle t}(t)\right\|_M^2$ and subsequently for $\left\|\tilde{V}_{\triangle t}(t)\right\|_M^2$ by employing the estimate of $\left\|\tilde{V}_{\triangle t}(t)\right\|_{L^2(\mathbb{R}_+)}^2$ in (7.33). We avoid the problem to estimate the second sub-solver in M when $k > 1$. Thus, we see that this is an advance of the new splitting approximation 7.29.

Following [6], we attempt to estimate the difference between $\tilde{V}_{\triangle t}$ and $V_{\triangle t}$ on each of a haft interval of $(t_n, t_{n+1}]$.
For $t \in \left(t_n, t_{n+1/2}\right]$, $V_{\triangle t}$ and $\tilde{V}_{\triangle t}$ are given as follows:

$$V_{\triangle t}(t) = S_{CD}(t - t_n)V^n,$$

and

$$\tilde{V}_{\triangle t} = \left[S_{CD}(t - t_n) \circ S_P(\triangle t)\right] V^n.$$

Thus,

$$\begin{aligned}\left\|V_{\triangle t} - \tilde{V}_{\triangle t}\right\|_{L^1(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} \left|V_{\triangle t} - \tilde{V}_{\triangle t}\right| \mathrm{d}x \\ &= \int_{\mathbb{R}_+} \left|S_{CD}(t-t_n)\Big(V^n - S_P(\triangle t)V^n\Big)\right| \mathrm{d}x.\end{aligned}$$

Multiplying the above equation by a fixed test function $\phi(x)$, we obtain

$$\int_{\mathbb{R}_+} \left|V_{\triangle t} - \tilde{V}_{\triangle t}\right|\phi(x)\mathrm{d}x = \int_{\mathbb{R}_+} \left|S_{CD}(t-t_n)\Big(V^n - S_P(\triangle t)V^n\Big)\right|\phi(x)\mathrm{d}x. \quad (7.34)$$

Now we need to estimate for $\left|V^n - S_P(\triangle t)V^n\right|$. Integrating the second sub-equation (7.5) against a test function $\phi(x)$ over $\left[t_n, t_{n+1/2}\right] \times \mathbb{R}_+$ yields the result

$$\begin{aligned}&\int_{\mathbb{R}_+} \big(V_{\triangle t}(t_{n+1/2}, x) - V_{\triangle t}(t_n, x)\big)\phi(x)\mathrm{d}x \\ &\qquad = \int_{t_n}^{t_{n+1/2}} \int_{\mathbb{R}_+} \frac{1}{\epsilon}[(g - V_{\triangle t})^+]^{1/k}\phi(x)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Introducing

$$U^n(t) = S_P(t)V^n, \quad t \in [0, \triangle t],$$

we have

$$\int_{\mathbb{R}_+} \big(S_P(\triangle t)U^n - U^n\big)\phi(x)\mathrm{d}x = \int_0^{\triangle t}\int_{\mathbb{R}_+} \frac{1}{\epsilon}\big[(g-U^n)^+\big]^{1/k}\phi(x)\mathrm{d}x\mathrm{d}t$$
$$\leq \mathcal{O}(1)\triangle t\|\phi\|_{L^\infty(\mathbb{R}_+)},$$

where we have used Lemma 7.0.4. Following the above estimate, we conclude that

$$\int_{\mathbb{R}_+}\Big(S_P(\triangle t)U^n - U^n\Big)\phi(x)\mathrm{d}x = \mathcal{O}(1)\triangle t\|\phi\|_{L^\infty(\mathbb{R}_+)} \tag{7.35}$$

Thus, (7.34) gives for $t \in \big(t_n, t_{n+1/2}\big]$

$$\int_{t_n}^{t_{n+1/2}} \big|V_{\triangle t} - \tilde{V}_{\triangle t}\big|\phi(x)\mathrm{d}x = \mathcal{O}(1)\triangle t\|\phi\|_{L^\infty(\mathbb{R}_+)}.$$

Furthermore, for $t \in \big(t_{n+1/2}, t_{n+1}\big]$, $V_{\triangle t}$ and $\tilde{V}_{\triangle t}$ are defined as follows

$$V_{\triangle t}(t) = S_P(t - t_{n+1/2})V^{n+1/2}$$

and

$$\tilde{V}_{\triangle t}(t) = \big[S_{CD}(t-t_n)\circ S_P(\triangle t)\big]V^n$$
$$= S_{CD}(t-t_n)V^{n+1/2}.$$

Thus,

$$\begin{aligned}
&\big\|V_{\triangle t} - \tilde{V}_{\triangle t}\big\|_{L^1(\mathbb{R}_+)} \\
&\quad = \int_{\mathbb{R}_+}\big|V_{\triangle t} - \tilde{V}_{\triangle t}\big|\mathrm{d}x \\
&\quad = \int_{\mathbb{R}_+}\Big|S_P(t-t_{n+1/2})V^{n+1/2} - \big[S_{CD}(t-t_n)\circ S_P(\triangle t)\big]V^n\Big|\mathrm{d}x \\
&\quad \leq \int_{\mathbb{R}_+}\Big|S_P(t-t_{n+1/2})V^{n+1/2} - S_P(\triangle t)V^n\Big|\mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}_+}\Big|\big[S_{CD}(t-t_n)\circ S_P(\triangle t)\big]V^n - S_P(\triangle t)V^n\Big|\mathrm{d}x.
\end{aligned}$$

Multiplying by a test function $\phi(x)$, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}_+}\big|V_{\triangle t} - \tilde{V}_{\triangle t}\big|\phi(x)\mathrm{d}x \\
&\qquad \leq \underbrace{\int_{\mathbb{R}_+}\Big|S_P(t-t_{n+1/2})V^{n+1/2} - S_P(\triangle t)V^n\Big|\phi(x)\mathrm{d}x}_{E_1} \\
&\qquad\qquad + \underbrace{\int_{\mathbb{R}_+}\Big|\big[S_{CD}(t-t_n)\circ S_P(\triangle t)\big]V^n - S_P(\triangle t)V^n\Big|\phi(x)\mathrm{d}x}_{E_2}.
\end{aligned} \tag{7.36}$$

Since E_1 was estimated above by (7.35), it remains to estimate E_2. We integrate the first sub-equation (7.4) against a test function $\phi(x) \in \big(t_{n+1/2}, t_{n+1}\big]$ yields

$$\int_{\mathbb{R}_+} \Big(V_{\triangle t}(t_{n+1}, x) - V_{\triangle t}(t_{n+1/2}, x) \Big) \phi(x) \mathrm{d}x + \int_{t_{n+1/2}}^{t_{n+1}} a(t; V_{\triangle t}, \phi) \mathrm{d}t = 0, \tag{7.37}$$

where the bilinear form $a(t; u, v)$ is given by (7.10). Introducing

$$W^n(t) = S_{CD}(t) V^n, \quad t \in [0, \triangle t],$$

Thus, (7.37) gives

$$\begin{aligned} \left| \int_{\mathbb{R}_+} \big(S_{CD} W^n - W^n \big) \phi(x) \mathrm{d}x \right| &= \left| - \int_{t_{n+1/2}}^{t_{n+1}} a(t; W^n, \phi) \mathrm{d}t \right| \\ &= \mathcal{O}(1) \triangle t \|\phi\|_M, \end{aligned}$$

since we have used the property of continuous of the bilinear form $a(t; ., .)$ and (7.21) from Lemma 7.0.4.

Consequently, the following weak continuous holds for $t \in (t_n, t_{n+1}]$

$$\int_{\mathbb{R}_+} \big| V_{\triangle t} - \tilde{V}_{\triangle t} \big| \phi(x) \mathrm{d}x = \mathcal{O}(1) \triangle t \left(\|\phi\|_{L^\infty(\mathbb{R}_+)} + \|\phi\|_M \right).$$

Next, let η_ϵ be a standard C_0^∞-mollifier with smoothing radius ϵ. We set

$$d(x) = V_{\triangle t} - \tilde{V}_{\triangle t}$$

Define also, for $r > \epsilon$

$$\beta(x) = \begin{cases} \operatorname{sign}\big(d(x)\big), & |x| \leq r - \epsilon, \\ 0, & |x| > r - \epsilon. \end{cases}$$

Moreover, define

$$\beta^\epsilon = \eta_\epsilon * \beta.$$

Note that, it follows from properties of mollifier that $\beta^\epsilon \in C^\infty$ with support in $[r, -r]$ and $\left\|\beta^\epsilon\right\|_{L^2(\mathbb{R}_+)} = \mathcal{O}(1/\epsilon)$ (A.0.22). Since $d(x)$ is bounded, the following inequality holds (see the proof of Properties of mollifiers in [5, p. 714])

$$\int_{-r}^{r} \Big| |d(x)| - \beta^\epsilon(x) d(x) \Big| \mathrm{d}x \leq C_1 \epsilon, \tag{7.38}$$

for some constant C_1 not depending on ϵ and r. Using this estimate and by choosing $\phi(x) = \beta^\epsilon$ in (7.38), it follows that

$$\begin{aligned} \int_{-r}^{r} \Big| V_{\triangle t} - \tilde{V}_{\triangle t}(t_n, x) \Big| \mathrm{d}x &= \int_{-r}^{r} \Big| |d(x)| - \beta^\epsilon(x) d(x) + \beta^\epsilon(x) d(x) \Big| \mathrm{d}x \\ &\leq \int_{-r}^{r} \Big| |d(x)| - \beta^\epsilon(x) d(x) \Big| \mathrm{d}x + \left| \int_{-r}^{r} \beta^\epsilon(x) d(x) \right| \mathrm{d}x \\ &= C_1 \epsilon + C_2 (\triangle t / \epsilon). \end{aligned}$$

Choosing $\epsilon = \sqrt{\triangle t}$ and letting $r \to \infty$, we find that

$$\left\| V_{\triangle t}(t,x) - \tilde{V}_{\triangle t}(t,x) \right\|_{L^1(\mathbb{R}_+)} = \mathcal{O}(\sqrt{\triangle t}).$$

Thus,

$$\left\| V_{\triangle t}(t,x) - \tilde{V}_{\triangle t}(t,x) \right\|_{L^1(Q_T)} = \mathcal{O}(\sqrt{\triangle t}),$$

where $Q_T = [0,T] \times \mathbb{R}_+$. It follows that

$$\left\| V_{\triangle t}(t,x) - \tilde{V}_{\triangle t}(t,x) \right\|_{L^2(Q_T)} = \mathcal{O}(\sqrt{\triangle t}). \tag{7.39}$$

According to Theorem 7.0.5, $V_{\triangle t}$ converge strongly to a limit function V in $L^2_{loc}(Q_T)$ as $\triangle t \downarrow 0$. Thus, it follows from (7.39) that

$$\tilde{V}_{\triangle t}(t,x) \longrightarrow V \text{ as } \triangle t \downarrow 0 \text{ in } L^2_{loc}(Q_T)$$

as we desired.

CHAPTER 8

Numerical Schemes

In this chapter we present and implement (in Matlab) a numerical scheme for solving the price of American put options. The program is based on the power penalty scheme (see [15]).

8.1 Power-penalty scheme

We begin by the following truncation of the unbounded domain $Q_T = [0,T]\times\mathbb{R}_+$ to a bounded domain $\Omega_T = [0,T] \times [0,L]$, where $0 < L < \infty$ is given. Let $\triangle x > 0$ be the spatial discretization parameters. For a fixed $L > 0$, we choose an integer J such that

$$J\triangle x = L.$$

The spatial domain $[0,L]$ is then discretized into grid cells

$$I_j = [x_j, x_{j+1}), \quad j = 1, \cdots, J-2,$$

where

$$x_l = l\triangle x, \quad l = 0, 1, \cdots, J.$$

Moreover, we set

$$I_J = [x_{J-1}, x_J].$$

Similarly, let $\triangle t > 0$ be temporal discretization parameters. For a fixed time $T > 0$, an integer N is chosen such that

$$N\triangle t = T.$$

We divide the time interval $[0,T]$ into time strips

$$I^n = [t^n, t^n + 1), \quad n = N-2, \cdots, 0,$$

where

$$t_n = n\triangle t, \quad n = 0, \cdots, N.$$

Furthermore, we set

$$I^{N-1} = [t^{N-1}, t^N]$$

We denote by R_j^n the rectangle $I^n \times I_j$. For $j = 0, \cdots, J$ and $n = N, N-1, \cdots, 0$, V_j^n denotes the power penalty approximate solution associated with the point

(t^n, x_j). We extend the difference solution $\{V_j^n\}$ to all of $\Omega_T = [0,T] \times [0,L]$ by setting

$$V_\triangle(t,x) = \begin{cases} V_j^n, & (t,x) \in R_j^n, \quad j = 0,\cdots,J \\ & \qquad\qquad\quad n = N-1,\cdots,0, \\ V_j^N, & t = T, x \in I_j, \quad j = 0,\cdots,J, \end{cases}$$

where $\triangle$ is used as short-hand notation for $\triangle x$.

Let us now introduce the expicit power-penalty scheme. To simplify the presentation, we use $\triangle_+$ and $\triangle_-$ to designate the difference operators in the x direction:

$$\triangle_+ V_j^n = V_{j+1}^n - V_j^n, \quad \triangle_- V_j^n = V_1^n - V_{j-1}^n.$$

Also introducing the *upwind numerical flux function* $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(a,b) = \begin{cases} b & \text{when } (r-d) \geq 0, \\ a & \text{when } (r-d) \geq 0. \end{cases}$$

The suggested numerical scheme for the American option pricing problem (2.6) takes the following form:
For $j = 0,\cdots,J-1, \quad n = N-1,\cdots,0$

$$V_j^n = V_j^{n+1} + \frac{1}{2}\sigma^2 x_j^2 \frac{\triangle t}{\triangle x^2} \triangle_+ \triangle_- V_j^{n+1} + (r-d)x_j \frac{\triangle t}{\triangle x} \triangle_- F\left(V_j^{n+1}, V_{j+1}^{n+1}\right) \\ -r\triangle t V_j^{n+1} + \frac{1}{\epsilon}\left[\left(g(x_j) - V_j^{n+1}\right)^+\right]^{1/k}, \tag{8.1}$$

where the terminal data is given by

$$V_j^N = g(x_j), \quad j = 0,\cdots,J.$$

We impose the following bounded conditions, such that:
At $x = 0$

$$V_0^n = g(0), \quad n = N-1,\cdots,0.$$

At $x = L$

$$V_J^n = g(L), \quad n = N-1,\cdots,0.$$

Note that it follows by [15, p. 40] due to stability of (8.1) in L_{loc}^∞, it suffices to impose a condition of lower bound for V_j^n. An improvement of (8.1) is then given as follows

$$\begin{cases} V_j^{n+1/2} = V_j^{n+1} + \frac{1}{2}\sigma^2 x_j^2 \frac{\triangle t}{\triangle x^2} \triangle_+ \triangle_- V_j^{n+1} + (r-d)x_j \frac{\triangle t}{\triangle x} \\ \qquad \triangle_- F\left(V_j^{n+1}, V_{j+1}^{n+1}\right) - r\triangle t V_j^{n+1} + \frac{1}{\epsilon}\left(\left[g(x_j) - V_j^{n+1}\right]^+\right)^{1/k}, \\ V_j^n = \max\left(g(x_j), V_j^{n+1/2}\right). \end{cases} \tag{8.2}$$

For simplicity, here we will consider the case $r - d \geq 0$. (8.2) becomes

$$\begin{cases} V_j^{n+1/2} = V_j^{n+1} + \frac{1}{2}\sigma^2 x_j^2 \frac{\triangle t}{\triangle x^2}\left(V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}\right) \\ \qquad +(r-d)x_j \frac{\triangle t}{\triangle x}\left(V_{j+1}^{n+1} - V_j^{n+1}\right) - r\triangle t V_j^{n+1} + \frac{1}{\epsilon}\left(\left[g(x_j) - V_j^{n+1}\right]^+\right)^{1/k}, \\ V_j^n = \max\left(g(x_j), V_j^{n+1/2}\right). \end{cases}$$

Furthermore, for the convergence analysis, let us assume that the following parabolic CFL holds [15, p. 51]:

$$\frac{\triangle t}{\triangle x}|r-d|L+\frac{\triangle t}{\triangle x^2}\sigma^2L^2+r\triangle t+\frac{1}{\epsilon k}\max\{K,L\}^{1/k-1}.\leq 1$$

It follows that when $\triangle(=\triangle x)\downarrow 0$, then also $\triangle t\downarrow 0$.

Finally, we suggest that the explicit power penalty scheme for solving the price of an American put option defined by (2.6) is given as follows:

1 The power penalty algorithm for valuing American put options

Input: L, K, r, d, T, N, J, ϵ, k, σ

$deltax \leftarrow L/J$
$deltat \leftarrow T/N$
for $j=1:J+1$
  $x(j) \leftarrow (j-1)*deltax$
  $gput(j) \leftarrow \max(K-x(j),0)$
end
$power \leftarrow gput$
for $n=1:N$
  $u \leftarrow power$
  $v(0) \leftarrow gput(0)$
  $v(J) \leftarrow gput(L)$
  for $j=2:J$
    $m \leftarrow \max\{g(x_j)-u_j,0\}$
    $temp \leftarrow u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+.5*sigma^2*(j-1)^2*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j)+\frac{1}{\epsilon}m^{1/k}$
    $v(j) \leftarrow \max(gput,temp)$
  end
  $power \leftarrow v$
end
Return: $gput$, x, $power$

Replacing $gput$ by $gcall=\max\big\{x(j)-K,0\big\}$, for $j=1,\cdots,J+1$ in the above algorithm, we obtain the numerical scheme based on the power penalty method for the price of American call options.

8.2 A numerical example

We test in Matlab the numerical schemes (1) for American and European put options. We choose the spatial parameter $\triangle x$ and then the temporal parameter $\triangle t$ that satisfies according to the following CFL condition

$$\frac{\triangle t}{\triangle x}|r-d|L+\frac{\triangle t}{\triangle x^2}\sigma^2L^2+r\triangle t+\frac{1}{\epsilon k}\max\{K,L\}^{1/k-1}.\leq 1.$$

We choose the following parameters:

$$r=0.1,\quad \sigma=0.2,\quad k=1,\quad T=1,\quad L=4,\quad \triangle x=0.0526,\quad T=0.0015.$$

The choice $\triangle x = 0.0526$ corresponds to 76 grid points. Furthermore, we specify $k = 10^3$ and $\epsilon = 6e - 4$. The numerical solutions is given in Figure 8.1 for American put options.

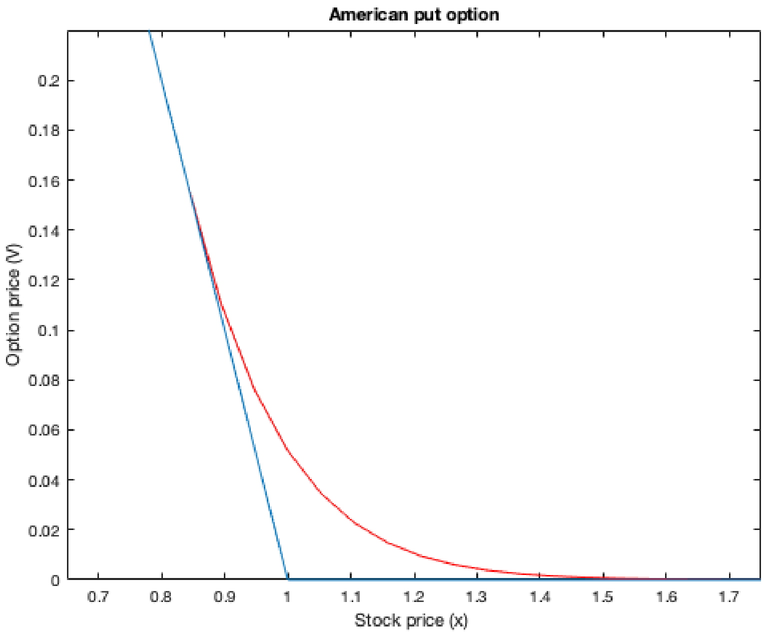

Figure 8.1: The price of an American put option with expiration time T=1 (red line) with the payoff function (solid line).

CHAPTER 9

Final comments

The thesis aims to use compactness method and functional analysis for proving that there exists a unique solution to the problem of pricing American option with variational method. For solving this problem, a penalised method is suggested. In particular, we have analyzed well-posedness result for the classical and the k-power penalty problem. First, we used energy estimates to derive a prior estimates for the solutions of the penalised problems. We then used functional analysis to extract a convergent subsequence. Passing to the limit in the weak formulations, then we inferred the convergence to the solution of the original problem.

While analyzing the problem, we have seen that standard theory for partial differential equation does not work because of the nonlinearity term in the penalty problems. But using compactness theory and functional analysis, we were successful in achieving the convergence. Moreover, when using the operator splitting method for solving the k-power penalty problem, we met an obstacle when $k > 1$ in the way we defined sub-solvers in each half of time step interval. In the end, we defined a new time interpolant hoping this can treat this obstacle. Our goal was to show that the new interpolant is sufficiently close to the old one, which implies convergence for the new approximation. We also wish to test the numerical solution for this convergence. Although I have not yet come to these final results, but for me, the operator splitting technique is very interesting since it helps us simplify complicated problems. This leads to many applications of this technique should be exploited.

Appendices

APPENDIX A

Theories

Definition A.0.1 (Probability Spaces)**.** If Ω is a given set, then a σ-algebra F on Ω is a family F of subsets of Ω with the following properties :

- $\emptyset \in F$
- $A \in F \Rightarrow A^C \in F$, where $A^C = \Omega F$ is the complement of F in Ω
- $A_1, A_2, \cdots \in F \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in F$

The pair (Ω, F) is called a measurable space. A probability measure P on a measurable space (Ω, F) is a function $P : F \longrightarrow [0, 1]$ such that

- $P(\emptyset) = 0, \quad P(\Omega) = 1$
- if $A_1, A_2, \cdots \in F$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e. $A_i \bigcap A_j = \emptyset$ if $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple (Ω, F, P) is called a probability space.

Definition A.0.2 (The filtration)**.** A filtration on (Ω, F) is a family $F = F_{t t \geq 0}$ of σ-algebra $F_t \subset F$ such that

$$0 \leq s \leq t \Rightarrow F_s \subset F_t$$

(i.e. F_t is increasing).

Definition A.0.3 (Stochastic process)**.** A stochastic process is collection of random variables parameterised by a set T.

$$\{X_t\}_{t \in [0,T]}$$

defined on a probability space $(\Omega; F; P)$ and assuming values in R^n.

Definition A.0.4 (Martingale)**.** A stochastic process $\{M_t\}$ on (Ω, F, P) is called a martingale with respect to a filtration $\{F_t\}_{t \geq 0}$ if

- $\{M_t\}_{t \in [0,T]}$ is adapted to the filtration $\{F_t\}_{t \in [0,T]}$.
- M_t is integrable, i.e, $\mathbb{E}[|M_t|] < \infty$, for all t.

- $\mathbb{E}[\, M_t|F_s \,] = M_s$, for all $s < t$.

Definition A.0.5 (Equivalent martingale measure or risk-neutral probability $\mathbb{Q}$)**.** A probability $\mathbb{Q}$ is called an equivalent martingale measure if there exists a random variable $Y > 0$ such that $\mathbb{Q}(A) = \mathbb{E}\,[\mathbb{1}_A Y]$ for all events A and $e^{-rt}S(t)$ is a martingale with respect to $\mathbb{Q}$.

Definition A.0.6 (Brownian motion)**.** A stochastic process B(t) is called a Brownian motion if it satisfies the following conditions:

1. $B(0) = 0$ almost surely.
2. $B(t)$ has independent increments: for $r < s \leq t < u$; then $B(u) - B(t)$ and $B(s) - B(r)$ are independent.
3. $B(t)$ has continuous trajectories with probability 1.
4. $B(t)$ has Gaussian increments: $B(t) - B(s) \sim N(0; t - s)$; for $0 \leq s <$t.

Definition A.0.7 (Adaptness)**.** Let $\{N_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $f(t; w) : [0; \infty) \times \Omega \to \mathbb{R}^n$ is called N_t-adapted if for each $t \geq 0$ the function

$$w \to f(t; w)$$

is N_t-measurable.

Proposition A.0.8. *(Dynamic programming principle)*

1. *For all stopping time θ taking values in $[t, T]$, we have*
$$v(t, x) \geq \mathbb{E}^{t,x}[e^{-r(\theta - t)} v(\theta, X(\theta))].$$
2. *Any stopping time $t \leq \theta \leq \tau_\epsilon$ satisfies*
$$v(t, x) = \mathbb{E}^{t,x}[e^{-r(\theta - t)} v(\theta, X(\theta))].$$
3. *t_0 is an optimal stopping time for $g(X(t))$, and $e^{-r(u\wedge t_0 - t)} v(u \wedge t_0, X(u \wedge t_0))$ is a martingale.*

Theorem A.0.9 (The 1-dimensonal Itô formula)**.** *Let X_t be an Itô process given by*

$$dX_t = udt + vdW_t$$

Let $g(t, x) \in C^2([0, \infty) \times R)$ (i.e. g is twice continuously differentiable on $[0, \infty) \times R$). Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{\partial^2 g}{\partial x^2}(t, X_t).(dX_s)^2.$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t).(\mathrm{d}X_t)$ is computed according to the rules

$$\mathrm{d}t.\mathrm{d}t = \mathrm{d}t.\mathrm{d}W_t = \mathrm{d}W_t.\mathrm{d}t = 0, \quad \mathrm{d}W_t.\mathrm{d}W_t = \mathrm{d}t.$$

Definition A.0.10 (Classical sub- and supersolution)**.**

- A function $v \in C^{1,2}(Q_T) \cap C(\bar{Q_T})$ is a *classical subsolution* of the American option valuation problem if $v(T, x) \leq g$ on $\mathbb{R}_+$ and the following inequalities hold on Q_T:

$$\begin{cases} \mathcal{L}_{BS}v(t,x) - rv(t,x) \geq 0, & v(t,x) > g(x), \\ \mathcal{L}_{BS}v(t,x) - rv(t,x) \geq -c(x), & v(t,x) \leq g(x). \end{cases}$$

- A function $v \in C^{1,2}(Q_T) \cap C(\bar{Q_T})$ is a *classical supersolution* of the American option valuation problem if $v(T, x) \geq g$ on $\mathbb{R}_+$ and the following inequalities hold on Q_T:

$$\begin{cases} \mathcal{L}_{BS}v(t,x) - rv(t,x) \leq 0, & v(t,x) \geq g(x), \\ \mathcal{L}_{BS}v(t,x) - rv(t,x) \leq -c(x), & v(t,x) < g(x). \end{cases}$$

- V is classical sub- or supersolution whenever it is smooth enough.

Definition A.0.11 (Compactness)**.** [5, p. 286] Let X and Y be Banach spaces, $X \cap Y$. We say that X is compactly embedded in Y, written

$$X \subset\subset Y,$$

provided

- $\|u\|_Y \leq C\|u\|_X$ $(u \in X)$ for some constants C
  and
- each bounded sequence in X is precompact in Y.

More precisely, condition (ii) means that if $\{u_k\}_{k=1}^{\infty}$ is a sequence in X with $\sup_k \|u_k\|_X < \infty$, then some subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subseteq \{u_k\}_{k=1}^{\infty}$ converges in Y to some limit u:

$$\lim_{j\to\infty} \|u_{k_j} - u\|_Y = 0.$$

Definition A.0.12. If $(.,.)$ is an inner product, the associated norm is

$$\|u\| := (u,u)^{1/2}, \quad u \in H.$$

The *Cauchy-Schwartz inequality* states

$$\|(u,v)\| \leq \|u\|\|v\|, \quad u, v \in H. \tag{A.1}$$

Definition A.0.13 (Weak convergence)**.** [5, p. 723] We say a sequence $u_{k}{}_{k=1}^{\infty} \subset X$ converges weakly to $u \in X$, written

$$u_k \rightharpoonup u,$$

if

$$\langle u^*, u_k \rangle \to \langle u^*, u \rangle$$

for each bounded linear funtional $u^* \in X*$, where $X*$ denote the collection of all bounded linear functionals on X, X^* is the dual space of X.

Theorem A.0.14 (Weak compactness)**.** *[5, p. 723] Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subseteq X$ is bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subseteq \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that*

$$u_{k_j} \text{ converges weakly to } u.$$

Theorem A.0.15. *Suppose that $v \in L^2(0,T;M)$, such that $\frac{\partial v}{\partial t} \in L^2(0,T;M')$. Then*

- $v \in C([0,T];L^2(\mathbb{R}_+))$.
- *The mapping $t \mapsto \|v(t)\|^2_{L^2(\mathbb{R}_+)}$ is absolutely continuous with*

$$\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R}_+)} = 2\left\langle \partial_t v, v\right\rangle_{M',M}$$

  for almost everywhere $0 \leq t \leq T$.
- *For C is a constant depends only on T, we have*

$$\max_{0\leq t\leq T}\|v(t)\|_{L^2(\mathbb{R}_+)} \leq C\left(\|v\|_{L^2(0,T;M)} + \|\partial_t v\|_{L^2(0,T;M')}\right). \tag{A.2}$$

Proof. Define the mollification of v such that $v^\varepsilon = \eta_\varepsilon * v$ (see [5, p. 714]). For any $\varepsilon, \delta > 0$, we have

$$\frac{d}{dt}\left\|v^\varepsilon(t) - v^\delta(t)\right\|^2_{L^2(\mathbb{R}_+)} = 2\left\langle \partial_t v^\varepsilon - \partial_t v^\delta, v^\varepsilon - v^\delta\right\rangle_{M',M}.$$

Thus,

$$\begin{aligned}\left\|v^\varepsilon(t) - v^\delta(t)\right\|^2_{L^2(\mathbb{R}_+)} &= \left\|v^\varepsilon(0) - v^\delta(0)\right\|^2_{L^2(\mathbb{R}_+)} \\ &\quad +2\int_0^t \left\langle \partial_s v^\varepsilon - \partial_s v^\delta, v^\varepsilon - v^\delta\right\rangle_{M',M} \mathrm{d}s.\end{aligned} \tag{A.3}$$

for $0 \leq t \leq T$. Fix $t \in (0,T)$ for which

$$v^\varepsilon(t) \to v(t) \quad \text{in } L^2(\mathbb{R}_+).$$

Consequently, (A.3) implies

$$\begin{aligned}&\limsup_{\varepsilon,\delta\to 0}\sup_{0\leq t\leq T}\left\|v^\varepsilon(t) - v^\delta(t)\right\|^2_{L^2(\mathbb{R}_+)} \\ \leq &\lim_{\varepsilon,\delta\to 0}\int_0^T \left(\left\|\partial_s v^\varepsilon - \partial_s v^\delta\right\|^2_{M'} + \left\|v^\varepsilon - v^\delta\right\|^2_M\right)\mathrm{d}s.\end{aligned}$$

Since $v \in L^2(0,T;M)$ and $\partial_t v \in L^2(0,T;M')$, it follows from definition that $\|v\|_{L^2(0,T;M)} \leq C$ and $\|\partial_t v\|_{L^2(0,T;M')} \leq C$ for a constant C. Therefore, for any $\varepsilon, \delta > 0$, we have

$$\lim_{\varepsilon,\delta\to 0}\int_0^T \left\|\partial_s v^\varepsilon - \partial_s v^\delta\right\|^2_{M'}\mathrm{d}s = 0$$

and

$$\lim_{\varepsilon,\delta\to 0}\int_0^T \left\|v^\varepsilon - v^\delta\right\|^2_M \mathrm{d}s = 0.$$

Thus,

$$\limsup_{\varepsilon,\delta\to 0} \sup_{0\le t\le T} \left\|v^{\varepsilon}(t) - v^{\delta}(t)\right\|^2_{L^2(\mathbb{R}_+)} = 0.$$

This implies that $\{v^{\varepsilon}\}_{0<\varepsilon\le 1}$ converges to u in $C([0,T];L^2(\mathbb{R}_+))$. Since we also know that $v^{\varepsilon}(t) \to v(t)$ almost every $t \in [0,T]$ as $\epsilon \to 0$, we deduce that $u = v$ almost everywhere.

For obtaining the second part, we have

$$\|v^{\epsilon}(t)\|^2_{L^2(\mathbb{R}_+)} = \|v^{\epsilon}(0)\|^2_{L^2(\mathbb{R}_+)} + 2\int_0^t \langle \partial_s v^{\epsilon}, v^{\epsilon}\rangle_{M',M}\,\mathrm{d}s.$$

Sending $\epsilon \to 0$, we know that v^{ε} converges to v almost every $t \in [0,T]$, thus

$$\|v(t)\|^2_{L^2(\mathbb{R}_+)} = \|v(0)\|^2_{L^2(\mathbb{R}_+)} + 2\int_0^t \langle \partial_s v, v\rangle_{M',M}\,\mathrm{d}s.$$

Finally, to obtain (A.2), we take maximum over $t \in [0,T]$, along with the inequality $|\langle \partial_t u, u\rangle|_{M',M} \le \|u\|_M \|\partial_t u\|_{M'}$, we get

$$\max_{0\le t\le T} \|v(t)\|^2_{L^2(\mathbb{R}_+)} \le 2\int_0^T \|\partial_t u\|_{M'} \|u\|_M \mathrm{d}t.$$

Applying Young's inequality ■

Definition A.0.16. The space $L^p(0,T;X)$ consists of all measurable function $v : [0,T] \to X$ with

- $\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|^p dt\right)^{\frac{1}{p}} < \infty, \quad$ for $1 \le p < \infty$.
- $\|v\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{0\le t\le T} \|v(t)\| < \infty$.

Definition A.0.17. The space $C([0,T],X)$ comprises all continuous function $v : [0,T] \to X$ with

$$\|v\|_{C([0,T],X)} = \max_{0\le t\le T} \|v(t)\| < \infty$$

Definition A.0.18. Let $v \in L^1(0,T;X)$. We say $u \in L^1(0,T;X)$ is the weak derivative of v. written

$$v' = u$$

provided

$$\int_0^T \phi'(t)v(t)dt = -\int_0^T \phi(t)u(t)dt$$

for all scalar test functions $\phi \in C_c^{\infty}(0,T)$.

Definition A.0.19.

- The Sobolev space $W^{1,p}(0,T;X)$ consists of all functions $v \in L^p(0,T;X)$ such that v' exists in the weak sense and belongs to $L^p(0,T;X)$. Furthermore,

$$\|u\|_{W^{1,p}(0,T;X)} = \begin{cases} \left(\int_0^T \|v(t)\|^p + \|v'(t)\|^p dt\right)^{\frac{1}{p}} & ,1 \le p < \infty \\ \operatorname{ess}\sup_{0\le t\le T} \|v(t)\| + \|v'(t)\| & ,p = \infty. \end{cases}$$

- We write $H^1(0, T; X) = W^{1,2}(0, T; X)$.

Let H denote a real linear space.

Definition A.0.20. A mapping $(\cdot, \cdot) : H \times H \to R$ is called an inner product if

1. $(u, v) = (v, u)$ for all $u, v \in H$,
2. the mapping $u \mapsto (u, v)$ is linear for each $v \in H$,
3. $(u, u) \geq 0$ for all $u \in H$,
4. $(u, u) = 0$ if and only if $u = 0$.

Definition A.0.21. A Hilbert space H *is a Banach space endowed with an inner product with generates the norm. Examples*

- The space $L^2(\Omega)$ is a Hilbert space, with
$$(f, g) = \int_\Omega fg dx.$$
- The Sobolev space $H^1(\Omega)$ is a Hilbert space, with
$$(f, g) = \int_\Omega fg + \partial f.\partial g dx.$$

Definition A.0.22 (Mollifiers)**.** [5, p. 713]

- Define $\eta \in C^\infty(\mathbb{R}^n)$ by
$$\eta(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$
for a constant C selected so that
$$\int_{\mathbb{R}^n} \eta \mathrm{d}x = 1.$$
- For each $\epsilon > 0$, set
$$\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta \left(\frac{x}{\epsilon}\right).$$
We call η the standard mollifier. The function η_ϵ are C^∞ and satisfy
$$\int_{\mathbb{R}^n} \eta_\epsilon \mathrm{d}x = 1, \quad \mathrm{spt}(\eta_\epsilon) \in B(0, \epsilon).$$

Theorem A.0.23 (Properties of mollifiers)**.** *[5, p. 714]*

- $f^\epsilon \in C^\infty(\Omega_\epsilon)$, *where* $\Omega_\epsilon := \left\{x \in \Omega \middle| \quad \mathrm{dist}(x, \partial\Omega) > \epsilon\right\}$.
- $f^\epsilon \to f$ *almost everywhere as* $\epsilon \downarrow 0$.

- *If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.*

- *If $1 \le p < \infty$ and $f \in L^p_{loc}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$.*

Proof. We refer to [5, p. 714] for the proof. ■

Finally, we list below some useful inequalities:

1. Cauchy's inequality.

$$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2, \qquad (a, b \in \mathbb{R}). \tag{A.4}$$

   Proof. We have $0 \le (a-b)^2 = a^2 - 2ab + b^2$. ■

2. Cauchy's inequality with ε

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2, \qquad (a, b > 0, \varepsilon > 0). \tag{A.5}$$

   Proof. Write

$$ab = \left((2\varepsilon)^{\frac{1}{2}}a\right)\left(\frac{b}{(2\varepsilon)^{\frac{1}{2}}}\right)$$

   and apply Cauchy's inequality (A.4). ■

3. Young's inequality. Let $1 < p, q < \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q, \qquad (a, b > 0). \tag{A.6}$$

   Proof. The mapping $x \mapsto e^x$ is convex, and consequently

$$\begin{aligned} ab = e^{\log a + \log b} &= e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \\ &\le \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} \\ &= \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

   ■

4. Hölder inequality.
   Assume $1 < p, q < \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have

$$\int_\Omega |uv| \mathrm{d}x \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{A.7}$$

   Proof. By homogeneity, we may assume $\|u\|_{L^p} = \|v\|_{L^q} = 1$. Then Young's inequality (A.6) implies for $1 < p, q < \infty$ that

$$\int_\Omega |uv| \mathrm{d}x \le \frac{1}{p}\int_\Omega \|u\|^p \mathrm{d}x + \frac{1}{q}\int_\Omega \|v\|^q \mathrm{d}x = 1 = \|u\|_{L^p}\|v\|_{L^q}.$$

   ■

5. Gronwall's inequality.

- (Differential form) Let $\eta(.)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for almost every t the differential inequality
$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \quad \forall t \in [0,T] \tag{A.8}$$
where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then
$$\eta(t) \leq e^{\int_0^t \phi(s)\mathrm{d}s}\left(\eta(0) + \int_0^t \psi(s)\mathrm{d}s\right)$$
for all $0 \leq t \leq T$.
- If
$$-\eta'(t) \leq C_1\eta(t) + C_2, \quad \forall t \in [0,T] \tag{A.9}$$
then
$$\eta(t) \leq e^{C_1 t}\left(\eta(T) + C_2 t\right).$$
- In particular, if
$$\eta(t) \leq C_1 \int_0^t \epsilon(s)\mathrm{d}s \tag{A.10}$$
for almost every time $0 \leq t \leq T$, then
$$\eta(t) = 0.$$

Proof.

- From (A.8) we see
$$\begin{aligned}\frac{d}{ds}\left(\eta(s)e^{-\int_0 s\phi(d)\mathrm{d}r}\right) &= e^{-\int_0 s\phi(d)\mathrm{d}r}\left(\eta'(s) - \phi(s)\eta(s)\right)\\ &\leq e^{-\int_0 s\phi(d)\mathrm{d}r}\psi(s)\end{aligned}$$
for almost every $0 \leq s \leq T$. Consequently for each $0 \leq t \leq T$, we have
$$\begin{aligned}\eta(t)e^{-\int_0 t\phi(d)\mathrm{d}r} &\leq \eta(0) + \int_0^t e^{-\int_0^s \phi(d)\mathrm{d}r}\psi(s)\mathrm{d}s\\ &\leq \eta(0) + \int_0^t \psi(s)\mathrm{d}s.\end{aligned}$$
This implies the result.
- Let $t = T - s$ for $s \in [0,T]$, then $s = T - t$. Rewriting (A.8), we have
$$\eta'(s) \leq C_1\eta(T-s) + C_2$$
Applying the differential form of Gronwall's inequality above
$$\eta(s) \leq e^{C_1 s}\left(\eta(T-s=0) + C_2 s\right).$$
Hence,
$$\eta(t) \leq e^{C_1 t}\left(\eta(T) + C_2 t\right), \quad t \in [0,T].$$

■

APPENDIX B

Notations

Sets

$\mathbb{R}$.	the real line
$\mathbb{R}_+$	the set of nonnegative reals
$[a,b]$	an closed interval in $\mathbb{R}$
(a,b)	an open interval in $\mathbb{R}$
N	the set of all nonnegative integers
Ω	a bounded domain in $\mathbb{R}$
$\overline{\Omega}$	the closure of Ω

Functional spaces

$C(\Omega)$	the space of functions, continuous in Ω
$C^k(\Omega)$	the space of functions whose derivative up to the order $k \in \mathbb{N}$ belong to $C(\Omega)$
$C^\infty(\Omega)$	$\cap_{k=0}^{\infty} C^k(\Omega)$
$C_0^\infty(\Omega)$	the space of infinitely differentiable functions with a compact support in Ω
N	the set of all nonnegative integers
$L^p(\Omega)$	the space of measurable functions in Ω such that $\int_\Omega \lvert v\rvert^p \mathrm{d}x < +\infty$, $p \in [1,\infty)$
$L^\infty(\Omega)$	the space of measurable functions in Ω such that $\sup_{x\in\Omega} \lvert v(x)\rvert < +\infty$
$\lVert \cdot \rVert_{L^p(\Omega)}$	the norm in $L^p(\Omega)$, $p \in [1,\infty]$
$W_0^{k,p}(\Omega)$	the set of measurable functions whose generalized derivatives up to the order k belong to $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega)$	$W^{k,2}(\Omega)$
$H_0^k(\Omega)$	$W_0^{k,2}(\Omega)$
$C^k\big([0,T];X\big)$.	the space of continuous X-valued functions in $[0,T]$ whose derivatives up to the order $k \in \mathbb{N}$ are continuous
$L^p(0,T;X)$	the space of measurable X-valued functions in $(0,T)$ such that $\int_0^T \lVert u\rVert_X^p \mathrm{d}t < +\infty$, $p \in [1,\infty)$
$L^\infty(0,T;X)$	the space of measurable X-valued functions in $(0,T)$ such that $\mathrm{ess}\sup\limits_{0\le t\le T} \lVert u(t)\rVert_X < +\infty$
$\mathrm{spt}(u)$	the support of the function u

Functions

$\partial_x V(t,x)$, $\frac{\partial V(t,x)}{\partial x}$ the derivative of $V(t,x)$ at x

APPENDIX C

Matlab program

The Matlab program is given below for solving the price of an American put option based on power penalty method.

```
function [gcall, gput, x, powercall, powerput] = ...
        PowerPenalty(r, d, sigma, epsilon, k, K, L, T, J, N)
    deltax = L/J;
    deltat = T/N;
    x = zeros(J+1);
    gcall = zeros(J+1);
    gput = zeros(J+1);

    for j = 1:(J+1)
        x(j) = (j-1) * deltax;
        gput(j) = max(K-x(j), 0);
        gcall(j) = max(x(j)-K, 0);
    end

    powercall = gcall;
    powerput = gput;

    for n = 1:N
        % put
        u = powerput;
        v(1) = K;
        v(J+1) = max(K-L, 0);

        for j = 2:J
            m = max(gput(j) - u(j), 0);
            temp = u(j) + (r-d) * (j-1) * deltat * (u(j+1) - u(j)) ...
                + .5 * sigma^2 * (j-1)^2 * deltat * (u(j+1) - 2 * u(j) + u(j
                     -1)) ...
                - r * deltat * u(j) + (1 / epsilon) * deltat * m^(1/k);
            v(j) = max(gput(j), temp);
        end
        powerput = v;

        % call
        u = wcall;
        temp(1) = 0;
        temp(J+1) = max(L-K, 0);

        for j = 2:J
            m = max(gcall(j) - u(j), 0);
            y = u(j) + (r-d) * (j-1) * deltat * (u(j+1) - u(j)) ...
                + .5 * sigma^2 * (j-1)^2 * deltat * (u(j+1) - 2 * u(j) + u(j
                     -1)) ...
```

```
                - r * deltat * u(j) + deltat * (1 / epsilon) * m^(1/k);
            temp(j) = max(gcall(j), y);
        end
        wcall = temp;
    end
end
```

Bibliography

[1] Achdou, Y. and Pironneau, O. *Computational Methods for Option Pricing (Frontiers in Applied Mathematics) (Frontiers in Applied Mathematics 30)*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2005.

[2] "Editors". In: *Applications of Variational Inequalities in Stochastic Control*. Ed. by Bensoussan, A. and Lions, J.-L. Vol. 12. Studies in Mathematics and Its Applications. Elsevier, 1982, p. ii.

[3] Benth, F., Karlsen, K., and Reikvam, K. "A semilinear Black and Scholes partial differential equation for valuing American options". In: *Finance and Stochastics* 7 (July 2003), pp. 277–298.

[4] Benth, F., Karlsen, K., and Reikvam, K. "A semilinear Black and Scholes partial differential equation for valuing American options: Approximate solutions and convergence". In: *Interfaces and Free Boundaries - INTERFACE FREE BOUND* 6 (Dec. 2004), pp. 379–404.

[5] Evans, L. C. *Partial Differential Equations: Second Edition (Graduate Studies in Mathematics)*. 2nd ed. Graduate Studies in Mathematics (Book 19). American Mathematical Society, Mar. 2010.

[6] Evje S. Hvistendahl Karlsen, K. N. "Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations". In: 83 (July 1999), pp. 107–137.

[7] F.E. Benth, K. H. K. and Reikvam, K. *On a semilinear Black and Scholes partial diferential equation for valuing American options.Part II: approximate solutions and convergence. Interface Free Bound.* Preprint 2001.

[8] Fredrik Nielsen, B., Skavhaug, O., and Tveito, A. "Penalty and front-fixing methods for the numerical solution of American option problems". In: *Journal of Computational Finance* 5 (Sept. 2001).

[9] Holden, H. et al. *Splitting Methods for Partial Differential Equations With Rough Solutions: Analysis and Matlabr Programs*. EMS Series of Lectures in Mathematics. Zürich: European Mathematical Society, Apr. 2010.

[10] Jaillet, P., Lamberton, D., and Lapeyre, B. "Variational inequalities and the pricing of American options". In: *Acta Applicandae Mathematica* 21.3 (Dec. 1990), pp. 263–289.

[11] McKean and Jr.H.P. *Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics*. Indust. Manage, 1965, Rev.6:32–39.

[12] Moerbeke, van and P. *On optimal stopping and free boundary problem*. Arch. Rational Mech. Anal., 1975-176, 60(2):101–148.

[13] R Zvan, P. F. and Vetzal, K. “Penalty methods for American options with stochastic volatility”. In: *Journal of Computational and Applied Mathematics* 91 (Dec. 1997), pp. 199–218.

[14] Wang, S., Yang, X. Q., and Teo, K. L. “Power Penalty Method for a Linear Complementarity Problem Arising from American Option Valuation”. In: *Journal of Optimization Theory and Applications* 129.2 (May 2006), pp. 227–254.

[15] Wasylewicz, A. E. “Analysis of the power penalty method for American options using viscosity solutions”. MA thesis. Oslo: University of Oslo, May 2008.

[16] Zhang, K., Yang, X., and Teo, K. “Convergence analysis of a monotonic penalty method for American option pricing”. In: *Journal of Mathematical Analysis and Applications - J MATH ANAL APPL* 348 (Dec. 2008), pp. 915–926.