Model Uncertainty Stochastic Mean-Field Control

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Abstract

We consider the problem of optimal control of a mean-field stochastic differential equation (SDE) under model uncertainty. The model uncertainty is represented by ambiguity about the law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t. For example, it could be the law $\mathcal{L}_{\mathbb{P}}(X(t))$ of $X(t)$ with respect to the given, underlying probability measure $\mathbb{P}$. This is the classical case when there is no model uncertainty. But it could also be the law $\mathcal{L}_{\mathbb{Q}}(X(t))$ with respect to some other probability measure $\mathbb{Q}$ or, more generally, any random measure $\mu(t)$ on $\mathbb{R}$ with total mass 1.

We represent this model uncertainty control problem as a *stochastic differential game* of a mean-field related type SDE with two players. The control of one of the players, representing the uncertainty of the law of the state, is a measure-valued stochastic process $\mu(t)$ and the control of the other player is a classical real-valued stochastic process $u(t)$. This optimal control problem with respect to random probability processes $\mu(t)$ in a non-Markovian setting is a new type of stochastic control problems that has not been studied before. By constructing a new Hilbert space $\mathcal{M}$ of measures, we obtain a sufficient and a necessary maximum principles for Nash equilibria for such games in the general nonzero-sum case, and for saddle points in zero-sum games.

As an application we find an explicit solution of the problem of optimal consumption under model uncertainty of a cash flow described by a mean-field related type SDE.

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1 Introduction

There are many ways of introducing model uncertainty. For example, in recent works of Øksendal and Sulem [17], [16], [15], the underlaying probability measure is not given a priori and there can be a family of possible probability measures to choose from.
The aim of this paper is to study stochastic optimal control under model uncertainty of a mean-field related type SDE driven by Brownian motion and an independent Poisson random measure. The model uncertainty is represented by ambiguity about the law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t. For example, it could be the law $\mathcal{L}_{\mathbb{P}}(X(t))$ of $X(t)$ with respect to the given, underlying probability measure $\mathbb{P}$. This is the classical case when there is no model uncertainty. But it could also be the law $\mathcal{L}_{\mathbb{Q}}(X(t))$ with respect to some other probability measure $\mathbb{Q}$ or, more generally, any random measure $\mu(t)$ on $\mathbb{R}$ with total mass 1.
We represent this model uncertainty control problem as a *stochastic differential game* of a mean-field related type SDE with two players. The control of one of the players, representing the uncertainty of the law of the state, is a measure-valued stochastic process $\mu(t)$, and the control of the other player is a classical real-valued stochastic process $u(t)$. We penalize $\mu(t)$ for being far away from the law $\mathcal{L}_{\mathbb{P}}(X(t))$ with respect to the original probability measure $\mathbb{P}$. This leads to a new type of mean-field stochastic control problems in which the control is random measure-valued stochastic process $\mu(t)$ on $\mathbb{R}$.

To the best of our knowledge this type of problem has not been studied before. By constructing a new Hilbert space $\mathcal{M}$ of measures, we obtain sufficient and necessary maximum principles for Nash equilibria for such games in the general nonzero-sum case, and saddle points for zero-sum games. As an application we find an explicit solution of the problem of optimal consumption under model uncertainty of a cash flow described by a mean-field related type SDE.

Mean-field games problems were first studied by Lasry and Lions [12] and Lions in [13] has proved the differentiability of functions of measures defined on a Wasserstein metric space $\mathcal{P}_2$ by using the lifting technics. Since then this type of problems has gained a lot attention, we can for example refer to Carmona *et al* [8], [7], Buckdahn *et al* [6], Bensoussan *et al* [4], Bayraktar *et al* [3], Corso and Pham [10], Djehiche and Hamadene [11], Pham and Wei [18] and Agram [1].

2 A weighted Sobolev space of random measures

In this section, we as in Agram and Øksendal [2] construct a Hilbert space $\mathcal{M}$ of random measures on $\mathbb{R}$. It is simpler to work with than the Wasserstein metric space that has been used by many authors previously.

Definition 1 *(Weighted Sobolev spaces of measures) For $k = 0, 1, 2, ...$ let $\tilde{\mathcal{M}}^{(k)}$ denote the set of random measures μ on $\mathbb{R}$ such that*

$$\mathbb{E}[\textstyle\int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy] < \infty, \tag{1}$$

where

$$\hat{\mu}(y) = \textstyle\int_{\mathbb{R}} e^{ixy} d\mu(x) \tag{2}$$

is the Fourier transform of the measure μ. If $\mu, \eta \in \tilde{\mathcal{M}}^{(k)}$ we define the inner product $\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}}$ by

$$\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}} = \mathbb{E}[\textstyle\int_{\mathbb{R}} \mathrm{Re}(\overline{\hat{\mu}}(y)\hat{\eta}(y)) |y|^k e^{-y^2} dy], \tag{3}$$

where, in general, $\mathrm{Re}(z)$ denotes the real part of the complex number z, and $\bar{z}$ denotes the complex conjugate of z. The norm $||\cdot||_{\tilde{\mathcal{M}}^{(k)}}$ associated to this inner product is given by

$$\|\mu\|^2_{\tilde{\mathcal{M}}^{(k)}} = \langle \mu, \mu \rangle_{\tilde{\mathcal{M}}^{(k)}} = \mathbb{E}[\textstyle\int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy]. \tag{4}$$

The space $\tilde{\mathcal{M}}^{(k)}$ equipped with the inner product $\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}}$ is a pre-Hilbert space. We let $\mathcal{M}^{(k)}$ denote the completion of this pre-Hilbert space. We denote by $\mathcal{M}_0^{(k)}$ the set of all deterministic elements of $\mathcal{M}^{(k)}$. For $k = 0$ we write $\mathcal{M}^{(0)} = \mathcal{M}$ and $\mathcal{M}_0^{(0)} = \mathcal{M}_0$.

There are several advantages with working with this Hilbert space $\mathcal{M}$, compared to the Wasserstein metric space:

- Our space of measures is easier to work with.
- A Hilbert space has a useful stronger structure than a metric space.
- The Wasserstein metric space $\mathcal{P}_2$ deals only with probability measures with finite second moment, while our Hilbert space deals with any (random) measure satisfying (1).
- With this norm we have the following useful estimate:

  Lemma 2 *Let $X^{(1)}$ and $X^{(2)}$ be two random variables in $L^2(\mathbb{P})$. Then*

  $$\left\|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\right\|^2_{\mathcal{M}_0} \leq \sqrt{\pi}\mathbb{E}[(X^{(1)} - X^{(2)})^2].$$

  We refer to [2] for a proof.

Let us give some examples of measures:

Example 3 (Measures)

1. *Suppose that* $\mu = \delta_{x_0}$*, the unit point mass at* $x_0 \in \mathbb{R}$*. Then* $\delta_{x_0} \in \mathcal{M}_0$ *and*

$$\int_{\mathbb{R}} e^{ixy} d\mu(x) = e^{ix_0 y},$$

and hence

$$\|\mu\|^2_{\mathcal{M}_0} = \int_{\mathbb{R}} |e^{ix_0 y}|^2 e^{-y^2} dy < \infty.$$

2. *Suppose* $d\mu(x) = f(x)dx$*, where* $f \in L^1(\mathbb{R})$*. Then* $\mu \in \mathcal{M}_0$ *and by Riemann-Lebesque lemma,* $\hat{\mu}(y) \in C_0(\mathbb{R})$*, i.e.* $\hat{\mu}$ *is continuous and* $\hat{\mu}(y) \to 0$ *when* $|y| \to \infty$*. In particular,* $|\hat{\mu}|$ *is bounded on* $\mathbb{R}$ *and hence*

$$\|\mu\|^2_{\mathcal{M}_0} = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty.$$

3. *Suppose that* μ *is any finite positive measure on* $\mathbb{R}$*. Then* $\mu \in \mathcal{M}_0^{(k)}$ *for all* k*, because*

$$|\hat{\mu}(y)| \leq \int_{\mathbb{R}} d\mu(y) = \mu(\mathbb{R}) < \infty, \text{ for all } y,$$

and hence

$$\|\mu\|^2_{\mathcal{M}_0^{(k)}} = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy \leq \mu^2(\mathbb{R}) \int_{\mathbb{R}} |y|^k e^{-y^2} dy < \infty.$$

4. *Next, suppose* $x_0 = x_0(\omega)$ *is random. Then* $\delta_{x_0(\omega)}$ *is a random measure in* $\mathcal{M}$*. Similarly, if* $f(x) = f(x,\omega)$ *is random, then* $d\mu(x,\omega) = f(x,\omega)dx$ *is a random measure in* $\mathcal{M}$*.*

2.1 t-absolute continuity and t-derivative of the law process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by a one-dimensional Brownian motion B and an independent Poisson random measure $N(dt, d\zeta)$. Let $\nu(d\zeta)dt$ denote the Lévy measure of N, and let $\tilde{N}(dt, d\zeta)$ denote the compensated Poisson random measure $N(dt, d\zeta) - \nu(d\zeta)dt$.
Suppose that $X(t) = X_t$ is an Itô-Lévy process of the form

$$\begin{cases} dX_t = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,\zeta)\tilde{N}(dt, d\zeta); \quad t \in [0,T], \\ X_0 = x \in \mathbb{R}, \end{cases} \tag{5}$$

where α, β and γ are bounded predictable processes.
Let $\varphi \in C^2$. Then under appropriate conditions on the coefficients, we get by the Itô formula

$$\mathbb{E}[\varphi(X_{t+h})] - \mathbb{E}[\varphi(X_t)] = \mathbb{E}[\int_t^{t+h} A\varphi(X_s)ds], \tag{6}$$

where

$$\begin{array}{rcl} A\varphi(X_s) & = & \alpha(s)\varphi'(X_s) + \frac{1}{2}\beta^2(s)\varphi''(X_s) \\ & & + \int_{\mathbb{R}_0} \{\varphi(X_s + \gamma(s,\zeta)) - \varphi(X_s) - \varphi'(X_s)\gamma(s,\zeta)\}\nu(d\zeta). \end{array}$$

In particular, if

$$\varphi(x) = \varphi_y(x) := \exp(ixy); \quad y \in \mathbb{R},$$

then

$$\begin{array}{rl} A\varphi_y(X_s) & = (iy\alpha(s) - \frac{1}{2}\beta^2(s)y^2 \\ & \quad + \int_{\mathbb{R}_0} \{\exp(i\gamma(s,\zeta)y) - 1 - iy\gamma(s,\zeta)\}\, \nu(d\zeta))\varphi_y(X_s), \end{array}$$

for all $y \in \mathbb{R}$.

Definition 4 (Law process) *From now on we use the notation*

$$M_t := M(t) := \mathcal{L}(X_t); \quad 0 \le t \le T$$

for the law process $\mathcal{L}(X_t)$ of $X_t = X(t)$ with respect to $\mathbb{P}$.

Lemma 5 *(i) The map $t \mapsto M_t : [0,T] \to \mathcal{M}_0$ is absolutely continuous, and the derivative*

$$M'(t) := \frac{d}{dt} M(t)$$

exists for all t.

(ii) There exists a constant $C < \infty$ such that

$$||M'(t)||_{\mathcal{M}_0} \le C||M(t)||_{\mathcal{M}_0^{(4)}} \text{ for all } t \in [0,T]; M(t) \in \mathcal{M}_0^{(4)}. \tag{7}$$

Proof. (i) Let $0 \le t < t+h \le T$. Then by (2) and (4) we get

$$\begin{array}{l} \|M_{t+h} - M_t\|^2_{\mathcal{M}_0} = \int_{\mathbb{R}} |\hat{M}_{t+h}(y) - \hat{M}_t(y)|^2 e^{-y^2} dy \\ = \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_{t+h}) - \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_t)(x)|^2 e^{-y^2} dy \\ = \int_{\mathbb{R}} |\mathbb{E}[\varphi_y(X_{t+h})] - \mathbb{E}[\varphi_y(X_t)]|^2 e^{-y^2} dy. \end{array} \tag{8}$$

The last equality holds by using that for any bounded function ψ we have

$$\mathbb{E}[\psi(X)] \;\; = \;\; \int_{\mathbb{R}} \psi(x) d\mathcal{L}(X)(x).$$

By (6), we obtain

$$\begin{array}{rl} \|M_{t+h} - M_t\|^2_{\mathcal{M}_0} & = \int_{\mathbb{R}} |\mathbb{E}[\int_t^{t+h} A\varphi_y(X(s))ds]|^2 e^{-y^2} dy \\ & \le \int_{\mathbb{R}} (\int_t^{t+h} \mathbb{E}[|A\varphi_y(X_s)|]ds)^2 e^{-y^2} dy \le C_1\ h^2, \end{array} \tag{9}$$

for some constant C_1 which does not depend on t and h.
We have proved that for different t and $t+h$, $\|M_{t+h} - M_t\|^2_{\mathcal{M}_0} \le C\ h^2$ and it is easy to

see that this holds for every finite disjoint partition of the interval $[0,T]$. Thus we get that $t \mapsto M(t)$ is absolutely continuous, and the derivative $M'(t) = \frac{d}{dt}M(t)$ exists for all t.

(ii) This follows from (9), using that the coefficients α, β, γ are bounded and that

$$\mathbb{E}[|A_{\varphi_y}(X_s)|] \leq const.y^2|\mathbb{E}[\exp(iyX_s)]| \leq const.y^2|\widehat{M_s}(y)|. \tag{10}$$

. □

From the lemma above we conclude the following:

Lemma 6 *If X_t is an Itô-Lévy process as in (5), then the derivative $M'_s := \frac{d}{ds}M_s$ exists in $\mathcal{M}_0$ for a.a. s, and we have*

$$M_t = M_0 + \int_0^t M'_s ds; \quad t \geq 0.$$

In the following we will apply this to the solutions $X(t)$ of the mean-field related type SDEs we consider below.

Example 7

(a) *Suppose that $X(t) = B(t)$ with $B(0) = 0$. Then*

$$d\mathcal{L}(X(t))(x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})dx,$$

i.e. $\mathcal{L}(X(t))$ has a density $\frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})$. Therefore $\frac{d}{dt}\mathcal{L}(X(t))$ is a measure with density

$$\frac{d}{dt}\frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t}) = (\frac{x^2-t}{2t^2})(\frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})).$$

(b) *Suppose $X(t) = N(t)$, a Poisson process with intensity $\bar{\lambda}$. Then for $k = 1, 2, ...$ we have*

$$\mathbb{P}(N(t) = k) = \frac{e^{-\bar{\lambda}t}(\bar{\lambda}t)^k}{k!}$$

and hence

$$\frac{d}{dt}\mathbb{P}(N(t) = k) = \frac{1}{k!}(\bar{\lambda}e^{-\bar{\lambda}t}(\lambda t)^{k-1}\{k - \bar{\lambda}t\}).$$

3 Preliminaries

We will recall some concepts and spaces which will be used on the sequel.
The probability $\mathbb{P}$ is a reference probability measure. We introduce two smaller filtrations $\mathbb{G}^{(i)}=(\mathcal{G}_t^{(i)})_{t\geq 0}$ such that $\mathcal{G}_t^{(i)} \subseteq \mathcal{F}_t$, for $i = 1, 2$ and for all $t \geq 0$. These filtrations represent the information available to player number i at time t.

3.1 Some basic concepts from Banach space theory

Since we deal with measures defined on an Hilbert space $\mathcal{M}$, we need the Fréchet derivative to differentiate functions of measures. Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$, respectively, and let $F : \mathcal{X} \to \mathcal{Y}$.

- We say that F has a directional derivative (or Gâteaux derivative) at $v \in \mathcal{X}$ in the direction $w \in \mathcal{X}$ if
$$D_w F(v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(v + \varepsilon w) - F(v))$$
exists in $\mathcal{Y}$.

- We say that F is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \to \mathcal{Y}$ such that
$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|_{\mathcal{X}}} \|F(v + h) - F(v) - A(h)\|_{\mathcal{Y}} = 0.$$
In this case we call A the *gradient* (or Fréchet derivative) of F at v and we write
$$A = \nabla_v F.$$

- If F is Fréchet differentiable at v with Fréchet derivative $\nabla_v F$, then F has a directional derivative in all directions $w \in \mathcal{X}$ and
$$D_w F(v) := \langle \nabla_v F, w \rangle = \nabla_v F(w) = \nabla_v F w.$$

In particular, note that if F is a linear operator, then $\nabla_v F = F$ for all v.

3.2 Spaces

Throughout this work, we will use the following spaces:

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted càdlàg processes $(X(t))_{t \in [0,T]}$ such that
$$\|X\|^2_{\mathcal{S}^2} := \mathbb{E}[\sup_{t \in [0,T]} |X(t)|^2] \ < \ \infty \ ,$$

- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-predictable processes $(Q(t))_{t \in [0,T]}$ such that
$$\|Q\|^2_{\mathbb{L}^2} := \mathbb{E}[\textstyle\int_0^T |Q(t)|^2 dt] < \ \infty \ .$$

- $L^2(\mathcal{F}_t)$ is the set of $\mathbb{R}$-valued square integrable $\mathcal{F}_t$-measurable random variables.

- $\mathbb{L}_\nu^2$ is the set of $\mathbb{F}$-predictable processes $R : [0,T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that

$$||R||^2_{\mathbb{L}_\nu^2} := \mathbb{E}[\textstyle\int_{\mathbb{R}_0} |R(t,\zeta)|^2 \nu(d\zeta)dt] \ < \ \infty \ .$$

- In general, for any given filtration $\mathbb{H}$, we say that the measure-valued process $\mu(t) = \mu(t,\omega) : [0,T] \times \Omega \to \mathcal{M}$ is adapted to $\mathbb{H}$ if $\mu(t)(V)$ is $\mathbb{H}$-adapted for all Borel sets $V \subseteq \mathbb{R}$. Let $\mathbb{M}_{\mathbb{G}} = \mathbb{M}_{\mathbb{G}^1}$ be a given set of $\mathcal{M}$-valued, $\mathbb{G}^1 = (\mathcal{G}_t^1)_{t \geq 0}$-predictable, stochastic processes $\mu(t)$. We call $\mathbb{M}_{\mathbb{G}}$ the set of admissible measure-valued control processes $\mu(\cdot)$.

- $\mathbb{M}_0$ is the set of t-differentiable $\mathcal{M}_0$-valued processes $m(t); t \in [0,T]$.
If $m \in \mathbb{M}_0$ we put $m'(t) = \frac{d}{dt} m(t)$.

- Let $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}^2}$ be a given set of real-valued, $\mathbb{G}^2 = (\mathcal{G}_t^2)_{t \geq 0}$-predictable, stochastic processes $u(t)$ required to have values in a given convex subset $\mathcal{U}$ of $\mathbb{R}$. We call $\mathcal{A}_{\mathbb{G}}$ the set of admissible real-valued control processes $u(\cdot)$.

- $\mathcal{R}$ is the set of measurable functions $r : \mathbb{R}_0 \to \mathbb{R}$.

- $C_a([0,T], \mathcal{M}_0)$ denotes the set of absolutely continuous functions $m : [0,T] \to \mathcal{M}_0$.

- $\mathbb{K}$ is the set of bounded linear functionals $K : \mathcal{M}_0 \to \mathbb{R}$ equipped with the operator norm

$$||K||_{\mathbb{K}} := \sup_{m \in \mathcal{M}_0, ||m||_{\mathcal{M}_0} \leq 1} |K(m)|.$$

- $\mathcal{S}_{\mathbb{K}}^2$ is the set of $\mathbb{F}$-adapted càdlàg processes $p : [0,T] \times \Omega \mapsto \mathbb{K}$ such that

$$||p||^2_{\mathcal{S}_{\mathbb{K}}} := \mathbb{E}[\sup_{t \in [0,T]} ||p(t)||^2_{\mathbb{K}}] < \infty.$$

- $\mathbb{L}_{\mathbb{K}}^2$ is the set of $\mathbb{F}$-predictable processes $q : [0,T] \times \Omega \mapsto \mathbb{K}$ such that

$$||q||^2_{\mathbb{L}^2_{\mathbb{K}}} := \mathbb{E}[\textstyle\int_0^T ||q(t)||^2_{\mathbb{K}} dt] < \infty.$$

- $\mathbb{L}_{\nu,\mathbb{K}}^2$ is the set of $\mathbb{F}$-predictable processes $r : [0,T] \times \mathbb{R}_0 \times \Omega \mapsto \mathbb{K}$ such that

$$||r||^2_{\mathbb{L}^2_{\nu,\mathbb{K}}} := \mathbb{E}[\textstyle\int_0^T \int_{\mathbb{R}_0} ||r(t,\zeta)||^2_{\mathbb{K}} \nu(d\zeta)dt] < \infty.$$

4 The model uncertainty stochastic optimal control problem

As pointed out in the Introduction, there are several ways to represent model uncertainty in a stochastic system. In this paper, we are interested in systems governed by controlled mean-field related type SDE $X^{\mu,u}(t) = X(t) \in \mathcal{S}^2$ on the form

$$\begin{cases} dX(t) &= b\,(t, X(t), \mu(t), u(t))\,dt + \sigma\,(t, X(t), \mu(t), u(t))\,dB(t) \\ & \quad +\int_{\mathbb{R}_0} \gamma\,(t, X(t), \mu(t), u(t), \zeta)\,\tilde{N}(dt, d\zeta);\; t \in [0, T]\,, \\ X\,(0) &= x \in \mathbb{R}. \end{cases} \tag{11}$$

The functions

$$\begin{array}{llll} b(t, x, \mu, u) & = b(t, x, \mu, u, \omega) & : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \Omega & \to \mathbb{R}, \\ \sigma(t, x, \mu, u) & = \sigma(t, x, \mu, u, \omega) & : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \Omega & \to \mathbb{R}, \\ \gamma(t, x, \mu, u, \zeta) & = \gamma(t, x, \mu, u, \zeta, \omega) & : [0, T] \times \mathbb{R} \times \mathcal{M} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega & \to \mathbb{R}, \end{array}$$

are supposed to be Lipschitz on $x \in \mathbb{R}$, uniformly with respect to t and ω for given $u \in \mathcal{U}$ and $\mu \in \mathcal{M}$. Then by e.g. Theorem 1.19 in Øksendal and Sulem [14], we have existence and uniqueness of the solution of $X(t)$. We may regard (11) as a perturbed version of the mean-field equation

$$\begin{cases} dX(t) &= b\,(t, X(t), \mathcal{L}(X(t)), u(t))\,dt + \sigma\,(t, X(t), \mathcal{L}(X(t)), u(t))\,dB(t) \\ & \quad +\int_{\mathbb{R}_0} \gamma\,(t, X(t), \mathcal{L}(X(t)), u(t), \zeta)\,\tilde{N}(dt, d\zeta);\; t \in [0, T]\,, \\ X\,(0) &= x \in \mathbb{R}. \end{cases} \tag{12}$$

For example, we could have $\mu(t) = \mathcal{L}_{\mathbb{Q}}(X(t))$ for some probability measure $\mathbb{Q} \neq \mathbb{P}$.
Thus the model uncertainty is represented by an uncertainty about what law $\mu(t)$ is influencing the coefficients of the system, and we are penalising the laws that are far away from $\mathcal{L}(X(t))$. See the application in Section 5.

Let us consider a performance functional of the form

$$J\,(\mu, u) \;=\; \mathbb{E}[g\,(X(T), M(T)) + \int_0^T \ell\,(s, X\,(s)\,, M(s), \mu(s), u\,(s))\,ds], \tag{13}$$

where $\ell(t, x, m, \mu, u) = \ell(t, x, m, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{M}_0 \times \Omega \to \mathbb{R}$ are given functions.
For fixed x, m, μ, u we assume that $\ell\,(s, \cdot)$ is $\mathcal{F}_s$-measurable for all $s \in [0, T]$ and $g(\cdot, \cdot)$ is $\mathcal{F}_T$-measurable. We also assume the following integrability condition

$$\mathbb{E}[\,|g\,(X(T), M(T))|^2 + \int_0^T |\ell\,(s, X\,(s)\,, M(s), \mu(s), u\,(s))|^2\,ds] < \infty,$$

for all $\mu \in \mathbb{M}_{\mathbb{G}}$ and $u \in \mathcal{A}_{\mathbb{G}}$.
Note that the system (11) and the performance (13) are not Markovian. However, recently a dynamic programming approaches to mean-field stochastic control problems have been introduced. See e.g. Bayraktar *et al* [3] and Pham and Wei [18]. In this paper we will use an approach based on a suitably modified stochastic maximum principle, which also works in partial information settings.

In the next section we study a stochastic differential game of two players, where one of the players is solving an optimal measure-valued control problem of the type described above, while the other player is solving a classical real-valued stochastic control problem. To the best of our knowledge this type of stochastic differential game has not been studied before.

4.1 Nonzero-sum games

We now proceed to a nonzero-sum maximum principle.
We consider the $\mathbb{R} \times \mathcal{M}_0$-valued process $(X(t), M(t))$ where $M(t) = \mathcal{L}(X(t))$, where $X(t)$ is given by (11) and

$$dM(t) = \beta(M(t))dt; \quad M(0) \in \mathcal{M}_0 \text{ given }, \tag{14}$$

where β is the operator on $\mathbb{M}_0$ given by

$$\beta(m(t)) = m'(t). \tag{15}$$

The cost functionals are assumed to be on the form

$$\begin{aligned} J_i(\mu, u) &= \mathbb{E}[g_i(X(T), M(T)) \\ &\quad + \textstyle\int_0^T \ell_i(s, X(s), M(s), \mu(s), u(s))\, ds]; \text{ for } i = 1, 2, \end{aligned} \tag{16}$$

where $M(s) := \mathcal{L}(X(s))$ and the functions

$$\begin{array}{llll} \ell_i(t, x, m, \mu, u) & = \ell_i(t, x, m, \mu, u, \omega) & : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega & \to \mathbb{R}, \\ g_i(x, m) & = g_i(x, m, \omega) & : \mathbb{R} \times \mathcal{M}_0 \times \Omega & \to \mathbb{R}, \end{array}$$

are continuously differentiable with respect to x, u and admit Fréchet derivatives with respect to m and μ.

Problem 8 *We consider the general nonzero-sum stochastic game to find $(\mu^*, u^*) \in \mathbb{M}_\mathbb{G} \times \mathcal{A}_\mathbb{G}$ such that*

$$\begin{array}{ll} J_1(\mu, u^*) \leq J_1(\mu^*, u^*), & \textit{for all } \mu \in \mathbb{M}_\mathbb{G}, \\ J_2(\mu^*, u) \leq J_2(\mu^*, u^*), & \textit{for all } u \in \mathcal{A}_\mathbb{G}. \end{array}$$

The pair (μ^*, u^*) is called a Nash equilibrium.

Definition 9 ***(The Hamiltonian)*** *For $i = 1, 2$ we define the Hamiltonian*

$$H_i : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0, T], \mathcal{M}_0) \to \mathbb{R}$$

by

$$\begin{aligned} H_i(t, x, m, \mu, u, p_i^0, q_i^0, r_i^0(\cdot), p_i^1) &= \ell_i(t, x, m, \mu, u) + p_i^0 b(t, x, \mu, u) + q_i^0 \sigma(t, x, \mu, u) \\ &\quad + \textstyle\int_{\mathbb{R}_0} r_i^0(\zeta) \gamma(t, x, \mu, u, \zeta)\, \nu(d\zeta) + \langle p_i^1, \beta(m) \rangle . \end{aligned} \tag{17}$$

We assume that H_i is continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.
For $u \in \mathcal{A}_\mathbb{G}, \mu \in \mathbb{M}_\mathbb{G}$ with corresponding solution $X = X^{\mu,u}$, define $p_i^0 = p_i^{0,\mu,u}, q_i^0 = q_i^{0,\mu,u}$ and $r_i^0 = r_i^{0,\mu,u}$ and $p_i^1 = p_i^{1,\mu,u}, q_i^1 = q_i^{1,\mu,u}$ and $r_i^1 = r_i^{1,\mu,u}$ for $i = 1, 2$ by the following set of adjoint equations:

- The real-valued BSDE in the unknown $(p_i^0, q_i^0, r_i^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$ is given by

$$\begin{cases} dp_i^0(t) & = & -\frac{\partial H_i}{\partial x}(t)dt + q_i^0(t)dB(t) + \int_{\mathbb{R}_0} r_i^0(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p_i^0(T) & = & \frac{\partial g_i}{\partial x}(X(T), M(T)), \end{cases} \tag{18}$$

- and the operator-valued BSDE in the unknown $(p_i^1, q_i^1, r_i^1) \in \mathcal{S}_{\mathbb{K}}^2 \times \mathbb{L}_{\mathbb{K}}^2 \times \mathbb{L}_{\nu,\mathbb{K}}^2$ is given by

$$\begin{cases} dp_i^1(t) & = & -\nabla_m H_i(t)dt + q_i^1(t)dB(t) + \int_{\mathbb{R}_0} r_i^1(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p^1(T) & = & \nabla_m g_i(X(T), M(T)), \end{cases} \tag{19}$$

where $H_i(t) = H_i(t, X(t), M(t), \mu(t), u(t), p_i^0(t), q_i^0(t), r_i^0(t,\cdot), p_i^1(t))$ etc.
We remark that the BSDEs (18) is linear, so whenever knowing the Hamiltonian H_i and the function g_i, we can get a solution explicitly. To remind the reader of this solution formula, let us consider the solution $(P,Q,R) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$ of the linear BSDE

$$\begin{cases} dP(t) & = -[\varphi(t) + \alpha(t)P(t) + \beta(t)Q(t) + \int_{\mathbb{R}_0} \phi(t,\zeta)R(t,\zeta)\nu(d\zeta)]dt \\ & \qquad + Q(t)dB(t) + \int_{\mathbb{R}_0} R(t,\zeta)\tilde{N}(dt,d\zeta); \ t \in [0,T], \\ P(T) & = \theta \in L^2(\mathcal{F}_T). \end{cases} \tag{20}$$

Here φ, α, β and ϕ are bounded predictable processes with ϕ is assumed to be an $\mathbb{R}$-valued process defined on $[0,T] \times \mathbb{R}_0 \times \Omega$. Then it is well-known (see e.g. Theorem 1.7 in Øksendal and Sulem [15]) that the component $P(t)$ of the solution of equation (20) can be written in closed form as follows:

$$P(t) \ = \mathbb{E}[\theta \tfrac{\Gamma(T)}{\Gamma(t)} + \int_t^T \tfrac{\Gamma(s)}{\Gamma(t)} \varphi(s) | \mathcal{F}_t]; \quad t \in [0,T], \tag{21}$$

where $\Gamma(t) \in \mathcal{S}^2$ is the solution of the linear SDE with jumps

$$\begin{cases} d\Gamma(t) & = & \Gamma(t^-)[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \phi(t,\zeta)\tilde{N}(dt,d\zeta)]; \ t \in [0,T], \\ \Gamma(0) & = & 1. \end{cases} \tag{22}$$

For notational convenience, we will employ the following short hand notations

$$\begin{aligned} \hat{H}_1(t) &= H_1(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_1^0(t), \hat{q}_1^0(t), \hat{r}_1^0(t,\cdot), \hat{p}_1^1(t)), \\ \check{H}_1(t) &= H_1(t, \hat{X}(t), \hat{M}(t), \mu(t), \hat{u}(t), \hat{p}_1^0(t), \hat{q}_1^0(t), \hat{r}_1^0(t,\cdot), \hat{p}_1^1(t)), \\ \bar{H}_2(t) &= H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_2^0(t), \hat{q}_2^0(t), \hat{r}_2^0(t,\cdot), \hat{p}_2^1(t)), \\ \breve{H}_2(t) &= H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), u(t), \hat{p}_2^0(t), \hat{q}_2^0(t), \hat{r}_2^0(t,\cdot), \hat{p}_2^1(t)). \end{aligned}$$

Similar notation is used for the derivatives of $H, \ell, g, b, \sigma, \gamma$ etc.
We now state a sufficient theorem for the nonzero-sum games.

Theorem 10 (Sufficient nonzero-sum maximum principle) *Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions $\hat{X}$, (p_i^0, q_i^0, r_i^0) and (p_i^1, q_i^1, r_i^1) of the forward and backward stochastic differential equations* (11)*,* (18) *and* (19) *respectively. Suppose that*

1. *(Concavity) The functions*

$$\begin{array}{ll} (x,m,\mu) & \mapsto H_1(t), \\ (x,m,u) & \mapsto H_2(t), \\ (x,m) & \mapsto g_i(x,m), \text{ for } i=1,2, \end{array}$$

are concave $\mathbb{P}.a.s$ *for each* $t \in [0,T]$.

2. *(Maximum conditions)*

$$\mathbb{E}[\hat{H}_1(t)|\mathcal{G}_t^{(1)}] = ess \sup_{\mu \in \mathbb{M}_{\mathbb{G}}} \mathbb{E}[\check{H}_1(t)|\mathcal{G}_t^{(1)}], \tag{23}$$

and

$$\mathbb{E}[\bar{H}_2(t)|\mathcal{G}_t^{(2)}] = ess \sup_{u \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\check{H}_2(t)|\mathcal{G}_t^{(2)}],$$

$\mathbb{P}.a.s$ *for each* $t \in [0,T]$.
Then $(\hat{\mu}, \hat{u})$ *is a Nash equilibrium for our problem.*

Proof. Let us first prove that $J_1(\mu, \hat{u}) \leq J_1(\hat{\mu}, \hat{u})$.
By the definition of the cost functional (16) we have for fixed $\hat{u} \in \mathcal{A}_{\mathbb{G}}$ and arbitrary $\mu \in \mathbb{M}_{\mathbb{G}}$

$$J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) = I_1 + I_2, \tag{24}$$

where

$$\begin{aligned} I_1 &= \mathbb{E}[\textstyle\int_0^T \{\check{\ell}_1(t) - \hat{\ell}_1(t)\} dt], \\ I_2 &= \mathbb{E}[\check{g}_1(X(T), M(T)) - \hat{g}_1(\hat{X}(T), \hat{M}(T))]. \end{aligned}$$

By the definition of the Hamiltonian (17) we have

$$I_1 = \mathbb{E}[\textstyle\int_0^T \check{H}_1(t) - \hat{H}_1(t) - \hat{p}_1^0(t)\tilde{b}(t) - \hat{q}_1^0(t)\tilde{\sigma}(t) - \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta) - \langle \hat{p}_1^1(t), \tilde{M}'(t)\rangle dt], \tag{25}$$

where $\tilde{b}(t) = \check{b}(t) - \hat{b}(t)$ etc. By the concavity of g_1 and the terminal values of the BSDEs (18), (19), we have

$$I_2 \leq \mathbb{E}[\tfrac{\partial g_1}{\partial x}(T)\tilde{X}(T) + \langle \nabla_m g_1(T), \tilde{M}(T)\rangle] = \mathbb{E}[\hat{p}_1^0(T)\tilde{X}(T) + \langle \hat{p}_1^1(T), \tilde{M}(T)\rangle].$$

Applying the Itô formula to $\hat{p}_1^0(t)\tilde{X}(t)$ and $\langle \hat{p}_1^1(t), \tilde{M}(t)\rangle$, we get

$$\begin{aligned} I_2 &\leq \mathbb{E}[\hat{p}_1^0(T)\tilde{X}(T) + \langle \hat{p}_1^1(T), \tilde{M}(T)\rangle] \\ &= \mathbb{E}[\textstyle\int_0^T \hat{p}_1^0(t) d\tilde{X}(t) + \int_0^T \tilde{X}(t) d\hat{p}_1^0(t) + \int_0^T \hat{q}_1^0(t)\tilde{\sigma}(t) dt + \int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta) dt] \\ &+ \mathbb{E}[\textstyle\int_0^T \langle \hat{p}_1^1(t), d\tilde{M}(t)\rangle + \int_0^T \tilde{M}(t) d\hat{p}_1^1(t)] \\ &= \mathbb{E}[\textstyle\int_0^T \hat{p}_1^0(t)\tilde{b}(t) dt - \int_0^T \frac{\partial \hat{H}_1}{\partial x}(t)\tilde{X}(t) dt + \int_0^T \hat{q}_1^0(t)\tilde{\sigma}(t) dt \\ &+ \textstyle\int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta) dt + \int_0^T \langle \hat{p}_1^1(t), \tilde{M}'\rangle dt \\ &- \textstyle\int_0^T \langle \nabla_m \hat{H}_1(t), \tilde{M}(t)\rangle dt], \end{aligned} \tag{26}$$

where we have used that the $dB(t)$ and $\tilde{N}(dt, d\zeta)$ integrals with the necessary integrability property are martingales and then have mean zero. Substituting (25) and (26) in (24), yields

$$\begin{aligned}&J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u})\\ &\leq \mathbb{E}[\int_0^T \{\check{H}_1(t) - \hat{H}_1(t) - \tfrac{\partial \hat{H}_1}{\partial x}(t)\tilde{X}(t) - \langle \nabla_m \hat{H}_1(t), \tilde{M}(t)\rangle\} dt].\end{aligned}$$

By the concavity of H_1 and the fact that the process μ is $\mathcal{G}_t^{(1)}$-adapted, we obtain

$$\begin{aligned}J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) &\leq \mathbb{E}[\int_0^T \tfrac{\partial \hat{H}_1}{\partial \mu}(t)\,(\mu(t) - \hat{\mu}(t))\,dt]\\ &= \mathbb{E}[\int_0^T \mathbb{E}(\tfrac{\partial \hat{H}_1}{\partial \mu}(t)\,(\mu(t) - \hat{\mu}(t))\,|\mathcal{G}_t^{(1)})dt]\\ &= \mathbb{E}[\int_0^T \mathbb{E}(\tfrac{\partial \hat{H}_1}{\partial \mu}(t)|\mathcal{G}_t^{(1)})\,(\mu(t) - \hat{\mu}(t))\,dt]\\ &\leq 0,\end{aligned}$$

where $\frac{\partial \hat{H}_1}{\partial \mu} = \nabla_\mu \hat{H}_1$. The last equality holds because of the maximum condition of $\hat{H}_1$ at $\mu = \hat{\mu}$.

Similar considerations apply to prove that $J_2(\hat{\mu}, u) \leq J_2(\hat{\mu}, \hat{u})$. For the sake of completeness, we give details in the Appendix. □

We now state and prove a necessary version of the maximum principle. We assume the following:

- Whenever $\mu \in \mathbb{M}_{\mathbb{G}}$ ($u \in \mathcal{A}_{\mathbb{G}}$) and $\eta \in \mathbb{M}_{\mathbb{G}}$ ($\pi \in \mathcal{A}_{\mathbb{G}}$) are bounded, there exists $\epsilon > 0$ such that
$$\mu + \lambda\eta \in \mathbb{M}_{\mathbb{G}}\ (u + \lambda\pi \in \mathcal{A}_{\mathbb{G}}),\ \text{for each}\ \lambda \in [-\epsilon, \epsilon]\,.$$
- For each $t_0 \in [0, T]$ and each bounded $\mathcal{G}_{t_0}^{(1)}$-measurable random measure α_1 and $\mathcal{G}_{t_0}^{(2)}$-measurable random variable α_2, the process
$$\eta\,(t) = \alpha_1 \mathbf{1}_{[t_0,T]}(t) \tag{27}$$
belongs to $\mathbb{M}_{\mathbb{G}}$ and the process
$$\pi\,(t) = \alpha_2 \mathbf{1}_{[t_0,T]}(t)$$
belongs to $\mathcal{A}_{\mathbb{G}}$.

Definition 11 *In general, if $K^u(t)$ is a process depending on u, we define the differential operator D on K by*

$$DK^u(t) := D^\pi K^u(t) = \tfrac{d}{d\lambda} K^{u+\lambda\pi}(t)|_{\lambda=0}$$

whenever the derivative exists.

The *derivative* of the state $X(t)$ defined by (11) is

$$DX^{\mu}(t) := \tfrac{d}{d\lambda}X^{\mu+\lambda\eta}|_{\lambda=0} = Z(t)$$

exists, and is given by

$$\begin{cases} dZ(t) & = & [\frac{\partial b}{\partial x}(t)\,Z(t) + \frac{\partial b}{\partial \mu}(t)\,\eta(t)]dt + [\frac{\partial \sigma}{\partial x}(t)\,Z(t) + \frac{\partial \sigma}{\partial \mu}(t)\,\eta(t)]dB(t) \\ & & +\int_{\mathbb{R}_0}[\frac{\partial \gamma}{\partial x}(t,\zeta)\,Z(t) + \frac{\partial \gamma}{\partial \mu}(t,\zeta)\,\eta(t)]\tilde{N}(dt,d\zeta); \quad t\in[0,T], \\ Z(0) & = & 0. \end{cases} \tag{28}$$

We remark that this derivative process is a linear SDE, then by assuming that b, σ and γ admit bounded partial derivatives with respect to x and μ, there is a unique solution $Z(t) \in \mathcal{S}^2$ of (28).
We want to prove that $Z(t)$ is exactly the derivative in $\mathbb{L}^2(\mathbb{P})$ of $X^{\mu+\lambda\eta}(t)$ with respect to λ at $\lambda = 0$. More precisely, we want to prove the following.

Lemma 12

$$\mathbb{E}[\int_0^T(\tfrac{X^{\mu+\lambda\eta}(t)-X^{\mu}(t)}{\lambda} - Z(t))^2dt] \to 0 \ \ as\ \lambda \to 0. \tag{29}$$

Proof. For notational convenience, we have here used the simplified notations

$$\mu^{\lambda} := \mu + \lambda\eta \tag{30}$$

and by $X^{\mu^{\lambda}}$ we mean the corresponding solution

$$X^{\mu^{\lambda}}(t) = x + \int_0^t\int_{\mathbb{R}_0}\gamma(s, X^{\mu^{\lambda}}(s), \mu^{\lambda}(s), \zeta)\tilde{N}(ds,d\zeta); \quad t\in[0,T],$$

when assuming that $b = \sigma = 0$, and because u is fixed we can omit it. Then, by the Itô-Lévy isometry, we get

$$\begin{aligned} &\mathbb{E}[\int_0^T(\tfrac{X^{\mu^{\lambda}}(t)-X(t)}{\lambda} - Z(t))^2dt] \\ &= \mathbb{E}[\int_0^T\int_{\mathbb{R}_0}\{\tfrac{\gamma(s,X^{\mu^{\lambda}}(s),\mu^{\lambda}(s),\zeta)-\gamma(s,X(s),\mu(s),\zeta)}{\lambda} - \tfrac{\partial\gamma}{\partial x}(s,\zeta)\,Z(t) - \tfrac{\partial\gamma}{\partial\mu}(s,\zeta)\,\eta(s)\}\tilde{N}(ds,d\zeta))^2dt] \\ &= \mathbb{E}[\int_0^T\int_{\mathbb{R}_0}\int_0^t(\tfrac{\gamma(s,X^{\mu^{\lambda}}(s),\mu^{\lambda}(s),\zeta)-\gamma(s,X(s),\mu(s),\zeta)}{\lambda} - \tfrac{\partial\gamma}{\partial x}(s,\zeta)\,Z(s) - \tfrac{\partial\gamma}{\partial\mu}(s,\zeta)\,\eta(s))^2\nu(d\zeta)dsdt]. \end{aligned}$$

This goes to 0 when λ goes to 0, by the bounded convergence theorem and our assumption on γ.

□

Theorem 13 ***(Necessary nonzero-sum maximum principle)*** *Let* $(\hat{\mu},\hat{u}) \in \mathbb{M}_{\mathbb{G}}\times\mathcal{A}_{\mathbb{G}}$ *with corresponding solutions* $\hat{X}$, (p_i^0, q_i^0, r_i^0) *and* (p_i^1, q_i^1, r_i^1) *of the forward and backward stochastic differential equations* (11) *and* (18)−(19)*, with the corresponding derivative process* $\hat{Z}$ *given by* (28). *Then the following (i) and (ii) are equivalent:*

(i) *For all μ, $\eta \in \mathbb{M}_{\mathbb{G}}$ and for all u, $\pi \in \mathcal{A}_{\mathbb{G}}$*

$$\tfrac{d}{d\lambda} J_1(\mu + \lambda\eta, u)|_{\lambda=0} = \tfrac{d}{ds} J_2(\mu, u + s\pi)|_{s=0} = 0,$$

(ii)

$$\mathbb{E}[\tfrac{\partial H_1}{\partial \mu}(t)|\mathcal{G}_t^{(1)}] = \mathbb{E}[\tfrac{\partial H_2}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Proof. First note that, by using the linearity of $\langle \cdot, \cdot \rangle$ and the fact that the Fréchet derivative of a linear operator is the same operator, we get, by interchanging the order of the derivatives $\frac{d}{dt}$ and ∇_m, that

$$\nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle = \langle p_1^1(t), \nabla_m \frac{d}{dt} m \rangle = \langle p_1^1(t), \frac{d}{dt} \nabla_m(m) \rangle = \langle p_1^1(t), \frac{d}{dt}(\cdot) \rangle,$$

and hence

$$\langle \nabla_m \langle p_1^1(t), \tfrac{d}{dt} m \rangle, DM(t) \rangle \;\; = \langle p_1^1(t), \tfrac{d}{dt} DM(t) \rangle \;\; = \langle p_1^1(t), DM'(t) \rangle$$

Also, note that

$$dDM(t) \;\; = DM'(t)dt.$$

Assume that (i) holds. Using the definition of J_1 (16), we get

$$\begin{aligned}
0 &= \tfrac{d}{d\lambda} J_1(\mu + \lambda\eta, u)|_{\lambda=0} \\
&= \mathbb{E}[\textstyle\int_0^T \{ \frac{\partial \ell_1}{\partial x}(t)\, Z(t) + \langle \nabla_m \ell_1(t), DM(t) \rangle + \frac{\partial \ell_1}{\partial \mu}(t)\, \eta(t) \} dt \\
&+ \tfrac{\partial g_1}{\partial x}(T)\, Z(T) + \langle \nabla_m g_1(T), DM(T) \rangle].
\end{aligned}$$

Hence, by the definition (17) of H_1, we have

$$\begin{aligned}
0 &= \tfrac{d}{d\lambda} J_1(\mu + \lambda\eta, u)|_{\lambda=0} \\
&= \mathbb{E}[\textstyle\int_0^T \{ \frac{\partial H_1}{\partial x}(t) - p_1^0(t) \frac{\partial b}{\partial x}(t) - q_1^0(t) \frac{\partial \sigma}{\partial x}(t) - \int_{\mathbb{R}_0} r_1^0(t,\zeta) \frac{\partial \gamma}{\partial x}(t,\zeta)\, \nu(d\zeta) \} Z(t) dt \\
&+ \textstyle\int_0^T \langle \nabla_m H_1(t), DM(t) \rangle dt \\
&- \textstyle\int_0^T \langle p_1^1(t), DM'(t) \rangle dt + \int_0^T \{ \frac{\partial H_1}{\partial \mu}(t) - p_1^0(t) \frac{\partial b}{\partial \mu}(t) \\
&- q_1^0(t) \tfrac{\partial \sigma}{\partial \mu}(t) - \textstyle\int_{\mathbb{R}_0} r_1^0(t,\zeta) \frac{\partial \gamma}{\partial \mu}(t,\zeta)\, \nu(d\zeta) \} \eta(t) dt + p_1^0(T) Z(T) + \langle p_1^1(T), DM(T) \rangle]. \qquad (31)
\end{aligned}$$

Applying now the Itô formula to both $p_1^0 Z$ and $\langle p_1^1, DM\rangle$, we get

$$\begin{aligned}
&\mathbb{E}[p_1^0(T)Z(T) + \langle p_1^1(T), DM(T)\rangle] \\
&= \mathbb{E}[\int_0^T p_1^0(t)dZ(t) + \int_0^T Z(t)dp_1^0(t) + \int_0^T q_1^0(t)(\tfrac{\partial\sigma}{\partial x}(t)\, Z(t) + \tfrac{\partial\sigma}{\partial\mu}(t)\, \eta(t))dt \\
&+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t,\zeta)(\tfrac{\partial\gamma}{\partial x}(t,\zeta)\, Z(t) + \tfrac{\partial\gamma}{\partial\mu}(t,\zeta)\, \eta(t))\nu(d\zeta)dt] \\
&+ \mathbb{E}[\int_0^T \langle p_1^1(t), DM'(t)\rangle dt + \int_0^T DM(t)dp_1^1(t)] \\
&= \mathbb{E}[\int_0^T p_1^0(t)(\tfrac{\partial b}{\partial x}(t)\, Z(t) + \tfrac{\partial b}{\partial\mu}(t)\, \eta(t))dt - \int_0^T \tfrac{\partial H_1}{\partial x}(t) Z(t)dt \\
&+ \int_0^T q_1^0(t)(\tfrac{\partial\sigma}{\partial x}(t)\, Z(t) + \tfrac{\partial\sigma}{\partial\mu}(t)\, \eta(t))dt \\
&+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t,\zeta)(\tfrac{\partial\gamma}{\partial x}(t,\zeta)\, Z(t) + \tfrac{\partial\gamma}{\partial\mu}(t,\zeta)\, \eta(t))\nu(d\zeta)dt \\
&+ \int_0^T \langle p_1^1(t), DM'(t)\rangle dt - \int_0^T \langle \nabla_m H_1(t), DM(t)\rangle dt].
\end{aligned} \tag{32}$$

Combining the above and recalling that η is of the form (27), we conclude that

$$0 = \mathbb{E}[\int_0^T \tfrac{\partial H_1}{\partial\mu}(t)\eta(t)dt] = \mathbb{E}[\int_s^T \tfrac{\partial H_1}{\partial\mu}(t)\alpha_1 dt];\ s \geq t_0.$$

Differentiating with respect to s we obtain

$$\begin{aligned}
0 &= \mathbb{E}[\tfrac{\partial H_1}{\partial\mu}(s)\alpha_1] \\
&= \mathbb{E}[\tfrac{\partial H_1}{\partial\mu}(t_0)|\mathcal{G}_{t_0}^{(1)}],
\end{aligned}$$

because this holds for all α_1 and all $s \geq t_0$.
This argument can be reversed, to prove that (ii)$\Longrightarrow$(i). We omit the details.
In the same manner, we can get the equivalence between

$$\tfrac{d}{ds} J_2(\mu, u + s\pi)|_{s=0} = 0$$

and

$$\mathbb{E}[\tfrac{\partial H_2}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

□

In the next section we will consider the zero-sum case, and find conditions for a saddle point of such games.

4.2 Zero-sum game

In this section, we proceed to study the maximum principle for the zero-sum game case. Let us then define the performance functional as

$$J(\mu, u) = \mathbb{E}[g(X(T), M(T)) + \int_0^T \ell(s, X(s), M(s), \mu(s), u(s))\, ds],$$

where the state $X(t)$ is the solution of a SDE (11) .
The functions

$$\ell(s,x,m,\mu,u) = \ell(s,x,m,\mu,u,\omega) : [0,T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega \to \mathbb{R}$$

and

$$g(x,m) = g(x,m,\omega) : \mathbb{R} \times \mathcal{M}_0 \times \Omega \to \mathbb{R}$$

are supposed to satisfy the following conditions:

(a) ℓ and g are continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.

(b) Moreover, the function

$$\mathbb{R} \times \mathcal{M}_0 \ni (x,m) \mapsto g(x,m)$$

is required to be affine $\mathbb{P}$-a.s.
We consider the stochastic zero-sum game to find (μ^*, u^*) such that

$$\sup_{u\in\mathcal{A}_{\mathbb{G}}} \inf_{\mu\in\mathbb{M}_{\mathbb{G}}} J(\mu,u) = \inf_{\mu\in\mathbb{M}_{\mathbb{G}}} \sup_{u\in\mathcal{A}_{\mathbb{G}}} J(\mu,u) = J(\mu^*,u^*).$$

We call (μ^*, u^*) a *saddle point* for $J(\mu,u)$.
In this case, let the Hamiltonian

$$H : [0,T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0,T],\mathcal{M}_0) \to \mathbb{R}$$

be given by

$$\begin{aligned} H(t,x,m,\mu,p^0,q^0,r^0(\cdot),p^1) &= \ell(t,x,m,\mu,u) + p^0 b(t,x,\mu,u) + q^0\sigma(t,x,\mu,u) \\ &\quad + \textstyle\int_{\mathbb{R}_0} r^0(\zeta)\gamma\,(t,x,\mu,u,\zeta)\,\nu(d\zeta) + \langle p^1, \beta(m)\rangle. \end{aligned}$$

We assume the following:

(c) H is continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.

(d) The Hamiltonian function

$$\mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \ni (x,m,\mu,u) \mapsto H(t,x,m,\mu,p^0,q^0,r^0(\cdot),p^1)$$

is *convex* with respect to (x,m,μ) and *concave* with respect to (x,m,u) $\mathbb{P}$.a.s and for each $t \in [0,T]$, $p^0, q^0, r^0(\cdot)$ and p^1.
For $u \in \mathcal{A}_{\mathbb{G}}, \mu \in \mathbb{M}_{\mathbb{G}}$ with corresponding solution $X = X^{\mu,u}$, define $p = p^{\mu,u}, q = q^{\mu,u}$ and $r = r^{\mu,u}$ by the adjoint equations: the real-BSDE in the unknown $(p^0, q^0, r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$ has the following form

$$\begin{cases} dp^0(t) &= -\frac{\partial H}{\partial x}(t)\,dt + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t,\zeta)\tilde{N}(dt,d\zeta);\ t \in [0,T]\,, \\ p^0(T) &= \frac{\partial g}{\partial x}(X(T),M(T)), \end{cases} \tag{33}$$

and the operator-valued BSDE for the unknown $(p^1, q^1, r^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}} \times \mathbb{L}^2_{\nu,\mathbb{K}}$ is given by

$$\begin{cases} dp^1(t) &= -\nabla_m H(t)dt + q^1(t)dB(t) + \int_{\mathbb{R}_0} r^1(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p^1(T) &= \nabla_m g(X(T),M(T)). \end{cases} \tag{34}$$

Theorem 14 (Sufficient zero-sum maximum principle) *Let $(\hat{\mu},\hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions $\hat{X}$ and (p^0,q^0,r^0), (p^1,q^1,r^1) of the forward and backward stochastic differential equations $(11)\,,(33)-(34)\,,$ respectively. Assume the following:*

- $$\mathbb{E}[\hat{H}(t)|\mathcal{G}_t^{(1)}] = ess \sup_{\mu\in\mathbb{M}_{\mathbb{G}}} \mathbb{E}[\check{H}(t)|\mathcal{G}_t^{(1)}],$$
- $$\mathbb{E}[\bar{H}(t)|\mathcal{G}_t^{(2)}] = ess \sup_{u\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}[\check{H}(t)|\mathcal{G}_t^{(2)}],$$

$\mathbb{P}$- a.s and for all $t \in [0,T]\,,$ and that assumptions (a)-(d) hold.

Then $(\hat{\mu},\hat{u})$ is a saddle point for $J(\mu,u)$.

This result will be applied in the next section.

Theorem 15 (Necessary zero-sum maximum principle) *Let $(\hat{\mu},\hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ with corresponding solutions $\hat{X}$, (p_i^0,q_i^0,r_i^0) and (p_i^1,q_i^1,r_i^1) of the forward and the backward stochastic differential equations (11) and $(33)-(34)$, respectively, with corresponding derivative process $\hat{Z}$ given by $(28)\,.$ Then we have equivalence between*

$$\tfrac{d}{d\lambda}J(\mu+\lambda\eta,u)|_{\lambda=0} = \tfrac{d}{ds}J(\mu,u+s\pi)|_{s=0} = 0,$$

and

$$\mathbb{E}[\tfrac{\partial H}{\partial \mu}(t)|\mathcal{G}_t^{(1)}] = \mathbb{E}[\tfrac{\partial H}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Proof. The same proof of both the sufficient and the necessary maximum principles for the nonzero-sum games works for the zero-sum case. □

5 Optimal consumption of a mean-field cash flow under uncertainty

Consider a net cash flow $X^{\mu,\rho} = X$ modeled by

$$\begin{cases} dX(t) = [\mu(t)(V) - \rho(t)]\, X(t)dt + \sigma\,(t)\, X(t)dB(t) + \int_{\mathbb{R}_0} \gamma\,(t,\zeta)\, X(t)\tilde{N}(dt,d\zeta);\ t \in [0,T]\,, \\ X\,(0) = x > 0, \end{cases}$$

where $\rho(t) \geq 0$ is our *relative consumption rate* at time t, assumed to be a càdlàg, $\mathcal{G}_t^{(2)}$-adapted process. Here V is a given Borel subset of $\mathbb{R}$. The value of $\mu(t)$ on V models the relative growth rate of the cash flow. The relative consumption rate $\rho(t)$ is our control process. We assume that $\int_0^T \rho(t)dt < \infty$ a.s. This implies that $X(t) > 0$ for all t, a.s. However, the measure-valued process $\mu(t)$ represents a kind of scenario uncertainty, and we want to maximise the total expected utility of the relative consumption rate ρ in the worst possible scenario μ. We penalize $\mu(\cdot)$ for being far away from the law process $\mathcal{L}(X(\cdot))$, in the sense that we introduce a quadratic cost rate $[(\mu(t) - M(t))(V)]^2$ in the performance functional. Hence we consider the zero-sum game

$$\sup_{\rho}\ \inf_{\mu}\ \mathbb{E}[\textstyle\int_0^T \{\log(\rho(t)X(t)) + [(\mu(t) - M(t))(V)]^2\}dt + \theta \log(X(T))],$$

where $\theta = \theta(\omega) > 0$ is a given bounded $\mathcal{F}_T$-measurable random variable, expressing the importance of the terminal value $X(T)$. *Here we have chosen a logarithmic utility because it is a central choice, and in many cases, as here, this leads to a nice explicit solution of the corresponding control problem.*

The Hamiltonian for this zero-sum game takes the form

$$\begin{aligned} H(t) = \log(\rho x) + (\mu(V) - m(V))^2 + p^0[\mu(V)x - \rho x] + q^0\sigma(t)x \\ + \textstyle\int_{\mathbb{R}_0} r^0(\zeta)\gamma(t,\zeta)x\nu(d\zeta) + \langle p^1, \beta(m)\rangle, \end{aligned}$$

and the adjoint processes $(p^0,q^0,r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_{\nu}$,$(p^1,q^1,r^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}} \times \mathbb{L}^2_{\nu,\mathbb{K}}$ are given by the BSDEs

- $$\begin{cases} dp^0(t) &= -[\frac{1}{X(t)} + p^0(t)[\mu(t)(V) - \rho(t)] + q^0(t)\sigma(t) + \int_{\mathbb{R}_0} r^0(t,\zeta)\gamma(t,\zeta)\nu(d\zeta)]dt \\ & \quad +q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T]\,, \\ p^0(T) &= \frac{\theta}{X(T)}, \end{cases}$$

- $$\begin{cases} dp^1(t) &= -\{2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\chi_V(\cdot) + < p^1(t), \beta(\cdot) > \}dt + q^1(t)dB(t) \\ & \quad + \int_{\mathbb{R}_0} r^1(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p^1(T) &= 0, \end{cases}$$

where $\chi_V(\cdot)$ is the operator which evaluates a given measure at V, i.e. $\langle \chi_V, \lambda \rangle = \lambda(V)$ for all $\lambda \in \mathcal{M}_0$. The first order condition for the optimal consumption rate $\hat{\rho}$ is

$$\mathbb{E}[\tfrac{1}{\hat{\rho}(t)} - \hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Since $\hat{\rho}(t)$ is $\mathcal{G}_t^{(2)}$-adapted, we have

$$\hat{\rho}(t) = \frac{1}{\mathbb{E}[\hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(2)}]}.$$

Now we use the minimum condition with respect to μ at $\mu = \hat{\mu}$ and get

$$\mathbb{E}[2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\lambda(V) + \hat{p}^0(t)\hat{X}(t)\lambda(V)|\mathcal{G}_t^{(1)}] = 0, \text{ for all } \lambda \in \mathcal{M}_0.$$

Using that $\hat{\mu}(t)$ is $\mathcal{G}_t^{(1)}$-adapted, we obtain

$$\hat{\mu}(t)(V) = \mathbb{E}[\hat{M}(t)(V) - \tfrac{1}{2}\hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(1)}].$$

It remains to find $\hat{p}^0(t)\hat{X}(t)$: We have by applying the Itô formula to $P(t) := \hat{p}^0(t)\hat{X}(t)$:

$$\begin{aligned} dP(t) &= \hat{p}^0(t)d\hat{X}(t) + \hat{X}(t)d\hat{p}^0(t) + d[\hat{p}^0, \hat{X}]_t \\ &= \hat{p}^0(t)([(\hat{\mu}(t)(V) - \rho(t))\,\hat{X}(t)]dt + \hat{\sigma}\,(t)\,\hat{X}(t)dB(t) + \textstyle\int_{\mathbb{R}_0} \hat{\gamma}\,(t,\zeta)\,\hat{X}(t)\tilde{N}(dt,d\zeta)) \\ &+ \hat{X}(t)[-\tfrac{1}{\hat{X}(t)} - \hat{p}^0(t)[\hat{\mu}(t)(V) - \rho(t)] - \hat{q}^{(0)}(t)\sigma(t) - \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\nu(d\zeta)]dt \\ &+ \hat{q}^0(t)\hat{X}(t)dB(t) + \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{X}(t)\tilde{N}(dt,d\zeta) + \hat{q}^0(t)\hat{\sigma}\,(t)\,\hat{X}(t)dt \\ &+ \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)N(dt,d\zeta). \end{aligned} \tag{35}$$

By definition

$$\begin{aligned} \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)\tilde{N}(dt,d\zeta) &= \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)N(dt,d\zeta) \\ &\quad - \textstyle\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)\nu(d\zeta)dt. \end{aligned} \tag{36}$$

Substituting (36) in (35) yields

$$\begin{aligned} dP(t) &= -dt + [P(t)\hat{\sigma}(t) + \hat{q}^0(t)\hat{X}(t)]dB(t) \\ &\quad + \textstyle\int_{\mathbb{R}_0} [P(t)\hat{\gamma}(t,\zeta) + \hat{r}^0(t,\zeta)\hat{X}(t)(1 + \hat{\gamma}(t,\zeta))]\tilde{N}(dt,d\zeta). \end{aligned}$$

Hence, if we put

$$\begin{aligned} P(t) &:= \hat{p}^0(t)\hat{X}(t), \\ Q(t) &:= P(t)\hat{\sigma}(t) + \hat{X}(t)\hat{q}^0(t), \\ R(t,\zeta) &:= P(t)\hat{\gamma}(t,\zeta) + \hat{r}^0(t,\zeta)\hat{X}(t)(1 + \hat{\gamma}(t,\zeta)). \end{aligned}$$

with $(P, Q, R) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$ satisfies the BSDE

$$\begin{cases} dP(t) &= -dt + Q(t)dB(t) + \int_{\mathbb{R}_0} R(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ P(T) &= \theta. \end{cases}$$

Solving this BSDE as in (21), we find the closed formula for $P(t)$ as

$$\begin{aligned} P(t) &= \mathbb{E}[\theta + \int_t^T ds|\mathcal{F}_t] \\ &= \mathbb{E}\left[\theta|\mathcal{F}_t\right] + T - t. \end{aligned}$$

Hence we have proved the following:

Theorem 16 *The optimal consumption rate $\hat{\rho}(t)$ and the optimal model uncertainty law $\hat{\mu}(t)$ are given respectively in feed-back form by*

$$\begin{aligned} \hat{\rho}(t) &= \frac{1}{T-t+\mathbb{E}[\theta|\mathcal{G}_t^{(2)}]}, \\ \hat{\mu}(t)(V) &= \hat{M}(t)(V) + T - t - \tfrac{1}{2}\mathbb{E}[\theta|\mathcal{G}_t^{(1)}]. \end{aligned}$$

6 Appendix

Let us give now the rest of the proof of Theorem 10. We want to prove that $J_2(\hat{\mu}, u) \leq J_2(\hat{\mu}, \hat{u})$. Using definition (16) gives for fixed $\hat{\mu} \in \mathbb{M}_{\mathbb{G}}$ and an arbitrary $u \in \mathcal{A}_{\mathbb{G}}$

$$J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) \;=\; j_1 + j_2, \tag{37}$$

where

$$\begin{aligned} j_1 &= \mathbb{E}[\int_0^T \left\{\breve{\ell}_2(t) - \bar{\ell}_2(t)\right\} dt], \\ j_2 &= \mathbb{E}[\breve{g}_2(X(T), M(T)) - \bar{g}_2(\hat{X}(T), \hat{M}(T))]. \end{aligned}$$

Applying the definition of the Hamiltonian (17) we have

$$\begin{aligned} j_1 = \mathbb{E}[\textstyle\int_0^T \{&\breve{H}_2(t) - \breve{H}_2(t) - \hat{p}_2^0(t)\tilde{b}(t) - \hat{q}_2^0(t)\tilde{\sigma}(t) \\ &- \textstyle\int_{\mathbb{R}_0} \hat{r}_2^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta) - \langle \hat{p}_2^1(t), \tilde{M}'(t)\rangle\}dt], \end{aligned} \tag{38}$$

where $\tilde{b}(t) = \breve{b}(t) - \bar{b}(t)$. etc., and

$$\tilde{M}'(t) = \frac{d\tilde{M}(t)}{dt}.$$

Concavity of g_2 and the definition of the terminal value of the BSDEs (18) and (19) shows that

$$\begin{aligned} j_2 &\leq \mathbb{E}[\tfrac{\partial g_2}{\partial x}(T)\tilde{X}(T) + \langle \nabla_m g_2(T), \tilde{M}(t)\rangle] \\ &= \mathbb{E}[\hat{p}_2^0(T)\tilde{X}(T) + \langle \hat{p}_2^1(T), \tilde{M}(t)\rangle]. \end{aligned} \tag{39}$$

Applying the Itô formula to $\hat{p}_2^0 \tilde{X}$ and $\langle \hat{p}_2^1, \tilde{M} \rangle$, we get

$$
\begin{aligned}
j_2 &\leq \mathbb{E}[\hat{p}_2^0(T)\tilde{X}(T) + \langle \hat{p}_2^1(T), \tilde{M}(T) \rangle] \\
&= \mathbb{E}[\int_0^T \hat{p}_2^0(t) d\tilde{X}(t) + \int_0^T \tilde{X}(t) d\hat{p}_2^0(t) + \int_0^T \hat{q}_2^0(t)\tilde{\sigma}(t)dt + \int_0^T \int_{\mathbb{R}_0} \hat{r}_2^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta)dt] \\
&+ \mathbb{E}[\int_0^T \langle \hat{p}_2^1(t), d\tilde{M}(t) \rangle + \int_0^T \tilde{M}(t) d\tilde{p}_2^1(t)] \\
&= \mathbb{E}[\int_0^T \hat{p}_2^0(t)\tilde{b}(t)dt - \int_0^T \tfrac{\partial \bar{H}_2}{\partial x}(t)\tilde{X}(t)dt + \int_0^T \hat{q}_2^0(t)\tilde{\sigma}(t)dt \\
&+ \int_0^T \int_{\mathbb{R}_0} \hat{r}_2^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta)dt + \int_0^T \langle \hat{p}_2^1(t), \tilde{M}'(t) \rangle dt - \int_0^T \langle \nabla_m \bar{H}_2(t), \tilde{M}(t) \rangle dt],
\end{aligned}
$$

where we have used that the $dB(t)$ and $\tilde{N}(dt, d\zeta)$ integrals have mean zero. Substituting (38) and (39) into (37), we obtain

$$
J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) \leq \mathbb{E}[\int_0^T \{\check{H}_2(t) - \bar{H}_2(t) - \tfrac{\partial \bar{H}_2}{\partial x}(t)\tilde{X}(t) - \langle \nabla_m \bar{H}_2(t), \tilde{M}(t) \rangle\} dt].
$$

Since H_2 is concave and the process u is $\mathcal{G}_t^{(2)}$-adapted, we have

$$
\begin{aligned}
J_2(\hat{\mu}, u) - J_2(\hat{\mu}, \hat{u}) &\leq \mathbb{E}[\int_0^T \tfrac{\partial \bar{H}_2}{\partial u}(t)\,(u(t) - \hat{u}(t))\,dt] \\
&= \mathbb{E}[\int_0^T \mathbb{E}[\tfrac{\partial \bar{H}_2}{\partial u}(t)|\mathcal{G}_t^{(2)}]\,(u(t) - \hat{u}(t))\,dt] \\
&\leq 0,
\end{aligned}
$$

because $\bar{H}_2$ has a maximum at $\hat{u}$. □

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