Crossed products by Hecke pairs

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To my parents

Contents

Abstract

We develop a theory of crossed products by actions of Hecke pairs (G, Γ), motivated by applications in non-abelian C^*-duality. Our approach gives back the usual crossed product construction whenever G/Γ is a group and retains many of the aspects of crossed products by groups. We start by laying the *-algebraic foundations of these crossed products by Hecke pairs and exploring their representation theory, and then proceed to study their different C^*-completions. We establish that our construction coincides with that of Laca, Larsen and Neshveyev **[15]** whenever they are both definable and, as an application of our theory, we prove a Stone-von Neumann theorem for Hecke pairs which encompasses the work of an Huef, Kaliszewski and Raeburn **[9]**.

Received by the editor May 8, 2013.

2010 *Mathematics Subject Classification.* Primary 46L55 Secondary 20C08.

Key words and phrases. crossed product, Hecke pair, Hecke algebra, C^*-dynamical system, Fell bundle, covariant representation.

Research supported by the Research Council of Norway and the Nordforsk research network "Operator Algebra and Dynamics".

Introduction

The goal of the present work is to develop a theory of crossed products by Hecke pairs with a view towards applications in non-abelian C^*-duality.

A *Hecke pair* (G, Γ) consists of a group G and a subgroup $\Gamma \subseteq G$ for which every double coset $\Gamma g \Gamma$ is the union of finitely many left cosets. In this case Γ is also said to be a *Hecke subgroup* of G. Examples of Hecke subgroups include finite subgroups, finite-index subgroups and normal subgroups. It is in fact many times insightful to think of this definition as a generalization of the notion of normality of a subgroup.

Given a Hecke pair (G, Γ) the *Hecke algebra* $\mathcal{H}(G, \Gamma)$ is a *-algebra of functions over the set of double cosets $\Gamma \backslash G / \Gamma$, with a suitable convolution product and involution. It generalizes the definition of the group algebra $\mathbb{C}(G/\Gamma)$ of the quotient group when Γ is a normal subgroup.

Heuristically, a crossed product of an algebra A by a Hecke pair (G, Γ) should be thought of as a crossed product (in the usual sense) of A by an "action" of G/Γ. The quest for a sound definition of crossed products by Hecke pairs may seem hopelessly flawed since G/Γ is not necessarily a group and thus it is unclear how it should "act" on A. It is the goal of this article to show that in some circumstances such a definition can be given in a meaningful way, recovering the original one whenever G/Γ is a group.

The term "crossed product by a Hecke pair" was first used by Tzanev **[22]** in order to give another perspective on the work of Connes and Marcolli **[3]**. This point of view was later formalized by Laca, Larsen and Neshveyev in **[15]**, where they defined a C^*-algebra which can be interpreted as a reduced C^*-crossed product of a commutative C^*-algebra by a Hecke pair.

It seems to be a very difficult task to define crossed products of *any* given algebra A by a Hecke pair, and for this reason we set as our goal to define a crossed product by a Hecke pair in a generality that will cover the following aspects:

- existence of a canonical spanning set of elements in the crossed product;
- possibility of defining covariant representations;
- the Hecke algebra must be a trivial example of a crossed product by a Hecke pair;
- the classical definition of a crossed product must be recovered whenever G/Γ is a group;
- our construction should agree with that of Laca, Larsen and Neshveyev, whenever they are both definable;
- our definition should be suitable for applications in non-abelian C^*-duality.

We develop a theory of crossed products of certain algebras A by Hecke pairs which takes into account the above requirements. Our approach makes sense when

A is a certain algebra of sections of a Fell bundle over a discrete groupoid. To summarize our set up: we start with a Hecke pair (G, Γ), a Fell bundle $\mathcal{A}$ over a discrete groupoid X and an action α of G on $\mathcal{A}$ satisfying some "nice" properties. From this we naturally give the space $\mathcal{A}/\Gamma$ of Γ-orbits of $\mathcal{A}$ a Fell bundle structure over the orbit space X/Γ, which under our assumptions on the action α is in fact a groupoid. We can then define a *-algebra

$$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\,,$$

which can be thought of as the crossed product of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ). We should point out that a *crossed product* for us is simply a *-algebra, which we can then complete with different C^*-norms or an L^1-norm. Hence, and so that no confusion arises, the symbol $\times^{alg}$ will always be used when talking about the (uncompleted) *-algebraic crossed product.

Our construction gives back the usual crossed product construction when Γ is a normal subgroup of G. Moreover, given any action of the group G/Γ on a Fell bundle $\mathcal{B}$ over a groupoid Y, the usual crossed product $C_c(\mathcal{B}) \times^{alg} G/\Gamma$ can be obtained via our setup as a crossed product by the Hecke pair (G, Γ).

Many of the features present in crossed products by discrete groups carry over to our setting. For instance, the role of the group G/Γ is played by the Hecke algebra $\mathcal{H}(G, \Gamma)$, which embeds in a natural way in the multiplier algebra of $C_c(\mathcal{A}/\Gamma) \times^{alg} G/\Gamma$. Additionally, just like a crossed product $A \times G$ by a discrete group is spanned by elements of the form $a * g$, with $a \in A$ and $g \in G$, our crossed products by Hecke pairs also admit a canonical spanning set of elements.

The representation theory of crossed products by Hecke pairs also has many similarities with the group case, but some distinctive new features arise. For instance, as it is well-known in the group case, there is a bijective correspondence between nondegenerate representations of a crossed product $A \times G$ and the so-called covariant representations of A and G, which are certain pairs of unitary representations of G and representations of A. We will show that something completely analogous occurs for Hecke pairs, but in this case one is obliged to consider *pre-representations* of the Hecke algebra, i.e. representations of $\mathcal{H}(G, \Gamma)$ as (possibly) unbounded operators. This consideration was unnecessary in the group case because unitary operators are automatically bounded.

In the second half of the present article we will study the different C^*-completions of our *-algebraic crossed products by Hecke pairs, with special emphasis on the reduced case which is technically more challenging to define, and explore some connections with non-abelian C^*-duality.

Reduced C^*-crossed products by groups are defined via the so-called regular representations. We will introduce a notion of a regular representation in the Hecke pair case by using the regular representation of the Hecke algebra on $\ell^2(G/\Gamma)$. The main novelty here is that we will have to start with a representation of a certain direct limit of algebras of the form $C_c(\mathcal{A}/H)$, where H is a finite intersection of conjugates of the Hecke subgroup Γ. In case Γ is normal, this direct limit is simply $C_c(\mathcal{A}/\Gamma)$ itself and we recover the usual notion of a covariant representation.

From regular representations it is then possible to define *reduced* C^*-crossed products. Since the algebra $C_c(\mathcal{A}/\Gamma)$ admits several C^*-completions there are several reduced C^*-crossed products that one can form, as for example $C_r^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}) \times_{\alpha,r} G/\Gamma$, each of these thought of as the reduced C^*-crossed product of $C_r^*(\mathcal{A}/\Gamma)$, respectively $C^*(\mathcal{A}/\Gamma)$, by the Hecke pair (G, Γ). These reduced

C^*-crossed products have always a faithful conditional expectation onto $C_r^*(\mathcal{A}/\Gamma)$ (respectively, $C^*(\mathcal{A}/\Gamma)$), a property that determines the reduced crossed product uniquely, just like in the case of groups.

Our construction of reduced C^*-crossed products by Hecke pairs is different from that of Laca, Larsen and Neshveyev in [**15**], being more particular in some sense (since we treat only discrete spectrum), but also more general (since it makes sense for certain non-commutative C^*-algebras). What we are going to show is that both constructions agree whenever they are both definable.

Complementing the reduced setting, one would like to form different *full* C^*-crossed products, as for example $C_r^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$, but in general their existence is not assured. They will always exist, however, if the Hecke algebra is a BG^*-algebra, which is a property that is satisfied by several classes of Hecke pairs, including most of those studied in the literature for which a full Hecke C^*-algebra is known to exist (see [**17**]).

This theory of crossed products by Hecke pairs is intended for applications in non-abelian duality theory. We develop completely a Stone-von Neumann type theorem for Hecke pairs which encompasses the work of an Huef, Kaliszewski and Raeburn [**9**], and we envisage for future work a form of Katayama duality with respect to Echterhoff-Quigg's "crossed product" [**5**].

The Stone-von Neumann theorem, in the language of crossed products by groups, states that for the action of translation of G on $C_0(G)$ we have

$$C_0(G) \times G \;\cong\; C_0(G) \times_r G \;\cong\; \mathcal{K}(\ell^2(G))\,.$$

In [**9**] an Huef, Kaliszweski and Raeburn introduced the notion of *covariant pairs* of representations of $C_0(G/\Gamma)$ and $\mathcal{H}(G,\Gamma)$, for a Hecke pair (G,Γ), and proved that all covariant pairs are amplifications of a certain "regular" covariant pair. Their result was proven without using or defining crossed products, and can also be thought of as a Stone-von Neumann theorem for Hecke pairs. Using our construction we express their result in the language of crossed products. We will show that the full crossed product $C_0(G/\Gamma) \times G/\Gamma$ always exists and one has

$$C_0(G/\Gamma) \times G/\Gamma \;\cong\; C_0(G/\Gamma) \times_r G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma))\,.$$

Moreover, our notion of a covariant representation coincides with the notion of a covariant pair of [**9**], and an Huef, Kaliszewski and Raeburn's result follows as a direct corollary of the above isomorphisms.

Our construction was very much influenced and developed with the wish of obtaining a form of Katayama duality for homogeneous spaces (those arising from Hecke pairs). Even though this has been left for future work, we shall nevertheless explain in Chapter 8 what we have in mind and how our set up is suitable for tackling this problem.

Katayama's duality theorem [**12**] is an analogue for coactions of the duality theorem of Imai and Takai. One version of it states the following: given a coaction δ of a group G on a C^*-algebra A and denoting by $A \times_\delta G$ the corresponding crossed product, we have a canonical isomorphism $A \times_\delta G \times_{\widehat{\delta},\omega} G \cong A \otimes \mathcal{K}(\ell^2(G))$, for some crossed product by the dual action $\widehat{\delta}$ of G. We would like to extend this result to homogeneous spaces coming from Hecke pairs. In spirit we hope to obtain an isomorphism of the type:

$$A \times_\delta G/\Gamma \times_{\widehat{\delta},\omega} G/\Gamma \cong A \otimes \mathcal{K}(\ell^2(G/\Gamma))\,.$$

The C^*-algebra $A \times_\delta G/\Gamma$ should be a crossed product by a coaction of the homogeneous space G/Γ, while the second crossed product should be by the "dual action" of the Hecke pair (G, Γ) in our sense. It does not make sense in general for a homogeneous space to coact on a C^*-algebra, but it is many times possible to define C^*-algebras which can be thought of as crossed products by coactions of homogeneous spaces ([**4**], [**5**]).

It is our point of view that $A \times_\delta G/\Gamma$ should be a certain C^*-completion of the *-algebra $C_c(\mathcal{A} \times G/\Gamma)$ defined by Echterhoff and Quigg [**5**], which we dub the Echterhoff and Quigg's crossed product (a terminology used in [**9**] for $C^*(\mathcal{A} \times G/\Gamma)$ in case of a maximal coaction). We explain in Chapter 8 how our set up for defining crossed products by Hecke pairs is suitable for achieving such a Katayama duality result for Echterhoff and Quigg's crossed product, and can therefore bring insight into the emerging theory of crossed products by coactions of homogeneous spaces.

This article is organized as follows. In Chapter 1 we set up the conventions and preliminary results to be used in the rest of the article.

Chapter 2 is dedicated to the development of the required set up for defining crossed products by Hecke pairs. Here we explain what type of actions are involved, how to define the orbit space groupoids X/H and the orbit bundles $\mathcal{A}/H$ out of $\mathcal{A}$, and how all the algebras $C_c(\mathcal{A}/H)$ are related with each other for different subgroups $H \subseteq G$.

In Chapter 3 we introduce the notion of a crossed product by a Hecke pair, explore some of its algebraic aspects and develop its representation theory. In the last part of this chapter we show how many of the formulas become much simpler in the case of free actions.

The direct limits of sectional algebras, crucial for defining regular representations, are defined in Chapter 4.

In Chapter 5 we define regular representations and reduced C^*-crossed products by Hecke pairs. The comparison between our approach and that of Laca, Larsen and Neshveyev is done in Section 5.4.

Full C^*-crossed products and other C^*-completions are discussed in detail in Chapter 6.

The last two chapters of the present article are devoted to the applications in non-abelian C^*-duality. In Chapter 7 we establish the Stone-von Neumann theorem for Hecke pairs and relate it to the work of an Heuf, Kaliszewski and Raeburn, while in Chapter 8 we explain how our setup is well adapted for establishing a Katayama duality result for Hecke pairs.

The present work is based on the author's Ph.D. thesis [**16**] written at the University of Oslo. There are a few differences between the present work and [**16**], notably the greater generality of the types of actions involved. This improvement follows a suggestion of Dana Williams and John Quigg.

The author would like to thank his advisor Nadia Larsen for the very helpful discussions, suggestions and comments during the elaboration of this work. A word of appreciation goes also to John Quigg, Dana Williams and Erik Bédos for some very helpful comments. Lastly, the author would like to thank the referee for having raised interesting questions on some of the assumptions we make in this work, and also for the various comments directed at improving this work both in content and presentation.

CHAPTER 1

Preliminaries

In this chapter we set up the conventions, notation, and background results which will be used throughout this work. We indicate the references where the reader can find more details, but we also provide proofs for those results which we could not find in the literature.

CONVENTION. *The following convention for displayed equations will be used throughout this work: if a displayed formula starts with the equality sign, it should be read as a continuation of the previously displayed formula.*

A typical example takes the following form:

$$\begin{aligned}(\text{expression } 1) &= (\text{expression } 2)\\ &= (\text{expression } 3)\,.\end{aligned}$$

By Theorem A and Lemma B it then follows that

$$\begin{aligned}&= (\text{expression } 4)\\ &= (\text{expression } 5)\,.\end{aligned}$$

Under our convention starting with the equality sign in the second array of equations simply means that (expression 3) *is equal to* (expression 4).

1.1. *-Algebras and (pre-)*-representations

Let $\mathscr{V}$ be an inner product space over $\mathbb{C}$. Recall that a function $T : \mathscr{V} \to \mathscr{V}$ is said to be *adjointable* if there exists a function $T^* : \mathscr{V} \to \mathscr{V}$ such that

$$\langle T\xi\,,\,\eta\rangle = \langle \xi\,,\,T^*\eta\rangle\,,$$

for all $\xi, \eta \in \mathscr{V}$. Recall also that every adjointable operator T is necessarily linear and that T^* is unique and adjointable with $T^{**} = T$. We will use the following notation:

- $L(\mathscr{V})$ denotes the *-algebra of all adjointable operators in $\mathscr{V}$
- $B(\mathscr{V})$ denotes the *-algebra of all bounded adjointable operators in $\mathscr{V}$.

Of course, we always have $B(\mathscr{V}) \subseteq L(\mathscr{V})$, with both *-algebras coinciding when $\mathscr{V}$ is a Hilbert space (see, for example, [**19**, Proposition 9.1.11]).

Following Palmer ([**18**],[**19**]) we will use the following definitions:

Definition 1.1.1 ([**19**], Def. 9.2.1). A *pre-*-representation* of a *-algebra A on an inner product space $\mathscr{V}$ is a *-homomorphism $\pi : A \to L(\mathscr{V})$. A **-representation* of A on a Hilbert space $\mathscr{H}$ is a *-homomorphism $\pi : A \to B(\mathscr{H})$.

Definition 1.1.2 ([**18**], Def. 4.2.1). A pre-*-representation $\pi : A \to L(\mathscr{V})$ is said to be *normed* if $\pi(A) \subseteq B(\mathscr{V})$, i.e. if $\pi(a)$ is a bounded operator for all $a \in A$.

Definition 1.1.3 ([**19**], Def. 10.1.17). A *-algebra A is called a *BG^*-algebra* if all pre-*-representations of A are normed.

We now introduce our notion of an *essential ideal*. Our definition is not the usual one, but this choice of terminology will be justified in what follows.

Definition 1.1.4. Let A be a *-algebra. An ideal $I \subseteq A$ is said to be *essential* if $aI \neq \{0\}$ for all $a \in A \setminus \{0\}$.

The usual definition of an essential ideal states that I is essential if it has nonzero intersection with every other nonzero ideal. Our definition is stronger, but coincides with the usual one for the general class of semiprime *-algebras. Before we prove this result we recall the definition of this class of *-algebras:

Definition 1.1.5 ([**18**], Definition 4.4.1). A *-algebra is said to be *semiprime* if $aAa = \{0\}$ implies $a = 0$, where $a \in A$.

The class of semiprime *-algebras is quite large, containing all *-algebras that have a faithful *-representation on a Hilbert space (in particular, all C^*-algebras) and many other classes of *-algebras (see [**19**, Theorem 9.7.21]).

Proposition 1.1.6. *Let A be an algebra and $I \subseteq A$ a nonzero ideal. We have*

i) *If I is essential, then I has a nonzero intersection with every other nonzero ideal of A.*
ii) *The converse of i) is true in case A is semiprime.*

Proof: *i)* Let I be an essential ideal of A. Let $J \subseteq A$ be a nonzero ideal and $a \in J \setminus \{0\}$. Since a is nonzero, then $aI \neq \{0\}$. Hence, $J \cdot I \neq \{0\}$, and since $J \cdot I \subseteq J \cap I$, we have $J \cap I \neq \{0\}$.

ii) Suppose A is semiprime. Suppose also that I is not essential. Thus, there is $a \in A \setminus \{0\}$ such that $aI = \{0\}$. Let $J_a \subseteq A$ be the ideal generated by a. We have $J_a \cdot I = \{0\}$. Since $(J_a \cap I)^2 \subseteq J_a \cdot I$ we have $(J_a \cap I)^2 = \{0\}$. Since A is semiprime this implies that $J_a \cap I = \{0\}$ (see [**18**, Theorem 4.4.3]). Hence, I has zero intersection with a nonzero ideal. $\square$

For C^*-algebras the focus is mostly on closed ideals. In this setting we still see that our definition is equivalent to the usual one ([**21**, Definition 2.35]):

Proposition 1.1.7. *Let A be a C^*-algebra and $I \subseteq A$ a closed ideal. The following are equivalent:*

i) *I is essential.*
ii) *I has nonzero intersection with every other nonzero ideal of A.*
iii) *I has nonzero intersection with every other nonzero closed ideal of A.*

Proof: *i)* $\Longleftrightarrow$ *ii)* This was established in Proposition 1.1.6, since C^*-algebras are automatically semiprime.

ii) $\Longrightarrow$ *iii)* This is obvious.

ii) $\Longleftarrow$ *iii)* Let J be a nonzero ideal of A and $\overline{J}$ its closure. From *iii)* we have $I \cap \overline{J} \neq \{0\}$. Since I and $\overline{J}$ are both closed, and A is a C^*-algebra, we have $I \cdot \overline{J} = I \cap \overline{J}$. Now, it is clear that $I \cdot J = \{0\}$ if and only if $I \cdot \overline{J} = \{0\}$. Hence, we necessarily have $I \cdot J \neq \{0\}$, which implies $I \cap J \neq \{0\}$. $\square$

We now introduce the notion of an *essential* *-algebra. The class of essential *-algebras seems to be the appropriate class of *-algebras for which one can a define a multiplier algebra (as we shall see in Section 1.2).

Definition 1.1.8. A *-algebra A is said to be *essential* if A is an essential ideal of itself, i.e. if $aA \neq \{0\}$ for all $a \in A \setminus \{0\}$.

Any unital *-algebra is obviously essential. Also, it is easy to see that a semiprime *-algebra is essential. The converse is false, so that essential *-algebras form a more general class than that of semiprime *-algebras:

Example 1.1.9. Let $\mathbb{C}[X]$ be the polynomial algebra in one selfadjoint variable X. For any $n \geq 2$ the algebra $\mathbb{C}[X]/\langle X^n \rangle$ is essential, because it is unital, but it is not semiprime because $[X^{n-1}]\Big(\mathbb{C}[X]/\langle X^n \rangle\Big)[X^{n-1}] = \{0\}$.

1.2. *-Algebraic multiplier algebras

Every C^*-algebra can be embedded in a unital C^*-algebra in a "maximal" way. These maximal unitizations of C^*-algebras enjoy a number of useful properties and certain concrete realizations of these algebras are commonly referred to as multiplier algebras. The reader is referred to [**21**] for an account.

The definition of a multiplier algebra is quite standard in C^*-algebra theory, but this notion is in fact more general and applicable for more general types of rings and algebras. For example, in [**1**, Section 1.1] it is explained how multiplier algebras can be defined for semiprime algebras.

In this section we are going to generalize this notion to the context of essential *-algebras and derive their basic properties. We believe that essential *-algebras are the appropriate class of *-algebras for which one can define multiplier algebras, since the property $aA = \{0\} \Rightarrow a = 0$, which characterizes an essential *-algebra, is constantly used in proofs.

Multiplier algebras are many times defined via the so-called double centralizers (see for example [**1**]), but since we are only interested in algebras with an involution a slightly simpler and more convenient approach can be given, analogue to the Hilbert C^*-module approach to C^*-multiplier algebras (presented in [**21**, Section 2.3]). This is the approach we follow.

Definition 1.2.1. Let $\mathcal{C}$ be a subclass of *-algebras. A *-algebra $A \in \mathcal{C}$ is said to have a *maximal unitization in* $\mathcal{C}$ if there exists a unital *-algebra $B \in \mathcal{C}$ (called the *maximal unitization* of A) and a *-embedding $i : A \hookrightarrow B$ for which $i(A)$ is an essential ideal of B and such that for every other *-embedding j of A as an essential ideal of a unital *-algebra $C \in \mathcal{C}$, there is a unique *-homomorphism $\phi : C \to B$ such that

$$\begin{tikzcd} & B \\ A \arrow[ur, "i"] \arrow[r, "j"'] & C \arrow[u, dotted, "\phi"'] \end{tikzcd}$$

commutes.

Lemma 1.2.2. *In the above diagram the *-homomorphism ϕ is always injective (even if C was not unital).*

Proof: We have that $j(A) \cap \operatorname{Ker}\phi = \{0\}$, because if $j(a) \in j(A) \cap \operatorname{Ker}\phi$, then $0 = \phi(j(a)) = i(a)$ and hence $a = 0$ and therefore $j(a) = 0$. Hence, since $j(A)$ is an essential ideal of C, it follows from Proposition 1.1.6 i) that $\operatorname{Ker}\phi = \{0\}$. $\square$

For C^*-algebras, one might expect to replace "ideal" by "closed ideal", in Definition 1.2.1. This condition, however, follows automatically since $i(A)$ and $j(A)$ are automatically closed. Hence, this definition encompasses the usual definition of a maximal unitization for a C^*-algebra.

Definition 1.2.3. Let A be a *-algebra. By a *right A-module* we mean a vector space X together with a mapping $X \times A \to X$ satisfying the usual consistency conditions. An *A-linear mapping* $T : X \to Y$ between A-modules is a linear mapping between the underlying vector spaces such that $T(xa) = T(x)a$, for all $x \in X$ and $a \in A$. We will often use the notation Tx, instead of $T(x)$.

Every *-algebra A is canonically a right A-module, with the action of right multiplication. This is the example we will use thoroughly in what follows.

Let $\langle \cdot, \cdot \rangle_A : A \times A \to A$ be the function

$$\langle a, b \rangle_A := a^* b\,.$$

The function $\langle \cdot, \cdot \rangle_A$ is an A-linear form, in the sense that the following properties are satisfied:

a) $\langle a\,,\, \lambda_1 b_1 + \lambda_2 b_2 \rangle_A = \lambda_1 \langle a, b_1 \rangle_A + \lambda_2 \langle a, b_2 \rangle_A\,,$
b) $\langle \lambda_1 a_1 + \lambda_2 a_2\,,\, b \rangle_A = \overline{\lambda_1} \langle a_1, b \rangle_A + \overline{\lambda_2} \langle a_2, b \rangle_A\,,$
c) $\langle a, bc \rangle_A = \langle a, b \rangle_A\, c\,,$

d) $\langle ac, b\rangle_A = c^* \langle a, b\rangle_A$,
e) $\langle a, b\rangle_A^* = \langle b, a\rangle_A$,

for all $a, a_1, a_2, b, b_1, b_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$.

If the *-algebra A is essential we also have:

f) If $\langle a\, ,\, b\rangle_A = 0$ for all $b \in A$, then $a = 0$.

Definition 1.2.4. Let A be a *-algebra. A function $T : A \to A$ is called *adjointable* if there is a function $T^* : A \to A$ such that

$$\langle T(a), b\rangle_A = \langle a, T^*(b)\rangle_A \,,$$

for all $a, b \in A$.

Proposition 1.2.5. *If A is an essential *-algebra, then every adjointable map $T : A \to A$ is A-linear. Moreover, the adjoint T^* is unique and adjointable with $T^{**} = T$.*

Proof: Let T be an adjointable map in A and $x_1, x_2, y \in A$. We have

$$\begin{aligned} \langle T(\lambda_1 x_1 + \lambda_2 x_2)\, ,\, y\rangle_A &= \langle \lambda_1 x_1 + \lambda_2 x_2\, ,\, T^*(y)\rangle_A \\ &= \overline{\lambda_1}\, \langle x_1\, ,\, T^*(y)\rangle_A + \overline{\lambda_2}\, \langle x_2\, ,\, T^*(y)\rangle_A \\ &= \overline{\lambda_1}\, \langle T(x_1)\, ,\, y\rangle_A + \overline{\lambda_2}\, \langle T(x_2)\, ,\, y\rangle_A \\ &= \langle \lambda_1 T(x_1) + \lambda_2 T(x_2)\, ,\, y\rangle_A \,. \end{aligned}$$

Hence, we have $\langle T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) + \lambda_2 T(x_2)\, ,\, y\rangle_A = 0$. We can then conclude from f) that

$$T(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T(x_1) + \lambda_2 T(x_2) = 0\,,$$

i.e. T is a linear map.

Let us now check that T is A-linear. For any $x, y, a \in A$ we have

$$\begin{aligned} \langle T(xa)\, ,\, y\rangle_A &= \langle xa\, ,\, T^*(y)\rangle_A = a^* \langle x\, ,\, T^*(y)\rangle_A \\ &= a^* \langle T(x)\, ,\, y\rangle_A = \langle T(x)a\, ,\, y\rangle_A \,. \end{aligned}$$

Hence, we have $\langle T(xa) - T(x)a\, ,\, y\rangle_A = 0$. We can then conclude from f) that $T(xa) - T(x)a = 0$, i.e. T is A-linear.

Let us now prove the uniqueness of the adjoint T^*. Suppose there was a function $S : A \to A$ such that

$$\langle x\, ,\, T^*(y)\rangle_A = \langle x\, ,\, S(y)\rangle_A \,.$$

for all $x, y \in A$. Then, $\langle T^*(y) - S(y)\, ,\, x\rangle_A = 0$. We can then conclude from f) that $T^*(y) - S(y) = 0$, i.e. $T^* = S$.

It remains to prove that T^* is adjointable with $T^{**} = T$. This follows easily from the equality

$$\langle T^* x\, ,\, y\rangle_A = \langle y\, ,\, T^* x\rangle_A^* = \langle Ty\, ,\, x\rangle_A^* = \langle x, Ty\rangle_A \,.$$

$\square$

Definition 1.2.6. Let A be an essential *-algebra. The set of all adjointable maps on A is called the *multiplier algebra* of A and is denoted by $M(A)$.

The multiplier algebra is in fact a *-algebra, and the proof of this fact is standard.

Proposition 1.2.7. *Let A be an essential *-algebra. The multiplier algebra of A is a unital *-algebra with the sum and multiplication given by pointwise sum and composition (respectively), and the involution given by the adjoint.*

Proposition 1.2.8. *Let A be an essential *-algebra. There is a *-embedding $L : A \to M(A)$ of A as an essential ideal of $M(A)$, given by*

$$a \mapsto L_a$$

where $L_a : A \to A$ is the left multiplication by a, i.e. $L_a(b) := ab$.

Proof: It is easy to see that, for every $a \in A$, L_a is adjointable with adjoint L_{a^*}, thus the mapping L is well-defined. Also clear is the fact that L is a *-homomorphism. Let us prove that it is injective: suppose $L_a = 0$ for some $a \in A$. Then, for all $b \in A$ we have $ab = L_a b = 0$ and since A is essential this implies $a = 0$. Thus, L is injective.

It remains to prove that $L(A)$ is an essential ideal of $M(A)$. Let us begin by proving that it is an ideal. Let $T \in M(A)$. For every $a, b \in A$ we have

$$TL_a(b) = T(ab) = T(a)b = L_{Ta}(b)\,,$$

and also

$$\begin{aligned} L_aT(b) &= aT(b) = \langle a^*, T(b)\rangle \\ &= \langle T^*(a^*), b\rangle = (T^*(a^*))^*b \\ &= L_{(T^*a^*)^*}(b)\,. \end{aligned}$$

Hence we have

$$TL_a = L_{Ta} \qquad \text{and} \qquad L_aT = L_{(T^*a^*)^*}\,, \tag{1.1}$$

from which it follows easily that $L(A)$ is an ideal of $M(A)$.

Let us now prove that this ideal is essential. Let $T \in M(A)$ be such that $TL(A) = \{0\}$. Then, in particular, $TL_a = 0$ for all $a \in A$, but as we have seen before $TL_a = L_{Ta}$, and since L is injective we must have $Ta = 0$ for all $a \in A$, i.e $T = 0$. $\square$

Remark 1.2.9. According to Proposition 1.2.8, an essential *-algebra A is canonically embedded in its multiplier algebra $M(A)$. We will often make no distinction of notation between A and its embedded image in $M(A)$, i.e. we will often just write a to denote an element of A and to denote the element $L(a)$ of $M(A)$. No confusion will arise from this because the left equality in (1.1) simply means, in this notation, that $T \cdot a = T(a)$.

THEOREM 1.2.10. *Let A be an essential *-algebra and $L : A \to M(A)$ the canonical *-embedding of A in $M(A)$. If $j : A \to C$ is a *-embedding of A as an ideal of a *-algebra C, then there exists a unique *-homomorphism $\phi : C \to M(A)$ such that the following diagram commutes*

$$\begin{tikzcd} & M(A) \\ A \arrow[ur, "L"] \arrow[r, "j"'] & C \arrow[u, dotted, "\phi"'] \end{tikzcd}$$

Moreover, if $j(A)$ is essential then ϕ is injective.

Proof: For simplicity of notation let us assume, without any loss of generality, that A itself is an ideal of a *-algebra C, so that we avoid any reference to j (or its inverse). Let $\phi : C \to M(A)$ be the function defined by

$$\phi(c) : A \to A$$
$$\phi(c)a := ca\,,$$

for every $c \in C$. It is a straightforward computation to check that $\phi(c) \in M(A)$ and that ϕ itself is a *-homomorphism. It is also easy to see that $\phi(a) = L_a$, for every $a \in A$. Hence, $\phi \circ j = L$. Let us now prove the uniqueness of ϕ relatively to this property. Suppose $\widetilde{\phi} : C \to M(A)$ is another *-homomorphism such that $\widetilde{\phi} \circ j = L$. Then, for all $c \in C$ and $a \in A$ we have

$$\begin{aligned} \big(\widetilde{\phi}(c) - \phi(c)\big)L_a &= \widetilde{\phi}(c)L_a - \phi(c)L_a \\ &= \widetilde{\phi}(c)\widetilde{\phi}(a) - \phi(c)\phi(a) \\ &= \widetilde{\phi}(ca) - \phi(ca) \\ &= L_{ca} - L_{ca} \\ &= 0\,. \end{aligned}$$

Since $L(A)$ is an essential ideal of $M(A)$ it follows that $\widetilde{\phi}(c) = \phi(c)$ for all $c \in C$, i.e. $\widetilde{\phi} = \phi$.

The last claim of the theorem, concerning injectivity of ϕ, was proven in Lemma 1.2.2. $\square$

COROLLARY 1.2.11. *The multiplier algebra $M(A)$ is a maximal unitization of A in the class of: essential *-algebras, semiprime *-algebras and C^*-algebras.*

Proof: By Theorem 1.2.10 we only need to check that if A is an essential *-algebra (respectively, semiprime *-algebra or C^*-algebra), then the multiplier algebra has the same property.

Suppose A is an essential *-algebra. Let $T \in M(A)$ be such that $TM(A) = \{0\}$. Then, by the embedding of A in $M(A)$ we have $Ta = 0$ for all $a \in A$, i.e. $T = 0$. Hence, $M(A)$ is also an essential *-algebra.

Suppose A is a semiprime *-algebra. Let $T \in M(A)$ be such that $TM(A)T = \{0\}$. Then, we also have that $TL_aM(A)TL_a = \{0\}$ for any $a \in A$, and therefore $L_{T(a)}M(A)L_{T(a)} = \{0\}$. Thus, in particular, $L_{T(a)}L(A)L_{T(a)} = \{0\}$, and since L

is injective this means that $T(a)AT(a) = \{0\}$. Since A is semiprime we conclude that $T(a) = 0$, and therefore $T = 0$. Hence, $M(A)$ is semiprime.

It is well-known that $M(A)$ is a C^*-algebra when A is a C^*-algebra. $\square$

EXAMPLE 1.2.12. If X is a locally compact space and $C_c(X)$ is the *-algebra of compactly supported continuous functions on X, then the multiplier algebra $M(C_c(X))$ is the *-algebra $C(X)$ of continuous (possibly unbounded) functions on X.

An important feature of C^*-multiplier algebras is that a nondegenerate *-representation of A extends uniquely to $M(A)$. This result does not hold in general for essential *-algebras. Nevertheless we can still extend a nondegenerate *-representation of A to a unique pre-*-representation of $M(A)$:

THEOREM 1.2.13. *Let A be an essential *-algebra, $\pi : A \to B(\mathscr{H})$ a nondegenerate *-representation of A on a Hilbert space $\mathscr{H}$ and $\mathscr{V} \subseteq \mathscr{H}$ the dense subspace*

$$\mathscr{V} := \pi(A)\mathscr{H} = \operatorname{span}\{\pi(a)\xi : a \in \mathcal{A}, \xi \in \mathscr{H}\}.$$

Then there is a unique pre-*-representation

$$\widetilde{\pi} : M(A) \to L(\mathscr{V})$$

such that $\widetilde{\pi}(a) = \pi(a)|_{\mathscr{V}}$ for every $a \in A$.

Proof: We define the pre-*-representation $\widetilde{\pi} : M(A) \to L(\mathscr{V})$ by

$$\widetilde{\pi}(T)\ \Big[\sum_{i=1}^{n} \pi(a_i)\xi_i\Big] := \sum_{i=1}^{n} \pi(Ta_i)\xi_i\,,$$

for $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $\xi_1, \ldots, \xi_n \in \mathscr{H}$. Let us first check that $\widetilde{\pi}$ is well-defined. Suppose $\sum_{i=1}^{n} \pi(a_i)\xi_i = \sum_{j=1}^{m} \pi(b_j)\eta_j$. Then, for every $z \in A$ we have

$$\begin{aligned}
\pi(z)\Big(\sum_{i=1}^{n} \pi(Ta_i)\xi_i - \sum_{j=1}^{m} \pi(Tb_i)\eta_j\Big) &= \sum_{i=1}^{n} \pi(zTa_i)\xi_i - \sum_{j=1}^{m} \pi(zTb_i)\eta_j \\
&= \pi(zT)\Big(\sum_{i=1}^{n} \pi(a_i)\xi_i - \sum_{j=1}^{m} \pi(b_i)\eta_j\Big) \\
&= 0\,.
\end{aligned}$$

Since the *-representation π is nondegenerate we necessarily have

$$\sum_{i=1}^{n} \pi(Ta_i)\xi_i - \sum_{j=1}^{m} \pi(Tb_i)\eta_j = 0\,,$$

which means that $\widetilde{\pi}(T)$ is well-defined.

Let us now check that $\widetilde{\pi}(T) \in L(\mathscr{V})$, i.e. that $\widetilde{\pi}(T)$ is indeed an adjointable operator in $\mathscr{V}$. We will in fact prove that $\widetilde{\pi}(T)^* = \widetilde{\pi}(T^*)$, which follows from the following equality

$$\begin{aligned}
\left\langle \widetilde{\pi}(T) \sum_{i=1}^{n} \pi(a_i)\xi_i \, , \, \sum_{j=1}^{m} \pi(b_i)\eta_j \right\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \pi(Ta_i)\xi_i \, , \, \pi(b_j)\eta_j \rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \xi_i \, , \, \pi(a_i^* T^*)\pi(b_j)\eta_j \rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \xi_i \, , \, \pi(a_i^* T^* b_j)\eta_j \rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \langle \pi(a_i)\xi_i \, , \, \pi(T^* b_j)\eta_j \rangle \\
&= \left\langle \sum_{i=1}^{n} \pi(a_i)\xi_i \, , \, \widetilde{\pi}(T^*) \sum_{j=1}^{m} \pi(b_i)\eta_j \right\rangle .
\end{aligned}$$

It is straightforward to see that $\widetilde{\pi}$ is linear, multiplicative and, as we have seen, $\widetilde{\pi}(T^*) = \widetilde{\pi}(T)^*$, hence $\widetilde{\pi}$ is a pre-*-representation of $M(A)$ on $\mathscr{V}$.

It is also clear that, for any $a \in A$, $\widetilde{\pi}(a)$ is just $\pi(a)$ restricted to $\mathscr{V}$, because of the equality

$$\widetilde{\pi}(a) \sum_{i=1}^{n} \pi(a_i)\xi_i = \sum_{i=1}^{n} \pi(aa_i)\xi_i = \pi(a) \sum_{i=1}^{n} \pi(a_i)\xi_i \, .$$

Let us now prove the uniqueness of $\widetilde{\pi}$. Suppose $\phi : M(A) \to L(\mathscr{V})$ is a pre-*-representation such that $\phi(a) = \pi(a)|_{\mathscr{V}}$. Then, for every $z \in A$ and $v \in \mathscr{V}$ we have

$$\begin{aligned}
\pi(z)(\phi(T)v - \widetilde{\pi}(T)v) &= \pi(z)\phi(T)v - \pi(z)\widetilde{\pi}(T)v \\
&= \phi(z)\phi(T)v - \widetilde{\pi}(z)\widetilde{\pi}(T)v \\
&= \phi(zT)v - \widetilde{\pi}(zT)v \\
&= \pi(zT)v - \pi(zT)v \\
&= 0 \, .
\end{aligned}$$

Since the *-representation π is nondegenerate, we necessarily have

$$\phi(T)v - \widetilde{\pi}(T)v = 0 \, ,$$

which means that $\phi(T) = \widetilde{\pi}(T)$, i.e. $\phi = \widetilde{\pi}$. □

REMARK 1.2.14. Theorem 1.2.13 can be interpreted in the following way: every nondegenerate *-representation $\pi : A \to B(\mathscr{H})$ can be extended to $M(A)$ by possibly unbounded operators, defined on the dense subspace $\pi(A)\mathscr{H}$.

DEFINITION 1.2.15. Let A be an essential *-algebra. We will denote by $M_B(A)$ the subset of $M(A)$ consisting of all the elements $T \in M(A)$ such that $\widetilde{\pi}(T) \in B(\mathscr{V})$ for all nondegenerate *-representations $\pi : A \to B(\mathscr{H})$, where $\mathscr{V} := \pi(A)\mathscr{H}$ and $\widetilde{\pi}$ is the unique pre-*-representation extending π as in Proposition 1.2.13.

As stated in the next result, $M_B(A)$ is a *-subalgebra of $M(A)$. The advantage of working with $M_B(A)$ over $M(A)$ is that nondegenerate *-representations of A always extend to *-representations of $M_B(A)$. Easy examples of elements of $M_B(A)$ that might not belong to A are the projections and unitaries of $M(A)$.

PROPOSITION 1.2.16. *Let A be an essential *-algebra. The set $M_B(A)$ is a *-subalgebra of $M(A)$ containing A. Moreover, if $\pi : A \to B(\mathscr{H})$ is a nondegenerate *-representation of A, then there exists a unique *-representation of $M_B(A)$ on $\mathscr{H}$ that extends π.*

Proof: Let $T, S \in M_B(A)$. Let $\pi : A \to B(\mathscr{H})$ be any nondegenerate *-representation of A and $\widetilde{\pi}$ its extension to $L(\mathscr{V})$, in the sense of Theorem 1.2.13, where $\mathscr{V} := \pi(A)\mathscr{H}$. By definition, $\widetilde{\pi}(T), \widetilde{\pi}(S) \in B(\mathscr{V})$, and therefore $\widetilde{\pi}(T + S), \widetilde{\pi}(TS), \widetilde{\pi}(T^*) \in B(\mathscr{V})$, since $B(\mathscr{V})$ is a *-algebra. Hence, $M_B(A)$ is a *-subalgebra of $M(A)$. Moreover, $A \subseteq M_B(A)$ since $\widetilde{\pi}(a) = \pi(a)|_{\mathscr{V}} \in B(\mathscr{V})$.

Let us now prove the last claim of this proposition. Let $\pi : A \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi} : M(A) \to L(\mathscr{V})$ its extension as in Theorem 1.2.13. Then we obtain by restriction a pre-*-representation $\widetilde{\pi} : M_B(A) \to L(\mathscr{V})$. By definition of $M_B(A)$ we actually have $\widetilde{\pi}(M_B(A)) \subseteq B(\mathscr{V})$. Hence $\widetilde{\pi}$ gives rise to a *-representation $\widetilde{\pi} : M_B(A) \to B(\mathscr{H})$, since $\mathscr{V}$ is dense in $\mathscr{H}$.

Let us now prove the uniqueness claim. Suppose φ is another representation of $M_B(A)$ that extends π. For $T \in M_B(A)$, $a \in A$ and $\xi \in \mathscr{H}$ we have

$$\begin{aligned} \varphi(T)\,\pi(a)\xi &= \varphi(T)\,\varphi(a)\xi = \varphi(Ta)\xi \\ &= \pi(Ta)\xi = \widetilde{\pi}(T)\,\pi(a)\xi\,. \end{aligned}$$

By linearity and density it follows that $\varphi(T) = \widetilde{\pi}(T)$, i.e. $\varphi = \widetilde{\pi}$. $\square$

The above result is a generalization of the well-known result for C^*-algebras which states that any nondegenerate *-representation can be extended to the multiplier algebra (see for example [**21**, Corollary 2.51]), because $M(A) = M_B(A)$ for any C^*-algebra A.

EXAMPLE 1.2.17. If X is a locally compact space then $M_B(C_c(X))$ is the *-algebra $C_b(X)$ of bounded continuous functions on X.

1.3. Hecke algebras

We start by establishing some notation and conventions concerning left coset spaces and double coset spaces and we prove two results which will be useful later on.

Let G be a group, B, C subgroups of G and $e \in G$ the identity element. The *double coset space* $B\backslash G/C$ is the set

$$B\backslash G/C := \{BgC \subseteq G : g \in G\}\,. \tag{1.2}$$

It is easy to see that the sets of the form BgC are either equal or disjoint, or in other words, we have an equivalence relation defined in G whose equivalence classes are precisely the sets BgC.

The *left coset space* G/C is the set

$$G/C := \{e\}\backslash G/C = \{gC \subseteq G : g \in G\} . \tag{1.3}$$

Given an element $g \in G$ and a double coset space $B\backslash G/C$ (which can in particular be a left coset space by taking $B = \{e\}$) we will denote by $[g]$ the double coset BgC. Thus, $[g]$ denotes the whole equivalence class for which $g \in G$ is a representative.

If A is a subset of G we define the double coset space $B\backslash A/C$ as the set of double cosets in $B\backslash G/C$ which have a representative in A, i.e.

$$B\backslash A/C := \{BaC \subseteq G : a \in A\} . \tag{1.4}$$

PROPOSITION 1.3.1. *Let A, B and C be subgroups of a group G. If $C \subseteq A$, then the following map is a bijective correspondence between the double coset spaces:*

$$\begin{aligned} B\backslash A/C &\longrightarrow (B \cap A)\backslash A/C \\ [a] &\mapsto [a] . \end{aligned} \tag{1.5}$$

Similarly, if $B \subseteq A$, then the following map is a bijective correspondence:

$$\begin{aligned} B\backslash A/C &\longrightarrow B\backslash A/(A \cap C) \\ [a] &\mapsto [a] . \end{aligned} \tag{1.6}$$

Proof: We first need to show that the map (1.5) is well defined, i.e. if $Ba_1C = Ba_2C$, for some $a_1, a_2 \in A$, then $(B \cap A)a_1C = (B \cap A)a_2C$. If $Ba_1C = Ba_2C$ then there exist $b \in B$ and $c \in C$ such that $a_1 = ba_2c$, from which it follows that $b = a_1c^{-1}a_2^{-1}$. Since A is a subgroup and $C \subseteq A$, it follows readily that $b \in B \cap A$, and therefore $a_1 \in (B \cap A)a_2C$, i.e. $(B \cap A)a_1C = (B \cap A)a_2C$.

The map defined in (1.5) is clearly surjective. It is also injective because if $(B \cap A)a_1C = (B \cap A)a_2C$, then clearly $Ba_1C = Ba_2C$.

A completely analogous argument shows that map defined in (1.6) is a bijection. $\square$

Suppose a group G acts (on the right) on a set X and let $x \in X$. We will henceforward denote by $\mathcal{S}_x$ the stabilizer of the point x, i.e.

$$\mathcal{S}_x := \{g \in G : xg = x\} . \tag{1.7}$$

Given a subset $Z \subseteq X$ and a subgroup $H \subseteq G$ we denote by Z/H the set of H-orbits which have representatives in Z, i.e.

$$Z/H := \{zH : z \in Z\} .$$

Suppose now that $H, K \subseteq G$ are subgroups and let $x \in X$ be a point. The following result establishes a correspondence between the set of H-orbits $(xK)/H$ and the double coset space $\mathcal{S}_x\backslash K/H$:

PROPOSITION 1.3.2. *Let G be a group which acts (on the right) on a set X. Let $x \in X$ be a point and $H, K \subseteq G$ be subgroups. We have a bijection*

$$(xK)/H \longrightarrow \mathcal{S}_x\backslash K/H ,$$

given by $xgH \mapsto \mathcal{S}_x gH$*, where* $g \in K$*.*

Proof: Let us first prove that the map $xgH \mapsto \mathcal{S}_x gH$ is well defined, i.e. if $xg_1H = xg_2H$, then $\mathcal{S}_x g_1 H = \mathcal{S}_x g_2 H$. If $xg_1H = xg_2H$, then there exists $h \in h$ such that $xg_1 = xg_2h$, which implies that $x = xg_2hg_1^{-1}$, from which it follows that $g_2hg_1^{-1} \in \mathcal{S}_x$. Thus we see that

$$\mathcal{S}_x g_1 H \quad = \quad \mathcal{S}_x g_2 h g_1^{-1} g_1 H \quad = \quad \mathcal{S}_x g_2 H\,.$$

We conclude that the map is well-defined. The map is obviously surjective. It is also injective because if $\mathcal{S}_x g_1 H = \mathcal{S}_x g_2 H$, then there exists $r \in \mathcal{S}_x$ and $h \in H$ such that $g_1 = rg_2h$, from which it follows that $xg_1H = xrg_2hH = xg_2H$. □

We will mostly follow **[13]** and **[10]** in what regards Hecke pairs and Hecke algebras and refer to these references for more details.

We start by establishing some notation which will be useful later on. Given a group G, a subgroup $\Gamma \subseteq G$ and $g \in G$, we will denote by Γ^g the subgroup

$$\Gamma^g := \Gamma \cap g\Gamma g^{-1}\,. \tag{1.8}$$

We now recall the definition of a Hecke pair:

Definition 1.3.3. Let G be a group and Γ a subgroup. The pair (G, Γ) is called a *Hecke pair* if every double coset $\Gamma g \Gamma$ is the union of finitely many right (and left) cosets. In this case, Γ is also called a *Hecke subgroup* of G.

Given a Hecke pair (G, Γ) we will denote by L and R, respectively, the left and right coset counting functions, i.e.

$$L(g) := |\Gamma g \Gamma / \Gamma| = [\Gamma : \Gamma^g] < \infty \tag{1.9}$$

$$R(g) := |\Gamma \backslash \Gamma g \Gamma| = [\Gamma : \Gamma^{g^{-1}}] < \infty\,. \tag{1.10}$$

We recall that L and R are Γ-biinvariant functions which satisfy $L(g) = R(g^{-1})$ for all $g \in G$. Moreover, the function $\Delta : G \to \mathbb{Q}^+$ given by

$$\Delta(g) := \frac{L(g)}{R(g)}\,, \tag{1.11}$$

is a group homomorphism (**[23]**, Proposition 2.1]), usually called the *modular function* of (G, Γ).

Definition 1.3.4. Given a Hecke pair (G, Γ), the *Hecke algebra* $\mathcal{H}(G, \Gamma)$ is the *-algebra of finitely supported $\mathbb{C}$-valued functions on the double coset space $\Gamma \backslash G / \Gamma$ with the product and involution defined by

$$(f_1 * f_2)(\Gamma g \Gamma) := \sum_{h\Gamma \in G/\Gamma} f_1(\Gamma h \Gamma) f_2(\Gamma h^{-1} g \Gamma)\,, \tag{1.12}$$

$$f^*(\Gamma g \Gamma) := \Delta(g^{-1}) \overline{f(\Gamma g^{-1} \Gamma)}\,. \tag{1.13}$$

Equivalently, we can define $\mathcal{H}(G,\Gamma)$ as the *-algebra of finitely supported Γ-left invariant functions $f: G/\Gamma \to \mathbb{C}$ with the product and involution operations given by

$$(f_1 * f_2)(g\Gamma) := \sum_{h\Gamma \in G/\Gamma} f_1(h\Gamma) f_2(h^{-1}g\Gamma)\,, \tag{1.14}$$

$$f^*(g\Gamma) := \Delta(g^{-1})\overline{f(g^{-1}\Gamma)}\,. \tag{1.15}$$

REMARK 1.3.5. Some authors, including Krieg [**13**], do not include the factor Δ in the involution. Here we adopt the convention of Kaliszewski, Landstad and Quigg [**10**] in doing so, as it gives rise to a more natural L^1-norm. We note, nevertheless, that there is no loss (or gain) in doing so, because these two different involutions give rise to *-isomorphic Hecke algebras.

The Hecke algebra has a natural basis, as a vector space, given by the characteristic functions of double cosets. We will henceforward identify a characteristic function of a double coset $1_{\Gamma g \Gamma}$ with the double coset $\Gamma g \Gamma$ itself.

The way in which a product of two double cosets is represented as sum of double cosets is well understood:

PROPOSITION 1.3.6. *Let (G,Γ) be a Hecke pair and $g, h \in G$. We have that*

$$\Gamma g\Gamma * \Gamma h \Gamma \;=\; \sum_{[v]\in \Gamma h\Gamma/\Gamma} \frac{L(g)}{L(gv)} \Gamma g v \Gamma \;=\; \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma h\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)} \Gamma u^{-1} v\Gamma\,.$$

Proof: The proof of the first equality can be found in [**10**, page 660]. Let us now prove the second equality. We have

$$\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma h\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)} \Gamma u^{-1} v\Gamma \;=\; \sum_{[\gamma]\in\Gamma^{g^{-1}}/\Gamma}\ \sum_{[v]\in\Gamma h\Gamma/\Gamma} \frac{\Delta(g)}{L(g\gamma^{-1}v)} \Gamma g\gamma^{-1} v\Gamma\,.$$

For any $\gamma \in \Gamma$ the mapping $[v] \mapsto [\gamma v]$ is a bijection of $\Gamma h\Gamma/\Gamma$. Hence

$$\begin{aligned} &= \sum_{[\gamma]\in\Gamma^{g^{-1}}/\Gamma}\ \sum_{[v]\in\Gamma h\Gamma/\Gamma} \frac{\Delta(g)}{L(gv)} \Gamma g v\Gamma = \sum_{[v]\in\Gamma h\Gamma/\Gamma} \frac{\Delta(g)L(g^{-1})}{L(gv)} \Gamma g v\Gamma \\ &= \sum_{[v]\in\Gamma h\Gamma/\Gamma} \frac{L(g)}{L(gv)} \Gamma g v\Gamma\,. \end{aligned}$$

This proves the second equality. □

As it is known, group algebras have two canonical C^*-completions, the reduced group C^*-algebra $C^*_r(G)$ and the full group C^*-algebra $C^*(G)$. For Hecke algebras the situation becomes more complicated, there being essentially four canonical C^*-completions. We will briefly review these completions in this subsection, but first we need to recall the definitions and basic facts about regular representations of Hecke algebras and L^1-norms.

DEFINITION 1.3.7. Let (G, Γ) be a Hecke pair. The mapping $\rho : \mathcal{H}(G,\Gamma) \to B(\ell^2(G/\Gamma))$ defined, for $f \in \mathcal{H}(G,\Gamma)$, $\xi \in \ell^2(G/\Gamma)$ and $g\Gamma \in G/\Gamma$, by

$$(\rho(f)\xi)(g\Gamma) := \sum_{[h] \in G/\Gamma} \Delta(h)^{\frac{1}{2}} f(\Gamma h \Gamma) \xi(gh\Gamma)\,, \tag{1.16}$$

is called the *right regular representation* of $\mathcal{H}(G,\Gamma)$.

It can be checked that ρ does define a *-representation of $\mathcal{H}(G,\Gamma)$. For the canonical vectors $\delta_{r\Gamma} \in \ell^2(G/\Gamma)$, expression (1.16) becomes:

$$\rho(f)\delta_{r\Gamma} = \sum_{[g] \in G/\Gamma} \Delta(g^{-1}r)^{\frac{1}{2}} f(\Gamma g^{-1} r \Gamma) \delta_{g\Gamma}\,, \tag{1.17}$$

and furthermore for f of the form $f := \Gamma d \Gamma$ we obtain:

$$\rho(\Gamma d \Gamma)\delta_{r\Gamma} = \sum_{t\Gamma \subseteq \Gamma d^{-1}\Gamma} \Delta(d)^{\frac{1}{2}} \delta_{rt\Gamma} \;=\; \Delta(d)^{\frac{1}{2}} \delta_{r\Gamma d^{-1}\Gamma}\,. \tag{1.18}$$

It can be easily checked, applying (1.17) to the vector δ_Γ for example, that ρ always defines a faithful *-representation.

One could in a similar fashion define a left regular representation of $\mathcal{H}(G,\Gamma)$, but in this work, however, it is the right regular representation the one that will play a central role.

We now recall the definition of the L^1-norm in a Hecke algebra (from [**10**]):

DEFINITION 1.3.8. The L^1*-norm* on $\mathcal{H}(G,\Gamma)$, denoted $\|\cdot\|_{L^1}$, is given by

$$\|f\|_{L^1} := \sum_{\Gamma g \Gamma \in \Gamma\backslash G/\Gamma} |f(\Gamma g \Gamma)|\, L(g) = \sum_{g\Gamma \in G/\Gamma} |f(\Gamma g \Gamma)|\,. \tag{1.19}$$

We will denote by $L^1(G,\Gamma)$ the completion of $\mathcal{H}(G,\Gamma)$ under this norm, which is a Banach *-algebra.

The fact that the L^1-norm is *-preserving can be easily seen on the basis elements $\Gamma g \Gamma$ of $\mathcal{H}(G,\Gamma)$ and then extended by conjugate-linearity for all elements of the Hecke algebra:

$$\begin{aligned} \|(\Gamma g \Gamma)^*\|_{L^1} &= \Delta(g)\|\Gamma g^{-1}\Gamma\|_{L^1} = \Delta(g) L(g^{-1}) = \frac{L(g)}{R(g)} R(g) \\ &= L(g) = \|\Gamma g \Gamma\|_{L^1}\,. \end{aligned}$$

There are several canonical C^*-completions of $\mathcal{H}(G,\Gamma)$ ([**10**], [**23**]) These are:

- $C^*_r(G,\Gamma)$ - Called the *reduced Hecke C^*-algebra*, it is the completion of $\mathcal{H}(G,\Gamma)$ under the C^*-norm arising from the right regular representation.

- $pC^*(\overline{G})p$ - The corner of the full group C^*-algebra $C^*(\overline{G})$ of the Schlichting completion $(\overline{G}, \overline{\Gamma})$ of the pair (G, Γ), by the projection $p := 1_{\overline{\Gamma}}$. We will not describe this construction here since it is well documented in the literature (see [**23**] and [**10**], for example) and because we will not make use of this C^*-completion in this work.
- $C^*(L^1(G, \Gamma))$ - The enveloping C^*-algebra of $L^1(G, \Gamma)$.
- $C^*(G, \Gamma)$ - The enveloping C^*-algebra (if it exists!) of $\mathcal{H}(G, \Gamma)$. When it exists, it is usually called the *full Hecke C^*-algebra*.

The various C^*-completions of $\mathcal{H}(G, \Gamma)$ are related in the following way, through canonical surjective maps:

$$C^*(G, \Gamma) \dashrightarrow C^*(L^1(G, \Gamma)) \longrightarrow pC^*(\overline{G})p \longrightarrow C^*_r(G, \Gamma)\,.$$

As was pointed out by Hall in [**8**, Proposition 2.21], the full Hecke C^*-algebra $C^*(G, \Gamma)$ does not have to exist in general, with the Hecke algebra of the pair $(SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p))$ being one such example, where p is a prime number and $\mathbb{Q}_p$, $\mathbb{Z}_p$ denote respectively the field of p-adic numbers and the ring of p-adic integers.

1.4. Fell bundles over discrete groupoids

Let X be a discrete groupoid. We will denote by X^0 the unit space of X and by $\mathbf{s}$ and $\mathbf{r}$ the source and range functions $X \to X^0$, respectively.

We will essentially follow [**14**] when it comes to Fell bundles over groupoids. All the groupoids in this work are assumed to be discrete, so that the theory of Fell bundles admits a few simplifications. Basically a *Fell bundle* over a discrete groupoid X consists of:

- a space $\mathcal{A}$ together with a surjective map $p : \mathcal{A} \to X$, such that each *fiber* $\mathcal{A}_x := p^{-1}(x)$ is a Banach space, for every $x \in X$;
- a multiplication operation between fibers over composable elements of the groupoid, which we suggestively write as $\mathcal{A}_x \cdot \mathcal{A}_y \subseteq \mathcal{A}_{xy}$;
- an involution $a \mapsto a^*$ which takes $\mathcal{A}_x$ onto $\mathcal{A}_{x^{-1}}$.

These operations and norms satisfy some consistency properties which we now describe (see [**14**, Section 2]):

- The multiplication operation $\mathcal{A}_x \times \mathcal{A}_y \longrightarrow \mathcal{A}_{xy}$ is bilinear, for all composable elements of the groupoid $x, y \in X$.
- Multiplication is associative whenever it is defined.
- $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$ where multiplication is defined.
- The involution map $\mathcal{A}_x \longrightarrow \mathcal{A}_{x^{-1}}$ is conjugate linear, and satisfies $a^{**} = a$ and $(ab)^* = b^*a^*$, for every $a, b \in \mathcal{A}$ where multiplication is defined.
- $\|a^*a\| = \|a\|^2$ for any $a \in \mathcal{A}$.
- $a^*a \geq 0$ for all $a \in \mathcal{A}$.

As it is well-known, it follows from the above conditions (without the last one) that each fiber over a unit element is naturally a C^*-algebra. This is why the last condition regarding positivity makes sense and that is how it should be interpreted.

Standing Assumption 1.4.1. Given a Fell bundle $\mathcal{A}$ over a discrete groupoid X we will always assume that the fibers over units are non-trivial, i.e. $\mathcal{A}_u \neq \{0\}$

for all $u \in X^0$.

Assumption 1.4.1 is not very restrictive. In fact, removing from the groupoid X all the units $u \in X^0$ for which $\mathcal{A}_u = \{0\}$ and also all the elements $x \in X$ such that $\mathbf{s}(x)$ or $\mathbf{r}(x)$ is u, we obtain a subgroupoid Y for which the assumption holds (relatively to the restriction $\mathcal{A}|_Y$ of $\mathcal{A}$ to Y). Moreover, and this is the important fact, the algebras of finitely supported sections (see Definition 1.4.4) are canonically isomorphic, i.e. $C_c(\mathcal{A}|_Y) \cong C_c(\mathcal{A})$.

The reason for us to follow Assumption 1.4.1 is because it will make our theory slightly simpler. Since we are interested mostly in algebras of sections, this assumption does not reduce the generality of the work in any way, as we observed in the previous paragraph.

Definition 1.4.2. Let $\mathcal{A}$ be a Fell bundle over a discrete groupoid X. An *automorphism* of $\mathcal{A}$ is a bijective map $\beta : \mathcal{A} \to \mathcal{A}$ which preserves the bundle structure, i.e. such that

i) β takes any fiber onto another fiber;
ii) β takes fibers over composable elements of X to fibers over composable elements;
iii) As a map between (two) fibers, β is a linear map;
iv) $\beta(a \cdot b) = \beta(a) \cdot \beta(b)$, whenever multiplication is defined;
v) $\beta(a^*) = \beta(a)^*$.

The set of all automorphisms of $\mathcal{A}$ forms a group under composition and will be denoted by $Aut(\mathcal{A})$.

It follows easily from *i)* and *ii)* above that every automorphism β of $\mathcal{A}$ entails a groupoid automorphism β_0 of X such that $\beta_0\big(p(a)\big) = p(\beta(a))$. We also note that, by being a groupoid automorphism, β_0 takes units into units.

Remark 1.4.3. The restricted map $\beta : \mathcal{A}_x \to \mathcal{A}_{\beta_0(x)}$ is an isometric linear map. Linearity was required in condition *iii)*, but the fact that the map is an isometry follows from the other axioms. To see this we note that

$$\|\beta(a)\| = \|\beta(a)^*\beta(a)\|^{\frac{1}{2}} = \|\beta(a^*a)\|^{\frac{1}{2}} .$$

Now $a^*a \in \mathcal{A}_{\mathbf{s}(x)}$ and $\mathbf{s}(x) \in X^0$. Thus, we also have $\beta_0(\mathbf{s}(x)) \in X^0$ and therefore both $\mathcal{A}_{\mathbf{s}(x)}$ and $\mathcal{A}_{\beta_0(\mathbf{s}(x))}$ are C^*-algebras. It follows from *iii)*, *iv)* and *v)* that the restricted map $\beta : \mathcal{A}_{\mathbf{s}(x)} \to \mathcal{A}_{\beta_0(\mathbf{s}(x))}$ is a C^*-isomorphism and is therefore isometric. Hence we have

$$\|\beta(a)\| = \|\beta(a^*a)\|^{\frac{1}{2}} = \|a^*a\|^{\frac{1}{2}} = \|a\| ,$$

which shows that $\beta : \mathcal{A}_x \to \mathcal{A}_{\beta_0(x)}$ is an isometry.

Definition 1.4.4. Given a Fell bundle $\mathcal{A}$ over a discrete groupoid X its *-*algebra of finitely supported sections* $C_c(\mathcal{A})$ is the space of functions $f : X \longrightarrow \mathcal{A}$ such that $f(x) \in \mathcal{A}_x$ for every $x \in X$ and $f(x) = 0$ for all but finitely many points $x \in X$.

The *-algebra of finitely supported sections $C_c(\mathcal{A})$ is indeed a *-algebra for the operations of pointwise sum and multiplication by scalars, and with multiplication and involution given by:

$$\begin{aligned} f * g(z) &= \sum_{\substack{x,y\in X\\ xy=z}} f(x)g(y)\,, \\ f^*(z) &= (f(z^{-1}))^*\,. \end{aligned}$$

The following notation will be used throughout the rest of this work: for $x \in X$ and $a \in \mathcal{A}_x$ the symbol a_x will always denote the element of $C_c(\mathcal{A})$ such that

$$a_x(y) := \begin{cases} a\,, & \text{if } y = x \\ 0, & \text{otherwise}\,. \end{cases} \tag{1.20}$$

According to the notation above we can then write any $f \in C_c(\mathcal{A})$ uniquely as a sum of the form

$$f = \sum_{x\in X} (f(x))_x\,. \tag{1.21}$$

For the elements of the form a_x in $C_c(\mathcal{A})$ the multiplication and involution operations are determined by:

$$\begin{aligned} a_x \cdot b_y &= \begin{cases} (ab)_{xy}\,, & \text{if } \mathbf{s}(x) = \mathbf{r}(y) \\ 0, & \text{otherwise}\,, \end{cases} \\ (a_x)^* &= (a^*)_{x^{-1}}\,, \end{aligned}$$

where $x, y \in X$ and $a \in \mathcal{A}_x$, $b \in \mathcal{A}_y$.

When a groupoid X is just a set, the fibers of a Fell bundle $\mathcal{A}$ over X are C^*-algebras. In this case we will frequently use the following terminology, which is standard in the literature, in order to stress the fact that the underlying groupoid is nothing but a set:

Definition 1.4.5. When a groupoid X is just a set, a Fell bundle $\mathcal{A}$ over X will be referred to as a *C^*-bundle* over X.

Given a Fell bundle $\mathcal{A}$ over a groupoid X we will denote by $\mathcal{A}^0$ the restricted bundle $\mathcal{A}|_{X^0}$ over the unit space X^0. Naturally, $\mathcal{A}^0$ is a C^*-bundle over X^0.

We will now briefly recall how the full and the reduced cross sectional algebras of a Fell bundle $\mathcal{A}$ over a groupoid X are defined.

Definition 1.4.6. The *full cross sectional algebra* of $\mathcal{A}$, denoted $C^*(\mathcal{A})$, is defined as the enveloping C^*-algebra of $C_c(\mathcal{A})$. If the groupoid X is just a set, in which case $\mathcal{A}$ is a C^*-bundle, we will use the notation $C_0(\mathcal{A})$ instead of $C^*(\mathcal{A})$.

The full cross sectional algebra $C^*(\mathcal{A})$ is known to exist always (see for example [**5**, Proposition 2.1]).

We now recall, from **[14]**, how the *reduced cross sectional algebra* $C_r^*(\mathcal{A})$ is defined. We see $C_c(\mathcal{A})$ as a pre-Hilbert $C_0(\mathcal{A}^0)$-module, where the inner product is defined by

$$\langle f_1, f_2\rangle_{C_c(\mathcal{A}^0)} := (f_1^* \cdot f_2)|_{X^0}\,, \qquad\qquad f_1, f_2 \in C_c(\mathcal{A})\,.$$

Its completion is a full Hilbert $C_0(\mathcal{A}^0)$-module, which we denote by $L^2(\mathcal{A})$. Now, the algebra $C_c(\mathcal{A})$ acts on itself by left multiplication, and moreover this action is continuous with respect to the norm induced by the inner product above, hence we get an injective *-homomorphism

$$C_c(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))\,. \tag{1.22}$$

DEFINITION 1.4.7. The *reduced cross sectional algebra* $C_r^*(\mathcal{A})$ is defined as the completion of $C_c(\mathcal{A})$ with respect to the operator norm in $\mathcal{L}(L^2(\mathcal{A}))$.

In this way we get a right-Hilbert bimodule ${}_{C_r^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$.

Since $C_r^*(\mathcal{A})$ is a completion of $C_c(\mathcal{A})$ we immediately get a canonical map $\Lambda : C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$. Also, the *-homomorphism above in (1.22) always completes to a *-homomorphism $C^*(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))$, and therefore gives rise to a right-Hilbert bimodule ${}_{C^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$. The image of $C^*(\mathcal{A})$ on $\mathcal{L}(L^2(\mathcal{A}))$ is then isomorphic to $C_r^*(\mathcal{B})$, or in other words, the kernel of the map $C^*(\mathcal{A}) \to \mathcal{L}(L^2(\mathcal{A}))$ is the same as the kernel of the canonical map $\Lambda : C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$.

CHAPTER 2

Orbit space groupoids and Fell bundles

In this chapter we present the basic set up which will enable us to define crossed products by Hecke pairs later in Chapter 3.

Our construction of a (*-algebraic) crossed product $A \times^{alg} G/\Gamma$ of an algebra A by a Hecke pair (G, Γ) will make sense when A is a certain algebra of sections of a Fell bundle over a discrete groupoid. In this chapter we show in detail what type of algebras A are involved in the crossed product and how they are obtained.

2.1. Group actions on Fell bundles

Throughout this section G will denote a discrete group. One of our ingredients for defining crossed products by Hecke pairs consists of a group action on a Fell bundle over a groupoid (a concept we borrow from [**11**, Section 6]). Such actions always carry an associated action on the corresponding groupoid (by groupoid automorphisms). Since we are primarily interested in *right* actions on groupoids, we start by recalling what they are:

DEFINITION 2.1.1. Let X be a groupoid. A *right action* of G on X is a mapping

$$\begin{aligned} X \times G &\to X \\ (x, g) &\mapsto xg\,, \end{aligned}$$

which is a right action of G on the underlying set of X, meaning that

1) $xe = x$, for all $x \in X$,
2) $x(g_1 g_2) = (xg_1)g_2$, for all $x \in X$, $g_1, g_2 \in G$,

which is compatible with the groupoid operations, meaning that

3) if x and y are composable in X, then so are xg and yg, for all $g \in G$, and moreover
$$(xg)(yg) = (xy)g\,,$$
4) $(xg)^{-1} = x^{-1}g$, for all $x \in X$ and $g \in G$.

In other words, a right action of G on X is a right action on the set X performed by groupoid automorphisms.

LEMMA 2.1.2. *Let X be a groupoid endowed with a right G-action. For every $x \in X$ and $g \in G$ we have*

$$\mathbf{s}(xg) = \mathbf{s}(x)g \qquad \textit{and} \qquad \mathbf{r}(xg) = \mathbf{r}(x)g\,.$$

In particular, G restricts to an action on the unit space X^0.

Proof: It follows easily from the definition of a right G-action that

$$\mathbf{s}(x)g = (x^{-1}x)g = (x^{-1}g)(xg) = (xg)^{-1}(xg) = \mathbf{s}(xg)\,,$$

and similarly for the range function. □

REMARK 2.1.3. Given elements x, y in a groupoid X endowed with a right G-action and given $g \in G$, we will often drop the brackets in expressions like $(xg)y$ and simply use the notation xgy. No confusion arises from this since G is only assumed to act on the right. On the other hand, we will never write an expression like xyg without brackets, since it can be confusing whether it means $x(yg)$ or $(xy)g$.

DEFINITION 2.1.4. [**11**, Section 6] Let G be a group and $\mathcal{A}$ a Fell bundle over a discrete groupoid X. An *action* of G on $\mathcal{A}$ consists of a homomorphism $\alpha : G \to Aut(\mathcal{A})$.

As observed in Section 1.4, each automorphism of $\mathcal{A}$ carries with it an associated automorphism of the underlying groupoid X. Hence, an action of a group G on $\mathcal{A}$ entails an action of G on X by groupoid automorphisms. Since we are interested only in right actions on groupoids, we just ensure that these associated actions are on the right simply by taking inverses. Moreover, even though we will typically denote by α the action of G on $\mathcal{A}$, we will simply write $(x, g) \mapsto xg$ to denote its associated action on X and it will be always assumed that this action comes from α. To summarize what we have said so far: given an action α of G on a Fell bundle $\mathcal{A}$ over a groupoid X, there is an associated right G-action $(x, g) \mapsto xg$ on X such that

$$p(\alpha_g(a)) = p(a)g^{-1}\,. \tag{2.1}$$

REMARK 2.1.5. Typically one would require the mapping $(a, g) \mapsto \alpha_g(a)$ to be continuous, but this is not necessary here since both G and X are discrete.

PROPOSITION 2.1.6. *Let α be an action of a group G on a Fell bundle $\mathcal{A}$ over a groupoid X. We have an associated action $\overline{\alpha} : G \to Aut(C_c(\mathcal{A}))$ of G on $C_c(\mathcal{A})$ given by*

$$\overline{\alpha}_g(f)\,(x) := \alpha_g(f(xg))\,,$$

for $g \in G$, $f \in C_c(\mathcal{A})$ and $x \in X$.

Proof: Let us first prove that the action is well-defined, i.e. $\overline{\alpha}_g(f) \in C_c(\mathcal{A})$. The fact that $\overline{\alpha}_g(f)$ is finitely supported is obvious, so the only thing one needs to check is that $\overline{\alpha}_g(f)$ is indeed a section of the bundle, i.e. $\alpha_g(f(xg)) \in \mathcal{A}_x$ for all $x \in X$, which is clear because $\alpha_g(\mathcal{A}_y) = \mathcal{A}_{yg^{-1}}$.

Let us now check that $\overline{\alpha}_g$ is indeed a *-homomorphism for all $g \in G$. Linearity of $\overline{\alpha}_g$ is obvious. Let $f, f_1, f_2 \in C_c(\mathcal{A})$. We have

$$\begin{aligned}
\overline{\alpha}_g(f_1 \cdot f_2)\,(x) &= \alpha_g((f_1 \cdot f_2)\,(xg)) \\
&= \sum_{\substack{y,z \in X \\ yz = xg}} \alpha_g(f_1(y) f_2(z)) \\
&= \sum_{\substack{y,z \in X \\ (yg^{-1})(zg^{-1}) = x}} \alpha_g(f_1(y))\alpha_g(f_2(z)) \\
&= \sum_{\substack{y,z \in X \\ yz = x}} \alpha_g(f_1(yg))\alpha_g(f_2(zg)) \\
&= \sum_{\substack{y,z \in X \\ yz = x}} \overline{\alpha}_g(f_1)(y)\overline{\alpha}_g(f_2)(z) \\
&= \big(\overline{\alpha}_g(f_1) \cdot \overline{\alpha}_g(f_2)\big)\,(x)\,.
\end{aligned}$$

Hence, $\overline{\alpha}_g(f_1 \cdot f_2) = \overline{\alpha}_g(f_1) \cdot \overline{\alpha}_g(f_2)$. Also,

$$\begin{aligned}
\overline{\alpha}_g(f^*)\,(x) &= \alpha_g(f^*(xg)) = \alpha_g(f(x^{-1}g))^* \\
&= \big(\overline{\alpha}_g(f)\,(x^{-1})\big)^* = \big(\overline{\alpha}_g(f)\big)^*\,(x)\,.
\end{aligned}$$

Hence, $\overline{\alpha}_g(f^*) = (\overline{\alpha}_g(f))^*$. The fact that $\overline{\alpha}_{g_1 g_2} = \overline{\alpha}_{g_1} \circ \overline{\alpha}_{g_2}$ for every $g_1, g_2 \in G$ is also easily checked. $\square$

Definition 2.1.7. Let α be a group action of G on a Fell bundle $\mathcal{A}$ over a groupoid X and let H be a subgroup of G. We will say that the G-action is *H-good* if for any $x \in X$ and $h \in H$ we have

$$\mathbf{s}(x)h = \mathbf{s}(x) \implies \alpha_{h^{-1}}(a) = a \qquad \forall a \in \mathcal{A}_x\,. \tag{2.2}$$

Also, a right G-action on a groupoid X is said to be *H-good* if for any $x \in X$ and $h \in H$ we have

$$\mathbf{s}(x)h = \mathbf{s}(x) \implies xh = x\,. \tag{2.3}$$

It is clear from the definitions that if the action α of G on $\mathcal{A}$ is H-good, then its associated right G-action on the underlying groupoid X is also H-good. We will mostly use actions on Fell bundles. However, some of our results (namely Proposition 2.1.10) are about groupoids only, and this is the reason for defining H-good actions for groupoids as well.

We now give equivalent definitions of a H-good action. For that we recall from (1.7) that given an action of G on a set X we denote by $\mathcal{S}_x$ the stabilizer of the point $x \in X$. We will also denote by $\mathcal{S}(\mathcal{A}_x)$ the set $\mathcal{S}(\mathcal{A}_x) := \{g \in G : \alpha_{g^{-1}}(a) = a\,, \forall a \in \mathcal{A}_x\}$.

Proposition 2.1.8. *Let α be an action of G on a Fell bundle $\mathcal{A}$ over a groupoid X. The following statements are equivalent:*

i) *The action α is H-good.*
ii) *For every $x \in X$ we have that $\mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}(\mathcal{A}_x) \cap H$.*

iii) *For any $x \in X$ we have*

$$\begin{aligned} \mathcal{S}_{\mathbf{s}(x)} \cap H \;&=\; \mathcal{S}_x \cap H \;=\; \mathcal{S}_{\mathbf{r}(x)} \cap H \;= \\ =\; \mathcal{S}(\mathcal{A}_{\mathbf{s}(x)}) \cap H \;&=\; \mathcal{S}(\mathcal{A}_x) \cap H \;=\; \mathcal{S}(\mathcal{A}_{\mathbf{r}(x)}) \cap H\,. \end{aligned} \tag{2.4}$$

iv) *The stabilizers of the H-actions on X and on the fibers of $\mathcal{A}$ are the same on composable pairs, i.e. if $x \in X$ and $y \in Y$ are composable, then*

$$\begin{aligned} \mathcal{S}_x \cap H = \mathcal{S}_y \cap H \;&= \\ =\; \mathcal{S}(\mathcal{A}_x) \cap H \;&=\; \mathcal{S}(\mathcal{A}_y) \cap H\,. \end{aligned}$$

Proof: *i)* $\Longrightarrow$ *ii)* Since the action is H-good we have, by definition, that $\mathcal{S}_{\mathbf{s}(x)} \cap H \subseteq \mathcal{S}(\mathcal{A}_x) \cap H$. Also, if $h \in \mathcal{S}(\mathcal{A}_x) \cap H$, then we necessarily have $xh = x$, and therefore by Lemma 2.1.2 we get $\mathbf{s}(x) = \mathbf{s}(xh) = \mathbf{s}(x)h$, from which we conclude that $h \in \mathcal{S}_{\mathbf{s}(x)} \cap H$. Hence we have $\mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}(\mathcal{A}_x) \cap H$.

ii) $\Longrightarrow$ *iii)* Repeating a little bit of what we did above: if $h \in \mathcal{S}(\mathcal{A}_x) \cap H$, then we necessarily have that $xh = h$, and therefore $h \in \mathcal{S}_x \cap H$. Moreover, if $h \in \mathcal{S}_x \cap H$, then it follows that by Lemma 2.1.2 that $h \in \mathcal{S}_{\mathbf{s}(x)} \cap H$. Thus, we have that

$$\mathcal{S}(\mathcal{A}_x) \cap H = \mathcal{S}_x \cap H = \mathcal{S}_{\mathbf{s}(x)} \cap H\,.$$

Since $\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$, we also have, directly by our assumption of *ii)*, that $\mathcal{S}_{\mathbf{s}(x)} \cap H = \mathcal{S}(\mathcal{A}_{\mathbf{s}(x)}) \cap H$.

Since we have $(xg)^{-1} = x^{-1}g$, it follows easily that $\mathcal{S}_x = \mathcal{S}_{x^{-1}}$. Similarly, since $\alpha_g(a)^* = \alpha_g(a^*)$, it follows easily that $\mathcal{S}(\mathcal{A}_x) = \mathcal{S}(\mathcal{A}_{x^{-1}})$. Observing that $\mathbf{s}(x^{-1}) = \mathbf{r}(x)$, equality (2.4) follows directly from what we proved above.

iii) $\Longrightarrow$ *iv)* Suppose $x \in X$ and $y \in X$ are composable. Then, $\mathbf{s}(x) = \mathbf{r}(y)$ and equality (2.4) immediately yelds that

$$\begin{aligned} \mathcal{S}_x \cap H = \mathcal{S}_y \cap H \;&= \\ =\; \mathcal{S}(\mathcal{A}_x) \cap H \;&=\; \mathcal{S}(\mathcal{A}_y) \cap H\,. \end{aligned}$$

iv) $\Longrightarrow$ *i)* Let $h \in H$ and $x \in X$ be such that $\mathbf{s}(x)h = \mathbf{s}(x)$. From *iv)* it follows that $h \in \mathcal{S}(\mathcal{A}_x) \cap H$. This means that the action is H-good. □

It is easy to see that any H-good action is also gHg^{-1}-good for any conjugate gHg^{-1}, and also K-good for any subgroup $K \subseteq H$.

The following property will also be important for defining crossed products by Hecke pairs:

Definition 2.1.9. Let X be a groupoid endowed with a right G-action and let H be a subgroup of G. We will say that the action has the *H-intersection property* if

$$uH \cap ugHg^{-1} = uH^g\,, \tag{2.5}$$

for every unit $u \in X^0$ and $g \in G$.

An action of G on a Fell bundle $\mathcal{A}$ is said to have the *H-intersection property* if its associated right G-action on the underlying groupoid has the H-intersection property.

We defer examples of H-good actions and actions with the H-intersection property for the next section. We now introduce one of the important ingredients for our definition of crossed products by Hecke pairs: the *orbit space groupoid*.

Let G be a group, $H \subseteq G$ a subgroup and X a groupoid endowed with a H-good right G-action. Then, the orbit space X/H becomes a groupoid in a canonical way which we will now describe. For that, and throughout this text, we will use the following notation: given elements x, y we define the set

$$H_{x,y} := \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y)\} . \tag{2.6}$$

The groupoid structure on X/H is described as follows:

- A pair $(xH, yH) \in (X/H)^2$ is composable if and only if $H_{x,y} \neq \emptyset$, or equivalently, $\mathbf{r}(y) \in \mathbf{s}(x)H$. This property is easily seen not to depend on the choice of representatives x, y from the orbits xH, yH respectively.
- Given a composable pair $(xH, yH) \in (X/H)^2$, their product is

$$xH\, yH := x\widetilde{h}yH\,, \tag{2.7}$$

  where $\widetilde{h}$ is any element of $H_{x,y}$. It will follow from the fact the action is H-good that $x\widetilde{h}$ does not depend on the representative $\widetilde{h}$ chosen from $H_{x,y}$. The result of the product $xH\, yH$ also does not depend on the choice of representatives x, y. We will prove this in the next result.
- The inverse of the element xH is simply the element $x^{-1}H$. It is also easy to see that this does not depend on the choice of representative x.

Proposition 2.1.10. *Let G be a group, $H \subseteq G$ a subgroup and X a groupoid endowed with a H-good right G-action. The operations above give the orbit space X/H the structure of a groupoid. Moreover, the unit space $(X/H)^0$ of this groupoid is $X^0/H = \{uH : u \in X^0\}$, where X^0 is the unit space of X, and the range and source functions satisfy*

$$\mathbf{s}(xH) = \mathbf{s}(x)H \qquad \text{and} \qquad \mathbf{r}(xH) = \mathbf{r}(x)H\,.$$

Proof: Let us first prove that the product is well-defined. Let $(xH, yH) \in (X/H)^2$ be a composable pair. The fact that $x\widetilde{h}$ does not depend on the representative $\widetilde{h}$ chosen from $H_{x,y}$ follows from the assumption that the action is H-good, since if $h_1, h_2 \in H_{x,y}$ then we have

$$\mathbf{s}(x)h_1 = \mathbf{r}(y) = \mathbf{s}(x)h_2\,,$$

and therefore $\mathbf{s}(x)h_1h_2^{-1} = \mathbf{s}(x)$, and because the action is H-good $xh_1h_2^{-1} = x$, i.e. $xh_1 = xh_2$.

Let us now prove that X/H is a groupoid with the operations above. We check associativity first. Suppose $xH, yH, zH \in X/H$ are such that (xH, yH) is composable and (yH, zH) is composable. We want to prove that $(xHyH, zH)$ and $(xH, yHzH)$ are also composable and moreover $(xHyH)zH = xH(yHzH)$. We have by definition that $xHyH = x\widetilde{h_1}yH$ and $yHzH = y\widetilde{h_2}zH$, where $\widetilde{h_1}$ is any element of $H_{x,y}$ and $\widetilde{h_2}$ is any element of $H_{y,z}$. We now notice that

$$H_{x\widetilde{h_1}y,z} = \{h \in H : \mathbf{s}(x\widetilde{h_1}y)h = \mathbf{r}(z)\} = \{h \in H : \mathbf{s}(y)h = \mathbf{r}(z)\} = H_{y,z}\,.$$

Since $H_{y,z} \neq \emptyset$ it follows that $H_{x\widetilde{h_1}y,z} \neq \emptyset$, and therefore $(x\widetilde{h_1}yH, zH)$ is composable. Similarly,

$$\begin{aligned} H_{x,y\widetilde{h_2}z} &= \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y\widetilde{h_2}z)\} \\ &= \{h \in H : \mathbf{s}(x)h = \mathbf{r}(y)\widetilde{h_2}\} \\ &= \{h \in H : \mathbf{s}(x)h\widetilde{h_2}^{-1} = \mathbf{r}(y)\} \\ &= H_{x,y}\,\widetilde{h_2}\,. \end{aligned}$$

Hence, since $H_{x,y} \neq \emptyset$ it follows that $H_{x,y\widetilde{h_2}z} \neq \emptyset$, and therefore $(xH, y\widetilde{h_2}zH)$ is composable.

As we saw above $H_{x\widetilde{h_1}y,z} = H_{y,z}$, and since $\widetilde{h_2} \in H_{y,z}$, we can write

$$\begin{aligned} (xHyH)zH &= x\widetilde{h_1}yHzH = (x\widetilde{h_1}y)\widetilde{h_2}zH \\ &= x\widetilde{h_1}\widetilde{h_2}y\widetilde{h_2}zH\,. \end{aligned}$$

Also seen above, we have that $H_{x,y\widetilde{h_2}z} = H_{x,y}\,\widetilde{h_2}$, so that $\widetilde{h_1}\widetilde{h_2} \in H_{x,y\widetilde{h_2}z}$. Hence, we conclude that

$$(xHyH)zH = xH(yHzH)\,.$$

We now check that for any element $xH \in X/H$ we have that $(xH, x^{-1}H)$ and $(x^{-1}H, xH)$ are composable pairs. We have that

$$H_{x,x^{-1}} = \{h \in H : \mathbf{s}(x)h = \mathbf{r}(x^{-1})\} = \{h \in H : \mathbf{s}(x)h = \mathbf{s}(x)\}\,,$$

and the identity element e obviously belongs to the latter set. Hence we conclude that $H_{x,x^{-1}} \neq \emptyset$, and therefore $(xH, x^{-1}H)$ is composable. A similar observation shows that $(x^{-1}H, xH)$ is also composable.

To prove that X/H is a groupoid it now remains to prove the inverse identities $xHyHy^{-1}H = xH$ and $y^{-1}HyHxH = xH$, in case (xH, yH) is composable (for the first identity) and (yH, xH) is composable (for the second identity). We first show that $yHy^{-1}H = \mathbf{r}(y)H$. We have that $yHy^{-1}H = y\widetilde{h}y^{-1}H$ for any element $\widetilde{h} \in H_{y,y^{-1}}$. Since, as we observed above, we always have $e \in H_{y,y^{-1}}$, it follows that we can take $\widetilde{h}$ as e. Thus, we get

$$yHy^{-1}H = yy^{-1}H = \mathbf{r}(y)H\,. \tag{2.8}$$

From this it follows that

$$xHyHy^{-1}H = xH\mathbf{r}(y)H = x\widetilde{h_1}\mathbf{r}(y)H\,,$$

where $\widetilde{h_1}$ is any element of $H_{x,\mathbf{r}(y)}$. By definition, $\widetilde{h_1}$ is such that $\mathbf{r}(y) = \mathbf{s}(x)\widetilde{h_1} = \mathbf{s}(x\widetilde{h_1})$. Hence we have that $x\widetilde{h_1}\mathbf{r}(y) = x\widetilde{h_1}$, and therefore

$$xHyHy^{-1}H = x\widetilde{h_1}H = xH\,.$$

The other identity $y^{-1}HyHxH = xH$ is proven in a similar fashion. Hence, we conclude that X/H is a groupoid.

From equality (2.8) it follows easily that the units of X/H are precisely the elements of the form uH where $u \in X^0$, so that we can write $(X/H)^0 = X^0/H$. Also from (2.8) it follows that the range function in X/H satisfies:

$$\mathbf{r}(xH) = \mathbf{r}(x)H\,.$$

The analogous result for the source function is proven in a similar fashion. □

The condition that the action is H-good is in fact necessary to define a "reasonable" groupoid structure on the orbit space X/H, for a given right G-action on X (by groupoid homomorphisms). In fact, if on X/H we require the product of elements xH and yH to be the product of the classes $(xH)(yH) = \{xh_1yh_2 \in X : h_1, h_2 \in H, (xh_1, yh_2) \text{ is composable}\}$, and in particular that the product $(xH)(yH)$ consists of only one class, then it follows that the action is H-good:

PROPOSITION 2.1.11. *Let X be a groupoid endowed with a right G-action. Let us assume that for every $x, y \in X$ and $h_1, h_2 \in H$ such that (xh_1, y) and (xh_2, y) are composable, there exists $h_3 \in H$ such that*

$$xh_1y = (xh_2y)h_3 .$$

Then the action is H-good.

Proof: Let $x \in X$ and $h \in H$ be such that $\mathbf{s}(x)h = \mathbf{s}(x)$. The pair (xh, x^{-1}) is then composable, and by our assumption there is $h_3 \in H$ such that

$$xhx^{-1} = (xex^{-1})h_3 = \mathbf{r}(x)h_3 . \tag{2.9}$$

However, $\mathbf{s}(xhx^{-1}) = \mathbf{s}(x^{-1}) = \mathbf{r}(x)$ and also $\mathbf{s}(\mathbf{r}(x)h_3) = \mathbf{r}(x)h_3$, since $\mathbf{r}(x)h_3$ is a unit. Thus, we have that $\mathbf{r}(x) = \mathbf{r}(x)h_3$. Hence, expression (2.9) now reads $xhx^{-1} = \mathbf{r}(x)$, which means $xh = x$. This shows that action is H-good. □

REMARK 2.1.12. A key ingredient in this proof is the fact that we assume that the action of G on X is by groupoid homomorphisms (what we called a G-action), as seen in the statement that $\mathbf{r}(x)h_3$ is a unit. The condition that the action is H-good is not necessary to form a groupoid X/H if one does not assume an action by groupoid homomorphisms. For example, G is a group and therefore it is also a groupoid, and when H is normal, with the action of right translation, G/H has a natural groupoid structure (the quotient group). The only H-good actions on G are the trivial ones, since there is only one unit, so right translation is not H-good. However, right translation is also easily seen not to be an action by groupoid homomorphisms.

In conclusion, a "reasonable" groupoid structure can be defined on X/H under milder assumptions than G-actions by groupoid homomorphisms and H-good actions. We will not need a result in greater generality though, as the only actions of interest to us here are by groupoid homomorphisms.

Let α be an action of G on a Fell bundle $\mathcal{A}$ over a groupoid X. Assume that the action is H-good, where H is a subgroup of G. We will now define a new Fell bundle $\mathcal{A}/H$ over the groupoid X/H. First we set some notation. The set of H-orbits of the action α on $\mathcal{A}$ gives us a partition of $\mathcal{A}$ into equivalence classes. We will denote by $[a]$ the equivalence class of the element $a \in \mathcal{A}$, i.e.

$$[a] := \{\alpha_h(a)\}_{h \in H} .$$

DEFINITION 2.1.13. We define $\mathcal{A}/H$ as the set of all the H-orbits in $\mathcal{A}$, i.e.

$$\mathcal{A}/H := \{[a] : a \in \mathcal{A}\}\,. \tag{2.10}$$

As we will now see, $\mathcal{A}/H$ is a Fell bundle over X/H in a natural way.

PROPOSITION 2.1.14. *Let α be an action of a group G on a Fell bundle $\mathcal{A}$ over a groupoid X and $H \subseteq G$ be a subgroup for which the G-action is H-good. The set of H-orbits $\mathcal{A}/H$ forms a Fell bundle over the groupoid X/H in the following way:*

- *The associated projection $p_H : \mathcal{A}/H \to X/H$ is defined by $p_H([a]) := p(a)H$, where p is the associated projection of the bundle $\mathcal{A}$.*
- *The vector space structure on each fiber $\big(\mathcal{A}/H\big)_{xH}$ is defined in the following way: if $a, b \in \mathcal{A}_x$ then $[a] + [b] := [a + b]$, and if $\lambda \in \mathbb{C}$ then $\lambda[a] := [\lambda a]$.*
- *The norm on $\mathcal{A}/H$ is defined by $\|[a]\| := \|a\|$.*
- *The multiplication maps $\big(\mathcal{A}/H\big)_{xH} \times \big(\mathcal{A}/H\big)_{yH} \to \big(\mathcal{A}/H\big)_{xH\cdot yH}$, for a composable pair (xH, yH), are defined in the following way: if $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$, then*

$$[a][b] = [\alpha_{\widetilde{h}^{-1}}(a)b]\,, \tag{2.11}$$

  where $\widetilde{h}$ is any element of $H_{x,y}$.
- *The involution map is defined by $[a]^* := [a^*]$.*

LEMMA 2.1.15. *Let α be an action of G on a Fell bundle $\mathcal{A}$ over a groupoid X and $H \subseteq G$ be a subgroup for which the G-action is H-good. Let $x \in X$ and $a \in \mathcal{A}_x$. Given any $y \in xH$ there exists a unique representative b of $[a]$ such that $b \in \mathcal{A}_y$.*

Proof: Given an element $y \in xH$ we have that $y = xh$ for some $h \in H$. The element $\alpha_{h^{-1}}(a)$ is then a representative of $[a]$ such that $\alpha_{h^{-1}}(a) \in \mathcal{A}_{xh} = \mathcal{A}_y$, thus existence is established.

The uniqueness claim follows from the fact the action is H-good. Suppose we have two representatives b and c of $[a]$ such that both b and c belong to $\mathcal{A}_y$. Being representatives of $[a]$ means that there are elements $h_1, h_2 \in H$ such that $b = \alpha_{h_1}(a)$ and $c = \alpha_{h_2}(a)$. Hence we have that

$$\alpha_{h_2 h_1^{-1}}(b) = c\,,$$

and therefore $h_2 h_1^{-1}$ takes $\mathcal{A}_y$ into $\mathcal{A}_y$. This means that $y h_1 h_2^{-1} = y$ and therefore $\mathbf{s}(y) h_1 h_2^{-1} = \mathbf{s}(y)$. Since the action is H-good it follows that $\alpha_{h_2 h_1^{-1}}(b) = b$, and therefore $b = c$. $\square$

Proof of Proposition 2.1.14: First, it is clear that the vector space structure on each fiber $\big(\mathcal{A}/H\big)_{xH}$ is well-defined. By this we mean two things: first, given two elements $[a], [b] \in \big(\mathcal{A}/H\big)_{xH}$ there exist unique representatives a, b such that $a, b \in \mathcal{A}_x$ for a given representative x of the orbit xH (Lemma 2.1.15); second, the sum $[a+b]$ still lies in $\big(\mathcal{A}/H\big)_{xH}$ and does not depend on the choice of representatives a and b (provided only that a and b are in the same fiber).

The norm on $\mathcal{A}/H$ is also easily seen to be well-defined, i.e. independent of the choice of representative. This is true because any other representative of $[a]$ is of the form $\alpha_h(a)$ for some $h \in H$, and by Remark 1.4.3 we know that α_h gives an isometry between fibers. It is also clear that each fiber $\big(\mathcal{A}/H\big)_{xH}$ is a Banach space under this norm.

The multiplication map is also easily seen to be well-defined: using the fact that the G-action on $\mathcal{A}$ is H-good we know that $\alpha_{\widetilde{h}^{-1}}(a)b$ does not depend on the choice of element $\widetilde{h} \in H_{x,y}$. Moreover, $\alpha_{\widetilde{h}^{-1}}(a)b \in \mathcal{A}_{x\widetilde{h}y}$ and therefore $[\alpha_{\widetilde{h}^{-1}}(a)b] \in \mathcal{A}_{x\widetilde{h}yH}$. The fact that the multiplication map does not depend on the chosen representatives of the orbits $[a]$ and $[b]$ is also easily checked.

It follows from a routine computation that map $\big(\mathcal{A}/H\big)_{xH} \times \big(\mathcal{A}/H\big)_{yH} \to \big(\mathcal{A}/H\big)_{xH\cdot yH}$ is bilinear. Moreover, for $[a] \in \big(\mathcal{A}/H\big)_{xH}$ and $[b] \in \big(\mathcal{A}/H\big)_{yH}$, where we assume without loss of generality that $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$, we have that

$$\begin{aligned} \|[a][b]\| &= \|\alpha_{\widetilde{h}^{-1}}(a)b\| \leq \|\alpha_{\widetilde{h}^{-1}}(a)\|\|b\| \\ &= \|a\|\|b\| = \|[a]\|\|[b]\| . \end{aligned}$$

We will now check associativity of the multiplication maps. Let (xH, yH) and (yH, zH) be two composable pairs in X/H, and let $[a] \in (\mathcal{A}/H)_{xH}$, $[b] \in (\mathcal{A}/H)_{yH}$ and $[c] \in (\mathcal{A}/H)_{zH}$, where we assume without loss of generality that $a \in \mathcal{A}_x$, $b \in \mathcal{A}_y$ and $c \in \mathcal{A}_z$. By definition, we have $[a][b] = [\alpha_{\widetilde{h_1}^{-1}}(a)b]$, where $\widetilde{h_1}$ is any element of $H_{x,y}$. Thus, we have

$$\begin{aligned} \big([a][b]\big)[c] &= [\alpha_{\widetilde{h_1}^{-1}}(a)b][c] = [\alpha_{\widetilde{h_2}^{-1}}\big(\alpha_{\widetilde{h_1}^{-1}}(a)b\big)c] \\ &= [\alpha_{\widetilde{h_2}^{-1}\widetilde{h_1}^{-1}}(a)\alpha_{\widetilde{h_2}^{-1}}(b)c] , \end{aligned}$$

where $\widetilde{h_2}$ is any element of $H_{x\widetilde{h_1}y,z}$. One can easily check (or see the proof of Proposition 2.1.10 where this is done) that $H_{x\widetilde{h_1}y,z} = H_{y,z}$ and moreover that $\widetilde{h_1}\widetilde{h_2} \in H_{x,y\widetilde{h_2}z}$. From this observations it follows that

$$\begin{aligned} \big([a][b]\big)[c] &= [\alpha_{\widetilde{h_2}^{-1}\widetilde{h_1}^{-1}}(a)\alpha_{\widetilde{h_2}^{-1}}(b)c] = [a][\alpha_{\widetilde{h_2}^{-1}}(b)c] \\ &= [a]\big([b][c]\big) . \end{aligned}$$

Hence, the multiplication maps are associative.

The involution on $\mathcal{A}/H$ is also easily seen not to depend on choice of representative of the orbit, since the maps α_h preserve the involution of $\mathcal{A}$. Moreover, it is easily checked that: if $[a] \in \big(\mathcal{A}/H\big)_{xH}$ then $[a]^* \in \big(\mathcal{A}/H\big)_{x^{-1}H}$, the associated map $\big(\mathcal{A}/H\big)_{xH} \to \big(\mathcal{A}/H\big)_{x^{-1}H}$ is conjugate linear, and $[a]^{**} = [a]$. Let us now check that $\big([a][b]\big)^* = [b]^*[a]^*$, whenever the multiplication is defined. Let us assume that $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$ and that (xH, yH) is composable. We have that

$$\big([a][b]\big)^* = [\alpha_{\widetilde{h}^{-1}}(a)b]^* = [b^*\alpha_{\widetilde{h}^{-1}}(a^*)] = [\alpha_{\widetilde{h}}(b^*)a^*] ,$$

where $\widetilde{h}$ is any element of $H_{x,y}$. It is easily seen that $\widetilde{h}^{-1} \in H_{y^{-1},x^{-1}}$, so that

$$\big([a][b]\big)^* = [\alpha_{\widetilde{h}}(b^*)a^*] = [b^*][a^*] = [b]^*[a]^* .$$

We also need to prove that $\|[a]^*[a]\| = \|[a]\|^2$. This is also easy because

$$\|[a]^*[a]\| = \|[a^*][a]\| = \|[a^*a]\| = \|a^*a\| = \|a\|^2 = \|[a]\|^2 .$$

The last thing we need to check is that if $[a] \in (\mathcal{A}/H)_{xH}$, then $[a]^*[a]$ is a positive element of $(\mathcal{A}/H)_{\mathbf{s}(x)H}$ (seen as a C^*-algebra). We have that $[a]^*[a] = [a^*a]$. We can assume without loss of generality that $a \in \mathcal{A}_x$, so that $a^*a \in \mathcal{A}_{\mathbf{s}(x)}$. Since $\mathcal{A}$ is a Fell bundle we have that a^*a is a positive element of $\mathcal{A}_{\mathbf{s}(x)}$ (seen as a C^*-algebra). Hence, there exists an element $b \in \mathcal{A}_{\mathbf{s}(x)}$ such that $a^*a = b^*b$. Moreover, $[b] \in (\mathcal{A}/H)_{\mathbf{s}(x)}$ and it is now clear that

$$[a]^*[a] = [a^*a] = [b^*b] = [b]^*[b] ,$$

i.e. $[a]^*[a]$ is a positive element of $(\mathcal{A}/H)_{\mathbf{s}(x)}$. This finishes our proof that $\mathcal{A}/H$ is a Fell bundle. □

CONVENTION. *For simplicity we will henceforward make the following convention. Given an orbit Fell bundle $\mathcal{A}/H$ as described in Proposition 2.1.14, if we write that an element $[a]$ belongs to some fiber $(\mathcal{A}/H)_{xH}$, we will always assume that the representative a of $[a]$ belongs to the fiber over the representative x of xH. In other words, if we write that $[a] \in (\mathcal{A}/H)_{xH}$, then we are implicitly assuming that $a \in \mathcal{A}_x$. This is possible and unambiguous by Lemma 2.1.15.*

We apply this convention also for elements of $C_c(\mathcal{A}/H)$, meaning that a canonical element $[a]_{xH} \in C_c(\mathcal{A}/H)$ is always assumed to be written in a way that $a \in \mathcal{A}_x$.

It is a straightforward fact that any function in $C_c(X/H)$ can also be seen as a complex-valued (H-invariant) function on X. This function on X is in general no longer finitely supported, but it still makes sense as a function in $C(X)$, the vector space of all complex-valued functions on X. We will now see that something analogous can be said for the elements of $C_c(\mathcal{A}/H)$.

Given an element $f \in C_c(\mathcal{A}/H)$ we define a function $\iota(f) \in C(\mathcal{A})$, where $C(\mathcal{A})$ is the vector space of all sections of $\mathcal{A}$, by the following rule:

$$\iota(f)(x) := R_x(f(xH)) , \tag{2.12}$$

where $R_x(f(xH))$ is the unique representative of $f(xH)$ such that $R_x(f(xH)) \in \mathcal{A}_x$, which is well-defined according to Lemma 2.1.15. It is then easy to see that the map ι is an injective linear map from $C_c(\mathcal{A}/H)$ to $C(\mathcal{A})$.

For ease of reading we will henceforward drop the symbol ι and use the same notation both for elements of $C_c(\mathcal{A}/H)$ and for their correspondents in $C(\mathcal{A})$. It will then be clear from context which one we are using.

Under this convention we can then write, for any $f \in C_c(\mathcal{A}/H)$ and $x \in X$, that $[f(x)] = f(xH)$. Moreover, the decomposition (1.21) of $f \in C_c(\mathcal{A}/H)$ as a sum of elements of the form $[a]_{xH}$ can now be written as:

$$f = \sum_{xH \in X/H} \big(f(xH)\big)_{xH} = \sum_{xH \in X/H} [f(x)]_{xH} . \tag{2.13}$$

2.2. Examples

In this section we give some examples of H-good actions and actions satisfying the H-intersection property. For the rest of the section we assume that $\mathcal{A}$ is a Fell bundle over a groupoid X where a group G acts and $H \subseteq G$ denotes a subgroup.

The first two examples (2.2.1 and 2.2.2) show that H-good actions that satisfy the H-intersection property are present in actions that have completely opposite behaviours, such as free actions and actions that fix every point.

Example 2.2.1. If the restricted action of H on the unit space X^0 is free, then the action is H-good and satisfies the H-intersection property.

Example 2.2.2. If the restricted action of H on $\mathcal{A}$ fixes every point, then the action is H-good and satisfies the H-intersection property.

The following example is one of the examples that motivated the development of this theory of crossed products by Hecke pairs. This example, and the study of the crossed products associated to it, seems to be valuable for obtaining a form of Katayama duality with respect to crossed products by "coactions" of discrete homogeneous spaces.

Example 2.2.3. Suppose X is the transformation groupoid $G \times G$. We recall that the multiplication and inversion operations on this groupoid are given by:

$$(s, tr)(t, r) = (st, r) \qquad \text{and} \qquad (s, t)^{-1} = (s^{-1}, st)\,.$$

Recall also that the source and range functions on $G \times G$ are defined by

$$\mathbf{s}(s, t) = (e, t) \qquad \text{and} \qquad \mathbf{r}(s, t) = (e, st)\,.$$

We observe that there is a natural right G-action on $G \times G$, given by

$$(s, t)g := (s, tg)\,. \tag{2.14}$$

Let δ be a coaction of G on a C^*-algebra B and $\mathcal{B}$ the associated Fell bundle. Following [**6**, Section 3], we will denote by $\mathcal{A} := \mathcal{B} \times G$ the corresponding Fell bundle over the groupoid $G \times G$. Elements of $\mathcal{A}$ have the form (b_s, t), where $b_s \in \mathcal{B}_s$ and $s, t \in G$. Any such element lies in the fiber $\mathcal{A}_{(s,t)}$ over (s, t).

It is easy to see that there is a canonical action α of G on $\mathcal{A}$, given by

$$\alpha_g(b_s, t) := (b_s, tg^{-1})\,.$$

This action of G on $\mathcal{A}$ entails the natural right action of G on $G \times G$, as described in (2.14). This G-action on $G \times G$ is free and therefore the action α is H-good and satisfies the H-intersection property with respect to any subgroup $H \subseteq G$.

The orbit space groupoid $(G \times G)/H$ can be canonically identified with the groupoid $G \times G/H$ of [**5**], whose operations are given by:

$$(s, trH)(t, rH) = (st, rH) \qquad \text{and} \qquad (s, tH)^{-1} = (s^{-1}, stH)\,.$$

Moreover, the orbit Fell bundle $\mathcal{A}/H$ is canonically identified with the Fell bundle $\mathcal{B} \times G/H$ over $G \times G/H$ defined in [**5**], and in this way $C_c(\mathcal{A}/H)$ is canonically isomorphic with the Echterhoff-Quigg algebra $C_c(\mathcal{B} \times G/H)$, also from [**5**].

EXAMPLE 2.2.4. Here we give an example of a G-action on a groupoid X which is not H-good. Let X be again the transformation groupoid $G \times G$, but now we consider the G-action given by conjugation:

$$(s,t)g := (g^{-1}sg, g^{-1}tg)\,. \tag{2.15}$$

A routine computation shows that (2.15) does indeed define a right action of G on X.

If a subgroup H is not in the center of G, then the action is not H-good. To see this, take two elements $h \in H$ and $g \in G$ that do not commute. We have that $\mathbf{s}(g,e) = (e,e)$, and therefore $\mathbf{s}(g,e)h = (e,e)h = (e,e) = \mathbf{s}(g,e)$, but $(g,e)h = (h^{-1}gh, e) \neq (g,e)$, so the action cannot be H-good.

2.3. The algebra $M(C_c(\mathcal{A}))$

We will assume for the rest of this section that G is a group, $H \subseteq G$ is a subgroup and $\mathcal{A}$ is a Fell bundle over a groupoid X endowed with a G-action α. We also assume that the action α is H-good. We recall that $\mathcal{A}/H$ stands for the orbit Fell bundle over the groupoid X/H, as defined in (2.10).

For the purpose of defining crossed products by Hecke pairs it is convenient to have a "large" algebra which contains the algebras $C_c(\mathcal{A}/H)$ for different subgroups $H \subseteq G$. In this way we are allowed to multiply elements of $C_c(\mathcal{A}/H)$ and $C_c(\mathcal{A}/K)$, for different subgroups $H, K \subseteq G$, in a meaningful way. This large algebra will be the multiplier algebra $M(C_c(\mathcal{A}))$. This section is thus devoted to show how algebras such as $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$ embed in $M(C_c(\mathcal{A}))$ in a canonical way.

Our first result shows that there is a natural inclusion $C_c(\mathcal{A}/H) \subseteq M(C_c(\mathcal{A}))$.

THEOREM 2.3.1. *There is an embedding ι of $C_c(\mathcal{A}/H)$ into $M(C_c(\mathcal{A}))$ determined by the following rule: for any $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$ we have*

$$\iota([a]_{xH})b_y := \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}\,, & \textit{if } H_{x,y} \neq \emptyset \\ 0, & \textit{otherwise,} \end{cases} \tag{2.16}$$

where $\widetilde{h}$ is any element of $H_{x,y}$.

REMARK 2.3.2. The above result allows us to see $C_c(\mathcal{A}/H)$ as a *-subalgebra of $M(C_c(\mathcal{A}))$. We shall henceforward drop the symbol ι and make no distinction of notation between an element of $C_c(\mathcal{A}/H)$ and its correspondent multiplier in $M(C_c(\mathcal{A}))$.

Proof of Theorem 2.3.1: Let us first show that expression (2.16) does indeed define an element of $M(C_c(\mathcal{A}))$. For this it is enough to check that $\langle \iota([a]_{xH})b_y, c_z\rangle = \langle b_y, \iota([a]^*_{x^{-1}H})\, c_z\rangle$, for all $b \in \mathcal{A}_y$ and $c \in \mathcal{A}_z$, with $y, z \in X$. For $\iota([a]_{xH})b_y$ to be non-zero, we must necessarily have $H_{x,y} \neq \emptyset$, and in this case $\iota([a]_{xH})b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}$, where $\widetilde{h} \in H_{x,y}$. Now,

$$\begin{aligned} \langle \iota([a]_{xH})b_y, c_z\rangle &= \langle (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}, c_z\rangle = (b^*\alpha_{\widetilde{h}^{-1}}(a)^*)_{y^{-1}(x^{-1}\widetilde{h})}c_z \\ &= b^*_{y^{-1}}\alpha_{\widetilde{h}^{-1}}(a)^*_{x^{-1}\widetilde{h}}\, c_z \end{aligned}$$

For $\alpha_{\widetilde{h}^{-1}}(a)^*_{x^{-1}\widetilde{h}}\, c_z$ to be non-zero we must necessarily have $\mathbf{r}(z) = \mathbf{s}(x^{-1})\widetilde{h}$, i.e. $\widetilde{h} \in H_{x^{-1},z}$. So, to summarize, for $\langle [a]_{xH} b_y, c_z\rangle$ to be non-zero we must have $H_{x,y} \cap H_{x^{-1},z} \neq \emptyset$ and in this case we obtain

$$\langle \iota([a]_{xH}) b_y, c_z\rangle = b^*_{y^{-1}} \alpha_{\widetilde{h}^{-1}}(a)^*_{x^{-1}\widetilde{h}}\, c_z \,,$$

where $\widetilde{h}$ is any element of $H_{x,y} \cap H_{x^{-1},z}$. A similar computation for $\langle b_y, \iota([a]^*_{x^{-1}H})\, c_z\rangle$ yelds the exact same result.

Recall from (1.21) that any $f \in C_c(\mathcal{A}/H)$ can be written as

$$f = \sum_{xH \in X/H} \big(f(xH)\big)_{xH} \,.$$

From this we are able to define a multiplier $\iota(f) \in M(C_c(\mathcal{A}))$, simply by extending expression (2.16) by linearity.

We want to show that ι is an injective *-homomorphism. First, we claim that given $[a]_{xH}, [b]_{yH} \in C_c(\mathcal{A}/H)$ we have

$$\iota([a]_{xH})\iota([b]_{yH}) = \iota([a]_{xH}[b]_{yH}) \,.$$

This amounts to proving that

$$\iota([a]_{xH})\iota([b]_{yH}) = \begin{cases} \iota([\alpha_{\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}yH}) \,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

with $\widetilde{h}$ being any element of $H_{x,y}$. To see this, let $c_z \in \mathcal{A}_z$, with $z \in X$. We have

$$\begin{aligned} \iota([a]_{xH})\iota([b]_{yH})c_z &= \begin{cases} \iota([a]_{xH})(\alpha_{h_0^{-1}}(b)c)_{yh_0z} \,, & \text{if } H_{y,z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} (\alpha_{h_1^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{xh_1yh_0z} \,, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,yh_0z} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

with $h_0 \in H_{y,z}$ and $h_1 \in H_{x,yh_0z}$. But $H_{x,yh_0z} = H_{x,yh_0} = H_{x,y}\, h_0$, hence the above can be written as

$$\begin{aligned} &= \begin{cases} (\alpha_{h_0^{-1}\widetilde{h}^{-1}}(a)\alpha_{h_0^{-1}}(b)c)_{x\widetilde{h}h_0yh_0z} \,, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \big(\alpha_{h_0^{-1}}(\alpha_{\widetilde{h}^{-1}}(a)b)c\big)_{(x\widetilde{h}y)h_0z} \,, & \text{if } H_{y,z} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $\widetilde{h} \in H_{x,y}$. Also, $H_{y,z} = H_{x\widetilde{h}y,z}$. Thus, we obtain

$$\begin{aligned} &= \begin{cases} \big(\alpha_{h_0^{-1}}(\alpha_{\widetilde{h}^{-1}}(a)b)c\big)_{(x\widetilde{h}y)h_0z} \,, & \text{if } H_{x\widetilde{h}y,z} \neq \emptyset \text{ and } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \iota([\alpha_{\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}yH})\, c_z \,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Since ι is linear and multiplicative on the elements of the form $[a]_{xH}$, it is necessarily a homomorphism. Now the fact that $\iota([a]_{xH})^* = \iota(([a]_{xH})^*) = \iota([a]^*_{x^{-1}H})$ follows directly from the computations in the beginning of this proof. Hence, ι is a *-homomorphism.

Let us now prove injectivity of ι. Suppose $f \in C_c(\mathcal{A}/H)$ is such that $\iota(f) = 0$. Decomposing f as a sum of elements of the form $[a]_{xH}$, following (2.13), we get

$$0 = \iota(f) = \sum_{xH \in X/H} \iota\Big(\big(f(xH)\big)_{xH}\Big) = \sum_{xH \in X/H} \iota\big([f(x)]_{xH}\big) .$$

For any $y \in X$ we then have

$$\begin{aligned} 0 &= \sum_{xH \in X/H} \iota\big([f(x)]_{xH}\big)(f(y)^*)_{y^{-1}} \\ &= \sum_{\substack{xH \in X/H \\ \mathbf{s}(y) \in \mathbf{s}(x)H}} \iota\big([f(x)]_{xH}\big)(f(y)^*)_{y^{-1}} \\ &= \sum_{\substack{xH \in X/H \\ \mathbf{s}(y) \in \mathbf{s}(x)H}} \big(\alpha_{\widetilde{h_x}^{-1}}(f(x))f(y)^*\big)_{x\widetilde{h_x}y^{-1}} , \end{aligned}$$

where $\widetilde{h_x}$ is any element of $H_{x,y^{-1}}$. Now the elements $x\widetilde{h_x}y^{-1}$ in the sum above are all different, because if we had $x_1\widetilde{h_{x_1}}y^{-1} = x_2\widetilde{h_{x_2}}y^{-1}$, then we would have $x_1\widetilde{h_{x_1}} = x_2\widetilde{h_{x_2}}$ and therefore $x_1H = x_2H$. Therefore each of the summands in the above sum is zero, and in particular we must have

$$\begin{aligned} 0 &= \big(\alpha_{\widetilde{h_y}^{-1}}(f(y))f(y)^*\big)_{y\widetilde{h_y}y^{-1}} \\ &= \big(f(y)f(y)^*\big)_{\mathbf{r}(y)} , \end{aligned}$$

and therefore $f(y)f(y)^* = 0$. Hence we get $f(y) = 0$, and since this is true for any $y \in X$, we have $f = 0$, i.e. ι is injective. □

PROPOSITION 2.3.3. *There is an embedding ι of $C_b(X^0)$ into $M(C_c(\mathcal{A}))$ defined by*

$$\iota(f)\, b_y := f(\mathbf{r}(y))b_y . \tag{2.17}$$

for every $f \in C_b(X^0)$, $y \in X$ and $b \in \mathcal{A}_y$.

REMARK 2.3.4. The above result allows us to see $C_b(X^0)$ as a *-subalgebra of $M(C_c(\mathcal{A}))$. We shall henceforward drop the symbol ι and make no distinction of notation between an element of $C_b(X^0)$ and its correspondent multiplier in $M(C_c(\mathcal{A}))$.

Proof of Proposition 2.3.3 : It is easy to see that $\langle \iota(f)b_y\, ,\, c_z\rangle = \langle b_y\, ,\, \iota(f^*)c_z\rangle$ for any $y, z \in X$, $b \in \mathcal{A}_y$ and $c \in \mathcal{A}_z$, so that the expression (2.17) does define an element of $M(C_c(\mathcal{A}))$.

Hence we get a linear map $\iota : C_b(X^0) \to M(C_c(\mathcal{A}))$. Given two elements $f_1, f_2 \in C_b(X^0)$, we have that

$$\iota(f_1)\iota(f_2)b_y = f_1(\mathbf{r}(y))f_2(\mathbf{r}(y))b_y = \iota(f_1f_2)b_y$$

for any $y \in X$ and$b \in \mathcal{A}_y$, so that ι is a *-homomorphism. Hence, we only need to prove that ι is injective. This is not difficult to see: given $f \in C_b(X^0)$ such that $\iota(f) = 0$ we have, for any unit $u \in X^0$ and $b \in \mathcal{A}_u$, that

$$0 \;=\; \iota(f)b_u \;=\; f(u)b_u .$$

Hence, $f(u) = 0$ because each fiber $\mathcal{A}_u$ is non-zero by our assumption on Fell bundles (see Assumption 1.4.1). Since this is true for any $u \in X^0$ we get $f = 0$, i.e. ι is injective. □

Recall, from Lemma 2.1.2, that the action of G on X restricts to an action of G on the set X^0. Thus it makes sense to talk about the commutative *-algebra

$$C_c(X^0/H) \subseteq C_b(X^0) .$$

Since there is a canonical embedding, given by Proposition 2.3.3, of $C_b(X^0)$ into $M(C_c(\mathcal{A}))$, we have in particular an embedding of $C_c(X^0/H)$ into $M(C_c(\mathcal{A}))$ which identifies an element $f \in C_c(X^0/H)$ with the multiplier $f \in M(C_c(\mathcal{A}))$ given by:

$$fb_y := f(\mathbf{r}(y)H)b_y .$$

Moreover Proposition 2.3.3 applied to the groupoid X/H and the Fell bundle $\mathcal{A}/H$ shows that there is a canonical embedding of $C_b(X^0/H)$ into $M(C_c(\mathcal{A}/H))$, which identifies an element $f \in C_b(X^0/H)$ with the multiplier $f \in M(C_c(\mathcal{A}/H))$ given by

$$f[b]_{yH} := f(\mathbf{r}(y)H)[b]_{yH} . \tag{2.18}$$

Since both $C_c(X^0/H)$ and $C_c(\mathcal{A}/H)$ are canonically embedded in $M(C_c(\mathcal{A}))$, it is convenient to understand what happens (inside $M(C_c(\mathcal{A}))$) when one multiplies an element of $C_c(X^0/H)$ by an element $C_c(\mathcal{A}/H)$. Perhaps unsurprisingly, this product is given exactly by expression (2.18), which models the action of $C_c(X^0/H)$ on $C_c(\mathcal{A}/H)$ as multipliers of the latter algebra. In other words, it makes no difference to view $C_c(X^0/H)$ inside $M(C_c(\mathcal{A}/H))$ or inside $M(C_c(\mathcal{A}))$ when it comes to multiplication by elements of $C_c(\mathcal{A}/H)$.

We will now show how the multiplication of elements of $C_c(\mathcal{A}/H)$ by elements of $C_c(X^0)$ is determined (inside $M(C_c(\mathcal{A}))$). Before we proceed we will first introduce some notation that will be used throughout this work: Given a set $A \subset X^0$ we will denote by $1_A \in C_b(X^0)$ the characteristic function of A. In case A is a singleton $\{u\}$ we will simply write 1_u.

Proposition 2.3.5. *Inside $M(C_c(\mathcal{A}))$ we have that, for $x \in X$, $a \in \mathcal{A}_x$ and $u \in X^0$,*

$$[a]_{xH}1_u = \begin{cases} \alpha_{\widetilde{h}^{-1}}(a)_{x\widetilde{h}} , & \text{if } H_{x,u} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{h}$ is any element of $H_{x,u}$.

Proof: Let $y \in X$ and $b \in \mathcal{A}_y$. For the product $[a]_{xH}1_u\, b_y$ to be non-zero we must necessarily have $u = \mathbf{r}(y)$ (from (2.17)), and in this case we obtain

$$[a]_{xH}1_u\, b_y = [a]_{xH}b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y} = \alpha_{\widetilde{h}^{-1}}(a)_{x\widetilde{h}}b_y ,$$

where $\widetilde{h}$ is any element of $H_{x,y}$. Since $u = \mathbf{r}(y)$, we have $H_{x,y} = H_{x,u}$, and this concludes the proof. □

It will be of particular importance to know how to multiply, inside $M(C_c(\mathcal{A}))$, elements of $C_c(\mathcal{A}/H)$ with elements of $C_c(\mathcal{A}/K)$ when $K \subseteq H$ is an arbitrary subgroup. It turns out that the algebra $C_c(\mathcal{A}/K)$ is preserved by multiplication by elements of $C_c(\mathcal{A}/H)$, as we show in the next result:

PROPOSITION 2.3.6. *Let $K \subseteq H$ be any subgroup. We have that*

$$[a]_{xH}[b]_{yK} = \begin{cases} [\alpha_{\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}yK}\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \tag{2.19}$$

where $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. In particular $C_c(\mathcal{A}/K)$ is invariant under multiplication by elements of $C_c(\mathcal{A}/H)$.

Proof: First we observe that since the action is assumed to be H-good, it is automatically K-good, so that we can form the groupoid X/K and the Fell bundle $\mathcal{A}/K$.

Let $z \in X$ and $c \in \mathcal{A}_z$. We have that

$$\begin{aligned}
[a]_{xH}[b]_{yK}c_z \;&=\; \begin{cases} [a]_{xH}(\alpha_{\widetilde{k}^{-1}}(b)c)_{y\widetilde{k}z}\,, & \text{if } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\
&=\; \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)\alpha_{\widetilde{k}^{-1}}(b)c)_{x\widetilde{h}y\widetilde{k}z}\,, & \text{if } H_{x,y\widetilde{k}z} \neq \emptyset \text{ and } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\
&=\; \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)\alpha_{\widetilde{k}^{-1}}(b)c)_{\left(x\widetilde{h}\widetilde{k}^{-1}y\right)\widetilde{k}z}\,, & \text{if } H_{x,y\widetilde{k}z} \neq \emptyset \text{ and } K_{y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where $\widetilde{k}$ is any element of $K_{y,z}$ and $\widetilde{h}$ is any element of $H_{x,y\widetilde{k}z}$. Now, since $H_{x,y\widetilde{k}z} = H_{x,y\widetilde{k}} = H_{x,y}\widetilde{k}$, it follows that $\widetilde{h}\widetilde{k}^{-1} \in H_{x,y}$, and moreover since $K_{y,z} = K_{x\widetilde{h}\widetilde{k}^{-1}y,z}$, we conclude that

$$\begin{aligned}
&=\; \begin{cases} (\alpha_{\widetilde{k}^{-1}}(\alpha_{\widetilde{k}\widetilde{h}^{-1}}(a)b)c)_{\left(x\widetilde{h}\widetilde{k}^{-1}y\right)\widetilde{k}z}\,, & \text{if } H_{x,y} \neq \emptyset \text{ and } K_{x\widetilde{h}\widetilde{k}^{-1}y,z} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases} \\
&=\; \begin{cases} [\alpha_{\widetilde{k}\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}\widetilde{k}^{-1}yK}c_z\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus (2.19) follows immediately (the element $\widetilde{h}$ in (2.19) is simply the element denoted by $\widetilde{h}\widetilde{k}^{-1}$ above). $\square$

In case the subgroup K has finite index in H we can strengthen Proposition 2.3.6 in the following way:

PROPOSITION 2.3.7. *Let $K \subseteq H$ be a subgroup such that $[H : K] < \infty$. Inside $M(C_c(\mathcal{A}))$ we have that*

$$[a]_{xH} \;=\; \sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK}\,, \tag{2.20}$$

for any $x \in X$ and $a \in \mathcal{A}_x$. In particular, inside $M(C_c(\mathcal{A}))$ we have that $C_c(\mathcal{A}/H)$ is a *-subalgebra of $C_c(\mathcal{A}/K)$.

Proof: First we notice that since $[H:K] < \infty$ we have that the right hand side of (2.20) is a finite sum and therefore does indeed define an element of $C_c(\mathcal{A}/K)$. To prove this result it suffices to show that

$$[a]_{xH} b_y = \sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK} b_y \,, \tag{2.21}$$

for all $y \in X$ and $b \in \mathcal{A}_y$. First we notice that both the right and left hand sides of (2.21) are zero unless $\mathbf{r}(y) \in \mathbf{s}(x)H$. In case $\mathbf{r}(y) \in \mathbf{s}(x)H$ we have

$$[a]_{xH} b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y} \,,$$

where $\widetilde{h}$ is any element of $H_{x,y}$.

Recall from Proposition 1.3.2 that there is a bijective correspondence between the set of K-orbits $(xH)/K$ and the double coset space $\mathcal{S}_x \backslash H/K$. It is clear that $[a]_{x\widetilde{h}K} b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}$. Moreover, for all the classes $[h] \neq [\widetilde{h}]$ in $\mathcal{S}_x \backslash H/K$ we have $\mathbf{r}(y) \notin \mathbf{s}(x)hK$, because $\mathbf{r}(y) \in \mathbf{s}(x)\widetilde{h}K$. Hence, for all the classes $[h] \neq [\widetilde{h}]$ in $\mathcal{S}_x \backslash H/K$ we have $[\alpha_{h^{-1}}(a)]_{xhK} b_y = 0$. We conclude that

$$\sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK} b_y = [\alpha_{\widetilde{h}^{-1}}(a)]_{x\widetilde{h}K} b_y = (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y} \,,$$

and equality (2.21) is proven. $\square$

Remark 2.3.8. In Proposition 2.3.7 the fact that $[H:K] < \infty$ was only used to ensure that the sum on the right hand side of (2.20) was finite. One could more generally just require that the sets $\mathcal{S}_x \backslash H/K$ are finite for all $x \in X$, but this generality will not be used here.

As we saw in Proposition 2.1.6 we have an action $\overline{\alpha}$ of G on $C_c(\mathcal{A})$. This action can be extended in a unique way to an action on $M(C_c(\mathcal{A}))$, which we will still denote by $\overline{\alpha}$, by the following formula:

$$\overline{\alpha}_g(T)f := \overline{\alpha}_g\big(T\overline{\alpha}_{g^{-1}}(f)\big) \,, \tag{2.22}$$

where $g \in G$, $T \in M(C_c(\mathcal{A}))$ and $f \in C_c(\mathcal{A})$. We will now show what this action on $M(C_c(\mathcal{A}))$ does to the algebras $C_b(X^0)$, $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$.

Proposition 2.3.9. *The extension of the action $\overline{\alpha}$ to $M(C_c(\mathcal{A}))$, also denoted by $\overline{\alpha}$, satisfies the following properties:*

(i) *The restriction of $\overline{\alpha}$ to $C_b(X^0)$ is precisely the action that comes from the G-action on X^0.*

(ii) *For any $g \in G$ the automorphism $\overline{\alpha}_g$ takes $C_c(X^0/H)$ to $C_c(X^0/gHg^{-1})$, by*

$$\overline{\alpha}_g(1_{xH}) = 1_{(xg^{-1})(gHg^{-1})} \,. \tag{2.23}$$

(iii) *For any $g \in G$ the automorphism $\overline{\alpha}_g$ takes $C_c(\mathcal{A}/H)$ to $C_c(\mathcal{A}/gHg^{-1})$, by*

$$\overline{\alpha}_g([a]_{xH}) = [\alpha_g(a)]_{(xg^{-1})(gHg^{-1})} \,. \tag{2.24}$$

(iv) *Both $C_c(\mathcal{A}/H)$ and $C_c(X^0/H)$ are contained in $M(C_c(\mathcal{A}))^H$, the algebra of H-fixed points.*

Proof: (i) Let $y \in X$, $b \in \mathcal{A}_y$ and $f \in C_b(X^0)$. For any $g \in G$ let us denote by $f_g \in C_b(X^0)$ the function defined by $f_g(x) = f(xg)$. By definition of the extension of α to $M(C_c(\mathcal{A}))$, we have

$$\begin{aligned}
\overline{\alpha}_g(f)\, b_y &= \overline{\alpha}_g(f \cdot \overline{\alpha}_g^{-1}(b_y)) \;=\; \overline{\alpha}_g(f \cdot \alpha_{g^{-1}}(b)_{yg}) \\
&= \overline{\alpha}_g(f(\mathbf{r}(yg))\alpha_{g^{-1}}(b)_{yg}) \;=\; \overline{\alpha}_g(f(\mathbf{r}(y)g)\alpha_{g^{-1}}(b)_{yg}) \\
&= f(\mathbf{r}(y)g)b_y \;=\; f_g(\mathbf{r}(y))b_y \\
&= f_g \cdot b_y\,.
\end{aligned}$$

Hence we conclude that $\overline{\alpha}_g(f) = f_g$ and therefore the action $\overline{\alpha}$ on $C_b(X^0)$ is just the action that comes from the G-action on X^0.

(ii) This follows directly from (i).

(iii) Let $y \in X$ and $b \in \mathcal{A}_y$. By definition of the extension of $\overline{\alpha}$ to $M(C_c(\mathcal{A}))$, we have

$$\overline{\alpha}_g([a]_{xH})\, b_y = \overline{\alpha}_g([a]_{xH}\overline{\alpha}_g^{-1}(b_y)) = \overline{\alpha}_g([a]_{xH}\alpha_{g^{-1}}(b)_{yg})\,.$$

Also, we can see that

$$\begin{aligned}
\overline{\alpha}_g([a]_{xH}\alpha_{g^{-1}}(b)_{yg}) &= \begin{cases} \overline{\alpha}_g\big((\alpha_{\widetilde{h}^{-1}}(a)\alpha_{g^{-1}}(b))_{x\widetilde{h}(yg)}\big)\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (\alpha_{g\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}g^{-1}y}\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (\alpha_{g\widetilde{h}^{-1}}(a)b)_{xg^{-1}g\widetilde{h}g^{-1}y}\,, & \text{if } H_{x,yg} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where $\widetilde{h} \in H_{x,yg}$. Now an easy computation shows that we have

$$H_{x,yg} = g^{-1}\big(gHg^{-1}\big)_{xg^{-1},y}\, g\,,$$

and thereby we obtain, for $t \in \big(gHg^{-1}\big)_{xg^{-1},y}$,

$$\begin{aligned}
\overline{\alpha}_g([a]_{xH})\, b_y &= \begin{cases} (\alpha_{gg^{-1}t^{-1}g}(a)b)_{xg^{-1}ty}\,, & \text{if } \big(gHg^{-1}\big)_{xg^{-1},y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (\alpha_{t^{-1}g}(a)b)_{xg^{-1}ty}\,, & \text{if } \big(gHg^{-1}\big)_{xg^{-1},y} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \\
&= [\alpha_g(a)]_{(xg^{-1})(gHg^{-1})}\, b_y\,.
\end{aligned}$$

(iv) This follows directly from (ii) and (iii). $\square$

It is important to know how to multiply an element of $C_c(\mathcal{A}/H)$ with an element of $C_c(X^0/gHg^{-1})$ inside $M(C_c(\mathcal{A}))$. This is easy if we are under the assumption that G-action satisfies the H-intersection property. We recall from (1.8) that H^g stands for the subgroup $H \cap gHg^{-1}$.

PROPOSITION 2.3.10. *If the G-action moreover satisfies the H-intersection property, then for every $x \in X$ and $g \in G$ the following equality holds in $M(C_c(\mathcal{A}))$:*

$$[a]_{xH}\, 1_{\mathbf{s}(x)gHg^{-1}} = [a]_{xH^g}\,.$$

Proof: For any $y \in X$ and $b \in \mathcal{A}_y$ we have

$$\begin{aligned}
& [a]_{xH}\, 1_{\mathbf{s}(x)gHg^{-1}}\, b_y \quad = \\
= \quad & \begin{cases} [a]_{xH} b_y\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)gHg^{-1} \\ 0, & \text{otherwise} \end{cases} \\
= \quad & \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)gHg^{-1} \text{ and } \mathbf{r}(y) \in \mathbf{s}(x)H \\ 0, & \text{otherwise} \end{cases} \\
= \quad & \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)H \cap \mathbf{s}(x)gHg^{-1} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where $\widetilde{h} \in H_{x,y}$. Now, by the H-intersection property, we obtain

$$= \begin{cases} (\alpha_{\widetilde{h}^{-1}}(a)b)_{x\widetilde{h}y}\,, & \text{if } \mathbf{r}(y) \in \mathbf{s}(x)H^g \\ 0, & \text{otherwise} \end{cases}\,.$$

Of course, we have $(H^g)_{x,y} \subseteq H_{x,y}$, and hence we can choose $\widetilde{h}$ as an element of $(H^g)_{x,y}$, thereby obtaining

$$= [a]_{xH^g}\, b_y\,,$$

which finishes the proof. □

CHAPTER 3

*-Algebraic crossed product by a Hecke pair

In this chapter we introduce our notion of a (*-algebraic) crossed product by a Hecke pair and we explore its basic properties and its representation theory. Throughout the rest of this work we impose the following standing assumption, based on the tools developed in Section 2.1.

STANDING ASSUMPTION 3.0.1. We assume from now on that (G, Γ) is a Hecke pair, $\mathcal{A}$ is a Fell bundle over a groupoid X endowed with a Γ-good right G-action α satisfying the Γ-intersection property.

3.1. Definition of the crossed product and basic properties

In this section we aim at defining the (*-algebraic) crossed product of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ). For that we are going to define some sort of a bundle over G/Γ, where the fiber over each $g\Gamma$ is precisely $C_c(\mathcal{A}/\Gamma^g)$. Recall that we denote by $\overline{\alpha}$ the associated action of G on $C_c(\mathcal{A})$ and also its extension to $M(C_c(\mathcal{A}))$.

DEFINITION 3.1.1. Let $B(\mathcal{A}, G, \Gamma)$ be the vector space of finitely supported functions $f : G/\Gamma \to M(C_c(\mathcal{A}))$ satisfying the following compatibility condition

$$f(\gamma g\Gamma) = \overline{\alpha}_\gamma(f(g\Gamma))\,, \tag{3.1}$$

for all $\gamma \in \Gamma$ and $g\Gamma \in G/\Gamma$.

LEMMA 3.1.2. *For every $f \in B(\mathcal{A}, G, \Gamma)$ and $g\Gamma \in G/\Gamma$ we have*

$$f(g\Gamma) \in M(C_c(\mathcal{A}))^{\Gamma^g} .$$

Proof : This follows directly from the compatibility condition (3.1), since for every $\gamma \in \Gamma^g$ we have $\overline{\alpha}_\gamma(f(g\Gamma)) = f(\gamma g\Gamma) = f(g\Gamma)$. $\square$

DEFINITION 3.1.3. The vector subspace of $B(\mathcal{A}, G, \Gamma)$ consisting of the functions $f : G/\Gamma \to M(C_c(\mathcal{A}))$ satisfying the compatibility condition (3.1) and the property

$$f(g\Gamma) \in C_c(\mathcal{A}/\Gamma^g)\,, \tag{3.2}$$

will be denoted by $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and will be called the **-algebraic crossed product* of $C_c(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ).

It is relevant to point out that the definitions of the spaces $B(\mathcal{A}, G, \Gamma)$ and $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ seem more suitable for Hecke pairs (G, Γ), as in general a function in $B(\mathcal{A}, G, \Gamma)$ could only have support on those elements $g\Gamma \in G/\Gamma$ such that $|\Gamma g \Gamma / \Gamma| < \infty$.

We now define a product and an involution in $B(\mathcal{A}, G, \Gamma)$ by:

$$(f_1 * f_2)(g\Gamma) := \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \overline{\alpha}_h(f_2(h^{-1}g\Gamma))\,, \tag{3.3}$$

$$(f^*)\,(g\Gamma) := \Delta(g^{-1}) \overline{\alpha}_g(f(g^{-1}\Gamma))^*\,. \tag{3.4}$$

Proposition 3.1.4. *$B(\mathcal{A}, G, \Gamma)$ becomes a unital *-algebra under the product and involution defined above, whose identity element is the function f such that $f(\Gamma) = 1$ and is zero in the remaining points of G/Γ.*

Proof: First, we claim that the expression for the product defined above is well-defined in $B(\mathcal{A}, G, \Gamma)$, i.e. for $f_1, f_2 \in B(\mathcal{A}, G, \Gamma)$ the expression

$$(f_1 * f_2)(g\Gamma) := \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \overline{\alpha}_h(f_2(h^{-1}g\Gamma))$$

is independent from the choice of the representatives $[h]$ and also that it has finitely many summands. Independence from the choice of the representatives $[h] \in G/\Gamma$ follows directly from the compatibility condition (3.1) and the fact that the sum is finite follows simply from the fact that f_1 has finite support.

Now we claim that $f_1 * f_2$ has also finite support, for $f_1, f_2 \in B(\mathcal{A}, G, \Gamma)$. Let $S_1, S_2 \subseteq G/\Gamma$ be the supports of the functions f_1 and f_2 respectively. We will regard S_1 and S_2 as subsets of G (being finite unions of left cosets). It is easy to check that the function $G \times G \to M(C_c(\mathcal{A}))$

$$(h, g) \mapsto f_1(h\Gamma)\overline{\alpha}_h(f_2(h^{-1}g\Gamma))$$

has support contained in $S_1 \times (S_1 \cdot S_2)$. Since (G, Γ) is a Hecke pair, the product $S_1 \cdot S_2$ is also a finite union of left cosets. Hence, $f_1 * f_2$ has finite support.

We also notice that $f_1 * f_2$ satisfies the compatibility condition (3.1), thus defining an element of $B(\mathcal{A}, G, \Gamma)$, since for any $\gamma \in \Gamma$ we have

$$\begin{aligned}
(f_1 * f_2)(\gamma g\Gamma) &= \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\, \overline{\alpha}_h(f_2(h^{-1}\gamma g\Gamma)) \\
&= \sum_{[h] \in G/\Gamma} f_1(\gamma h\Gamma)\, \overline{\alpha}_{\gamma h}(f_2(h^{-1}g\Gamma)) \\
&= \sum_{[h] \in G/\Gamma} \overline{\alpha}_\gamma(f_1(h\Gamma))\, \overline{\alpha}_\gamma \circ \overline{\alpha}_h(f_2(h^{-1}g\Gamma)) \\
&= \overline{\alpha}_\gamma\big((f_1 * f_2)(g\Gamma)\big)\,.
\end{aligned}$$

In a similar way we can see that the expression that defines the involution is well-defined in $B(\mathcal{A}, G, \Gamma)$. There are now a few things that need to be checked before we can say that $B(\mathcal{A}, G, \Gamma)$ is a *-algebra, namely that the product is associative and the involution is indeed an involution relatively to this product (the fact that the product is distributive and the properties concerning multiplication by scalars are obvious). The proofs of these facts are essentially just a mimic of the corresponding proofs for "classical" crossed products by groups. Thus, we can say that $B(\mathcal{A}, G, \Gamma)$ is *-algebra under this product and involution. $\square$

THEOREM 3.1.5. *$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-ideal of $B(\mathcal{A}, G, \Gamma)$. In particular it is a *-algebra for the above operations.*

Proof: It is easy to see that the space $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is invariant for the involution, i.e.

$$f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \implies f^* \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma .$$

Thus, to prove that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a (two-sided) *-ideal of $B(\mathcal{A}, G, \Gamma)$ it is enough to prove that it is a right ideal, i.e. if $f_1 \in B(\mathcal{A}, G, \Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ then $f_1 * f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, because any right *-ideal is automatically two-sided. Hence, all we need to prove is that $(f_1 * f_2)(g\Gamma) \in C_c(\mathcal{A}/\Gamma^g)$, for every $f_1 \in B(\mathcal{A}, G, \Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. The proof of this fact will follow the following steps:

1) Prove that: given a subgroup $H \subseteq G$, $f \in C_c(\mathcal{A}/H)$ and a unit $u \in X^0$, we have $f \cdot 1_u \in C_c(\mathcal{A})$.
2) Let $T := (f_1 * f_2)(g\Gamma) = \sum_{[h] \in G/\Gamma} f_1(h\Gamma)\overline{\alpha}_h(f_2(h^{-1}g\Gamma))$. Use 1) to show that $T \cdot 1_u \in C_c(\mathcal{A})$ for any unit $u \in X^0$.
3) Fix a unit $u \in X^0$. By 2) we have $T\, 1_u = \sum_i (a_i)_{x_i}$, where the elements $x_i \in X$ are such that $\mathbf{s}(x_i) = u$. Show that $T\, 1_{u\Gamma^g} = \sum_i [a_i]_{x_i\Gamma^g}$, and conclude that $T\, 1_{u\Gamma^g} \in C_c(\mathcal{A}/\Gamma^g)$.
4) Prove that there exists a finite set of units $\{u_1, \dots, u_n\} \subseteq X^0$ such that $T = \sum_{i=1}^n T\, 1_{u_i\Gamma^g}$. Conclude that $T \in C_c(\mathcal{A}/\Gamma^g)$.

- Proof of 1) : This follows immediately from Proposition 2.3.5.
- Proof of 2) : We know that $f_2(h^{-1}g\Gamma) \in C_c(\mathcal{A}/\Gamma^{h^{-1}g})$. Thus, from Proposition 2.3.9, we conclude that $\overline{\alpha}_h(f_2(h^{-1}g)) \in C_c\big(\mathcal{A}/h\Gamma h^{-1} \cap g\Gamma g^{-1}\big)$. Now, using 1), we see that $\overline{\alpha}_h(f_2(h^{-1}g))\, 1_u \in C_c(\mathcal{A})$ and consequently $f_1(h\Gamma)\overline{\alpha}_h(f_2(h^{-1}g))\, 1_u \in C_c(\mathcal{A})$. Hence, $T\, 1_u \in C_c(\mathcal{A})$.
- Proof of 3) : For any $\gamma \in \Gamma^g$ we have, using Lemma 3.1.2,

$$\begin{aligned} T\, 1_{u\gamma} &= \overline{\alpha}_{\gamma^{-1}}(T)\, 1_{u\gamma} = \overline{\alpha}_{\gamma^{-1}}\big(T\overline{\alpha}_\gamma(1_{u\gamma})\big) \\ &= \overline{\alpha}_{\gamma^{-1}}\big(T\, 1_u\big) = \sum_i \alpha_{\gamma^{-1}}(a_i)_{x_i\gamma} . \end{aligned}$$

  Let $y \in X$ and $b \in \mathcal{A}_y$. We have

$$T\, 1_{u\Gamma^g}\, b_y = \begin{cases} Tb_y , & \text{if } \mathbf{r}(y) \in u\Gamma^g \\ 0, & \text{otherwise} . \end{cases}$$

  Assume now that $\mathbf{r}(y) \in u\Gamma^g$ and let $\widetilde{\gamma} \in \Gamma^g$ be such that $\mathbf{r}(y) = u\widetilde{\gamma}$. We then have

$$Tb_y = T\, 1_{u\widetilde{\gamma}}\, b_y = \sum_i \alpha_{\widetilde{\gamma}^{-1}}(a_i)_{x_i\widetilde{\gamma}}\, b_y .$$

Since $\mathbf{s}(x_i) = u$, we have $\mathbf{s}(x_i\widetilde{\gamma}) = u\widetilde{\gamma} = \mathbf{r}(y)$. Hence,

$$Tb_y \;=\; \sum_i (\alpha_{\widetilde{\gamma}^{-1}}(a_i)b)_{x_i\widetilde{\gamma}y}\,.$$

We conclude that

$$\begin{aligned} T\,1_{u\Gamma^g}\,b_y &= \begin{cases} \sum_i (\alpha_{\widetilde{\gamma}^{-1}}(a_i)b)_{x_i\widetilde{\gamma}y}\,, & \text{if } \mathbf{r}(y) \in u\Gamma^g \\ 0, & \text{otherwise}\,. \end{cases} \\ &= \sum_i [a_i]_{x_i\Gamma^g}\,b_y\,. \end{aligned}$$

Thus, $T\,1_{u\Gamma^g} = \sum_i [a_i]_{x_i\Gamma^g} \in C_c(\mathcal{A}/\Gamma^g)$.

- Proof of 4) : For easiness of reading of this last part of the proof we introduce the following definition: given $F \in M(C_c(\mathcal{A}))$ we define the *support* of F to be the set $\{u \in X^0 : F\,1_u \neq 0\}$. Notice in particular that the support of an element $[a]_{xH}$, with $a \neq 0$, is the set $\mathbf{s}(x)H$.

  Since $\overline{\alpha}_h(f_2(h^{-1}g\Gamma)) \in C_c(\mathcal{A}/h\Gamma h^{-1} \cap g\Gamma g^{-1})$, there exists a finite number of units $v_1, \dots, v_k \in X^0$ such that $\overline{\alpha}_h(f_2(h^{-1}g\Gamma))$ has support in

$$\bigcup_{i=1}^{k} v_i\big(h\Gamma h^{-1} \cap g\Gamma g^{-1}\big) \;\subseteq\; \bigcup_{i=1}^{k} v_i g\Gamma g^{-1}\,.$$

  Hence, there is a finite number of units $w_1, \dots, w_l \in X^0$ such that T has support contained in

$$\bigcup_{i=1}^{l} w_i g\Gamma g^{-1}\,.$$

  Therefore, T has support contained in

$$\bigcup_{i=1}^{l}\bigcup_{j=1}^{m} w_i\theta_j\Gamma^g\,,$$

  where $\theta_1, \dots, \theta_m$ are representatives of the classes of $g\Gamma g^{-1}/\Gamma^g$ (being a finite number because (G,Γ) is a Hecke pair). Thus, we have proven that there is a finite number of units $u_1, \dots, u_n \in X^0$ such that T has support inside $\bigcup_{i=1}^{n} u_i\Gamma^g$. Moreover, we can suppose we have chosen the units $u_1, \dots, u_n$ such that the corresponding orbits $u_i\Gamma^g$ are mutually disjoint. It is now easy to see that we have $T = \sum_{i=1}^{n} T\,1_{u_i\Gamma^g}$. Indeed, given $y \in X$ and $b \in \mathcal{A}_y$, if $\mathbf{r}(y) \notin \bigcup_{i=1}^{n} u_i\Gamma^g$, then

$$Tb_y = T\,1_{\mathbf{r}(y)}\,b_y = 0 = \sum_{i=1}^{n} T\,1_{u_i\Gamma^g}\,b_y\,,$$

  and if $\mathbf{r}(y) \in \bigcup_{i=1}^{n} u_i\Gamma^g$, then $\mathbf{r}(y)$ belongs to precisely one of the orbits, say $u_{i_0}\Gamma^g$, and we have

$$\sum_{i=1}^{n} T\,1_{u_i\Gamma^g}\,b_y = T1_{u_{i_0}\Gamma^g}\,b_y = Tb_y\,.$$

  Hence, we must have $T = \sum_{i=1}^{n} T\,1_{u_i\Gamma^g}$, and by 3) we conclude that $T \in C_c(\mathcal{A}/\Gamma^g)$. □

As it is well-known, when working with crossed products $A \times G$ by discrete groups, one always has an embedded copy of A inside the crossed product. Something analogous happens in the case of crossed products by Hecke pairs, where $C_c(\mathcal{A}/\Gamma)$ is canonically embedded in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, as is stated in the next result (whose proof amounts to routine verification).

PROPOSITION 3.1.6. *There is a natural embedding of the* **-algebra* $C_c(\mathcal{A}/\Gamma)$ *in* $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$*, which identifies an element* $f \in C_c(\mathcal{A}/\Gamma)$ *with the function* $\iota(f) \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *such that*

$$\iota(f)(\Gamma) = f \qquad \text{and} \qquad \iota(f) \text{ is zero elsewhere}.$$

REMARK 3.1.7. The above result says that we can identify $C_c(\mathcal{A}/\Gamma)$ with the functions of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with support in Γ. We shall, henceforward, make no distinctions in notation between an element of $C_c(\mathcal{A}/\Gamma)$ and its correspondent in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.

THEOREM 3.1.8. $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *is an essential* **-ideal of* $B(\mathcal{A}, G, \Gamma)$*. In particular,* $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *is an essential* **-algebra. Moreover, there are natural embeddings*

$$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \hookrightarrow B(\mathcal{A}, G, \Gamma) \hookrightarrow M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\,,$$

that make the following diagram commute

$$\begin{tikzcd} & M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) \\ C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \arrow[ur, "L"] \arrow[r] & B(\mathcal{A}, G, \Gamma)\,. \arrow[u] \end{tikzcd}$$

Proof: We have already proven that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-ideal of $B(\mathcal{A}, G, \Gamma)$, thus we only need to check that this ideal is in fact essential. Suppose $f \in B(\mathcal{A}, G, \Gamma)$ is such that $f * \big(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\big) = \{0\}$. Then, in particular, using Proposition 3.1.6, we must have $f * \big(C_c(\mathcal{A}/\Gamma)\big) = \{0\}$. Let $g \in G$ and take $[a]_{x\Gamma} \in C_c(\mathcal{A}/\Gamma)$, we then have

$$0 = \Big(f * [a]_{x\Gamma}\Big)(g\Gamma) = f(g\Gamma)\overline{\alpha}_g([a]_{x\Gamma}) = f(g\Gamma)[\alpha_g(a)]_{xg^{-1}g\Gamma g^{-1}}\,.$$

Thus, multiplying by $1_{\mathbf{s}(x)g^{-1}} \in M(C_c(\mathcal{A}))$ we get

$$0 = f(g\Gamma)[\alpha_g(a)]_{xg^{-1}g\Gamma g^{-1}} 1_{\mathbf{s}(x)g^{-1}} = f(g\Gamma)\alpha_g(a)_{xg^{-1}} = f(g\Gamma)\overline{\alpha}_g(a_x)\,.$$

Since this true for all $a \in \mathcal{A}_x$ and $x \in X$ and given that α takes fibers of $\mathcal{A}$ bijectively into fibers of $\mathcal{A}$, we must have $f(g\Gamma)b_y = 0$ for all $b \in \mathcal{A}_y$ and $y \in X$. Hence, we must have $f(g\Gamma) = 0$. Thus, $f = 0$ and we conclude that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is an essential *-ideal of $B(\mathcal{A}, G, \Gamma)$.

Since $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is a *-subalgebra of $B(\mathcal{A}, G, \Gamma)$, we immediately conclude that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is an essential *-algebra.

The embedding of $B(\mathcal{A}, G, \Gamma)$ in $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$ then follows from the universal property of multiplier algebras, Theorem 1.2.10. $\square$

In the theory of crossed products $A \times G$ by groups, one always has an embedded copy of the group algebra $\mathbb{C}(G)$ inside the multiplier algebra $M(A \times G)$. Something analogous happens in the case of crossed products by Hecke pairs, where the Hecke algebra $\mathcal{H}(G, \Gamma)$ is canonically embedded in the multiplier algebra $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$, as is stated in the next result (whose proof amounts to routine verification).

PROPOSITION 3.1.9. *The Hecke *-algebra $\mathcal{H}(G, \Gamma)$ embeds in $B(\mathcal{A}, G, \Gamma)$ in the following way: an element $f \in \mathcal{H}(G, \Gamma)$ is identified with the element $\widetilde{f} \in B(\mathcal{A}, G, \Gamma)$ given by*

$$\widetilde{f}(g\Gamma) := f(g\Gamma)\mathbf{1}\,,$$

where $\mathbf{1}$ is the unit of $M(C_c(\mathcal{A}))$.

The next result does not typically play an essential role in the case of crossed products by groups, but will be extremely important for us in case of crossed products by Hecke pairs. The proof is also just routine verification.

PROPOSITION 3.1.10. *The algebra $C_c(X^0/\Gamma)$ embeds in $B(\mathcal{A}, G, \Gamma)$ in the following way: an element $f \in C_c(X^0/\Gamma)$ is identified with the function $\iota(f) \in B(\mathcal{A}, G, \Gamma)$ given by*

$$\iota(f)(\Gamma) = f \qquad \textit{and} \qquad \iota(f) \ \textit{is zero elsewhere}\,.$$

REMARK 3.1.11. Propositions 3.1.9 and 3.1.10 allow us to view both the Hecke *-algebra $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ as *-subalgebras of $B(\mathcal{A}, G, \Gamma)$. We shall henceforward make no distinctions in notation between an element of $\mathcal{H}(G, \Gamma)$ or $C_c(X^0/\Gamma)$ and its correspondent in $B(\mathcal{A}, G, \Gamma)$.

The purpose of the following diagram is to illustrate, in a more condensed form, all the canonical embeddings we have been considering so far:

$$\begin{tikzcd}
C_c(\mathcal{A}/\Gamma) \arrow[r] & C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \arrow[dr] & & \\
\mathcal{H}(G, \Gamma) \arrow[rr] & & B(\mathcal{A}, G, \Gamma) \arrow[r] & M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)) \\
C_c(X^0/\Gamma) \arrow[urr] & & &
\end{tikzcd}$$

REMARK 3.1.12. The reason for considering the algebra $B(\mathcal{A}, G, \Gamma)$ is two-fold. On one side $B(\mathcal{A}, G, \Gamma)$ made it easier to make sure the convolution product (3.3) was well-defined in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. On the other (perhaps more important) side, the fact that both $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ are canonically embedded in $B(\mathcal{A}, G, \Gamma)$ allows us to treat the elements of $\mathcal{H}(G, \Gamma)$ and $C_c(X^0/\Gamma)$ both as multipliers in $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma))$, but also allows us to operate these elements with the convolution product and involution expressions (3.3) and (3.4), as these are defined in $B(\mathcal{A}, G, \Gamma)$.

As it is well-known in the theory of crossed products by discrete groups, a (*-algebraic) crossed product $A \times G$ is spanned by elements of the form $a * g$, where $a \in A$ and $g \in G$ (here g is seen as an element of the group algebra $\mathbb{C}(G) \subseteq M(A \times G)$). We will now explore something analogous in the case of crossed products by Hecke pairs. It turns out that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form $[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$, where $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$, as we show in the next result.

THEOREM 3.1.13. *For any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have*

$$f = \sum_{[g] \in \Gamma \backslash G/\Gamma} \sum_{x\Gamma^g \in X/\Gamma^g} \Big[f(g\Gamma)(x) \Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \,. \tag{3.5}$$

In particular, $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form

$$[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \,,$$

with $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$.

The following lemma is needed in order to prove the above result:

LEMMA 3.1.14. *Let $x \in X$, $a \in \mathcal{A}_x$ and $g\Gamma \in G/\Gamma$. We have*

$$[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \ (h\Gamma) = \begin{cases} [\alpha_\gamma(a)]_{x\gamma^{-1}\Gamma^{\gamma g}} \,, & \text{if } h\Gamma = \gamma g \Gamma, \ \text{ with } \gamma \in \Gamma \\ 0, & \text{otherwise} \ . \end{cases}$$

In particular,

$$[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \ (g\Gamma) = [a]_{x\Gamma^g} \,.$$

Proof: An easy computation yields

$$[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \ (h\Gamma) \ = \ [a]_{x\Gamma} \cdot \Gamma g \Gamma(h\Gamma) \cdot \overline{\alpha}_h(1_{\mathbf{s}(x)g\Gamma}) \,,$$

from which we conclude that $[a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$ is supported in the double coset $\Gamma g \Gamma$. Now, evaluating at the point $g\Gamma \in G/\Gamma$ we get

$$\begin{aligned} [a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma} \ (g\Gamma) \ &= \ [a]_{x\Gamma} \cdot \Gamma g \Gamma(g\Gamma) \cdot \overline{\alpha}_g(1_{\mathbf{s}(x)g\Gamma}) \\ &= \ [a]_{x\Gamma} \cdot \overline{\alpha}_g(1_{\mathbf{s}(x)g\Gamma}) \\ &= \ [a]_{x\Gamma} \cdot 1_{\mathbf{s}(x)g\Gamma g^{-1}} \\ &= \ [a]_{x\Gamma^g} \,, \end{aligned}$$

where the last equality comes from Proposition 2.3.10. From the compatibility condition (3.1) and Proposition 2.3.9 it then follows that, for $\gamma \in \Gamma$,

$$\begin{aligned} [a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g\Gamma}\ (\gamma g \Gamma) &= \overline{\alpha}_\gamma([a]_{x\Gamma^g}) \\ &= [\alpha_\gamma(a)]_{x\gamma^{-1}\Gamma^{\gamma g}} \,. \end{aligned}$$

□

Proof of Theorem 3.1.13: Let us first prove that the expression on the right hand side of (3.5) is well-defined. It is easy to see that for every $g \in G$, the expression

$$\sum_{x\Gamma^g \in X/\Gamma^g} \Big[f(g\Gamma)(x)\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g \Gamma}$$

does not depend on the choice of the representative x of $x\Gamma^g$. Now, let us see that it also does not depend on the choice of the representative g in $\Gamma g \Gamma$. Let $\gamma g \theta$, with $\gamma, \theta \in \Gamma$, be any other representative. We have

$$\begin{aligned} &\sum_{x\Gamma^{\gamma g \theta} \in X/\Gamma^{\gamma g \theta}} \Big[f(\gamma g \theta \Gamma)(x)\Big]_{x\Gamma} * \Gamma \gamma g \theta \Gamma * 1_{\mathbf{s}(x) \gamma g \theta \Gamma} = \\ = &\sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big[f(\gamma g \Gamma)(x)\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) \gamma g \Gamma} \\ = &\sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big[\overline{\alpha}_\gamma(f(g\Gamma))(x)\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) \gamma g \Gamma} \\ = &\sum_{x\Gamma^{\gamma g} \in X/\Gamma^{\gamma g}} \Big[\alpha_\gamma(f(g\Gamma)(x\gamma))\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) \gamma g \Gamma} \end{aligned}$$

We notice that there is a well-defined bijective correspondence $X/\Gamma^g \to X/\Gamma^{\gamma g}$ given by $x\Gamma^g \mapsto x\gamma^{-1}\Gamma^{\gamma g}$. Thus, we get

$$\begin{aligned} = &\sum_{x\Gamma^g \in X/\Gamma^g} \Big[\alpha_\gamma(f(g\Gamma)(x))\Big]_{x\gamma^{-1}\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x\gamma^{-1}) \gamma g \Gamma} \\ = &\sum_{x\Gamma^g \in X/\Gamma^g} \Big[f(g\Gamma)(x)\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g \Gamma} \,. \end{aligned}$$

Hence, the expression in (3.5) is well-defined. Let us now prove the decomposition in question. For any $t\Gamma \in G/\Gamma$ we have

$$\begin{aligned} &\sum_{[g] \in \Gamma \backslash G/\Gamma} \ \sum_{x\Gamma^g \in X/\Gamma^g} \Big[f(g\Gamma)(x)\Big]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x) g \Gamma}\ (t\Gamma) = \\ = &\sum_{x\Gamma^t \in X/\Gamma^t} \Big[f(t\Gamma)(x)\Big]_{x\Gamma} * \Gamma t \Gamma * 1_{\mathbf{s}(x) t \Gamma}\ (t\Gamma) \,. \end{aligned}$$

By Lemma 3.1.14 it follows that

$$\begin{aligned} &= \sum_{x\Gamma^t \in X/\Gamma^t} \Big[f(t\Gamma)(x)\Big]_{x\Gamma^t} \\ &= f(t\Gamma) \,, \end{aligned}$$

and this finishes the proof. □

In the following result we collect some useful equalities concerning products in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, which will be useful later on. One should observe the similarities between the equalities (3.8) and (3.9) and the equalities obtained by an Huef, Kaliszewski and Raeburn in [**9**, Lemma 1.3 (i) and (ii)] if in their setting one was allowed to somehow "drop" the representations. The similarity is more than a coincidence as we will see later in Chapter **??**.

PROPOSITION 3.1.15. *In $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ the following equalities hold:*

$$\big([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^* = \Delta(g)\ [\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}\,, \tag{3.6}$$

$$1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma} = [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\,, \tag{3.7}$$

$$[a]_{x\Gamma} * \Gamma g\Gamma = \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\,. \tag{3.8}$$

$$\Gamma g\Gamma * [a]_{x\Gamma} = \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\gamma g^{-1}\Gamma} * \Gamma g\Gamma * [a]_{x\Gamma}\,. \tag{3.9}$$

In particular, from (3.7) we see that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is also spanned by all elements of the form $1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [a]_{xg\Gamma}$, with $g \in G$, $x \in X$ and $a \in \mathcal{A}_x$.

Proof: Let us first prove equality (3.6). First we notice that

$$\big([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^* = \Delta(g)\ 1_{\mathbf{s}(x)g\Gamma} * \Gamma g^{-1}\Gamma * [a^*]_{x^{-1}\Gamma}\,,$$

which means that $\big([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^*$ has support in the double coset $\Gamma g^{-1}\Gamma$. Now evaluating this element on $g^{-1}\Gamma$ we get,

$$\begin{aligned}
&\big([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)^* (g^{-1}\Gamma) \;=\\
=\;& \Delta(g)\ \overline{\alpha}_{g^{-1}}\big(\big([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}\big)(g\Gamma)\big)^*\\
=\;& \Delta(g)\ \overline{\alpha}_{g^{-1}}([a]_{x\Gamma^g})^*\\
=\;& \Delta(g)\ [\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma^{g^{-1}}}\\
=\;& \Delta(g)\ \big([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}\big)(g^{-1}\Gamma)\,.
\end{aligned}$$

Let us now prove equality (3.7). We have

$$\begin{aligned}
1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma} &= \Delta(g)\ \big([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma}\big)^*\\
&= \Delta(g)\ \big([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1}g)g^{-1}\Gamma}\big)^*\,,
\end{aligned}$$

which together with (3.6) yields

$$\begin{aligned}
&= \Delta(g)\Delta(g^{-1})\ [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(xg)\Gamma}\\
&= [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(xg)\Gamma}\,.
\end{aligned}$$

Let us now prove (3.8). An easy computation yields

$$[a]_{x\Gamma} * \Gamma g\Gamma\ (h\Gamma) \;=\; [a]_{x\Gamma} \cdot \Gamma g\Gamma(h\Gamma)\,,$$

from which we conclude that $[a]_{x\Gamma} * \Gamma g\Gamma$ has support in $\Gamma g\Gamma$. Evaluating this element on the point $g\Gamma$ we get

$$[a]_{x\Gamma} * \Gamma g\Gamma\ (g\Gamma) \;=\; [a]_{x\Gamma} \cdot \Gamma g\Gamma(g\Gamma) \;=\; [a]_{x\Gamma}\,.$$

From Proposition 2.3.7 one always has the following decomposition

$$[a]_{x\Gamma} \;=\; \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [\alpha_{\gamma^{-1}}(a)]_{x\gamma\Gamma^g}\,.$$

Together with Lemma 3.1.14 we get

$$\begin{aligned}
[a]_{x\Gamma} * \Gamma g\Gamma\;(g\Gamma) &= [a]_{x\Gamma} \\
&= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [\alpha_{\gamma^{-1}}(a)]_{x\gamma\Gamma^g} \\
&= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [\alpha_{\gamma^{-1}}(a)]_{x\gamma\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\;(g\Gamma) \\
&= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\;(g\Gamma)\,,
\end{aligned}$$

and equality (3.8) is proven.

Equality (3.9) follows easily from (3.8) by taking the involution and using the fact that $\mathcal{S}_x = \mathcal{S}_{x^{-1}}$.

The last claim of this proposition follows simply from (3.7) and Proposition 3.1.13. □

In the theory of crossed products $A \times G$ by discrete groups one has a "covariance relation" of the form $g * a * g^{-1} = \alpha_g(a)$. This relation is essential in the passage from covariant representations of the system (A, G, α) to representations of the crossed product. More generally, the following relation holds in $A \times G$:

$$g * a * h \;=\; \alpha_g(a) * gh\,.$$

We will now explore how this generalizes to the setting of crossed products by Hecke pairs. What we are aiming for is a description of how products of the form $\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma$ can be expressed by the canonical spanning set of elements of the form $[b]_{y\Gamma} * \Gamma h\Gamma * 1_{\mathbf{s}(x)h\Gamma}$ (according to Theorem 3.1.13). This will be achieved in Corollary 3.1.18 below and will play an important role in the representation theory of crossed products by Hecke pairs, particularly in the definition of covariant representations. One should observe the similarities between the expressions we obtain both in Theorem 3.1.16 and Corollary 3.1.18 and the expression provided by an Huef, Kaliszewski and Raeburn in [**9**, Definition 1.1] (if one "forgets" the representations in their setting). Once again, this is more than a coincidence as we will see in Chapter **??**. In fact, an Huef, Kaliszewski and Raeburn's definition served as a guiding line for our results below and for the definition of a covariant representation (Definition 3.3.1) which we shall present in the next section.

Before we establish the results we are aiming for we need to establish some notation, which will be used throughout this work. For $w, v \in G$ and a unit $y \in X^0$ we define the sets

$$\mathfrak{n}^y_{w,v} := \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; yw^{-1} \in y\Gamma r^{-1}\right\}, \tag{3.10}$$

$$\mathfrak{d}^y_{w,v} := \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \; yw^{-1} \in y\Gamma r^{-1}\Gamma^{wv}\right\}. \tag{3.11}$$

and the numbers

$$n^y_{w,v} := \#\, \mathfrak{n}^y_{w,v}\,, \tag{3.12}$$

$$d^y_{w,v} := \#\, \mathfrak{d}^y_{w,v}\,, \tag{3.13}$$

$$N^y_{w,v} := \frac{n^y_{w,v}}{d^y_{w,v}}\,. \tag{3.14}$$

We will also denote by $E^y_{u,v}$ the double coset space

$$E^y_{u,v} := \mathcal{S}_y \backslash \Gamma / (u\Gamma u^{-1} \cap v\Gamma v^{-1})\,. \tag{3.15}$$

THEOREM 3.1.16. *Let $g, s \in G$ and $y \in X^0$. We have that*

$$\begin{aligned}
\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma &= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)\\
&= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g) N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)\\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right).
\end{aligned}$$

In order to prove the above result we will need the following lemma, which gives some properties of the numbers $n^y_{w,v}$ and $d^y_{w,v}$.

LEMMA 3.1.17. *Let $w, v, \in G$, $\theta \in \Gamma$ and $y \in X^0$. The numbers $n^y_{w,v}$ and $d^y_{w,v}$ satisfy the following properties:*

i) $n^y_{w,v\theta} = n^y_{w,v}$ *i′)* $d^y_{w,v\theta} = d^y_{w,v}$
ii) $n^y_{\theta w,v} = n^y_{w,v}$ *ii′)* $d^y_{\theta w,v} = d^y_{w,v}$
iii) $n^{y\theta}_{w,\theta^{-1}v} = n^y_{w\theta^{-1},v}$ *iii′)* $d^{y\theta}_{w,\theta^{-1}v} = d^y_{w\theta^{-1},v}$

More generally, if $\widetilde{w}, \widetilde{v} \in G$ and $\widetilde{y} \in X^0$ are such that $\Gamma\widetilde{w}\Gamma = \Gamma w\Gamma$, $\Gamma\widetilde{v}\Gamma = \Gamma v\Gamma$, $\widetilde{y}\Gamma = y\Gamma$, $\widetilde{w}\widetilde{v}\Gamma = wv\Gamma$ and $\widetilde{y}\widetilde{w}^{-1}\Gamma^{wv} = yw^{-1}\Gamma^{wv}$, then

iv) $n^y_{w,v} = n^{\widetilde{y}}_{\widetilde{w},\widetilde{v}}$ *iv′)* $d^y_{w,v} = d^{\widetilde{y}}_{\widetilde{w},\widetilde{v}}$

Proof: Assertions *i)* and *i′)* are obvious.

Assertion *ii)* follows from the observation that $[r] \mapsto [\theta^{-1}r]$ establishes a bijection between the sets $\mathfrak{n}^y_{w,v}$ and $\mathfrak{n}^y_{\theta w,v}$.

Assertion *ii′)* is proven in a similar fashion as assertion *ii)*.

To prove assertion *iv)*, let $\theta \in \Gamma^{wv}$ be such that $\widetilde{y}\widetilde{w}^{-1} = yw^{-1}\theta$. We have

$$\begin{aligned}
\mathfrak{n}^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} &= \left\{[r] \in \Gamma\widetilde{w}\Gamma/\Gamma : r^{-1}\widetilde{w}\widetilde{v}\Gamma \subseteq \Gamma\widetilde{v}\Gamma \text{ and } \ \widetilde{y}\widetilde{w}^{-1} \in \widetilde{y}\Gamma r^{-1}\right\}\\
&= \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ yw^{-1}\theta \in y\Gamma r^{-1}\right\}.
\end{aligned}$$

Since $\theta \in \Gamma^{wv}$ we have $\theta wv\Gamma = wv\Gamma$, so that

$$\begin{aligned} &= \big\{[r] \in \Gamma\theta^{-1}w\Gamma/\Gamma : r^{-1}\theta^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } yw^{-1}\theta \in y\Gamma r^{-1}\big\} \\ &= \mathfrak{n}^{y}_{\theta^{-1}w,v}\,. \end{aligned}$$

Now, from assertion *ii*), it follows that $n^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} = n^{y}_{\theta^{-1}w,v} = n^{y}_{w,v}$.

As for assertion *iv'*), taking $\theta \in \Gamma^{wv}$ again as such that $\widetilde{y}\widetilde{w}^{-1} = yw^{-1}\theta$, we notice that

$$\begin{aligned} \mathfrak{d}^{\widetilde{y}}_{\widetilde{w},\widetilde{v}} &= \big\{[r] \in \Gamma\widetilde{w}\Gamma/\Gamma : r^{-1}\widetilde{w}\widetilde{v}\Gamma \subseteq \Gamma\widetilde{v}\Gamma \text{ and } \widetilde{y}\widetilde{w}^{-1} \in \widetilde{y}\Gamma r^{-1}\Gamma^{\widetilde{w}\widetilde{v}}\big\} \\ &= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } yw^{-1}\theta \in y\Gamma r^{-1}\Gamma^{wv}\big\} \\ &= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } yw^{-1} \in y\Gamma r^{-1}\Gamma^{wv}\big\} \\ &= \mathfrak{d}^{y}_{w,v}\,. \end{aligned}$$

Assertions *iii*) and *iii'*) are a direct consequence of *iv*) and *iv'*). $\square$

Proof of Theorem 3.1.16: We have

$$\begin{aligned} \Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma\ (t\Gamma) &= \sum_{[w]\in G/\Gamma} \Gamma g\Gamma(w\Gamma)\,\overline{\alpha}_w\big((1_{y\Gamma} * \Gamma s\Gamma)\ (w^{-1}t\Gamma)\big) \\ &= \sum_{[w]\in \Gamma g\Gamma/\Gamma} \overline{\alpha}_w\big((1_{y\Gamma} * \Gamma s\Gamma)\ (w^{-1}t\Gamma)\big) \\ &= \sum_{[w]\in \Gamma g\Gamma/\Gamma} \overline{\alpha}_w\big(1_{y\Gamma}\cdot \Gamma s\Gamma(w^{-1}t\Gamma)\big) \\ &= \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \overline{\alpha}_w(1_{y\Gamma}) \\ &= \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}} \end{aligned}$$

We now claim that

$$\sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}} \;=\; \sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}}\ \sum_{[\gamma]\in E^{y}_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^{t}}\,. \tag{3.16}$$

To see this, we will evaluate both the right and left expressions above on all points $x \in X^0$ and see that we obtain the same value. First, we note that if $x \in X^0$ is not of the form $y\theta\widetilde{w}^{-1}$, for some $\theta \in \Gamma$ and $\widetilde{w} \in \Gamma g\Gamma$ such that $\widetilde{w}^{-1}t\Gamma \subseteq \Gamma s\Gamma$, then both expressions are zero. Suppose now that $x = y\theta\widetilde{w}^{-1}$ for some $\widetilde{w} \in \Gamma g\Gamma$ such that $\widetilde{w}^{-1}t\Gamma \subseteq \Gamma s\Gamma$. Evaluating the left expression we get

$$\sum_{\substack{[w]\in \Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}}(y\theta\widetilde{w}^{-1}) = \sum_{\substack{[w]\in \Gamma \widetilde{w}\Gamma/\Gamma \\ w^{-1}\widetilde{w}\widetilde{w}^{-1}t\Gamma\subseteq\Gamma\widetilde{w}^{-1}t\Gamma}} 1_{y\Gamma w^{-1}}(y\theta\widetilde{w}^{-1}) = n^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\,.$$

As for the right expression, first we observe that if $y\theta\widetilde{w}^{-1} \in y\gamma w^{-1}\Gamma^{t}$, then by Lemma 3.1.17 *iv*) and *iv'*) we have $N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} = N^{y\gamma}_{w,w^{-1}t}$. Thus, evaluating the right expression we get

$$\begin{aligned}
& \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) = \\
= & \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
= & N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1})
\end{aligned}$$

Using Proposition 1.3.2 we notice that

$$\begin{aligned}
\sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t} &= \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma(w^{-1}\Gamma w\cap w^{-1}t\Gamma t^{-1}w)w^{-1}} \\
&= 1_{y\Gamma w^{-1}\Gamma^t}\,,
\end{aligned}$$

from which we obtain that,

$$\begin{aligned}
& N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} 1_{y\gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) = \\
= & N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} 1_{y\Gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
= & N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \sum_{\substack{[w]\in\Gamma \widetilde{w}\Gamma/\Gamma \\ w^{-1}\widetilde{w}\widetilde{w}^{-1}t\Gamma\subseteq\Gamma\widetilde{w}^{-1}t\Gamma}} 1_{y\Gamma w^{-1}\Gamma^t}\,(y\theta\widetilde{w}^{-1}) \\
= & N^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\, d^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t} \\
= & n^{y\theta}_{\widetilde{w},\widetilde{w}^{-1}t}\,.
\end{aligned}$$

So, equality (3.16) is established.

Now, by Proposition 3.1.14, we see that

$$\begin{aligned}
& \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, 1_{y\gamma w^{-1}\Gamma^t} = \\
= & \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ w^{-1}t\Gamma\subseteq\Gamma s\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma t\Gamma * 1_{y\gamma w^{-1}t\Gamma}\big)(t\Gamma)
\end{aligned}$$

Now, using the fact that condition $w^{-1}t\Gamma \subseteq \Gamma s\Gamma$ means that there exists a (necessarily unique) element $[v]\in\Gamma s\Gamma/\Gamma$ such that $w^{-1}t\Gamma = v\Gamma$, or equivalently, $t\Gamma = wv\Gamma$, we obtain

$$
\begin{aligned}
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},w^{-1}t}} N^{y\gamma}_{w,w^{-1}t}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma t\Gamma * 1_{y\gamma w^{-1}t\Gamma}\big)(t\Gamma) \\
&\qquad = \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma)\,.
\end{aligned}
$$

We now claim that

$$
\begin{aligned}
&\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma) = \\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma)
\end{aligned}
$$

To prove this we note that, given any $[w]\in\Gamma g\Gamma/\Gamma$ and $[v]\in\Gamma s\Gamma/\Gamma$, the element $\big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma)$ is nonzero if and only if $\Gamma t\Gamma = \Gamma wv\Gamma$, so that we can write

$$
\begin{aligned}
&\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma) = \\
&= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma\subseteq \Gamma t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma) \\
&= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma = \theta t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma) \\
&= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ \theta wv\Gamma = \theta t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1}\theta^{-1},v}} \frac{N^{y\gamma}_{\theta w,v}}{L(\theta wv)}\, \big(1_{y\gamma w^{-1}\theta^{-1}\Gamma} * \Gamma \theta wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma) \\
&= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma = t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{\theta w,v}}{L(wv)}\, \big(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\big)(t\Gamma)
\end{aligned}
$$

By Lemma 3.1.17 *ii*) and *ii′*) we know that $N^{y\gamma}_{\theta w,v} = N^{y\gamma}_{w,v}$, hence

$$= \sum_{[\theta]\in\Gamma/\Gamma^t} \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

$$= L(t) \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)$$

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ wv\Gamma=t\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} N^{y\gamma}_{w,v} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)(t\Gamma)\,.$$

Hence, we have proven that

$$\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right).$$

Also,

$$\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{w^{-1},v}} \frac{N^{y\gamma}_{w,v}}{L(wv)} \left(1_{y\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^g \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1}\theta^{-1},v}} \frac{N^{y\gamma}_{\theta g,v}}{L(\theta gv)} \left(1_{y\gamma g^{-1}\theta^{-1}\Gamma} * \Gamma\theta gv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^g \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right).$$

Moreover, we also have

$$\sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{L(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= L(g^{-1}) \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{\Delta(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}} \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1},v}} \frac{\Delta(g)N^{y\gamma}_{g,v}}{L(gv)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}} \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{g^{-1},\theta^{-1}v}} \frac{\Delta(g)N^{y\gamma}_{g,\theta^{-1}v}}{L(g\theta^{-1}v)} \left(1_{y\gamma g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma\theta^{-1}v\Gamma}\right),$$

but since there is a well-defined bijection $E^y_{\theta g^{-1},v} \to E^y_{g^{-1},\theta^{-1}v}$ given by $[\gamma] \mapsto [\gamma\theta]$, we obtain

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}} \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{\theta g^{-1},v}} \frac{\Delta(g) N^{y\gamma\theta}_{g,\theta^{-1}v}}{L(g\theta^{-1}v)} \left(1_{y\gamma\theta g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma\theta\theta^{-1}v\Gamma}\right)$$

and from Lemma 3.1.17 we get $N^{y\gamma\theta}_{g,\theta^{-1}v} = N^{y\gamma}_{g\theta^{-1},v}$, thus

$$= \sum_{\substack{[\theta]\in\Gamma/\Gamma^{g^{-1}} \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{\theta g^{-1},v}} \frac{\Delta(g) N^{y\gamma}_{g\theta^{-1},v}}{L(g\theta^{-1}v)} \left(1_{y\gamma\theta g^{-1}\Gamma} * \Gamma g\theta^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right).$$

□

COROLLARY 3.1.18. *Similarly, for $a \in \mathcal{A}_x$ with $x \in X$, we have*

$$\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma \quad =$$

$$= \sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)} \left([\alpha_{w\gamma^{-1}}(a)]_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right)$$

$$= \sum_{[v]\in\Gamma s\Gamma/\Gamma} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},v}} \frac{L(g) N^{\mathbf{s}(x)\gamma}_{g,v}}{L(gv)} \left([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma} * \Gamma gv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right)$$

$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\right).$$

Proof: According to equality (3.9) in Proposition 3.1.15 we have

$$\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma \quad =$$

$$= \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\theta g^{-1}\Gamma} * \Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma$$

$$= \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\theta g^{-1}\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(\alpha_{g\theta^{-1}}(a))]_{x\theta g^{-1}g\Gamma} * \Gamma s\Gamma$$

and by (3.7) in the same proposition we get

$$= \sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} [\alpha_{g\theta^{-1}}(a)]_{x\theta g^{-1}\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma} * \Gamma s\Gamma\,,$$

and by Theorem 3.1.16 we obtain

$$= \sum_{\substack{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}} \\ [w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, [\alpha_{g\theta^{-1}}(a)]_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,.$$

For each fixed w, v and γ all the summands in the expression

$$\sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, [\alpha_{g\theta^{-1}}(a)]_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,,$$

are zero except precisely for one summand and we have

$$\begin{aligned} &\sum_{[\theta]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, [\alpha_{g\theta^{-1}}(a)]_{x\theta g^{-1}\Gamma} * 1_{\mathbf{s}(x)\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma} \\ = &\ \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, [\alpha_{w\gamma^{-1}}(a)]_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\Gamma g\Gamma * a_{x\Gamma} * \Gamma s\Gamma \ = \\ = &\sum_{\substack{[w]\in\Gamma g\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{w^{-1},v}} \frac{N^{\mathbf{s}(x)\gamma}_{w,v}}{L(wv)}\, [\alpha_{w\gamma^{-1}}(a)]_{x\gamma w^{-1}\Gamma} * \Gamma wv\Gamma * 1_{\mathbf{s}(x)\gamma v\Gamma}\,. \end{aligned}$$

The remaining equalities in the statement of this corollary are proven in a similar fashion. ☐

3.2. Basic Examples

Example 3.2.1. We will now show that when Γ is a normal subgroup of G our notion of a crossed product by the Hecke pair (G,Γ) is precisely the usual crossed product by the group G/Γ. Normality of the subgroup Γ implies that the G-action $\overline{\alpha}$ on $M(C_c(\mathcal{A}))$ gives rise to an action of G/Γ on $C_c(\mathcal{A}/\Gamma)$. Moreover, we have $\Gamma^g = \Gamma$ for all $g \in G$, and it follows easily from the definitions that $C_c(\mathcal{A}/\Gamma) \times^{alg} G/\Gamma$ is nothing but the usual crossed product by the action of the group G/Γ.

It is also interesting to observe that any usual crossed product $C_c(\mathcal{B}) \times^{alg} G/\Gamma$ coming from an action of the group G/Γ on a Fell bundle $\mathcal{B}$ over a groupoid Y is actually a crossed product by the Hecke pair (G,Γ) in our sense. To see this we note that the action of G/Γ on $\mathcal{B}$ lifts to an action of G on $\mathcal{B}$. In this lifted action the subgroup Γ acts trivially, so that the action is Γ-good. Moreover, since Γ is normal in G, the Γ-intersection property is also trivially satisfied. It is clear that Y/Γ is just Y and $\mathcal{B}/\Gamma$ coincides with $\mathcal{B}$. Thus, forming the crossed product by the Hecke pair (G,Γ) will give nothing but the usual crossed product by G/Γ, i.e. $C_c(\mathcal{B}/\Gamma) \times^{alg} G/\Gamma \cong C_c(\mathcal{B}) \times^{alg} G/\Gamma$.

EXAMPLE 3.2.2. We will now explain how the Hecke algebra $\mathcal{H}(G,\Gamma)$ is an example of a crossed product by a Hecke pair, namely $\mathcal{H}(G,\Gamma) \cong \mathbb{C} \times^{alg} G/\Gamma$, just like group algebras are examples of crossed products by groups.

We start with a groupoid X consisting of only one element, i.e. $X = \{*\}$, and we take $\mathbb{C}$ as the Fell $\mathcal{A}$ bundle over X, i.e. $\mathcal{A}_* = \mathbb{C}$. We take also the trivial G-action α on $\mathcal{A}$. Since the G-action fixes every element of $\mathcal{A}$, it is indeed Γ-good and in this case we have $X/\Gamma = X = \{*\}$. For the orbit bundle we have that $\mathcal{A}/\Gamma = \mathcal{A}$, and moreover

$$C_c(\mathcal{A}/\Gamma) \cong C_c(X/\Gamma) \cong C_c(X) \cong \mathbb{C}\,.$$

Hence, we are in the conditions of the Standing Assumption 3.0.1 and we can form the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, which we will simply write as $\mathbb{C} \times_\alpha^{alg} G/\Gamma$.

Since $\mathbb{C}$ is unital the definitions of $B(\mathcal{A}, G, \Gamma)$ and $\mathbb{C} \times_\alpha^{alg} G/\Gamma$ coincide in this case. Moreover Definition 3.1.3 reads that $\mathbb{C} \times_\alpha^{alg} G/\Gamma$ is the set of functions $f : G/\Gamma \to \mathbb{C}$ satisfying the compatibility condition (3.1). Since the action $\overline{\alpha}$ is trivial, the compatibility condition simply says that $\mathbb{C} \times_\alpha^{alg} G/\Gamma$ consists of all the functions $f : G/\Gamma \to \mathbb{C}$ which are left Γ-invariant. Morever, the product and involution expressions become respectively

$$(f_1 * f_2)(g\Gamma) := \sum_{[h]\in G/\Gamma} f_1(h\Gamma)\, f_2(h^{-1}g\Gamma)\,,$$
$$(f^*)\,(g\Gamma) := \Delta(g^{-1})\,\overline{f(g^{-1}\Gamma)}\,.$$

Hence, it is clear that $\mathbb{C} \times_\alpha^{alg} G/\Gamma$ is nothing but the Hecke algebra $\mathcal{H}(G,\Gamma)$.

It follows from this that the product $\Gamma g\Gamma * 1_{*\Gamma} * \Gamma s\Gamma$ is just the product of the double cosets $\Gamma g\Gamma$ and $\Gamma s\Gamma$ inside the Hecke algebra, since $1_{*\Gamma}$ is the identity element. It is interesting to note in this regard that the expression for this product described in Theorem 3.1.16 is a familiar expression for the product $\Gamma g\Gamma * \Gamma s\Gamma$ in $\mathcal{H}(G,\Gamma)$. To see this, we note that the stabilizer $\mathcal{S}_*$ of $*$ is the whole group G, and therefore $E^*_{u,v}$ consists only of the class $[e]$. Moreover, the numbers $n^*_{u^{-1},v}$ and $d^*_{u^{-1},v}$, defined in (3.12) and (3.13), are equal, so that $N^*_{u^{-1},v} = 1$. Thus, the expression described in Theorem 3.1.16 is just the familiar expression from Proposition 1.3.6

$$\Gamma g\Gamma * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)}\,\Gamma u^{-1}v\Gamma\,,$$

EXAMPLE 3.2.3. As a generalization of Example 3.2.2 we will now show that if the G-action fixes every element of the bundle $\mathcal{A}$, then $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is isomorphic to the *-algebraic tensor product of $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$. This result also has a known analogue in the theory of crossed products by groups.

PROPOSITION 3.2.4. *If the G-action fixes every element of $\mathcal{A}$, then we have*

$$C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \;\cong\; C_c(\mathcal{A}/\Gamma) \odot \mathcal{H}(G,\Gamma)\,,$$

*where $\odot$ is the symbol that denotes the *-algebraic tensor product.*

Proof: Given that we have canonical embeddings of $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$ into $M(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$ we have a canonical linear map from $C_c(\mathcal{A}/\Gamma)\odot\mathcal{H}(G,\Gamma)$ to $M(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$ determined by

$$f_1\otimes f_2\mapsto f_1 * f_2\,, \tag{3.17}$$

where $f_1\in C_c(\mathcal{A}/\Gamma)$ and $f_2\in\mathcal{H}(G,\Gamma)$. Standard arguments can be used to show that this mapping is injective (since the mappings from both $C_c(\mathcal{A}/\Gamma)$ and $\mathcal{H}(G,\Gamma)$ into the multiplier algebra of the crossed product are injections). It is also clear that the image of the map determined by (3.17) is contained in $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. Let us now check that this mapping is surjective. First we will show that the elements of $C_c(\mathcal{A}/\Gamma)$ commute with elements of $\mathcal{H}(G,\Gamma)$ inside $M(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$. It follows from expressions (3.8) and (3.7) that

$$\begin{aligned}[a]_{x\Gamma} * \Gamma g\Gamma &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\gamma g\Gamma}\\ &= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^g} 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}\gamma^{-1}}(a)]_{x\gamma g\Gamma}\,.\end{aligned}$$

Since every point of X is fixed by the associated G-action on X, we have that $\mathcal{S}_x = G$, and therefore $\mathcal{S}_x\backslash\Gamma/\Gamma^g$ consists only of the class $[e]$, so that we can write

$$= \quad 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma}\,.$$

Moreover, since the G-actions on $\mathcal{A}$ and X are trivial we can furthermore write

$$= \quad 1_{\mathbf{r}(x)g^{-1}\Gamma} * \Gamma g\Gamma * [a]_{x\Gamma}\,.$$

Now, by the same reasoning as above and using expression (3.9) we have

$$\begin{aligned}&= \sum_{[\gamma]\in\mathcal{S}_x\backslash\Gamma/\Gamma^{g^{-1}}} 1_{\mathbf{r}(x)\gamma g^{-1}\Gamma} * \Gamma g\Gamma * [a]_{x\Gamma}\\ &= \Gamma g\Gamma * [a]_{x\Gamma}\,.\end{aligned}$$

Thus we conclude that $[a]_{x\Gamma} * \Gamma g\Gamma = \Gamma g\Gamma * [a]_{x\Gamma}$. By Theorem 3.1.13 we know that elements of the form $[a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ span $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$, and from the commutation relation we just proved it follows that

$$\begin{aligned}[a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma} &= \Gamma g\Gamma * [a]_{x\Gamma} * 1_{\mathbf{s}(x)g\Gamma}\\ &= \Gamma g\Gamma * [a]_{x\Gamma} * 1_{\mathbf{s}(x)\Gamma}\\ &= \Gamma g\Gamma * [a]_{x\Gamma}\\ &= [a]_{x\Gamma} * \Gamma g\Gamma\,,\end{aligned}$$

so that $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ is spanned by elements of the form $a_{x\Gamma} * \Gamma g\Gamma$. We now conclude that the image of the map (3.17) is the whole $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$.

The fact that this map is a *-homomorphism also follows directly from the commutation relation proved above. □

3.3. Representation theory

In this section we develop the representation theory of crossed products by Hecke pairs. We will introduce the notion of a *covariant pre-representation* and show that there is a bijective correspondence between covariant pre-representations and representations of the crossed product, in a similar fashion to the theory of crossed products by groups.

Recall from Proposition 1.2.16 that every nondegenerate *-representation $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ extends uniquely to a *-representation

$$\widetilde{\pi} : M_B(C_c(\mathcal{A}/\Gamma)) \to B(\mathscr{H})\,.$$

We will use the notation $\widetilde{\pi}$ to denote this extension throughout this section, many times without any reference. Since $C_c(X^0/\Gamma)$ is spanned by projections, it is a BG^*-algebra (recall Definition 1.1.3) and therefore we naturally have $C_c(X^0/\Gamma) \subseteq M_B(C_c(\mathcal{A}/\Gamma))$.

Definition 3.3.1. Let π be a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$ on a Hilbert space $\mathscr{H}$ and $\widetilde{\pi}$ its unique extension to a *-representation of $M_B(C_c(\mathcal{A}/\Gamma))$. Let μ be a unital pre-*-representation of $\mathcal{H}(G,\Gamma)$ on the inner product space $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We say that (π,μ) is a *covariant pre-*-representation* if the following equality

$$\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma s\Gamma) \; = \tag{3.18}$$
$$= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \; \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \, \widetilde{\pi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma})\, \mu(\Gamma u^{-1}v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\,,$$

holds on $L(\mathscr{W})$, for all $g,s\in G$ and $x\in X$.

Condition (3.18) simply says that the pair (π,μ) must preserve the structure of products of the form $\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma$, when expressed in terms of the canonical spanning set of elements of the form $[b]_{y\Gamma} * \Gamma d\Gamma * 1_{\mathbf{s}(y)d\Gamma}$, as explicitly described in Corollary 3.1.18.

The reader should note the similarity between our definition of a covariant pre-*-representation and the *covariant pairs* of an Huef, Kaliszewski and Raeburn in [**9**, Definition 1.1]. Their notion of covariant pairs served as a motivation for us and is actually a particular case of our Definition 3.3.1, as we shall see in Chapter **??**.

The operators $\widetilde{\pi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma})\, \mu(\Gamma u^{-1}v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})$ in expression (3.18) are all bounded, as we will now show, and are therefore defined in the whole Hilbert space $\mathscr{H}$.

Theorem 3.3.2. *Let $\pi : C_(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\mu : \mathcal{H}(G,\Gamma) \to L(\mathscr{W})$ a pre-*-representation on the inner product space $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))$. Every element of the form*

$$\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,,$$

is a bounded operator on $\mathscr{W}$ and therefore extends uniquely to the whole Hilbert space $\mathscr{H}$.

We will need some preliminary facts and lemmas in order to prove Theorem 3.3.2. These auxiliary results will also be useful later in this section.

Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi}$ its extension to $M_B(C_c(\mathcal{A}/\Gamma))$. For any unit $u \in X^0$ the operator $\widetilde{\pi}(1_{u\Gamma}) \in B(\mathscr{H})$ is a projection, and therefore $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ is a Hilbert subspace. The fiber $(\mathcal{A}/\Gamma)_{u\Gamma}$ is a C^*-algebra which we can naturally identify with the *-subalgebra

$$\{[a]_{u\Gamma} \in C_c(\mathcal{A}/\Gamma) : [a] \in (\mathcal{A}/\Gamma)_{u\Gamma}\} \subseteq C_c(\mathcal{A}/\Gamma)\,,$$

under the identification given by

$$(\mathcal{A}/\Gamma)_{u\Gamma} \ni [a] \;\longleftrightarrow\; [a]_{u\Gamma} \in C_c(\mathcal{A}/\Gamma)\,.$$

The *-representation $\widetilde{\pi}$ when restricted to $(\mathcal{A}/\Gamma)_{u\Gamma}$, under the above identification, leaves the subspace $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ invariant, because

$$\widetilde{\pi}([a]_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = \widetilde{\pi}([a]_{u\Gamma})\xi = \widetilde{\pi}(1_{u\Gamma})\widetilde{\pi}([a]_{u\Gamma})\xi\,.$$

The following lemma assures that this restriction is nondegenerate.

Lemma 3.3.3. *Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\pi}$ its unique extension to $M_B(C_c(\mathcal{A}/\Gamma))$. The *-representation of $(\mathcal{A}/\Gamma)_{u\Gamma}$ on the Hilbert space $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$, as above, is nondegenerate.*

Proof: Let $\widetilde{\pi}(1_{u\Gamma})\xi$ be an element of $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ such that

$$\widetilde{\pi}([a]_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = 0\,,$$

for all $[a] \in (\mathcal{A}/\Gamma)_{u\Gamma}$. We want to prove that $\widetilde{\pi}(1_{u\Gamma})\xi = 0$. To see this, let $x \in X$ and $[b] \in (\mathcal{A}/\Gamma)_{x\Gamma}$. We have two alternatives: either $\mathbf{s}(x)\Gamma \neq u\Gamma$ or $\mathbf{s}(x)\Gamma = u\Gamma$. In the first case we see that

$$\widetilde{\pi}([b]_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = \widetilde{\pi}([b]_{x\Gamma} \cdot 1_{u\Gamma})\xi = 0\,,$$

whereas for the second we see that

$$\begin{aligned}
\|\widetilde{\pi}([b]_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\|^2 &= \langle \widetilde{\pi}([b]_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\, \widetilde{\pi}([b]_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\rangle \\
&= \langle \widetilde{\pi}([b^*b]_{\mathbf{s}(x)\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\, \widetilde{\pi}(1_{u\Gamma})\xi\rangle \\
&= \langle \widetilde{\pi}([b^*b]_{u\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi\,,\, \widetilde{\pi}(1_{u\Gamma})\xi\rangle \\
&= 0\,,
\end{aligned}$$

by assumption. Thus, in any case we have $\widetilde{\pi}([b]_{x\Gamma})\widetilde{\pi}(1_{u\Gamma})\xi = 0$ for all $x \in X$ and $[b] \in (\mathcal{A}/\Gamma)_{x\Gamma}$. By nondegeneracy of π, this implies that $\widetilde{\pi}(1_{u\Gamma})\xi = 0$, as we wanted to prove. ☐

Lemma 3.3.4. *Let π be a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$ on a Hilbert space $\mathscr{H}$. We have that $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H} = \widetilde{\pi}(C_c(X^0/\Gamma))\mathscr{H}$.*

Proof: It is clear that $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \subseteq \widetilde{\pi}(C_c(X^0/\Gamma))\mathscr{H}$ since for any element of the form $[a]_{x\Gamma}$ in $C_c(\mathcal{A}/\Gamma)$ and $\xi \in \mathscr{H}$ we have $\pi([a]_{x\Gamma})\xi = \pi(1_{\mathbf{r}(x)\Gamma}[a]_{x\Gamma})\xi = \widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\pi([a]_{x\Gamma})\xi$.

Let us now prove that $\widetilde{\pi}(C_c(X^0/\Gamma))\mathscr{H} \subseteq \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. Let $u\Gamma \in X^0/\Gamma$ and $\xi \in \mathscr{H}$. We know, by Lemma 3.3.3, that π gives a nondegenerate *-representation of $(\mathcal{A}/\Gamma)_{u\Gamma}$ on $\widetilde{\pi}(1_{u\Gamma})\mathscr{H}$. Since $(\mathcal{A}/\Gamma)_{u\Gamma}$ is a C^*-algebra we have, by the general version of Cohen's factorization theorem ([**18**, Theorem 5.2.2]), that there exists $[c] \in (\mathcal{A}/\Gamma)_{u\Gamma}$ and $\eta \in \widetilde{\pi}(1_{u\Gamma})\mathscr{H}$ such that

$$\widetilde{\pi}(1_{u\Gamma})\xi = \pi([c]_{u\Gamma})\eta\,,$$

which means that $\widetilde{\pi}(1_{u\Gamma})\xi \in \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. This finishes the proof. $\square$

Proof of Theorem 3.3.2: The operator $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})$ is clearly defined on the inner product space $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. By Lemma 3.3.4 this operator is then defined on the space $\widetilde{\pi}(C_c(X^0/\Gamma))\mathscr{H}$. Since

$$\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma}) = \pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,,$$

it follows that the operator $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})$ is actually defined in the whole Hilbert space $\mathscr{H}$ (or in other words, it extends canonically to $\mathscr{H}$).

A similar argument shows that $\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\mu((\Gamma g\Gamma)^*)\pi([a^*]_{x^{-1}\Gamma})$ is also defined in the whole Hilbert space $\mathscr{H}$ and it is easy to see that $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})$ is an adjointable operator on $\mathscr{H}$, whose adjoint is $\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\mu((\Gamma g\Gamma)^*)\pi([a^*]_{x^{-1}\Gamma})$. Since adjointable operators on a Hilbert space are necessarily bounded (see [**19**, Proposition 9.1.11]), it follows that $\pi([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})$ is a bounded operator. $\square$

The striking feature that we actually have to consider pre-representations of $\mathcal{H}(G,\Gamma)$, and not just representations, was not present in the theory of crossed products by groups because a group algebra $\mathbb{C}(G)$ of a discrete group is always a BG^*-algebra and therefore all of its pre-representations come from true representations (see further Remark 3.3.8).

It will be useful to distinguish between covariant pre-*-representations and covariant *-representations, so we will treat them in separate definitions. As will be discussed below we will see covariant *-representations as a particular type of covariant pre-*-representations.

DEFINITION 3.3.5. Let π be a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$ on a Hilbert space $\mathscr{H}$ and μ a unital *-representation of $\mathcal{H}(G,\Gamma)$ on $\mathscr{H}$. We say that (π,μ) is a *covariant *-representation* if equality (3.18) holds in $B(\mathscr{H})$ for all $g,s \in G$ and $x \in X$.

LEMMA 3.3.6. *Let (π,μ) be a covariant *-representation on a Hilbert space $\mathscr{H}$. Then μ leaves the subspace $\mathscr{W} := \pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ invariant.*

Proof: Consider elements of the form $\pi([a]_{x\Gamma})\xi$, whose span gives $\mathscr{W}$. Using the fact that μ is unital and the covariance relation (3.18) we see that

$$\begin{aligned}
&\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\xi \;= \\
&= \mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma)\xi \\
&= \sum_{[u]\in\Gamma g^{-1}\Gamma/\Gamma}\ \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,e}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},e}}{L(u^{-1})}\,\widetilde{\pi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma\Gamma})\xi\,.
\end{aligned}$$

Hence, $\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\xi \in \mathscr{W}$, and consequently $\mu(\Gamma g\Gamma)$ leaves $\mathscr{W}$ invariant. This finishes the proof. $\square$

From a covariant *-representation (π,μ) one can obtain canonically a covariant pre-*-representation (π,μ), just by restricting μ to the dense subspace $\mathscr{W} :=$

$\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ (which is an invariant subspace by Lemma 3.3.6). So we can regard covariant *-representations as a special kind of covariant pre-*-representations: they are exactly those for which μ is normed. As we shall see later in Example 3.4.1, there are covariant pre-*-representations which are not covariant *-representations, thus in general the latter form a proper subclass of the former. We shall also see examples where they actually coincide.

Remark 3.3.7. Equivalently, one could define covariant (pre-)*-representation using any other of the equalities in Corollary 3.1.18 and substituting with the appropriate (pre-)*-representations. It is easy to see, using completely analogous arguments as in the proof of Corollary 3.1.18 or Theorem 3.1.16, that all three expressions yield the same result.

Remark 3.3.8. Even though it might not be entirely clear from the start, when Γ is a normal subgroup of G the definition of a covariant pre-representation is nothing but the usual definition of covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$. We recall that a covariant representation of $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ is a pair (π, U) consisting of a nondegenerate *-representation π of $C_c(\mathcal{A}/\Gamma)$ and a unitary representation U of G/Γ satisfying the relation

$$\pi(\overline{\alpha}_{g\Gamma}(f)) = U_{g\Gamma}\pi(f)U_{g^{-1}\Gamma}\,,$$

for all $f \in C_c(\mathcal{A}/\Gamma)$ and $g\Gamma \in G/\Gamma$. Now, as it is well known, every unitary representation U of G/Γ is associated in a canonical way to a unital *-representation μ of the group algebra $\mathbb{C}(G/\Gamma)$, so that we can write the covariance condition as $\pi(\overline{\alpha}_{g\Gamma}(f)) = \mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma)$. As a consequence we have that for any $g\Gamma, s\Gamma \in G/\Gamma$, $x \in X$ and $a \in \mathcal{A}_x$:

$$\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(s\Gamma) = \pi([\alpha_g(a)]_{xg^{-1}\Gamma})\mu(g^{-1}s\Gamma)\,.$$

We want to check that covariant representations of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ are the same as covariant pre-*-representations as in Definition 3.3.1.

Given a covariant pre-*-representation (π, μ) on some Hilbert space $\mathscr{H}$ in the sense of Definition 3.3.1, we have that μ is a pre-*-representation of $\mathbb{C}(G/\Gamma)$, which is normed since any group algebra of a discrete group is a BG^*-algebra, and thus we can see μ as a true *-representation on $\mathscr{H}$. We then have that

$$\begin{aligned}
& \mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma) \\
= \; & \mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma g^{-1}\Gamma) \\
= \; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma g^{-1}\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \,\widetilde{\pi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\widetilde{\pi}([\alpha_g(a)]_{xg^{-1}\Gamma})\,\mu(gg^{-1}\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g^{-1}\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\widetilde{\pi}([\alpha_g(a)]_{xg^{-1}\Gamma}\cdot 1_{\mathbf{s}(x)g^{-1}\Gamma}) \\
= \; & \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}} N^{\mathbf{s}(x)\gamma}_{g,g^{-1}}\,\pi([\alpha_g(a)]_{xg^{-1}\Gamma})\,.
\end{aligned}$$

It is clear from the normality of Γ that $E^{\mathbf{s}(x)}_{g^{-1},g^{-1}}$ consists only of the class $[e]$ and moreover $N^{\mathbf{s}(x)}_{g,g^{-1}} = 1$, so that

$$\mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma) \quad = \quad \pi([\alpha_g(a)]_{xg^{-1}\Gamma})\,.$$

By linearity it follows that $\mu(g\Gamma)\pi(f)\mu(g^{-1}\Gamma) = \pi(\overline{\alpha}_{g\Gamma}(f))$ for any $f \in C_c(\mathcal{A}/\Gamma)$. Thus, with U being the unitary representation of G/Γ associated to μ, we see that (π, U) is covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$.

For the other direction, let (π, U) be a covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma)$ and let μ be the *-representation of $\mathbb{C}(G/\Gamma)$ associated to U, which we restrict to the inner product space $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We want to prove that (π, μ) is a covariant pre-*-representation in the sense of Definition 3.3.1. We have

$$\begin{aligned}
& \mu(g\Gamma)\pi([a]_{x\Gamma})\mu(s\Gamma) \\
= \; & \mu(g\Gamma)\pi([a]_{x\Gamma})\mu(g^{-1}\Gamma)\mu(gs\Gamma) \\
= \; & \pi([\alpha_g(a)]_{xg^{-1}\Gamma})\mu(gs\Gamma) \\
= \; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \,\widetilde{\pi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\,,
\end{aligned}$$

where the last equality is obtained following analogous computations as those above. Thus, (π, μ) is a covariant pre-*-representation in the sense of Definition 3.3.1.

The following result makes it clear that some of the relations we have inside the crossed product (see Proposition 3.1.15) are preserved upon taking covariant pre-*-representations. This is expected since, as we stated before, we will prove that covariant pre-representations give rise to representations of the crossed product, and this result is the first step in that direction:

PROPOSITION 3.3.9. *Let (π, μ) be a covariant pre-*-representation. The following two equalities hold:*

$$\widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([\alpha_{g^{-1}}(a)]_{xg\Gamma}) \;=\; \widetilde{\pi}([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,. \tag{3.19}$$

$$\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma}) = \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma})\,. \tag{3.20}$$

Proof: Since (π, μ) is a covariant pre-*-representation we have

$$\begin{aligned}
\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma}) &= \mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma})\mu(\Gamma)\\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} N^{\mathbf{s}(x)\gamma}_{g,e}\,\widetilde{\pi}([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\gamma\Gamma})\\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,,
\end{aligned}$$

where the last equality comes from the fact that $n^{\mathbf{s}(x)\gamma}_{g,e} = 1 = d^{\mathbf{s}(x)\gamma}_{g,e}$, and thus $N^{\mathbf{s}(x)\gamma}_{g,e} = 1$. From this it follows that

$$\begin{aligned}
&\widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma}) \;=\\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\widetilde{\pi}([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot[\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,.
\end{aligned}$$

Now the product $1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot[\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma}$ is nonzero only when $\mathbf{r}(x)g^{-1}\Gamma = \mathbf{r}(x)\gamma g^{-1}\Gamma$, from which one readily concludes that $\mathbf{r}(x)\gamma \in \mathbf{r}(x)g^{-1}\Gamma g$. Since one trivially has $\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma$ we conclude that

$$\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma \cap \mathbf{r}(x)g^{-1}\Gamma g\,,$$

and by the Γ-intersection property we have $\mathbf{r}(x)\gamma \in \mathbf{r}(x)\Gamma^{g^{-1}}$. From Proposition 1.3.2 this means that $[\gamma] = [e]$ in $E^{\mathbf{r}(x)}_{g^{-1},e}$. We recall that $E^{\mathbf{r}(x)}_{g^{-1},e} = \mathcal{S}_{\mathbf{r}(x)}\backslash\Gamma/\Gamma^{g^{-1}}$, and since $\Gamma^{g^{-1}} \subseteq \Gamma$ we have by Proposition 1.3.1 that $[\gamma] \to [\gamma]$ defines a canonical bijection between $E^{\mathbf{r}(x)}_{g^{-1},e}$ and $(\mathcal{S}_{\mathbf{r}(x)}\cap\Gamma)\backslash\Gamma/\Gamma^{g^{-1}}$. Since the G-action is Γ-good we necessarily have $\mathcal{S}_{\mathbf{s}(x)}\cap\Gamma = \mathcal{S}_x\cap\Gamma = \mathcal{S}_{\mathbf{r}(x)}\cap\Gamma$, and therefore using Proposition 1.3.1 one more time we can say that $E^{\mathbf{r}(x)}_{g^{-1},e} = E^{\mathbf{s}(x)}_{g^{-1},e}$. Hence, we can say that $[\gamma] = [e]$ in $E^{\mathbf{s}(x)}_{g^{-1},e}$. We conclude that

$$\begin{aligned}
\widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma}) &= \widetilde{\pi}(1_{\mathbf{r}(x)g^{-1}\Gamma}\cdot[\alpha_g(a)]_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\\
&= \widetilde{\pi}([\alpha_g(a)]_{xg^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma})\,.
\end{aligned}$$

Since the last expression is valid for any $x \in X$ and $[a] \in (\mathcal{A}/\Gamma)_{x\Gamma}$, if we take x to be xg and $[a]$ to be $[\alpha_{g^{-1}}(a)]$ we obtain the desired equality (3.19):

$$\widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([\alpha_{g^{-1}}(a)]_{xg\Gamma}) = \widetilde{\pi}([a]_{x\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

Let us now prove equality (3.20). Using the equality in beginning of this proof and equality (3.19) which we have just proven, we get precisely

$$\begin{aligned}
\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma}) &= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}([\alpha_{g\gamma^{-1}}(a)]_{x\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x\gamma g^{-1})g\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([\alpha_{\gamma^{-1}}(a)]_{x\gamma\Gamma}) \\
&= \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g^{-1}\Gamma})\mu(\Gamma g\Gamma)\widetilde{\pi}([a]_{x\Gamma})\,.
\end{aligned}$$

This finishes the proof. □

The passage from a covariant pre-representation (π, μ) to a representation of $C_c(\mathcal{A}/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ is done via the so-called *integrated form* $\pi \times \mu$, which we now describe:

DEFINITION 3.3.10. Let (π, μ) be a covariant pre-*-representation on a Hilbert space $\mathscr{H}$. We define the *integrated form* of (π, μ) as the function $\pi\times\mu : C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma \to B(\mathscr{H})$ defined by

$$[\pi \times \mu](f) := \sum_{[g]\in\Gamma\backslash G/\Gamma} \; \sum_{x\Gamma^g\in X/\Gamma^g} \widetilde{\pi}\Big(\big[f(g\Gamma)(x)\big]_{x\Gamma}\Big)\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

REMARK 3.3.11. For f of the form $f = a_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ we have

$$[\pi \times \mu](f) = \widetilde{\pi}([a]_{x\Gamma})\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\,.$$

Moreover, from equality (3.19), for f' of the form $f' = 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma}$ we have

$$[\pi \times \mu](f') = \widetilde{\pi}(1_{\mathbf{r}(x)\Gamma})\,\mu(\Gamma g\Gamma)\,\widetilde{\pi}([\alpha_{g^{-1}}(a)]_{xg\Gamma})\,.$$

PROPOSITION 3.3.12. *The integrated form $\pi\times\mu$ of a covariant pre-*-representation (π, μ) is a well-defined nondegenerate *-representation.*

Proof: First we need to check that the expression that defines $[\pi \times \mu](f)$ for a given $f \in C_c(\mathcal{A}/\Gamma)\times^{alg}_{\alpha} G/\Gamma$ is well-defined. This is proven in an entirely analogous way as in the proof that the expression (3.5) in Proposition 3.1.13 is well-defined. Secondly, we need to show that $[\pi \times \mu](f)$ makes sense as an element of $B(\mathscr{H})$. From Theorem 3.3.2 we have that

$$\widetilde{\pi}\Big(\big[f(g\Gamma)(x)\big]_{x\Gamma}\Big)\mu(\Gamma g\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma}) \in B(\mathscr{W})\,,$$

thus, it follows that $[\pi \times \mu](f) \in B(\mathscr{W})$, and therefore $[\pi \times \mu](f)$ admits a unique extension to $B(\mathscr{H})$.

Now, it is obvious that $\pi \times \mu$ is a linear transformation. Let us check that it preserves the involution. It is then enough to check it for elements of the form $f = [a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$. Since (π, μ) is a covariant pre-*-representation we have, by Propositions 3.3.9 and 3.1.15,

$$\begin{aligned}
\big([\pi \times \mu](f)\big)^* &= \Delta(g)\, \widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}([a^*]_{x^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}(1_{\mathbf{r}(x^{-1})g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}([a^*]_{x^{-1}gg^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x^{-1})gg^{-1}\Gamma}) \\
&= \Delta(g)\, \widetilde{\pi}([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma})\, \mu(\Gamma g^{-1}\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x^{-1})\Gamma}) \\
&= [\pi \times \mu]\, \big(\Delta(g)\, [\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{s}(x^{-1})\Gamma}\big) \\
&= [\pi \times \mu]\, (f^*)\,.
\end{aligned}$$

Let us now prove that $\pi \times \mu$ preserves products. We will start by proving that

$$[\pi \times \mu](f_1 * f_2) = [\pi \times \mu](f_1)\, [\pi \times \mu](f_2)\,, \tag{3.21}$$

for $f_1 := [a]_{x\Gamma} * \Gamma g \Gamma * 1_{\mathbf{s}(x)g\Gamma}$ and $f_2 := [b]_{y\Gamma} * \Gamma s \Gamma * 1_{\mathbf{s}(y)s\Gamma}$. Let us compute the expression on the left side of (3.21). First, we notice that for the product $f_1 * f_2$ to be non-zero one must have $\mathbf{r}(y) \in \mathbf{s}(x)g\Gamma$, and in this case we obtain

$$f_1 * f_2 = [a]_{x\Gamma} * \Gamma g \Gamma * [b]_{y\Gamma} * \Gamma s \Gamma * 1_{\mathbf{s}(y)s\Gamma}$$

which by Corollary 3.1.18 gives

$$= \sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v}}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\, [a]_{x\Gamma} * [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma} * \Gamma u^{-1} v \Gamma * 1_{\mathbf{s}(y)\gamma v\Gamma} * 1_{\mathbf{s}(y)s\Gamma}$$

$$= \sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v} \\ \mathbf{s}(y)s\Gamma = \mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\, [a]_{x\Gamma} * [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma} * \Gamma u^{-1} v \Gamma * 1_{\mathbf{s}(y)\gamma v\Gamma}$$

$$= \sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v} \\ \mathbf{s}(y)s\Gamma = \mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\, [a]_{x\Gamma} * [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma} * \Gamma u^{-1} v \Gamma * 1_{\mathbf{s}(y\gamma u)u^{-1}v\Gamma}$$

The product $[a]_{x\Gamma} * [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma}$ is always either zero or of the form $[c]_{(x\theta)(y\gamma u)\Gamma}$, for some $\theta \in \Gamma$ and $c \in \mathcal{A}_{(x\theta)(y\gamma u)}$. The point is that $\mathbf{s}\big((x\theta)(y\gamma u)\big) = \mathbf{s}(y\gamma u)$, so that each non-zero summand in the last sum above is actually of the form

$$[c]_{z\Gamma} * \Gamma d \Gamma * 1_{\mathbf{s}(z)d\Gamma}\,,$$

for appropriate $[c] \in (\mathcal{A}/\Gamma)_{z\Gamma}$, $z \in X$ and $d \in G$. Thus, by linearity of $\pi \times \mu$ and Remark 3.3.11 we obtain

$$\begin{aligned}&[\pi \times \mu](f_1 * f_2) = \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\, \widetilde{\pi}([a]_{x\Gamma} \cdot [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma})\, \mu(\Gamma u^{-1} v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma})\,.\end{aligned}$$

Let us now compute the expression on the right side of (3.21). We have

$$[\pi \times \mu](f_1)\, [\pi \times \mu](f_2) = \quad \widetilde{\pi}([a]_{x\Gamma})\, \mu(\Gamma g\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(x)g\Gamma})\, \widetilde{\pi}([b]_{y\Gamma})\, \mu(\Gamma s\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma})\,.$$

For $1_{\mathbf{s}(x)g\Gamma} \cdot [b]_{y\Gamma}$ to be non-zero we must have $\mathbf{r}(y) \in \mathbf{s}(x)g\Gamma$, and in this case we obtain, using the definition of a covariant pre-*-representation,

$$\begin{aligned}&[\pi \times \mu](f_1)\, [\pi \times \mu](f_2) = \\ &= \widetilde{\pi}([a]_{x\Gamma})\, \mu(\Gamma g\Gamma)\, \widetilde{\pi}([b]_{y\Gamma})\, \mu(\Gamma s\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma}) \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v}}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \widetilde{\pi}([a]_{x\Gamma}[\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma}) \mu(\Gamma u^{-1} v\Gamma) \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma}) \widetilde{\pi}(1_{\mathbf{s}(y)s\Gamma}) \\ &= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma \\ [\gamma]\in E^{\mathbf{s}(y)}_{u,v} \\ \mathbf{s}(y)s\Gamma=\mathbf{s}(y)\gamma v\Gamma}} \frac{\Delta(g) N^{\mathbf{s}(y)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \widetilde{\pi}([a]_{x\Gamma} \cdot [\alpha_{u^{-1}\gamma^{-1}}(b)]_{y\gamma u\Gamma})\, \mu(\Gamma u^{-1} v\Gamma)\, \widetilde{\pi}(1_{\mathbf{s}(y)\gamma v\Gamma})\,.\end{aligned}$$

Hence, we have proven equality (3.21) for the special case of f_1 and f_2 being $f_1 := [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ and $f_2 := [b]_{y\Gamma} * \Gamma s\Gamma * 1_{\mathbf{s}(y)s\Gamma}$. Using this we will now show that equality (3.21) holds for any $f_1, f_2 \in C_c(\mathcal{A}/\Gamma) \times^{alg}_{\alpha} G/\Gamma$. In fact, by Proposition 3.1.13, f_1 and f_2 can be written as sums

$$f_1 = \sum_i v_i\,, \qquad\qquad f_2 = \sum_j w_j\,,$$

where each v_i and w_j is of the form $[a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$, for some $g\Gamma \in G/\Gamma$, $x \in X$ and $a \in \mathcal{A}_x$. Since $\pi \times \mu$ is a linear mapping we have

$$\begin{aligned}[\pi \times \mu](f_1 * f_2) &= [\pi \times \mu]\Big(\big(\sum_i v_i\big) * \big(\sum_j w_j\big)\Big) \\ &= [\pi \times \mu]\Big(\sum_{i,j} v_i * w_j\Big) \\ &= \sum_{i,j} [\pi \times \mu](v_i * w_j)\,,\end{aligned}$$

and by the special case of equality (3.21) we have just proven we get

$$\begin{aligned}
[\pi\times\mu](f_1 * f_2) &= \sum_{i,j}[\pi\times\mu](v_i)[\pi\times\mu](w_j)\\
&= \Big(\sum_i[\pi\times\mu](v_i)\Big)\Big(\sum_j[\pi\times\mu](w_j)\Big)\\
&= [\pi\times\mu]\big(\sum_i v_i\big)[\pi\times\mu]\big(\sum_j w_j\big)\\
&= [\pi\times\mu](f_1)[\pi\times\mu](f_2)\,.
\end{aligned}$$

Hence, $\pi\times\mu$ is a *-representation. To finish the proof we now only need to show that $\pi\times\mu$ is nondegenerate. The restriction of $\pi\times\mu$ to the *-subalgebra $C_c(\mathcal{A}/\Gamma)$ is precisely the representation π. Since π is assumed to be nondegenerate it follows that $\pi\times\mu$ must be nondegenerate as well. □

The next result shows how from a representation of the crossed product one can naturally form a covariant pre-representation.

PROPOSITION 3.3.13. *Let $\Phi : C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation. Consider the pair $(\Phi|,\omega_\Phi)$ defined by*

- *$\Phi|$ is the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$.*
- *Let $\widetilde{\Phi}$ be the extension of Φ to a pre-*-representation (via Proposition 1.2.13) of $M(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$ on the inner product space $\Phi(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)\mathscr{H}$. We define ω_Φ to be the restriction of $\widetilde{\Phi}$ to $\mathcal{H}(G,\Gamma)$.*

The pair $(\Phi|,\omega_\Phi)$ is a covariant pre-*-representation.

We will need some preliminary lemmas in order to prove Proposition 3.3.13.

LEMMA 3.3.14. *If $\Phi : C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ is a nondegenerate *-representation, then its restriction to $C_c(\mathcal{A}/\Gamma)$ is also nondegenerate.*

Proof: Let $\xi\in\mathscr{H}$ be such that $\Phi(C_c(\mathcal{A}/\Gamma))\,\xi=\{0\}$. We want to show that $\xi=0$. Since Φ is nondegenerate, it is then enough to prove that $\Phi(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)\,\xi=\{0\}$. Thus, by virtue of Proposition 3.1.15, it suffices to prove that $\Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi = 0$ for all $g\in G$, $x\in X$, $a\in\mathcal{A}_x$. We have

$$\begin{aligned}
&\|\Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi\|^2 \;=\\
&= \Delta(g)\langle\Phi([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma} * \Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi\,,\,\xi\rangle\\
&= \Delta(g)\langle\Phi([\alpha_{g^{-1}}(a^*)]_{x^{-1}g\Gamma})\Phi(\Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi\,,\,\xi\rangle\\
&= \Delta(g)\langle\Phi(\Gamma g^{-1}\Gamma * 1_{\mathbf{r}(x)\Gamma} * \Gamma g\Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi\,,\,\Phi([\alpha_{g^{-1}}(a)]_{xg\Gamma})\xi\rangle\\
&= 0\,.
\end{aligned}$$

Hence $\xi=0$ and therefore Φ restricted to $C_c(\mathcal{A}/\Gamma)$ is nondegenerate. □

LEMMA 3.3.15. *Let $\Phi : C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation and $\widetilde{\Phi}$ its unique extension to $M_B(C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma)$ (via Proposition 1.2.16). Let $\Phi|$ be the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$ and $\widetilde{\Phi|}$ its unique extension* *to $M_B(C_c(\mathcal{A}/\Gamma))$. We have that*

$$\widetilde{\Phi}(f) = \widetilde{\Phi|}(f)\,,$$

for all $f \in C_c(X^0/\Gamma)$. In other words, the two *-representations $\widetilde{\Phi}$ and $\widetilde{\Phi|}$ are the same in $C_c(X^0/\Gamma)$.

Proof: By Lemma 3.3.14 the subspace $\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$ is dense in $\mathscr{H}$, so that it is enough to check that $\widetilde{\Phi}(f)\Phi(f_2)\xi = \widetilde{\Phi|}(f)\Phi(f_2)\xi$, for all $f_2 \in C_c(\mathcal{A}/\Gamma)$ and $\xi \in \mathscr{H}$. By definition of the extension $\widetilde{\Phi}$ (see Proposition 1.2.16) we have

$$\widetilde{\Phi}(f)\Phi(f_2)\xi \quad = \quad \Phi(f * f_2)\xi\,,$$

where $f * f_2$ is the product of f and f_2, which lies inside $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. Since both f and f_2 are elements of $B(\mathcal{A}, G, \Gamma)$ we see the product $f * f_2$ as taking place in $B(\mathcal{A}, G, \Gamma)$. By definition of the embeddings of $C_c(X^0/\Gamma)$ and $C_c(\mathcal{A}/\Gamma)$ in $B(\mathcal{A}, G, \Gamma)$ we have that $f * f_2$ is nothing but the element $f \cdot f_2$, where the product is just the product of f and f_2 inside $M(C_c(\mathcal{A}))$. As we observed in Section 2.3, this product is exactly same as the product of f and f_2 in $M(C_c(\mathcal{A}/\Gamma))$. Thus, the following computation makes sense:

$$\begin{aligned}\widetilde{\Phi}(f)\Phi(f_2)\xi &= \Phi(f * f_2)\xi = \Phi(f \cdot f_2)\xi \\ &= \Phi|(f \cdot f_2)\xi = \widetilde{\Phi|}(f)\Phi|(f_2)\xi\,.\end{aligned}$$

This finishes the proof. □

LEMMA 3.3.16. *Let $\Phi : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \to B(\mathscr{H})$ be a nondegenerate *-representation. We have that*

$$\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \;=\; \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}\,.$$

Proof: The inclusion $\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \subseteq \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}$ is obvious. To check the converse inclusion it is enough to prove that

$$\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)\Gamma})\xi \in \Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}\,,$$

for all $x \in X$, $a \in \mathcal{A}_x$, $g \in G$ and $\xi \in \mathscr{H}$. Let $\widetilde{\Phi} : M_B(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) \to B(\mathscr{H})$ be the unique extension of Φ to a *-representation of $M_B(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$, as in Proposition 1.2.16. We then get

$$\begin{aligned}\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi &= \Phi(1_{\mathbf{r}(x)\Gamma} * [a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi \\ &= \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma})\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi\,.\end{aligned}$$

Denoting by $\Phi|$ the restriction of Φ to $C_c(\mathcal{A}/\Gamma)$ we have, by Lemma 3.3.15, that

$$= \quad \widetilde{\Phi|}(1_{\mathbf{r}(x)\Gamma})\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi\,,$$

i.e. $\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi \in \widetilde{\Phi|}(C_c(X/\Gamma))\mathscr{H}$. By Lemma 3.3.4 it then follows that $\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\xi \in \Phi|(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. □

Proof of Proposition 3.3.13: First of all, by Lemma 3.3.14, $\Phi|$ is indeed a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$. Secondly, from Lemma 3.3.16, we have

$$\Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H} \ = \ \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H} \, .$$

Thus, ω_Φ is a pre-*-representation of $\mathcal{H}(G,\Gamma)$ on $\mathscr{W} := \Phi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We now only need to check covariance. We have

$$\begin{aligned}
& \omega_\Phi(\Gamma g \Gamma)\Phi|([a]_{x\Gamma})\omega_\Phi(\Gamma s \Gamma) \ = \\
= \ & \widetilde{\Phi}(\Gamma g \Gamma)\widetilde{\Phi}([a]_{x\Gamma})\widetilde{\Phi}(\Gamma s \Gamma) \\
= \ & \widetilde{\Phi}(\Gamma g \Gamma * [a]_{x\Gamma} * \Gamma s \Gamma) \\
= \ & \widetilde{\Phi}\Big(\sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v] \in \Gamma s \Gamma/\Gamma}} \ \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \, [\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u \Gamma} * \Gamma u^{-1} v \Gamma * 1_{\mathbf{s}(x)\gamma v \Gamma} \Big) \\
= & \sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v] \in \Gamma s \Gamma/\Gamma}} \ \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \, \widetilde{\Phi}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u \Gamma})\widetilde{\Phi}(\Gamma u^{-1} v \Gamma)\widetilde{\Phi}(1_{\mathbf{s}(x)\gamma v \Gamma}) \, .
\end{aligned}$$

Denoting by $\widetilde{\Phi|}$ the unique extension of $\Phi|$ to $M_B(C_c(\mathcal{A}/\Gamma))$ we have, by Lemma 3.3.15, that

$$= \sum_{\substack{[u]\in \Gamma g^{-1}\Gamma/\Gamma \\ [v] \in \Gamma s \Gamma/\Gamma}} \ \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g) N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)} \, \widetilde{\Phi|}([\alpha_{u^{-1}\gamma^{-1}}(a)]_{x\gamma u \Gamma})\omega_\Phi(\Gamma u^{-1} v \Gamma)\widetilde{\Phi|}(1_{\mathbf{s}(x)\gamma v \Gamma}) \, .$$

This finishes the proof. □

THEOREM 3.3.17. *There is a bijective correspondence between nondegenerate *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and covariant pre-*-representations. This bijection is given by $(\pi,\mu) \longmapsto \pi \times \mu$, with inverse given by $\Phi \longmapsto (\Phi|, \omega_\Phi)$.*

Proof: We have to prove that the composition of these maps, in both orders, is the identity.

Let (π,μ) be a covariant pre-*-representation and $\pi \times \mu$ its integrated form. We want to show that

$$\big((\pi\times\mu)|,\, \omega_{\pi\times\mu}\big) = (\pi,\mu) \, .$$

By definition of the integrated form we readily have $(\pi\times\mu)| = \pi$. This also implies, via Lemma 3.3.14, that the inner product spaces on which μ and $\omega_{\pi\times\mu}$ are defined are actually the same. Thus, it remains to be checked that $\omega_{\pi\times\mu} = \mu$. Let $\pi([a]_{x\Gamma})\xi$ be one of the generators of $\pi(C_c(\mathcal{A}/\Gamma))\mathscr{H}$. We have

$$\begin{aligned}
& \omega_{\pi\times\mu}(\Gamma g \Gamma)\, \pi([a]_{x\Gamma})\xi \ = \\
= \ & \widetilde{[\pi\times\mu]}(\Gamma g \Gamma)\, \pi([a]_{x\Gamma})\xi \\
= \ & [\pi\times\mu](\Gamma g \Gamma * [a]_{x\Gamma})\xi \, ,
\end{aligned}$$

and using Proposition 3.1.15, Remark 3.3.11 and Proposition 3.3.9 we obtain

$$\begin{aligned}
&= [\pi \times \mu]\Big(\sum_{[\gamma] \in E^{\mathbf{s}(x)}_{g^{-1},e}} 1_{\mathbf{r}(x)\gamma g\Gamma} * \Gamma g \Gamma * [a]_{x\Gamma} \Big) \xi \\
&= \sum_{[\gamma] \in E^{\mathbf{s}(x)}_{g^{-1},e}} \widetilde{\pi}(1_{\mathbf{r}(x)\gamma g\Gamma}) \mu(\Gamma g \Gamma) \widetilde{\pi}([a]_{x\Gamma}) \xi \\
&= \mu(\Gamma g \Gamma)\, \pi([a]_{x\Gamma}) \xi
\end{aligned}$$

Hence, we conclude that $\omega_{\pi \times \mu} = \mu$.

Now let Φ be a *-representation of $C_c(\mathcal{A}/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ and $(\Phi|, \omega_\Phi)$ its corresponding covariant pre-*-representation. We want to prove that

$$\Phi| \times \omega_\Phi = \Phi\,.$$

Let $1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma}$ be one of the spanning elements of $C_c(\mathcal{A}/\Gamma) \times^{alg}_{\alpha} G/\Gamma$ and $\xi \in \mathscr{H}$. We have

$$[\Phi| \times \omega_\Phi]\,(1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\, \xi = \widetilde{\Phi|}(1_{\mathbf{r}(x)\Gamma}) \omega_\Phi(\Gamma g \Gamma) \widetilde{\Phi|}([\alpha_{g^{-1}}(a)]_{xg\Gamma})\, \xi\,,$$

which by Lemma 3.3.15 gives that

$$\begin{aligned}
&= \widetilde{\Phi}(1_{\mathbf{r}(x)\Gamma}) \widetilde{\Phi}(\Gamma g \Gamma) \widetilde{\Phi}([\alpha_{g^{-1}}(a)]_{xg\Gamma})\, \xi \\
&= \Phi(1_{\mathbf{r}(x)\Gamma} * \Gamma g \Gamma * [\alpha_{g^{-1}}(a)]_{xg\Gamma})\, \xi\,.
\end{aligned}$$

Thus, $\Phi| \times \omega_\Phi = \Phi$. ☐

We will now show that the bijective correspondence between covariant pre-*-representations and nondegenerate *-representations of the crossed product behaves as expected regarding unitary equivalence. First however we make the following remark/definition:

Remark 3.3.18. Let (π, μ) be a covariant pre-*-representation on a Hilbert space $\mathscr{H}$. If $\mathscr{H}_0$ is another Hilbert space and $U : \mathscr{H} \to \mathscr{H}_0$ is a unitary, then it is easily seen that $(U\pi U^*, U\mu U^*)$ is also a covariant pre-*-representation. We will henceforward say that two covariant pre-*-representations (π_1, μ_1) and (π_2, μ_2) are *unitarily equivalent* if there exists a unitary U between the underlying Hilbert spaces such that $(\pi_1, \mu_1) = (U\pi_2 U^*, U\mu_2 U^*)$.

Proposition 3.3.19. *Suppose that (π_1, μ_1) and (π_2, μ_2) are two covariant pre-*-representations. Then (π_1, μ_1) is unitarily equivalent to (π_2, μ_2) if and only if $\pi_1 \times \mu_1$ is unitarily equivalent to $\pi_2 \times \mu_2$.*

Proof: ($\Longrightarrow$) This direction is straightforward from the definition of the integrated form and from the following computation, where U is a unitary which establishes an equivalence between (π_1, μ_1) and (π_2, μ_2) :

$$\begin{aligned} & [U(\pi_1 \times \mu_1)U^*]([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}) \;= \\ =\;& U\pi_1([a]_{x\Gamma})\mu_1(\Gamma g\Gamma)\widetilde{\pi_1}(1_{\mathbf{s}(x)g\Gamma})U^* \\ =\;& U\pi_1([a]_{x\Gamma})U^*U\mu_1(\Gamma g\Gamma)U^*U\widetilde{\pi_1}(1_{\mathbf{s}(x)g\Gamma})U^* \\ =\;& [U\pi_1U^* \times U\mu_1U^*]([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}) \\ =\;& [\pi_2 \times \mu_2]([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}) \,. \end{aligned}$$

($\Longleftarrow$) Suppose that $\pi_1 \times \mu_1$ and $\pi_2 \times \mu_2$ are unitarily equivalent and let U be a unitary which establishes this equivalence. Then, since π_1 and π_2 are just the restrictions of, respectively, $\pi_1 \times \mu_1$ and $\pi_2 \times \mu_2$ we automatically have that $U\pi_1U^* = \pi_2$. To see that $U\mu_1U^* = \mu_2$ we just note that U canonically establishes a unitary equivalence between the associated pre-*-representations $\widetilde{\pi_1 \times \mu_1}$ and $\widetilde{\pi_2 \times \mu_2}$ of the multiplier algebra $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$. □

3.4. More on covariant pre-*-representations

In the previous section we introduced the notion of covariant pre-*-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ (Definition 3.3.1) and a particular instance of these which we called covariant *-representations (Definition 3.3.5).

In this section we will see that the class of covariant pre-*-representations is in general strictly larger than the class of covariant *-representations. It is thus unavoidable, in general, to consider pre-representations of the Hecke algebra in the representation theory of crossed products by Hecke pairs. We shall also see, nevertheless, that in many interesting situations every covariant pre-*-representation is actually a covariant *-representation.

Example 3.4.1. Let (G, Γ) be a Hecke pair such that its corresponding Hecke algebra $\mathcal{H}(G, \Gamma)$ does not have an enveloping C^*-algebra (it is well known that such pairs exist, as for example $(G, \Gamma) = (SL_2(\mathbb{Q}_p), SL_2(\mathbb{Z}_p))$ as discussed in [**8**]). The fact that the Hecke algebra does not have an enveloping C^*-algebra implies that there is a sequence of *-representations $\{\mu_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}(G, \Gamma)$ on Hilbert spaces $\{\mathscr{H}_n\}_{n\in\mathbb{N}}$ and an element $f \in \mathcal{H}(G, \Gamma)$ such that $\|\mu_n(f)\| \to \infty$. Let $\mathscr{V}$ be the inner product space $\mathscr{V} := \bigoplus_{n\in\mathbb{N}} \mathscr{H}_n$ and $\mu : \mathcal{H}(G, \Gamma) \to L(\mathscr{V})$ the diagonal pre-*-representation

$$\mu := \bigoplus_{n\in\mathbb{N}} \mu_n \,,$$

which of course is not normed. Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set, with the trivial groupoid structure, i.e. X is just a set. We consider the Fell bundle $\mathcal{A}$ over X whose fibers are the complex numbers, i.e. $\mathcal{A}_x = \mathbb{C}$ for every $x \in X$, and we consider the trivial action of G on $\mathcal{A}$, i.e. the action that fixes every element of $\mathcal{A}$. Thus, the action is Γ-good and has the Γ-intersection property. We also have that

$$C_c(\mathcal{A}/\Gamma) = C_c(X) = C_c(X^0/\Gamma) \,.$$

Let $\pi : C_c(X) \to B(\overline{\mathscr{V}})$ be the *-representation on the Hilbert space completion $\overline{\mathscr{V}}$ of $\mathscr{V}$ such that $\pi(1_{x_n})$ is the projection onto the subspace $\mathscr{H}_n$.

We claim that (π, μ) is a covariant pre-*-representation of $C_c(X) \times_\alpha^{alg} G/\Gamma$. To see this, first we notice that π is obviously nondegenerate and moreover $\pi(C_c(X))\overline{\mathscr{V}} = \mathscr{V}$, which is the inner product space where μ is defined. Next we notice that for every $x_n \in X$ and $g \in G$, the operators $\pi(1_{x_n})$ and $\mu(\Gamma g \Gamma)$ commute. Moreover, we have

$$\pi(1_{x_n})\mu(\Gamma g\Gamma)\pi(1_{x_n}) = \mu_n(\Gamma g \Gamma)\,,$$

on the subspace $\mathscr{H}_n$. Also we have

$$\begin{aligned}
& \mu(\Gamma g\Gamma)\pi(1_{x_n})\mu(\Gamma s\Gamma) \;\; = \\
= \;\; & \mu(\Gamma g\Gamma)\mu(\Gamma s\Gamma)\pi(1_{x_n}) \\
= \;\; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)} \;\mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n}) \\
= \;\; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\Delta(g)}{L(u^{-1}v)} \;\pi(1_{x_n})\mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n}) \\
= \;\; & \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E_{u,v}^{x_n}} \frac{\Delta(g)N_{u^{-1},v}^{x_n\gamma}}{L(u^{-1}v)} \;\pi(1_{x_n\gamma u})\mu(\Gamma u^{-1}v\Gamma)\,\pi(1_{x_n\gamma v})\,,
\end{aligned}$$

where the last equality comes from the fact that since $\mathcal{S}_{x_n} = G$ we must have that $E_{u,v}^{x_n}$ consists only of the class $[e]$, $N_{u^{-1},v}^{x_n} = 1$ and also that $1_{x_n\gamma u} = 1_{x_n} = 1_{x_n\gamma v}$.

So we have established that (π, μ) is indeed a covariant pre-*-representation. Nevertheless, μ is not normed, so that (π, μ) is not a covariant *-representation.

It is worth noting that here we are in the conditions of Example 3.2.3, so that $C_c(X) \times_\alpha^{alg} G/\Gamma \;\;\cong\;\; C_c(X) \odot \mathcal{H}(G,\Gamma)$.

Example 3.4.1 shows that there can be more covariant pre-*-representations than covariant *-representations. Nevertheless, the two classes actually coincide in many cases. One such case is when $C_c(\mathcal{A}/\Gamma)$ has an identity element:

Proposition 3.4.2. *If the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element (equivalently, if $C_c(\mathcal{A}/\Gamma)$ has an identity element), then every covariant pre-*-representation is a covariant *-representation.*

Proof: Let us assume that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element (equivalently, $C_c(\mathcal{A}/\Gamma)$ has an identity element).

Let (π, μ) be a covariant pre-*-representation. As it was shown in Theorem 3.3.17, the integrated form $\pi\times\mu$ is a *-representation of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ such that $\mu = \omega_{\pi\times\mu}$, where $\omega_{\pi\times\mu}$ is the pre-*-representation which is obtained by extending $\pi \times \mu$ to the multiplier algebra $M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)$ and then restricting it to $\mathcal{H}(G,\Gamma)$. Since the crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ has an identity element, we have

$$M(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) = C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma\,,$$

and therefore $\omega_{\pi\times\mu}$ is just the restriction of $\pi \times \mu$ to the the Hecke algebra $\mathcal{H}(G,\Gamma)$. Hence, $\mu = \omega_{\pi\times\mu}$ is a true *-representation. $\square$

Another interesting situation where covariant pre-*-representations coincide with covariant *-representations is when $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra. This is known to be the case for many classes of Hecke pairs (G,Γ) as we proved in [**17**]. Actually, the author does not know of any Hecke pair (G,Γ) for which the full Hecke C^*-algebra exists but $\mathcal{H}(G,\Gamma)$ is not BG^*-algebra. It would be interesting to know if a counter-example exists (as was already asked in [**17**, Section 7, point 4]).

PROPOSITION 3.4.3. *If $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra, then every covariant pre-*-representation is a covariant *-representation.*

Proof: If $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra, then every pre-*-representation of $\mathcal{H}(G,\Gamma)$ is automatically normed and hence arises from a true *-representation. $\square$

3.5. Crossed product in the case of free actions

In this section we will see that when the associated G-action on X is free the expressions for the products of the form $\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma$, described in Corollary 3.1.18, as well as the definition of a covariant pre-*-representation become much simpler and even more similar to the notion of *covariant pairs* of [**9**].

THEOREM 3.5.1. *If the action of G on X is free, then*

$$\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} 1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma} \tag{3.22}$$

and similarly,

$$\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} [\alpha_{u^{-1}}(a)]_{xu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)v\Gamma}\,. \tag{3.23}$$

We recall from (3.10) and (3.11) the definitions of the sets $\mathfrak{n}^y_{w,v}$ and $\mathfrak{d}^y_{w,v}$, and from (3.12) and (3.13) the definitions of the numbers $n^y_{w,v}$ and $d^y_{w,v}$.

LEMMA 3.5.2. *If the action of G on X is free, then*

$$n^y_{w,v} = 1 \qquad \textit{and} \qquad d^y_{w,v} = [\Gamma^{wv} : \Gamma^{wv} \cap w\Gamma w^{-1}]\,.$$

Proof: We have

$$\begin{aligned}
\mathfrak{n}^y_{w,v} &= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ yw^{-1} \in y\Gamma r^{-1}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ w^{-1} \in \Gamma r^{-1}\big\} \\
&= \big\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ r\Gamma = w\Gamma\big\} \\
&= \{w\Gamma\}\,.
\end{aligned}$$

Thus, $n^y_{w,v} = 1$. Also,

$$\begin{aligned}\mathfrak{d}^y_{w,v} &= \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ yw^{-1} \in y\Gamma r^{-1}\Gamma^{wv}\right\} \\ &= \left\{[r] \in \Gamma w\Gamma/\Gamma : r^{-1}wv\Gamma \subseteq \Gamma v\Gamma \text{ and } \ w^{-1} \in \Gamma r^{-1}\Gamma^{wv}\right\}.\end{aligned}$$

Now we notice that in the above set the condition $r^{-1}wv\Gamma \subseteq \Gamma v\Gamma$ is automatically satisfied from the second condition $w^{-1} \in \Gamma r^{-1}\Gamma^{wv}$, because the latter means that $r^{-1} = \theta_1 w^{-1}\theta_2$ for some $\theta_1 \in \Gamma$ and $\theta_2 \in \Gamma^{wv}$. Thus, we get

$$\begin{aligned}\mathfrak{d}^y_{w,v} &= \left\{[r] \in \Gamma w\Gamma/\Gamma : w^{-1} \in \Gamma r^{-1}\Gamma^{wv}\right\} \\ &= \left\{[r] \in \Gamma w\Gamma/\Gamma : r \in \Gamma^{wv}w\Gamma\right\} \\ &= \Gamma^{wv}w\Gamma/\Gamma.\end{aligned}$$

Thus, we obtain $d^y_{w,v} = \left|\Gamma^{wv}w\Gamma/\Gamma\right| = [\Gamma^{wv} : \Gamma^{wv} \cap w\Gamma w^{-1}]$. □

Proof of Theorem 3.5.1: We have seen in Theorem 3.1.16 that

$$\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g)N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right)$$

It follows from Lemma 3.5.2 that

$$N^{y\gamma}_{u^{-1},v} = \frac{1}{[\Gamma^{u^{-1}v} : \Gamma^{u^{-1}v} \cap u^{-1}\Gamma u]}.$$

Moreover, freeness of the action also implies that

$$\begin{aligned}E^y_{u,v} &= S_y\backslash\Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}) \\ &= \Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}).\end{aligned}$$

Now, we have the following well-defined bijective correspondence

$$\begin{aligned}\Gamma/(\Gamma^u \cap \Gamma^v) &\longrightarrow \Gamma/(v\Gamma v^{-1} \cap u\Gamma u^{-1}) \\ [\theta] &\mapsto [\theta],\end{aligned}$$

given by Proposition 1.3.1. Note that $\Gamma^u \cap \Gamma^v$ is simply the subgroup $u\Gamma u^{-1} \cap v\Gamma v^{-1} \cap \Gamma$, but in the following we will take preference on the notation $\Gamma^u \cap \Gamma^v$ for being shorter.

Consider now the action of Γ on $G/\Gamma \times G/\Gamma$ by left multiplication and denote by $\mathcal{O}_{h_1,h_2}$ the orbit of the element $(h_1\Gamma, h_2\Gamma) \in G/\Gamma \times G/\Gamma$. It is easy to see that the map

$$\begin{aligned}\Gamma/(\Gamma^{h_1} \cap \Gamma^{h_2}) &\longrightarrow \mathcal{O}_{h_1,h_2} \\ [\theta] &\mapsto (\theta h_1\Gamma, \theta h_2\Gamma)\end{aligned}$$

is also well-defined and is a bijection. We will denote by $\mathcal{C}$ the set of all orbits contained in $\Gamma g^{-1}\Gamma/\Gamma \times \Gamma s\Gamma/\Gamma$ (note that this set is Γ-invariant, so that it is a union of orbits). We then have

$$
\begin{aligned}
&\Gamma g\Gamma * 1_{y\Gamma} * \Gamma s\Gamma = \\
= &\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^y_{u,v}} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right) \\
= &\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in\Gamma/(\Gamma^u\cap\Gamma^v)} \frac{\Delta(g) N^{y\gamma}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}v\Gamma * 1_{y\gamma v\Gamma}\right) \\
= &\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in\Gamma/(\Gamma^u\cap\Gamma^v)} \frac{\Delta(g) N^{y}_{u^{-1}\gamma^{-1},\gamma v}}{L(u^{-1}\gamma^{-1}\gamma v)} \left(1_{y\gamma u\Gamma} * \Gamma u^{-1}\gamma^{-1}\gamma v\Gamma * 1_{y\gamma v\Gamma}\right)
\end{aligned}
$$

where the last equality comes from the fact that $N^{y\gamma}_{u^{-1},v} = N^{y}_{u^{-1}\gamma^{-1},\gamma v}$, which is a consequence of Lemma 3.1.17 *iii*), or simply by Lemma 3.5.2. Using now the bijection between $\Gamma/(\Gamma^u\cap\Gamma^v)$ and the orbit space $\mathcal{O}_{u,v}$ as described above, we obtain

$$
\begin{aligned}
= &\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{([r],[t])\in\mathcal{O}_{u,v}} \frac{\Delta(g) N^{y}_{r^{-1},t}}{L(r^{-1}t)} \left(1_{yr\Gamma} * \Gamma r^{-1}t\Gamma * 1_{yt\Gamma}\right) \\
= &\sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \sum_{([r],[t])\in\mathcal{O}_{u,v}} \frac{\Delta(g) N^{y}_{r^{-1},t}}{L(r^{-1}t)} \left(1_{yr\Gamma} * \Gamma r^{-1}t\Gamma * 1_{yt\Gamma}\right) \\
= &\sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \sum_{([r],[t])\in\mathcal{O}} \frac{\Delta(g) N^{y}_{r^{-1},t}}{L(r^{-1}t)} \left(1_{yr\Gamma} * \Gamma r^{-1}t\Gamma * 1_{yt\Gamma}\right) \\
= &\sum_{\mathcal{O}\in\mathcal{C}} \sum_{([r],[t])\in\mathcal{O}} \frac{\#\mathcal{O}\,\Delta(g) N^{y}_{r^{-1},t}}{L(r^{-1}t)} \left(1_{yr\Gamma} * \Gamma r^{-1}t\Gamma * 1_{yt\Gamma}\right),
\end{aligned}
$$

where $\#\mathcal{O}$ denotes the total number of elements of the given orbit $\mathcal{O}$. Changing the names of the variables (r to u and t to v) we get

$$
\begin{aligned}
= &\sum_{\mathcal{O}\in\mathcal{C}} \sum_{([u],[v])\in\mathcal{O}} \frac{\#\mathcal{O}\,\Delta(g) N^{y}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma}\right) \\
= &\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma \\ [v]\in\Gamma s\Gamma/\Gamma}} \frac{\#\mathcal{O}_{u,v}\,\Delta(g) N^{y}_{u^{-1},v}}{L(u^{-1}v)} \left(1_{yu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{yv\Gamma}\right).
\end{aligned}
$$

We are now going to prove that the coefficients satisfy

$$\frac{\#\mathcal{O}_{u,v}\,\Delta(g) N^{y}_{u^{-1},v}}{L(u^{-1}v)} = 1\,.$$

This follows from the following computation:

$$\begin{aligned}
\frac{\#\mathcal{O}_{u,v}}{L(u^{-1}v)}\, N^{y}_{u^{-1},v}\, \Delta(g) &= \frac{[\Gamma:\Gamma^{u}\cap\Gamma^{v}]}{[\Gamma:\Gamma^{u^{-1}v}]}\cdot\frac{1}{[\Gamma^{u^{-1}v}:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]}\cdot\frac{[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^{u}]}\\
&= \frac{[\Gamma:\Gamma^{u}\cap\Gamma^{v}]\,[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u][\Gamma:\Gamma^{u}]}\\
&= \frac{[\Gamma^{u}:\Gamma^{u}\cap\Gamma^{v}]\,[\Gamma:\Gamma^{u^{-1}}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]}\\
&= \frac{[\Gamma^{u}:\Gamma^{u}\cap\Gamma^{v}]\,[u\Gamma u^{-1}:\Gamma^{u}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]}\\
&= \frac{[u\Gamma u^{-1}:\Gamma^{u}\cap\Gamma^{v}]}{[\Gamma:\Gamma^{u^{-1}v}\cap u^{-1}\Gamma u]}\\
&= \frac{[u\Gamma u^{-1}:\Gamma^{u}\cap\Gamma^{v}]}{[u\Gamma u^{-1}:\Gamma^{u}\cap\Gamma^{v}]}\\
&= 1\,.
\end{aligned}$$

This finishes the first claim of the theorem. The second claim, concerning the product $\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma$, is proven in a completely similar fashion. $\square$

PROPOSITION 3.5.3. *Let $\pi : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H})$ be a nondegenerate *-representation, $\mu : \mathcal{H}(G,\Gamma) \to L(\pi(C_c(\mathcal{A}/\Gamma)\mathscr{H})$ a unital pre-*-representation, and let us assume that the associated G-action on X is free. The pair (π,μ) is a covariant pre-*-representation if and only if the following equality*

$$\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma s\Gamma) = \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \pi([\alpha_{u^{-1}}(a)]_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,. \tag{3.24}$$

holds for all $g,s\in G$, $x\in X$ and $a\in\mathcal{A}_x$.

Proof: $(\Longrightarrow)$ Assume that (π,μ) is a covariant pre-*-representation. Then we have

$$\begin{aligned}
\mu(\Gamma g\Gamma)\pi([a]_{x\Gamma})\mu(\Gamma s\Gamma) &= [\pi\times\mu](\Gamma g\Gamma * [a]_{x\Gamma} * \Gamma s\Gamma)\\
&= [\pi\times\mu]\Big(\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} [\alpha_{u^{-1}}(a)]_{xu\Gamma} * \Gamma u^{-1}v\Gamma * 1_{\mathbf{s}(x)v\Gamma}\Big)\\
&= \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \pi([\alpha_{u^{-1}}(a)]_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,.
\end{aligned}$$

$(\Longleftarrow)$ In order to prove equality (3.18) one just needs to show that

$$\begin{aligned}
&\sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \sum_{[\gamma]\in E^{\mathbf{s}(x)}_{u,v}} \frac{\Delta(g)N^{\mathbf{s}(x)\gamma}_{u^{-1},v}}{L(u^{-1}v)}\,\widetilde{\pi}([\alpha_{u^{-1}}(a)]_{x\gamma u\Gamma})\,\mu(\Gamma u^{-1}v\Gamma)\,\widetilde{\pi}(1_{\mathbf{s}(x)\gamma v\Gamma})\\
=\;& \sum_{\substack{[u]\in\Gamma g^{-1}\Gamma/\Gamma\\ [v]\in\Gamma s\Gamma/\Gamma}} \widetilde{\pi}([\alpha_{u^{-1}}(a)]_{xu\Gamma})\mu(\Gamma u^{-1}v\Gamma)\widetilde{\pi}(1_{\mathbf{s}(x)v\Gamma})\,,
\end{aligned}$$

and this is proven in a completely analogous way as in the proof of Theorem 3.5.1. $\square$

CHAPTER 4

Direct limits of sectional algebras

Having defined *-algebraic crossed products by Hecke pairs in the previous chapter, the goal is now to complete them with appropriate C^*-norms and the current chapter contains the preliminary ideas and results to achieve that goal.

In this chapter we will see how, for a finite index subgroup inclusion $K \subseteq H$, the algebra $C_c(\mathcal{A}/H)$ embeds canonically inside $C_c(\mathcal{A}/K)$. All these inclusions are compatible with each other, so that we are able to form a certain direct limit $\mathcal{D}(\mathcal{A})$ which will play an essential role for defining C^*-crossed products by Hecke pairs.

Throughout this chapter $\mathcal{A}$ denotes a Fell bundle over a discrete groupoid X, endowed with an action α of a group G. We will always assume in every statement of this chapter that the subgroup denoted by $H \subseteq G$ is such that the action α is H-good.

PROPOSITION 4.0.1. *Suppose $K \subseteq H \subseteq G$ are subgroups such that $[H : K] < \infty$. Then, there is an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ determined by*

$$[a]_{xH} \longmapsto \sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK} \,.$$

REMARK 4.0.2. We have shown in Proposition 2.3.7 that inside the multiplier algebra $M(C_c(\mathcal{A}))$ the element $[a]_{xH}$ decomposes as a sum of elements of $C_c(\mathcal{A}/K)$ as above. The point of Proposition 4.0.1 is that this decomposition really defines an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$. Moreover, here we are not working inside $M(C_c(\mathcal{A}))$ anymore. Nevertheless this embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ is compatible with the embeddings of these algebras into $M(C_c(\mathcal{A}))$ as we will see at the end of this chapter.

Proof of Propostion 4.0.1: It is clear that the expression above is well-defined, since $[H : K] < \infty$, and it determines a linear map $\Phi : C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)$. Moreover, it follows directly from Proposition1.3.2 that this map is injective. The fact that Φ preserves the involution follows from the following computation

$$\begin{aligned}
\Phi(([a]_{xH})^*) &= \Phi([a^*]_{x^{-1}H}) = \sum_{[h] \in \mathcal{S}_{x^{-1}} \backslash H/K} [\alpha_{h^{-1}}(a^*)]_{x^{-1}hK} \\
&= \sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a^*)]_{x^{-1}hK} = \Big(\sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK} \Big)^* \\
&= \Phi([a]_{xH})^* \,.
\end{aligned}$$

Let us now check that Φ preserves products. If the pair (xH, yH) is not composable, then no pair of the form (xuK, ytK), with $u, t \in H$, is composable. Hence, in this case we have

$$\Phi([a]_{xH}[b]_{yH}) \;=\; 0 \;=\; \Phi([a]_{xH})\Phi([b]_{yH})\,.$$

Suppose now the pair (xH, yH) is composable, and let $\widetilde{h} \in H_{x,y}$. We have

$$\begin{aligned}\Phi([a]_{xH})\Phi([b]_{yH}) &= \Big(\sum_{[u]\in\mathcal{S}_x\backslash H/K}[\alpha_{u^{-1}}(a)]_{xuK}\Big)\Big(\sum_{[t]\in\mathcal{S}_y\backslash H/K}[\alpha_{t^{-1}}(b)]_{ytK}\Big)\\ &= \sum_{[t]\in\mathcal{S}_y\backslash H/K}\;\sum_{[u]\in\mathcal{S}_x\backslash H/K}[\alpha_{u^{-1}}(a)]_{xuK}[\alpha_{t^{-1}}(b)]_{ytK}\,.\end{aligned}$$

We now claim that

$$\sum_{[u]\in\mathcal{S}_x\backslash H/K}[\alpha_{u^{-1}}(a)]_{xuK}[\alpha_{t^{-1}}(b)]_{ytK} = [\alpha_{t^{-1}\widetilde{h}^{-1}}(a)\alpha_{t^{-1}}(b)]_{(x\widetilde{h}t)(yt)K}\,.$$

To see this we notice that for $u = \widetilde{h}t$ we do have that the pair $(x\widetilde{h}tK, ytK)$ is composable and $[\alpha_{t^{-1}\widetilde{h}^{-1}}(a)]_{x\widetilde{h}tK}[\alpha_{t^{-1}}(b)]_{ytK} = [\alpha_{t^{-1}\widetilde{h}^{-1}}(a)\alpha_{t^{-1}}(b)]_{(x\widetilde{h}t)(yt)K}$. Now if $[u] \in \mathcal{S}_x\backslash H/K$ is such that the pair (xuK, ytK) is composable, then $\mathbf{s}(x)uK = \mathbf{r}(y)tK$. Since the pair $(x\widetilde{h}tK, ytK)$ is composable we also have $\mathbf{s}(x)\widetilde{h}tK = \mathbf{r}(y)tK$. Thus, $\mathbf{s}(x)uK = \mathbf{s}(x)\widetilde{h}tK$, i.e. $[u] = [\widetilde{h}t]$ by Proposition 1.3.2. This proves our claim and therefore we get

$$\begin{aligned}\Phi([a]_{xH})\Phi([b]_{yH}) &= \sum_{[t]\in\mathcal{S}_y\backslash H/K}\;\sum_{[u]\in\mathcal{S}_x\backslash H/K}[\alpha_{u^{-1}}(a)]_{xuK}[\alpha_{t^{-1}}(b)]_{ytK}\\ &= \sum_{[t]\in\mathcal{S}_y\backslash H/K}[\alpha_{t^{-1}\widetilde{h}^{-1}}(a)\alpha_{t^{-1}}(b)]_{(x\widetilde{h}t)(yt)K}\\ &= \sum_{[t]\in\mathcal{S}_y\backslash H/K}[\alpha_{t^{-1}}(\alpha_{\widetilde{h}^{-1}}(a)b)]_{(x\widetilde{h}y)tK}\,.\end{aligned}$$

Recall that since the G-action on $\mathcal{A}$ is H-good we have

$$\mathcal{S}_y \cap H = \mathcal{S}_{\mathbf{s}(y)} \cap H = \mathcal{S}_{x\widetilde{h}y} \cap H\,.$$

Hence, using Proposition 1.3.1, we have bijections

$$\mathcal{S}_y\backslash H/K \;\cong\; (\mathcal{S}_y \cap H)\backslash H/K \;\cong\; (\mathcal{S}_{x\widetilde{h}y}\cap H)\backslash H/K \;\cong\; \mathcal{S}_{x\widetilde{h}y}\backslash H/K\,,$$

determined by the maps $[t] \to [t]$, where $[t]$ denotes the double coset with representative t in the appropriate double coset space. Therefore we get

$$\begin{aligned}\Phi([a]_{xH})\Phi([b]_{yH}) &= \sum_{[t]\in\mathcal{S}_{x\widetilde{h}y}\backslash H/K}[\alpha_{t^{-1}}(\alpha_{\widetilde{h}^{-1}}(a)b)]_{(x\widetilde{h}y)tK}\\ &= \Phi([\alpha_{\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}yH})\\ &= \Phi([a]_{xH}[b]_{yH})\,.\end{aligned}$$

Hence, Φ is an embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$. □

The canonical embeddings described in Proposition 4.0.1 are all compatible, as the following result shows:

PROPOSITION 4.0.3. *Suppose that $L \subseteq K \subseteq H$ are subgroups of G such that $[H : L] < \infty$. The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/L)$ factors through the canonical embeddings of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$, and $C_c(\mathcal{A}/K)$ into $C_c(\mathcal{A}/L)$. In other words, the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C_c(\mathcal{A}/H) \arrow[r] \arrow[rr, bend right] & C_c(\mathcal{A}/K) \arrow[r] & C_c(\mathcal{A}/L) \end{tikzcd} .$$

Proof: Let us denote by $\Phi_1 : C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)$, $\Phi_2 : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/L)$ and $\Phi_3 : C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/L)$ the canonical embeddings. We want to prove that $\Phi_3 = \Phi_2 \circ \Phi_1$. For this it is enough to check this equality on elements of the form $[a]_{xH}$. We have

$$\begin{aligned} \Phi_2 \circ \Phi_1([a]_{xH}) &= \sum_{[h]\in \mathcal{S}_x\backslash H/K} \Phi_2([\alpha_{h^{-1}}(a)]_{xhK}) \\ &= \sum_{[h]\in \mathcal{S}_x\backslash H/K} \sum_{[k]\in \mathcal{S}_{xh}\backslash K/L} [\alpha_{k^{-1}h^{-1}}(a)]_{xhkL} \,. \end{aligned}$$

We claim that if $h_1, \dots, h_n \in H$ is a set of representatives for $\mathcal{S}_x\backslash H/K$, and if $k_1^i, \dots, k_{r_i}^i$ is a set of representatives of $\mathcal{S}_{xh_i}\backslash K/L$ for each $i = 1, \dots, n$, then the set of all products of the form $h_i k_j^i$ is a set of representatives for $\mathcal{S}_x\backslash H/L$. Let us start by proving that every two such products correspond to distinct elements of $\mathcal{S}_x\backslash H/L$. In other words, we want to show that if $[h_i k_j^i] = [h_l k_p^l]$ in $S_x\backslash H/L$, then $h_i = h_l$ and $k_j^i = k_p^l$. To see this we notice that the equality $[h_i k_j^i] = [h_l k_p^l]$ means that $xh_i k_j^i L = xh_l k_p^l L$ (see Proposition 1.3.2), and therefore $xh_i K = xh_l K$, i.e. $[h_i] = [h_l]$ in $S_x\backslash H/K$, hence $h_i = h_l$ because these form a set a of representatives of $\mathcal{S}_x\backslash H/K$. Now, the equality $xh_i k_j^i L = xh_i k_p^i L$ means that $k_j^i = k_p^i$ for the same reasons. Now it remains to prove that any element of $[h] \in \mathcal{S}_x\backslash H/L$ has a representative of the form $h^i k_j^i$. To see this, first we take h_i such that $xhK = xh_iK$, and we consider an element $k \in K$ such that $xh = xh_i k$, obtaining $xhL = xh_i kL$. Now we take k_j^i such that $xh_i kL = xh_i k_j^i L$, and the result follows.

After proving the above claim we can now write

$$\begin{aligned} \Phi_2 \circ \Phi_1([a]_{xH}) &= \sum_{[h]\in \mathcal{S}_x\backslash H/K} \sum_{[k]\in \mathcal{S}_{xh}\backslash K/L} [\alpha_{k^{-1}h^{-1}}(a)]_{xhkL} \\ &= \sum_{[\widetilde{h}]\in \mathcal{S}_x\backslash H/L} [\alpha_{\widetilde{h}^{-1}}(a)]_{x\widetilde{h}L} \\ &= \Phi_3([a]_{xH}) \,. \end{aligned}$$

This finishes the proof. $\square$

Suppose now that (G, Γ) is a Hecke pair for which the G-action on the Fell bundle $\mathcal{A}$ is Γ-good. We define the set $\mathcal{C}$ as the set of all finite intersections of conjugates of Γ, i.e.

$$\mathcal{C} := \Big\{ \bigcap_{i=1}^{n} g_i \Gamma g_i^{-1} : n \in \mathbb{N}, g_1, \dots, g_n \in G \Big\} . \tag{4.1}$$

The set $\mathcal{C}$ becomes a directed set with respect to the partial order given by reverse inclusion of subgroups, i.e. $H_1 \leq H_2 \Leftrightarrow H_1 \supseteq H_2$, for any $H_1, H_2 \in \mathcal{C}$.

Since we are assuming that (G, Γ) is a Hecke pair it is not difficult to see that for any $H_1, H_2 \in \mathcal{C}$ we have

$$H_1 \leq H_2 \implies [H_1 : H_2] < \infty\,.$$

Also, since we are assuming that the G-action on $\mathcal{A}$ is Γ-good and this property passes to conjugates and subgroups, it follows automatically that the action is also H-good, for any $H \in \mathcal{C}$.

The observations in the previous paragraph together with Proposition 4.0.3 imply that $\{C_c(\mathcal{A}/H)\}_{H\in\mathcal{C}}$ is a directed system of *-algebras. Let us denote by $D(\mathcal{A})$ the *-algebraic direct limit of this directed system, i.e.

$$D(\mathcal{A}) := \lim_{\substack{\longrightarrow \\ H\in\mathcal{C}}} C_c(\mathcal{A}/H)\,. \tag{4.2}$$

There is an equivalent way of defining the algebra $D(\mathcal{A})$, by viewing it as the *-subalgebra of $M(C_c(\mathcal{A}))$ generated by all the $C_c(\mathcal{A}/H)$ with $H \in \mathcal{C}$, as we prove in the next result. This characterization of $D(\mathcal{A})$ is also a very useful one.

Proposition 4.0.4. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Then the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C_c(\mathcal{A}/H) \arrow[r] \arrow[dr] & C_c(\mathcal{A}/K) \arrow[d] \\ & M(C_c(\mathcal{A}))\,. \end{tikzcd} \tag{4.3}$$

As a consequence, $D(\mathcal{A})$ is *-isomorphic to the *-subalgebra of $M(C_c(\mathcal{A}))$ generated by all the $C_c(\mathcal{A}/H)$ with $H \in \mathcal{C}$.

Proof: We have to show that, inside $M(C_c(\mathcal{A}))$, we have

$$[a]_{xH} = \sum_{[h]\in\mathcal{S}_x\backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK}\,,$$

for all $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. This was proven in Proposition 2.3.7.

Commutativity of the diagram (4.3) then implies, by universal properties, that there exists a *-homomorphism from $D(\mathcal{A})$ to $M(C_c(\mathcal{A}))$ whose image is precisely the *-subalgebra generated by all $C_c(\mathcal{A}/H)$, with $H \in \mathcal{C}$. This *-homomorphism is injective since all the maps in the diagram (4.3) are injective. □

It is clear that the action $\overline{\alpha}$ gives rise to an action of G on $\mathcal{D}(\mathcal{A})$, which we will still denote by $\overline{\alpha}$. This can be seen either directly, or simply by noticing that the action $\overline{\alpha}$ on $M(C_c(\mathcal{A}))$ takes $\mathcal{D}(\mathcal{A})$ to itself (since for a given $g \in G$ it takes $C_c(\mathcal{A}/H)$ to $C_c(\mathcal{A}/gHg^{-1})$).

The algebra $\mathcal{D}(\mathcal{A})$ will play an essential role in the definition of the various C^*-crossed products by Hecke pairs, particularly the reduced ones. There are two reduced C^*-crossed products by Hecke pairs which are of particular interest to us, and these are $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. These will be defined and studied, in a single approach, in Section 5.2, but for that we need first to understand how the canonical embeddings

$$C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/K)\,, \tag{4.4}$$

defined in Proposition 4.0.1 for $K \subseteq H$ such that $[H : K] < \infty$, behave with respect to the full and reduced C^*-completions. The goal of next subsections is exactly to show that these embeddings always give rise to embeddings in the two canonical C^*-completions

$$C_r^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/K) \qquad \text{and} \qquad C^*(\mathcal{A}/H) \to C^*(\mathcal{A}/K)\,,$$

so that we are able to form the useful C^*-direct limits $\lim_{H \in C} C_r^*(\mathcal{A}/H)$ and $\lim_{H \in C} C^*(\mathcal{A}/H)$.

4.1. Reduced completions $C_r^*(\mathcal{A}/H)$

The purpose of this subsection is to prove the following result:

THEOREM 4.1.1. *Let $K \subseteq H \subseteq G$ be subgroups such that $[H : K] < \infty$. The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to an embedding of $C_r^*(\mathcal{A}/H)$ into $C_r^*(\mathcal{A}/K)$.*

In order to prove this result we need to establish some notation and some lemmas first. Even though Theorem 4.1.1 is stated for subgroups $K \subseteq H$ for which we have a finite index $[H : K]$ we will state and prove the two following lemmas in greater generality, as it will be convenient later on.

Recall, from Proposition 2.3.7, that for any two subgroups $K \subseteq H$ of G for which the G-action is H-good we have that, inside $M(C_c(\mathcal{A}))$, the algebra $C_c(\mathcal{A}/H)$ acts on $C_c(\mathcal{A}/K)$ in the following way:

$$[a]_{xH}[b]_{yK} = \begin{cases} [\alpha_{\widetilde{h}^{-1}}(a)b]_{x\widetilde{h}yK}\,, & \text{if } H_{x,y} \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{h}$ is any element of $H_{x,y}$. As a consequence, this action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)$ defines a *-homomorphism

$$C_c(\mathcal{A}/H) \to M(C_c(\mathcal{A}/K))\,.$$

It could be proven (in the same fashion as Theorem 2.3.1) that the *-homomorphism above is in fact an embedding, but we will not need this fact here. We now make the following definition:

DEFINITION 4.1.2. Suppose A is *-algebra and B is a C^*-algebra. A *right $A-B$ bimodule* X is a (right) inner product B-module (in the sense of [**21**, Definition 2.1]) which is also a left A-module satisfying:

$$a(xb) = (ax)b\,,$$
$$\langle ax, y\rangle_B = \langle x, a^*y\rangle_B\,,$$

for all $x, y \in X$, $a \in A$ and $b \in B$.

Given a right $A - B$ bimodule X we will say that A *acts by bounded operators* on X if for any $a \in A$ there exists $C > 0$ such that

$$\|ax\|_B \leq C\|x\|_B\,,$$

for every $x \in X$, where $\|\cdot\|_B$ is the norm induced by $\langle \cdot\,, \cdot\rangle_B$.

If A is a *-algebra which has an enveloping C^*-algebra $C^*(A)$, then any right $A - B$ bimodule where A acts by bounded operators can be completed to a right-Hilbert $C^*(A) - B$ bimodule.

LEMMA 4.1.3. *Let $K \subseteq H$ be subgroups of G and let D be a C^*-algebra. Suppose $C_c(\mathcal{A}/K)$ is an inner product D-module, denoted by $C_c(\mathcal{A}/K)_D$. Assume furthermore that $C_c(\mathcal{A}/K)_D$ is a right $C_c(\mathcal{A}/K) - D$ bimodule and also a right $C_c(\mathcal{A}/H) - D$ bimodule, where $C_c(\mathcal{A}/K)$ acts on itself by right multiplication and $C_c(\mathcal{A}/H)$ acts on $C_c(\mathcal{A}/K)$ in the canonical way.*

If $C_c(\mathcal{A}/K)$ acts on $C_c(\mathcal{A}/K)_D$ by bounded operators, then $C_c(\mathcal{A}/H)$ also acts on $C_c(\mathcal{A}/K)_D$ by bounded operators.

Proof: Suppose that $C_c(\mathcal{A}/K)$ acts on $C_c(\mathcal{A}/K)_D$ by bounded operators. We need to show that $C_c(\mathcal{A}/H)$ also acts on $C_c(\mathcal{A}/K)_D$ by bounded operators, with respect to the norm $\|\cdot\|_D$ induced by the D-valued inner product in $C_c(\mathcal{A}/K)_D$. For this it is enough to prove that the maps

$$[a]_{xH} : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/K)\,,$$

are bounded with respect to the norm $\|\cdot\|_D$. Moreover, from the fact that $([a]_{xH})^*[a]_{xH} = ([a^*a])_{s(x)H}$ it actually suffices to show that for any unit $u \in X^0$ the mapping $[a]_{uH} : C_c(\mathcal{A}/K) \to C_c(\mathcal{A}/K)$ is bounded with respect to the norm $\|\cdot\|_D$, .

As we have seen at the end of Section 2.1 we can write any element $f \in C_c(\mathcal{A}/K)$ as a sum of the form $f = \sum_{yK\in X/K}[f(y)]_{yK}$. Furthermore, we can split the sum according to the ranges of elements, i.e.

$$f = \sum_{yK\in X/K}[f(y)]_{yK} = \sum_{vK\in X^0/K}\ \sum_{\substack{yK\in X/K\\ \mathbf{r}(y)K=vK}}[f(y)]_{yK}\,.$$

Applying the multiplier $[a]_{uH}$ to this element we get

$$\begin{aligned}
[a]_{uH}f &= [a]_{uH}\sum_{vK\in X^0/K}\ \sum_{\substack{yK\in X/K\\ \mathbf{r}(y)K=vK}}[f(y)]_{yK}\\
&= \sum_{vK\in X^0/K}\ \sum_{\substack{yK\in X/K\\ \mathbf{r}(y)K=vK}}[a]_{uH}[f(y)]_{yK}\\
&= \sum_{vK\in X^0/K}\ \sum_{\substack{yK\in X/K\\ \mathbf{r}(y)K=vK}}[\alpha_{\widetilde{h_v}^{-1}}(a)]_{vH}[f(y)]_{yK}\,,
\end{aligned}$$

where h_v is any element of $H_{u,v}$. Hence, if $k_{v,y}$ is any element of $K_{v,y} \subseteq H_{v,y}$ we get

$$
\begin{aligned}
&= \sum_{vK\subseteq uH} \sum_{\substack{yK\in X/K \\ \mathbf{r}(y)K=vK}} [\alpha_{\widetilde{k_{v,y}}^{-1}}(\alpha_{\widetilde{h_v}^{-1}}(a))f(y)]_{yK} \\
&= \sum_{vK\subseteq uH} \sum_{\substack{yK\in X/K \\ \mathbf{r}(y)K=vK}} [\alpha_{\widetilde{h_v}^{-1}}(a)]_{vK}[f(y)]_{yK} \,.
\end{aligned}
$$

Since f has compact support, there are a finite number of elements $v_1K, \dots, v_nK \subseteq uH$ such that

$$
\begin{aligned}
[a]_{uH}f &= \sum_{i=1}^{n} \sum_{\substack{yK\in X/K \\ \mathbf{r}(y)K=v_iK}} [\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}[f(y)]_{yK} \\
&= \Big(\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}\Big)\Big(\sum_{yK\in X/K}[f(y)]_{yK}\Big) \\
&= \Big(\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}\Big)\, f \,.
\end{aligned}
$$

Our assumptions say that left multiplication by elements of $C_c(\mathcal{A}/K)$ is continuous with respect to $\|\cdot\|_D$. Denoting by $\overline{C_c(\mathcal{A}/K)_D}$ the completion of $C_c(\mathcal{A}/K)_D$ as a Hilbert D-module, we have that every element of $C_c(\mathcal{A}/K)$ uniquely defines an element of $\mathcal{L}\big(\overline{C_c(\mathcal{A}/K)_D}\big)$. Denoting by $\|\cdot\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A}/K)_D}\right)}$ the operator norm in $\mathcal{L}\big(\overline{C_c(\mathcal{A}/K)_D}\big)$, we have

$$
\begin{aligned}
\|[a]_{uH}f\,\|_D &= \|\Big(\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}\Big)f\|_D \\
&\leq \|\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A}/K)_D}\right)}\,\|f\|_D \,,
\end{aligned}
$$

Now we notice that we can canonically see $\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}$ as an element of the direct sum of C^*-algebras $(\mathcal{A}/K)_{v_1K}\oplus\cdots\oplus(\mathcal{A}/K)_{v_nK}$, from which we must have, by uniqueness of C^*-norms on C^*-algebras,

$$
\|\sum_{i=1}^{n}[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]_{v_iK}\|_{\mathcal{L}\left(\overline{C_c(\mathcal{A}/K)_D}\right)} = \max_i \|[\alpha_{\widetilde{h_{v_i}}^{-1}}(a)]\| = \max_i \|\alpha_{\widetilde{h_{v_i}}^{-1}}(a)\| = \|a\| \,.
$$

Hence we conclude that $\|[a]_{uH}f\|_D \leq \|a\|\,\|f\|_D$, i.e. $[a]_{uH}$ is bounded. $\square$

Let us now consider $C_c(\mathcal{A}/K)$ as the right $C_c(\mathcal{A}/K) - C_0(\mathcal{A}^0/K)$ bimodule whose completion is the right-Hilbert bimodule ${}_{C^*(\mathcal{A}/K)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$. We claim that the canonical action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)$ makes $C_c(\mathcal{A}/K)$ into a right $C_c(\mathcal{A}/H) - C_0(\mathcal{A}/K)$ bimodule. The fact that $f_1(\xi f_2) = (f_1\xi)f_2$, for any $f_1 \in C_c(\mathcal{A}/H)$, $\xi \in C_c(\mathcal{A}/K)$ and $f_2 \in C_0(\mathcal{A}^0/K)$, is obvious. Thus, we only need to check that $\langle f\xi\,,\,\eta\rangle_{C_0(\mathcal{A}^0/K)} = \langle \xi\,,\,f^*\eta\rangle_{C_0(\mathcal{A}^0/K)}$, for any $f \in C_c(\mathcal{A}/H)$ and

$\xi, \eta \in C_c(\mathcal{A}/K)$. This is also easy to see because, by definition,

$$\begin{aligned} \langle f\xi\,,\,\eta\rangle_{C_0(\mathcal{A}^0/K)} &= ((f\xi)^*\eta)|_{C_0(\mathcal{A}/K)} \\ &= (\xi^*(f^*\eta))|_{C_0(\mathcal{A}/K)} \\ &= \langle \xi\,,\, f^*\eta\rangle_{C_0(\mathcal{A}^0/K)}\,. \end{aligned}$$

Hence, we are under the conditions of Lemma 4.1.3, and therefore the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$ is by bounded operators. Hence, the right $C_c(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodule $C_c(\mathcal{A}/K)$ can be completed to a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$.

LEMMA 4.1.4. *The* **-homomorphism* $\Phi : C^*(\mathcal{A}/H) \to \mathcal{L}(L^2(\mathcal{A}/K))$ *associated with the right-Hilbert bimodule* ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}$ *has the same kernel as the canonical map* $\Lambda : C^*(\mathcal{A}/H) \to C^*_r(\mathcal{A}/H)$.

Proof: The proof of this fact is essentially an adaptation of the proof of [**5**, Proposition 2.10], and is achieved by exhibiting two isomorphic right-Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodules Y and Z such that the *-homomorphisms from $C^*(\mathcal{A}/H)$ into $\mathcal{L}(Y)$ and $\mathcal{L}(Z)$ have the same kernels as Λ and Φ respectively.

We naturally have a right-Hilbert bimodule ${}_{C_0(\mathcal{A}^0/H)}C_0(\mathcal{A}^0/K)_{C_0(\mathcal{A}^0/K)}$, where the action of $C_0(\mathcal{A}^0/H)$ on $C_0(\mathcal{A}^0/K)$ extends the action of $C_c(\mathcal{A}^0/H)$ on $C_c(\mathcal{A}^0/K)$. We define Y as the balanced tensor product of the right-Hilbert bimodules ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/H)_{C_0(\mathcal{A}^0/H)}$ and ${}_{C_0(\mathcal{A}^0/H)}C_0(\mathcal{A}^0/K)_{C_0(\mathcal{A}^0)/K}$, i.e.

$$Y := L^2(\mathcal{A}/H) \otimes_{C_0(\mathcal{A}^0/H)} C_0(\mathcal{A}^0/K)\,.$$

Since $C_0(\mathcal{A}^0/H)$ acts faithfully on $C_0(\mathcal{A}^0/K)$, the associated *-homomorphism of $C^*(\mathcal{A}/H)$ to $\mathcal{L}(Y)$ has the same kernel as Λ. We define Z simply as

$${}_{C^*(\mathcal{A}/H)}Z_{C_0(\mathcal{A}^0/K)} := {}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/K)_{C_0(\mathcal{A}^0/K)}\,.$$

We now want to define an isomorphism $\Psi : L^2(\mathcal{A}/H) \otimes_{C_0(\mathcal{A}^0/H)} C_0(\mathcal{A}^0/K) \to L^2(\mathcal{A}/K)$ of Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodules. We start by defining

$$\begin{aligned} \Psi_0 : C_c(\mathcal{A}/H) \otimes_{C_c(\mathcal{A}^0/H)} C_c(\mathcal{A}^0/K) &\longrightarrow L^2(\mathcal{A}/K)\,, \\ \Psi_0(f_1 \otimes f_2) &:= f_1 \cdot f_2\,. \end{aligned}$$

It is easy to see that Ψ_0 is well-defined. To see that Ψ_0 preserves the inner products it is enough to check on the generators. So let $[a]_{xH}, [b]_{yH} \in C_c(\mathcal{A}/H)$ and $[c]_{uK}, [d]_{vK} \in C_c(\mathcal{A}^0/K)$, with $u, v \in X^0$. We have

$$\begin{aligned} &\langle \Psi_0([a]_{xH} \otimes [c]_{uK})\,,\, \Psi_0([b]_{yH} \otimes [d]_{vK})\rangle_{C_0(\mathcal{A}^0/K)} = \\ &= \langle [a]_{xH}[c]_{uK}\,,\, [b]_{yH}[d]_{vK}\rangle_{C_0(\mathcal{A}^0/K)} \\ &= ([c^*]_{uK}[a^*]_{x^{-1}H}[b]_{yH}[d]_{vK})|_{C_0(\mathcal{A}^0/K)}\,. \end{aligned}$$

Now the product $([c^*]_{uK}[a^*]_{x^{-1}H}[b]_{yH}[d]_{vK})|_{C_0(\mathcal{A}^0/K)}$ is automatically zero unless $vK = uK$, $xH = yH$ and $vK \subseteq \mathbf{s}(y)H$, in which case we necessarily have that $([c^*]_{uK}[a^*]_{x^{-1}H}[b]_{yH}[d]_{vK})|_{C_0(\mathcal{A}^0/K)} = [c^*]_{uK}[a^*]_{x^{-1}H}[b]_{yH}[d]_{vK}$. On the other hand,

$$\begin{aligned} &\langle [a]_{xH} \otimes [c]_{uK}\,,\, [b]_{yH} \otimes [d]_{vK}\rangle_{C_0(\mathcal{A}^0/K)} = \\ &= \langle [c]_{uK}\,,\, \langle [a]_{xH}\,,\, [b]_{yH}\rangle_{C_0(\mathcal{A}^0/H)}\, [d]_{vK}\rangle_{C_0(\mathcal{A}^0/K)} \\ &= [c^*]_{uK}([a^*]_{x^{-1}H}[b]_{yH})|_{C_0(\mathcal{A}^0/H)}\, [d]_{vK} \end{aligned}$$

Now the product $[c^*]_{uK}([a^*]_{x^{-1}H}[b]_{yH})|_{C_0(\mathcal{A}^0/H)}\,[d]_{vK}$ is automatically zero unless $vK = uK$, $xH = yH$ and $vK \subseteq \mathbf{s}(y)H$, in which case we necessarily have that $[c^*]_{uK}([a^*]_{x^{-1}H}[b]_{yH})|_{C_0(\mathcal{A}^0/H)}\,[d]_{vK} = [c^*]_{uK}[a^*]_{x^{-1}H}[b]_{yH}[d]_{vK}$. Hence, we conclude that Ψ_0 preserves the inner products.

Now, if $f_1, f_2 \in C_c(\mathcal{A}/H)$ and $f_3 \in C_c(\mathcal{A}^0/K)$ we have

$$\Psi_0(f_1(f_2 \otimes f_3)) = \Psi_0(f_1 f_2 \otimes f_3) = f_1 f_2 f_3 = f_1 \Psi_0(f_2 \otimes f_3)\,.$$

Thus, Ψ_0 preserves the left module actions. Let us now check that Ψ_0 has a dense image in $L^2(\mathcal{A}/K)$. It is enough to prove that all generators $[a]_{xK} \in C_c(\mathcal{A}/K)$ are in closure of the image of Ψ_0, since their span is dense in $L^2(\mathcal{A}/K)$. To see this, let $\{e^\lambda\}_\lambda$ be an approximate identity of $\mathcal{A}_{\mathbf{s}(x)}$. We have

$$\Psi_0([a]_{xH} \otimes [e^\lambda]_{\mathbf{s}(x)K}) = [a]_{xH}[e^\lambda]_{\mathbf{s}(x)K} = [ae^\lambda]_{xK}\,.$$

We then get

$$\begin{aligned}
\|[ae^\lambda]_{xK} - [a]_{xK}\|^2_{L^2(\mathcal{A}/K)} &= \|[ae^\lambda - a]_{xK}\|^2_{L^2(\mathcal{A}/K)}\\
&= \|\big([ae^\lambda - a]^*[ae^\lambda - a]\big)_{\mathbf{s}(x)K}\|_{C_0(\mathcal{A}^0/K)}\\
&= \|[ae^\lambda - a]^*[ae^\lambda - a]\|\\
&= \|e^\lambda a^* a e^\lambda - e^\lambda a^* a - a^* a e^\lambda + a^* a\|\,.
\end{aligned}$$

Noticing that $a^*a \in \mathcal{A}_{\mathbf{s}(x)}$, we then have that

$$\begin{aligned}
&\leq \|e^\lambda a^* a e^\lambda - e^\lambda a^* a\| + \|-a^* a e^\lambda + a^* a\|\\
&\leq \|a^* a e^\lambda - a^* a\| + \|-a^* a e^\lambda + a^* a\|\\
&\longrightarrow 0\,.
\end{aligned}$$

Thus, we conclude that Ψ_0 has dense range. Hence, from [**5**, Lemma 2.9], it follows that Ψ_0 extends to an isomorphism of the right-Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0/K)$ bimodules Y and Z. □

Proof of Theorem 4.1.1: The image of $C^*(\mathcal{A}/H)$ in $\mathcal{L}(L^2(\mathcal{A}/K))$ is isomorphic to $C_r^*(\mathcal{A}/H)$ by Lemma 4.1.4. On the other hand, the image of $C^*(\mathcal{A}/H)$ in $\mathcal{L}(L^2(\mathcal{A}/K))$ is simply the completion of $C_c(\mathcal{A}/H)$ as a subalgebra of $C_r^*(\mathcal{A}/K)$. Hence, we conclude that the canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to an embedding of $C_r^*(\mathcal{A}/H)$ into $C_r^*(\mathcal{A}/K)$. □

It follows from Theorem 4.1.1 and Proposition 4.0.3 that for any subgroups $L \subseteq K \subseteq H$ such that $[H : L] < \infty$ the following diagram of canonical embeddings commutes

$$\begin{tikzcd}
C_r^*(\mathcal{A}/H) \arrow[r] \arrow[rr, bend right] & C_r^*(\mathcal{A}/K) \arrow[r] & C_r^*(\mathcal{A}/L)
\end{tikzcd}\,.$$

Hence, we have a direct system of C^*-algebras $\{C_r^*(\mathcal{A}/H)\}_{H\in\mathcal{C}}$. Let us denote by $\mathcal{D}_r(\mathcal{A})$ its corresponding C^*-algebraic direct limit

$$\mathcal{D}_r(\mathcal{A}) := \lim_{H\in\mathcal{C}} C_r^*(\mathcal{A}/H)\,. \tag{4.5}$$

We notice that the algebra $\mathcal{D}(\mathcal{A})$ is a dense *-subalgebra of $\mathcal{D}_r(\mathcal{A})$. We now want to show that the action $\overline{\alpha}$ of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_r(\mathcal{A})$.

THEOREM 4.1.5. *The action $\overline{\alpha}$ of G on $\mathcal{D}(\mathcal{A})$ extends uniquely to an action of G on $\mathcal{D}_r(\mathcal{A})$ and is such that $\overline{\alpha}_g$ takes $C^*_r(\mathcal{A}/H)$ to $C^*_r(\mathcal{A}/gHg^{-1})$, for any $g \in G$.*

Proof: We have a canonical isomorphism between the right-Hilbert bimodules ${}_{C^*(\mathcal{A}/H)}L^2(\mathcal{A}/H)_{C_0(\mathcal{A}^0/H)}$ and ${}_{C^*(\mathcal{A}/gHg^{-1})}L^2(\mathcal{A}/gHg^{-1})_{C_0(\mathcal{A}^0/gHg^{-1})}$, that is determined by the canonical isomorphisms $C_c(\mathcal{A}/H) \to C_c(\mathcal{A}/gHg^{-1})$ and $C_c(\mathcal{A}^0/H) \to C_c(\mathcal{A}^0/gHg^{-1})$ defined by $\overline{\alpha}_g$, i.e. defined respectively by

$$[a]_{xH} \mapsto [\alpha_g(a)]_{xg^{-1}gHg^{-1}}\,, \qquad \text{and} \qquad [b]_{uH} \mapsto [\alpha_g(b)]_{ug^{-1}gHg^{-1}}\,,$$

where $x \in X$, $u \in X^0$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_u$. Since $C^*_r(\mathcal{A}/H)$ is the image of $C^*(\mathcal{A}/H)$ inside $\mathcal{L}(L^2(\mathcal{A}/H))$, and similarly for $C^*_r(\mathcal{A}/gHg^{-1})$, we conclude that the isomorphism $C_c(\mathcal{A}/H) \cong C_c(\mathcal{A}/gHg^{-1})$ defined by $\overline{\alpha}_g$ extends to an isomorphism $C^*_r(\mathcal{A}/H) \cong C^*_r(\mathcal{A}/gHg^{-1})$. Since $C^*_r(\mathcal{A}/gHg^{-1})$ is embedded in $D_r(\mathcal{A})$, we can see $\overline{\alpha}_g$ as an injective *-homomorphism from $C^*_r(\mathcal{A}/H)$ into $D_r(\mathcal{A})$.

A routine computation shows that the following diagram of canonical injections commutes:

$$\begin{tikzcd} C^*_r(\mathcal{A}/H) \arrow[r] \arrow[dr, "\overline{\alpha}_g"'] & C^*_r(\mathcal{A}/K) \arrow[d, "\overline{\alpha}_g"] \\ & D_r(\mathcal{A})\,. \end{tikzcd}$$

Hence, we obtain an injective *-homomorphism from $\mathcal{D}_r(\mathcal{A})$ to itself, which we still denote by $\overline{\alpha}_g$, and which extends the usual map $\overline{\alpha}_g$ from $D(\mathcal{A})$ to itself. It is also clear that this map is surjective, and that for $g, h \in G$ we have $\overline{\alpha}_{gh} = \overline{\alpha}_g \circ \overline{\alpha}_h$, so that we get an action of G on $D_r(\mathcal{A})$ which extends the usual action of G on $D(\mathcal{A})$. $\square$

4.2. Maximal completions $C^*(\mathcal{A}/H)$

The purpose of this subsection is to prove the following result:

THEOREM 4.2.1. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. The canonical embedding of $C_c(\mathcal{A}/H)$ into $C_c(\mathcal{A}/K)$ completes to a nondegenerate embedding of $C^*(\mathcal{A}/H)$ into $C^*(\mathcal{A}/K)$.*

In order to prove this result we will need to know how to "extend" a representation of $C_c(\mathcal{A}/H)$ to a representation of $C_c(\mathcal{A}/K)$ on a larger Hilbert space.

DEFINITION 4.2.2. Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Let $\pi : C_c(\mathcal{A}/H) \to B(\mathscr{H})$ be a *-representation. We define the map $\pi^K : C_c(\mathcal{A}/K) \to B(\mathscr{H} \otimes \ell^2(X^0/K))$ by

$$\pi^K([a]_{xK})\,(\xi \otimes \delta_{uK}) := \begin{cases} \pi([a]_{xH})\xi \otimes \delta_{\mathbf{r}(x)K}\,, & \text{if } uK = \mathbf{s}(x)K \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

PROPOSITION 4.2.3. *The map π^K is a well-defined *-representation.*

Proof: It is clear that the expression that defines $\pi^K([a]_{xH})$ defines a linear operator in the inner product space $\mathscr{H} \otimes C_c(X^0/K)$, which is easily observed to be bounded. Thus, $\pi^K([a]_{xH}) \in B(\mathscr{H} \otimes \ell^2(X^0/K))$.

It is clear that expression (4.6) defines a linear mapping π^K on $C_c(\mathcal{A}/K)$, so that we only need to see that it preserves products and the involution. To see that it preserves products, consider two elements of the form $[a]_{xK}$ and $[b]_{yK}$. There are two cases to consider: either $\mathbf{r}(y) \in \mathbf{s}(x)K$ or $\mathbf{r}(y) \notin \mathbf{s}(x)K$.

In the second case, we have $[a]_{xK}[b]_{yK} = 0$ and thus $\pi^K([a]_{xK}[b]_{yK}) = 0$. But also $\pi^K([a]_{xK})\pi^K([b]_{yK}) = 0$, because for any vector $\xi \otimes \delta_{uK}$ we have that $\pi^K([b]_{yK})(\xi \otimes \delta_{uK})$ is either zero or equal to $\pi([b]_{yK})\xi \otimes \delta_{\mathbf{r}(y)K}$, and therefore we always have $\pi^K([a]_{xK})\pi^K([b]_{yK})(\xi \otimes \delta_{uK}) = 0$.

In the first case we have

$$
\begin{aligned}
& \pi^K([a]_{xK}[b]_{yK})\,(\xi \otimes \delta_{uK}) \;= \\
= \; & \pi^K([\alpha_{\widetilde{k}^{-1}}(a)b]_{x\widetilde{k}yK})\,(\xi \otimes \delta_{uK}) \\
= \; & \begin{cases} \pi([\alpha_{\widetilde{k}^{-1}}(a)b]_{x\widetilde{k}yH})\xi \otimes \delta_{\mathbf{r}(x\widetilde{k}y)K}\,, & \text{if } uK = \mathbf{s}(x\widetilde{k}y)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \pi([a]_{xH})\pi([b]_{yH})\xi \otimes \delta_{\mathbf{r}(x)K}\,, & \text{if } uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \pi^K([a]_{xK})\,(\pi([b]_{yH})\xi \otimes \delta_{\mathbf{s}(x)K})\,, & \text{if } uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \pi^K([a]_{xK})\,(\pi([b]_{yH})\xi \otimes \delta_{\mathbf{r}(y)K})\,, & \text{if } uK = \mathbf{s}(y)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \pi^K([a]_{xK})\pi^K([b]_{yK})\,(\xi \otimes \delta_{uK})\,.
\end{aligned}
$$

In both cases we have $\pi^K([a]_{xK}[b]_{yK}) = \pi^K([a]_{xK})\pi^K([b]_{yK})$, hence π^K preserves products. Let us now check that it preserves the involution. We have

$$
\begin{aligned}
& \langle \pi^K([a]_{xK})\,(\xi \otimes \delta_{uK})\,,\, \eta \otimes \delta_{vK}\rangle \;= \\
= \; & \begin{cases} \langle \pi([a]_{xH})\xi \otimes \delta_{\mathbf{r}(x)K}\,,\, \eta \otimes \delta_{vK}\rangle\,, & \text{if } uK = \mathbf{s}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \langle \pi([a]_{xH})\xi\,,\, \eta\rangle\,, & \text{if } uK = \mathbf{s}(x)K \text{ and } vK = \mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \langle \xi\,,\, \pi([a^*]_{x^{-1}H})\eta\rangle\,, & \text{if } uK = \mathbf{s}(x)K \text{ and } vK = \mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \begin{cases} \langle \xi \otimes \delta_{uK}\,,\, \pi([a^*]_{x^{-1}H})\eta \otimes \delta_{\mathbf{s}(x)K}\rangle\,, & \text{if } vK = \mathbf{r}(x)K \\ 0, & \text{otherwise.} \end{cases} \\
= \; & \langle \xi \otimes \delta_{uK}\,,\, \pi^K([a^*]_{x^{-1}K})\,(\eta \otimes \delta_{vK})\rangle\,.
\end{aligned}
$$

Hence, we conclude that $\pi^K([a]_{xK})^* = \pi^K(([a]_{xK})^*)$, and therefore π^K preserves the involution. Hence, π^K is a *-representation. □

LEMMA 4.2.4. *Let us denote by $\delta_{uH} \in \ell^2(X^0/K)$ the vector*

$$\delta_{uH} := \sum_{[h]\in \mathcal{S}_u\backslash H/K} \delta_{uhK}\,. \tag{4.7}$$

The map π^K satisfies

$$\pi^K([a]_{xH})\,(\xi\otimes\delta_{uH}) := \begin{cases} \pi([a]_{xH})\xi\otimes\delta_{\mathbf{r}(x)H}\,, & \text{if } uH = \mathbf{s}(x)H,\\ 0, & \text{otherwise.}\end{cases}$$

Proof: We have

$$\pi^K([a]_{xH})\,(\xi\otimes\delta_{uH}) \;=\; \sum_{[h]\in\mathcal{S}_x\backslash H/K}\;\sum_{[h']\in\mathcal{S}_u\backslash H/K} \pi^K([\alpha_{h^{-1}}(a)]_{xhK})\,(\xi\otimes\delta_{uh'K})\,,$$

from which we see that, if $uH \neq \mathbf{s}(x)H$ then $\pi^K([a]_{xH})\,(\xi\otimes\delta_{uH}) = 0$. On the other hand, if $uH = \mathbf{s}(x)H$, then we have

$$\begin{aligned}
&\pi^K([a]_{xH})\,(\xi\otimes\delta_{\mathbf{s}(x)H}) \;=\\
=\;& \sum_{[h]\in\mathcal{S}_x\backslash H/K}\;\sum_{[h']\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi^K([\alpha_{h^{-1}}(a)]_{xhK})\,(\xi\otimes\delta_{\mathbf{s}(x)h'K})\\
=\;& \sum_{[h]\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K}\;\sum_{[h']\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi^K([\alpha_{h^{-1}}(a)]_{xhK})\,(\xi\otimes\delta_{\mathbf{s}(x)h'K})\\
=\;& \sum_{[h]\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi([\alpha_{h^{-1}}(a)]_{xhH})\xi\otimes\delta_{\mathbf{r}(x)hK}\\
=\;& \sum_{[h]\in\mathcal{S}_{\mathbf{s}(x)}\backslash H/K} \pi([a]_{xH})\xi\otimes\delta_{\mathbf{r}(x)hK}\\
=\;& \sum_{[h]\in\mathcal{S}_{\mathbf{r}(x)}\backslash H/K} \pi([a]_{xH})\xi\otimes\delta_{\mathbf{r}(x)hK}\\
=\;& \pi([a]_{xH})\xi\otimes\delta_{\mathbf{r}(x)H}\,.
\end{aligned}$$

This finishes the proof. □

Proof of Theorem 4.2.1: In order to prove this statement we have to show that for any $f\in C_c(\mathcal{A}/H)$ we have $\|f\|_{C^*(\mathcal{A}/K)} = \|f\|_{C^*(\mathcal{A}/H)}$. Since we are viewing $C_c(\mathcal{A}/H)$ as a *-subalgebra of $C_c(\mathcal{A}/K)$ we automatically have the inequality

$$\|f\|_{C^*(\mathcal{A}/K)} \leq \|f\|_{C^*(\mathcal{A}/H)}\,.$$

In order to prove the converse inequality, it suffices to prove that

$$\|\pi(f)\| \leq \|\pi^K(f)\|\,, \tag{4.8}$$

for any nondegenerate *-representation π of $C_c(\mathcal{A}/H)$, because, since π is arbitrary, this clearly implies that $\|f\|_{C^*(\mathcal{A}/H)} \leq \|f\|_{C^*(\mathcal{A}/K)}$. Let us then prove inequality (4.8).

We can write any element $f \in C_c(\mathcal{A}/H)$ as $f = \sum_{xH\in X/H}[f(x)]_{xH}$. Furthermore we can split this sum according to the ranges of elements, i.e.

$$f = \sum_{xH\in X/H} [f(x)]_{xH} = \sum_{vH\in X^0/H} \sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} [f(x)]_{xH}\,.$$

Suppose $\pi : C_c(\mathcal{A}/H) \to B(\mathscr{H})$ is a *-representation and $\xi \in \mathscr{H}$ is a vector of norm one. We have

$$\|\pi\Big(\sum_{xH\in X/H} [f(x)]_{xH}\Big)\xi\|^2 = \|\sum_{vH\in X^0/H} \pi\Big(\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} [f(x)]_{xH}\Big)\xi\|^2\,.$$

For different units $vH \in X^0/H$, the elements $\pi\big(\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} [f(x)]_{xH}\big)\xi$ are easily seen to be orthogonal, so that

$$\begin{aligned}
&= \sum_{vH\in X^0/H} \|\pi\Big(\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} [f(x)]_{xH}\Big)\xi\|^2 \\
&= \sum_{vH\in X^0/H} \|\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} \pi([f(x)]_{xH})\xi\|^2
\end{aligned}$$

In the notation of (4.7), let $\delta_{uH} := \sum_{[h]\in\mathcal{S}_u\backslash H/K} \delta_{uhK}$. Let us denote by C_u the number of elements of $\mathcal{S}_u\backslash H/K$. It is not difficult to check that for any $r \in H$ the map $[h] \mapsto [r^{-1}h]$ is a well-defined bijection between $\mathcal{S}_u\backslash H/K$ and $\mathcal{S}_{ur}\backslash H/K$, so that $C_u = C_{ur}$. We have

$$\begin{aligned}
&= \sum_{vH\in X^0/H} \frac{1}{C_v}\|\sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} \pi([f(x)]_{xH})\xi\otimes\delta_{vH}\|^2 \\
&= \|\sum_{vH\in X^0/H} \sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} \frac{1}{C_v}\pi([f(x)]_{xH})\xi\otimes\delta_{vH}\|^2 \\
&= \|\sum_{vH\in X^0/H} \sum_{\substack{xH\in X/H \\ \mathbf{r}(x)H=vH}} \frac{1}{C_{\mathbf{r}(x)}}\pi([f(x)]_{xH})\xi\otimes\delta_{\mathbf{r}(x)H}\|^2 \\
&= \|\sum_{xH\in X/H} \frac{1}{C_{\mathbf{r}(x)}}\pi([f(x)]_{xH})\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{r}(x)H}\|^2\,.
\end{aligned}$$

By Lemma 4.2.4 we have that

$$= \|\sum_{xH\in X/H} \frac{1}{C_{\mathbf{r}(x)}}\pi^K([f(x)]_{xH})\big(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\big)\|^2\,,$$

and since $\mathcal{S}_{\mathbf{s}(x)}\backslash H/K = \mathcal{S}_{\mathbf{r}(x)}\backslash H/K$, we get that $C_{\mathbf{s}(x)} = C_{\mathbf{r}(x)}$. Thus,

$$\begin{aligned}
&= \|\sum_{xH\in X/H} \frac{1}{C_{\mathbf{s}(x)}}\,\pi^K([f(x)]_{xH})\left(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2\\
&= \|\sum_{xH\in X/H} \frac{1}{C_{\mathbf{s}(x)}}\,\pi^K([f(x)]_{xH})\left(\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2\\
&= \|\sum_{xH\in X/H} \pi^K([f(x)]_{xH})\left(\frac{1}{C_{s(x)}}\,\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2 .
\end{aligned}$$

Similarly as we did for ranges, we can split the sum $\sum_{xH\in X/H}[f(x)]_{xH}$ according to sources. In this way, since this sum is finite, there is a finite number of units $u_1H,\dots,u_nH \in X^0/H$, which we assume to be pairwise different, such that we can write

$$\sum_{xH\in X/H}[f(x)]_{xH} = \sum_{i=1}^{n}\sum_{\substack{xH\in X/H\\ \mathbf{s}(x)H=u_iH}}[f(x)]_{xH} .$$

By Lemma 4.2.4 we see that $\pi^K([f(x)]_{xH})\Big(\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH}\Big) = 0$ unless $\mathbf{s}(x)H = u_iH$. Hence we get

$$\begin{aligned}
&\|\sum_{xH\in X/H} \pi^K([f(x)]_{xH})\left(\frac{1}{C_{s(x)}}\,\widetilde{\pi}(1_{\mathbf{s}(x)H})\xi\otimes\delta_{\mathbf{s}(x)H}\right)\|^2\\
= &\|\sum_{xH\in X/H} \pi^K([f(x)]_{xH})\left(\sum_{i=1}^{n}\frac{1}{C_{u_i}}\,\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH}\right)\|^2\\
= &\|\pi^K\Big(\sum_{xH\in X/H}[f(x)]_{xH}\Big)\left(\sum_{i=1}^{n}\frac{1}{C_{u_i}}\,\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH}\right)\|^2 .
\end{aligned}$$

We now notice that, since we are assuming ξ to be of norm one, it follows that the vector

$$\sum_{i=1}^{n}\frac{1}{C_{u_i}}\,\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH} ,$$

also has norm less or equal to one, because

$$\begin{aligned}
\|\sum_{i=1}^{n}\frac{1}{C_{u_i}}\,\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH}\|^2 &= \sum_{i=1}^{n}\|\frac{1}{C_{u_i}}\,\widetilde{\pi}(1_{u_iH})\xi\otimes\delta_{u_iH}\|^2\\
&= \sum_{i=1}^{n}\|\widetilde{\pi}(1_{u_iH})\xi\|^2\\
&= \|\widetilde{\pi}\Big(\sum_{i=1}^{n}1_{u_iH}\Big)\xi\|^2\\
&\leq \|\xi\|^2\\
&= 1 .
\end{aligned}$$

Hence, taking the supremum over vectors ξ of norm one, we immediately get the inequality

$$\|\pi(f)\| \leq \|\pi^K(f)\|.$$

As we explained earlier, this proves that we get an embedding of $C^*(\mathcal{A}/H)$ into $C^*(\mathcal{A}/K)$. $\square$

It follows from 4.2.1 that $\{C^*(\mathcal{A}/H)\}_{H\in\mathcal{C}}$ is a direct system of C^*-algebras. Let us denote by $\mathcal{D}_{\max}(\mathcal{A})$ its corresponding C^*-algebraic direct limit

$$\mathcal{D}_{\max}(\mathcal{A}) := \lim_{H\in\mathcal{C}} C^*(\mathcal{A}/H)\,, \tag{4.9}$$

We notice that the algebra $\mathcal{D}(\mathcal{A})$ is a dense *-subalgebra of $\mathcal{D}_{\max}(\mathcal{A})$. We now want to show that the action $\overline{\alpha}$ of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_{\max}(\mathcal{A})$.

THEOREM 4.2.5. *The action $\overline{\alpha}$ of G on $\mathcal{D}(\mathcal{A})$ extends uniquely to an action of G on $\mathcal{D}_{\max}(\mathcal{A})$ and is such that $\overline{\alpha}_g$ takes $C^*(\mathcal{A}/H)$ to $C^*(\mathcal{A}/gHg^{-1})$, for any $g\in G$.*

Proof: Since $\overline{\alpha}_g$ is a *-isomorphism between $C_c(\mathcal{A}/H)$ and $C_c(\mathcal{A}/gHg^{-1})$, it necessarily extends to a *-isomorphism between the enveloping C^*-algebras $C^*(\mathcal{A}/H)$ and $C^*(\mathcal{A}/gHg^{-1})$. Since $C^*(\mathcal{A}/gHg^{-1})$ is embedded in $D_{\max}(\mathcal{A})$, we can see $\overline{\alpha}_g$ as an injective *-homomorphism from $C^*(\mathcal{A}/H)$ into $D_{\max}(\mathcal{A})$.

A routine computation shows that the following diagram of canonical injections commutes:

$$\begin{tikzcd} C^*(\mathcal{A}/H) \arrow[r] \arrow[dr, "\overline{\alpha}_g"'] & C^*(\mathcal{A}/K) \arrow[d, "\overline{\alpha}_g"] \\ & D_{\max}(\mathcal{A})\,. \end{tikzcd}$$

Hence, we obtain an injective *-homomorphism from $\mathcal{D}_{\max}(\mathcal{A})$ to itself, which we still denote by $\overline{\alpha}_g$, and which extends the usual map $\overline{\alpha}_g$ from $D(\mathcal{A})$ to itself. It is also clear that this map is surjective, and that for $g,h\in G$ we have $\overline{\alpha}_{gh}=\overline{\alpha}_g\circ\overline{\alpha}_h$, so that we get an action of G on $D_{\max}(\mathcal{A})$ which extends the usual action of G on $D(\mathcal{A})$. $\square$

CHAPTER 5

Reduced C^*-crossed products

In this chapter we define reduced C^*-crossed products by Hecke pairs and study some of their properties. Since the algebra $C_c(\mathcal{A}/\Gamma)$ admits several possible C^*-completions, we will be able to form several reduced C^*-crossed products, such as $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. As we shall see, many of the main properties of reduced C^*-crossed products by groups hold also in the Hecke pair case.

In Section 5.4 we also compare our construction of a reduced crossed product by a Hecke pair with that of Laca, Larsen and Neshveyev in **[15]**, and show that they agree whenever they are both definable.

5.1. Regular representations

In this subsection we introduce the notion of *regular representations* in the context of crossed products by Hecke pairs. These are concrete *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ involving the regular representation of the Hecke algebra $\mathcal{H}(G,\Gamma)$ and are indispensable for defining reduced C^*-crossed products.

In the theory of crossed products by groups $A \times G$, regular representations are the integrated forms of certain covariant representations involving the regular representation of G. They are defined in the following way: one starts with a nondegenerate representation π of A on some Hilbert space $\mathscr{H}$ and constructs a new representation π_α of A on the Hilbert $\mathscr{H} \otimes \ell^2(G)$, defined in an appropriate way, such that π_α together with the regular representation of G form a covariant representation. Their integrated form is then called a *regular representation*.

We are now going to make an analogous construction in the case of Hecke pairs. The main novelty here is that we have to start with a representation π of $\mathcal{D}(\mathcal{A})$, instead of $C_c(\mathcal{A}/\Gamma)$, so that we can construct the new representation π_α of $C_c(\mathcal{A}/\Gamma)$. This is because we need to take into account all algebras of the form $C_c(\mathcal{A}/H)$, where $H = g_1\Gamma g_1^{-1} \cap \cdots \cap g_n\Gamma g_n^{-1}$ is a finite intersection of conjugates of Γ. Naturally, when Γ is a normal subgroup, $\mathcal{D}(\mathcal{A})$ is nothing but the algebra $C_c(\mathcal{A}/\Gamma)$ itself, so that we will recover the original definition of a regular representation for crossed products by groups.

Definition 5.1.1. Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. We define the map $\pi_\alpha : C_c(\mathcal{A}/\Gamma) \to B(\mathscr{H} \otimes \ell^2(G/\Gamma))$ by

$$\pi_\alpha(f)\ (\xi \otimes \delta_{h\Gamma}) := \pi(\overline{\alpha}_h(f))\xi \otimes \delta_{h\Gamma}\,.$$

PROPOSITION 5.1.2. *Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. Then, the map π_α is a nondegenerate *-representation of $C_c(\mathcal{A}/\Gamma)$.*

LEMMA 5.1.3. *Let $\pi : D(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. Then the restriction of π to $C_c(\mathcal{A}/H)$ is nondegenerate, for any $H \in \mathcal{C}$.*

Proof: Let $\xi \in \mathscr{H}$ be such that $\pi(C_c(\mathcal{A}/H))\xi = 0$. Take any $x \in X$, $a \in \mathcal{A}_x$ and $K \in \mathcal{C}$ such that $K \subseteq H$. We have that

$$\begin{aligned}
\|\pi([a]_{xK})\xi\|^2 &= \langle \pi([a^*a]_{\mathbf{s}(x)K})\xi \, , \, \xi\rangle \\
&= \langle \pi([a^*]_{x^{-1}K} \cdot [a]_{xH})\xi \, , \, \xi\rangle \\
&= \langle \pi([a^*]_{x^{-1}K})\pi([a]_{xH})\xi \, , \, \xi\rangle \\
&= 0\,.
\end{aligned}$$

From this we conclude that $\pi(C_c(\mathcal{A}/K))\xi = 0$, for any $K \in \mathcal{C}$ such that $K \subseteq H$. Since for any subgroup $L \in \mathcal{C}$ we have $C_c(\mathcal{A}/L) \subseteq C_c(\mathcal{A}/(L \cap H))$, and obviously $L \cap H \subseteq H$, we can in fact conclude that $\pi(C_c(\mathcal{A}/L))\xi = 0$ for all $L \in \mathcal{C}$. In other words, we have proven that $\pi(D(\mathcal{A}))\xi = 0$, which by nondegeneracy of π implies that $\xi = 0$. □

Proof of Proposition 5.1.2: It is clear that the expression that defines $\pi_\alpha(f)$, for $f \in C_c(\mathcal{A}/\Gamma)$, defines a linear operator on the inner product space $\mathscr{H} \otimes C_c(G/\Gamma)$. Let us first check that this operator is indeed bounded. We have

$$\begin{aligned}
\|\pi_\alpha(f)\Big(\sum_{[h]\in G/\Gamma} \xi_{h\Gamma} \otimes \delta_{h\Gamma}\Big)\|^2 &= \|\sum_{[h]\in G/\Gamma} \pi(\overline{\alpha}_h(f))\xi_{h\Gamma} \otimes \delta_{h\Gamma}\|^2 \\
&= \sum_{[h]\in G/\Gamma} \|\pi(\overline{\alpha}_h(f))\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma} \|\pi(\overline{\alpha}_h(f))\|^2 \|\xi_{h\Gamma}\|^2 \\
&\leq \sum_{[h]\in G/\Gamma} \|\overline{\alpha}_h(f)\|^2_{C^*(\mathcal{A}/h\Gamma h^{-1})} \|\xi_{h\Gamma}\|^2\,.
\end{aligned}$$

Since $\overline{\alpha}_h$ gives an isomorphism between $C^*(\mathcal{A}/\Gamma)$ and $C^*(\mathcal{A}/h\Gamma h^{-1})$ we get

$$\begin{aligned}
&= \sum_{[h]\in G/\Gamma} \|f\|^2_{C^*(\mathcal{A}/\Gamma)} \|\xi_{h\Gamma}\|^2 \\
&= \|f\|^2_{C^*(\mathcal{A}/\Gamma)} \|\sum_{[h]\in G/\Gamma} \xi_{h\Gamma} \otimes \delta_{h\Gamma}\|^2\,.
\end{aligned}$$

Hence, $\pi_\alpha(f)$ is bounded and thus defines uniquely an operator in $B(\mathscr{H} \otimes \ell^2(G/\Gamma))$. It is simple to check that π is linear and preserves products. Let us then see that it preserves the involution. We have

$$\begin{aligned}\langle\pi_\alpha(f)\,(\xi\otimes\delta_{h\Gamma})\,,\,\eta\otimes\delta_{g\Gamma}\rangle &= \langle\pi(\overline{\alpha}_h(f))\xi\otimes\delta_{h\Gamma}\,,\,\eta\otimes\delta_{g\Gamma}\rangle\\ &= \langle\pi(\overline{\alpha}_h(f))\xi\,,\,\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle\\ &= \langle\xi\,,\,\pi(\overline{\alpha}_h(f^*))\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle\\ &= \langle\xi\,,\,\pi(\overline{\alpha}_g(f^*))\eta\rangle\langle\delta_{h\Gamma}\,,\,\delta_{g\Gamma}\rangle\\ &= \langle\xi\otimes\delta_{h\Gamma}\,,\,\pi_\alpha(f^*)\,(\eta\otimes\delta_{g\Gamma})\rangle\,.\end{aligned}$$

Thus, $\pi_\alpha(f)^* = \pi_\alpha(f^*)$, and therefore π_α defines a *-representation. It remains to check that this *-representation is nondegenerate. To see this, we start by canonically identifying $\mathscr{H}\otimes\ell^2(G/\Gamma)$ with the Hilbert space $\ell^2(G/\Gamma,\mathscr{H})$. On this Hilbert space, it is easy to see that $\pi_\alpha(f)$ is given by

$$[\pi_\alpha(f)\,(\zeta)]\,(h\Gamma) = \pi(\overline{\alpha}_h(f))\,\zeta(h\Gamma)$$

for $\zeta\in\ell^2(G/\Gamma,\mathscr{H})$. Suppose now that $\zeta\in\ell^2(G/\Gamma,\mathscr{H})$ is such that $\pi_\alpha(f)\,\zeta = 0$ for all $f\in C_c(\mathcal{A}/\Gamma)$. Thus, for each $h\Gamma\in G/\Gamma$ we have $\pi(\overline{\alpha}_h(f))\,\zeta(h\Gamma) = 0$ for all $f\in C_c(\mathcal{A}/\Gamma)$. This can be expressed equivalently as $\pi(f)\,\zeta(h\Gamma) = 0$ for all $f\in C_c(\mathcal{A}/h\Gamma h^{-1})$. By Lemma 5.1.3 the restriction of π to $C_c(\mathcal{A}/h\Gamma h^{-1})$ is nondegenerate and therefore we have $\zeta(h\Gamma) = 0$. Thus, π_α is nondegenerate. □

Definition 5.1.4. Let $\pi : D(\mathcal{A})\to B(\mathscr{H})$ be a nondegenerate *-representation and $\rho : \mathcal{H}(G,\Gamma)\to B(\ell^2(G/\Gamma))$ the right regular representation of the Hecke algebra. The pair $(\pi_\alpha, 1\otimes\rho)$ is called a *regular covariant representation*.

Remark 5.1.5. We observe that when Γ is a normal subgroup of G we have $g\Gamma g^{-1} = \Gamma$ for all $g\in G$, so that the algebra $D(\mathcal{A})$ coincides with $C_c(\mathcal{A}/\Gamma)$. For this reason our notion of a regular representation coincides with the usual notion of a regular covariant representation of the system $(C_c(\mathcal{A}/\Gamma), G/\Gamma, \overline{\alpha})$.

Theorem 5.1.6. *Every regular covariant representation* $(\pi_\alpha, 1\otimes\rho)$ *is a covariant *-representation. Moreover, its integrated form is given by*

$$[\pi_\alpha\times(1\otimes\rho)](f)\;(\xi\otimes\delta_{h\Gamma}) = \sum_{[g]\in G/\Gamma}\Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\xi\otimes\delta_{g\Gamma}\,, \tag{5.1}$$

for every $f\in C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$.

Proof: We shall first check that the expression (5.1) does indeed define a *-representation of $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$. Afterwards we will show that the covariant pre-*-representation associated to it is precisely $(\pi_\alpha, 1\otimes\rho)$.

Let $\pi_{reg} : C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma\to B(\mathscr{H}\otimes\ell^2(G/\Gamma))$ be defined by

$$\pi_{reg}(f)\;(\xi\otimes\delta_{h\Gamma}) := \sum_{[g]\in G/\Gamma}\Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\xi\otimes\delta_{g\Gamma}\,.$$

It is not evident that π_{reg} is a bounded operator for all $f\in C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$, but it is clear that $\pi_{reg}(f)$ is well-defined as a linear operator on the inner product space $\mathscr{H} \otimes C_c(G/\Gamma)$. Under the identification of $\mathscr{H} \otimes C_c(G/\Gamma)$ with $C_c(G/\Gamma, \mathscr{H})$, it is easy to see that $\pi_{reg}(f)$ is given by

$$[\pi_{reg}(f)\,\eta]\,(g\Gamma) = \sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\eta(h\Gamma)\,,$$

for any $\eta \in C_c(G/\Gamma, \mathscr{H})$. Let us now check that $\pi_{reg}(f)$ is indeed bounded. For any vector $\eta \in C_c(G/\Gamma, \mathscr{H})$ we have

$$\begin{aligned}
\|\pi_{reg}(f)\,\eta\|^2 &= \sum_{[g]\in G/\Gamma} \|[\pi_{reg}(f)\,\eta](g\Gamma)\|^2 \\
&= \sum_{[g]\in G/\Gamma} \|\sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\eta(h\Gamma)\ \|^2 \\
&\leq \sum_{[g]\in G/\Gamma} \Big(\sum_{[h]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|\|\eta(h\Gamma)\|\Big)^2 .
\end{aligned}$$

For each $h\Gamma \in G/\Gamma$ let us define $T^{h\Gamma} \in C_c(G/\Gamma)$ by

$$T^{h\Gamma}(g\Gamma) := \Delta(g^{-1}h)^{\frac{1}{2}}\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|\|\eta(h\Gamma)\|\,,$$

and $T \in C_c(G/\Gamma)$ by $T := \sum_{[h]\in G/\Gamma} T^{h\Gamma}$, which is clearly a finite sum since η has finite support. Thus, we have

$$\begin{aligned}
\|\pi_{reg}(f)\,\eta\|^2 &\leq \sum_{[g]\in G/\Gamma} \Big(\sum_{[h]\in G/\Gamma} T^{h\Gamma}(g\Gamma)\Big)^2 \\
&= \sum_{[g]\in G/\Gamma} (T(g\Gamma))^2 \\
&= \|T\|^2_{\ell^2(G/\Gamma)} \\
&= \|\sum_{[h]\in G/\Gamma} T^{h\Gamma}\ \|^2_{\ell^2(G/\Gamma)} \\
&\leq \Big(\sum_{[h]\in G/\Gamma} \|T^{h\Gamma}\|_{\ell^2(G/\Gamma)}\Big)^2 \\
&= \Bigg(\sum_{[h]\in G/\Gamma} \sqrt{\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|^2\|\eta(h\Gamma)\|^2}\ \Bigg)^2 \\
&= \Bigg(\sum_{[h]\in G/\Gamma} \|\eta(h\Gamma)\| \sqrt{\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|^2}\ \Bigg)^2 .
\end{aligned}$$

By the Cauchy-Schwarz inequality in $\ell^2(G/\Gamma)$ we get

$$\begin{aligned}
&\leq \Big(\sum_{[h]\in G/\Gamma} \|\eta(h\Gamma)\|^2\Big)\Big(\sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|^2\ \Big) \\
&= \Big(\sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\,\|\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\|^2\ \Big)\,\|\eta\|^2\,,
\end{aligned}$$

which shows that $\pi_{reg}(f)$ is bounded.

Let us now check that π_{reg} preserves products and the involution. Let $f_1, f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$. We have

$$\begin{aligned}
&\pi_{reg}(f_1 * f_2)\left(\xi \otimes \delta_{h\Gamma}\right) = \\
&= \sum_{[g]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g\big((f_1 * f_2)(g^{-1}h\Gamma)\big)\big)\xi \otimes \delta_{g\Gamma} \\
&= \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g\big((f_1(s\Gamma)\overline{\alpha}_s(f_2(s^{-1}g^{-1}h\Gamma))\big)\big)\xi \otimes \delta_{g\Gamma} \\
&= \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g(f_1(s\Gamma))\overline{\alpha}_{gs}(f_2(s^{-1}g^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
&= \sum_{[g]\in G/\Gamma} \sum_{[s]\in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g(f_1(g^{-1}s\Gamma))\overline{\alpha}_{s}(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
&= \sum_{[s]\in G/\Gamma} \sum_{[g]\in G/\Gamma} \Delta(g^{-1}s)^{\frac{1}{2}}\Delta(s^{-1}h)^{\frac{1}{2}} \pi\big(\overline{\alpha}_g(f_1(g^{-1}s\Gamma))\big)\pi\big(\overline{\alpha}_{s}(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma} \\
&= \sum_{[s]\in G/\Gamma} \pi_{reg}(f_1)\left(\Delta(s^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_{s}(f_2(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma}\right) \\
&= \pi_{reg}(f_1)\pi_{reg}(f_2)\left(\xi \otimes \delta_{h\Gamma}\right).
\end{aligned}$$

Hence we conclude that $\pi_{reg}(f_1 * f_2) = \pi_{reg}(f_1)\pi_{reg}(f_2)$. Let us now check that π_{reg} preserves the involution. For $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ we have

$$\begin{aligned}
&\left\langle \pi_{reg}(f^*)\left(\xi \otimes \delta_{h\Gamma}\right), \eta \otimes \delta_{s\Gamma}\right\rangle = \\
= \quad & \sum_{[g]\in G/\Gamma} \left\langle \Delta(g^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g(f^*(g^{-1}h\Gamma))\big)\xi \otimes \delta_{g\Gamma}, \eta \otimes \delta_{s\Gamma}\right\rangle \\
= \quad & \sum_{[g]\in G/\Gamma} \left\langle \Delta(g^{-1}h)^{\frac{1}{2}}\Delta(h^{-1}g)\, \pi\big(\overline{\alpha}_g(\overline{\alpha}_{g^{-1}h}(f(h^{-1}g\Gamma))^*)\big)\xi \otimes \delta_{g\Gamma}, \eta \otimes \delta_{s\Gamma}\right\rangle \\
= \quad & \sum_{[g]\in G/\Gamma} \left\langle \Delta(h^{-1}g)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_h(f(h^{-1}g\Gamma))\big)^*\xi, \eta\right\rangle \left\langle \delta_{g\Gamma}, \delta_{s\Gamma}\right\rangle \\
= \quad & \left\langle \xi, \Delta(h^{-1}s)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_h(f(h^{-1}s\Gamma))\big)\eta\right\rangle.
\end{aligned}$$

On the other side we also have

$$\begin{aligned}
&\left\langle \xi \otimes \delta_{h\Gamma}, \pi_{reg}(f)\left(\eta \otimes \delta_{s\Gamma}\right)\right\rangle = \\
= \quad & \sum_{[g]\in G/\Gamma} \left\langle \xi \otimes \delta_{h\Gamma}, \Delta(g^{-1}s)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g(f(g^{-1}s\Gamma))\big)\eta \otimes \delta_{g\Gamma}\right\rangle \\
= \quad & \sum_{[g]\in G/\Gamma} \left\langle \xi, \Delta(g^{-1}s)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_g(f(g^{-1}s\Gamma))\big)\eta\right\rangle \left\langle \delta_{h\Gamma}, \delta_{g\Gamma}\right\rangle \\
= \quad & \left\langle \xi, \Delta(h^{-1}s)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_h(f(h^{-1}s\Gamma))\big)\eta\right\rangle.
\end{aligned}$$

Therefore we can conclude that $\pi_{reg}(f^*) = \pi_{reg}(f)^*$. Hence, π_{reg} is a *-representation.

The restriction of π_{reg} to $C_c(\mathcal{A}/\Gamma)$ is precisely π_α, and since π_α is nondegenerate, then so is π_{reg}. Hence, it follows from Theorem 3.3.17 that π_{reg} is the integrated form of a covariant pre-*-representation $(\pi_{reg}|, \omega_{\pi_{reg}})$, as defined in Proposition 3.3.13. As we pointed out above, $\pi_{reg}| = \pi_\alpha$. Thus, to finish the proof we only need to prove that $\omega_{\pi_{reg}} = 1 \otimes \rho$. For a vector of the form $\pi_\alpha([a]_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \in \pi_\alpha(C_c(\mathcal{A}/\Gamma))(\mathscr{H} \otimes \ell^2(G/\Gamma))$ and a double coset $\Gamma g \Gamma$ we have

$$\begin{aligned} \omega_{\pi_{reg}}(\Gamma g \Gamma)\, \pi_\alpha([a]_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) &= \widetilde{\pi_{reg}}(\Gamma g \Gamma)\, \pi_\alpha([a]_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \\ &= \widetilde{\pi_{reg}}(\Gamma g \Gamma) \pi_{reg}([a]_{x\Gamma})(\xi \otimes \delta_{h\Gamma}) \\ &= \pi_{reg}(\Gamma g \Gamma * [a]_{x\Gamma})(\xi \otimes \delta_{h\Gamma})\,. \end{aligned}$$

Let us now compute $\pi_{reg}(f)(\xi \otimes \delta_{h\Gamma})$ for $f := \Gamma g \Gamma * [a]_{x\Gamma}$. By definition

$$\pi_{reg}(f)\ (\xi \otimes \delta_{h\Gamma}) = \sum_{[s] \in G/\Gamma} \Delta(s^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_s(f(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma}\,.$$

It is clear that $f(s^{-1}h\Gamma)$ is nonzero if and only if $s^{-1}h\Gamma \subseteq \Gamma g \Gamma$, which is equivalent to $s\Gamma \subseteq h\Gamma g^{-1}\Gamma$. Hence,

$$= \sum_{[s] \in\ h\Gamma g^{-1}\Gamma/\Gamma} \Delta(s^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_s(f(s^{-1}h\Gamma))\big)\xi \otimes \delta_{s\Gamma}\,.$$

It is easy to see that $[\theta] \mapsto [h\theta g^{-1}]$ establishes a well-defined bijection between $\Gamma/\Gamma^{g^{-1}}$ and $h\Gamma g^{-1}\Gamma/\Gamma$, so that

$$\begin{aligned} &= \sum_{[\theta] \in \Gamma/\Gamma^{g^{-1}}} \Delta(g\theta^{-1}h^{-1}h)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_{h\theta g^{-1}}(f(g\theta^{-1}h^{-1}h\Gamma))\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\ &= \sum_{[\theta] \in \Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_{h\theta g^{-1}}(f(g\Gamma))\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma}\,. \end{aligned}$$

Now, it is easily seen that $f(g\Gamma) = [\alpha_g(a)]_{xg^{-1}g\Gamma g^{-1}}$. Hence, we get

$$\begin{aligned} &= \sum_{[\theta] \in \Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_{h\theta g^{-1}}([\alpha_g(a)]_{xg^{-1}g\Gamma g^{-1}})\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\ &= \sum_{[\theta] \in \Gamma/\Gamma^{g^{-1}}} \Delta(g)^{\frac{1}{2}}\, \pi\big(\overline{\alpha}_h([a]_{x\Gamma})\big)\xi \otimes \delta_{h\theta g^{-1}\Gamma} \\ &= (1 \otimes \rho)(\Gamma g \Gamma)\, \Big(\pi\big(\overline{\alpha}_h([a]_{x\Gamma})\big)\xi \otimes \delta_{h\Gamma}\Big) \\ &= (1 \otimes \rho)(\Gamma g \Gamma)\, \pi_\alpha([a]_{x\Gamma})\, \big(\xi \otimes \delta_{h\Gamma}\big)\,. \end{aligned}$$

This shows that $\omega_{\pi_{reg}} = 1 \otimes \rho$ in $\pi_\alpha(C_c(\mathcal{A}/\Gamma))(\mathscr{H} \otimes \ell^2(G/\Gamma))$ and finishes the proof. ∎

Remark 5.1.7. The proof of Theorem 5.1.6 may seem odd, since we did not first prove that the pair $(\pi_\alpha, 1 \otimes \rho)$ is a covariant *-representation and then deduce that its integrated form $\pi_\alpha \times (1 \otimes \rho)$ is a *-representation and is given by (5.1). Instead we followed the opposite approach. This is because we do not know how to prove directly that $(\pi_\alpha, 1 \otimes \rho)$ is a covariant *-representation, due to the several difficult technicalities that arise in the computations.

5.2. Reduced C^*-crossed products

We now want to define reduced C^*-norms in the *-algebraic crossed product $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. Since $C_c(\mathcal{A}/\Gamma)$ admits several canonical C^*-completions one should expect that there are several reduced C^*-norms we can give to $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg}$

G/Γ, which give rise to different reduced C^*-crossed products, as for example $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. We will treat in this section all these different reduced C^*-norms (and reduced C^*-crossed products) in a single approach, and for that the notion we need is that of a $\overline{\alpha}$*-permissible* C^*-norm on $\mathcal{D}(\mathcal{A})$:

Definition 5.2.1. A C^*-norm $\|\cdot\|_\tau$ in $\mathcal{D}(\mathcal{A})$ is said to be $\overline{\alpha}$*-permissible* if the action $\overline{\alpha}$ of G on $\mathcal{D}(\mathcal{A})$ extends to $\mathcal{D}_\tau(\mathcal{A})$, the completion of $\mathcal{D}(\mathcal{A})$ with respect to the norm $\|\cdot\|_\tau$. In other words, if for every $g \in G$ the automorphism $\overline{\alpha}_g$ of $\mathcal{D}(\mathcal{A})$ is continuous with respect to $\|\cdot\|_\tau$.

Definition 5.2.2. Let $\|\cdot\|_\tau$ be an $\overline{\alpha}$-permissible C^*-norm in $\mathcal{D}(\mathcal{A})$ and let us denote by $\mathcal{D}_\tau(\mathcal{A})$ and $C^*_\tau(\mathcal{A}/\Gamma)$ the completions of $\mathcal{D}(\mathcal{A})$ and $C_c(\mathcal{A}/\Gamma)$, respectively, with respect to the norm $\|\cdot\|_\tau$. We define the norm $\|\cdot\|_{\tau,r}$ in $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ by

$$\|f\|_{\tau,r} := \sup_{\pi \in R(\mathcal{D}_\tau(\mathcal{A}))} \|[\pi_\alpha \times (1 \otimes \rho)](f)\|\,,$$

where the supremum is taken over the class $R(\mathcal{D}_\tau(\mathcal{A}))$ of all nondegenerate *-representations of $\mathcal{D}_\tau(\mathcal{A})$. The completion of $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ with respect to this norm shall be denoted by $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and referred to as the *reduced crossed product* of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G, Γ).

Before we prove that $\|\cdot\|_\tau$ is indeed a C^*-norm, let us first look at the two main instances we have in mind, which arise when $C^*_\tau(\mathcal{A}/\Gamma)$ is $C^*_r(\mathcal{A}/\Gamma)$ or $C^*(\mathcal{A}/\Gamma)$. It is not obvious from the start that there exists a C^*-norm $\|\cdot\|_\tau$ in $\mathcal{D}(\mathcal{A})$ whose restriction to $C_c(\mathcal{A}/\Gamma)$ will give the reduced or the maximal C^*-norm in $C_c(\mathcal{A}/\Gamma)$, but this is indeed the case from what we proved in the preliminary sections 4.1 and 4.2:

- *For* $C^*_r(\mathcal{A}/\Gamma)$*:*
  As described in Section 4.1, we can form the C^*-algebraic direct limit $\mathcal{D}_r(\mathcal{A}) = \lim_{H \in \mathcal{C}} C^*_r(\mathcal{A}/H)$, which contains $\mathcal{D}(\mathcal{A})$ as a dense *-subalgebra. Taking $\|\cdot\|_\tau$ to be the C^*-norm $\|\cdot\|_r$ of $\mathcal{D}_r(\mathcal{A})$, we see that $C^*_\tau(\mathcal{A}/\Gamma) = C^*_r(\mathcal{A}/\Gamma)$. The norm $\|\cdot\|_r$ is $\overline{\alpha}$-permissible because of Theorem 4.1.5.
- *For* $C^*(\mathcal{A}/\Gamma)$*:*
  As described in Section 4.2, we can form the C^*-algebraic direct limit $\mathcal{D}_{\max}(\mathcal{A}) = \lim_{H \in \mathcal{C}} C^*(\mathcal{A}/H)$, which contains $\mathcal{D}(\mathcal{A})$ as a dense *-subalgebra. Taking $\|\cdot\|_\tau$ to be the C^*-norm $\|\cdot\|_{\max}$ of $\mathcal{D}_{\max}(\mathcal{A})$, we see that $C^*_\tau(\mathcal{A}/\Gamma) = C^*(\mathcal{A}/\Gamma)$. The norm $\|\cdot\|_{\max}$ is $\overline{\alpha}$-permissible because of Theorem 4.2.5.

Lemma 5.2.3. *If* $\pi : \mathcal{D}(\mathcal{A}) \to B(\mathscr{H})$ *is a nondegenerate* **-representation which is continuous with respect to an* $\overline{\alpha}$*-permissible norm* $\|\cdot\|_\tau$ *in* $\mathcal{D}(\mathcal{A})$*, then* π_α *is a representation of* $C_c(\mathcal{A}/\Gamma)$ *which is continuous with respect to the norm* $\|\cdot\|_\tau$ *as well.*

Proof: Let $f \in C_c(\mathcal{A}/\Gamma)$. We have

$$\begin{aligned}\|\pi_\alpha(f)\big(\sum_{[h]\in G/\Gamma}\xi_{h\Gamma}\otimes\delta_{h\Gamma}\big)\|^2 &= \|\sum_{[h]\in G/\Gamma}\pi(\overline{\alpha}_h(f))\xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2\\ &= \sum_{[h]\in G/\Gamma}\|\pi(\overline{\alpha}_h(f))\xi_{h\Gamma}\|^2\\ &\leq \sum_{[h]\in G/\Gamma}\|\pi(\overline{\alpha}_h(f))\|^2\|\xi_{h\Gamma}\|^2\\ &\leq \sum_{[h]\in G/\Gamma}\|\overline{\alpha}_h(f)\|_\tau^2\,\|\xi_{h\Gamma}\|^2\,.\end{aligned}$$

Since $\|\cdot\|_\tau$ is $\overline{\alpha}$-permissible we have that $\|\overline{\alpha}_h(f)\|_\tau = \|f\|_\tau$. Hence we have

$$\begin{aligned}\|\pi_\alpha(f)\big(\sum_{[h]\in G/\Gamma}\xi_{h\Gamma}\otimes\delta_{h\Gamma}\big)\|^2 &\leq \sum_{[h]\in G/\Gamma}\|f\|_\tau^2\|\xi_{h\Gamma}\|^2\\ &= \|f\|_\tau^2\,\|\sum_{[h]\in G/\Gamma}\xi_{h\Gamma}\otimes\delta_{h\Gamma}\|^2\,.\end{aligned}$$

Hence, π_α is continuous with respect to the norm $\|\cdot\|_\tau$. □

PROPOSITION 5.2.4. *$\|\cdot\|_{\tau,r}$ is a well-defined C^*-norm on $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$.*

Proof: First we must show that the supremum in the definition of $\|\cdot\|_{\tau,r}$ is bounded. Given a *-representation π of $\mathcal{D}_\tau$ we have, by Lemma 5.2.3, that

$$\begin{aligned}&\|[\pi_\alpha\times(1\otimes\rho)](f)\| \leq\\ \leq\; &\sum_{[g]\in\Gamma\backslash G/\Gamma}\;\sum_{x\Gamma^g\in X/\Gamma^g}\|\pi_\alpha\big(\big[f(g\Gamma)(x)\big]_{x\Gamma}\big)\|\|(1\otimes\rho)\,(\Gamma g\Gamma)\|\|\widetilde{\pi_\alpha}(1_{xg\Gamma})\|\\ \leq\; &\sum_{[g]\in\Gamma\backslash G/\Gamma}\;\sum_{x\Gamma^g\in X/\Gamma^g}\|\big[f(g\Gamma)(x)\big]_{x\Gamma}\|_\tau\;\|\Gamma g\Gamma\|_{C_r^*(G,\Gamma)}\,.\end{aligned}$$

Thus, since $\|[\pi_\alpha\times(1\otimes\rho)](f)\|$ is finite and bounded by a number that does not depend on π, we conclude that $\|f\|_{\tau,r}$ is bounded by this same number.

It is clear from the definition and the above paragraph that $\|\cdot\|_{\tau,r}$ is C^*-seminorm. To prove that it is actually a C^*-norm it is enough to prove that if π is a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$, then $\pi_\alpha\times(1\otimes\rho)$ is a faithful *-representation of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. Let us then prove this claim. Suppose $f\in C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ is such that $[\pi_\alpha\times(1\otimes\rho)](f)=0$. Then, for every $\xi\otimes\delta_{h\Gamma}\in\mathscr{H}\otimes\ell^2(G/\Gamma)$ we have

$$0 \;=\; [\pi_\alpha\times(1\otimes\rho)](f)(\xi\otimes\delta_{h\Gamma}) \;=\; \sum_{[g]\in G/\Gamma}\Delta(g^{-1}h)^{\frac{1}{2}}\,\pi\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\xi\otimes\delta_{g\Gamma}\,.$$

In particular, for $g\Gamma=\Gamma$, we have $\pi(f(h\Gamma))\xi=0$, and since this holds for every $\xi\in\mathscr{H}$ we have $\pi(f(h\Gamma))=0$. Now, since π is a faithful *-representation, it follows that $f(h\Gamma)=0$. Since this holds for every $h\Gamma\in G/\Gamma$, we have $f=0$, i.e. $\pi_\alpha\times(1\otimes\rho)$ is injective. □

The next result explains why we call the completion of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ under the norm $\|\cdot\|_{\tau,r}$ the reduced crossed product of $C_\tau^*(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ)

and justifies also the notation $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ chosen to denote this completion.

PROPOSITION 5.2.5. *The restriction of the norm* $\|\cdot\|_{\tau,r}$ *of* $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *to* $C_c(\mathcal{A}/\Gamma)$ *is precisely the norm* $\|\cdot\|_\tau$. *Hence, the embedding* $C_c(\mathcal{A}/\Gamma) \to C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ *completes to an embedding* $C^*_\tau(\mathcal{A}/\Gamma) \to C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

Proof: Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. From Lemma 5.2.3 we have, for every $f \in C_c(\mathcal{A}/\Gamma)$,

$$\|[\pi_\alpha \times (1 \otimes \rho)](f)\| = \|\pi_\alpha(f)\| \leq \|f\|_\tau \,,$$

and therefore

$$\|f\|_{\tau,r} \leq \|f\|_\tau \,.$$

We now wish to prove the converse inequality. Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a faithful nondegenerate *-representation. For any $f \in C_c(\mathcal{A}/\Gamma)$ we have

$$\begin{aligned} \|f\|_\tau &= \|\pi(f)\| = \sup_{\|\xi\|=1} \|\pi(f)\xi\| \\ &= \sup_{\|\xi\|=1} \|\pi_\alpha(f)(\xi \otimes \delta_\Gamma)\| \\ &\leq \sup_{\|\zeta\|=1} \|\pi_\alpha(f)\zeta\| = \|\pi_\alpha(f)\| \\ &= \|[\pi_\alpha \times (1 \otimes \rho)](f)\| \leq \|f\|_{\tau,r} \,, \end{aligned}$$

thus proving the converse inequality. We conclude that

$$\|f\|_{\tau,r} = \|f\|_\tau \,,$$

for any $f \in C_c(\mathcal{A}/\Gamma)$ and this finishes the proof. □

An important feature of reduced crossed products by groups $A \times_r G$ is the existence of a faithful conditional expectation onto A. We will now explain how this holds as well for reduced crossed products by Hecke pairs, with somewhat analogous proofs. The goal is to prove Theorem 5.2.7 bellow, and for that we follow closely the approach presented in **[20]** in the case of groups.

PROPOSITION 5.2.6. *For every* $g\Gamma \in G/\Gamma$ *the map* $E_{g\Gamma}$ *defined by*

$$\begin{aligned} E_{g\Gamma} : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma &\longrightarrow C^*_\tau(\mathcal{A}/\Gamma^g) \\ E_{g\Gamma}(f) &:= f(g\Gamma) \,. \end{aligned}$$

is linear and continuous with respect to the norm $\|\cdot\|_{\tau,r}$.

Before we give a proof of the result above we need to set some notation. For each element $g\Gamma \in G/\Gamma$ we will denote by $\sigma_{g\Gamma}$ the Hilbert space isometry $\sigma_{g\Gamma} : \mathscr{H} \to \mathscr{H} \otimes \ell^2(G/\Gamma)$ defined by

$$\sigma_{g\Gamma}(\xi) := \xi \otimes \delta_{g\Gamma} \,. \tag{5.2}$$

Proof of Proposition 5.2.6: Let π be a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. It is easily seen that $\sigma_\Gamma^*\,[\pi \times (1 \otimes \rho)](f)\,\sigma_{g\Gamma} = \Delta(g)^{\frac{1}{2}}\,\pi(f(g\Gamma))$. Hence we have

$$\begin{aligned} \|E_{g\Gamma}(f)\|_\tau &= \|f(g\Gamma)\|_\tau = \|\pi(f(g\Gamma))\| \\ &= \|\Delta(g^{-1})^{\frac{1}{2}}\,\sigma_\Gamma^*\,[\pi_\alpha \times (1 \otimes \rho)](f)\,\sigma_{g\Gamma}\| \\ &\leq \Delta(g^{-1})^{\frac{1}{2}}\,\|[\pi_\alpha \times (1 \otimes \rho)](f)\| \\ &\leq \Delta(g^{-1})^{\frac{1}{2}}\,\|f\|_{\tau,r}\,. \end{aligned}$$

This finishes the proof. $\square$

We shall henceforward make no distinction of notation between the maps $E_{g\Gamma}$ defined on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ and their extension to $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

The following result is of particular importance in theory of reduced C^*-crossed products. Analogously to the case of groups, it reveals two important features of reduced C^*-crossed products by Hecke pairs: the fact that every element of a reduced crossed product is uniquely described in terms of its coefficients (determined by the $E_{g\Gamma}$ maps); and the fact that E_Γ is a faithful conditional expectation.

THEOREM 5.2.7. *We have*

i) *If $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ and $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$, then $f = 0$.*
ii) *E_Γ is a faithful conditional expectation of $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ onto $C_\tau^*(\mathcal{A}/\Gamma)$.*

We start with the following auxiliary result:

LEMMA 5.2.8. *Let π be a nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. For all $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ we have*

$$\sigma_{g\Gamma}^*\,[\pi \times (1 \otimes \rho)](f)\,\sigma_{h\Gamma} = \Delta(g^{-1}h)^{\frac{1}{2}}\,\pi(\overline{\alpha}_g(E_{g^{-1}h\Gamma}(f)))\,. \tag{5.3}$$

Proof: We notice that equality (5.3) above holds for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, following the definitions of the maps $E_{t\Gamma}$, $[\pi_\alpha \times (1 \otimes \rho)](f)$ and $\sigma_{t\Gamma}$, with $t\Gamma \in G/\Gamma$. By continuity, it follows readily that the equality must hold for every $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. $\square$

Proof of Theorem 5.2.7: *i*) Let $f \in C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Suppose $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$. Then, for any given nondegenerate *-representation π of $\mathcal{D}_\tau(\mathcal{A})$ we have, by Lemma 5.2.8, that $\sigma_{g\Gamma}^*\,[\pi_\alpha \times (1 \otimes \rho)](f)\,\sigma_{h\Gamma} = 0$ for all $g\Gamma, h\Gamma \in G/\Gamma$. Hence, $[\pi_\alpha \times (1 \otimes \rho)](f) = 0$. Since, this is true for any π, we must have $\|f\|_{\tau,r} = 0$, i.e. $f = 0$.

ii) Let us first prove that E_Γ is a conditional expectation, i.e. E_Γ is an idempotent, positive, $C_\tau^*(\mathcal{A}/\Gamma)$-linear map.

If $f \in C_c(\mathcal{A}/\Gamma)$ then it is clear that $E_\Gamma(f) = f$. By continuity and Proposition 5.2.5 it follows that $E_\Gamma(f) = f$ for all $f \in C_\tau^*(\mathcal{A}/\Gamma)$. Thus, E_Γ is idempotent.

Suppose now that $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have

$$\begin{aligned} E_\Gamma(f^* * f) &= (f^* * f)(\Gamma) = \sum_{[h]\in G/\Gamma} f^*(h\Gamma)\overline{\alpha}_h(f(h^{-1}\Gamma)) \\ &= \sum_{[h]\in G/\Gamma} \Delta(h^{-1})\,\overline{\alpha}_h(f(h^{-1}\Gamma))^*\overline{\alpha}_h(f(h^{-1}\Gamma)) \geq 0 \end{aligned}$$

By continuity it follows that $E_\Gamma(f^* * f) \geq 0$ for all $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, i.e. E_Γ is positive. It remains to show that E_Γ is $C^*_\tau(\mathcal{A}/\Gamma)$-linear. We recall that we see $C_c(\mathcal{A}/\Gamma)$ as a *-subalgebra of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ in the following way: an element $f \in C_c(\mathcal{A}/\Gamma)$ is identified with the element $F \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with support in Γ and such that $F(\Gamma) = f$. For any $f \in C_c(\mathcal{A}/\Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have

$$\begin{aligned} E_\Gamma(f * f_2) &= (F * f_2)(\Gamma) = \sum_{[h]\in G/\Gamma} F(h\Gamma)\overline{\alpha}_h(f_2(h^{-1}\Gamma)) \\ &= F(\Gamma)f_2(\Gamma) = f\,E_\Gamma(f_2)\,, \end{aligned}$$

and similarly we get $E_\Gamma(f_2 * f) = E_\Gamma(f_2)\,f$. Once again by continuity we conclude that the same equalities hold for $f \in C^*_\tau(\mathcal{A}/\Gamma)$ and $f_2 \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Thus, E_Γ is a conditional expectation.

Let us now prove that E_Γ is faithful. For any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have (where the first equality was computed above):

$$\begin{aligned} E_\Gamma(f^* * f) &= \sum_{[h]\in G/\Gamma} \Delta(h^{-1})\,\overline{\alpha}_h(f(h^{-1}\Gamma))^*\overline{\alpha}_h(f(h^{-1}\Gamma)) \\ &= \sum_{[h]\in G/\Gamma} \Delta(h^{-1})\,\overline{\alpha}_h(E_{h^{-1}\Gamma}(f))^*\overline{\alpha}_h(E_{h^{-1}\Gamma}(f))\,. \end{aligned}$$

Hence, we have $E_\Gamma(f^* * f) \geq \Delta(h^{-1})\,\overline{\alpha}_h(E_{h^{-1}\Gamma}(f))^*\overline{\alpha}_h(E_{h^{-1}\Gamma}(f))$ for each $h\Gamma \in G/\Gamma$,. By continuity this inequality holds for every $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, and therefore if $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ is such that $E_\Gamma(f^* * f) = 0$, then $E_{g\Gamma}(f) = 0$ for all $g\Gamma \in G/\Gamma$. Hence, by part i), we conclude that $f = 0$. Thus, E_Γ is faithful. $\square$

The next result shows, like in crossed products by groups, that to define the norm $\|\cdot\|_{\tau,r}$ of the reduced crossed product $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ we only need to start with a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$, instead of taking the supremum over all nondegenerate *-representations of $\mathcal{D}_\tau(\mathcal{A})$.

THEOREM 5.2.9. *Let $\pi : \mathcal{D}_\tau(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation. We have that*

i) *If $\pi_\alpha : C^*_\tau(\mathcal{A}/\Gamma) \to B(\mathscr{H} \otimes \ell^2(G/\Gamma))$ is faithful, then $[\pi_\alpha \times (1 \otimes \rho)]$ is a faithful *-representation of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Consequently,*

$$\|f\|_{\tau,r} = \|[\pi_\alpha \times (1 \otimes \rho)](f)\|\,,$$

for all $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

ii) *If π is faithful, then π_α is faithful.*

Proof: Let us prove *i*) first. Suppose π_α is faithful as a *-representation of $C^*_\tau(\mathcal{A}/\Gamma)$. Let $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ be such that $[\pi \times (1 \otimes \rho)](f) = 0$. Then, of course, $[\pi_\alpha \times (1 \otimes \rho)](f^* * f) = 0$ and we have

$$\begin{aligned} 0 &= \sigma^*_{g\Gamma}\,[\pi_\alpha \times (1 \otimes \rho)](f^* * f)\,\sigma_{g\Gamma} \;=\; \pi(\overline{\alpha}_g(E_\Gamma(f^* * f))) \\ &= \sigma^*_{g\Gamma}\,\pi_\alpha(E_\Gamma(f^* * f))\,\sigma_{g\Gamma}\,. \end{aligned}$$

This implies that $\pi_\alpha(E_\Gamma(f^* * f)) = 0$, i.e. $E_\Gamma(f^* * f) = 0$, and since E_Γ is a faithful conditional expectation we have $f^* * f = 0$, i.e. $f = 0$. Thus, $\pi_\alpha \times (1 \otimes \rho)$ is faithful.

Let us now prove claim *ii*). We know that π_α, as a *-representation of $C_c(\mathcal{A}/\Gamma)$, is given by

$$\pi_\alpha(f)\,(\xi \otimes \delta_{g\Gamma}) = \pi(\overline{\alpha}_g(f))\xi \otimes \delta_{g\Gamma}\,,$$

By continuity the same expression holds for $f \in C^*_\tau(\mathcal{A}/\Gamma)$. Now suppose that $\pi_\alpha(f) = 0$ for some $f \in C^*_\tau(\mathcal{A}/\Gamma)$. Then, by the above expression, we have $\pi(f) = 0$. Since π is faithful we must have $f = 0$. Thus, π_α is faithful. $\square$

Another feature of reduced C^*-crossed products by groups $A \times_r G$ is the fact that the reduced C^*-algebra of the group is always canonically embedded in the multiplier algebra $M(A \times_r G)$. The same is true in the Hecke pair case as we now show:

Proposition 5.2.10. *There is a unique embedding of the reduced Hecke C^*-algebra $C^*_r(G,\Gamma)$ into $M(C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$ extending the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$.*

Proof: Let us first see that the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ is continuous with respect to the norm $\|\cdot\|_{\tau,r}$, so that it extends uniquely to an action of $\mathcal{H}(G,\Gamma)$ on $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$.

Let π be a faithful nondegenerate *-representation of $\mathcal{D}_\tau(\mathcal{A})$. From Theorem 5.2.9 we know that $\pi_\alpha \times (1 \otimes \rho)$ is also faithful. For $f_1 \in \mathcal{H}(G,\Gamma)$ and $f_2 \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$, we have

$$\begin{aligned} \|f_1 * f_2\|_{\tau,r} &= \|[\pi_\alpha \times (1 \otimes \rho)](f_1 * f_2)\| \\ &\leq \|(1 \otimes \rho)(f_1)\|\|[\pi_\alpha \times (1 \otimes \rho)](f_2)\| \\ &= \|\rho(f_1)\|\|f_2\|_{\tau,r}\,. \end{aligned}$$

Thus, the action of $\mathcal{H}(G,\Gamma)$ on $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ extends uniquely to an action on $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, or in other words, we have an embedding of $\mathcal{H}(G,\Gamma)$ into $M(C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$. We now want to prove that this embedding extends to an embedding of $C^*_r(G,\Gamma)$ into the same multiplier algebra. For that it is enough to prove that, for any $f \in \mathcal{H}(G,\Gamma)$, we have

$$\|f\|_{M(C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)} = \|f\|_{C^*_r(G,\Gamma)}\,.$$

Let $\pi_\alpha \widetilde{\times (1 \otimes \rho)}$ denote the extension of $\pi_\alpha \times (1 \otimes \rho)$ to $M(C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)$, which is faithful since $\pi_\alpha \times (1 \otimes \rho)$ is faithful on $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. We have that $\pi_\alpha \widetilde{\times (1 \otimes \rho)}$ and $(1 \otimes \rho)$ coincide in $\mathcal{H}(G,\Gamma)$ since they are given by the same expression on the dense subspace $[\pi_\alpha \times (1 \otimes \rho)](C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma)\mathscr{H}$. Thus, we have

$$[\pi_\alpha \widetilde{\times (1 \otimes} \rho)](f) = (1 \otimes \rho)(f)\,,$$

for any $f \in \mathcal{H}(G, \Gamma)$. It then follows that

$$\begin{aligned} \|f\|_{M(C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma)} &= \|[\pi_\alpha \widetilde{\times (1 \otimes} \rho)](f)\| = \|(1 \otimes \rho)(f)\| \\ &= \|\rho(f)\| = \|f\|_{C_\tau^*(G,\Gamma)}\,. \end{aligned}$$

This finishes the proof. ☐

As it is known, reduced C^*-crossed products by discrete groups satisfy a universal property among all the C^*-completions of the *-algebraic crossed product that have a certain conditional expectation. This universal property says that every such completion has a canonical surjective map onto the reduced C^*-crossed product. As a consequence, the reduced C^*-crossed product is the only C^*-completion of the *-algebraic crossed product that has a certain faithful conditional expectation.

The next result explains how this holds in the Hecke pair case.

THEOREM 5.2.11. *Let $\|\cdot\|_\tau$ be an $\overline{\alpha}$-permissible C^*-norm on $D(\mathcal{A})$ and $\|\cdot\|_\omega$ a C^*-norm on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ whose restriction to $C_c(\mathcal{A}/\Gamma)$ is just the norm $\|\cdot\|_\tau$. Let us denote by $C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ the completion of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ under the norm $\|\cdot\|_\omega$.*

If there exists a bounded linear map $F : C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma \to C_\tau^*(\mathcal{A}/\Gamma)$ such that

$$F(f) = f(\Gamma)\,,$$

for all $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, then:

a) *There exists a surjective *-homomorphism*

$$\Lambda : C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma \to C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma\,,$$

such that Λ is the identity on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.

b) *F is a conditional expectation.*

c) *F is faithful if and only if Λ is an isomorphism.*

Proof: Let X_0 be the space $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. It is easily seen that X_0 is a (right) inner product $C_c(\mathcal{A}/\Gamma)$-module, where $C_c(\mathcal{A}/\Gamma)$ acts on X_0 by right multiplication and the inner product is given by

$$\langle f_1\,,\, f_2 \rangle := (f_1^* * f_2)(\Gamma)\,.$$

Since for any $f \in X_0$ and $f_1 \in C_c(\mathcal{A}/\Gamma)$ we have

$$\begin{aligned} \|\langle f * f_1\,,\, f * f_1 \rangle\|_\tau &= \|((f * f_1)^* * (f * f_1))(\Gamma)\|_\tau \\ &= \|(f_1^* * f^* * f * f_1)(\Gamma)\|_\tau \\ &= \|f_1^*((f^* * f)(\Gamma))f_1\|_\tau \\ &= \|f_1\|_\tau^2 \|\langle f\,,\, f \rangle\|_\tau\,, \end{aligned}$$

it follows that we can complete X_0 to a (right) Hilbert $C_\tau^*(\mathcal{A}/\Gamma)$-module, which we will denote by X. The inner product on X, which extends the inner product $\langle \cdot\,,\, \cdot \rangle$ above, will be denoted by $\langle \cdot\,,\, \cdot \rangle_{C_\tau^*(\mathcal{A}/\Gamma)}$.

The *-algebra $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ acts on X_0 by left multiplication and therefore it is easily seen that this action is compatible with the right module structure. Moreover, $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ acts on X_0 by bounded operators, relatively to the norm induced by the inner product $\langle \cdot \,,\, \cdot \rangle_{C^*_\tau(\mathcal{A}/\Gamma)}$, as we now show. For this we recall the conditional expectation E_Γ of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ onto $C^*_\tau(\mathcal{A}/\Gamma)$ as defined in Proposition 5.2.6. For any $f, f_1 \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have that inside $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ the following holds:

$$\begin{aligned}
\langle f * f_1 \,,\, f * f_1 \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} &= ((f * f_1)^* * (f * f_1))(\Gamma) \\
&= E_\Gamma((f * f_1)^* * (f * f_1)) \\
&= E_\Gamma(f_1^* * f^* * f * f_1) \\
&\leq \|f\|^2_{\tau,r} E_\Gamma(f_1^* * f_1) \\
&= \|f\|^2_{\tau,r} \langle f_1 \,,\, f_1 \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} \,,
\end{aligned}$$

where we used the positivity of E_Γ in $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Since the norm $\| \cdot \|_\tau$ is just the restriction of the norm $\| \cdot \|_{\tau,r}$ we get

$$\|\langle f * f_1 \,,\, f * f_1 \rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau \leq \|f\|^2_{\tau,r} \|\langle f_1 \,,\, f_1 \rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau \,, \tag{5.4}$$

which shows that $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ acts on X_0 by bounded operators. Moreover, inequality (5.4) shows that this action extends to an action of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ on X and thus gives rise to a *-homomorphism $\Phi : C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma \to \mathcal{L}(X)$. We will now show that Φ is injective. Firstly, we will prove that Φ is injective on $C^*_\tau(\mathcal{A}/\Gamma)$, which is the same as to show that

$$\|\Phi(f)\|_{\mathcal{L}(X)} = \|f\|_\tau \,, \tag{5.5}$$

for all $f \in C_c(\mathcal{A}/\Gamma)$. It is clear from inequality (5.4) that $\|\Phi(f)\|_{\mathcal{L}(X)} \leq \|f\|_\tau$. The converse inequality follows from the fact that, for any $f, f_1 \in C_c(\mathcal{A}/\Gamma)$, we have

$$\|\langle f * f_1 \,,\, f * f_1 \rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau = \|f \cdot f_1\|_\tau^{\frac{1}{2}} \,.$$

Before we prove that Φ is injective in the whole of $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ we need to establish some notation and results.

As usual, $Y := C^*_\tau(\mathcal{A}/\Gamma)$ is a Hilbert module over itself. We define the map $j_\Gamma : Y \to X$ simply by inclusion, i.e. $j_\Gamma(f) := f$. It is then easy to see that j_Γ is adjointable with adjoint $j_\Gamma^* : X \to Y$ given by $j_\Gamma(f) = f(\Gamma)$, for any $f \in X_0$. It is also easy to see that, for any $f \in C_c(\mathcal{A}/\Gamma)$ we have

$$\langle j_\Gamma(f) \,,\, j_\Gamma(f) \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} = \langle f \,,\, f \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} \,,$$

where the inner product on the left (respectively, right) hand side corresponds to the inner product in X (respectively, in Y). Thus, j_Γ is an isometry between Y and X and has therefore norm 1.

Let $\widehat{E} : \Phi(C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma) \to C^*_\tau(\mathcal{A}/\Gamma)$ be the map defined by

$$\widehat{E}(\Phi(f)) := \Phi(f(\Gamma)) \,.$$

First let us say a few words about why $\widehat{E}$ is well-defined. This is the case because Φ is injective on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$, which is easily seen to be true because $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is an essential *-algebra (Theorem 3.1.8).

We claim that $\widehat{E}$ is continuous with respect to the norm of $\mathcal{L}(X)$. First we notice that for any $f \in C_c(\mathcal{A}/\Gamma)$ we have that (as elements of $\mathcal{L}(Y)$)

$$f = j_\Gamma^* \Phi(f) j_\Gamma \,.$$

Let $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have

$$\|\widehat{E}(\Phi(f))\|_{\mathcal{L}(X)} = \|\Phi(f(\Gamma))\|_{\mathcal{L}(X)} \,.$$

As we proved in (5.5), the norm $\|\cdot\|_{\mathcal{L}(X)}$ when restricted to $\Phi(C_c(\mathcal{A}/\Gamma))$ is such that $\|\Phi(g)\|_{\mathcal{L}(X)} = \|g\|_\tau$, and moreover the norm $\|\cdot\|_\tau$ coincides with the norm $\|\cdot\|_{\mathcal{L}(Y)}$, since $\mathcal{L}(Y) = M(C^*_\tau(\mathcal{A}/\Gamma))$. Hence we have:

$$\begin{aligned} \|\widehat{E}(\Phi(f))\|_{\mathcal{L}(X)} &= \|\Phi(f(\Gamma))\|_{\mathcal{L}(X)} = \|f(\Gamma)\|_{\mathcal{L}(Y)} \\ &= \|j_\Gamma^* \Phi(f) j_\Gamma\|_{\mathcal{L}(Y)} \leq \|\Phi(f)\|_{\mathcal{L}(X)} \,, \end{aligned}$$

which shows that $\widehat{E}$ is continuous with respect to the norm of $\mathcal{L}(X)$.

We can now prove that Φ is injective. First we notice that for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have $\widehat{E}(\Phi(f)) = \Phi(E_\Gamma(f))$. By continuity, this equality then holds for any $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Suppose now that $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ is such that $\Phi(f) = 0$. Then we have

$$0 = \widehat{E}(\Phi(f^* * f)) = \Phi(E_\Gamma(f^* * f)) \,.$$

Since Φ is faithful on $C^*_\tau(\mathcal{A}/\Gamma)$, it then follows that $E_\Gamma(f^* * f) = 0$, and since E_Γ is faithful this implies that $f^* * f = 0$, i.e. $f = 0$. Thus, Φ is injective.

We will first prove part *b*) of the theorem and only afterwards prove part *a*). For that we need to show that F is an idempotent, positive, $C^*_\tau(\mathcal{A}/\Gamma)$-linear map. The fact that F is idempotent is obvious. Now, let $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have that

$$\begin{aligned} F(f^* * f) &= (f^* * f)(\Gamma) \\ &= \sum_{[h] \in G/\Gamma} f^*(h\Gamma) \overline{\alpha}_h(f(h^{-1}\Gamma)) \\ &= \sum_{[h] \in G/\Gamma} \Delta(h^{-1})\, \overline{\alpha}_h(f(h^{-1}\Gamma))^* \overline{\alpha}_h(f(h^{-1}\Gamma)) \,, \end{aligned}$$

which by continuity means that F is positive. Moreover, for $f_1 \in C_c(\mathcal{A}/\Gamma)$ we have that

$$\begin{aligned} F(f_1 * f) &= (f_1 * f)(\Gamma) = f_1 \cdot f(\Gamma) \\ &= f_1 \cdot F(f) \,, \end{aligned}$$

and similarly $F(f * f_1) = F(f) \cdot f_1$. By continuity of F, it follows that $F(f_1 * f) = f_1 \cdot F(f)$ and $F(f * f_1) = F(f) \cdot f_1$ for any $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ and $f_1 \in C^*_\tau(\mathcal{A}/\Gamma)$. Hence we have shown that F is a conditional expectation, and therefore *b*) is proven.

Now, let $f, g \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. We have that inside $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ the following holds:

$$\begin{aligned} \langle f * g \,,\, f * g \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} &= ((f * g)^* * (f * g))(\Gamma) \\ &= F((f * g)^* * (f * g)) \\ &= F(g^* * f^* * f * g) \\ &\leq \|f\|^2_\omega F(g^* * g) \\ &= \|f\|^2_\omega \langle g \,,\, g \rangle_{C^*_\tau(\mathcal{A}/\Gamma)} \,, \end{aligned}$$

where we have used to the positivity of F. Since the norm $\|\cdot\|_\tau$ is just the restriction of the norm $\|\cdot\|_\omega$ we get

$$\|\langle f * g\,,\, f * g\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau \quad\leq\quad \|f\|^2_\omega \|\langle g\,,\, g\rangle_{C^*_\tau(\mathcal{A}/\Gamma)}\|_\tau\,, \tag{5.6}$$

which shows that the action of $C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ on X_0 extends to an action of $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,\omega} G/\Gamma$ on X and thus gives rise to a *-homomorphism from $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,\omega} G/\Gamma$ to $\mathcal{L}(X)$. As the injectivity of Φ shows, the closure of the image of $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ in $\mathcal{L}(X)$ is isomorphic to $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$. Hence, we conclude that there is a map $\Lambda : C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma \to C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$ such that $\Lambda(f) = f$, for $f \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$, and so part $a)$ is proven.

Let us now prove $c)$. The direction $(\Longleftarrow)$ is clear, because F is then nothing but the conditional expectation E_Γ, which is faithful. Let us now prove the direction $(\Longrightarrow)$. For any $f \in C_c(\mathcal{A}/\Gamma) \times^{alg}_\alpha G/\Gamma$ we have that $E_\Gamma \circ \Lambda(f^* * f) = F(f^* * f)$. By continuity this formula holds for any $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$. Let $f \in C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,\omega} G/\Gamma$ be such that $\Lambda(f) = 0$. Then we necessarily have that $0 = E_\Gamma \circ \Lambda(f^* * f) = F(f^* * f)$, and since F is faithful we have that $f^* * f = 0$, i.e. $f = 0$. ☐

5.3. Alternative definition of $C^*_r(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$

The C^*-direct limit $D_r(\mathcal{A})$ played a key role in the definition of the reduced crossed product $C^*_r(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$. In this section we will see that instead of $D_r(\mathcal{A})$ one can use the more natural C^*-algebra $C^*_r(\mathcal{A})$ to define the reduced crossed product $C^*_r(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$. The algebra $C^*_r(\mathcal{A})$ has several advantages over $D_r(\mathcal{A})$. For instance $C^*_r(\mathcal{A})$ appears more naturally in the setup for defining crossed products (recall that we start with the bundle $\mathcal{A}$ and then we form the various bundles $\mathcal{A}/\Gamma^g$ from it). Also, $C^*_r(\mathcal{A})$, being a cross sectional algebra of a Fell bundle, seems to be structurally simpler than $D_r(\mathcal{A})$, which is a direct limit of cross sectional algebras of Fell bundles.

The question one might ask at this point is: can one similarly use $C^*(\mathcal{A})$ instead of $D_{\max}(\mathcal{A})$ in order to define $C^*(\mathcal{A}/\Gamma) \rtimes_{\alpha,r} G/\Gamma$? As we shall also see in this section, this is not possible in general. At the core of this problem lies the fact that one has always an embedding

$$C^*_r(\mathcal{A}/H) \to M(C^*_r(\mathcal{A}))\,,$$

extending the natural embedding of $C_c(\mathcal{A}/H)$ into $M(C_c(\mathcal{A}))$, whereas the analogous map

$$C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))\,,$$

is not always injective. This implies that the while the algebra $D_r(\mathcal{A})$ embeds naturally in $M(C_r(\mathcal{A}))$, the analogous map from $D_{\max}(\mathcal{A})$ to $M(C^*(\mathcal{A}))$ is not an embedding in general.

We start with the following general result:

Proposition 5.3.1. *Let $\|\cdot\|_\tau$ be any C^*-norm on $C_c(\mathcal{A})$ and $C^*_\tau(\mathcal{A})$ its completion. There is a unique mapping $C^*(\mathcal{A}/H) \to M(C^*_\tau(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$.*

Proof: As is known $C_\tau^*(\mathcal{A})$ is naturally a Hilbert $C_\tau^*(\mathcal{A})$-module, whose algebra of adjointable operators $\mathcal{L}(C_\tau^*(\mathcal{A}))$ is precisely the multiplier algebra $M(C_\tau^*(\mathcal{A}))$. In particular $X := C_c(\mathcal{A})$ is an inner product $C_\tau^*(\mathcal{A})$-module. Moreover, X is also a right $C_c(\mathcal{A}) - C_\tau^*(\mathcal{A})$ bimodule and a right $C_c(\mathcal{A}/H) - C_\tau^*(\mathcal{A})$ bimodule (in the sense of Definition 4.1.2), under the canonical actions of $C_c(\mathcal{A})$ and $C_c(\mathcal{A}/H)$ on X. Since $C_c(\mathcal{A})$ acts on X by bounded operators, it then follows from Lemma 4.1.3 (taking $K = \{e\}$) that $C_c(\mathcal{A}/H)$ acts on X by bounded operators. Thus, by completion, we obtain a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}C_\tau^*(\mathcal{A})_{C_\tau^*(\mathcal{A})}$. Hence obtain a unique map $C^*(\mathcal{A}/H) \to M(C_\tau^*(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$. $\square$

As shall see later in this section the map $C^*(\mathcal{A}/H) \to M(C_\tau^*(\mathcal{A}))$ is not an embedding in general, not even when $C_\tau^*(\mathcal{A}) = C^*(\mathcal{A})$. Nevertheless for the reduced norms we have the following result:

THEOREM 5.3.2. *There is a unique embedding of $C_r^*(\mathcal{A}/H)$ into $M(C_r^*(\mathcal{A}))$ which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$.*

Proof: From Proposition 5.3.1 we know that there exists a unique *-homomorphism from $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$, which extends the action of $C_c(\mathcal{A}/H)$ on $C_c(\mathcal{A})$. Thus, we have a right-Hilbert bimodule ${}_{C^*(\mathcal{A}/H)}C_r^*(\mathcal{A})_{C_r^*(\mathcal{A})}$. Taking the balanced tensor product of this right-Hilbert bimodule with ${}_{C_r^*(\mathcal{A})}L^2(\mathcal{A})_{C_0(\mathcal{A}^0)}$ we get a $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0)$ right-Hilbert bimodule

$$ {}_{C^*(\mathcal{A}/H)}\Big(C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})\Big)_{C_0(\mathcal{A}^0)} . $$

Since the action of $C_r^*(\mathcal{A})$ on $L^2(\mathcal{A})$ is faithful, the kernels of the maps from $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$ and $\mathcal{L}\Big(C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})\Big)$ are the same.

Now, $C_r^*(\mathcal{A}) \otimes_{C_r^*(\mathcal{A})} L^2(\mathcal{A})$ is isomorphic to $L^2(\mathcal{A})$ as a Hilbert $C^*(\mathcal{A}/H) - C_0(\mathcal{A}^0)$ bimodule. Hence, it follows that the kernel of the map from $C^*(\mathcal{A}/H)$ to $M(C_r^*(\mathcal{A}))$ is the same as the kernel of the map from $C^*(\mathcal{A}/H)$ to $\mathcal{L}(L^2(\mathcal{A}))$. Now, the latter map has the same kernel as the canonical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$, by Lemma 4.1.4 applied when K is the trivial subgroup. Thus, this gives an embedding of $C_r^*(\mathcal{A}/H)$ into $M(C_r^*(\mathcal{A}))$. $\square$

The next result is a generalization of [**5**, Proposition 2.10] (see Example 2.2.3). Its proof relies ultimately on Lemma 4.1.4, whose proof, we recall, was essentially an adaptation of the proof [**5**, Proposition 2.10] itself.

COROLLARY 5.3.3. *Suppose $\mathcal{A}$ is amenable. Then, the kernel of the canonical map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ is the same as the kernel of the canonical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$.*

Proof: In the proof of Proposition 5.3.2 we established that the kernel of the canonnical map $\Lambda : C^*(\mathcal{A}/H) \to C_r^*(\mathcal{A}/H)$ is the same as the kernel of the map $C^*(\mathcal{A}/H) \to M(C_r^*(\mathcal{A}))$, which is the same as the map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ by amenability of $\mathcal{A}$. □

We now give an example where the map $C^*(\mathcal{A}/H) \to M(C^*(\mathcal{A}))$ is not injective:

EXAMPLE 5.3.4. Let $\mathcal{B}$ be a non-amenable Fell bundle over the group G, and let $\mathcal{A} := \mathcal{B} \times G$ be the associated Fell bundle over the transformation groupoid $G \times G$. Following Example 2.2.3, we have a right G-action on $\mathcal{A}$ which entails the action on the groupoid $G \times G$, given by $(s,t)g := (s, tg)$. Moreover, since the G-action is free, it is H-good and satisfies the H-intersection property, for any subgroup $H \subseteq G$. In this example we will consider H to be the whole group G. In this case the orbit groupoid $(G \times G)/G$ can be naturally identified with the group G, and moreover, the Fell bundle $\mathcal{A}/G$ is naturally identified with $\mathcal{B}$.

It is known that the bundle $\mathcal{A}$ is always amenable (see [**5**, Remark 2.11]), and therefore by Corollary 5.3.3 we have that the kernel of the map $C^*(\mathcal{A}/G) \to M(C^*(\mathcal{A}))$ is the same as the kernel of the canonical map $C^*(\mathcal{A}/G) \to C^*_r(\mathcal{A}/G)$. As we pointed out above, the bundle $\mathcal{A}/G$ is just $\mathcal{B}$, which is non-amenable by assumption. Hence, the canonical map $C^*(\mathcal{A}/G) \to C^*_r(\mathcal{A}/G)$ has a non-trivial kernel, and therefore the map $C^*(\mathcal{A}/G) \to M(C^*(\mathcal{A}))$ is not injective.

We will now see that $D_r(\mathcal{A})$ is canonically embedded in $M(C^*_r(\mathcal{A}))$, being the C^*-algebra generated by all the images of $C^*_r(\mathcal{A}/H)$ inside $M(C^*_r(\mathcal{A}))$, as in Proposition 5.3.2, with $H \in \mathcal{C}$.

PROPOSITION 5.3.5. *Let $K \subseteq H$ be subgroups of G such that $[H : K] < \infty$. Then, the following diagram of canonical embeddings commutes:*

$$\begin{tikzcd} C^*_r(\mathcal{A}/H) \arrow[r] \arrow[dr] & C^*_r(\mathcal{A}/K) \arrow[d] \\ & M(C^*_r(\mathcal{A}))\,. \end{tikzcd} \tag{5.7}$$

As a consequence $D_r(\mathcal{A})$ embedds in $M(C^*_r(\mathcal{A}))$, being *-isomorphic to the subalgebra of $M(C^*_r(\mathcal{A}))$ generated by all the $C^*_r(\mathcal{A}/H)$, with $H \in \mathcal{C}$.

Proof: We have already proven in Proposition 4.0.4 that

$$[a]_{xH} b_y = \sum_{[h] \in \mathcal{S}_x \backslash H/K} [\alpha_{h^{-1}}(a)]_{xhK} b_y\,, \tag{5.8}$$

for any $x, y \in X$, $a \in \mathcal{A}_x$ and $b \in \mathcal{A}_y$. Hence, by linearity, density and continuity, we conclude that diagram (5.7) commutes. By the universal property of $D_r(\mathcal{A})$ we then have a *-homomorphism from $D_r(\mathcal{A})$ to $M(C^*_r(\mathcal{A}))$, whose image is generated by all the images of $C^*_r(\mathcal{A}/H)$ inside $M(C^*_r(\mathcal{A}))$, for any $H \in \mathcal{C}$. This *-homomorphism from $D_r(\mathcal{A})$ to $M(C^*_r(\mathcal{A}))$ is injective because all the maps in diagram (5.7) are injective. □

We can now give an equivalent definition for the reduced crossed product $C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$, using the algebra $C^*_r(\mathcal{A})$ instead of $D^*_r(\mathcal{A})$. This can be advantageous as we observed in the opening paragraph of this subsection. Also, this equivalence of definitions will make the connection between our definition of a reduced crossed product by a Hecke pair and that of Laca, Larsen and Neshveyev in **[15]** more clear, as we shall see in the next subsection.

THEOREM 5.3.6. *Let $\pi : C^*_r(\mathcal{A}) \to B(\mathscr{H})$ be a nondegenerate *-representation, and $\widetilde{\pi}$ its extension to $M(C^*_r(\mathcal{A}))$. We have that*

i) *If $\widetilde{\pi}_\alpha : C^*_r(\mathcal{A}/\Gamma) \to B(\mathcal{H} \otimes \ell^2(G/\Gamma))$ is faithful, then $\widetilde{\pi}_\alpha \times (1 \otimes \rho)$ is a faithful representation of $C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$. Consequently,*

$$\|f\|_{r,r} := \|[\widetilde{\pi}_\alpha \times (1 \otimes \rho)](f)\|\,,$$

for all $f \in C^*_r(\mathcal{A}/\Gamma) \times_{r,\alpha} G/\Gamma$.

ii) *If π is faithful, then $\widetilde{\pi}_\alpha$ is faithful.*

Proof: By Proposition 5.3.5 $D_r(\mathcal{A})$ is canonically embedded in $M(C^*_r(\mathcal{A}))$, so that $\widetilde{\pi}$ restricts to a *-representation of $D_r(\mathcal{A})$. This restriction is nondegenerate, because the restriction to $C^*_r(\mathcal{A}/\Gamma)$ is already nondegenerate, as follows from the following argument. Let $\xi \in \mathscr{H}$ be such that $\widetilde{\pi}(C^*_r(\mathcal{A}/\Gamma))\xi = 0$. For any $x \in X$ and $a \in \mathcal{A}_x$ we have

$$\begin{aligned} \|\pi(a_x)\xi\|^2 &= \langle \pi((a^*a)_{\mathbf{s}(x)})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}} \cdot [a]_{x\Gamma})\xi\,,\,\xi\rangle \\ &= \langle \pi(a^*_{x^{-1}})\widetilde{\pi}([a]_{x\Gamma})\xi\,,\,\xi\rangle \\ &= 0\,. \end{aligned}$$

Thus, by nondegeneracy of π we get that $\xi = 0$, and therefore $\widetilde{\pi}$ restricted to $C^*_r(\mathcal{A}/\Gamma)$, and hence also $D_r(\mathcal{A})$, is nondegenerate. We are now in the conditions of Theorem 5.2.9.

Claim *ii)* also follows from Theorem 5.2.9, given the fact that a faithfull nondegenerate *-representation of $C^*_r(\mathcal{A})$ extends faithfully to $M(C^*_r(\mathcal{A}))$. □

5.4. Comparison with Laca-Larsen-Neshveyev construction

In **[15]**, Laca, Larsen and Neshveyev, based on the work of Connes-Marcolli **[3]** and Tzanev **[22]**, introduced an algebra which can be thought of as a reduced crossed product of an abelian algebra by an action of a Hecke pair.

The construction introduced by Laca, Larsen and Neshveyev was one of the motivations behind our definition of a crossed product by a Hecke pair. However, the setup for their construction in **[15]** is slightly different from ours, being on one side more particular, as it only allows one to take a crossed product by an abelian algebra, but also more general, as the underlying space is not assumed in **[15]** to be discrete. We will show in this section that when both setups agree, our crossed product is canonically isomorphic to the crossed product of **[15]**.

We will first briefly recall the setup and construction presented in [**15**, Section 1]. In order to make a coherent and more meaningful comparison between our construction and that of **[15]** we will have to make a few simple modifications in the latter. Essentially, we will consider right actions of G instead of left ones, and make the appropriate changes in the construction of [**15**] according to this.

Let G be a group acting on the right on a locally compact space X. Let $\Gamma \subseteq G$ be a Hecke subgroup and consider the (right) action of $\Gamma \times \Gamma$ on $X \times G$, given by:

$$(x, g)(\gamma_1, \gamma_2) := (x\gamma_1, \gamma_1^{-1} g \gamma_2)\,. \tag{5.9}$$

Define $X \times_\Gamma G/\Gamma$ to be the quotient space of $X \times G$ by the action of $\Gamma \times \Gamma$. We assume that the space $X \times_\Gamma G/\Gamma$ is Hausdorff.

Remark 5.4.1. In [**15**] the original assumption was that the action of Γ on X was proper (hence implying that $X \times_\Gamma G/\Gamma$ is Hausdorff), but as it was observed in [**15**, Remark 1.4], requiring that $X \times_\Gamma G/\Gamma$ is Hausdorff was actually enough for the construction to make sense, and this is an important detail for us as the actions we consider are not proper in general (see Remark 5.4.2).

Remark 5.4.2. Let $X = \{*\}$ be a space with just one point and (G, Γ) a Hecke pair where Γ is infinite. We consider the trivial action of G on X. In this case $X \times_\Gamma G/\Gamma$ is the space $\Gamma\backslash G/\Gamma$ with the discrete topology, which is Hausdorff. Nevertheless, since Γ is infinite, neither the action of Γ on X, nor the action of $\Gamma \times \Gamma$ on $X \times G$, is proper. To see this, notice that the pre-image of the compact sets $\{*\}$ and $\{(*, e)\}$, in $X \times \Gamma$ and $X \times G \times \Gamma \times \Gamma$ repectively, are infinite.

Let $C_c(X \times_\Gamma G/\Gamma)$ be the space of compactly supported continuous functions on $X \times_\Gamma G/\Gamma$. We will view the elements of $C_c(X \times_\Gamma G/\Gamma)$ as $(\Gamma \times \Gamma)$-invariant functions on $X \times G$. One can define a convolution product and involution in $C_c(X \times_\Gamma G/\Gamma)$ according to the following formulas:

$$(f_1 * f_2)(x, g) := \sum_{[h] \in G/\Gamma} f_1(x, h) f_2(xh, h^{-1} g)\,, \tag{5.10}$$

$$f^*(x, g) := \overline{f(xg, g^{-1})}\,. \tag{5.11}$$

For each given $x \in X$ we can define a *-representation $\pi_x : C_c(X \times_\Gamma G/\Gamma) \to B(\ell^2(G/\Gamma))$ by

$$\pi_x(f)\delta_{h\Gamma} := \sum_{g \in G/\Gamma} f(xg, g^{-1}h)\delta_{g\Gamma}\,. \tag{5.12}$$

The C^*-algebra $C_r^*(X \times_\Gamma G/\Gamma)$ is defined as the completion of $C_c(X \times_\Gamma G/\Gamma)$ in the norm

$$\|f\| := \sup_{x \in X} \|\pi_x(f)\|\,. \tag{5.13}$$

The setup behind this construction differs slightly from our own, so we will compare both constructions under the following assumptions:

- (G, Γ) is a Hecke pair;
- X is a set (seen as both a discrete space and a discrete groupoid);
- There is a right action of G on X;
- The G-action satisfies the Γ-intersection property.

We notice that since X and G are discrete the space $X \times_\Gamma G/\Gamma$ is also discrete and therefore Hausdorff, so that the necessary assumptions for the construction of **[15]** are satisfied. Also, since X is just a set, the action G on X is necessarily Γ-good. Thus, the assumptions for our construction (Standing Assumption 3.0.1) are satisfied with respect to the trivial Fell bundle $\mathcal{A}$ over X in which every fiber $\mathcal{A}_x$ is just $\mathbb{C}$. Recall that in this case $C_c(\mathcal{A}) = C_c(X)$ and $C_c(\mathcal{A}/\Gamma) = C_c(X/\Gamma)$.

THEOREM 5.4.3. *Let (G, Γ) be a Hecke pair and X a set. Assume that there is a right G-action on X which satisfies the Γ-intersection property. Then, the map $\Phi : C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma \to C_c(X \times_\Gamma G/\Gamma)$ given by*

$$\Phi(f)\,(x, g) := \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)(x)\,,$$

is a *-isomorphism. This map extends to a *-isomorphism between the reduced completions $\Phi : C_0(X/\Gamma) \times_{\alpha,r} G/\Gamma \to C_r^*(X \times_\Gamma G/\Gamma)$. Moreover, under the *-isomorphism Φ, the *-representation π_x is just $(\widetilde{\varphi_x})_\alpha \times \rho$, where φ_x is the *-representation of $C_0(X)$ given by evaluation at x, i.e. $\varphi_x(f) = f(x)$.

Proof: Let us first check that Φ is well-defined, i.e. $\Phi(f)$ is a $(\Gamma \times \Gamma)$-invariant function in $G \times X$, with compact support (as a function on $X \times_\Gamma G/\Gamma$). To see this, let $\gamma_1, \gamma_2 \in \Gamma$. We have that

$$\begin{aligned}
\Phi(f)\,(x\gamma_1, \gamma_1^{-1} g \gamma_2) &= \Delta(\gamma_1^{-1} g \gamma_2)^{\frac{1}{2}}\, f(\gamma_1^{-1} g \gamma_2 \Gamma)(x\gamma_1) \\
&= \Delta(g)^{\frac{1}{2}}\, \overline{\alpha}_{\gamma_1^{-1}}(f(g\Gamma))\,(x\gamma_1) \\
&= \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)\,(x) \\
&= \Phi(f)\,(x, g)\,,
\end{aligned}$$

so that $\Phi(f)$ is $\Gamma \times \Gamma$-invariant. It is easy to see that $\Phi(f)$ has compact support (as a function on $X \times_\Gamma G/\Gamma$). Thus, Φ is well-defined.

Let us now prove that Φ is a *-homomorphism. It is clear that Φ is linear, so that we only need to check that Φ preserves products and the involution. For $f_1, f_2 \in C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have that

$$\begin{aligned}
& \Phi(f_1 * f_2)\,(x, g) \;= \\
=\;& \Delta(g)^{\frac{1}{2}}\,(f_1 * f_2)(g\Gamma)\,(x) \\
=\;& \sum_{[h] \in G/\Gamma} \Delta(g)^{\frac{1}{2}}\, f_1(h\Gamma)\overline{\alpha}_h(f_2(h^{-1}g\Gamma))\,(x) \\
=\;& \sum_{[h] \in G/\Gamma} \Big(\Delta(h)^{\frac{1}{2}}\, f_1(h\Gamma)\,(x)\Big)\Big(\Delta(h^{-1}g)^{\frac{1}{2}}\, \overline{\alpha}_h(f_2(h^{-1}g\Gamma))\,(x)\Big) \\
=\;& \sum_{[h] \in G/\Gamma} \Big(\Phi(f_1)\,(x, h)\Big)\Big(\Delta(h^{-1}g)^{\frac{1}{2}}\, f_2(h^{-1}g\Gamma)\,(xh)\Big) \\
=\;& \sum_{[h] \in G/\Gamma} \Big(\Phi(f_1)\,(x, h)\Big)\Big(\Phi(f_2)(xh, h^{-1}g)\Big) \\
=\;& \Phi(f_1) * \Phi(f_2)\,(x, g)\,.
\end{aligned}$$

Also for $f \in C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ we have

$$\begin{aligned}\Phi(f^*)\,(x,g) &= \Delta(g)^{\frac{1}{2}}\, f^*(g\Gamma)\,(x) \;=\; \Delta(g)^{\frac{1}{2}}\Delta(g^{-1})\,\overline{\overline{\alpha}_g(f(g^{-1}\Gamma))\,(x)} \\ &= \Delta(g^{-1})^{\frac{1}{2}}\,\overline{f(g^{-1}\Gamma)\,(xg)} \;=\; \overline{\Phi(f)\,(xg,g^{-1})} \\ &= (\Phi(f))^*\,(x,g)\,.\end{aligned}$$

Hence, Φ is a *-homomorphism. Let us now prove that Φ is injective. Suppose $\Phi(f)=0$. Then for every $g \in G$ and $x \in X$ we have

$$0 \;=\; \Phi(f)\,(x,g) \;=\; \Delta(g)^{\frac{1}{2}}\, f(g\Gamma)\,(x)\,.$$

Hence, we conclude that $f(g\Gamma)=0$ for all $g \in G$, and therefore $f=0$, i.e. Φ is injective.

Let us now prove the surjectivity of Φ. The elements of $C_c(X \times_\Gamma G/\Gamma)$ are simply linear combinations of characteristic functions of elements of $X \times_\Gamma G/\Gamma$, so in order to prove that Φ is surjective we only need to check that each of these characteristic functions belongs the image of Φ. Let $[(x,g)] \in X \times_\Gamma G/\Gamma$. We claim that $\Phi(\Delta(g)^{-\frac{1}{2}}\, 1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma}) = 1_{[(x,g)]}$. To see this, we recall Lemma 3.1.14 and notice that

$$\Phi(1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma})\,(x,g) \;=\; \Delta(g)^{\frac{1}{2}}\,.$$

It is not difficult to see that $\Phi(1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma})\,(y,h)=0$ if (y,h) does not belong to the $\Gamma\times\Gamma$-orbit of (x,g), so that $\Phi(\Delta(g)^{-\frac{1}{2}}\, 1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma}) = 1_{[(x,g)]}$. Hence, we can conclude that Φ is surjective and therefore establishes a *-isomorphism between $C_c(X/\Gamma) \times_\alpha^{alg} G/\Gamma$ and $C_c(X \times_\Gamma G/\Gamma)$.

We will now see that under the *-isomorphism Φ, the *-representation π_x is just $(\widetilde{\varphi_x})_\alpha \times \rho$, in other words $\pi_x \circ \Phi = (\widetilde{\varphi_x})_\alpha \times \rho$. This follows from the following computation:

$$\begin{aligned}\pi_x \circ \Phi(f)\,\delta_{h\Gamma} &= \sum_{g\Gamma \in G/\Gamma} \Phi(f)(xg, g^{-1}h)\,\delta_{g\Gamma} \\ &= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, f(g^{-1}h\Gamma)(xg)\,\delta_{g\Gamma} \\ &= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \overline{\alpha}_g(f(g^{-1}h\Gamma))(x)\,\delta_{g\Gamma} \\ &= \sum_{g\Gamma \in G/\Gamma} \Delta(g^{-1}h)^{\frac{1}{2}}\, \widetilde{\varphi_x}\big(\overline{\alpha}_g(f(g^{-1}h\Gamma))\big)\,\delta_{g\Gamma} \\ &= [(\widetilde{\varphi_x})_\alpha \times \rho](f)\,\delta_{g\Gamma}\,.\end{aligned}$$

Let us now prove that the *-isomorphism Φ extends to a *-isomorphism between $C_0(X/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C_r^*(X \times_\Gamma G/\Gamma)$. Let $\pi : C_c(X \times_\Gamma G/\Gamma) \longrightarrow B(\ell^2(X))$ be the direct sum *-representation $\pi := \bigoplus_{x\in X} \pi_x$ on the Hilbert space $\bigoplus_{x \in X} \mathbb{C} \cong \ell^2(X)$.

We then have that

$$\begin{aligned}
\pi \circ \Phi(f) &= \bigoplus_{x \in X} \pi_x(\Phi(f)) \\
&= \bigoplus_{x \in X} [(\widetilde{\varphi_x})_\alpha \times \rho](f) \\
&= [(\bigoplus_{x \in X} \widetilde{\varphi_x})_\alpha \times \rho](f) \\
&= [(\widetilde{\bigoplus_{x \in X} \varphi_x})_\alpha \times \rho](f)\,.
\end{aligned}$$

Now the *-representation $\bigoplus_{x \in X} \varphi_x$ of $C_0(X)$ is obviously injective. Hence, by Theorem 5.3.6 *ii*), it follows that $\pi \circ \Phi$ extends to a faithful *-representation of $C_0(X/\Gamma) \times_{\alpha,r} G/\Gamma$. This implies that Φ extends to an isomorphism between $C_0(X/\Gamma) \times_{\alpha,r} G/\Gamma$ and $C_r^*(X \times_\Gamma G/\Gamma)$, because

$$\begin{aligned}
\|\Phi(f)\| &= \sup_{x \in X} \|\pi_x(\Phi(f))\| = \|\pi \circ \Phi(f)\| \\
&= \|f\|_{r,r}\,.
\end{aligned}$$

$\square$

CHAPTER 6

Other completions

Just like there are several canonical C^*-completions of a Hecke algebra, one can also consider different C^*-completions of crossed products by Hecke pairs. Especially interesting for this work are full C^*-crossed products, but we will also take a look at C^*-completions arising from a L^1-norm.

6.1. Full C^*-crossed products

In this section we define and study full C^*-crossed products by Hecke pairs. Just like in the reduced case, several full C^*-crossed products can be considered, such as $C^*_r(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ and $C^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ where each of these is thought of as the full C^*-crossed product of $C^*_r(\mathcal{A}/\Gamma)$, respectively $C^*(\mathcal{A}/\Gamma)$, by the Hecke pair (G, Γ). As is the case for Hecke algebras, full crossed products by Hecke pairs do not have to exist in general.

Definition 6.1.1. Let $\| \cdot \|_\tau$ be an $\overline{\alpha}$-permissible C^*-norm in $D(\mathcal{A})$. We will denote by $\| \cdot \|_{\tau,u} : C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma \longrightarrow \mathbb{R}_0^+ \cup \{\infty\}$ the function defined by

$$\|f\|_{\tau,u} := \sup_{\Phi \in R_\tau} \|\Phi(f)\| , \tag{6.1}$$

where the supremum is taken over the class R_τ of *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ whose restrictions to $C_c(\mathcal{A}/\Gamma)$ are continuous with respect to $\| \cdot \|_\tau$.

Proposition 6.1.2. *We have that $\| \cdot \|_{\tau,u}$ is a C^*-norm in $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ if and only if $\|f\|_{\tau,u} < \infty$ for all $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$.*

Proof: $(\Longrightarrow)$: This direction is trivial since a norm must take values in $\mathbb{R}_0^+$.
$(\Longleftarrow)$: It is clear in this case that $\| \cdot \|_{\tau,u}$ defines a C^*-seminorm. To check that it is a true C^*-norm it is enough to find a faithful *-representation $\Phi \in R_\tau$. This is easy because since $\| \cdot \|_\tau$ is $\overline{\alpha}$-permissible we can take any nondegenerate faithful *-representation π of $D_\tau(\mathcal{A})$ and take $\Phi := \pi_\alpha \times (1 \otimes \rho)$, which is a faithful *-representation by Theorem 5.2.9. We have that $\Phi \in R_\tau$ because its restriction to $C_c(\mathcal{A}/\Gamma)$ is just π_α, which is continuous with respect to $\| \cdot \|_\tau$ by Lemma 5.2.3. $\square$

Definition 6.1.3. Let $\| \cdot \|_\tau$ be an $\overline{\alpha}$-permissible C^*-norm in $D(\mathcal{A})$. When $\| \cdot \|_{\tau,u}$ is a C^*-norm we will call it *the universal norm associated to* $\| \cdot \|_\tau$. The completion of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ with respect to this norm will be denoted by $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ and referred to as the *full crossed product* of $C^*_\tau(\mathcal{A}/\Gamma)$ by the

Hecke pair (G,Γ).

It is clear that $\|\cdot\|_{\tau,r} \leq \|\cdot\|_{\tau,u}$ so that the identity map on $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ extends to a surjective *-homomorphism

$$C_\tau^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma \longrightarrow C_\tau^*(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma\,, \tag{6.2}$$

in case $\|\cdot\|_{\tau,u}$ is a norm.

In general, full crossed products do not necessarily exist, as it is already clear from the fact that a Hecke algebra (which is a particular case of crossed product by a Hecke pair) does not need to have an enveloping C^*-algebra. Nevertheless, for Hecke pairs whose Hecke algebras are BG^*-algebras one can always assure the existence of full C^*-crossed products, as we show below. We recall that a *-algebra is called a *BG^*-algebra* if all of its pre-*-representations are normed. Most Hecke algebras for which it is known that a full Hecke C^*-algebra exists are known to be BG^*-algebras, as we discussed in [**17**].

Theorem 6.1.4. *If $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra, then the full crossed product $C_\tau^*(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ always exists, for any $\overline{\alpha}$-permissible norm $\|\cdot\|_\tau$.*

Proof: We will prove that when $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra we have

$$\sup_\Phi \|\Phi(f)\| < \infty\,, \tag{6.3}$$

where the supremum runs over the class of all *-representations of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. To see this we first notice that it is enough to consider nondegenerate *-representations. Secondly, from Theorem 3.3.17, any nondegenerate *-representation Φ of $C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$ is the integrated form of a covariant pre-*-representation $(\Phi|, \omega_\Phi)$, so that we can write $\Phi = \Phi| \times \omega_\Phi$. Taking any element $[a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma}$ of the canonical spanning set of elements of the crossed product we then have

$$\begin{aligned} \|\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\| &= \|\Phi|([a]_{x\Gamma})\omega_\Phi(\Gamma g\Gamma)\widetilde{\Phi|}(1_{\mathbf{s}(x)g\Gamma})\| \\ &\leq \|\Phi|([a]_{x\Gamma})\omega_\Phi(\Gamma g\Gamma)\|\,. \end{aligned}$$

Now, since $\mathcal{H}(G,\Gamma)$ is a BG^*-algebra we have that ω_Φ is normed, i.e. $\omega_\Phi(\Gamma g\Gamma)$ is a bounded operator. Moreover, because it is a BG^*-algebra, $\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra. Hence, we conclude that

$$\begin{aligned} &\leq \|\Phi|([a]_{x\Gamma})\|\|\omega_\Phi(\Gamma g\Gamma)\| \\ &\leq \|[a]_{x\Gamma}\|_{C^*(\mathcal{A}/\Gamma)}\|\Gamma g\Gamma\|_{C^*(G,\Gamma)}\,. \end{aligned}$$

Thus, it is clear that

$$\sup_\Phi \|\Phi([a]_{x\Gamma} * \Gamma g\Gamma * 1_{\mathbf{s}(x)g\Gamma})\| < \infty\,.$$

Since this is true for the elements of the canonical spanning set, it follows that (6.3) holds for any $f \in C_c(\mathcal{A}/\Gamma) \times_\alpha^{alg} G/\Gamma$. □

Any BG^*-algebra necessarily has an enveloping C^*-algebra. Is it then possible to weaken the assumptions on Theorem 6.1.4 to cover all Hecke algebras with an enveloping C^*-algebra? In other words:

Open Question 6.1.5. If $\mathcal{H}(G,\Gamma)$ has an enveloping C^*-algebra, do the full crossed products $C^*_\tau(\mathcal{A}/\Gamma)\times_\alpha G/\Gamma$ always exist?

We do not know the answer to this question. In fact we do not even have an example of a Hecke algebra which has an enveloping C^*-algebra and is not a BG^*-algebra. More generally even, the author does not know any example of a *-algebra that can be faithfully represented on a Hilbert space and has an enveloping C^*-algebra, but which is not a BG^*-algebra.

Regarding the existence of full crossed products we will show, in the next chapter, that they can exist for Hecke pairs for which the Hecke algebra does not have an enveloping C^*-algebra. Namely, the full crossed product $C_0(G/\Gamma)\times_\alpha G/\Gamma$, arising from the action of G on itself by translation, exists for all Hecke pairs (G,Γ).

6.2. L^1-norm and associated C^*-completion

We now define a L^1-norm on $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$, whose corresponding enveloping C^*-algebra can still be understood as a crossed product of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ), for a $\overline{\alpha}$-permissible norm $\|\cdot\|_\tau$.

Definition 6.2.1. Let $\|\cdot\|_\tau$ be an $\overline{\alpha}$-permissible C^*-norm on $D(\mathcal{A})$. We define the norm $\|\cdot\|_{\tau,L^1}$ in $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ by:

$$\|f\|_{\tau,L^1} := \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\,\|f(g\Gamma)\|_\tau\,. \tag{6.4}$$

Before we prove that $\|\cdot\|_{\tau,L^1}$ is a norm we observe that $\|\cdot\|_{\tau,L^1}$ is well-defined, i.e. it does not depend on the chosen representative g of $[g]$, because for any $\gamma\in\Gamma$ we have, using the fact that the $\|\cdot\|_\tau$ is $\overline{\alpha}$-permissible,

$$\|f(\gamma g\Gamma)\|_\tau \;=\; \|\overline{\alpha}_\gamma(f(g\Gamma))\|_\tau \;=\; \|f(g\Gamma)\|_\tau\,.$$

With this observation at hand we can easily derive another formula for $\|\cdot\|_{\tau,L^1}$, for which we have

$$\|f\|_{\tau,L^1} = \sum_{[g]\in G/\Gamma} \|f(g\Gamma)\|_\tau\,. \tag{6.5}$$

Proposition 6.2.2. *The function $\|\cdot\|_{\tau,L^1}$ is a norm for which*

$$\|f_1 * f_2\|_{\tau,L^1} \leq \|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1} \qquad \text{and} \qquad \|f^*\|_{\tau,L^1} = \|f\|_{\tau,L^1}\,.$$

Thus, under this norm $C_c(\mathcal{A}/\Gamma)\times^{alg}_\alpha G/\Gamma$ becomes a normed *-algebra.

Proof: It is easy to check that $\|\cdot\|_{\tau,L^1}$ is a vector space norm in $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$. Let us prove first that $\|f^*\|_{\tau,L^1}=\|f\|_{\tau,L^1}$. We have

$$\begin{aligned}\|f^*\|_{\tau,L^1} &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\,\|f^*(g\Gamma)\|_\tau\\ &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\Delta(g^{-1})\,\|\overline{\alpha}_g(f(g^{-1}\Gamma))^*\|_\tau\\ &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g^{-1})\,\|f(g^{-1}\Gamma)\|_\tau\,.\end{aligned}$$

Since $[g]\mapsto[g^{-1}]$ is a bijection of the set $\Gamma\backslash G/\Gamma$, we get

$$\begin{aligned} &= \sum_{[g]\in\Gamma\backslash G/\Gamma} L(g)\,\|f(g\Gamma)\|_\tau\\ &= \|f\|_{\tau,L^1}\,.\end{aligned}$$

Let us now prove that $\|f_1*f_2\|_{\tau,L^1}\leq\|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1}$. For this we will use the formula for $\|\cdot\|_{\tau,L^1}$ given by (6.5). We have that

$$\begin{aligned}\|f_1*f_2\|_{\tau,L^1} &= \sum_{[g]\in G/\Gamma}\|(f_1*f_2)(g\Gamma)\|_\tau\\ &\leq \sum_{[g]\in G/\Gamma}\sum_{[h]\in G/\Gamma}\|f_1(h\Gamma)\|_\tau\|\overline{\alpha}_h(f_2(h^{-1}g\Gamma))\|_\tau\,.\end{aligned}$$

Using the fact that $\|\cdot\|_\tau$ is $\overline{\alpha}$-permissible we have

$$\begin{aligned} &= \sum_{[g]\in G/\Gamma}\sum_{[h]\in G/\Gamma}\|f_1(h\Gamma)\|_\tau\|f_2(h^{-1}g\Gamma)\|_\tau\\ &= \sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma}\|f_1(h\Gamma)\|_\tau\|f_2(h^{-1}g\Gamma)\|_\tau\\ &= \sum_{[h]\in G/\Gamma}\sum_{[g]\in G/\Gamma}\|f_1(h\Gamma)\|_\tau\|f_2(g\Gamma)\|_\tau\\ &= \Big(\sum_{[h]\in G/\Gamma}\|f_1(h\Gamma)\|_\tau\Big)\Big(\sum_{[g]\in G/\Gamma}\|f_2(g\Gamma)\|_\tau\Big)\\ &= \|f_1\|_{\tau,L^1}\|f_2\|_{\tau,L^1}\,.\end{aligned}$$

□

Completing $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$ in the norm $\|\cdot\|_{\tau,L^1}$ we obtain a Banach *-algebra, and taking the enveloping C^*-algebra of this Banach *-algebra we obtain a C^*-completion of $C_c(\mathcal{A}/\Gamma)\times_\alpha^{alg} G/\Gamma$, which we denote by $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,L^1} G/\Gamma$. We notice that the restriction of the norm $\|\cdot\|_{\tau,L^1}$ to $C_c(\mathcal{A}/\Gamma)$ is precisely the norm $\|\cdot\|_\tau$, from which we can conclude that $\|\cdot\|_{\tau,u}$ is always greater or equal to the C^*-norm of $C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,L^1} G/\Gamma$. This means that, if $\|\cdot\|_{\tau,u}$ is a norm, there is canonical map

$$C^*_\tau(\mathcal{A}/\Gamma)\times_\alpha G/\Gamma\to C^*_\tau(\mathcal{A}/\Gamma)\times_{\alpha,L^1} G/\Gamma\,. \tag{6.6}$$

In case the crossed product is just the Hecke algebra itself, the map (6.6) is just the usual map

$$C^*(G,\Gamma) \to C^*(L^1(G,\Gamma)) .$$

So far we have seen three canonical C^*-crossed products of $C^*_\tau(\mathcal{A}/\Gamma)$ by the Hecke pair (G,Γ), and these are $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,r} G/\Gamma$, $C^*_\tau(\mathcal{A}/\Gamma) \times_{\alpha,L^1} G/\Gamma$ and $C^*_\tau(\mathcal{A}/\Gamma) \times_\alpha G/\Gamma$ if it exists. Each one of these corresponds respectively, in the Hecke algebra case, to the completions $C^*_r(G,\Gamma)$, $C^*(L^1(G,\Gamma))$ and $C^*(G,\Gamma)$. It is an interesting problem, which we will not explore here, to understand how the Schlichting completion construction and the remaining Hecke C^*-algebra $pC^*(\overline{G})p$ carry over to the setting of crossed products by Hecke pairs.

CHAPTER 7

Stone-von Neumann theorem for Hecke pairs

A modern version of the Stone-von Neumann theorem in the language of crossed products by groups states that (see [**21**, Theorem C.34])

$$C_0(G) \times_\alpha G \;\cong\; C_0(G) \times_{\alpha,r} G \;\cong\; \mathcal{K}(\ell^2(G))\,.$$

More precisely, if α is the action of G on $C_0(G)$ by right translation, $M : C_0(G) \to B(\ell^2(G))$ the *-representation by pointwise multiplication and ρ the right regular representation of G on $\ell^2(G)$, then (M, ρ) is a covariant representation of the system $(C_0(G), G)$ and $M \times \rho$ is a faithful *-representation of $C_0(G) \times_\alpha G$ with range $\mathcal{K}(\ell^2(G))$.

It follows from this result that any covariant representation of $(C_0(G), G)$ is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ), since the algebra of compact operators has a trivial representation theory ([**21**, Remark C.35]).

The goal of this chapter is to show how the Stone-von Neumann theorem generalizes to the setting of Hecke pairs and their crossed products. In the process we recover an Huef, Kaliszewski and Raeburn's notion of a *covariant pair* [**9**] and their version of the Stone-von Neumann theorem for Hecke pairs, which did not make use of crossed products and which we will now recall.

In [**9**, Definition 1.1], an Huef, Kaliszewski and Raeburn introduced the notion of a *covariant pair* (π, μ) consisting of a nondegenerate *-representation $\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ and a unital *-representation $\mu : \mathcal{H}(G, \Gamma) \to B(\mathscr{H})$ satisfying

$$\mu(\Gamma g \Gamma)\pi(1_{x\Gamma})\mu(\Gamma s \Gamma) = \sum_{\substack{[u] \in \Gamma g^{-1}\Gamma/\Gamma \\ [v] \in \Gamma s \Gamma/\Gamma}} \pi(1_{xu\Gamma})\mu(\Gamma u^{-1} v \Gamma)\pi(1_{xv\Gamma})\,. \tag{7.1}$$

The basic example of a covariant pair, computed in [**9**, Example 1.5], is that of (M, ρ) where $M : C_0(G/\Gamma) \to B(\ell^2(G/\Gamma))$ is the *-representation by pointwise multiplication and ρ is the right regular representation of $\mathcal{H}(G, \Gamma)$.

Remark 7.0.1. One should note that the definition of the right regular representation ρ used in [**9**] differs from ours, since in [**9**] the factor $\Delta^{\frac{1}{2}}$ is absent. Nevertheless, (M, ρ) is still a covariant pair with our definition of ρ. Moreover, the results of [**9**] remain valid for our ρ as well, up to multiplication by some factor in some of them.

It was proven in [**9**, Theorem 1.6] that all covariant pairs are unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ), which can be seen as an analogue for Hecke pairs of the Stone-von Neumann theorem. It should be noted that this result was proven without any crossed product construction behind it.

In the following we will prove a Stone-von Neumann theorem for Hecke pairs in the language of crossed products, stating that

$$C_0(G/\Gamma) \times G/\Gamma \;\cong\; C_0(G/\Gamma) \times_r G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma))\,.$$

We will also show that the covariant pairs of [**9**] coincide with our notion of a covariant *-representation and we will recover an Huef, Kaliszewski and Raeburn's version of the Stone-von Neumann theorem ([**9**, Theorem 1.6]) as a consequence of the above isomorphisms.

The case under consideration now is that when the groupoid X is the set G and $\mathcal{A}$ is the Fell bundle over (the set) G whose fibers are $\mathbb{C}$. In this case we have $C_c(\mathcal{A}) = C_c(G)$ and, naturally, $C_r^*(\mathcal{A}) = C^*(\mathcal{A}) = C_0(G)$. We consider the action α of G on $\mathcal{A}$ induced by the right multiplication of G on itself. Since this action is free, it is Γ-good and satisfies the Γ-intersection property. Moreover the induced action $\overline{\alpha}$ of G on $C_c(G)$ is simply the action by right translation. In this setting the groupoid X/Γ is then nothing but the orbit set G/Γ, and $C_c(\mathcal{A}/\Gamma) = C_c(G/\Gamma)$. Moreover, $C_r^*(\mathcal{A}/\Gamma) = C^*(\mathcal{A}/\Gamma) = C_0(G/\Gamma)$.

Proposition 7.0.2. *Let $T_{g\Gamma,h\Gamma} \in C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$ be the element*

$$T_{g\Gamma,h\Gamma} := 1_{g\Gamma} * \Gamma g^{-1} h\Gamma * 1_{h\Gamma}\,.$$

Then $\{T_{g\Gamma,h\Gamma}\}_{g\Gamma,h\Gamma\in G/\Gamma}$ is a set of matrix units that span $C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$.

Proof: It is clear that $T^*_{g\Gamma,h\Gamma} = T_{h\Gamma,g\Gamma}$. Let us now compute the product $T_{g\Gamma,h\Gamma} * T_{s\Gamma,t\Gamma}$. If $h\Gamma \neq s\Gamma$, then $T_{g\Gamma,h\Gamma} * T_{s\Gamma,t\Gamma} = 0$. In case $h\Gamma = s\Gamma$, we get

$$\begin{aligned} T_{g\Gamma,h\Gamma} * T_{h\Gamma,t\Gamma} &= 1_{g\Gamma} * \Gamma g^{-1} h\Gamma * 1_{h\Gamma} * \Gamma h^{-1} t\Gamma * 1_{t\Gamma} \\ &= 1_{g\Gamma} * \Big(\sum_{\substack{[u]\in\Gamma h^{-1}g\Gamma/\Gamma \\ [v]\in\Gamma h^{-1}t\Gamma/\Gamma}} 1_{hu\Gamma} * \Gamma u^{-1} v\Gamma * 1_{hv\Gamma} \Big) * 1_{t\Gamma}\,. \end{aligned}$$

Now for the product $1_{g\Gamma}1_{hu\Gamma}$ to be non-zero, we must have $hu\Gamma = g\Gamma$, i.e. $u\Gamma = h^{-1}g\Gamma$. Similarly, for the product $1_{hv\Gamma}1_{t\Gamma}$ to be non-zero we must have $hv\Gamma = t\Gamma$, i.e. $v\Gamma = h^{-1}t\Gamma$. Thus,

$$\begin{aligned} T_{g\Gamma,h\Gamma} * T_{h\Gamma,t\Gamma} &= 1_{g\Gamma} * 1_{hh^{-1}g\Gamma} * \Gamma(h^{-1}g)^{-1} h^{-1} t\Gamma * 1_{hh^{-1}t\Gamma} * 1_{t\Gamma} \\ &= 1_{g\Gamma} * \Gamma g^{-1} t\Gamma * 1_{t\Gamma} \\ &= T_{g\Gamma,t\Gamma}\,. \end{aligned}$$

Hence, $\{T_{g\Gamma,h\Gamma}\}_{g\Gamma,h\Gamma}$ is a set of matrix units. The fact that this set spans $C_c(G/\Gamma)\times_\alpha^{alg} G/\Gamma$ follows readily from Theorem 3.1.13, noting that for $x \in G$ and $g\Gamma \in G/\Gamma$ we have

$$1_{x\Gamma} * \Gamma g\Gamma * 1_{xg\Gamma} = T_{x\Gamma,xg\Gamma}\,.$$

This finishes the proof. □

Theorem 7.0.3. *The full crossed product $C_0(G/\Gamma) \times_\alpha G/\Gamma$ exists and moreover*

$$C_0(G/\Gamma) \times_\alpha G/\Gamma \;\;\cong\;\; C_0(G/\Gamma) \times_{\alpha,r} G/\Gamma \;\;\cong\;\; \mathcal{K}(\ell^2(G/\Gamma))\,.$$

Denoting by $M : C_0(G/\Gamma) \to B(\ell^2(G))$ the *-representation by pointwise multiplication, we have that (M, ρ) is a covariant *-representation and $M \times \rho$ is a faithful *-representation of $C_0(G/\Gamma) \times_\alpha G/\Gamma$ with range $\mathcal{K}(\ell^2(G/\Gamma))$.

Proof: By Proposition 7.0.2 we have that $\{T_{g\Gamma, h\Gamma}\}_{g\Gamma, h\Gamma}$ is a set of matrix units that span $C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$. Hence, the enveloping C^*-algebra of $C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$ must exist. As it is known, there exists only one C^*-algebra, up to isomorphism, generated by a set of matrix units indexed by G/Γ, and that is $\mathcal{K}(\ell^2(G/\Gamma))$. Hence, we necessarily have

$$C_0(G/\Gamma) \times_\alpha G/\Gamma \;\cong\; C_0(G/\Gamma) \times_{\alpha, r} G/\Gamma \;\cong\; \mathcal{K}(\ell^2(G/\Gamma)) \,.$$

It has been shown in [**9**, Example 1.5] that (M, ρ) is a covariant pair, so that equality (7.1) holds. Since the action of G on itself is free, it follows readily from Proposition 3.5.3 that this means that (M, ρ) is a covariant *-representation.

Let us denote by $\phi : C_0(G) \to \mathbb{C}$ the *-representation given by evaluation at the identity element, i.e.

$$\phi(f) := f(e) \,,$$

and let $\widetilde{\phi}$ be its extension to $M(C_0(G)) \cong C_b(G)$. We claim that $\widetilde{\phi}_\alpha$ restricted to $C_c(G/\Gamma)$ is nothing but the representation by multiplication, i.e. $\widetilde{\phi}_\alpha = M$, and this follows from the following computation, where $f \in C_c(G/\Gamma)$:

$$\begin{aligned} \widetilde{\phi}_\alpha(f)\delta_{h\Gamma} &= \widetilde{\phi}(\alpha_h(f))\delta_{h\Gamma} \;=\; \alpha_h(f)\,(e)\,\delta_{h\Gamma} \\ &= f(h\Gamma)\delta_{h\Gamma} \;=\; M(f)\delta_{h\Gamma} \,. \end{aligned}$$

Since $M = \widetilde{\phi}_\alpha$ is faithful, it now follows from Theorem 5.3.6 that $M \times \rho$ is a faithful *-representation of $C_0(G/\Gamma) \times_{\alpha, r} G/\Gamma \cong C_0(G/\Gamma) \times_\alpha G/\Gamma$ in $B(\ell^2(G/\Gamma))$, whose image must necessarily be $\mathcal{K}(\ell^2(G/\Gamma))$. $\square$

As a corollary of our Stone-von-Neumann theorem we recover [**9**, Theorem 1.6] and we show that the covariant pre-*-representations of $C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$ coincide with the covariant pairs of [**9**].

COROLLARY 7.0.4. *Let (G, Γ) be a Hecke pair, $\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ a nondegenerate *-representation and $\mu : \mathcal{H}(G, \Gamma) \to L(\pi(C_c(G/\Gamma))\mathscr{H})$ a unital pre-*-representation. Then (π, μ) is a covariant pre-*-representation if and only if it is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ). In particular we have*

- i) *All covariant pre-*-representation are covariant *-representations, and these are the same as the covariant pairs of* [**9**].
- ii) *A *-representation π is equivalent to an amplification of M if and only if there exists a *-representation μ of $\mathcal{H}(G, \Gamma)$ such that (π, μ) is a covariant *-representation.*

Proof: Let (π, μ) be a covariant pre-*-representation of $C_c(G/\Gamma) \times_\alpha^{alg} G/\Gamma$. Then its integrated form $\pi \times \mu$ extends to a nondegenerate *-representation of $C_0(G/\Gamma) \times_\alpha G/\Gamma$. By Theorem 7.0.3 $M \times \rho$ is a *-isomorphism between $C_0(G/\Gamma) \times_\alpha G/\Gamma$ and $\mathcal{K}(\ell^2(G/\Gamma))$, so that $(\pi \times \mu) \circ (M \times \rho)^{-1}$ is a nondegenerate *-representation of $\mathcal{K}(\ell^2(G/\Gamma))$. Since the algebra of compact operators has a trivial representation theory (see for example [**21**, Lemma B.34]) there exists a Hilbert space $\mathscr{H}$ such that $(\pi \times \mu) \circ (M \times \rho)^{-1}$ is unitarily equivalent to the representation $1 \otimes \mathrm{id}$ in $\mathscr{H} \otimes \ell^2(G/\Gamma)$. Hence, $(\pi \times \mu)$ is unitarily equivalent to $1 \otimes (M \times \rho)$. Now given the fact that (M, ρ) is a covariant *-representation, it is not difficult to see that $(1 \otimes M, 1 \otimes \rho)$ is also a covariant *-representation and moreover

$$1 \otimes (M \times \rho) = (1 \otimes M) \times (1 \otimes \rho)\,.$$

By Proposition 3.3.19 it follows that (π, μ) is unitarily equivalent to $(1 \otimes M, 1 \otimes \rho)$.

The converse is easier: suppose now that (π, μ) is equivalent ot an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ). Since $(1 \otimes M, 1 \otimes \rho)$ is a covariant *-representation, it follows that (π, μ) must also be a covariant *-representation.

Let us now check $i)$. As we have just proven, every covariant pre-*-representation is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ). Since, $(1 \otimes M, 1 \otimes \rho)$ is a covariant *-representation it follows that every covariant pre-*-representation is actually a covariant *-representation.

Let us now prove $ii)$. Suppose $\pi : C_0(G/\Gamma) \to B(\mathscr{H})$ is equivalent to an amplification of M, i.e. there exists a Hilbert space $\mathscr{H}_0$ and a unitary $U : \mathscr{H} \to \mathscr{H}_0 \otimes \ell^2(G/\Gamma)$ such that $\pi = U(1 \otimes M)U^*$. As $(U(1 \otimes M)U^*, U(1 \otimes \rho)U^*)$ is a covariant *-representation, we conclude that there exists a *-representation μ such that (π, μ) is a covariant *-representation. The converse follows easily from what we proved above: if there exists a *-representation μ of $\mathcal{H}(G, \Gamma)$ such that (π, μ) is a covariant *-representation, then (π, μ) is unitarily equivalent to an amplification $(1 \otimes M, 1 \otimes \rho)$ of (M, ρ), and therefore π is unitarily equivalent to an amplification of M. $\square$

CHAPTER 8

Towards Katayama duality

The theory of crossed products by Hecke pairs we have developed is intended for applications in non-abelian crossed product duality. We have already taken the first step in this direction, having established a Stone-von Neumann theorem for Hecke pairs which reflects the results of an Huef, Kaliszewski and Raeburn [**9**]. We believe that this theory of crossed products by Hecke pairs can be further applied and bring insight into the emerging theory of crossed products by coactions of homogeneous spaces ([**5**], [**4**]). The basic idea here is to obtain duality results for "actions" and "coactions" of homogeneous spaces (those coming from Hecke pairs).

In this chapter we will explain how our construction of a crossed product of a Hecke pair seems very suitable for obtaining a form of Katayama duality for homogeneous spaces arising from Hecke pairs, with respect to what we would call the *Echterhoff-Quigg crossed product.* This is work in progress and we have nearly finished a paper where we show this duality. The goal of this chapter is merely to show that our setup is suitable for obtaining such a duality result.

Let δ be a coaction of a discrete group G on a C^*-algebra B and $B \times_\delta G$ the corresponding crossed product. We follow the conventions and notation of [**5**] for coactions and their crossed products. As it is known, there is an action $\widehat{\delta}$ of G on $B \times_\delta G$, called the *dual action*, determined by

$$\widehat{\delta}_s\big(j_B(a)j_G(f)\big) := j_B(a)j_G(\sigma_s(f))\,, \qquad \forall a \in B, f \in C_0(G), s \in G\,,$$

where σ denotes the action of right translation on $C_0(G)$, i.e. $\sigma_s(f)(t) := f(ts)$.

Katayama's duality theorem (the original version comes from [**12**, Theorem 8]) is an analogue for coactions of the duality theorem of Imai and Takai. A general version of it states that we have a canonical isomorphism

$$(B \times_\delta G) \times_{\widehat{\delta},\omega} G \cong B \otimes \mathcal{K}(\ell^2(G))\,, \tag{8.1}$$

for some C^*-completion of the *-algebraic crossed product $(B \times_\delta G) \times_{\widehat{\delta}}^{alg} G$. This C^*-completion $(B \times_\delta G) \times_{\widehat{\delta},\omega} G$ lies in between the full and the reduced crossed products, and the coaction δ is called *maximal* (respectively, *normal*) if this C^*-crossed product is the full (respectively, the reduced) crossed product.

We would like to extend this duality result for coactions of homogeneous spaces G/Γ. In this spirit we should obtain an isomorphism of the type

$$(B \times_\delta G/\Gamma) \times_{\widehat{\delta},\omega} G/\Gamma \;\cong\; B \otimes \mathcal{K}(\ell^2(G/\Gamma))\,. \tag{8.2}$$

Of course, the expression on the left hand side makes no sense unless Γ is normal in G (in which case, the above is just Katayama's result), and there are a few reasons for that. First, it does not make sense in general for a homogeneous space to coact on a C^*-algebra, which consequently makes it difficult to give meaning to $B \times_\delta G/\Gamma$.

Secondly, it also does not make sense in general for a homogeneous space G/Γ to act (namely, by $\widehat{\delta}$) on a C^*-algebra.

The second objection can be overcome by simply using our definition of a crossed product by (an "action" of) a Hecke pair (G, Γ). The first objection can be overcomed because, even though there is no definition of a coaction of a homogeneous space, it is possible to define C^*-algebras $B \times_\delta G/\Gamma$ which can be thought of as crossed products of B by a coaction of G/Γ ([**5**], [**4**]). In this way the iterated crossed product in expression (8.2) may have a true meaning. This is the approach we suggest towards a generalization of Katayama's result.

It is our point of view that such a Katayama duality result can hold when $B \times_\delta G/\Gamma$ is a certain C^*-completion of the algebra $C_c(\mathcal{B} \times G/\Gamma)$ defined by Echterhoff and Quigg in [**5**]. The full completion $C^*(\mathcal{B} \times G/\Gamma)$ has already been dubbed the *Echterhoff and Quigg's crossed product* by the restricted coaction of G/Γ in [**9**] (in case we start with a maximal coaction of G on B).

In the remaining part of this chapter we will show that the Echterhoff and Quigg's algebra $C_c(\mathcal{B} \times G/\Gamma)$ falls in our set up for defining crossed products by Hecke pairs. In other words, we will show that it makes sense to define the crossed product $C_c(\mathcal{B} \times G/\Gamma) \times_{\widehat{\delta}}^{alg} G/\Gamma$ by the dual action $\widehat{\delta}$ of the Hecke pair (G, Γ).

We recall briefly the construction of Echterhoff and Quigg, and the reader is advised to read our Example 2.2.3 again. We start with a coaction δ of a discrete group G on a C^*-algebra B, and we denote by $\mathcal{B}$ its associated Fell bundle. Following [**6**, Section 3] we denote by $\mathcal{B} \times G$ the corresponding Fell bundle over the groupoid $G \times G$. Elements of $\mathcal{B} \times G$ have the form (b_s, t), with $b_s \in \mathcal{B}_s$ and $s, t \in G$. Any such element lies in the fiber $(\mathcal{B} \times G)_{(s,t)}$ over $(s, t) \in G \times G$.

We recall that the multiplication and inversion in $G \times G$ are given by

$$(s, tr)(t, r) = (st, r) \qquad \text{and} \qquad (s, t)^{-1} = (s^{-1}, st)\,,$$

and the corresponding multiplication and involution on $\mathcal{B} \times G$ are given by

$$(b_s, tr)(c_t, r) = ((bc)_{st}, r) \qquad \text{and} \qquad (b_s, t)^{-1} = (b^*_{s^{-1}}, st)\,.$$

An important property of $C_c(\mathcal{B} \times G/\Gamma)$ is that it embeds densely in the coaction crossed product $B \times_\delta G$, by identifying (a_s, t) with $j_B(a) j_G(1_t)$. In this setting we have that $B \times_\delta G \cong C^*(\mathcal{B} \times G) \cong C^*_r(\mathcal{B} \times G)$, as stated in [**6**, Corollary 3.4].

The dual action $\widehat{\delta}$ of G on $B \times_\delta G$ is determined by $\widehat{\delta}_g(j_B(a) j_G(1_t)) = j_B(a) j_G(1_{tg^{-1}})$, which on the generators of $C_c(\mathcal{B} \times G)$ means

$$\widehat{\delta}_g(a_s, t) := (a_s, tg^{-1})\,. \tag{8.3}$$

Now let $H \subseteq G$ be a subgroup. Following [**5**], one can define a Fell bundle $\mathcal{B} \times G/H$ over the groupoid $G \times G/H$. We recall from [**5**] that the operations on the groupoid $G \times G/H$ are defined by

$$(s, trH)(t, rH) = (st, rH) \qquad \text{and} \qquad (s, tH)^{-1} = (s^{-1}, stH)\,,$$

and the corresponding operations on the Fell bundle $\mathcal{B} \times G/H$ are defined by

$$(b_s, trH)(c_t, rH) = ((bc)_{st}, rH) \qquad \text{and} \qquad (b_s, tH)^{-1} = (b_{s^{-1}}, stH)\,.$$

The *Echterhoff and Quigg algebra* is defined as the algebra $C_c(\mathcal{B} \times G/H)$ of finitely supported sections of this Fell bundle.

Let us now consider the case of a Hecke pair (G, Γ) to see that the conditions of our definition of crossed products by Hecke pairs are met, and see that it makes sense to define $C_c(\mathcal{B} \times G/\Gamma) \times^{alg}_{\widehat{\delta}} G/\Gamma$.

For this we take the bundle $\mathcal{A} := \mathcal{B} \times G$ over the groupoid $X := G \times G$, as above. We observe that there is a natural G-action $\widehat{\delta}$ on $\mathcal{A}$ given by (8.3), which of course gives precisely the dual action of G on $C_c(\mathcal{A})$. This action also entails the canonical right action of G on the groupoid X, given by

$$(s,t)g := (s,tg)\,. \tag{8.4}$$

Since this action is free, it is H-good and satisfies the H-intersection property for any subgroup $H \subseteq G$. Moreover, the orbit groupoid X/H is canonically identified with the groupoid $G \times G/H$, simply by $(s,t)H \mapsto (s,tH)$. This canonical identification is easily seen to be a groupoid isomorphism, so that X/H and $G \times G/H$ are "the same" groupoid. Under this identification, the Fell bundle $\mathcal{A}/H$ is just the Fell bundle $\mathcal{B} \times G/H$, and therefore we can canonically identify $C_c(\mathcal{A}/H)$ with $C_c(\mathcal{B} \times G/H)$.

We can now conclude that all our conditions are met and therefore we can define the *-algebraic crossed product $C_c(\mathcal{B} \times G/\Gamma) \times^{alg}_{\widehat{\delta}} G/\Gamma$. We expect that there is a C^*-completion of the Echterhoff and Quigg algebra $C_c(\mathcal{B} \times G/\Gamma)$, which we would like to call *the* Echterhoff and Quigg's crossed product, for which a form of Katayama duality as in (8.2) holds.

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