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# Component importance in multistate directed network flow systems

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The front page depicts a section of the root system of the exceptional Lie group $E_8$, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

## Abstract

There seem to be two main reasons for coming up with a measure of importance of system components. Firstly, it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Secondly, it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow. It should be noted that no measure of importance can be expected to be universally best irrespective of usage purpose. In this project we will present a new and general approach to importance measures related to multistate systems. Particular focus is put on a class of repairable directed network flow systems.

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# Contents

# 1 Introduction

In reliability theory a main problem is to find out how the reliability of a complex system can be determined from knowledge of the reliabilities of its components. One weakness of traditional binary reliability theory is that the system and the components are always described just as functioning or failed. This approach represents an oversimplification in many real-life situations where the system and their components are capable of assuming a whole range of levels of performance, varying from perfect functioning to complete failure. The basic reliability theory for such multistate systems of multistate components has been established by the mid 1980s, and has been developed continuously. In particular, many different tools for analyzing the importance of components in multistate systems have been developed in recent years. There are two main reasons for coming up with a measure of importance of system components. Firstly, it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Secondly, it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow. Over the years different measures have been suggested. More recent work in this area includes extensions of the binary case of some well-established measures of component importance to the class of multistate repairable systems. Furthermore, the importance of the components in a multistate system have been studied from some new perspectives, introducing various approaches applied to real-life situations.

The classical approaches to importance measures include Birnbaum, Barlow-Proschan and Natvig measures of component importance, including the dual extensions of the latter measures. The measures are treated in details in [5], [6], [9], [10], [11], [12]. Furthermore, a number of applications have been proposed, e.g in [8] the extended Natvig measure is applied to an offshore oil and gas production system. In particular, the Birnbaum measure of component importance provides a dynamic approach to determining importance of the components in the system. The Birnbaum measure of a given component is defined as the probability that such component is critical to the functioning of the system. Furthermore, time-independent Barlow-Proschan measure of importance can be expressed as a weighted average of the Birnbaum measure with respect to the component lifetime densities. The Barlow-Proschan measure implies that components with long lifetimes compared to the system lifetime will have a large reliability importance. An alternative measure can be defined by instead saying that components which greatly reduce the remaining system lifetime by failing, are the most important components. This is reflected by Natvig measure. The Natvig type measures focus on how a change in the component state affects the expected system uptime and downtime relative to the given system state.

Furthermore, a series of new approaches to component importance measures for multistate system has been developed in recent years. In particular, several different measures applied to real life situations have been derived. For instance, a cost-based importance measure, as an extension of Birnbaum measure has been proposed in [17]. In particular, it is pointed out that existing importance measures have paid little attention to the costs incurred by maintaining

a system and its components. Hence, this paper considers costs of improving component reliability, costs due to component failure and cost of system failure, and provides possible extensions and applications of the importance measure.
Paper [16] introduces a component state based integrated importance measure for multistate systems. Importance measures are used to identify weak components and states in contributing to the reliable functioning of a system. Traditionally, importance measures do not consider the possible effect of groups of transition rates among different component states, which, however, has great effect on the component probability distribution and should therefore be taken into consideration. Thus, a generalization of the integrated importance measure describes in which state it is most worthy to keep the component to provide the desired level of system performance, and which component is the most important to keep in some state and above for improving the performance of the system. An application to an oil transportation system is presented to illustrate the use of the suggested importance measure.
Furthermore, paper [18] introduces component maintenance priority importance measure. Time on performing preventive maintenance on a component in a system may affect system availability if system operation needs stopping for the maintenance. To avoid such an availability reduction, one may adopt the following method: if a component fails, preventive maintenance is carried out on a number of the other components while the failed component is being repaired. Hence, the importance measure can be used to select components for the preventive maintenance.

To conclude our brief overview of various component importance measures, we remark that no measure can be said to be universally best irrespective of usage purpose. Still comparing different measures is often of interest. In this project we will provide further generic extensions of the Birnbaum measure for binary systems, applied to multistate systems. Particular focus will be put on class of repairable directed network flow systems. Thus, we consider asymptotic Birnbaum measure as a generalization of the Birnbaum measure to multistate repairable systems, and two importance measures based on asymptotic availability and mean state of the system, respectively. In particular, one way of improving the system is to increase the time spent by the components in the higher level states. This can be modelled as an increase in the corresponding expected waiting time. Hence, the alternative family of importance measures aims to assess change in the asymptotic system availability at given system level with respect to change in expected waiting time in higher level states. Alternatively, an asymptotic mean state of the system can be used as a basis for importance measure. For the binary systems, the Birnbaum measure of importance can be obtained by differentiating the reliability function of the system with respect to component reliability. Thus, the two suggested importance measures, derived as partial derivatives of asymptotic availability and mean state of the system, propose an alternative way of reflecting this relation for the class of multistate systems.

This project has the following structure: first, section (2) provides an introduction to the binary systems and an extension of a few basic concepts to multistate systems, providing a necessary framework for the study. Furthermore, section (3) introduces the component importance measures. Finally, numerical examples applied to directed network flow systems are presented in section (4).

# 2 Binary and multistate monotone systems

The first subsection gives a brief introduction to the simple concept of the binary monotone systems. The further subsections extend basic results to multistate systems, as well as a few definitions are introduced, providing a necessary framework for this project. Also, the class of directed network flow systems is introduced.

## 2.1 Binary monotone systems

Assume that $(C, \phi)$ is a binary monotone system, where $C = \{1, ..., n\}$ is the component set, and $\phi$ is the structure function. Moreover, let $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))$ where $X_i(t)$ is the state variable of component $i$ at time $t$, where $i \in C$, i.e., $X_i(t) = 1$ if component $i$ is functioning at time $t$, and $X_i(t) = 0$ otherwise. Then a component $i \in C$ is said to be *critical* for the system if:

$$\phi(0_i, \boldsymbol{X}(t)) = 0, \text{ and } \phi(1_i, \boldsymbol{X}(t)) = 1. \tag{1}$$

For the explanation of the notation, see [7]. Since $(C, \phi)$ is a binary monotone system, the structure function $\phi$ is binary and non-decreasing. Hence we always have: $0 \leq \phi(0_i, \boldsymbol{X}(t)) \leq \phi(1_i, \boldsymbol{X}(t)) \leq 1$. From this i follows that (1) is equivalent to:

$$\phi(0_i, \boldsymbol{X}(t)) < \phi(1_i, \boldsymbol{X}(t)). \tag{2}$$

In fact, due to monotonicity, we may equivalently say that component $i$ is critical at time $t$ if:

$$\phi(0_i, \boldsymbol{X}(t)) \neq \phi(1_i, \boldsymbol{X}(t)). \tag{3}$$

The condition (3) can be rewritten even further in a way that makes it easy to extend it to more general settings. In order to do so we introduce:

$$X_i^+(t) = \begin{cases} 0, & \text{for } X_i(t) = 1 \\ 1 & \text{for } X_i(t) = 0 \end{cases}$$

We observe that for repairable components $X_i^+(t)$ represents the *upcoming* state of component $i$ at time $t$. Using this notation, we may rewrite (3) as:

$$\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t)). \tag{4}$$

Hence, component $i$ is critical at time $t$ if a state change of component $i$ at time $t$, implies a system state change at time $t$ as well.

According to [2] the Birnbaum measure of importance of component $i \in C$ at time $t$, denoted by $I_B^{(i)}(t)$, is the probability that the component is critical at time $t$. By (4) this implies that we have:

$$I_B^{(i)}(t) = P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{5}$$

We also introduce the *asymptotic* Birnbaum measure of importance of component $i \in C$ as:

$$I_B^{(i)} = \lim_{t\to\infty} I_B^{(i)}(t) = \lim_{t\to\infty} P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{6}$$

Assuming that the component state processes $\{X_1(t), ..., X_n(t)\}$ are independent, and that the limiting distributions for these processes exists, we introduce $\boldsymbol{p} = (p_1, ..., p_n)$, where:

$$p_i = \lim_{t \to \infty} P(X_i(t) = 1), \quad i \in C.$$

We also introduce the reliability of the system:

$$h = h(\boldsymbol{p}) = \lim_{t \to \infty} P(\phi(\boldsymbol{X}(t)) = 1).$$

It is then very well-known that we have (see [7]):

$$I_B^{(i)} = \frac{\partial h}{\partial p_i}(\boldsymbol{p}), \quad i \in C. \tag{7}$$

This implies that the asymptotic importance of component $i$ may be interpreted as the change rate of the asymptotic system availability with respect to a small change in the asymptotic component availability. Thus, if one looks for ways to improve the asymptotic system availability, one should focus on the component with the highest asymptotic importance.

## 2.2 Multistate monotone systems

For an extensive introduction to multistate monotone systems we refer to [10]. In this context we define a multistate monotone system similar to a binary monotone system as an ordered pair $(C, \phi)$, where $C = \{1, .., n\}$ is the component set, and $\phi$ is the structure function. Moreover, we let $\boldsymbol{X}(t) = (X_1(t), ..., X_n(t))$ where $X_i(t)$ is the state variable of component $i$ at time $t$. Contrary to the binary system, however, both the components and the system may be in multiple states (not just 0 and 1). More specifically, if $i \in C$, we let $S_i = \{0, 1, ..., r_i\}$ denote the set of states for component $i$. Assume that each component starts out at its top-level state, and then at random points of time transits downwards through the state set until it reaches state 0. At this stage the component is replaced by a new component, and a new life cycle begins.

For each component $i \in C$ we also introduce a function $f_i : S_i \to \mathbb{R}$ representing the *physical state* of the component as a function of the state. Thus, if $X_i(t) = x_i \in S_i$, then the physical state of component $i$ at time $t$ is $f_i(X_i(t))$. E.g., if component $i$ is a pipeline, then the physical state of the component at a given point of time may be the capacity of the pipeline at this point of time. Being a physical property of the pipeline, this may be any arbitrary non-negative number depending on the state of the component, and the function $f_i$ provides a convenient way of encoding this directly into the model. Note that the functions $f_1, ..., f_n$ do not necessary need to be monotone. In particular, such assumption introduces additional flexibility to the modeling of component states within the predefined life cycle of a component. E.g, it permits performing minimal repairs on the components, that is maintenance or improvement of the component before it reaches its failure state. Also, in many real-life situations one may think of several possibilities for component states ordering. In particular, consider some kind of machine part or an engine that becomes more efficient after some time functioning, that is so called burn-in cases, where a new component starts its life cycle at some intermediate state before reaching its perfect functioning state.

The structure function $\phi$ represents the state of the system expressed as a function of the states of the components. It is common in multistate reliability theory to assume that $\phi$ also assumes values in a set of non-negative integers. In this context, however, we let the structure function represent the physical state of the system. Moreover, we assume that $\phi$ can be written as:

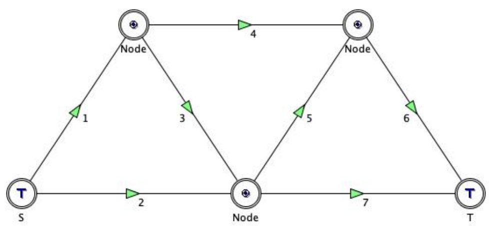
$$\phi(\boldsymbol{X}(t)) = \phi(f_1(X_1(t)), ..., f_n(X_n(t)))$$

This assumption implies that the physical state of the system is a function of the physical states of the components. Furthermore, it seems reasonable to allow the physical state of the system to be expressed as a non-decreasing function of the physical states of the components, reflecting the physical monotonicity of the system. It should be noted that $\phi$ does not necessarily need to be non-decreasing in component states $X_i$, $i \in C$. Hence, assume that $\phi$ is a non-decreasing function of the physical state functions $f_1, ..., f_n$. The advantage with this approach is that the system state is expressed in terms of physical quantities rather than being encoded more abstractly as non-negative integers.

## 2.3 Network flow systems

An important class of multistate systems which can be handled within this framework is the class of *directed network flow systems*. A directed network flow system contains of a set of points, referred to as *nodes*, and a set of lines between these points, referred to as *edges*. See Figure 1.

Figure 1: A directed network flow system.




The edges of a directed network flow system are *directed* (indicated by an arrow), implying that flow can only pass through an edge according to the direction of the edge. From now and throughout this study we only consider simple *source-to-terminal* flow networks where one of the nodes is the *source node*, while another node is the *terminal node*. The components of the system are the edges of the network, and the state of the system is defined as the amount of flow (of some kind) that can be transmitted from the source node to the terminal node through the network.

In figure (1) the node $S$ is the source node, while the node $T$ is the terminal node. The component set of the system is the set of edges, i.e. $C = \{1, ..., 7\}$. In order to determine the amount of flow that can be transmitted from $S$ to $T$, we consider a subset $K$ of the edge set $C$, e.g., $K = \{4, 5, 7\}$. We observe that if we remove all edges in $K$ from the network, the connection between $S$ and $T$ through the network is broken. Thus, $K$ is referred to as a minimal cut set of the network. We also note that the amount of flow that can be transmitted from $S$ to $T$ is limited by the sum of the capacities of the components in $K$. Hence, we have:

$$\phi(\boldsymbol{X}(t)) \leq \sum_{i \in K} f_i(X_i(t)).$$

The same holds true for any minimal cut set in the network. Thus, if we identify all minimal cut sets of the system, say $K_1, ..., K_k$, we must have:

$$\phi(\boldsymbol{X}(t)) \leq \min_{1 \leq j \leq k} \sum_{i \in K_j} f_i(X_i(t)).$$

According to the so-called *max-flow-min-cut* theorem (see [3]), we actually have:

$$\phi(\boldsymbol{X}(t)) = \min_{1 \leq j \leq k} \sum_{i \in K_j} f_i(X_i(t)).$$

This result gives us an easy way of determining the state of a directed network flow system. Moreover, [3] also provides an efficient algorithm for determining the maximal flow.

## 2.4 Stationary probability distribution in multistate systems

In this section we introduce a few basic concepts used in multistate system reliability theory. The components of the system are assumed to be stochastically independent. For $i = 1, ..., n$ and $j \in S_i = \{0, 1, ..., r_i\}$ introduce the marginal probability distributions for the component state variables:

$$q_{ij}(t) = P[X_i(t) = j], \tag{8}$$

$$\boldsymbol{q}_i(t) = (q_{i0}(t), q_{i1}(t), ..., q_{ir_i}(t)). \tag{9}$$

Where the vector $\boldsymbol{q}_i(t)$, $i = 1, ..., n$, contains the probability distribution for $X_i(t)$, with the following property:

$$\sum_{j \in S_i} q_{ij}(t) = 1, \quad i = 1, ..., n. \tag{10}$$

Finally, introduce the vector $\boldsymbol{q}(t)$ containing probability distributions for all the component state variables:

$$\boldsymbol{q}(t) = (\boldsymbol{q}_1(t), ..., \boldsymbol{q}_n(t)). \tag{11}$$

Using standard renewal theory and under mild restrictions on the waiting time distributions it is well known that that for $i = 1, ..., n$ the stationary probabilities for the states of component $i$ are given by:

$$q_{ij} = \lim_{t \to \infty} q_{ij}(t) = \frac{\mu_{ij}}{\sum_{k \in S_i} \mu_{ik}}, \tag{12}$$

where $\mu_{ij}$ denotes the expected waiting time for component $i$ in state $j$, $i \in C$, $j \in S_i$. Introduce the stationary vectors:

$$\boldsymbol{q}_i = (q_{i0}, ..., q_{ir_i}), \quad i = 1, ..., n,$$

$$\boldsymbol{q} = (\boldsymbol{q}_1, ..., \boldsymbol{q}_n).$$

Let the $i$th component have an absolutely continuous distribution $F_{ij}(t)$ of time spent in state $j$, before jumping downwards to state $j-1$, with density $f_{ij}(t)$. It is assumed that all these times spent in the various states are independent. Finally, introduce the following notation:

$$(s_i, \boldsymbol{x}) = (x_1, ..., x_{i-1}, s, x_{i+1}, ..., x_n). \tag{13}$$

that is, the vector $(s_i, \boldsymbol{x})$ is obtained by replacing the $i$th coordinate by $s$ for any vector $\boldsymbol{x}$. Also, introduce the $(r_i + 1)$- dimensional row vector $\boldsymbol{e}_{ij}$, with coordinates indexed from 0 up to $r_i$, such that $j$th coordinate is 1 and the rest of coordinates are zero:

$$\boldsymbol{e}_{ij} = (1_j, \boldsymbol{0}), \quad i = 1, ..., n, j = 0, ..., r_i. \tag{14}$$

Thus, $\boldsymbol{e}_{ij}$ can be interpreted as a vector representing the conditional probability distribution for $X_i(t)$ given that $X_i(t) = j$.

# 3 Measures of component importance

The following section presents three different measures of component importance in multistate systems. In particular, a necessary framework is established, followed by numerical study of the measures presented in the last section (4). First, a generalization of the well-known Birnbaum measure of component importance is introduced. Furthermore, an alternative family of importance measures for multistate systems is suggested. In particular, we introduce two new alternative approaches to component importance measures, based on the asymptotic availability of the system and asymptotic mean state of the system.

## 3.1 Birnbaum importance measure

In order to define criticality and importance we start out by considering a multistate monotone system $(C, \phi)$. Now let $i \in C$, and let $S_i = \{0, 1, ..., r_i\}$ be the set of states for this component. We then introduce:

$$X_i^+(t) = \begin{cases} X_i(t) - 1, & \text{for } X_i(t) > 0 \\ s_i & \text{for } X_i(t) = 0 \end{cases} \tag{15}$$

Thus, as for the binary case $X_i^+(t)$ represents the *upcoming* state of component $i$ at time $t$. We then say, as in the binary case, that component $i$ is critical at time $t$ if:

$$\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t)) \tag{16}$$

Hence, component $i$ is critical at time $t$ if a state change of component $i$ at time $t$ implies a system state change at time $t$ as well. As before we define the Birnbaum measure of importance of component $i$ at time $t$, denoted by $I_B^{(i)}(t)$, as the probability that the component is critical at time $t$. By (16) this implies that we still have:

$$I_B^{(i)}(t) = P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{17}$$

We also introduce the *asymptotic* Birnbaum measure of importance of component $i \in C$ as:

$$I_B^{(i)} = \lim_{t\to\infty} I_B^{(i)}(t) = \lim_{t\to\infty} P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))]. \tag{18}$$

We observe that by following the above path the generalization of the Birnbaum to multistate systems is very straightforward. However, if one wants to derive a version of this measure expressed as a partial derivative similar to (7), it is not obvious how this can be done. In fact, there are many alternative solutions to this problem. The following subsections present two particular methods for computing the component importance measures with expressions equivalent to the derivative for the binary case.

## 3.2 Asymptotic availability of the system

Introduce:

$$\mu_{ij} = \text{Expected waiting time for component } i \text{ in state} j, i \in C, j \in S_i.$$

We also let $\boldsymbol{\mu_i} = (\mu_{i0}, \mu_{i1}, ..., \mu_{ir_i}), i = 1, ..., n$, and $\boldsymbol{\mu} = (\boldsymbol{\mu_1}, ...., \boldsymbol{\mu_n})$. Under mild restrictions on the waiting time distributions for the components it can be shown that the limiting distribution for the component state process $\{X_i(t)\}$ depends only on $\boldsymbol{\mu_i}, i \in C$. Moreover, assuming that the component state processes are independent, the limiting distribution for the system state process $\{\phi(\boldsymbol{X}(t))\}$ depends only on $\boldsymbol{\mu_1}, ..., \boldsymbol{\mu_n}$. Now let $M$ be given by:

$$M = \max_{\boldsymbol{x}} \phi(\boldsymbol{x}),$$

where the maximum is taken over all possible component state vectors $\boldsymbol{x} = (x_1, ..., x_n)$, where $x_i \in S_i, i \in C$. We then introduce the asymptotic system availability at level $m$ given by:

$$h_m = \lim_{t\to\infty} P(\phi(\boldsymbol{X}(t)) \geq m), \quad m \in (0, M]. \tag{19}$$

Then it follows by the above arguments that we may write:

$$h_m = h_m(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_n) = h_m(\boldsymbol{\mu}), \quad m \in (0, M].$$

One way of improving the system is to increase the time spent by the components in the higher level states. This can be modelled as an increase in the corresponding expected waiting time. Motivated by this we may define alternative family of importance measures:

$$I_{B1}^{(i,j)}(m) = \frac{\partial h_m}{\partial \mu_{ij}}(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_n), \quad i \in C, j \in M_i, m \in (0, M]. \tag{20}$$

This implies that the measure defined above can be interpreted as the change rate of the asymptotic system availability at level $m$ with respect to a small change in expected waiting time in state $j$ of component $i$.

One of the basic tools for calculating reliability of binary monotone systems is pivotal decomposition. The method allows us to simplify the structure and reliability functions by dividing the problem into two simpler problems and reduce the order of the functions:

$$\begin{aligned}
&\phi(\boldsymbol{x}(t)) = x_i(t)\phi(1_i, \boldsymbol{x}(t)) + (1 - x_i(t))\phi(0_i, \boldsymbol{x}(t)),\\
&h(\boldsymbol{p}) = \lim_{t\to\infty} P(\phi(\boldsymbol{X}(t)) = 1) = p_i h(1_i, \boldsymbol{p}) + (1 - p_i)h(0_i, \boldsymbol{p}).
\end{aligned}$$

An equivalent expression can be obtained for multistate systems. This can be done, in similar way as for the binary case, by conditioning on the state $j \in S_i$ of a component $i \in C$. Hence, the asymptotic system availability at level $m \in (0, M]$ can be decomposed as follows:

$$\begin{aligned}
h_m &= \lim_{t\to\infty} P(\phi(\boldsymbol{X}(t)) \geq m)\\
&= \lim_{t\to\infty} \sum_{j\in S_i} P\big(\phi(\boldsymbol{X}(t)) \geq m \mid X_i(t) = j\big) \cdot P(X_i(t) = j)\\
&= \sum_{j\in S_i} \frac{\mu_{ij}}{(\sum_{k\in S_i} \mu_{ik})} \cdot h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})\\
&= \Big(\sum_{j\in S_i} \mu_{ij}\Big)^{-1} \Big(\sum_{j\in S_i} \Big(\mu_{ij} \cdot h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})\Big)\Big),
\end{aligned} \tag{21}$$

where, for $i \in C$, $j \in S_i$, $m \in (0, M]$, and by (12) we have:

$$\lim_{t\to\infty} P(X_i(t) = j) = q_{ij} = \frac{\mu_{ij}}{\sum_{k\in S_i} \mu_{ik}},$$
$$\lim_{t\to\infty} P\big(\phi(\boldsymbol{X}(t)) \geq m \mid X_i(t) = j\big) = h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}),$$

and where $\big((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}\big) = \big(\boldsymbol{\mu_1}, ..., \boldsymbol{\mu_{i-1}}, (\boldsymbol{e}_{ij})_i, \boldsymbol{\mu_{i+1}}, ..., \boldsymbol{\mu_n}\big)$, a vector representing the conditional probability distribution. Thus, $h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ represents the conditional asymptotic system availability at level $m$, given component $i$ in state $j$.
Furthermore, for $i = 1, ..., n$, $j = 0, 1, ..., r_i$ and $m \in (0, M]$ we have:

$$\begin{aligned} I_{B1}^{(i,j)}(m) &= \frac{\partial h_m}{\partial \mu_{ij}}(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_n) \\ &= \Big(\sum_{k\in S_i} \mu_{ik}\Big)^{-2} \bigg(\Big(\sum_{k\neq j} \mu_{ik}\Big) \cdot h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k\neq j} \Big(\mu_{ik} \cdot h_m((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\bigg). \end{aligned} \tag{22}$$

Note that (22) can also be expressed as a function of the stationary probabilities:

$$I_{B1}^{(i,j)}(m) = \Big(\sum_{k\in S_i} \mu_{ik}\Big)^{-1} \bigg(\Big(\sum_{k\neq j} q_{ik}\Big) \cdot h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k\neq j} \Big(q_{ik} \cdot h_m((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\bigg). \tag{23}$$

Hence, $I_{B1}^{(i,j)}(m)$ can be expressed as a sum of conditional asymptotic availabilities of the system, weighted by the stationary probabilities. Furthermore, observe that $\sum_{j\in S_i} \mu_{ij}$ may be interpreted as the expected amount of time it takes for component $i$ to complete one full life cycle by deteriorating through all states from the perfect functioning state until the complete failure state. Hence, the $I_{B1}^{(i,j)}(m)$ measure depends both on the stationary probability distributions of component waiting times, and expected time of full life cycle of component $i$. Thus, the measure is sensitive to the choice of scale of the expected waiting times in states $j$ of component $i$, and an adjustment of the measure might be necessary in certain cases. This relation is examined and discussed further in subsection (4.2).

The $I_{B1}^{(i,j)}(m)$ importance measure indicates how the change in expected waiting time in state $j$ of component $i$ affects the asymptotic availability of the system at level $m$. In particular, we will see that the measures can both be negative and positive. Hence, the absolute values of the measures are used to rank the importance of the system components $i$ for each state $j$ and at level $m$.

## 3.3 Asymptotic mean state of the system

Instead of using the asymptotic system availability as a basis for an importance measure we may alternatively use the asymptotic mean state of the system:

$$\xi = \lim_{t\to\infty} E[\phi(\boldsymbol{X}(t))],$$

and define:

$$I_{B2}^{(i,j)} = \frac{\partial \xi}{\partial \mu_{ij}}(\boldsymbol{\mu_1}, ..., \boldsymbol{\mu_n}), \quad i \in C, j \in S_i. \tag{24}$$

Hence, the importance measure $I_{B2}^{(i,j)}$ can be interpreted as the change rate of the asymptotic mean state of the system with respect to a small change in expected waiting time in state $j$ of component $i$.

Similarly to $I_{B1}^{(i,j)}$, by using pivotal decomposition we obtain the following expression:

$$\begin{aligned}
\xi &= \lim_{t\to\infty} E[\phi(\boldsymbol{X}(t))] \\
&= \lim_{t\to\infty} E\big[E[\phi(\boldsymbol{X}(t)) \mid X_i(t) = j]\big] \\
&= \lim_{t\to\infty} \sum_{j\in S_i} E[\phi(\boldsymbol{X}(t)) \mid X_i(t) = j] \cdot P(X_i(t) = j) \\
&= \sum_{j\in S_i} \frac{\mu_{ij}}{(\sum_{k\in S_i} \mu_{ik})} \cdot \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) \\
&= \Big(\sum_{j\in S_i} \mu_{ij}\Big)^{-1} \Big(\sum_{j\in S_i} \Big(\mu_{ij} \cdot \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})\Big)\Big),
\end{aligned} \tag{25}$$

where, for $i \in C$ and $j \in S_i$ we have:

$$\lim_{t\to\infty} E\big[\phi(\boldsymbol{X}(t)) \mid X_i(t) = j\big] = \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}),$$

and where $\xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ represents the conditional asymptotic mean state of the system, given component $i$ in state $j$. Furthermore,

$$\begin{aligned}
I_{B2}^{(i,j)} &= \frac{\partial \xi}{\partial \mu_{ij}}(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_n) \\
&= \Big(\sum_{k\in S_i} \mu_{ik}\Big)^{-2} \Big(\Big(\sum_{k\neq j} \mu_{ik}\Big) \cdot \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k\neq j} \Big(\mu_{ik} \cdot \xi((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\Big).
\end{aligned} \tag{26}$$

Note that (26) can also be expressed as a function of the stationary probabilities:

$$I_{B2}^{(i,j)}(m) = \Big(\sum_{k\in S_i} \mu_{ik}\Big)^{-1} \Big(\Big(\sum_{k\neq j} q_{ik}\Big) \cdot \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k\neq j} \Big(q_{ik} \cdot \xi((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\Big). \tag{27}$$

Hence, $I_{B2}^{(i,j)}$ can be expressed as a sum of conditional asymptotic mean states of the system, weighted by the stationary probabilities. Similarly as for the $I_{B1}^{(i,j)}(m)$ measure, the $I_{B2}^{(i,j)}$ measure depends on both the stationary probability distributions of component waiting times, and expected time of full life cycle of component $i$. Thus, the measure is sensitive to the choice of scale of the component mean waiting times.

The $I_{B2}^{(i,j)}$ importance measure indicates how the change in expected waiting time in state $j$ of component $i$ affects the asymptotic mean state of the system.

Similarly as for the $I_{B1}^{(i,j)}(m)$ measure, the absolute values of the $I_{B2}^{(i,j)}$ measures are considered in order to rank the importance of the system components $i$ for each state $j$.

# 4 Numerical study of importance measures

In this section we will present different examples of directed network flows systems. Subsection (4.1) gives a general introduction to discrete event simulation and describes the simulation process. Furthermore, the three suggested importance measures $I_B^{(i)}, I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ are computed and compared. Each of the presented examples, analyzed in subsections (4.2) - (4.4) looks at different properties of the measures and highlights possible issues, differences and similarities between the measures. In particular, we investigate sensitivity of the measures with respect to scaling of mean expected waiting times in component states. Furthermore, we examine how the ordering of the component states affects the three measures. Finally, a more complex system with several bridge components is considered and the sensitivity of component importance with respect to direction of the component flow is analyzed. The final subsection (4.5) gives a brief summary of the observations and conclusions.

## 4.1 Discrete event simulation

Discrete event models are frequently used in simulation studies to model and analyze pure jump processes. A discrete event model can be viewed as a system consisting of a collection of stochastic processes (the *elementary processes* of the system), where the states of the individual processes change as results of various kinds of events occurring at random points of time. We always assume that each event only affects one of the elementary processes. Between these events the states of the processes are considered to be constant.

In the context of multistate systems, we assume that the life cycle of the $i$'th component starts out with the component being in its perfect functioning state. Then the state of the component degrades through all intermediate states, and ends up in the complete failure state. After that the component is replaced or repaired back to its perfect functioning state again, and a new life cycle begins. In order to model this, introduce the following random variable:

$$U_{ij}^k = \text{The } k\text{'th time spent by the } i\text{'th component in state } j,$$

where $i = 1, ..., n$, $j = 0, 1, ..., r_i$ and $k = 1, 2, ....$ All these random variables are assumed to be stochastically independent. This implies in particular that the component states $X_1(t), ..., X_n(t)$ are independent for each $t \geq 0$. Furthermore, we assume that $U_{ij}^{(1)}, U_{ij}^{(2)}, ...$ are identically distributed with an absolutely continuous distribution $F_{ij}(t)$ with a positive mean value $\mu_{ij} < \infty$, $i = 1, ..., n$, $j = 0, 1, ..., r_i$. Thus, component objects are equipped with methods for generating state change events according to the distribution of $U_{ij}$'s.

### 4.1.1 Pure jump processes

As before we consider a multistate system with component set $C = 1, ..., n$ and structure function $\phi$. Moreover, for $i = 1, ..., n$ we let $X_i(t)$ denote the state of the $i$'th component at time $t \geq 0$, and let the events affecting the i'th component be denoted by $E_{i1}, E_{i2}, ...$, listed in chronological order. Since we assumed that the times spent in each state have absolutely continuous distributions, all these events happen at distinct points of time almost surely. We let $T_{i1} < T_{i2}, ...$

be the corresponding points of time for these events. We also let $T_{i0} = 0$, $i = 1, ..., n$. Then, the component state processes can then be expressed as:

$$X_i(t) = X_i(0) + \sum_{k=0}^{\infty} I(T_{ik} \leq t) J_{ik}, \qquad t \geq 0, i = 1, ..., n, \tag{28}$$

where the jumps $J_{ik}$ are the changes in state values as result of the respective events $E_{ik}$. We assume that all components start out by being in their perfect state. Thus, for $i = 1, ..., n$ we have $X_i(0) = s_{ir_i}$, while the jumps are given by:

$$J_{i1} = s_{i(r_i-1)} - s_{ir_i}, \quad J_{i2} = s_{i(r_i-2)} - s_{i(r_i-1)}, \quad \cdots \quad,$$
$$J_{ik} = s_{i(r_i-k)} - s_{i(r_i-(k-1))}, \quad \cdots \quad, J_{ir_i} = s_{i0} - s_{i1}.$$

Hence, component $i \in C$ starts at its top level state at time $T_{i0} = 0$, and accordingly to the jumps deteriorates by going through all states from the perfect functioning state until the complete failure state, denoted by the last jump in the life cycle, $J_{ir_i}$. Then, the component is replaced by a new one and a new life cycle begins, following the same pattern with the jumps as described above. For $i = 1, ..., n$ we also introduce the times between the events defined as:

$$\Delta_{ik} = T_{ik} - T_{i(k-1)}, \quad k = 1, 2, ...,$$

Thus, for $i = 1, ..., n$ we have:

$$\Delta_{i1} = U_{ir_i}^{(1)}, \quad \Delta_{i2} = U_{ir_{i-1}}^{(1)}, \quad \cdots$$

Since $U_{ij}^{(1)}, U_{ij}^{(2)}, ...$ are independent and identically distributed with positive mean value $\mu_{ij}$, it follows that $X_i$ is a pure jump process, $i = 1, ..., n$. In particular, observe that the infinite sum in (28) indicates that the number of events occurring in the interval $[0, t]$ is unbounded. The possibility of having an infinite number of events in $[0, t]$, however, may cause various technical difficulties. In particular, this may cause simulations to break down since an infinite number of events need to be generated and handled. To avoid these difficulties, we always assume that the number of events occurring in any finite interval is finite with probability one. A pure jump process satisfying this assumption is said to be *regular*. For more details we refer to ([6]). Furthermore, it can be shown that the system states $\phi = \phi(\boldsymbol{X})$ as well as all the criticality states are regular pure jump processes.

Stationary statistical properties of a system, can easily be estimated by running a single discrete event simulation on the system over a sufficiently long time horizon, or by working directly on the stationary probability distributions of the elementary processes. Sometimes, however, one needs to estimate how the statistical properties of the system evolve over time. In such cases it is necessary to run many simulations to obtain stable results.

### 4.1.2 Estimating availability, mean state and importance

The systems presented in this study are analyzed using the software Multicue$^{\text{TM}}$, developed at University of Oslo. In particular, in order to compute the importance measures, we estimate the system availability $\hat{h}_m(t)$, mean state of the system $\hat{\xi}$ and the Birnbaum importance measure $I_B^{(i)}$ as functions of time $t$, for $i = 1, ..., n, j = 1, ..., r_i$, and $m \in (0, M]$. All calculations are carried out using discrete event simulation. We run $N$ simulations on the system, where each simulation covers the time interval $[0, T]$. In each simulation we sample the values of the system state and the criticality states at each sampling point $t_1, t_2, ..., t_H = T$. Then, the asymptotic estimates are obtained by running the simulations over a sufficiently long time horizon $T$.

In order to obtain stable results, a reasonable length of the time horizon $T$ and sufficiently large number of simulations $N$ is required. In order to explain how $N$ can be chosen, we will consider pointwise estimates of the system availability at level $m$ and Birnbaum importance measure for component $i$. Denote the $k$th simulated value of the component state vector process at time $t \geq 0$ by $\boldsymbol{X}_k(t)$, $k = 1, 2, ..., N$. Then, for $i \in C$, $m \in (0, M]$ and $h = 1, 2..., H$ we introduce the following pointwise estimates:

$$\hat{h}_m(t_h) = \frac{1}{N} \sum_{k=1}^{N} I\big[\phi(\boldsymbol{X}_k(t_h)) \geq m\big],$$

$$\hat{I}_B^{(i)}(t_h) = \frac{1}{N} \sum_{k=1}^{N} I\big[\phi\big(X_i^+(t_h), \boldsymbol{X}_k(t_h)\big) \neq \phi(\boldsymbol{X}_k(t_h))\big].$$

Note that for $N = 1$, $\hat{h}_m(t_h)$ can be considered a Bernoulli variable with mean $h_m(t_h)$. Then, by repeating the simulation $N$ times, we obtain $\sum_{k=1}^{N} I\big[\phi(\boldsymbol{X}_k(t_h)) \geq m\big] \sim \text{Binomial}(N, h_m(t_h))$. Thus, we can derive an upper bound for the variance of the pointwise estimates:

$$\text{Var}\Big(\frac{1}{N} \sum_{k=1}^{N} I\big[\phi(\boldsymbol{X}_k(t_h)) \geq m\big]\Big) = \frac{h_m(t_h)(1 - h_m(t_h))}{N} \leq \frac{1}{4N}.$$

Hence, increasing the number of simulations $N$ provides an effective way of stabilizing the results for the pointwise estimates, by reducing the standard deviation of the estimates. For more details on this matter we refer to [6] and [7].

Furthermore, one can derive $T$ by computing the expected number of full cycles of component $i$, that is the number of times a component $i$, $i \in C$ deteriorates through all states $j \in S_i$. Hence, by defining the following relation:

$$\text{Expected number of cycles of component } i := \frac{T}{\sum_{j \in S_i} \mu_{ij}},$$

a reasonable time horizon can be found by choosing a sensible number of cycles for the component with the longest life cycle, $\max_{i \in C} \sum_{j \in S_i} \mu_{ij}$. In particular, it can be shown that the system availability converges fast towards its stationary value for certain distributions of the waiting times of component states. According to an example applied to a binary system, shown in [7], we have that all the component availabilities converge very fast towards their common stationary value, when all the components in the system have exponential lifetime distributions with equal means. As a result of this the system availability converges very fast towards its stationary value, and the same conclusion applies to the

Birnbaum measures of importance. A lower bound for the time horizon of the simulations $T$, and hence number of cycles is derived. Thus, it can be shown that approximately 10 component life cycles is sensible for the binary case with exponentially distributed component waiting times. This result provides a general idea for the lower bound of $T$. E.g for $\max_{i \in C} \sum_{j \in S_i} \mu_{ij} = 31$, we conclude that $T > 310$, is the lower bound for the time horizon of the simulations.

The main purpose of this study is to give an introduction to the $I_{B1}^{(i,j)}(m)$, $I_{B2}^{(i,j)}$ and $I_B^{(i)}$ importance measures and to illustrate how these are calculated. In particular, the numerical examples presented in this section focus mostly on the rank of the component importance. Hence, both $N$ and $T$ are chosen accordingly, compromising the computation time and the aim of the study. In order to further analyze differences in the simulated rates $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$, particularly when the purpose of the study is is to compare approximately equal rates for components $i \in C$, it is recommended to further reduce standard deviation of the estimates by increasing the number of simulations $N$. Furthermore, the choice of time horizon $T$ can be easily evaluated by inspecting the plots of the resulting estimates, and verifying that the stability in the estimates is reached.

The software Multicue$^{\mathrm{TM}}$has a built-in support for the calculation of the asymptotic Birnbaum measure of importance, $I_B^{(i)}$ for arbitrary complex directed network flow systems. The measure is simulated using the definition described by the formula (18), that is for each simulation at time $t \in [0, T]$, a probability that component $i$ is critical for the system is sampled. The simulated values at $t = T$ for the analyzed systems are reported. Due to the stochastic nature of the waiting times in component states, the simulated probabilities fluctuate for each $t \in [0, T]$, and the mean value of the simulated measure, denoted by $\bar{I}_B^{(i)} = \frac{1}{T} \int_0^T I_B^{(i)}(t) dt$, is reported along with $I_B^{(i)}$. Note that in order to compute $\bar{I}_B^{(i)}$, a longer time horizon is required. At time $t = 0$ and $i \in C$, we have that $I_B^{(i)}(0)$ is either equal to 1 or 0, affecting the computed mean value. Additional simulations with longer time horizon are performed in order to obtain more reasonable results for $\bar{I}_B^{(i)}$. In particular, for the numerical examples examined in in subsections (4.2) - (4.4), we perform an additional simulation of the $\bar{I}_B^{(i)}$ measure with a time horizon $T^{'} = 10T$.

The asymptotic system availabilities for component $i$ in state $j$ and at level $m$, $h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ and asymptotic mean state of the system for component $i$ in state $j$, $\xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$, $i \in C$, $j \in S_i$, $m \in (0, M]$, are computed by applying pivotal decomposition, that is by conditioning on the state of the component and running the simulations over the sufficiently long time horizon. In particular, for each component $i$, a simulation is performed by conditioning on state $j$, returning the mean state of the system $\xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ and the probabilities $h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ for alle the levels $m$. Such simulation is done by replacing the distribution of the mean times spent in each state by a fixed number equal to maximal time horizon of the simulation, $T$ for the state $j$ we condition on, and a small number close to zero, e.g 0.01 for the latter states. Simulations are then repeated for each component $i$ and state $j$.

In sections (3.2) and (3.3) we derived an explicit formula for computing the values of importance measures $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$. After simulating the asymptotic availabilities and mean state of the system, the calculation of the measures is straight forward. For all the examples of the directed network flow systems analyzed in this section, we will assume the following set of possible states for component $i \in C$, $S_i = \{0, 1, 2\}$, where state $j = 0$ is the complete failure state, and $j = 2$ is the perfect functioning state. Thus, for component $i$, $i \in C$, state $j = 1$ and given level $m \in (0, M]$ we have by the derived formula (22):

$$I_{B1}^{(i,1)}(m) = \Big(\sum_{k \in S_i} \mu_{ik}\Big)^{-2} \bigg(\Big(\sum_{k \neq j} \mu_{ik}\Big) \cdot h_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k \neq j} \Big(\mu_{ik} \cdot h_m((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\bigg)$$

$$= \Big(\sum_{j=0}^{2} \mu_{ij}\Big)^{-2} \bigg(\Big(\mu_{i0} + \mu_{i2}\Big) \cdot \hat{h}_m((\boldsymbol{e}_{i1})_i, \boldsymbol{\mu}) - \mu_{i0} \cdot \hat{h}_m((\boldsymbol{e}_{i0})_i, \boldsymbol{\mu}) - \mu_{i2} \cdot \hat{h}_m((\boldsymbol{e}_{i2})_i, \boldsymbol{\mu})\bigg)$$

$$= \Big(\sum_{j=0}^{2} \mu_{ij}\Big)^{-1} \bigg(\Big(q_{i0} + q_{i2}\Big) \cdot \hat{h}_m((\boldsymbol{e}_{i1})_i, \boldsymbol{\mu}) - q_{i0} \cdot \hat{h}_m((\boldsymbol{e}_{i0})_i, \boldsymbol{\mu}) - q_{i2} \cdot \hat{h}_m((\boldsymbol{e}_{i2})_i, \boldsymbol{\mu})\bigg).$$

Similarly, for component $i$, $i \in C$ and state $j = 1$ we have by (26):

$$I_{B2}^{(i,1)} = \Big(\sum_{k \in S_i} \mu_{ik}\Big)^{-2} \bigg(\Big(\sum_{k \neq j} \mu_{ik}\Big) \cdot \xi((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu}) - \sum_{k \neq j} \Big(\mu_{ik} \cdot \xi((\boldsymbol{e}_{ik})_i, \boldsymbol{\mu})\Big)\bigg)$$

$$= \Big(\sum_{j=0}^{2} \mu_{ij}\Big)^{-2} \bigg(\Big(\mu_{i0} + \mu_{i2}\Big) \cdot \hat{\xi}((\boldsymbol{e}_{i1})_i, \boldsymbol{\mu}) - \mu_{i0} \cdot \hat{\xi}((\boldsymbol{e}_{i0})_i, \boldsymbol{\mu}) - \mu_{i2} \cdot \hat{\xi}((\boldsymbol{e}_{i2})_i, \boldsymbol{\mu})\bigg)$$

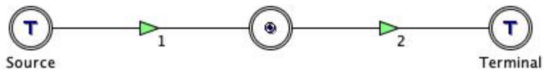
$$= \Big(\sum_{j=0}^{2} \mu_{ij}\Big)^{-1} \bigg(\Big(q_{i0} + q_{i2}\Big) \cdot \hat{\xi}((\boldsymbol{e}_{i1})_i, \boldsymbol{\mu}) - q_{i0} \cdot \hat{\xi}((\boldsymbol{e}_{i0})_i, \boldsymbol{\mu}) - q_{i2} \cdot \hat{\xi}((\boldsymbol{e}_{i2})_i, \boldsymbol{\mu})\bigg).$$

The following sections present examples of numerical studies of the three importance $I_{B1}^{(i,j)}(m)$, $I_{B2}^{(i,j)}$ and $I_B^{(i)}$, applied to directed network flow systems. The main purpose is to examine various properties of the measures. In particular, each examined case illustrates different type of sensitivity of the measures.

## 4.2 Scaling of mean waiting times of component states

As we have seen in sections (3.2) and (3.3), the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(j,j)}$ importance measures depend on the stationary probabilities for the component waiting times, $q_{ij}$'s. Thus, then main purpose of this example is to examine how this dependency emerges when the stationary probabilities for the component waiting times, $q_{ij}$'s, are equal in each equivalent state $j$ for all components $i \in C$, but a different scale for the mean waiting times $\mu_{ij}$'s, is applied for each component $i$. That is, we examine the sensitivity of the importance measures with respect to scaling of the mean waiting times of component states. In particular, we will see that an adjustment of the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(j,j)}$ measures is necessary to obtain reasonable results. In order to study such case, we consider a simple series system of two components $i \in C = \{1, 2\}$, shown in Figure (2).

Figure 2: A simple series system with two components




Denote the set of states for component $i \in C$ as $S_i = \{0, 1, 2\}$. In particular, assume that the times spent in state $j$ are exponentially distributed, with the expected waiting times for components $i = 1, 2$ as shown in Table (1). Hence, for each state $j$, the expected waiting times are ten times lager for component $i = 1$ in each state $j$. Furthermore, observe that the system state takes all values in the set $S_\phi = \{0, 1, 2, \}$, with $M = 2$ being the maximal flow of the system. The number of simulations is $N = 3000$, and the time horizon of the simulations, $T = 20000$ is chosen such that a reasonable number of expected life cycles of the component $i = 1$ with the largest expected time of one life cycle is ensured. In particular, the component $i = 1$ is expected to perform approximately 65 life cycles thorough the total simulation time.

Table 1: Distribution of times spent in state $j$ for component $i \in C$

| Order | State $j$ | Distribution $i = 1$ | Distribution $i = 2$ |
|---|---|---|---|
| 2 | 2 | Expon(200) | Expon(20) |
| 1 | 1 | Expon(100) | Expon(10) |
| 0 | 0 | Expon(10) | Expon(1) |

The stationary probabilities, $q_{ij} = \frac{\mu_{ij}}{\sum_{l=0}^{r_i} \mu_{il}}$ of component $i$ are equal in each $j$ for both components, i.e for $i = 1, 2$ we have $q_{i2} \approx 0.65$, $q_{i1} \approx 0.32$ and $q_{i0} \approx 0.03$. Hence, we expect the two components in series to be equally important for the system reliability. In particular, we will se how this assumption is reflected by the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(j,j)}$ importance measures.

Table 2: A simple series system with two components - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m = 1$ | | $m = 2$ | |
|---|---|---|---|---|
| | $j = 1$ | $j = 2$ | $j = 1$ | $j = 2$ |
| 1 | 0.0001 | 0.0001 | -0.0013 | 0.0007 |
| 2 | 0.0010 | 0.0010 | -0.0135 | 0.0074 |

Table 3: A simple series system with two components - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ |
|---|---|---|
| $j=1$ | -0.0012 | -0.0124 |
| $j=2$ | 0.0008 | 0.0084 |

Tables (2), (3) display the results form the simulations for $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(j,j)}$ measures. The absolute values of the rates are higher for the component $i=2$ for all the values of component state $j=1,2$ and system level $m=1,2$, implying higher importance of the second component. However, the asymptotic system availabilities $h_m$ and asymptotic mean state of the system $\xi$ depend only on the stationary distribution of component states, thus the simulated values $\hat{h}_m((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ and $\hat{\xi}((\boldsymbol{e}_{ij})_i, \boldsymbol{\mu})$ are approximately equal for both components in each $j \in S_i$ and system level $m \in (0,2]$.

Recall the derived expression for the measures, (23) and (27). Observe that the only term differentiating the calculated measures for each $i$ is the sum of expected waiting times in states $j\ S_i$. Thus, we have:

$$\Big(\sum_{j\in S_i} \mu_{ij}\Big)^{-1} = \frac{1}{310} = 0.0032 \quad \text{for} \quad i=1,$$
$$\Big(\sum_{j\in S_i} \mu_{ij}\Big)^{-1} = \frac{1}{31} = 0.032 \quad \text{for} \quad i=2.$$

Hence, the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(j,j)}$ measures are affected by the choice of scale of the expected waiting times in the component states. In order to avoid this problem, we introduce a scaled version of the importance measure,

$$I_{B1}^{*(i,j)}(m) = \big(\sum_{k\in S_i} \mu_{ik}\big) \cdot I_{B1}^{(i,j)}(m). \tag{29}$$

Similarly as for the $I_{B1}^{(i,j)}(m)$ measure, introduce a scaled version of the importance measure,

$$I_{B2}^{*(i,j)} = \big(\sum_{k\in S_i} \mu_{ik}\big) \cdot I_{B2}^{(i,j)}. \tag{30}$$

Tables (4), (5) show the computed values of the scaled measures. Finally, the simulated asymptotic Birnbaum importance measures, $I_B^{(i)}$, $i=1,2$ are shown in Table (6) and Figure (3). The simulated probabilities are approximately equal for both components, indicated in the figure by the the smooth lines representing the values of $\bar{I}_B^{(i)}$.

Hence, the scaled measures $I_{B1}^{*(i,j)}(m)$ and $I_{B2}^{*(i,j)}$ along with the $I_B^{(i)}$ measure lead to the same conclusion that the two components with equal stationary probabilities for each of the equivalent states $j$ indeed are equally important for

Table 4: A simple series system with two components - $I_{B1}^{*(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m=1$ | | $m=2$ | |
|---|---|---|---|---|
| | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| 1 | 0.0312 | 0.0312 | -0.4164 | 0.2290 |
| 2 | 0.0312 | 0.0312 | -0.4170 | 0.2294 |

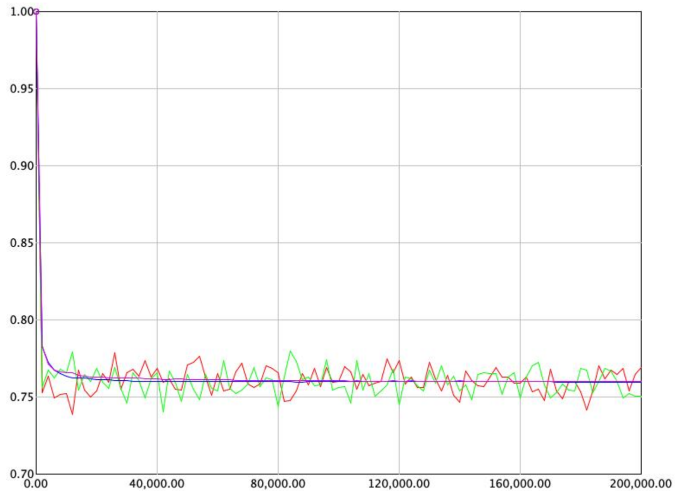
Table 5: A simple series system with two components - $I_{B2}^{*(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ |
|---|---|---|
| $j=1$ | -0.3852 | -0.3857 |
| $j=2$ | 0.2603 | 0.2605 |

Table 6: A simple series system with two components - $I_B^{(i)}$ measure for component $i$

| | $i=1$ | $i=2$ |
|---|---|---|
| $I_B^{(i)}$ | 0.7690 | 0.7507 |
| $\bar{I}_B^{(i)}$ | 0.7598 | 0.7601 |

Figure 3: A simple series system with two components - $I_B^{(i)}$ measures for $i=1,2$

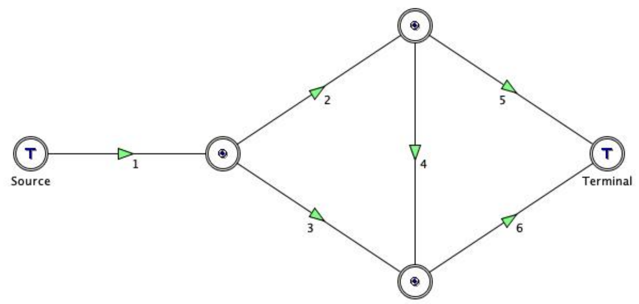


the system reliability. In particular, we have seen that the original measures $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ demonstrate sensitivity with respect to scaling of the mean waiting times in component states, while the asymptotic Birnbaum measure, $I_B^{(i)}$ does not have this property. Hence, the scaled versions of the measures are applied in order to determine the rank of component importance. Thus, when modelling waiting times of component states with different values of $\sum_{j \in S_i} \mu_{ij}$ for components $i \in C$, use of the scaled version of the measures is necessary. Otherwise, the unscaled measures can be used.

## 4.3 Ordering of component states

We will now consider the component importance sensitivity with respect to ordering of the component states. In particular, we will examine whether the three measures, $I_{B1}^{(i,j)}(m)$, $I_{B2}^{(i,j)}$ and $I_B^{(i)}$ are affected by change in component state ordering. The Figure (4) below shows the analyzed system, a bridge system with a series component. We will investigate two cases of ordering of the component states. The first case, where we assume a natural ordering of the component states, that is we assume that each component $i \in C$ starts at its perfect functioning state $j = 2$, followed by a transition to the intermediate state $j = 1$, until it reaches the failure state $j = 0$. After that the component is repaired back to its top level state again, and a new life cycle begins. The case is denoted by the ordering {2-1-0}. In the second case, for the component in series $i = 1$, we assume that the first and second order of the states switch places, that is component $i = 1$ start its life cycle at level $j = 1$, then after the first transition reaches its top level state $j = 2$, followed by the transition to failure state $j = 0$ at the end of the cycle, denoted by the ordering {1-2-0}).

Figure 4: Bridge system with a series component




The component set is given by $C = \{1, 2, 3, 4, 5, 6\}$. Denote set of states for component $i \in C$ as $S_i = \{0, 1, 2\}$. In particular, assume that the times spent in state $j$ are exponentially distributed, with the expected mean waiting times for components $i \in C$ as shown in Tables (7), (8). Also, observe that the system state takes all values in the set $S_\phi = \{0, 1, 2, \}$, with $M = 2$ being the maximal flow of the system. The time horizon of the simulations is set to be $T = 3000$ and number of simulation $N = 5000$.

Table 7: Distribution of times spent in state $j$ for component $i$, $i \in C$, with ordering {2-1-0}

| Order | State $j$ | Distribution |
|---|---|---|
| 2 | 2 | Expon(20) |
| 1 | 1 | Expon(10) |
| 0 | 0 | Expon(2) |

Table 8: Distribution of times spent in state $j$ for component $i = 1$, with ordering {1-2-0}

| Order | State $j$ | Distribution |
|---|---|---|
| 2 | 1 | Expon(10) |
| 1 | 2 | Expon(20) |
| 0 | 0 | Expon(2) |

The results presented in Tables (9) and (10) show the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ measures of component importance for the system. Recall that the two measures depend only on the stationary distribution of the component states. Thus, the asymptotic availabilities of the system $h_m$, and the asymptotic mean state of the system, $\xi$, for $i \in C$, $j = 1, 2$, $m \in (0, 2]$, and hence the corresponding importance measures are not affected by the ordering of the component states. Therefore, the simulated measures apply to both cases, that is the case when states of the components are assumed to transit from the top state ($j = 2$) to failure state ($j = 0$), and the case where the ordering of the states of component $i = 1$ is changed.

Furthermore, we expect the component in series, $i = 1$, to have the highest importance, and hence the highest absolute values of $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ measures for all states $j$ and system levels $m$. The simulated values show the symmetrical property of the system, with approximately equal importance for component pairs $i = 2$ and $i = 6$, $i = 3$ and $i = 5$, and where the bridge component $i = 4$ has the lowest importance. Thus, the simulated rates seem sensible. The rank of the two measures is presented in Table (11).

The simulated values of the $I_B^{(i)}$ measure, presented in Tables (12) and (13), show that the ordering of the component state transitions affects the asymptotic Birnbaum measure. In particular, the rank of component importance remains the same, but the simulated probability for component $i = 1$ with changed order of the state transitions is increased. Thus, the $I_B^{(i)}$ measure is sensitive with respect to ordering of the component states. Figure (5), corresponding to the result in Tables (12), (13), illustrates the difference in $I_B^{(i)}$ measure for component $i = 1$ (the red curve), with a visible upwards shift for the $I_B^{(1)}$ measure corresponding

Table 9: Bridge system with a series component - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m = 1$ | | $m = 2$ | |
|---|---|---|---|---|
| | $j = 1$ | $j = 2$ | $j = 1$ | $j = 2$ |
| 1 | 0.0019 | 0.0019 | -0.0174 | 0.0104 |
| 2 | 0.0002 | 0.0002 | -0.0006 | 0.0013 |
| 3 | 0.0001 | 0.0001 | -0.0002 | 0.0007 |
| 4 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 5 | 0.0001 | 0.0001 | -0.0001 | 0.0007 |
| 6 | 0.0002 | 0.0002 | -0.0006 | 0.0013 |

Table 10: Bridge system with a series component - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ |
|---|---|---|---|---|---|---|
| $j = 1$ | -0.0154 | -0.0004 | 0.0000 | 0.0000 | 0.0000 | -0.0004 |
| $j = 2$ | 0.0124 | 0.0015 | 0.0008 | 0.0000 | 0.0008 | 0.0015 |

Table 11: Bridge system with a series component - the ranks of the component importance measures corresponding to the results in Tables (9), (10)

| | Rank for $m = 1, 2$ and $j = 1, 2$ |
|---|---|
| $I_{B1}^{(i,j)}(m)$ | $1 > 2 \approx 6 > 3 \approx 5 > 4$ |
| $I_{B2}^{(i,j)}$ | $1 > 2 \approx 6 > 3 \approx 5 > 4$ |

to the ordering {1-2-0}.

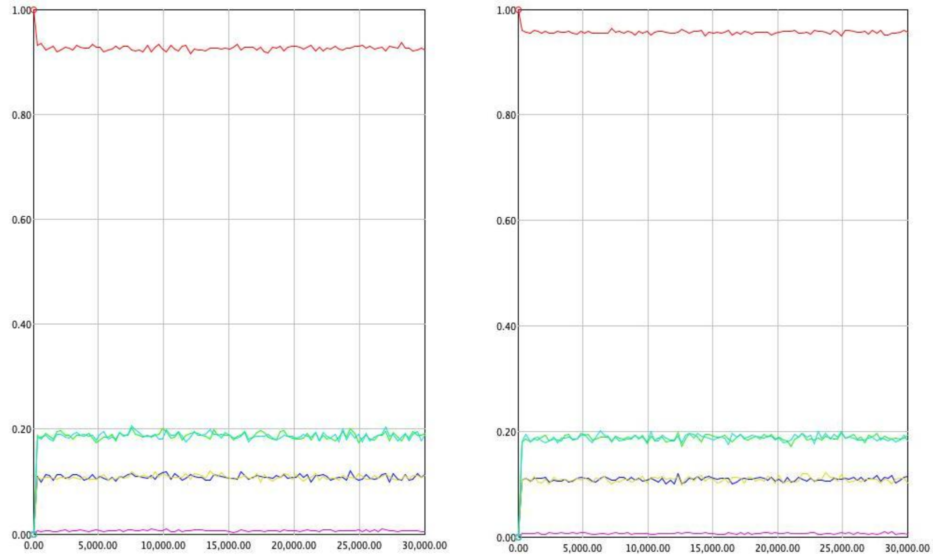
Table 12: Bridge system with a series component - $I_B^{(i)}$ measure with ordering {2-1-0} $\forall i \in C$

| | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ |
|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.9234 | 0.1932 | 0.1138 | 0.0060 | 0.1124 | 0.1892 |
| $\bar{I}_B^{(i)}$ | 0.9267 | 0.1879 | 0.1093 | 0.0072 | 0.1094 | 0.1881 |

Table 13: Bridge system with a series component - $I_B^{(i)}$ measure with ordering {1-2-0} for $i = 1$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ |
|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.9566 | 0.1936 | 0.1158 | 0.0056 | 0.1032 | 0.1832 |
| $\bar{I}_B^{(i)}$ | 0.9574 | 0.1874 | 0.1095 | 0.0071 | 0.1093 | 0.1875 |

Figure 5: Bridge system with a series component - $I_B^{(i)}$ measures - ordering {2-1-0} $\forall i \in C$ to the left, ordering {1-2-0} for $i = 1$ to the right




To see why this happen, we will examine how the Birnbaum measure of importance is calculated for component $i = 1$ for each case of the ordering of component states. Denote $Y(t)$ as flow through a bridge system at time $t$, that is the bridge module in Figure (4) consisting of components $i = 2, 3, 4, 5, 6$. Observe that the variable $Y(t)$ takes all values in the set $S_Y = \{0, 1, 2, 3, 4\}$. Thus, we now consider a simple series system with component state variables $X_1(t)$ and $Y(t)$ at time $t$. Recall the definition of the Birnbaum importance measure as the probability that component $i$ is critical at time $t$, i.e probability that a state change of component $i$ at time $t$ implies a system state change at time $t$ as well:

$$I_B^{(i)}(t) = P[\phi(X_i^+(t), \boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))].$$

First, observe that state of a series system of two components, denoted from now by $\phi(\boldsymbol{X}(t)) = \phi(X_1(t), Y(t))$, can be obtained by the following combinations of component states:

$$\begin{aligned}
&\phi(0,0)=\phi(1,0)=\phi(2,0)=\phi(0,1)=\phi(0,2)=\phi(0,3)=\phi(0,4)=0,\\
&\phi(1,1)=\phi(2,1)=\phi(1,2)=\phi(1,3)=\phi(1,4)=1,\\
&\phi(2,2)=\phi(2,3)=\phi(2,4)=2.
\end{aligned}$$

Hence, $S_\phi = \{0,1,2\}$ are the possible states of the system. Furthermore, observe that by the definition of upcoming state of component $i$ at time $t$, given by (15), we have for the {2-1-0} ordering of component state transitions:

$$\begin{aligned}
&\text{If } X_i(t)=2, \text{ then } X_i^+(t)=1,\\
&\text{If } X_i(t)=1, \text{ then } X_i^+(t)=0,\\
&\text{If } X_i(t)=0, \text{ then } X_i^+(t)=2.
\end{aligned}$$

For the {1-2-0} ordering of component state transitions we have:

$$\begin{aligned}
&\text{If } X_i(t)=1, \text{ then } X_i^+(t)=2,\\
&\text{If } X_i(t)=2, \text{ then } X_i^+(t)=0,\\
&\text{If } X_i(t)=0, \text{ then } X_i^+(t)=1.
\end{aligned}$$

Furthermore, observe that the Birnbaum measure of component importance can be decomposed by conditioning on the state $j$ of component $i$ at time $t$:

$$\begin{aligned}
&P\big[\phi(X_i^+(t),\boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t))\big] =\\
&=\sum_{j\in S_i} P\big[\phi(X_i^+(t),\boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t)) \mid X_i(t)=j\big]\cdot P\big[X_i(t)=j\big].
\end{aligned} \tag{31}$$

By conditioning on the state $y \in S_Y = \{0,1,2,3,4\}$ of the flow of the bridge module, we obtain:

$$\begin{aligned}
&P\big[\phi(X_i^+(t),\boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t)) \mid X_i(t)=j\big] =\\
&=\sum_{y\in S_Y} P\big[\phi(X_i^+(t),\boldsymbol{X}(t)) \neq \phi(\boldsymbol{X}(t)) \mid X_i(t)=j, Y(t)=y\big]\cdot P\big[Y(t)=y\big].
\end{aligned} \tag{32}$$

First, we will see how the Birnbaum importance measure is calculated for component $i=1$ when the component states transitions follow the {2-1-0} ordering. For $X_1(t)=2$ and $Y(t)=2$ we have:

$$\begin{aligned}
&P[\phi(X_1^+(t),Y(t)) \neq \phi(X_1(t),Y(t)) \mid X_1(t)=2, Y(t)=2)] =\\
&= P[\phi(1,2) \neq \phi(2,2) \mid X_1(t)=2, Y(t)=2)] = 1.
\end{aligned}$$

Observe that this holds for all $Y(t) \geq 2$, and we have:

$$P[\phi(X_1^+(t),Y(t)) \neq \phi(X_1(t),Y(t)) \mid X_1(t)=2, Y(t)\geq 2)] = 1.$$

Hence, for component $i=1$ in state $j=2$ and the bridge module $Y$ in state $y \geq 2$, the change of the state of component $i=1$ will cause change of the system state as well with probability one. In particular, $1 = \phi(1,2) \neq \phi(2,2) = 2$. Similarly, for $X_1(t) = 2$, and $Y(t) = 1$ and $Y(t) = 0$ respectively we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 2, Y(t) = 1] = 0,$$
$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 2, Y(t) = 0] = 0.$$

That is, for component $i = 1$ in state $j = 2$ and states of the bridge module $y = 0, 1$, the change of state of component $i = 1$ will not generate system state change. Thus, by the formula (32) we have:

$$\begin{aligned} &P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 2] = \\ &= 1 \cdot P[Y(t) \geq 2] + 0 \cdot P[Y(t) = 1] + 0 \cdot P[Y(t) = 0] = \\ &= P[Y(t) \geq 2]. \end{aligned}$$

Furthermore, for $X_1(t) = 1$ and $y \in S_Y$ we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 1, Y(t) = y] = \begin{cases} 1, & \text{if } Y(t) \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by the formula (32) we get:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 1] = P[Y(t) \geq 1].$$

Similarly, for $X_1(t) = 0$ and $y \in S_Y$ we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 1, Y(t) = y] = \begin{cases} 1, & \text{if } Y(t) \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by the formula (32) we get:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 0] = P[Y(t) \geq 1].$$

Finally, by combining the conditional probabilities defined in the expression (31), we obtain the following probability that component $i = 1$ with component state ordering {2-1-0} is critical at time $t$:

$$\begin{aligned} &I_B^{(i)}(t) = P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t))] = \\ &= q_{12}(t) \cdot P[Y(t) \geq 2] + q_{11}(t) \cdot P[Y(t) \geq 1] + q_{10}(t) \cdot P[Y(t) \geq 1] = \\ &= \big(q_{12}(t) + q_{11}(t)\big) \cdot P[Y(t) \geq 2] + q_{11}(t) \cdot P[Y(t) = 1] + q_{10}(t) \cdot P[Y(t) \geq 1]. \end{aligned} \tag{33}$$

where $q_{ij}(t) = P[X_i(t) = j]$, $j \in S_i$.

Similarly, the Birnbaum importance measure is calculated for component $i = 1$ when the component states transitions follow the {1-2-0} ordering. For $X_1(t) = 2$ and $y \in S_Y$ we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 2, Y(t) = y] = \begin{cases} 1, & \text{if } Y(t) \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by the formula (32) we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 2] = P[Y(t) \geq 1].$$

Furthermore, for $X_1(t) = 1$ and $y \in S_Y$ we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 1, Y(t) = y] = \begin{cases} 1, & \text{if } Y(t) \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by the formula (32) we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 1] = P[Y(t) \geq 2].$$

Furthermore, for $X_1(t) = 0$ and $y \in S_Y$ we have:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 0, Y(t) = y] = \begin{cases} 1, & \text{if } Y(t) \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, by the formula (32) we get:

$$P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t)) \mid X_1(t) = 0] = P[Y(t) \geq 1].$$

Finally, by combining the conditional probabilities defined in the expression (31), we obtain the following probability that component $i = 1$ with component state ordering {1-2-0} is critical at time $t$:

$$\begin{aligned} I_B^{(i)}(t) &= P[\phi(X_1^+(t), Y(t)) \neq \phi(X_1(t), Y(t))] = \\ &= q_{12}(t) \cdot P[Y(t) \geq 1] + q_{11}(t) \cdot P[Y(t) \geq 2] + q_{10}(t) \cdot P[Y(t) \geq 1] = \\ &= \big(q_{12}(t) + q_{11}(t)\big) \cdot P[Y(t) \geq 2] + q_{12}(t) \cdot P[Y(t) = 1] + q_{10}(t) \cdot P[Y(t) \geq 1]. \end{aligned} \tag{34}$$

We can now compare the two derived expressions for Birnbaum importance measure of component $i = 1$ at time $t$, shown by (33) and (34) for the the ordering {2-1-0} and {1-2-0}, respectively. In particular, we omit terms that are equal for both expressions, and recognize one particular term in each expression differentiating the two measures: $q_{11}(t) \cdot P[Y(t) = 1]$ for the {2-1-0} ordering of states of component $i = 1$, and $q_{12}(t) \cdot P[Y(t) = 1]$ for the {1-2-0} ordering of states of component $i = 1$. Recall that the stationary probabilities of waiting times in state $j$ are given by $q_{i2} \approx 0.63 > q_{i1} \approx 0.31 > q_{i0} \approx 0.06$. Hence, when $t \to \infty$, the calculated asymptotic Birnbaum measure of importance of component $i = 1$ having the {1-2-0} ordering of component states is greater that the importance measure computed for the {2-1-0} ordering.

## 4.4 Direction of the component flow

The purpose of this section is to investigate sensitivity of the three importance measures with respect to direction of the component flow. In particular, we will analyze a more complex system with two bridge components, denoted by $i = 3$ and $i = 5$. Subsection (4.4.1) examines how the importance of the components changes when the direction of the flow of the two bridge components varies. In total four different cases, denoted by Cases 1.1-1.4, are analyzed and compared.

Figures (6), (9), (12) and (15) illustrate the four systems we will investigate. Furthermore, subsection (4.4.2) takes a closer look at a special case of the system, where the component importance behavior is analyzed when component $i = 4$ is assumed to always be in its top level state.

For all the systems analyzed in this section assume the following distribution of times spent in state $j \in S_i$ of component $i \in C = \{1, 2, 3, 4, 5, 6, 7\}$, presented in Table (14). The time horizon of the simulations is set to be $T = 3000$ and number of simulations $N = 5000$.

Table 14: Distribution of times spent in state $j \in S_i$ for component $i$, $i \in C$

| Order | State $j$ | Distribution |
|---|---|---|
| 2 | 2 | Expon(20) |
| 1 | 1 | Expon(10) |
| 0 | 0 | Expon(1) |

We regard the system as a flow network and let the system state be the amount of the flow that can be transported through the network. In order to determine this we start out by identifying the binary minimal cut sets of the network, $K_j,\ j = 1, .., k.$, that is the minimal sets of components the removal of which will break the connection between the endpoints of the network. For example, for the first case shown in Figure (6), the minimal cut sets are $K_1 = \{1, 2\}, K_2 = \{2, 4\}, K_3 = \{4, 7\}$ and $K_4 = \{6, 7\}$, $K_5 = \{1, 3, 7\}$, $K_6 = \{2, 5, 6\}$ such that for the top level state $j = 2$ we have:

$$\sum_{i \in K_j} X_i(t) = 4 \quad \text{for} \quad j = 1, 2, 3, 4$$
$$\sum_{i \in K_j} X_i(t) = 6 \quad \text{for} \quad j = 5, 6.$$

We then apply the well-known max-flow-min-cut theorem, as described in section (2.3):

$$\phi(\boldsymbol{X}(t)) = \min_{1 \le j \le k} \sum_{i \in K_j} X_i(t) = 4.$$

We can easily verify this for the three other cases as well. Hence, for the directed network flow systems presented in this section, by considering all possible combinations of component states, it is easy to see that $\phi$ takes all values in the set $S_\phi = \{0, 1, 2, 3, 4\}$, with $M = 4$ being the maximal flow of the system.

### 4.4.1 Direction of component flow in complex system with bridge components

Figure 6: Complex system with bridge components - Case 1.1

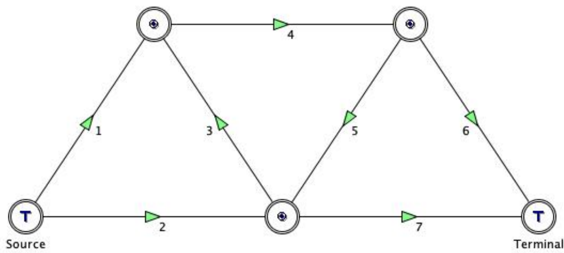



Figure (6) shows the first analyzed system (Case 1.1). Tables (15), (16) and (17) display the results from the simulations for the three measures. First, observe that the resulting values can be both positive or negative. Recall the interpretation of the $I_{B1}^{(i,j)}(m)$ importance measure as the change rate of the asymptotic system availability at level $m$ with respect to a small change in expected waiting time in state $j$ of component $i$. E.g, the simulated value $I_{B1}^{(7,1)}(2) = -0.00053$ represents the change in the asymptotic system availability at level $m = 2$ as a result of a small change, $\Delta$, in expected waiting time in state $j = 1$ of component $i = 7$. Hence, a small increase in $\mu_{71}$, $\Delta > 0$, will result in reduction in asymptotic system availability at level $m = 2$. Similarly, for the top level state $j = 2$, the measure $I_{B1}^{(7,2)}(2) = 0.00111 > 0$ reflects that the increased mean waiting time will improve the asymptotic system availability at level $m = 2$. Finally, $I_{B2}^{(i,j)}$ importance measure can be interpreted as the change rate of the asymptotic mean state of the system with respect to a small change in expected waiting time in state $j$ of component $i$.

Table 15: Complex system - Case 1.1 - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m = 1$ | | $m = 2$ | | $m = 3$ | | $m = 4$ | |
|---|---|---|---|---|---|---|---|---|
| | $j = 1$ | $j = 2$ | $j = 1$ | $j = 2$ | $j = 1$ | $j = 2$ | $j = 1$ | $j = 2$ |
| 1 | 0.00003 | 0.00003 | -0.00010 | 0.00056 | -0.00313 | 0.00249 | -0.00363 | 0.00200 |
| 2 | 0.00007 | 0.00007 | -0.00054 | 0.00112 | -0.00732 | 0.00458 | -0.00363 | 0.00200 |
| 3 | 0.00000 | 0.00000 | 0.00000 | 0.00002 | 0.00003 | 0.00003 | 0.00001 | 0.00000 |
| 4 | 0.00006 | 0.00006 | -0.00030 | 0.00097 | -0.00582 | 0.00383 | -0.00364 | 0.00200 |
| 5 | 0.00000 | 0.00000 | 0.00001 | 0.00001 | 0.00003 | 0.00003 | -0.00001 | 0.00000 |
| 6 | 0.00003 | 0.00003 | -0.00010 | 0.00056 | -0.00313 | 0.00249 | -0.00364 | 0.00200 |
| 7 | 0.00007 | 0.00007 | -0.00053 | 0.00111 | -0.00731 | 0.00458 | -0.00364 | 0.00200 |

Table 16: Complex system - Case 1.1 - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $j=1$ | -0.00683 | -0.01142 | 0.00004 | -0.00969 | 0.00002 | -0.00684 | -0.01142 |
| $j=2$ | 0.00508 | 0.00776 | 0.00004 | 0.00687 | 0.00005 | 0.00508 | 0.00776 |

Table 17: Complex system - Case 1.1 - $I_B^{(i)}$ measure for component $i$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.5222 | 0.7046 | 0.0112 | 0.6738 | 0.0158 | 0.5258 | 0.7066 |
| $\bar{I}_B^{(i)}$ | 0.5280 | 0.7084 | 0.0141 | 0.6690 | 0.0141 | 0.5281 | 0.7086 |

Figures (7) and (8) show the the simulated asymptotic Birnbaum importance measure, $I_B^{(i)}$. Note the different scale of the probabilities shown on the y-axis in both figures. The values for the simulated $I_B^{(i)}$ measure reported in Table (17) correspond to the component criticality probabilities at time $T^{'} = 30000$. The probabilities fluctuate as a result of random variation in waiting times in each state. Therefore, $\bar{I}_B^{(i)}$, mean value of the simulated measure, might be useful when the difference between the simulated values of $I_B^{(i)}$ for different components is small. $\bar{I}_B^{(i)}$ is shown in figure (8) (the smooth lines) among with $I_B^{(i)}$.

Figure 7: Complex system - Case 1.1 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$

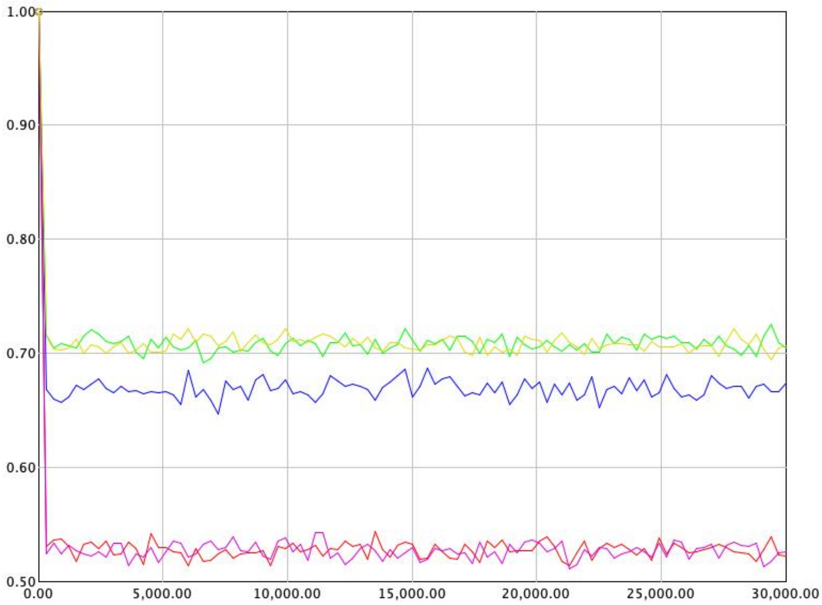


Figure 8: Complex system - Case 1.1 - $I_B^{(i)}$ measure of components $i = 3, 5$


By considering the absolute values of the simulated change rates, conclusions might be drawn about the impact on system availability or mean state of the system for different values of state $j$ and level $m$, and hence the importance of each component in each case. Tables (18) and (19) show the resulting ranks of the measures for the components $i \in C$. Furthermore, observe that for the $I_{B1}^{(i,j)}(m)$ measure, the overall rank of component importance may vary for different values of level $m$, that is the differences in component importance are not equally large for certain system levels. For example, for system level $m = 4$, the component importance is approximately equal for components $i = 1, 2, 7, 4, 6$. In particular, the $I_{B1}^{(i,j)}(m)$ importance measure provides more detailed information about component importance for different component states $j$ and system levels $m$. Such information may be useful when different ways of improving the system are considered, e.g. when costs of component improvements, resulting in increased expected waiting times in state $j$, vary for different components and component states. Thus, the rates may be used to support the selection of the component the analyst should focus on.

In general, the ranks of component importance lead to equivalent conclusions about component importance for the three measures. The asymptotic Birnbaum measure, $I_B^{(i)}$ provides a more general information about the component importance for the system reliability. Finally, observe that the two bridge components $i = 3$ and $i = 5$ form a directed cycle with component $i = 4$. Thus, the probabilities that the two bridge components are critical for the system are approximately equal, and the simulated values of $I_B^{(i)}$ are relatively low, compared to the other components in the system.

Table 18: Complex system - Case 1.1 - The ranks of the $I_{B1}^{(i,j)}(m)$ measure at system level $m$ and state $j = 1, 2$, corresponding to the results in Table (15)

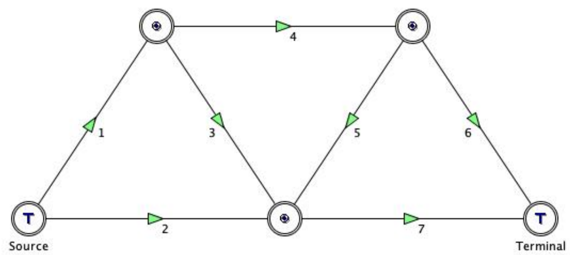
| | Rank state $j = 1, 2$ |
|---|---|
| $m = 1$ | $2 \approx 7 > 4 > 1 \approx 6 > 3 \approx 5$ |
| $m = 2$ | $2 \approx 7 > 4 > 1 \approx 6 > 3 \approx 5$ |
| $m = 3$ | $2 \approx 7 > 4 > 1 \approx 6 > 3 \approx 5$ |
| $m = 4$ | $2 \approx 7 \approx 4 \approx 1 \approx 6 > 3 \approx 5$ |

Table 19: Complex system - Case 1.1 - The ranks of the component importance measures corresponding to the results in Tables (16) and (17)

| | Rank state $j = 1, 2$ |
|---|---|
| $I_{B2}^{(i,j)}$ | $2 \approx 7 > 4 > 1 \approx 6 > 3 \approx 5$ |
| $I_B^{(i)}$ | $2 \approx 7 > 4 > 1 \approx 6 > 3 \approx 5$ |

Consider now the second analyzed system (Case 1.2), shown by Figure (9). In particular observe that the direction of the flow of component $i = 3$ is changed. Tables (20), (21) and (22) display the results from the simulations for the three importance measures.

Figure 9: Complex system with bridge components - Case 1.2




Figures (10) and (11) show the the simulated asymptotic Birnbaum importance measure, $I_B^{(i)}$. Note the different scale of the probabilities shown on the y-axis in both figures. The values for the simulated $I_B^{(i)}$ measure reported in Table (22) correspond to the component criticality probabilities at time $T^{'} = 30000$.

Table 20: Complex system - Case 1.2 - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m=1$ | | $m=2$ | | $m=3$ | | $m=4$ | |
|---|---|---|---|---|---|---|---|---|
| | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| 1 | 0.00006 | 0.00006 | -0.00038 | 0.00102 | -0.00630 | 0.00409 | -0.00363 | 0.00200 |
| 2 | 0.00003 | 0.00003 | -0.00012 | 0.00058 | -0.00367 | 0.00278 | -0.00364 | 0.00200 |
| 3 | 0.00000 | 0.00000 | 0.00000 | 0.00002 | 0.00003 | 0.00005 | 0.00000 | 0.00000 |
| 4 | 0.00003 | 0.00003 | 0.00003 | 0.00048 | -0.00225 | 0.00207 | -0.00364 | 0.00200 |
| 5 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 6 | 0.00003 | 0.00003 | 0.00003 | 0.00046 | -0.00217 | 0.00203 | -0.00364 | 0.00200 |
| 7 | 0.00010 | 0.00010 | -0.00103 | 0.00158 | -0.01049 | 0.00619 | -0.00364 | 0.00200 |

Table 21: Complex system - Case 1.2 - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $j=1$ | -0.01024 | -0.00740 | 0.00003 | -0.00583 | 0.00000 | -0.00574 | -0.01506 |
| $j=2$ | 0.00717 | 0.00540 | 0.00007 | 0.00459 | 0.00000 | 0.00452 | 0.00987 |

Table 22: Complex system - Case 1.2 - $I_B^{(i)}$ measure for component $i$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.6922 | 0.5520 | 0.0196 | 0.5112 | 0.0006 | 0.5000 | 0.8392 |
| $\bar{I}_B^{(i)}$ | 0.6859 | 0.5455 | 0.0206 | 0.5117 | 0.0005 | 0.5057 | 0.8460 |

Tables (23) and (24) show the resulting ranks of the measures for the components $i \in C$. In general, the ranks of component importance lead to equivalent conclusions about component importance for the three measures. Observe that the change in the direction of the flow of component $i = 3$ influence the rank of all the remaining components in the system. In particular, the importance of the two bridge components $i = 3$ and $i = 5$ with respect to each other is changed, such that $3 > 5$.

Figure 10: Complex system - Case 1.2 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$


Figure 11: Complex system - Case 1.2 - $I_B^{(i)}$ measure of components $i = 3, 5$

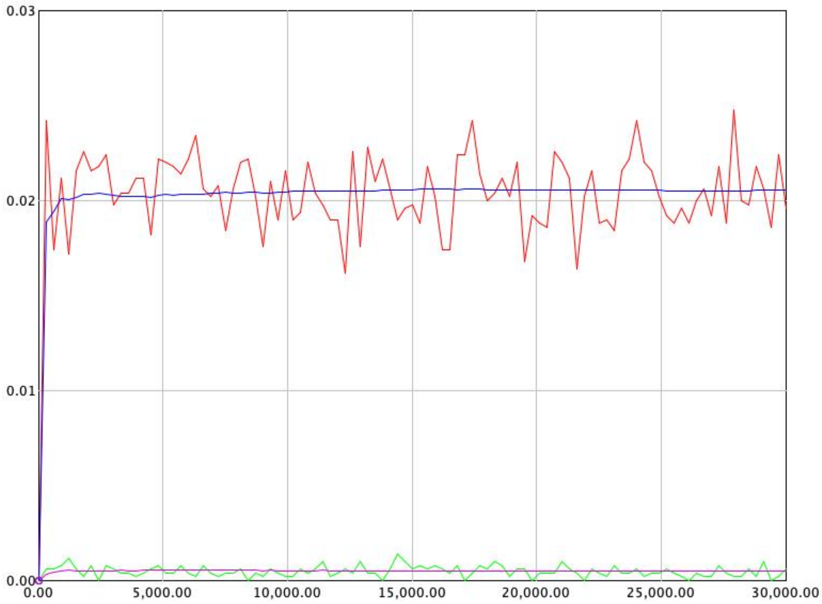


Table 23: Complex system - Case 1.2 - The ranks of the $I_{B1}^{(i,j)}(m)$ measure at system level $m$ and state $j = 1, 2$, corresponding to the results in Table (20)

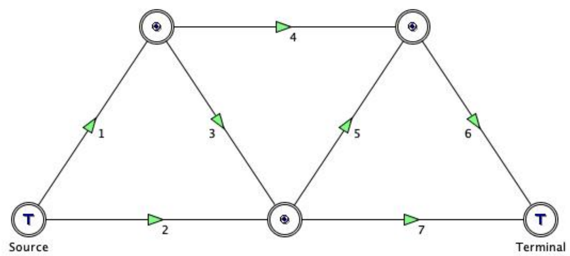
| | Rank state $j = 1, 2$ |
|---|---|
| $m = 1$ | $7 > 1 > 2 \approx 4 \approx 6 > 3 \approx 5$ |
| $m = 2$ | $7 > 1 > 2 > 4 \approx 6 > 3 \approx 5$ |
| $m = 3$ | $7 > 1 > 2 > 4 > 6 > 3 > 5$ |
| $m = 4$ | $7 \approx 1 \approx 2 \approx 4 \approx 6 > 3 \approx 5$ |

Table 24: Complex system - Case 1.2 - The ranks of the component importance measures corresponding to the results in Tables 21) and 22)

| | Rank state $j = 1, 2$ |
|---|---|
| $I_{B2}^{(i,j)}$ | $7 > 1 > 2 > 4 > 6 > 3 > 5$ |
| $I_{B}^{(i)}$ | $7 > 1 > 2 > 4 > 6 > 3 > 5$ |

Consider now the second analyzed system (Case 1.3), shown by Figure (12). In particular observe that the direction of the flow of component $i = 5$ is changed, compared with the preceding example. Note also that the directions of the flow of components $i = 3$ and $i = 5$ are the opposite to the system presented in Case 1.1. Tables (25), (26) and (27) display the results from the simulations for the three importance measures.

Figure 12: Complex system with bridge components - Case 1.3




Figures (13) and (14) show the the simulated asymptotic Birnbaum importance measure, $I_B^{(i)}$. Note the different scale of the probabilities shown on the y-axis in both figures. The values for the simulated $I_B^{(i)}$ measure reported in Table (27) correspond to the component criticality probabilities at time $T^{'} = 30000$.

Tables (28) and (29) show the resulting ranks of the measures for the compo-

Table 25: Complex system - Case 1.3 - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m=1$ | | $m=2$ | | $m=3$ | | $m=4$ | |
|---|---|---|---|---|---|---|---|---|
| | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| 1 | 0.00003 | 0.00003 | -0.00031 | 0.00069 | -0.00490 | 0.00355 | -0.00539 | 0.00297 |
| 2 | 0.00004 | 0.00004 | -0.00033 | 0.00072 | -0.00522 | 0.00371 | -0.00541 | 0.00297 |
| 3 | 0.00000 | 0.00000 | 0.00001 | 0.00004 | 0.00001 | 0.00015 | -0.00002 | 0.00010 |
| 4 | 0.00000 | 0.00000 | 0.00002 | 0.00004 | -0.00003 | 0.00024 | -0.00013 | 0.00022 |
| 5 | 0.00000 | 0.00000 | 0.00000 | 0.00004 | 0.00000 | 0.00016 | -0.00002 | 0.00010 |
| 6 | 0.00004 | 0.00003 | -0.00030 | 0.00069 | -0.00491 | 0.00356 | -0.00541 | 0.00297 |
| 7 | 0.00004 | 0.00004 | -0.00033 | 0.00072 | -0.00524 | 0.00372 | -0.00539 | 0.00296 |

Table 26: Complex system - Case 1.3 - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $j=1$ | -0.01056 | -0.01091 | 0.00000 | -0.00014 | -0.00001 | -0.01058 | -0.01092 |
| $j=2$ | 0.00724 | 0.00744 | 0.00029 | 0.00050 | 0.00030 | 0.00725 | 0.00744 |

Table 27: Complex system - Case 1.3 - $I_B^{(i)}$ measure for component $i$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.6786 | 0.6904 | 0.0718 | 0.1054 | 0.0730 | 0.6666 | 0.6764 |
| $\bar{I}_B^{(i)}$ | 0.6710 | 0.6829 | 0.0703 | 0.1053 | 0.0705 | 0.6717 | 0.6835 |

Table 28: Complex system - Case 1.3 - The ranks of the $I_{B1}^{(i,j)}(m)$ measure at system level $m$ and state $j=1,2$, corresponding to the results in Table (25)

| | Rank state $j=1,2$ |
|---|---|
| $m=1$ | $2 \approx 7 \approx 1 \approx 6 > 4 \approx 3 \approx 5$ |
| $m=2$ | $2 \approx 7 > 1 \approx 6 > 4 \approx 3 \approx 5$ |
| $m=3$ | $2 \approx 7 > 1 \approx 6 > 4 > 3 \approx 5$ |
| $m=4$ | $2 \approx 7 \approx 1 \approx 6 > 4 > 3 \approx 5$ |

nents $i \in C$. In general, the ranks of component importance lead to equivalent

Figure 13: Complex system - Case 1.3 - $I_B^{(i)}$ measure of components $i = 1, 2, 6, 7$


Figure 14: Complex system - Case 1.3 - $I_B^{(i)}$ measure of components $i = 3, 4, 5$


conclusions about component importance for the three measures. Observe that the change in the direction of the flow of component $i = 5$ affects the rank of all the remaining components in the system. In particular, the two bridge components $i = 3$ and $i = 5$ together with component $i = 4$ form a parallell structure with respect to each other. Hence, compared with Case 1.1 with the opposite

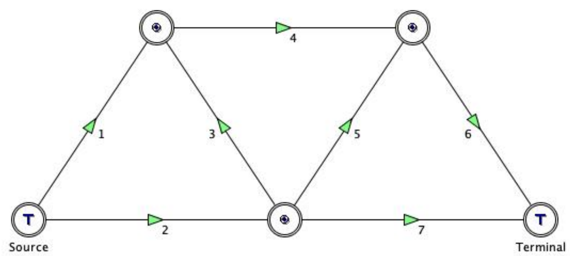
Table 29: Complex system - Case 1.3 - The ranks of the component importance measures corresponding to the results in Tables (26) and (27)

| | Rank state $j = 1, 2$ |
|---|---|
| $I_{B2}^{(i,j)}$ | $2 \approx 7 > 1 \approx 6 > 4 > 3 \approx 5$ |
| $I_B^{(i)}$ | $2 \approx 7 > 1 \approx 6 > 4 > 3 \approx 5$ |

direction of the flow of the two bridge components, we have that component $i = 4$ becomes less important than the component pair $i = 1, 6$. Also, similarly as for the first case we have $3 \approx 5$.

Consider now the last analyzed system (Case 1.4), shown by Figure (15). In particular observe that the direction of the flow of component $i = 3$ is changed, compared with the preceding example. Note also that the directions of the flow of components $i = 3$ and $i = 5$ are the opposite to the system presented in Case 1.2. Tables (30), (31) and (32) display the results from the simulations for the three importance measures.

Figure 15: Complex system with bridge components - Case 1.4




Figures (16) and (17) show the the simulated asymptotic Birnbaum importance measure, $I_B^{(i)}$. Note the different scale of the probabilities shown on the y-axis in both figures. The values for the simulated $I_B^{(i)}$ measure reported in Table (32) correspond to the component criticality probabilities at time $T^{'} = 30000$.

Tables (28) and (29) show the resulting ranks of the measures for the components $i \in C$. In general, the ranks of component importance lead to equivalent conclusions about component importance for the three measures. Observe that the change in the direction of the flow of component $i = 3$ again affects the rank of all the remaining components in the system. In particular, the importance of the two bridge components $i = 3$ and $i = 5$ with respect to each other is changed. Hence, compared with Case 1.2 with the opposite direction of the flow of the two bridge components, we have that $5 > 3$.

Table 30: Complex system - Case 1.4 - $I_{B1}^{(i,j)}(m)$ measure for component $i$ in state $j$ at level $m$

| $i$ | $m=1$ | | $m=2$ | | $m=3$ | | $m=4$ | |
|---|---|---|---|---|---|---|---|---|
| | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ | $j=1$ | $j=2$ |
| 1 | 0.00003 | 0.00003 | 0-00003 | 0.00047 | -0.00218 | 0.00204 | -0.00364 | 0.00200 |
| 2 | 0.00010 | 0.00010 | -0.00102 | 0.00157 | -0.01048 | 0.00618 | -0.00362 | 0.00199 |
| 3 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | -0.00002 | 0.00001 | -0.00001 | 0.00001 |
| 4 | 0.00003 | 0.00003 | 0.00003 | 0.00048 | -0.00226 | 0.00208 | -0.00363 | 0.00200 |
| 5 | 0.00000 | 0.00000 | 0.00001 | 0.00002 | 0.00005 | 0.00004 | 0.00001 | 0.00000 |
| 6 | 0.00006 | 0.00006 | -0.00039 | 0.00102 | -0.00629 | 0.00409 | -0.00364 | 0.00200 |
| 7 | 0.00003 | 0.00003 | -0.00012 | 0.00058 | -0.00365 | 0.00277 | -0.00363 | 0.00200 |

Table 31: Complex system - Case 1.4 - $I_{B2}^{(i,j)}$ measure for component $i$ in state $j$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $j=1$ | -0.00576 | -0.01502 | -0.00004 | -0.00584 | 0.00007 | -0.01025 | -0.00737 |
| $j=2$ | 0.00454 | 0.00984 | 0.00002 | 0.00459 | 0.00006 | 0.00717 | 0.00538 |

Table 32: Complex system - Case 1.4 - $I_B^{(i)}$ measure for component $i$

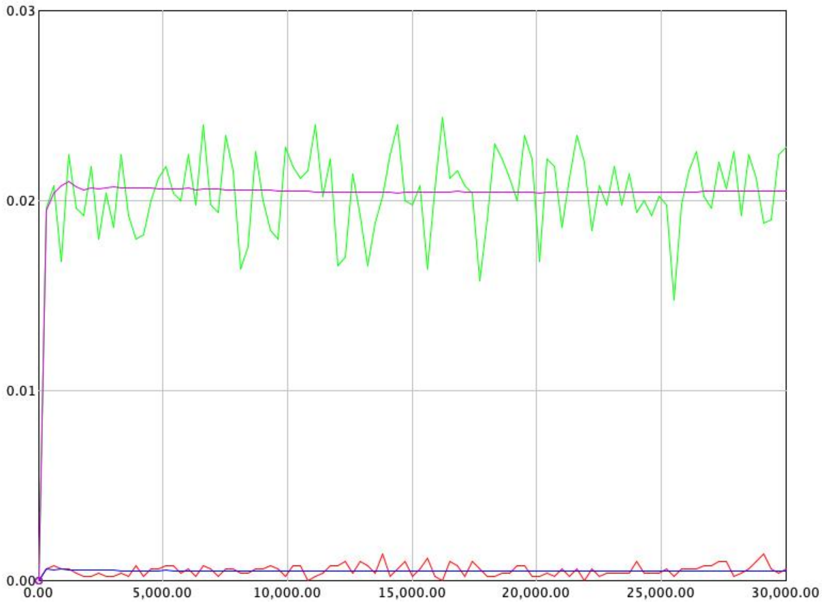
| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.5066 | 0.8492 | 0.0006 | 0.5138 | 0.0208 | 0.6876 | 0.5400 |
| $\bar{I}_B^{(i)}$ | 0.5058 | 0.8460 | 0.0005 | 0.5117 | 0.0205 | 0.6856 | 0.5456 |

Table 33: Complex system - Case 1.4 - The ranks of the $I_{B1}^{(i,j)}(m)$ measure at system level $m$ and state $j=1,2$, corresponding to the results in Table (30)

| | Rank state $j=1,2$ |
|---|---|
| $m=1$ | $2>6>7\approx 4\approx 1>5\approx 3$ |
| $m=2$ | $2>6>7>4\approx 1>5>3$ |
| $m=3$ | $2>6>7>4>1>5>3$ |
| $m=4$ | $2\approx 6\approx 7\approx 4\approx 1>5\approx 3$ |

Figure 16: Complex system - Case 1.4 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$


Figure 17: Complex system - Case 1.4 - $I_B^{(i)}$ measure of components $i = 3, 5$




Hence, the three importance measures, asymptotic Birnbaum measure $I_B^{(i)}$, as well as the two alternative importance measures $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$, provide equivalent conclusions about rank of component importance. However, note that the $I_{B1}^{(i,j)}(m)$ measure does not imply significant differences in component

Table 34: Complex system - Case 1.4 - The ranks of the component importance measures corresponding to the results in Tables (31) and (32)

| | Rank state $j = 1, 2$ |
|---|---|
| $I_{B2}^{(i,j)}$ | $2 > 6 > 7 > 4 > 1 > 5 > 3$ |
| $I_B^{(i)}$ | $2 > 6 > 7 > 4 > 1 > 5 > 3$ |

importance for all system levels $m$. In particular, for all the examined cases the ranking of $I_{B1}^{(i,j)}(m)$ is identical to the rank of the $I_{B2}^{(i,j)}$ and $I_B^{(i)}$ measures only at level $m = 3$. Summarizing the discussion we may simplify this by saying that in general, the three measures of component importance in a directed network flow system are sensitive with respect to the choice of direction of the component flow.

### 4.4.2 Further analysis of importance of the bridge components

For the four different cases of the system examined in the preceding section, we have seen that the importance of the bridge components $i = 3, 5$ vary for different directions of the flow of these components. In particular, the state of component $i = 4$ seems to have a significant effect on the importance of the bridge components. Thus, we will now examine a special case of the complex system shown in the preceding subsection, where the influence of component $i = 4$ is neutralized. For all the four cases of different flow directions of components $i = 3, 5$, assume that component $i = 4$ is always in its perfect functioning state, $j = 2$, that is $X_4(t) = 2$, $\forall\ t \in [0, T]$. Hence, the system becomes a bridge system with two bridge components $i = 3$ and $i = 5$, and observe that the components are in parallell or anti-parallell, depending on the direction of the flow of the components with respect to each other. We will investigate how this assumption affects the ranking of the component importance for the four different cases equivalent to the cases analyzed in the preceding subsection. In particular, we will see how the importance of the bridge components is affected.

Consider now a special case of the system shown in Figure (6), equivalent to the Case 1.1 with the additional assumption about state of component $i = 4$. Table (35) shows the simulated asymptotic Birnbaum measures for the system. The measures for this case, denoted by Case 2.1, are then presented in Figures (18) and (19), and the resulting rank of the component importance is shown in Table (36).

Hence, the analyzed system is symmetrical, reflected by the approximately identical values of the $I_B^{(i)}$ importance measure for components $i = 1, 2, 6, 7$. Furthermore, the two bridge components in anti-parallell can be considered as a one bridge component, with possible flow in both directions. Hence, the simulated probabilities for the two bridge components $i = 3, 5$ are approximately equal as well, similarly to the Case 1.1.

Table 35: Complex system - Case 2.1 - $I_B^{(i)}$ measure for component $i$

| | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.6966 | 0.6978 | 0.0190 | 0.5118 | 0.0176 | 0.6748 | 0.6726 |
| $\bar{I}_B^{(i)}$ | 0.6849 | 0.6849 | 0.0206 | 0.5137 | 0.0204 | 0.6852 | 0.6852 |

Figure 18: Complex system - Case 2.1 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$

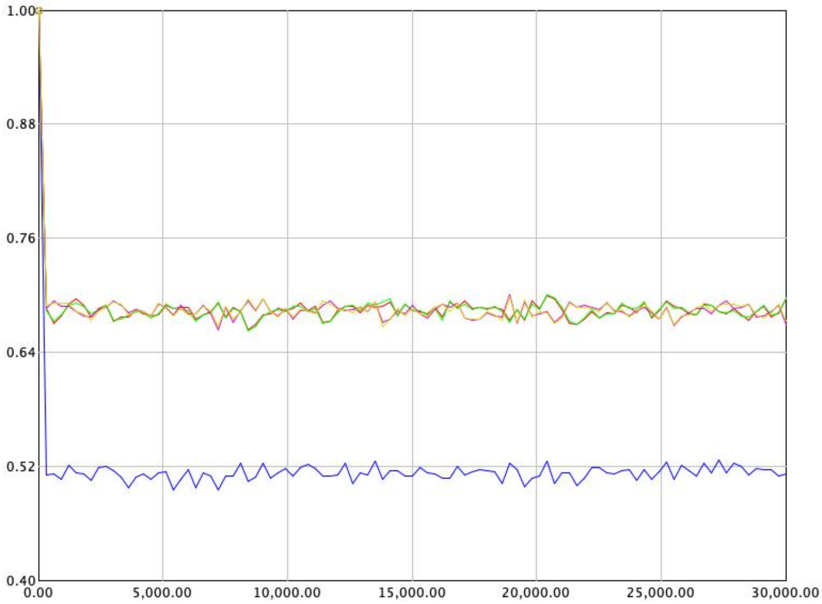



Table 36: Complex system - Case 2.1 - The rank of the $I_B^{(i)}$ measure corresponding to the results in Table (35)

| | Rank |
|---|---|
| $I_B^{(i)}$ | $1 \approx 2 \approx 6 \approx 7 > 4 > 3 \approx 5$ |

Consider now a special case of the system shown in Figure (9), equivalent to the Case 1.2 with the additional assumption about state of component $i = 4$. Again, the direction of the flow of the component $i = 3$ is changed, compared to the previous example. Table (37) shows the simulated asymptotic Birnbaum measures for the system. The measures for this case, denoted by Case 2.2, are then presented in Figures (20) and (21), and the resulting rank of the component importance is shown in Table (38).

Figure 19: Complex system - Case 2.1 - $I_B^{(i)}$ measure of components $i = 3, 5$


Table 37: Complex system - Case 2.2 - $I_B^{(i)}$ measure for component $i$

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $i=5$ | $i=6$ | $i=7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.8214 | 0.6378 | 0.0006 | 0.3180 | 0.0002 | 0.6238 | 0.8104 |
| $\bar{I}_B^{(i)}$ | 0.8124 | 0.6280 | 0.0007 | 0.3221 | 0.0007 | 0.6286 | 0.8129 |

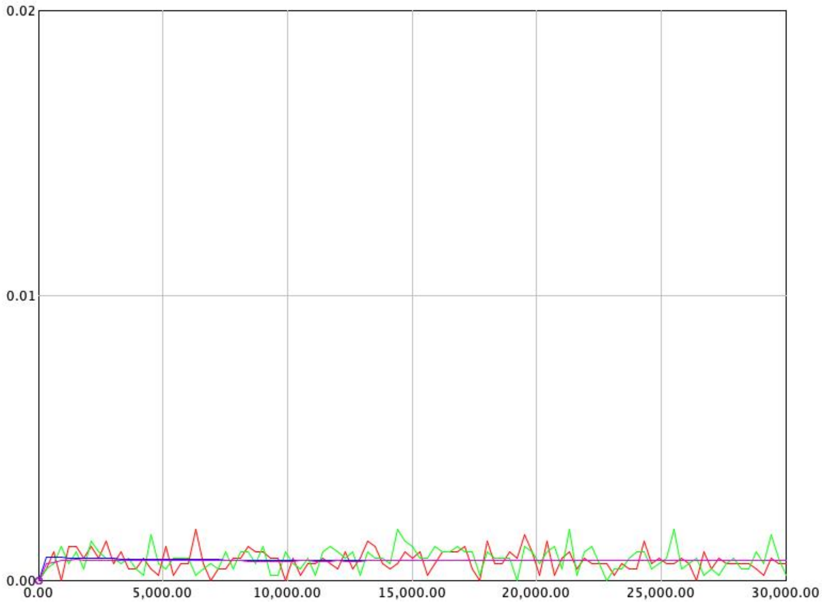
Table 38: Complex system - Case 2.2 - The rank of the $I_B^{(i)}$ measure corresponding to the results in Table (37)

| | Rank |
|---|---|
| $I_B^{(i)}$ | $1 \approx 7 > 2 \approx 6 > 4 > 3 \approx 5$ |

The analyzed system is symmetrical, reflected by the approximately identical values of the importance measures for component pairs $i = 1, 7$ and $i = 2, 6$. Observe that the two bridge components in parallell, with identical direction of the flow, can be considered as a one bridge component. The simulated measures of components $i = 3, 5$ are relatively low compared to the remaining components

Figure 20: Complex system - Case 2.2 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$


Figure 21: Complex system - Case 2.2 - $I_B^{(i)}$ measure of components $i = 3, 5$



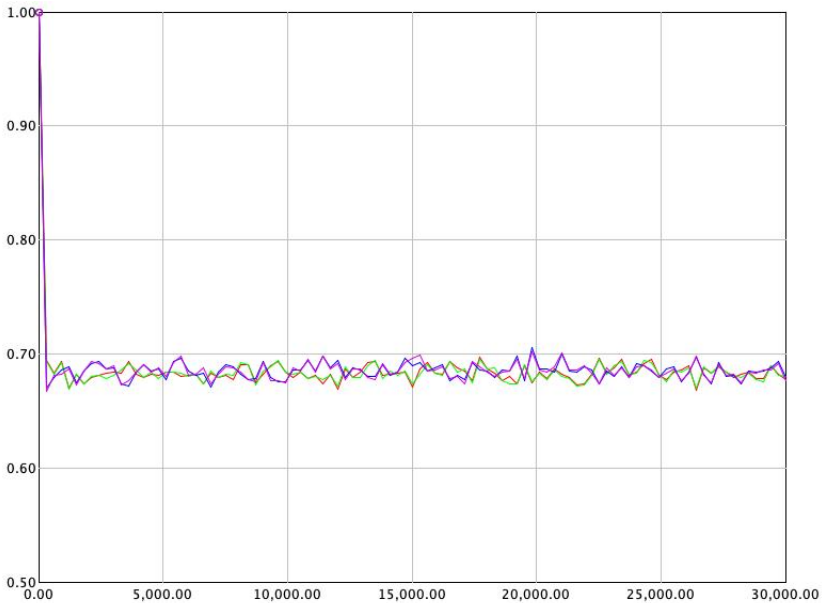

in the system, and approximately equal for the two bridge components. Hence, components $i = 3, 5$ are nearly irrelevant for the reliability of the system when component $i = 4$ is assumed to be functioning at its top level state for all $t \in [0, T]$.

Consider now a special case of the system shown in Figure (12), equivalent to the Case 1.3 with the additional assumption about state of component $i = 4$. Again, the direction of the flow of the component $i = 5$ is changed, compared to the previous example. Table (39) shows the simulated asymptotic Birnbaum measures for the system. The measures for this case, denoted by Case 2.3, are then presented in Figures (22) and (23), and the resulting rank of the component importance is shown in Table (40).

Table 39: Complex system - Case 2.3 - $I_B^{(i)}$ measure for component $i$

| | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.6782 | 0.6810 | 0.0220 | 0.0190 | 0.0218 | 0.6796 | 0.6770 |
| $\bar{I}_B^{(i)}$ | 0.6850 | 0.6851 | 0.0205 | 0.0202 | 0.0206 | 0.6849 | 0.6849 |

Figure 22: Complex system - Case 2.3 - $I_B^{(i)}$ measure of components $i = 1, 2, 6, 7$




Similarly as for the Case 2.1, the two bridge components in anti-parallell can be considered as a one bridge component, with possible flow in both directions. Hence, the analyzed system is symmetrical, reflected by the approximately identical values of the $I_B^{(i)}$ importance measure for components $i = 1, 2, 6, 7$. Furthermore, the simulated probabilities for the two bridge components $i = 3, 5$ are approximately equal as well, similarly to the Case 1.3.

Figure 23: Complex system - Case 2.3 - $I_B^{(i)}$ measure of components $i = 3, 4, 5$


Table 40: Complex system - Case 2.3 - The rank of the $I_B^{(i)}$ measure corresponding to the results in Table (39)

| | Rank |
|---|---|
| $I_B^{(i)}$ | $1 \approx 2 \approx 6 \approx 7 > 3 \approx 5 > 4$ |

Finally, consider a special case of the system shown in Figure (15), equivalent to the Case 1.4 with the additional assumption about state of component $i = 4$. Again, the direction of the flow of the component $i = 3$ is changed, compared to the previous example. Table (41) shows the simulated asymptotic Birnbaum measures for the system. The measures for this case, denoted by Case 2.4, are then presented in Figures (24) and (25), and the resulting rank of the component importance is shown in Table (42).

Table 41: Complex system - Case 2.4 - $I_B^{(i)}$ measure for component $i$

| | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $i = 5$ | $i = 6$ | $i = 7$ |
|---|---|---|---|---|---|---|---|
| $I_B^{(i)}$ | 0.6352 | 0.8176 | 0.0008 | 0.3256 | 0.0006 | 0.8110 | 0.6274 |
| $\bar{I}_B^{(i)}$ | 0.6284 | 0.8129 | 0.0007 | 0.3221 | 0.0007 | 0.8126 | 0.6287 |

Figure 24: Complex system - Case 2.4 - $I_B^{(i)}$ measure of components $i = 1, 2, 4, 6, 7$

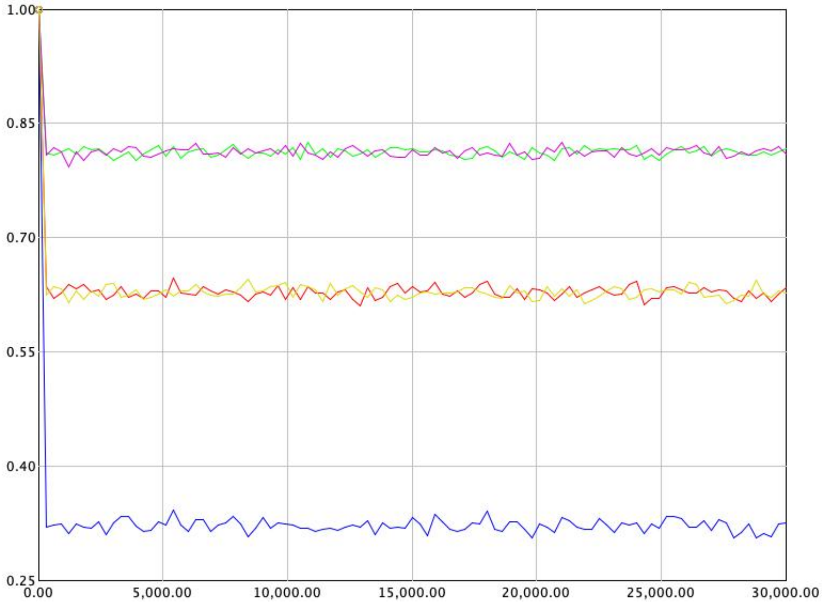
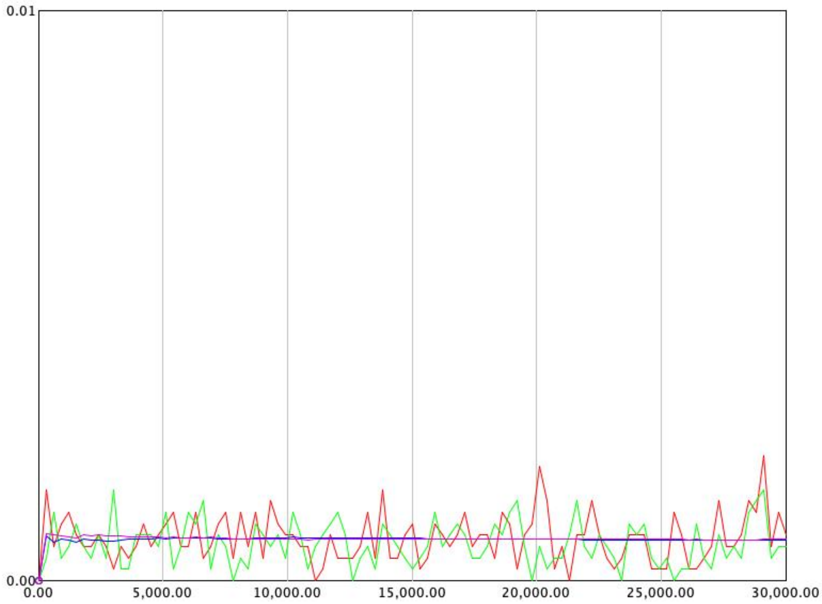



Figure 25: Complex system - Case 2.4 - $I_B^{(i)}$ measure of components $i = 3, 5$




The analyzed system is symmetrical, reflected by the approximately identical values of the importance measures for component pairs $i = 2, 6$ and $i = 1, 7$. Observe that the two bridge components in parallell, with identical direction of the flow, can be considered as a one bridge component. The simulated measures

Table 42: Complex system - Case 2.4 - The rank of the $I_B^{(i)}$ measure corresponding to the results in Table (41)

| | Rank |
|---|---|
| $I_B^{(i)}$ | $2 \approx 6 > 1 \approx 7 > 4 > 3 \approx 5$ |

of components $i = 3, 5$ are relatively low compared to the remaining components in the system, and approximately equal for the two bridge components. Hence, components $i = 3, 5$ are nearly irrelevant for the reliability of the system when component $i = 4$ is assumed to be functioning at its top level state for all $t \in [0, T]$.

Hence, by adding the assumption about the state of component $i = 4$, the importance of the two bridge components $i = 3, 5$ becomes approximately equal for the system reliability in the four examined cases. In particular, this implies that the state of component $i = 4$ does indeed affect importance of the two bridge components. Hence, when the effect of component $i = 4$ is neutralized, the direction of the flow for the two bridge components does not affect the rank of the two components with respect to each other. However, the overall rank of component importance, i.e the rank of the remaining components $i = 1, 2, 4, 6, 7$ changes as the assumption on component $i = 4$ is added, which further highlights the sensitivity of the $I_B^{(i)}$ measure with respect to the direction of the component flow.

## 4.5 Concluding remarks

In this study we have introduced a new and general approach to importance measures applied to multistate systems, with a special focus on directed network flow systems. In particular, we have introduced a generalization of the Birnbaum importance measure to multistate systems, $I_B^{(i)}$, defined as an asymptotic probability that the component is critical. Furthermore, two new importance measures for multistate systems $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ have been suggested. The two measures provide a further extension of the Birnbaum importance measure for multistate systems, expressed as partial derivatives of the asymptotic system availability and mean state of the system, respectively. Thus, the two measures can be interpreted as a change rate of the asymptotic system availability at level $m$ or mean state of the system with respect to a small change in expected waiting time in state $j$ of component $i$.

It should be noted that no measure of importance can be expected to be universally best irrespective of usage purpose, and comparing different measures is often of interest. In particular, the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ measures provides more detailed information about component importance. For the $I_{B1}^{(i,j)}(m)$ measure, the importance of a component is calculated separately for each component state and system level. It may happen that a component is very important at one state, and less important, or even irrelevant, at another. The asymptotic

Birnbaum measure provides a more unified measure of component criticality as $t \to \infty$.

In in section (4) we have shown how discrete event simulations can be a very useful tool in the study of repairable multistate systems. These methods are particularly convenient in the study of advanced importance measures. Such measures can be very complicated, and thus impossible to calculate analytically. By using discrete event simulations, however, this can be done in a very natural and intuitive way. Furthermore, the numerical examples presented in this section highlight different properties of the three measures. In particular, we have examined sensitivity of the measures with respect to scaling of expected waiting times in component states, ordering of the component states and direction of the component flow. We have seen an example of how the $I_B^{(i)}$ measure can be derived analytically, and why the measure is affected by ordering of component states, while the two latter measures do not have this property. Furthermore, we have concluded that the $I_{B1}^{(i,j)}(m)$ and $I_{B2}^{(i,j)}$ are sensitive with respect to scaling of the mean waiting times of component states, and proposed scaled versions of the measures in order to avoid this problem. The last numerical example provides a more general study of the usage and calculation of the importance measures, applied to a specific case of flow system, where the direction of the component flow varies.

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