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Pricing of Unit-Linked Insurance Policies with Respect to Turbulent Stock Markets

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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Chapter 1

Introduction

The objective of this master thesis is to conduct a survey on valuation theories for unit-linked life insurance policies based on economic theories. Life insurance companies are subjected to two major sources of uncertainty, namely mortality risk and financial risk. In traditional actuarial practice, financial risk is not treated explicitly. In recent years, new life insurance products are being introduced. The so-called unit-linked insurance is the kind of insurance where the amount of benefits for the policy-holder is linked to some specified reference portfolio whose market value fluctuates rather randomly. Thus we model the amount of such benefits by a stochastic process in connection to the price of stocks. Since the work of Black and Scholes [3], the classical way is to apply the Black-Scholes market dynamic where the volatility process is constant or deterministic. However it has become clear from observed prices that the Black-Scholes model is inconsistent with the reality of the market. Gradually, more sophisticated models have been introduced such as local volatility models [5] [8] where the volatility process is taken as a deterministic function of time and asset price. While the Hull and White model [14], the Heston model [12] and the SABR model [11] considered the volatility as an Itô process driven by an additional Brownian motion. In terms of the smoothness of the volatility process, these models have sample paths close as that of a Brownian motion. However, the recent result in [9] shows that based on statistical analysis of historical data that the volatility is much rougher. In fact, the authors showed that the rough fractional stochastic volatility model, where the log-volatility process is modeled by a fractional Brownian motion with Hurst parameter H of order 0.1, consistently reflects the behavior of financial time series data.

As a self-contained survey, we will first introduce all the necessary concepts and knowledge. Chapter 2 covers the foundation of probability theory, stochastic analysis, mathematical finance and related theories. Chapter 3 serves as a refresher of Life insurance mathematics. Chapter 4 will explain theories of simulation of fractional Brownian motion.

The rough fractional stochastic volatility model (RFSV) and related concepts will be introduced in Chapter 5. Chapter 6 carries out the simulation and the discussion of results. And finally we will review possible future work in Chapter 7.

Chapter 2

Preliminaries

In this chapter, we shall give an introduction to parts of probability theory, stochastic analysis and some basic concepts about insurance market that are relevant for this thesis. It can be skipped for readers familiar with such concepts.

2.1 Measure Theory and Probability Theory

Definition 2.1. *(σ-algebra) A σ-algebra $\mathfrak{A}$ is a family of subsets of a given non-empty set Ω with properties:*

(i) $\emptyset \in \mathfrak{A}$.

(ii) If $A \in \mathfrak{A}$ *, then* $A^{\complement} \in \mathfrak{A}$.

(iii) If $A_1, A_2, \ldots \in \mathfrak{A}$ *and* $A_i \neq A_j$ *for* $i \neq j$*, then* $A \equiv \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

Remark 2.2. *In the life insurance setting, we could consider $\mathfrak{A}$ as the entirety of market information at a certain time t, and the elements $A \in \mathfrak{A}$ are called market events*

Definition 2.3. *(σ-algebra generated by $\mathcal{U}$) Consider any family $\mathcal{U}$ of subsets of Ω, the smallest σ-algebra*

$$\mathcal{H}_{\mathcal{U}} := \bigcap \{\mathcal{H}; \mathcal{H}\sigma\text{-algebra of } \Omega,\ \mathcal{U} \subset \mathcal{H}\}$$

is called the σ-algebra generated by $\mathcal{U}$.

Example 2.4. *(Borel σ-algebra) The σ-algebra generated by all open sets of a topological space Ω is called the Borel σ-algebra on Ω, and is denoted $\mathcal{B}(\Omega)$. Its elements are called Borel sets.*

Example 2.5. *(σ-algebra generated by a function) For all$X : \Omega \to \mathbb{R}^n$, the σ-algebra generated by X is*

$$\{X^{-1}(B); B \in \mathcal{B}\}.$$

Definition 2.6. *(Measure, measurable space, measure space, probability measure and probability space) Let Ω be a set and $\mathfrak{A}$ a σ-algebra of subsets of Ω. A measure μ on $\mathfrak{A}$ is an extended real-valued function with properties:*

(i) $\mu(A) \geq 0$, *for all* $A \in \mathfrak{A}$.

(ii) $\mu(\emptyset) = 0$.

(iii) If $A_1, A_2, \ldots \in \mathfrak{A}$ *and* $A_i \cap A_j = \emptyset$ *for* $i \neq j$, *then* $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

The pair $(\Omega, \mathfrak{A})$ *is called a measurable space and the triplet* $(\Omega, \mathfrak{A}, \mu)$ *is called a measure space. If* $\mu(\Omega) = 1$, *we call* μ *a probability measure and denote it by* P, *and* $(\Omega, \mathfrak{A}, P)$ *is then called a probability space.*

Example 2.7. *(Dirac measure)For a set* A, *the so-called Dirac measure is defined by:*

$$\delta_\omega(A) := \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Example 2.8. *(Lebesque-Borel measure on* $[0,1]$*) Let* $\Omega = [0,1], \mathfrak{A} = \{A \cap [0,1] : A \in \mathcal{B}(\mathbb{R})\}$ *where* $\mathcal{B}(\mathbb{R})$ *is the borel* σ*-algebra on* $\mathbb{R}$, *then there exists a unique probability measure*

$$\lambda : \mathfrak{A} \to [0,1]$$

such that

$$\lambda([a,b]) = b - a, \qquad \text{for all } 0 \leq a \leq b \leq 1.$$

Remark 2.9. *The law of a uniformly distributed random variable on* $[0,1]$ *is the Lebesgue measure on* $[0,1]$.

Definition 2.10. *(Measurable function) If* $(\Omega, \mathfrak{A}, P)$ *is a given probability space, then a function* $X : \Omega \to \mathbb{R}^n$ *is called* $\mathfrak{A}$*-measurable if*

$$X^{-1}(B) := \{\omega \in \Omega; X(\omega) \in B\} \in \mathfrak{A}$$

for all borel sets $B \in \mathbb{R}^n$.

Definition 2.11. *(Random variable) A random variable is an* $\mathfrak{A}$*-measurable function* $X : \Omega \to \mathbb{R}^n$. *The probability law of X, denoted* $\mu_X(B)$, *is defined by*

$$\mu_X(B) = P(X^{-1}(B)).$$

Definition 2.12. *(Integral with respect to a measure) Let* X *be a positive random variable, then*

$$\begin{aligned} E[X] &:= \int_\Omega X(\omega) P(\mathrm{d}\omega) \\ &:= \lim_{n\to\infty} \sum_{i=1}^{n\cdot 2^n} i \cdot 2^{-n} P(A_{i,n}), \end{aligned}$$

where

$$A_{i,n} := \begin{cases} \{(i+1)2^{-n} > X \geq i \cdot 2^{-n}\}, & i = 0, \cdots, n \cdot 2^n - 1, \\ \{X \geq n\}, & i = n \cdot 2^n. \end{cases}$$

For general X, *we can always write* $X = X^+ - X^-$ *where*

$$X^+ = max(X, 0), X^- = max(-X, 0),$$

if $E[X^+], E[X^-] < \infty$ then we define:

$$E[X] := \int_\Omega X(\omega)P(\mathrm{d}\omega)$$

and X is called P-integrable

Definition 2.13. *(Expectation) if $\int_\Omega |X(\omega)|\,\mathrm{d}P(\omega) < \infty$ for a random variable X, then the expectation of X is*

$$E[X] = \int_\Omega X(\omega)\,\mathrm{d}P(\omega) = \int_{\mathbb{R}^n} x\,\mathrm{d}\mu_X(x)$$

Definition 2.14. *(Independence)*

- *Two events $A, B \in \mathcal{F}$ are called independent if*

$$P(A \cap B) = P(A) \cdot P(B).$$

- *A collection of families $\mathcal{H}_i$ of measurable sets is independent if*

$$P(H_{i_1}, \cap \cdots \cap H_{i_k}) = P(H_{i_1}) \cdots P(H_{i_k}), \quad \textit{for all } i, k$$

  where i is the index for the families and k is the index of the sets within each family i.

- *A collection of random variablesX_i is independent if the collection of generated σ-algebras $\mathcal{H}_{X_i}$ is independent.*

Definition 2.15. *(Stochastic process) A stochastic process is a parameterized collection of random variables*

$$\{X_t\}_{t \in \mathbb{T}}$$

defined on a probability space $(\Omega, \mathfrak{A}, P)$. $\mathbb{T}$ is the parameter space, and we consider a finite timeline$[0, T]$ where $T > 0$ is usually the exercise time of a stock.

Note that for each

- $t \in \mathbb{T}$ fixed, we have a random variable

$$\omega \to X_t(\omega); \qquad \omega \in \Omega.$$

- $\omega \in \Omega$ fixed, we have a function

$$t \to X_t(\omega); \qquad t \in \mathbb{T}$$

  which is called a path of X_t .

Remark 2.16. *Intuitively, we can think of t as time and ω as an individual of "particle", therefore $X_t(\omega)$ would then represent the position of the particle ω at time t. It is sometimes cognitively convenient to think of $X_t(\omega)$ as a function of two variables $X(t, \omega)$ from $\mathbb{T} \times \Omega$ into $\mathbb{R}^n$.*

Theorem 2.17. *(Kolmogorov's extension theorem) Let $v_{t_1,\dots,t_k}$ be probability measures on $\mathbb{R}^{nk}$ for all $t_1,\dots,t_k \in \mathbb{T}, k \in \mathbb{N}$, if*

(i) $v_{t_{\sigma(1)},\dots,t_{\sigma(k)}}(A_1 \times \cdots \times A_k) = v_{t_1,\dots,t_k}(A_{\sigma^{-1}(1)} \times \cdots \times A_{\sigma^{-1}(k)})$
for all permutations σ on $\{1,2,\dots,k\}$.

(ii) $v_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = v_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}}(A_1 \times \cdots \times A_k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n)$, *for all $m \in \mathbb{N}$.*

then there exists a probability space $(\Omega, \mathfrak{A}, P)$ and a stochastic process $\{X_t\}$ on $\Omega, X_t : \Omega \to \mathbb{R}^n$ such that

$$v_{t_1,\dots,t_k}(A_1 \times \cdots \times A_k) = P(X_{t_1} \in A_1, \cdots, X_{t_k} \in A_k)$$

for all $t_i \in T, k \in \mathbb{N}$ and all Borel sets A_i.

2.2 Martingale Theory

Definition 2.18. *(Filtration and filtered space) A filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathfrak{A})$ is a family of σ-algebras $\mathcal{F}_t \subset \mathfrak{A}$ such that*

$$\mathcal{F}_s \subset \mathcal{F}_t \qquad \text{`for all}$$

i.e. $\{\mathcal{F}_t\}$ is increasing.

A probability space $(\Omega, \mathfrak{A}, P)$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is called a filtered probability space and denoted $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathfrak{A}, P)$.

Definition 2.19. *(Adaptedness) A stochastic process $\{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathfrak{A}, P)$ is called $\mathcal{F}_t$-adapted if, for each $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.*

Definition 2.20. *(Conditional expectation) Let$(\Omega, \mathfrak{A}, P)$ be a probability space, $X : \Omega \to \mathbb{R}^n$ be a random variable such that $E[|X|] < \infty$. If $\mathcal{H} \subset \mathfrak{A}$ is a σ-algebra, the conditional expectation of X given $\mathcal{H}$ is a function, denoted by $E[X|\mathcal{H}]$,from Ω to $\mathbb{R}^n$ such that*

(i) $E[X|\mathcal{H}]$ *is $\mathcal{H}$-measurable.*

(ii) $\int_H E[X|\mathcal{H}]\,\mathrm{d}P = \int_H X\,\mathrm{d}P$, *for all $H \in \mathcal{H}$*

Consider another random variable $Y : \Omega \to \mathbb{R}^n$ such that $E[|Y|] < \infty$, $a, b \in \mathbb{R}$ and $\mathcal{G}$ a σ-algebra such that $\mathcal{G} \subset \mathcal{H}$. The conditional expectation has following properties:

(i) $E[aX + bY|\mathcal{H}] = aE[X|\mathcal{H}] + bE[Y|\mathcal{H}]$

(ii) $E[E[X|\mathcal{H}]] = E[X]$

(iii) $E[X|\mathcal{H}] = E[X]$ if X is independent of $\mathcal{H}$

(iv) $E[X|\mathcal{H}] = X$ if X is $\mathcal{H}$ measurable

(v) $E[Y \cdot X|\mathcal{H}] = Y \cdot E[X|\mathcal{H}]$ if Y is $\mathcal{H}$ measurable

(vi) $E[X|\mathcal{G}] = E[E[X|\mathcal{H}]|\mathcal{G}]$

Definition 2.21. *(Martingale) An n-dimensional stochastic process $\{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathfrak{A}, P)$ is called a martingale if*

(i) $E[|X_t|] < \infty$*, for all t.*

(ii) $E[X_s|\mathcal{F}_t] = X_t$*, for all* $s \geq t$

Definition 2.22. *(Stopping time) Consider a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$, a random variable $\tau : \Omega \to \mathbb{R}_{\geq 0}$ is called a $\mathcal{F}$-stopping time, if*

$$\{\tau \leq t\} = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t; \quad \text{for all } t \in \mathbb{R}_{\geq 0}$$

Definition 2.23. *(Local martingale) A stochastic process $\{X_t\}$ on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathfrak{A}, P)$ is called a local martingale if there exists an increasing sequence of $\mathcal{F}$-stopping times τ_k such that*

$$\tau_k \to \infty \quad \text{a.s. as } k \to \infty$$

and

$$X(t \wedge \tau_k) \text{ is an } \mathcal{F}\text{-martingale , for all } k.$$

Definition 2.24. *(Semi-martingale) A stochastic process $X(t)$ is called a semi-martingale if there exist two adapted process $\{u_t, t \in \mathbb{T}\}$ and $\{v_t, t \in \mathbb{T}\}$ such that $\int_0^t u(s)^2 \,\mathrm{d}s \leq \infty$, $\int_0^t |v(s)| \,\mathrm{d}s \leq \infty$ a.s. for all $t \in \mathbb{T} = [0, T]$, and*

$$X_t = X_0 + \int_0^t u_s \,\mathrm{d}B_s + \int_0^t v_s \,\mathrm{d}s, \qquad X_0 \in \mathbb{R}$$

$M_t = \int_0^t u_s \,\mathrm{d}B_s$ and $V_t = \int_0^t v_s \,\mathrm{d}s$ are the local martingale part and bounded variation part of X, respectively.

2.3 Brownian Motion and Fractional Brownian Motion

2.3.1 Brownian Motion

Definition 2.25. *(Brownian motion) A Brownian motion is a stochastic process $\{B_t\}_{t\geq 0}$ on Ω for a probability space $(\Omega, \mathfrak{A}, P)$ such that*

(i) $B_0 = 0$ *almost surely.*

(ii) B_t *has continuous paths* $t \to B_t(\omega)$.

(iii) B_t *has independent increments.*

$B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ *are independent for all* $0 \leq t_1 < t_2 < \cdots < t_k$.

(iv) B_t *has stationary Gaussian increments and is a Gaussian process*

- $(B_t - B_s) \overset{d}{=} B_{t-s} \sim N(0, t-s)$*, for all* $0 \leq s \leq t \leq T$
- *For all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, the random variable $Z = (B_{t_1}, \ldots, B_{t_k}) \in \mathbb{R}^{nk}$ has a multi-normal distribution*

2.3.2 Fractional Brownian Motion

Fractional Brownian motion was originally studied by Kolmogorov with in a Hilbert space framework. It's properties makes it especially suitable for modeling fractional noise in mathematical finance. We will give an introduction of it's definition and properties here.

Definition 2.26. *(Fractional Brownian motion) A centered and continuous Gaussian Process $B^H = \{B_t^H, t \leq 0\}$ is called a fractional Brownian motion(fBm) of Hurst parameter $H \in (0,1)$ if it has the covariance*

$$\rho_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{2.1}$$

Fractional Brownian motion has the following properties:

(i) $B_0^H = 0$ and $E[B_t^H] = 0$,for all $t \geq 0$.

(ii) B_t^H is self-similar.

For any constant $a > 0$, the process $\{a^{-H}B_{at}^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same distribution.

(iii) B_t^H has stationary increments.

It can be shown that the variance of increment in an interval $[s,t]$ is:

$$E(|B_t^H - B_s^H|^2) = |t-s|^{2H}$$

i.e. $B_t^H - B_s^H$ has the same distribution as B_{t-s}^H, for all $0 \leq s \leq t \leq T$.

(iv) B_t^H is a Gaussian process with variance $E[(B_t^H)^2] = t^{2H}$, for all $t \geq 0$.

In addition to these properties, we also note that

(i) B_t^H does not have independent increments for $H \neq \frac{1}{2}$.

For $H = \frac{1}{2}$, the covariance can be written as $\rho_{\frac{1}{2}}(t,s) = t \wedge s$, and the process $B^{\frac{1}{2}}$ is a standard Brownian motion.

However for $H \neq \frac{1}{2}$, set $X_n = B_n - B_{n-1}, n \geq 1$,the increment process $\{X_n\}_{n\geq 1}$is then a Gaussian stationary sequence with covariance function

$$\rho_H(n) = \frac{1}{2}((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}).$$

called fractional Gaussian noise.

This implies

- For $H > \frac{1}{2}$, the increments are positively related, and X_n exhibits long range dependence, i.e.

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1$$

and

$$\sum_{n=1}^{\infty} \rho_H(n) = \infty.$$

- For $H < \frac{1}{2}$, the increments are negatively related and

$$\sum_{n=1}^{\infty} |\rho_H(n)| \leq \infty.$$

(ii) B_t^H is not a semi-martingale for $H \neq \frac{1}{2}$.
The proof is omitted. As a consequence, the usual Itô stochastic calculus does not apply. The concept of Itô stochastic calculus will be introduced later.

2.4 Stochastic Analysis

2.4.1 Itô Integral

Consider a generalized model of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = b(t, X_t) + \sigma(t, X_t) \cdot \text{“}noise\text{”},$$

where b and σ are given functions.The random noise can be represented by a stochastic process W_t, then we have

$$\frac{\mathrm{d}X}{\mathrm{d}t} = b(t, X_t) + \sigma(t, X_t) \cdot W_t. \tag{2.2}$$

ideally for real life applications, we would want W_t to have the properties

(i) W_{t_i} and W_{t_j} are independent for $i \neq j$

(ii) $\{W_t\}$ is stationary, i.e. the joint distribution of $\{W_{t_1+t}, \cdots, W_{t_k+t}\}$ does not depend on t.

(iii) $E[W_t] = 0$ for all t.

However, it can be proven that such stochastic process does not have continuous paths. We then consider a discrete version of (2.2) for $0 < t_0 < t_1 < \cdots < t_m = t$ such that

$$X_{k-1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k) W_k \Delta t_k, \tag{2.3}$$

where

$$X_j = X(t_j), \quad W_k = W_{t_k}, \quad \Delta t_k = t_{k-1} - t_k$$

in hope to replace $W_k \Delta t_k$ with $\Delta V_k = V_{t_{k+1}} - V_{t_k}$ where $\{V_t\}_{t \geq 0}$ is a suitable stochastic process. It turns out that the Brownian motion B_t is exactly such process that satisfies the requirement of having stationary independent increments with mean 0. By replacing V_t with B_t, from equation(2.3), we have

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j. \tag{2.4}$$

It can then be proven that the limit of the right hand side of equation(2.4) exist in probability and we write

$$\int_0^t \sigma(s, X_s)\, \mathrm{d}B_s.$$

We obtain

$$X_t = X_0 + \int_0^t b(s, X_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s)\, \mathrm{d}B_s. \tag{2.5}$$

2.4.2 Itô Formula

We will utilize Itô formula to calculate Itô integral explicitly.

Definition 2.27. *(Itô process) Let B_t be a 1-dimensional Brownian motion on $(\Omega, \mathfrak{A}, P)$. An Itô process is a stochastic process X_t on $(\Omega, \mathfrak{A}, P)$ of the form*

$$X_t = X_0 + \int_0^t u(s,\omega)\,\mathrm{d}s + \int_0^t v(s,\omega)\,\mathrm{d}B_s \tag{2.6}$$

where

$$P\left[\int_0^t v(s,\omega)^2\,\mathrm{d}s < \infty \; for\ all\ t \geq 0\right] = 1$$

and

$$P\left[\int_0^t |u(s,\omega)|\,\mathrm{d}s < \infty \; for\ all\ t \geq 0\right] = 1$$

X_t can be written in the differential form, that is

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t \tag{2.7}$$

Definition 2.28. *(Itô formula) Let X_t be an Itô process of the form*

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t.$$

Let g be a twice continuously differentiable function on $[0,\infty) \times \mathbb{R}$. Then

$$Y_t = g(t, X_t)$$

is also an Itô process, and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}\,\mathrm{d}t + \frac{\partial g}{\partial x}\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} \cdot (\mathrm{d}X_t)^2, \tag{2.8}$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t) \cdot (\mathrm{d}X_t)$ is computed in accord with the rules

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_t = \mathrm{d}B_t \cdot \mathrm{d}t = 0, \quad \mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t.$$

The proof is omitted here.

2.4.3 SDEs with respect to Brownian Motion

Black and Scholes assumed in their work that the price of the underlying stock varies like the price of a bond, which yields a continuously compounding rate of return μ that is then randomly distorted by $\sigma B(t)$. This model is called geometric Brownian motion, to put it in mathematical terms, we have the following definition:

Definition 2.29. *(Black-Scholes model for stock prices) For $t \in \mathbb{T}$, let S_t be given by*

$$S_t = S_0 \exp(\mu \mathrm{t} + \sigma \mathrm{B_t}), \tag{2.9}$$

where μ is the drift of the stock and σ is the volatility. The current stock price is S_0, and B_t is standard Brownian motion.

Consider the geometric Brownian motion in Black-Scholes model

$$S_t = S_0 \exp(\mu t + \sigma B_t),$$

apply Itô formula to it, note

$$S_t = f(t, B_t),$$

and

$$\frac{\partial f(t,B_t)}{\partial t} = \mu f(t,B_t), \quad \frac{\partial f(t,B_t)}{\partial B_t} = \sigma f(t,B_t), \quad \frac{\partial^2 f(t,B_t)}{\partial B_t^2} = \sigma^2 f(t,B_t),$$

we have

$$\mathrm{d}f(t,B_t) = \mu f(t,B_t) + \sigma f(t,B_t)\,\mathrm{d}B_t + \frac{1}{2}\sigma^2 f(t,B_t)(\mathrm{d}B_t)^2.$$

Since $(\mathrm{d}B_t)^2 = \mathrm{d}t$, we then have

$$\mathrm{d}S_t = \left(\mu + \frac{1}{2}\sigma^2\right) S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t. \tag{2.10}$$

However, if we instead look for the solution to the stochastic differential equation of the form

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t,$$

a similar calculation will result in

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right). \tag{2.11}$$

2.5 Finance

2.5.1 Mathematical Finance

Definition 2.30. *(Contingent T-claim) A contingent T-claim as a financial contract that pays the holder a random amount Y at time T, is a lower-bounded $\mathcal{F}_T$-adapted random variable,where T is called the exercise time of the contingent claim and $\mathcal{F}_T$ is a general filtration.*

Note that insurance policies of the form $Y = f(S_t)$, where f is some function, are contingent claims. Y is $\mathcal{F}_T$-adapted since Y is dependent on S_t and therefore B_t.

Consider a financial market consisting of a stock, a bond and a contingent claim, where the price of the stock is modeled by the Black-Scholes model (2.9) i.e.

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}B_t,$$

the price of the bond by

$$\mathrm{d}R_t = rR_t\,\mathrm{d}t, \qquad R_0 = 1,$$

where r is the continuously compounding rate of return, which translate to an actuarial rate of return of

$$r_a = \exp(r) - 1,$$

and let Y_t denote an adapted stochastic process of the price of the contingent claim.

A portfolio strategy (a, b, c) consists of a_t, b_t and c_t numbers of stocks, bonds and contingent claims respectively, has the value at time t

$$V_t = a_t S_t + b_t R_t + c_t Y_t.$$

V_t is then also an adapted stochastic process.

Now we will consider the more general and formal definitions.

Definition 2.31. *(Financial market) A market is an $\mathcal{F}_t$-adapted $(n+1)$-dimensional Itô process $\{X_t\}_{t\in\mathbb{T}} = (X_t^{(0)}, X_t^{(1)}, \cdots, X_t^{(n)}); t \in \mathbb{T}$, which is assumed to be of the form*

$$\mathrm{d}X_t^{(0)} = \rho(t,\omega) X_t^{(0)}\,\mathrm{d}t; \qquad X_0^{(0)} = 1$$

and

$$\begin{aligned}\mathrm{d}X_t^{(i)} &= \mu_i(t,\omega)\,\mathrm{d}t + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)\,\mathrm{d}B_{j_t} \\ &= \mu_i(t,\omega)\,\mathrm{d}t + \sigma_i(t,\omega)\,\mathrm{d}B_t; \qquad X_t^{(i)} = x_i,\end{aligned}$$

where σ_i is row number i of the $n \times M$ matrix $[\sigma_{ij}]; 1 \le i \le n$.

The market is called normalized if $X_t^{(0)} \equiv 1$

$X_t^{(i)} = X_i(t,\omega)$ represents the price of asset number i at time t. The asset number 0 is called a risk-free asset and represents for example an investment in bonds, the return rate of it $\rho(t,\omega)$ is assumed to be bounded for simplicity. The assets number $1, \cdots, n$ are called risky assets representing for example investments in stocks.

If we regard the price of the risk-free assets as the unit price, and calculate the prices of other assets accordingly, we can always normalize the market. Since

$$X_t^{(0)} = \exp\left(\int_0^t \rho(s,\omega)\,\mathrm{d}s\right)$$

define

$$\xi(t) = X_t^{(0)^{-1}} = \exp\left(-\int_0^t \rho(s,\omega)\,\mathrm{d}s\right)$$

we have

$$\{\overline{X}_t\}_{t\in\mathbb{T}} = \xi(t)\{X_t\}_{t\in\mathbb{T}} = (1, \overline{X}_t^{(0)}, \cdots, \overline{X}_t^{(n)}); \quad t \in \mathbb{T}$$

and

$$\mathrm{d}\overline{X}_t^{(i)} = \mathrm{d}(\xi(t) X_t^{(i)}) = \xi(t)[(\mu_i - \rho X_i)\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_t]; \qquad 1 \le i \le n.$$

Definition 2.32. *(Self-financing portfolio) A portfolio in the market $\{X_t\}_{t\in\mathbb{T}}$ is a $(n+1)$-dimensional (t,ω)-measurable and $\mathcal{F}_t$-adapted stochastic process*

$$\{\theta_t\}_{t\in\mathbb{T}} = (\theta_0(t,\omega), \theta_1(t,\omega), \cdots, \theta_n(t,\omega)).$$

The value of a portfolio θ_t at time t is also a (t,ω)-measurable and $\mathcal{F}_t$-adapted stochastic process

$$V^\theta(t,\omega) = \theta_t \cdot X_t = \sum_{i=0}^{n} \theta_t^{(i)} X_t^{(i)}.$$

We refer to it as the value process.

The portfolio is called self financing if

$$\mathrm{d}V_t = \theta_t \cdot \mathrm{d}X_t = \sum_{i=0}^{n} \theta_t^{(i)}\, \mathrm{d}X_t^{(i)}$$

i.e.

$$V_t = V_0 + \int_0^t \theta_s \cdot \mathrm{d}X_s = V_0 + \int_0^t \sum_{i=1}^{n} \theta_t^{(i)}\, \mathrm{d}X_s^{(i)}.$$

The term $\int_0^t \theta_s \cdot \mathrm{d}X_s$ is also a (t,ω)-measurable and $\mathcal{F}_t$-adapted stochastic process refereed to as the return process.

Definition 2.33. *(Admissibility) A self-financing portfolio $\{\theta_t\}_{t\in\mathbb{T}}$ is called admissible if the corresponding value process V_t is a.s. bounded from below w.r.t. (t,ω), i.e. there exists $K_\theta < \infty$ such that*

$$V^\theta(t,\omega) \geq -K_\theta \qquad \textit{for } (t,\omega) \in \mathbb{T}\times\Omega.$$

Definition 2.34. *(Arbitrage) An admissible portfolio $\{\theta_t\}_{t\in\mathbb{T}}$ is called an arbitrage if the corresponding value process V_t satisfies*

$$V_0 = 0,\ V_t \geq 0\ a.s.\ and\ P[V_t > 0] > 0$$

Theorem 2.35. *(Arbitrage-free market) if there exist a measure Q on $\mathcal{F}_T$ such that $P \sim Q$ and the normalized market $\{\overline{X}_t\}_{t\in\mathbb{T}}$ is a local martingale w.r.t. Q. Then the market $\{X_t\}_{t\in\mathbb{T}}$ has no arbitrage. And Q is called an equivalent martingale measure.*

The goal is to find self-financing portfolios to hedge or replicate the contingent claims in an arbitrage-free market.

Lemma 2.36. *Let $\overline{X}_t = \xi(t)X_t$ be a normalized price process, if θ_t is an admissible portfolio in the market $\{X_t\}_{t\in\mathbb{T}}$ with value process*

$$V_t^\theta = \theta_t \cdot X_t = V_0^\theta + \int_0^t \theta_s \cdot \mathrm{d}X_s; \quad t \in \mathbb{T}$$

then θ_t is also an admissible portfolio for the normalized market $\{\overline{X}_t\}_{t\in\mathbb{T}}$ with value process

$$\overline{V}_t^\theta = \theta_t \cdot \overline{X}_t = \xi(t)V_t^\theta = V_0^\theta + \int_0^t \theta_s \cdot \mathrm{d}\overline{X}_s; \quad t \in \mathbb{T}$$

and vise versa.

Theorem 2.37. *Suppose a process $u(t,\omega)$ satisfies*

a)

$$E\left[\exp\left(\frac{1}{2}\int_0^{\mathrm{T}} \mathrm{u}^2(\mathrm{s},\omega)\,\mathrm{ds}\right)\right] < \infty$$

b)

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)X_t \quad \textit{for a.a. } (t,\omega)$$

with the market $X_t = (X_t^{(0)}, X_t^{(1)}, \cdots, X_t^{(n)})$

Define measure Q on $\mathcal{F}_T^{(m)}$ by

$$\mathrm{d}Q(\omega) = \exp\left(-\int_0^{\mathrm{T}} \mathrm{u(t,\omega)\,dB_t} - \frac{1}{2}\int_0^{\mathrm{T}} \mathrm{u^2(t,\omega)\,dt}\right)\mathrm{dP}(\omega)$$

then

(i) The market $X_t = (X_t^{(0)}, X_t^{(1)}, \cdots, X_t^{(n)})$ has no arbitrage.

(ii)

$$\tilde{B}_t := \int_0^t u(s,\omega)\,\mathrm{d}s + B_t$$

is a Brownian motion w.r.t Q.

(iii) The normalized market $\{\overline{X}_t\}_{t\in\mathbb{T}} = \xi(t)\{X_t\}_{t\in\mathbb{T}} = (1, \overline{X}_t^{(0)}, \cdots, \overline{X}_t^{(n)})$ has representation in terms of $\tilde{B}$

$$\begin{aligned}\mathrm{d}\overline{X}_t^{(0)} &= 0\\ \mathrm{d}\overline{X}_t^{(i)} &= \xi(t)\sigma_t^{(i)}\,\mathrm{d}\tilde{B}_t; \quad 1 \le i \le n.\end{aligned}$$

Definition 2.38. *(Attainability, replicating portfolio and completeness)*

- *The contingent claim Y is called attainable if there exists an admissible portfolio θ_t and a real number z such that*

$$Y = V_z^\theta(T,\omega) := z + \int_0^T \theta_t \cdot \mathrm{d}X_t \quad a.s.$$

  and

$$\begin{aligned}\overline{V}_z^\theta(T,\omega) &= z + \int_0^t \theta_s \cdot \mathrm{d}\overline{X}_s, \text{which by calculation}\\ &= z + \int_0^t \xi(t)\sum_{i=1}^n \theta_s^{(i)}\sigma_s^{(i)}\,\mathrm{d}\tilde{B}_s; \quad t \in \mathbb{T}\ is\ a\ Q\text{-}martingale\end{aligned}$$

  where

$$\tilde{B}_t := \int_0^t u(s,\omega)\,\mathrm{d}s + B_t$$

  is a $\mathcal{F}_t$-Brownian motion w.r.t Q.

- *If such θ_t exists, it is called a replicating or hedging portfolio for Y.*

- *The market $\{X_t\}_{t\in\mathbb{T}}$ is complete if every T-claim is attainable.*

2.5.2 Pricing Theory

Consider two kinds of European options for a claim Y with strike price K:

- the European call option, where

$$Y(\omega) = (X_i(T, \omega) - K)^+.$$

  This option gives the owner the right to buy one unit of security i at strike price K at time T. For each security i, if $X_i(T, \omega) > K$ at T, the owner will sell the security and obtain $X_i(T, \omega) - K$, otherwise not selling is the logical choice and results in a payoff of 0.

- the European put option, where

$$Y(\omega) = (K - X_i(T, \omega))^+.$$

  This option gives the owner the right to sell one unit of security i at strike price K at time T. The calculation of payoff is similar.

As we can see, the European options guarantee an amount of $Y(\omega)$ for the owner at time T. Suppose the buyer is willing to pay the price y for the claim, thus go into a debt of $-y$ at time 0, the debt can then be hedged to time T with value $V_{-y}^{\theta}(T, \omega)$. For the investment to not be unprofitable, it requires

$$V_{-y}^{\theta}(T, \omega) + Y(\omega) \geq 0 \quad \text{a.s.}$$

Therefore, the maximum price $p(Y)$ the buyer is willing to pay is

$$\sup\{y | \ \exists \varphi \text{ such that } V_{-y}^{\varphi}(T, \omega) := -y + \int_0^T \varphi_s \, \mathrm{d}X_s \geq -Y(\omega) \text{ a.s.}\}.$$

On the other hand, the seller sells the claim for the price z and the fortune can then be hedged to time T with value $V_z^{\theta}(T, \omega)$. Since the seller will need to pay the buyer $Y(\omega)$ at time T, for the transaction to not be unprofitable, it requires

$$V_z^{\theta}(T, \omega) \geq Y(\omega) \quad \text{a.s}$$

and thus the minimum price $q(Y)$ the seller is willing to accept is

$$\inf\{z | \ \exists \psi \text{ such that } V_z^{\psi}(T, \omega) := z + \int_0^T \psi_s \, \mathrm{d}X_s \geq Y(\omega) \text{ a.s.}\}$$

.

Theorem 2.39. *Let $\{X_t\}_{t \in \mathbb{T}}$ be a complete market. Suppose the conditions in Theorem 2.37 are met and we have Q, B as in the theorem. Let Y be a European claim such that $E_Q[\xi(t)Y] < \infty$. Then the price of Y is*

$$p(Y) = q(Y) = E_Q[\xi(t)Y].$$

2.6 Monte Carlo Simulation

Later on in this thesis, we will rely on Monte Carlo method to simulate various processes. We will give a brief introduction of the method here. The fundamental principle of Monte Carlo method is the law of large numbers.

Theorem 2.40. *Let $\{X_n\}_{n\in\mathbb{N}}$ be an infinite sequence of i.i.d. random variables where*

$$E[X_n] = \mu, \quad \textit{for all } n \geq 0.$$

Then

$$\overline{X}_n = \frac{1}{i}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu \quad \textit{as } n \to \infty.$$

In practice, we seek to reduce the problem we are interested in to the expected value of some transformation $\phi(\cdot)$ of $\{X_n\}_{n\in\mathbb{N}}$, such that

$$E[\phi(X_n)] \approx \frac{1}{i}\sum_{i=1}^{n} \phi(X_i),$$

for a large n.

We give a simple algorithm here:

Algorithm 2.1: Monte Carlo

Input: function $\phi(\cdot)$, distribution Θ, a large $n \in \mathbb{N}$

1 Draw $X_i \sim \Theta, \quad i \in [0, n]$

2 $\overline{\phi} \leftarrow \frac{1}{i}\sum_{i=1}^{n} \phi(X_i)$

3 return $\overline{\phi}$

Chapter 3

Life Insurance

Life insurance is a contract between the insurer and the holder of the insurance policy, upon which the insured pays premiums (or a single premium) to the insurer in exchange for the promise of payment when certain time is reached and/or conditions are met. Life insurance provides financial security for the policyholder and/or the beneficiaries of the policy against life-contingent events. It is characteristic of life insurance that the payment stream is strongly related to the state (e.g. alive or deceased, active or disabled) the insured is in. For example, consider the state space $\mathbb{S} = \{*, \diamond, \dagger\}$ where $*, \diamond, \dagger$ represent that the insured is alive, disabled and deceased respectively. Life insurance requires the assessment and managements of life-contingent risks which relies heavily on probability theory and mathematical finance. We have introduced basic concepts in both fields in Chapter 2, and will in this chapter focus specifically on introducing concepts most central to life insurance.

First we will give some examples of the most common types of insurance offered by typical life insurance companies

- Life insurance

  Consider a person x years of age, future life time of the individual is denoted by

  $$T = T(x),$$

  the age of death is then $x + T$. Under a life insurance contract, the time and amount of the payment(s) given to the dependant of the insured are subject to the random source T, and can be regarded as random variables.

  Most commonly, we will consider:

  Example 3.1. *(Pure endowment) The insured pays premium for a certain period of time, and receives a lump sum payment if he/she survives till the age of maturity of the policy. There is no payment otherwise.*

  Example 3.2. *(Term/permanent life insurance) In contrast to a pure endowment, the beneficiary of the insured receives payments if the insured dies before the age of maturity of the policy. There is no payment in the case of term life insurance if the insured survived till the age of maturity of the policy. However, for permanent life insurance there is always a payout no matter the time of death of the insured.*

  Example 3.3. *(Endowment) A combination of pure endowment and term/permanent life insurance which covers early death and also yields a payment if the insured reaches the age of maturity of the policy.*

Life insurance contract provides financial security to the dependant of the insured.

- Pension Insurance

  Example 3.4. *(Pension) The insured pays premium till the age of maturity of the policy and is thereafter payed a stream of payments referred to as life annuities by the insurer till death.*

  Pension insurance contract finances the living standard of the insured after retirement.

- Disability insurance

  Disability insurance provides indemnity on disablement instead of death. The difference here is that an initial waiting period is often introduced since in most cases the person recovers quickly from the temporary disability after an accident or illness. The introduction of the waiting period reduces the price of such policies.

Modern life insurance deals increasingly with the so-called unit-linked insurance. In an unit-linked policy, some or all of the premium is invested in an underlying stock index or fund, the death benefit depends on the performance of the equity investment but no lower than the benefit floor(investment guarantee). Thus the policyholder gets to take advantage of the equity investment in conjunction with mortality protection. In this case, we can consider it as if the insurance company issues a put option on its fund, and the pricing of such policies in essentially the pricing of put options.

Example 3.5. *(Guaranteed Minimum Maturity Benefit (GMMB))*
In GMMB, the payoff for the policyholder is

$$max(G, F_t)$$

where G is the guaranteed minimum return and F_t is the value of the fund at exercise time t. And then insurer's liability is

$$max(G - F_t, 0),$$

which is equivalent to a put option with strike price G.

In this thesis, we consider mostly regular insurance models

Definition 3.6. *(Regular insurance model) A regular insurance model would consist of*

- *a regular Markov chain $\{X_t\}_{t\in\mathbb{T}}$ with a state space $\mathbb{S}$,*
- *payout functions $a_{ij}(t)$ and $a_i(t)$,*
- *right continuous interest intensities ρ_t of bounded variation.*

In the following sections, the definition of relevant concepts will be introduced.

3.1 Markov Chains

If we consider an insured person who is in a state at time t, the state he/she is in can be represented by a stochastic process takes value in $\mathbb{S}$, where $\mathbb{S}$ is the state space of the insurance policy with all the possible relevant states an insured could be in.

Definition 3.7. *(Markov chain) Let $\{X_t\}_{t\in\mathbb{T}}$ be a stochastic process on a probability space $(\Omega, \mathfrak{A}, P)$ with state space $\mathbb{S}$ and parameter space $\mathbb{T} = [0, T] \in \mathbb{R}$. It is called a Markov chain if*

(i) $P\Big(X_{t_0} = s_0, \cdots, X_{t_n} = s_n\Big) > 0;$

(ii) $P\Big(X_{t_n} = s_n | X_{t_0} = s_0, \cdots, X_{t_{n-1}} = s_{n-1}\Big)$
$= P\Big(X_{t_n} = s_n | X_{t_{n-1}} = s_{n-1}\Big).$

for all $t_0 < \cdots < t_n \in \mathbb{T},\ s_0, \cdots, s_n \in \mathbb{S}$.

That is, the conditional probability of the state the insured would be in only depends on the last state, but not the path before. We say that Markov chain is a process "without memory".

Definition 3.8. *(Homogeneous Markov chain) A Markov chain $\{X_t\}_{t\in\mathbb{T}}$ is called homogeneous if*

$$P\Big(X_{s+h} = j | X_s = i\Big) = P\Big(X_{t+h} = j | X_t = i\Big) \qquad h > 0$$

for all s, t and $i, j \in \mathbb{S}$, provided that $P(X_s = i), P(X_t = i) \neq 0$

Definition 3.9. *(Transition probability) Let $\{X_t\}_{t\in\mathbb{T}}$ be a stochastic process on $(\Omega, \mathfrak{A}, P)$, then*

$$p_{ij}(s, t) := P\Big(X_t = j | X_s = i\Big), \quad 0 \leq s \leq t \leq T,\ i, j \in \mathbb{S}$$

is called the transition probability from state i at time s to state j at time t.

Theorem 3.10. *(Chapman-Kolmogorov equation) Let $\{X_t\}_{t\in\mathbb{T}}$ be a Markov chain with state space $\mathbb{S}$ and parameter space $\mathbb{T} = [0, T] \in \mathbb{R}$, if $P(X_s = i) > 0$ for $0 \leq s \leq t \leq u \leq T, i, k \in \mathbb{S}$, then*

$$p_{ik}(s, u) = \sum_{j\in\mathbb{S}} p_{ij}(s, t) p_{jk}(t, u)$$

In matrix form it can be written as

$$\mathbf{P}(s, u) = \mathbf{P}(s, t) \times \mathbf{P}(t, u).$$

See e.g. [17, p. 11] for proof.

Definition 3.11. *(Transition probability matrix) The matrix $\mathbf{P}(s, t) = (p_{ij}(s, t))_{i,j\in\mathbb{S}}$ is called transition (probability) matrix if*

(i) $p_{ij} \geq 0$.

(ii) $\sum_{j\in\mathbb{S}} p_{ij}(s, t) = 1$.

(iii) $p_{ij}(s,s) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$ *if* $P(X_s = i) > 0$.

(iv) $p_{ik}(s,u) = \sum_{j \in \mathbb{S}} p_{ij}(s,t)p_{jk}(t,u)$ *for* $0 \leq s \leq t \leq u \leq T, i, k \in \mathbb{S}$ *and* $P[X_s = i] > 0$.

Definition 3.12. *(Transition rate) Let* $\{X_t\}_{t\in\mathbb{T}}$ *be a Markov chain on a finite state space* $\mathbb{S}$. $\{X_t\}_{t\in\mathbb{T}}$ *is called regular, if*

$$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(t, t + \Delta t)}{\Delta t}, \quad i \neq j, i, j \in \mathbb{S}$$

$$q_i(t) = \sum_{j \neq i, j \in \mathbb{S}} q_{ij}(t) = \lim_{\Delta t \to 0} \frac{1 - p_{ii}(t, t + \Delta t)}{\Delta t}, \quad i \in \mathbb{S}$$

are well defined and continuous in time.

Furthermore

$$q_{ii}(t) := -q_i(t), \quad i \in \mathbb{S}.$$

The functions $q_{ij}(t)$ *and* $q_i(t)$ *are called transition rates of the Markov chain.*

Remark 3.13. $q_{ij}(t)$ *can be perceived as the probability of transitioning from state* i *to state* j *at the instant of time t, and* $q_i(t)$ *can be perceived as the probability of leaving state* i *at the instant of* t.

Definition 3.14. *(Transition rate matrix) The matrix* $\mathbf{Q}(t) = (q_{ij})_{i,j\in\mathbb{S}}$ *is called transition rate matrix if*

- $q_{ij} \geq 0$ *for* $i \neq j$.
- $0 \leq -q_{ii} \leq \infty$.
- $\sum_{j\in\mathbb{S}} q_{ij} = 0$.

Theorem 3.15. *(Kolmogorov equations) Let* $\{X_t\}_{t\in\mathbb{T}}$ *be a Markov chain on a finite state space* $\mathbb{S}$. *Then we have*

- *(Backward differential equation)*

$$\frac{\mathrm{d}}{\mathrm{d}s} p_{ij} = q_i(s)p_{ij}(s,t) - \sum_{k \neq i} q_{ik}(s)p_{kj}(s,t),$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{P}(s,t) = -\mathbf{Q}(s)\mathbf{P}(s,t).$$

- *(Forward differential equation)*

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(s,t) = -p_{ij}(s,t)q_j(t) + \sum_{k \neq i} p_{ik}(s,t)q_{kj}(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(s,t) = \mathbf{P}(s,t)\mathbf{Q}(t).$$

See e.g. [17, p. 16] for proof.

Remark 3.16. *Theorem 3.15 bridges the transition probabilities with the the transition rates. In practice, transition rate(e.g. mortality rate in life insurance) can be estimated based on historical data, and we can obtain the transition probabilities thereafter.*

3.2 Interest Rate

It is of great importance for the insured to know the guaranteed interest of their insurance policies promised by the insurer and the so-called technical interest rate plays a significant role in determining the premiums. It can be modeled by a deterministic or a stochastic model. If it is deterministic, we denote it by $i_t, t \geq 0$. It is a stochastic process $\{i_t\}_{t\geq 0}$ otherwise.

Definition 3.17. *(Interest rate, conversion period, nominal interest rate, effective interest rate) Interest is credited at the end of every conversion period. Interest rate is the percentage of money on a deposit or loan that is credited. If the conversion period coincide with the basic time unit(e.g. most commonly a year), the interest rate is called effective. Otherwise, the interest rate is called nominal.*

Example 3.18. *Consider an effective annual interest i_k at year k, an assets of value V_k will, by the end of year k, have value*

$$V_{k+1} = (1 + i_k) \times V_k.$$

However, if instead i_k is nominal, and for example the conversion period is 3 months, then

$$V_{k+1} = (1 + \frac{i_k}{4})^4 \times V_k$$

Definition 3.19. *(Discount rate) Let i_k be the interest rate at year k, then*

$$\xi_k = \frac{1}{1 + i_k}$$

is called the discount rate at year k.

In continuous time, we assume that the interest in payed continuously and we define the concept of interest intensity.

Definition 3.20. *(Interest intensity and continuous discount rate) The interest intensity at time t is denoted by ρ_t such that an asset of value V_t will have value*

$$V_{t+\Delta t} = \exp\left(\int_t^{t+\Delta t} \rho_\tau \, \mathrm{d}\tau\right) V_t.$$

The discount rate in continuous time is then

$$\xi_t = \exp\left(-\int_t^{t+\Delta t} \rho_\tau \, \mathrm{d}\tau\right).$$

In addition we can define the discount rate from t to 0,

$$\xi(t) = \exp\left(-\int_0^{t} \rho_\tau \, \mathrm{d}\tau\right).$$

Remark 3.21. *We can interpret $\rho(t)$ as the nominal interest rate with respect to the time period $[t, t + \Delta t]$. And for a stochastic interest rate, the notation $\rho_t = \rho(t, \omega)$ would represent a stochastic process.*

Remark 3.22. *Note the notion ξ_t denotes only the discount from time $t + \Delta t$ to time t, while $\xi(t)$ discounts all the way back to 0.*

For discrete case we can define a similar concept in the form of

$$\xi(t) = \xi_0 \cdot \xi_1 \cdots \xi_{t-1},$$

where the interest rate need not to be constant, and we will later encounter the case where the interest rate depends on the state of a Markov chain.

3.3 Cash Flow

3.3.1 Deterministic Cash Flow

Definition 3.23. *(Function of bounded variation) Let $[a, b] \subset \mathbb{R}$ be a bounded interval partitioned by*

$$a \le a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n \le b.$$

For a function $f : [a, b] \to \mathbb{R}$, the total variation of f over all partitions of $[a, b]$ is

$$V\Big[f, [a, b]\Big] := \sup \sum_{i=1}^{n} |f(b_i) - f(a_i)|$$

.

The function f is of bounded variation on $[a, b]$, if $V\Big[f, [a, b]\Big] < \infty$.

In a life insurance setting, functions are mostly defined on a finite time line $\mathbb{T} = [0, T]$, where T is the age the insured is last alive.

A function of bounded variation $f : [a, b] \to \mathbb{R}$ has following properties:

(i) f corresponds to a measure on $\mathbb{R}$. In life insurance setting, we refer to it as payout measure.

(ii) there exist two positive, increasing and bounded functions g, h such that $f = g - h$. We can interpret them as inflow and outflow of cash respectively in life insurance setting.

(iii) f as the measure can be decomposed uniquely into a discrete measure μ and a continuous measure ν.

(iv) If $\mathbb{T} \in [a, b]$, then $f \times \mathbb{1}_T$ is also a function of bounded variation, where $\mathbb{1}_T$ is the indicator function defined as

$$\mathbb{1}_T(t) := \begin{cases} 1, & \text{if } t \in \mathbb{T}, \\ 0, & \text{if } t \notin \mathbb{T}. \end{cases}$$

Definition 3.24. *(Payout function) A deterministic payout function$A_t : \mathbb{T} \to \mathbb{R}$ is a right continuous function of bounded variation. It represents the total payment the insurer has payed to the insured up to time t.*

Definition 3.25. *(Value of a cash flow) Let A be a deterministic cash flow and $t \in \mathbb{T} = [0, T] \subset \mathbb{R}$, the value of A at time t is*

$$V(t, A) := \frac{1}{\xi(t)} \int_0^\infty \xi(\tau)\, \mathrm{d}A_\tau.$$

The value of the future cash flow is

$$V^+(t,A) := V(t, A \times \mathbb{1}_{(t,\infty]}) = \frac{1}{\xi(t)} \int_t^\infty \xi(\tau)\, \mathrm{d}A_\tau.$$

which is also referred to as prospective reserve.

3.3.2 Stochastic Cash Flow

Definition 3.26. *(Stochastic cash flow) A stochastic cash flow is a stochastic process $\{A_t\}_{t\in\mathbb{T}}$ on $(\Omega, \mathfrak{A}, P)$ where almost all sample paths i.e. $t \mapsto A_t(\omega)$, for all $\omega \in \Omega$ are functions of bounded variation.*

Let $F : \mathbb{R} \times \Omega \to \mathbb{R}$ be a bounded and product measurable function, then

$$(F \cdot A)_t(\omega) = \int_0^t F(\tau,\omega)\, \mathrm{d}A_\tau(\omega).$$

Definition 3.27. *(Policy functions)*

1. *Continuous time:*
   - *(i) Generalized pension payments $a_i(t)$:*
     $a_i(t)$ is the sum of the payments to the insured up to time t given that the insured has always been in state i.
   - *(ii) Generalized capital benefits $a_{ij}(t)$:*
     $a_{ij}(t)$ is the payment at time t in case of a state change from i to j, $i,j \in \mathbb{S}$ at time t
2. *Discrete time:*
   - *(i) Generalized pension payments $a_i^{Pre}(t)$:*
     $a_i^{pre}(t)$ is the payment at time t given that the insured is in state i at time t.
   - *(ii) Generalized capital benefits $a_{ij}^{Post}(t)$:*
     $a_{ij}^{Post}(t)$ is the payment at time $t+1$ in care of a state change from i to j, $i,j \in \mathbb{S}$ at time t.

Definition 3.28. *(Policy cash flow) The stochastic cash flow corresponding to an insurance policy with state space $\mathbb{S}$ and bounded payout functions $a_{ij}(t)$ and $a_i(t)$ is defined as*

$$A(t,\omega) = \sum_{i\in\mathbb{S}} A_i(t,\omega) + \sum_{(i,j)\in\mathbb{S}\times\mathbb{S}, i\neq j} \mathrm{d}A_{ij}(t,\omega),$$

where

- *For continuous time*

$$A_{ij}(t,\omega) := \int_0^t a_{ij}(s)\, \mathrm{d}N_{ij}(s,\omega),$$

is the increase of liabilities caused by transition from state i to j, and $N_{ij}(t,\omega)$ is the number of jumps from i to j in the time interval $[0,t]$ defined as

$$N_{ij}(t,\omega) = \sum_{s>0}^{t} \mathbb{1}_{\{X_{s^-}=i, X_s=j\}}, \qquad X_{s^-} := \lim_{n\to\infty} X_{s-\frac{1}{n}}$$

and

$$A_i(t,\omega) := \int_0^t \mathbb{1}_i(s,\omega)\,\mathrm{d}a_i(s)$$

is the liability caused by the insured staying in the state i, $\mathbb{1}_i(t,\omega)$ is the indicator function with respect to a stochastic process $\{X_t\}_{t\in\mathbb{T}}$ defined as

$$\mathbb{1}_i(t,\omega) := \begin{cases} 1, & \text{if } X_t(\omega) = i, \\ 0, & \text{if } X_t(\omega) \neq i. \end{cases}$$

- *For discrete time, similarly*

$$A_{ij}(t,\omega) := \sum_{s=0}^{t} \mathbb{1}_{\{X_s=i, X_{s+1}=j\}} a_{ij}^{Post}(s),$$

$$A_i(t,\omega) := \sum_{s=0}^{t} \mathbb{1}_i(s,\omega) a_i^{Pre}(s).$$

Definition 3.29. *(Stochastic prospective reserve) If A and ξ are stochastic processes on*
$(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathfrak{A}, P)$, then the prospective reserve is defined by

$$V_{\mathcal{F}}^{+}(t,A) = E\left[V^{+}(t,A)|\mathcal{F}_t\right]$$

Remark 3.30. *For a Markov chain, the conditional expectation with respect to $\mathcal{F}_t$ depends only on the state at time t, in this case*

$$V_j^{+}(t,A) = E[V^{+}(t,A)|X_t = j]$$

3.4 Mathematical Reserve

Mathematical reserve is the present value of the future cash flows of insurance policies and the total expected liability which insurer is required to reserve offsetting assets for. With the definition of cash flows, we can define the mathematical reserve.

Definition 3.31. *(Mathematical reserve)*

- *For continuous time, the mathematical reserve of being in state g is the expected present value of all payments in time interval $\mathbb{T} \subset \mathbb{R}$ given $X_t = j$ and defined by*

$$V_j(t, A_g) = E\left[\xi^{-1}(t) \int_{\mathbb{T}} \xi(\tau)\,\mathrm{d}A_g(\tau)|X_t = j\right].$$

Similarly, the reserve of transitioning from g to $h \in \mathbb{S}$ is

$$V_j(t, A_{gh}) = E\Big[\xi^{-1}(t) \int_{\mathbb{T}} \xi(\tau)\,\mathrm{d}A_{gh}(\tau)|X_t = j\Big].$$

The total mathematical reserve of a certain state j is

$$V_j(t, A) = \sum_{g\in\mathbb{S}} V_j(t, A_g) + \sum_{g,h\in\mathbb{S},g\neq h} V_j(t, A_{gh}).$$

where $\xi(t)$ is the continuous discount rate defined as in Definition 3.20.

- *For discrete time, similarly we have*

$$V_j(t, A_g^{Pre}) = E\Big[\xi^{-1}(t) \sum_{i=1}^{n} \xi(\tau_i)\Delta A_g(\tau_i)|X_t = j\Big]$$

$$V_j(t, A_{gh}^{Post}) = E\Big[\xi^{-1}(t) \sum_{i=1}^{n} \xi(\tau_{i+1})\Delta A_{gh}(\tau_i)|X_t = j\Big].$$

$$V_j(t, A) = \sum_{g\in\mathbb{S}} V_j(t, A_g^{Pre}) + \sum_{g,h\in\mathbb{S},g\neq h} V_j(t, A_{gh}^{Post}).$$

In order to calculate the mathematical reserve we need the following result

Theorem 3.32. *Let $\{X_t\}_{t\in\mathbb{T}}$ be a regular Markov chain on $(\Omega, \mathfrak{A}, P)$ where $s < t$ and $\mathbb{T} \subset [s, \infty]$, then for $i, j, k \in \mathbb{S}$ and function of bounded variation $A(t)$*

(i) $E[\int_{\mathbb{T}} a(\tau)\,\mathrm{d}N_{jk}(\tau)|X_s = i] = \int_{\mathbb{T}} a(\tau)p_{ij}(s, \tau)q_{jk}(\tau)\,\mathrm{d}\tau$

(ii) $E[\int_{\mathbb{T}} \mathbb{1}_i(\tau, \omega)\,\mathrm{d}A(\tau)|X_s = i] = \int_{\mathbb{T}} p_{ij}(s, \tau)\,\mathrm{d}A(\tau)$

For the discrete time case,

(i) $E[\sum_{i=1}^{n} a(\tau_i)\Delta N_{jk}(\tau_i)|X_s = i] = \sum_{i=1}^{n} a(\tau_i)p_{ij}(s, \tau_i)p_{jk}(\tau_i, \tau_{i+1})$

(ii) $E[\sum_{j=1}^{n} \mathbb{1}_j(\tau_i)\Delta A_{\tau_i}|X_s = i] = \sum_{i=1}^{n} p_{ij}(s, \tau_i)\Delta A(\tau_i)$

where $\Delta N_{jk}(\tau_i) := N_{jk}(\tau_{i+1}) - N_{jk}(\tau_i)$ and $\Delta A_{\tau_i} = A_{\tau_{i+1}} - A_{\tau_i}$.

Based on these results, we have the following theorem:

Theorem 3.33. *(Explicit formula for the mathematical reserves) Consider a regular insurance model with deterministic interest intensities,*

- *For continuous time,*

(i) $V_i(t, A_j) = \xi^{-1}(t) \int\limits_{\mathbb{T}} \xi(\tau) p_{ij}(t, \tau) \, \mathrm{d}a_j(\tau).$

(ii) $V_i(t, A_{jk}) = \xi^{-1}(t) \int\limits_{\mathbb{T}} \xi(\tau) a_{jk}(\tau) p_{ij}(t, \tau) q_{jk}(\tau) \, \mathrm{d}\tau.$

- *For discrete time,*

(i) $V_i(t, A_j) = \xi^{-1}(t) \sum_{i=1}^{n} \xi(\tau_i) p_{ij}(t, \tau_i) a_j^{Pre}(\tau_i).$

(ii) $V_i(t, A_{jk}) = \xi^{-1}(t) \sum_{i=1}^{n} \xi(\tau_{i+1}) p_{ij}(t, \tau_i) p_{jk}(\tau_i, \tau_{i+1}) a_{jk}^{Post}(\tau_i).$

Correspondingly we have the following result:

Theorem 3.34. *(Explicit formula for the prospective reserves)*

$$V_j^+(t) = \xi^{-1}(t) \int_{(t,\infty)} \xi(\tau) \sum_{g \in \mathbb{S}} p_{ig}(t, \tau) \times \left\{ \mathrm{d}a_g(\tau) + \sum_{\mathbb{S} \ni h \neq g} a_{gh}(\tau) q_{gh}(\tau) \, \mathrm{d}\tau \right\}.$$

However, this formula is in general difficult to evaluate numerically in practice, one possible solution is to utilize difference or differential equations.

Theorem 3.35. *(Recursion formula) Let $\{X_t\}_{t \in \mathbb{T}}$be a regular insurance model with deterministic interest intensities and state space$\mathbb{S}$, $j \in \mathbb{S}$ and $s < t < u$, then*

$$W_j^+(t) = \sum_{g \in \mathbb{S}} p_{jg}(t, u) W_g^+(u) + \int_{(t,u]} \xi(\tau) \sum_{g \in \mathbb{S}} p_{ig}(t, \tau) \times \left\{ \mathrm{d}a_g(\tau) + \sum_{\mathbb{S} \ni h \neq g} a_{gh}(\tau) \mu_{gh}(\tau) \, \mathrm{d}\tau \right\}.$$

where

$$W_j^+(t) := \xi(t) V_j^+(t).$$

Theorem 3.36. *(Thiele's difference equation) For a discrete time Markov model,*

$$V_i^+(t) = a_i^{Pre}(t) + \sum_{j \in \mathbb{S}} \xi_t p_{ij}(t, t+1) \left\{ a_{ij}^{Post}(t) + V_j^+(t+1) \right\}.$$

Theorem 3.37. *(Thiele's differential equation) Let $\{X_t\}_{t \in \mathbb{T}}$ be a regular insurance model and the payout function $A_g(t)$ be continuous, assume a deterministic interest intensity, then*

(i) $W_g^+(t)$ *continuous for all* $g \in \mathbb{S}$.

(ii) Thiele's differential equation

$$\begin{aligned} \frac{\partial}{\partial t} W_j^+(t) = &- \xi(t) \{ a_j(t) + \sum_{g \in \mathbb{S}, g \neq j} \mu_{jg}(t) a_{jg}(t) \} + \mu_j(t) W_j^+(t) \\ &- \sum_{g \in \mathbb{S}, g \neq j} \mu_{jg}(t) W_g^+(t). \end{aligned}$$

(iii)

$$V_j^+(t) = \xi(t)^{-1}\left[\int_t^u \xi(s)\overline{p}_{jj}(t,\tau)\left\{a_j(\tau) + \sum_{g\in\mathbb{S},g\neq j}\mu_{jg}(\tau)[a_{jg}(\tau)+V_g^+(\tau)]\right\}\mathrm{d}s + \xi(u)\overline{p}_{jj}(t,u)V_j^+(u)\right].$$

Remark 3.38. *We can interpret parts in Theorem 3.37 (iii) as*

- $a_j(\tau)$ *: reserve for pension and premiums for being in state* j.
- $\mu_{jg}(\tau)a_{jg}(\tau)$ *: transition cost from state* j *to* g.
- $\mu_{jg}(\tau)[a_{jg}(\tau)+V_g^+(\tau)]$ *: the total reserve needed for the transition from state* j *to* g.
- $\xi(u)\overline{p}_{jj}(t,u)V_j^+(u)$ *: reserve needed for the case where the insured stayed in state* j *after* $[t,u]$.

3.4.1 Distribution of Mathematical Reserves

The calculation of premium reserves is based on the assumption that an expected average value of total losses could be sufficient to cover the insurer's liabilities. However, if the variance of the total loss is extreme, the reserve calculated with expectation may not be enough to meet the requirement at certain time points. Thus it is important for the insurer to measure the risk in the variation of the total loss or more generally to be aware of the distributional characteristics of the mathematical reserve.

From Definition 3.29 we know that the prospective reserve of the discrete Markov model is given by

$$V_t^+ = \xi^{-1}(t)\sum_{n\geq t}\left\{\sum_{j\in\mathbb{S}}\xi(n)\mathbb{1}_j(n)a_j^{Pre}(n) + \sum_{j,l\in\mathbb{S}}\xi(n+1)\mathbb{1}_{\{X_n=j,X_{n+1}=l\}}a_{jl}^{Post}\right\}$$

where

$$\xi(t) = \xi_0\cdot\xi_1\cdots\xi_{t-1}$$

and

$$\xi_t = \sum_{j\in\mathbb{S}}\mathbb{1}_j(t)\xi_t^{(j)} \qquad \xi_t^{(j)} := \frac{1}{1+i_t^{(j)}},$$ where the interest rate depends on the state j

Now consider the distribution of V_t^+, given $X_t = i$:

$$\begin{aligned}
P_i(t,u) :&= P\Big(V_t^+ < u|X_t = i\Big) && (3.1)\\
&= \frac{1}{P(X_t=i)}\left(\sum_{l\in\mathbb{S}}P\Big(v_t^+ < u, X_t = i, X_{t+1} = l\Big)\right)\\
&= \frac{1}{P(X_t=i)}\left(\sum_{l\in\mathbb{S}}P\Big(v_t^+ < u, X_t = i, X_{t+1} = l\Big)\right)\cdot\frac{P\big(X_t=i,X_{t+1}=l\big)}{P\big(X_t=i,X_{t+1}=l\big)}\\
&= \sum_{l\in\mathbb{S}}p_{il}(t,t+1)P\Big(V_t^+ < u\Big|X_t=i,X_{t+1}=l\Big). && (3.2)
\end{aligned}$$

We know from the definition of V_t^+ that

$$V_t^+ = \frac{\xi(t+1)}{\xi(t)} \cdot V_{t+1}^+ + \sum_{j \in \mathbb{S}} \mathbb{1}_j(t) a_j^{Pre}(t) + \sum_{j,k \in \mathbb{S}} \frac{\xi(t+1)}{\xi(t)} \mathbb{1}_{\{X_t = j, X_{t+1} = k\}} a_{jk}^{Post}, \tag{3.3}$$

where

$$\frac{\xi(t+1)}{\xi(t)} = \xi_t = \sum_{j \in \mathbb{S}} \mathbb{1}_j(t) \xi_t^{(j)}.$$

Combining (3.2) and (3.3) we have,

$$\begin{aligned} P_i(t,u) &= \sum_{l \in \mathbb{S}} p_{il}(t,t+1) P\Big(\xi_t^{(i)} \cdot V_{t+1}^+ + a_i^{Pre}(t) + \xi_t^{(i)} a_{il}^{Post}(t) < u \Big| X_t = i, X_{t+1} = l\Big) \\ &= \sum_{l \in \mathbb{S}} p_{il}(t,t+1) P\bigg(V_{t+1}^+ < (\xi_t^{(i)})^{-1}\Big(u - a_i^{Pre}(t)\Big) - a_{il}^{Post}(t) \bigg| X_t = i, X_{t+1} = l\bigg) \\ &= \sum_{l \in \mathbb{S}} p_{il}(t,t+1) P\bigg(V_{t+1}^+ < (\xi_t^{(i)})^{-1}\Big(u - a_i^{Pre}(t)\Big) - a_{il}^{Post}(t) \bigg| X_t = i\bigg) \\ &= \sum_{l \in \mathbb{S}} p_{il}(t,t+1) P_l\bigg(t+1, (\xi_t^{(i)})^{-1}\Big(u - a_i^{Pre}(t)\Big) - a_{il}^{Post}(t)\bigg) \end{aligned}$$

Note the third equality is achieved by the Markov property and the last equality is achieved by equation (3.1. We thereby have the recursion:

Theorem 3.39. *(Thiele's difference equation for distribution of the reserves)*

$$P_i(t,u) = \sum_{l \in \mathbb{S}} p_{il}(t,t+1) P_l\bigg(t+1, (\xi_t^{(i)})^{-1}\Big(u - a_i^{Pre}(t)\Big) - a_{il}^{Post}(t)\bigg).$$

Remark 3.40. *Depending on the different types of insurance, various boundary conditions are required.*

Similarly for the continuous case, we introduce without proof:

Theorem 3.41. *(Thiele's differential equation for distribution of the reserves)*
Let

$$P_j(t,u) := P\Big(\int_t^\infty \exp(-\int_t^s \rho_\tau \,\mathrm{d}\tau) \,\mathrm{d}A(s) \le u \Big| X_t = j\Big)$$

be the distribution function of the prospective reserve given $X_t = j$, then

$$\begin{aligned} P_j(t,u) = \sum_{k \in \mathbb{S}, k \neq j} \int_t^\infty & \bigg\{ \exp\Big(-\int_t^s \sum_{l \neq j} q_{jl}(\tau) \,\mathrm{d}\tau\Big) q_{jk}(s) \\ & \cdot P_k\bigg(s, \exp\Big(\rho_j(s-t)\Big)u - \int_t^s \exp\Big(\rho_j(s-\tau)\Big) \,\mathrm{d}a_j(\tau) - a_{jk}(s)\bigg)\bigg\} \,\mathrm{d}s \\ & + \exp\Big(-\int_t^\infty \sum_{l \neq j} q_{jk}(\tau) \,\mathrm{d}\tau\Big) \cdot \mathbb{1}_{\left\{\int_t^n \exp\left(-\rho_j(\tau-t)\right) \mathrm{d}a_j(\tau) \le u\right\}}. \end{aligned}$$

As we can see, the equations for the continuous case are very cumbersome and difficult to handle, it is merely introduced as the counterpart for the discrete case.

3.5 Unit-linked Policies

It is mentioned above that modern life insurance, with increasing interest, deals with unit-linked policies where the policyholder participates in the performance of an underlying fund. The pricing of unit-linked policies is central to this thesis, we will give a more detailed introduction of such policies in this section.

Recall as in Definition 3.29, that the prospective reserve is the present value of all pension and benefit payments with respect to the time interval $[t, \infty]$ under the scenario of various outcomes of X_s for $s > t$, defined as

$$V_{\mathcal{F}}^{+}(t, A) = E\left[V^{+}(t, A)|\mathcal{F}_t\right]$$

where

$$\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$$

is the smallest σ-algebra containing all information of X_s up to time t. In a Markovian setting,

$$V_j^{+}(t, A) = E[V^{+}(t, A)|X_t = j]$$

And

$$\begin{aligned} V^{+}(t, A) :&= \xi^{-1}(t) \int_t^\infty \xi(s)\, \mathrm{d}A(s) \\ &= \sum_{i \in \mathbb{S}} \int_t^\infty \mathbb{1}_i(s) \pi_t^{(i)}(s)\, \mathrm{d}s + \sum_{(i,j) \in \mathbb{S} \times \mathbb{S}, i \neq j} \int_t^\infty \pi_t^{(ij)}(s)\, \mathrm{d}N_{ij}(s) \end{aligned} \tag{3.4}$$

where

$$\pi_t^{(i)} := \frac{\xi(s)}{\xi(t)} \cdot \mathrm{d}a_i(s), \text{ and } \pi_t^{(ij)}(s) := \frac{\xi(s)}{\xi(t)} a_{ij}(s)$$

are the present values at time t of the deterministic generalized pension payments $\mathrm{d}a_j(s)$ and the generalized benefit payments $a_{ij}(s)$ respectively, for $s > t$.

Example 3.42. *Consider a term insurance policy with state space $\mathbb{S} = \{*, \dagger\}$, where $*, \dagger$ represents the insured being alive and deceased respectively, then by Theorem 3.33 we have*

$$V_*^{+}(0) = \int_0^\infty \pi_0(s) p_{**}(0, s) q_{*\dagger}(s)\, \mathrm{d}s$$

where

$$\pi_0(s) := \frac{\xi(s)}{\xi(0)} C(s)$$

and

$$C(s) := a_{*\dagger}(s).$$

However, in the case that the term insurance is unit-linked, $C(t)$ is tied to the value of a fund or stock $S(s)$, for $0 \leq s \leq t$. For example,

$$C(t) = max\big(S(t), G(t)\big),$$

where

$$G(t) := \int_0^t \exp\big(\gamma(t - s)\big)\, \mathrm{d}a_*(s)$$

is the refund guarantee of the paid premiums with an additional interest rate of γ. $C(t)$ is no longer deterministic but stochastic.

For the rest of the section, when considering the prospective value, we will assume the following,

(i) $\mathcal{F}_t = \sigma(\mathcal{G}_t, \mathcal{H}_t)$ where
$$\mathcal{H}_t = \sigma(X_s, 0 \le s \le t)$$
is the collection of information of the insurance events, and
$$\mathcal{G}_t = \sigma(S_s, 0 \le s \le t, \mathcal{N})$$
with
$$\mathcal{N} := \{N : P(N) = 0\}$$
is the collection of all information of the market events including stock prices and extremely rare events.

(ii) Insurance events are independent of financial events.

Example 3.43. *The prospective reserves of a unit-linked term insurance is defined as*

$$\begin{aligned} V^+_{\mathcal{F}}(t, A) &= E_X[\int_t^\infty \pi_t^{*\dagger}(s)\,\mathrm{d}N_{*\dagger}|X_t] \\ &= \int_t^\infty \pi_t^{*\dagger}(s) p_{**}(t,s) q_{*\dagger}(s)\,\mathrm{d}s \end{aligned}$$

where E_X denotes the expectation with respect to X_s (i.e. $\pi^{*\dagger}$ is regarded as deterministic here).

Unit-linked policy usually are handled with a single premium, by setting $t = 0$, we obtain such premium given by

$$V^+_*(0) = V^+_{\mathcal{F}}(0, A) = \int_0^\infty \pi_0^{*\dagger}(s) p_{**}(0,s) q_{*\dagger}(s)\,\mathrm{d}s$$

.

Chapter 4

Simulation of Fractional Brownian Motion

As introduced in Chapter 2, a fractional Brownian motion is a zero-mean Gaussian process with a certain covariance structure. Because of the non-Markovian nature of fractional Brownian motion, many traditional techniques do not apply and the need of simulation scheme arises. In this section, we will give an introduction on the exact methods that capture the true covariance structure of fractional Brownian motion with a focus on fast Fourier transformation method due to its higher efficiency.

4.1 Hosking Method

The Hosking method (also known as the Durbin or Levinson method) generates a general stationary Gaussian process through the conditional distribution of the multivariate Gaussian distribution and can also be applied to any stationary Gaussian process other than fBms. In this case, the fractional Brownian motion sample is obtained by taking cumulative sums of the generated fractional Gaussian noise sample $\{X_n\}$.

Let $\gamma(k)$ be the autocovariance function of a stationary zero-mean Gaussian process seen as in Chapter 2:

$$\gamma(k) := E[X_n X_{n+k}] = \frac{1}{2}((k+1)^{2H} + (k-1)^{2H} - 2k^{2H})$$

for $n, k = 0, 1, \cdots$, and $\gamma(0) := 1$. Furthermore, let $\Gamma(n) = [\gamma(i-j)]_{i,j=0,1,\cdots,n}$ be the autocovariance matrix, then we have the following recursion

$$\begin{aligned}
\Gamma(n+1) &= \begin{bmatrix} 1 & \gamma(1) & \gamma(2) & \dots & \gamma(n+1) \\ \gamma(1) & 1 & \gamma(1) & \dots & \gamma(n) \\ \gamma(2) & \gamma(1) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(n+1) & \gamma(n) & \dots & \dots & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & c(n)' \\ c(n) & \Gamma(n) \end{bmatrix} \\
&= \begin{bmatrix} \Gamma(n) & F(n)c(n) \\ c(n)'F(n) & 1 \end{bmatrix}
\end{aligned}$$

where the prime denotes vector transpose, $c(n)$ is the $(n+1)$-column vector with elements $c(n)_k = \gamma(k+1),\ k = 0, \ldots, n$, and $F(n) = [\mathbb{1}_{i=n-j}]_{i,j=0,1,\cdots,n}$ is the matrix

$$F(n) = \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We seek to compute the conditional distribution of X_{n+1} given $X_n, \ldots, X_0$. It can be shown that this distribution is Gaussian with expectation μ_n and variance σ_n^2 given by

$$\mu_n := c(n)'\Gamma(n)^{-1} \begin{bmatrix} X_n \\ \vdots \\ X_1 \\ X_0 \end{bmatrix}, \qquad \sigma_n^2 := 1 - c(n)'\Gamma(n)^{-1}c(n).$$

Define $d(n) = \Gamma(n)^{-1}c(n)$, then the inverse of $\Gamma(n+1)$ satisfies:

$$\begin{aligned} \Gamma(n+1)^{-1} &= \frac{1}{\sigma_n^2} \begin{bmatrix} 1 & -d(n)' \\ -d(n) & \sigma_n^2\Gamma(n)^{-1} + d(n)d(n)' \end{bmatrix} \\ &= \frac{1}{\sigma_n^2} \begin{bmatrix} \sigma_n^2\Gamma(n)^{-1} + F(n)d(n)d(n)'F(n) & -F(n)d(n) \\ -d(n)'F(n) & 1 \end{bmatrix}. \end{aligned}$$

With the distribution known, we obtain the fGn sample by generating a standard normal random variable X_0 and simulate $X_1, \ldots, X_n$ recursively. Taking the matrix inversion every step is obviously computationally expensive, the algorithm proposed by Hosking [13] avoids doing so by computing $d(n)$ recursively. Assume μ_n, σ_n^2 and $\tau_n := d(n)'F(n)c(n) = c(n)'F(n)d(n)$ are known, then σ_n^2 satisfies the recursion

$$\sigma_{n+1}^2 = \sigma_n^2 - \frac{(\gamma(n+2) - \tau_n)^2}{\sigma_n^2}$$

and the recursion for $d(n+1) = \Gamma(n+1)^{-1}c(n+1)$ is:

$$d(n+1) = \begin{bmatrix} , d(n) - \phi_n F(n)d(n) \\ \phi_n \end{bmatrix}$$

where

$$\phi_n = \frac{\gamma(n+1) - \tau_n}{\sigma_n^2}.$$

Start the recursion with $\mu_0 = \gamma(1)X_0$, $\sigma_0^2 = 1 - \gamma(1)^2$ and $\tau_0 = \gamma(1)^2$, $\mu_{n+1}, \sigma_{n+1}^2$ and τ_{n+1} can be computed readily. And the fBm sample is thereafter the cumulative sum of the fGn sample. The advantage of this algorithm is that it's rather simple to implement and can generate sample path without needing to know the sample size beforehand. However, its complexity of order N^2 is prohibitive.

4.2 Cholesky Method

With the covariance structure given in matrix form, it is a natural approach to take advantage of the Cholesky decomposition. The idea is to decompose the covariance matrix $\Gamma(n)$ into the product of an $(n+1)\times(n+1)$ lower triangular matrix $L(n)$ and its conjugate transpose $L(n)'$. $L(n)$ is called lower triangular if its element $l_{ij} > 0$ for all $i > j$ and $i, j = 0, \ldots, n$. It can be proven such decomposition exist if $\Gamma(n)$ is a symmetric positive definite matrix.

Let

$$L(n) = \begin{bmatrix} l_{00} & 0 & 0 & \ldots & 0 \\ l_{10} & l_{11} & 0 & \ldots & 0 \\ l_{20} & l_{21} & l_{22} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n0} & l_{n1} & l_{n2} & \vdots & l_{nn} \end{bmatrix} \qquad \text{then } L(n)' = \begin{bmatrix} l_{00} & l_{10} & l_{20} & \ldots & l_{n0} \\ 0 & l_{11} & l_{21} & \ldots l_{n1} & \\ 0 & 0 & l_{22} & \ldots & l_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & l_{nn} \end{bmatrix}$$

and their product being

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \ldots & \gamma(n) \\ \gamma(1) & \gamma(0) & \gamma(1) & \ldots & \gamma(n-1) \\ \gamma(2) & \gamma(1) & \gamma(0) & \ldots & \gamma(n-2) \\ \vdots & \vdots & \vdots & \ddots \vdots & \\ \gamma(n) & \gamma(n-1) & \gamma(n-2) & \ldots & \gamma(0) \end{bmatrix}$$

gives the easy realization that $l_{00} = \gamma(0)$ for $i = j = 0$. Furthermore, $l_{10}l_{00} = \gamma(1)$ and $l_{10}^2 + l_{11}^2 = \gamma(0)$ for $i = 1$. And the rest of entries can be calculated thereafter for $i \geq 1$ by

$$l_{i0} = \frac{\gamma(i)}{l_{00}},$$
$$l_{ij} = \frac{1}{l_{jj}}\left(\gamma(i-j) - \sum_{k=0}^{j-1} l_{ik}l_{jk}\right), \qquad 0 < j \leq n.$$

Let$\{Z_k\}$be an $(n+1)$-column vector of i.i.d. standard normal random variables, the we can simulate the fGn by

$$X_{n+1} = \sum_{k=0}^{n+1} l_{n+1,k} \cdot Z_k$$

or in matrix form by $X(n) = L(n)Z(n)$. And the covariance of $X(n)$ satisfies

$$Cov(X(n)) = Cov(L(n)Z(n)) = L(n)Cov(Z(n))L(n)' = L(n)L(n)' = \Gamma(n).$$

The Cholesky method enjoys pleasant theoretical simplicity and is easy to implement, however in practice to store $L(n)$ in every step of the recursion is computationally uneconomical. We will introduce one more exact method who shares the fundamental idea as the Cholesky method, but with a more efficient decomposition.

4.3 Fast Fourier Transformation Method

Using Cholesky decomposition to simulate Gaussian process with a given covariance matrix is the most intuitive approach, but it is slow and inefficient. To improve upon the speed, Davies and Harte [4] proposed the idea of utilizing fast Fourier transformation, the algorithm was developed further by Dietrich and Newsam [7] and Wood and Chan [24].

Similar to the Cholesky method, the Fast Fourier Transformation(FFT) method aims at finding the decomposition $\Gamma = GG'$, then the sample is generated by $X = GZ$ for a standard normal random variable Z. The covariance of X satisfies

$$Cov(X) = Cov(GZ) = GCov(Z)G' = GG' = \Gamma.$$

The idea is to embed Γ in a so-called circulant matrix.

Definition 4.1. *(Circulant Matrix) A circulant martix is an $n \times n$ matrix of the form*

$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \dots & c_1 \\ c_1 & c_0 & c_{n-1} & \dots & c_2 \\ c_2 & c_1 & c_0 & \dots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-1} & \dots & c_1 & c_0 \end{bmatrix}$$

which is specified by the column vector c as the first column of C.

The polynomial $P(x) = c_0 + c_1 x + \dots + c_{n-1}x^{n-1}$is called the associated polynomial of C.

Definition 4.2. *(Generating circulant matrix) The circulant matrix G is defined by*

$$G := \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

Remark 4.3. *With simple calculation we can see,*

$$G^2 = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix}.$$

Tt can be viewed that G^2 is achieved by shifting elements in each column by one element down. Arbitrary powers of G can be obtained accordingly.

C is then generated by

$$P(C) = c_0(I)^n + c_1 G + c_2 G^2 + \dots + c_{n-1}G^{n-1}.$$

Definition 4.4. *(Fourier matrix) The Fourier matrix is defined as*

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^{2\times 2} & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} \end{bmatrix}$$

where ω is the conjugate of the n-th unity root $e^{-2\pi i \frac{1}{n}}$, $i = \sqrt{-1}$.

Remark 4.5. *Note $\omega^n = e^{-2\pi i} = 1$, F_N can be written as*

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^{2\times 2} & \dots & \omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{n-2} & \dots & \omega \end{bmatrix}.$$

Multiply the Fourier matrix with the generating circulant matrix G, we then have

$$\begin{aligned} FG &= \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ \omega & \omega^2 & \dots & \omega^{n-1} & 1 \\ \omega^2 & \omega^{2\times 2} & \dots & \omega^{n-2} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{n-1} & \omega^{n-2} & \dots & \omega & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^{2\times 2} & \dots & \omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{n-2} & \dots & \omega \end{bmatrix} \\ &= \Lambda F, \end{aligned}$$

where Λ is a diagonal matrix with k-th diagonal valued ω^k for $0 \le k \le n-1$.
Furthermore

$$FGF^{-1} = \Lambda$$

means that F diagonalizes G with eigenvalues $\{\omega^k\}_{0 \le k \le n-1}$.

Since $C = P(G)$, we have the following theorem:

Theorem 4.6. *(Fourier decomposition of general circulant matrix) The circulant matrix C is decomposable by the Fourier matrix F with eigenvalue matrix $\Lambda = \{P(\omega^k)\}_{k=0,\dots,n-1}$. i.e.*

$$FCF^{-1} = \Lambda = \begin{bmatrix} P(1) & 0 & 0 & \dots & 0 \\ 0 & P(\omega) & 0 & \dots & 0 \\ 0 & 0 & P(\omega^2) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p(\omega^{n-1}) \end{bmatrix}.$$

Theorem 4.6 provides the theoretical foundation for the FFT method. In practice,following the methodology of Davies and Harte, for a sample of size $N = 2^g$, $g \in \mathbb{N}$ (the power of 2 is required by the calculation of FFT) we embed the covariance matrix

$$\Gamma = \begin{bmatrix} \gamma(0) & \gamma(1) & \dots & \gamma(N-1) \\ \gamma(1) & \gamma(0) & \dots & \gamma(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(N-1) & \gamma(N-2) & \dots & \gamma(0) \end{bmatrix}$$

into a $M \times M$ [1] circulant covariance matrix C defined by

$$C := \begin{bmatrix} \gamma(0) & \gamma(1) & \dots & \gamma(N-1) & 0 & \gamma(N-1) & \dots & \gamma(1) \\ \gamma(1) & \gamma(0) & \dots & \gamma(N-2) & \gamma(N-1) & 0 & \dots & \gamma(2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(N-1) & \gamma(N-2) & \dots & \gamma(0) & \gamma(1) & \gamma(2) & \dots & 0 \\ 0 & \gamma(N-1) & \dots & \gamma(1) & \gamma(0) & \gamma(1) & \dots & \gamma(N-1) \\ \gamma(N-1) & 0 & \dots & \gamma(2) & \gamma(1) & \gamma(0) & \dots & \gamma(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma(1) & \gamma(2) & \dots & 0 & \gamma(N-1) & \gamma(N-2) & \dots & \gamma(0) \end{bmatrix},$$

where each row is a cyclic permutation achieved by moving the elements in the previous row one element right and Γ is embedded in the top left corner of C.

C can then be decomposed into $Q\Lambda Q^*$ given Q the unitary matrix defined by

$$(Q)_{j,k} := \frac{1}{\sqrt{2N}} \exp\left(-2\pi i \frac{jk}{2N}\right) \qquad \text{for } j, k = 0, \dots, 2N-1$$

and Λ the eigenvalue matrix defined by the eigenvalues:

$$\lambda_k = \sum_{j=0}^{2N-1} (C)_{1,j+1} \exp\left(2\pi i \frac{jk}{2N}\right) \qquad \text{for } k = 0, \dots, 2N-1.$$

Assuming that C is positive definite and symmetric, the resulting eigenvalues will be positive and real. Let $S = Q\Lambda^{\frac{1}{2}}Q^*$ where $\Lambda^{\frac{1}{2}}$ is the matrix with eigenvalues $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_{2N-1}}$, since Q is unitary we readily see that $SS* = SS' = C$. Thus we finally obtain the matrix we search for, and are ready to simulate.

Algorithm 4.1: Simulation of fGn with FFT

Input : Hurst parameter$H$, Number of simulations N
Output: A path of fBm with hurst parameter H of size N

Generate the covariance matrix Γ
Construct the circulant matrix C with Γ
Compute the eigenvalues $\lambda_k,\ k = 0, \dots, 2N-1$ using FFT
Calculate $W = Q * V$
Compute $Z = Q\Lambda^{\frac{1}{2}}W$
Recover fBm form fGn using recursion.
Return fBm

The main advantage of the FFT method is the speed, the number of computation required for a sample of size N is only of order $Nlog(N)$, it is ideal for simulation of financial data that require many paths with limited computing power.

Figure 4.1 shows examples of fractional Brownian motion with various H simulated using FFT method.

[1] $M \geq 2(N-1)$, see [22], choosing $M = 2N$ for simplicity here

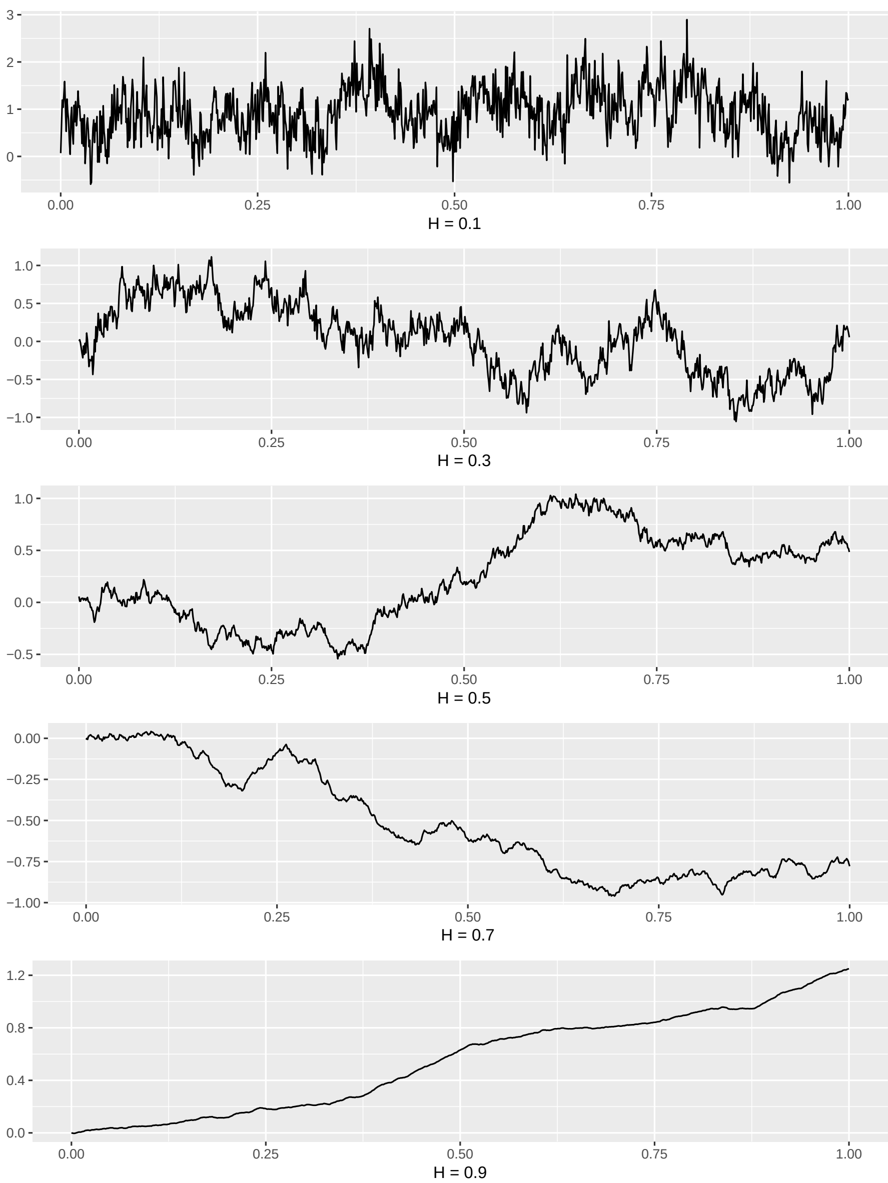


Figure 4.1: fBms with different H

Chapter 5

Rough Fractional Stochastic Volatility Model

In this master thesis we propose a model for pricing unit-linked life insurance policies in connection with stock prices with stochastic volatility to better capture the dynamics of stock prices in the so-called "turbulent" stock markets.

Recall the classical Black-Scholes model for stock prices S_t of the form:

$$S_t = S_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\Big), \qquad 0 \leq t \leq T,$$

where S_0 is the initial value of the stock, μ the mean return, σ the volatility and $B_t, 0 \leq t \leq T$ a one-dimensional Brownian motion.

The risk-neutral dynamics of the stock prices S_t under a risk-neutral measure is given by

$$S_t = S_0 \exp\Big((r - \frac{1}{2}\sigma^2)t + \sigma B_t\Big), \qquad 0 \leq t \leq T,$$

where r is a constant overnight market interest rate.

In this thesis, we will consider a model where the volatility is stochastic, the resulting model is thus

$$S_t = S_0 \exp\left(\int_0^t \left(r - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s\right) \tag{5.1}$$

where the dynamics of the stochastic volatility σ_t is modeled by

$$\sigma_t = g(X_t)$$

for a non-negative Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ and a (unique weak) solution $\{X_t\}_{t\in\mathbb{T}}$ to the SDE:

$$X_t = y + \int_0^t b(X_u)\,\mathrm{d}u + \beta B_t^H \tag{5.2}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, $y \in \mathbb{R}$ and $\beta > 0$. Furthermore $B_t^H,\ 0 \leq t \leq T$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ assumed to be independent of $B_t,\ 0 \leq t \leq T$ in (5.1) .

Recall the covariance function of B_t^H is given by

$$\rho_H(t, s) = E(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$$

and that for $H \neq \frac{1}{2}$ the increments of the process are correlated. The process is non-Markovian and with memory, Thiele's differential equation no longer applies, and we must resort to Monte Carlo simulation. It is also known that the paths of B_t^H become rougher as the Hurst parameter H becomes smaller (see e.g. [18]).In order to capture the rough behavior of the dynamics of stochastic volatility in turbulent stock markets, it makes sense to choose a small Hurst parameter $H < \frac{1}{2}$, in our study, we choose $H = 0.14$ based on the work in [9].

We then proceed to the calculation of net single premium and premium reserves of unit-linked life insurance policies with respect to the stock prices. For example in the case of a unit-linked term insurance premium reserves can be computed by means of the present value of the death benefits

$$C(s) = \max(S(s), G(s)),$$

where $G(s)$ is a deterministic refund guarantee (at time s) of the paid premiums. Using techniques from life insurance mathematics, the present value $\pi_0(s)$ of the death benefit can be defined as

$$\pi_0(s) = E[\xi(s)C(s)],$$

where

$$\xi(t) := \exp(-tr)$$

is the continuous discount rate.

We will discuss the specifications of the model in the next section.

5.1 Specifications of the Model

In [9], it is shown with empirical evidence that the increments of the log-volatility has a scaling property with a constant smoothing parameter and their distribution is close to Gaussian. This suggest a natural choice of $g(X_t)$

$$g(X_t) = \exp(\{X_t\}_{t\in\mathbb{T}})$$

where X_t satisfies equation (5.2). Setting $b(X_t) = 0$ for simplicity, we have

$$\sigma_t = \eta \exp\bigl(\beta B_t^H\bigr) \tag{5.3}$$

where $\eta = \exp(y)$.

To calculate the present value of future claims, we will consider the following:

(i) The Guarantee is constant i.e. $G(t) \equiv G$ for all $t \in \mathbb{T}$, which can be seen as setting the interest rate to 0.

(ii) The most common case in practice where the insured receives a minimum return R promised by the insurer on their investment i.e.

$$G(t) := G_0 \exp(Rt)$$

for $t \in \mathbb{T}$.

Chapter 6

Implementation of the Model and Results

6.1 Simulation Procedure

We wish to calculate the single net premium of a unit-linked life insurance policy in relation to the stock prices under stochastic volatility, the procedure is as follows:

1. For different number of iterations M, we generate M paths of the fractional Brownian motion with Hurst parameter $H = 0.14$ on the interval $\mathbb{T} = [0, T] = [0, 10]$ where $\mathbb{T}$ is partitioned into $N := 2520$ business days with $\Delta t = \frac{\mathbb{T}}{N}$ representing one business day in the span of 10 years.

2. We then calculate the paths of volatility process under different model. For the classic Black-scholes model the volatility is set to constant σ. For the model proposed by this thesis, we will set η and β to different values. For the model proposed in [9], we take the parameters suggested by the authors as $\nu = 0.3, m = -5$ and $\alpha = 5 \times 10^{-4}$.

3. Next we calculate $\int_0^s (r - \frac{1}{2}\sigma_u^{(i)}) \, \mathrm{d}u$ for all paths of volatility $\sigma_t^{(i)}$, for $i = 1, \ldots, M$. Here we take the overnight market interest r as 0.01.

4. We proceed to calculating $\int_0^s \sigma_u^{(i)} \, \mathrm{d}B_u$ for all paths of volatility $\sigma_t^{(i)}$, with M generated paths of standard Brownian motion $B_t^{(i)}$. This is achieved by assume the approximation

$$\int_0^s \sigma_u^{(i)} \, \mathrm{d}B_u \approx \sum_{j=1}^{N} \sigma_{u_j}^{(i)} (B_{u_{j+1}} - B_{u_j})$$

where

$$0 = u_1 < u_2 < \cdots < u_N = T.$$

5. We can then calculate the paths of stock price $S^{(i)}(s)$ where $S(t)$ is defined as in (5.1) with $S_0 = 50000$.

6. The paths of present value of future claims can be calculated by

$$\pi_0^{*\dagger}(s) = E[e^{-r\cdot s}\max(S(s), G(s))] \tag{6.1}$$

$$\approx \frac{i=1}{M}\sum_{i}^{M} e^{-r\cdot s}\max((S^{(i)}(s), G^{(i)}(s)) \tag{6.2}$$

where G takes different forms as discussed in the previous chapter with $G = G_0 = 50000$, the return rate R will take different values as we observe its effect. For the case with no guarantee,

$$\pi_0^{*\dagger}(s) \approx \frac{1}{M}\sum_{i=1}^{M} e^{-r\cdot s} S^{(i)}(s)$$

.

7. Finally we calculate the premium reserve by

$$V_*^{+}(0) = \int_0^T \pi_0^{*\dagger}(s) p_{**}(0,s) q_{*\dagger}(s)\,\mathrm{d}s$$

Here we consider a 50 years old Norwegian man in year 2016 as example. Using the data from the world health organization, the mortality rate $q_{*\dagger}(t)$ from age 50 to 54 is constant at 0.003 and from age 55 to 59 is constant at 0.005, we take 0.004 as the constant mortality rate for age 50-59. The surviving probability is calculated by

$$p_{**}(0,t) = \exp(-t\cdot 0.004)$$

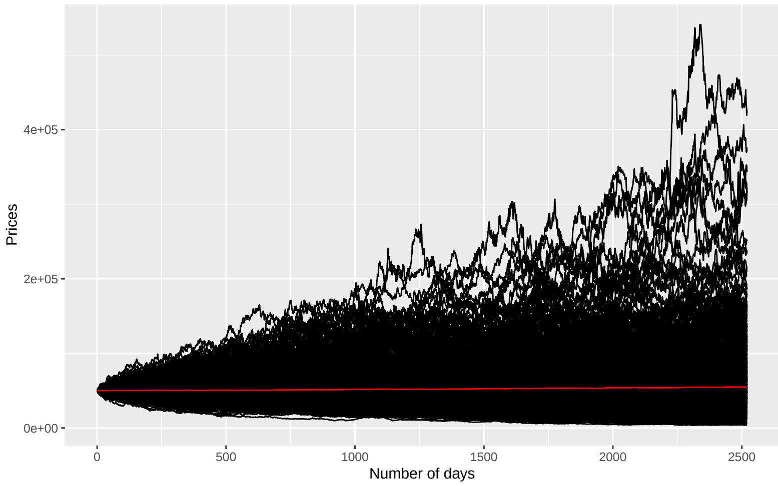
We repeat this process for different numbers of iterations of M to test the convergence of the Monte Carlo simulation.

6.2 Results

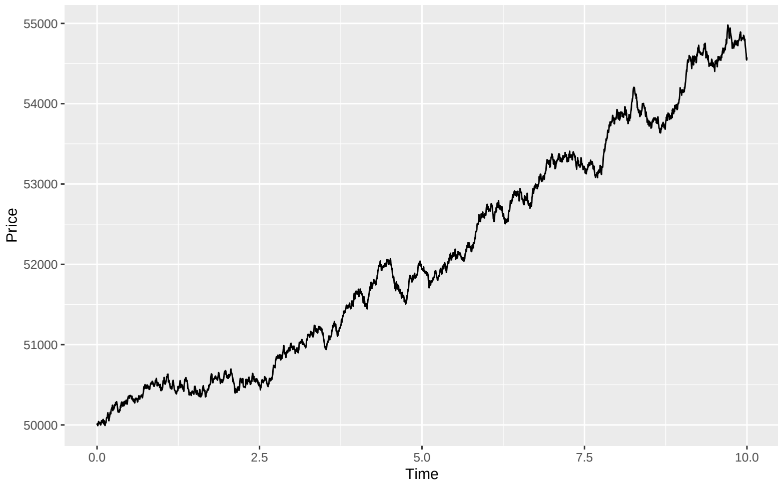
First we consider the proposed model with $\eta = 0.2$ and $\beta = 0.3$. The results are shown in Figure 6.1. The uppermost subplot shows the generated paths of stock prices $S^{(i)}(t)$ and the red line is the average of the simulated prices. It is presented in the middle subplot for a clearer view. Averaging over 1000 paths, the extreme outliers have rather mitigated effect on the process. The corresponding path of the present value of claims is shown in the lowermost subplot with return rate promised by the insurance company set equal to the market overnight interest rate at 0.01. For comparison, we simulated 1000 paths each for the case of extreme volatility values where $\eta = 0.01$ and $\eta = 0.5$ respectively. The results are presented in Figure 6.2.

Figure 6.3 shows the premium reserve required under these different values of η. As shown in the graphs, in a less volatile market, the present value of future claims is more predictable and a smaller premium reserve is required, and vice versa.

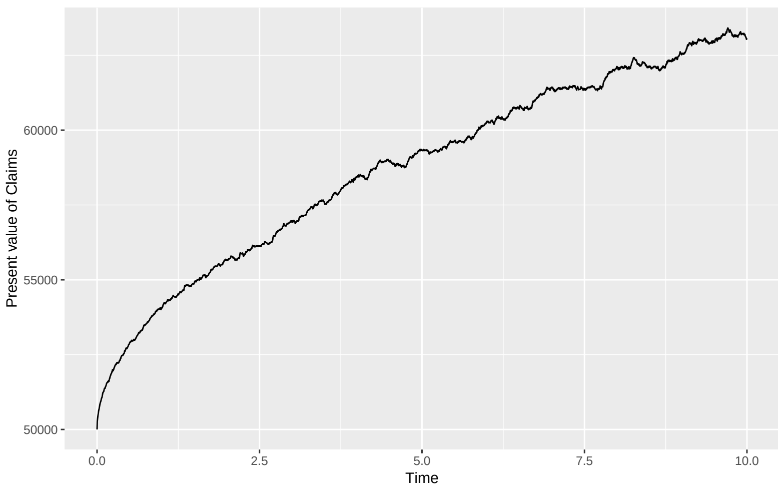
Next we investigate the effect of return rate R on the present value of claims. We simulate 10000 paths of prices with $\eta = 0.2, \beta = 0.3$, then for these 10000 paths, we calculate the paths of present value of claims for each $R = 0, 0.01, 0.025, 0.05, 0.1$, together with the case of no guarantee. The corresponding premium reserves are shown in Figure 6.5. As we can see, the guarantee smooth out the curve significantly. Figure 6.5 further shows that the return R is a significant factor in determining the premium reserve.



Simulated prices of assets

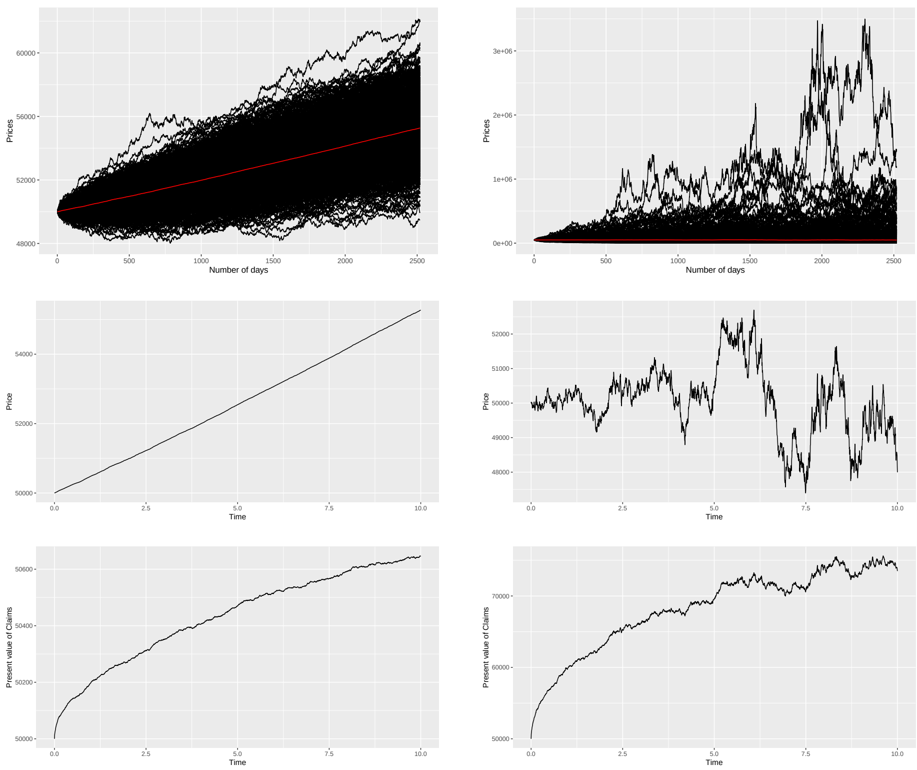


The average of simulated stock prices



Path of the present value of claims

Figure 6.1: Simulation with $\eta = 0.2, \beta = 0.3$



$\eta = 0.01$ $\eta = 0.5$

Figure 6.2: Comparison of extreme values of η

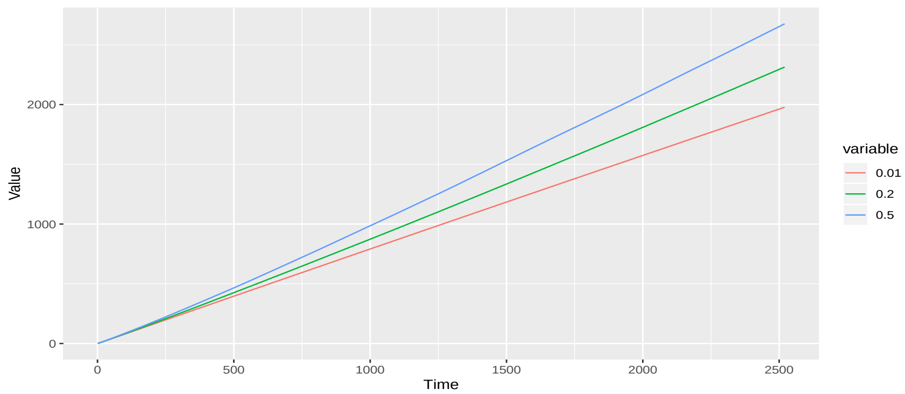
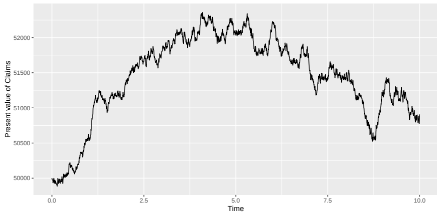
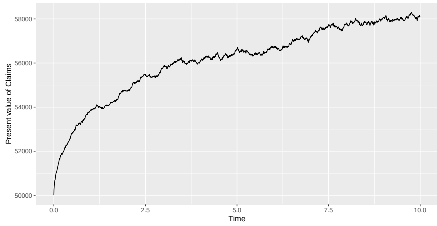


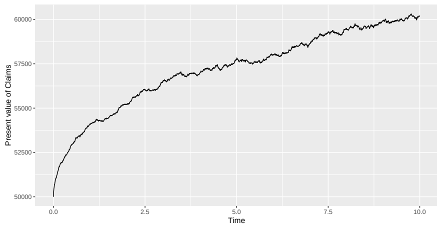
Figure 6.3: The premium reserve under different values of η



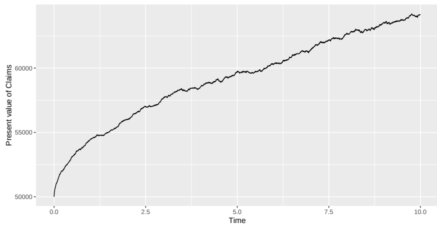
No gurantee



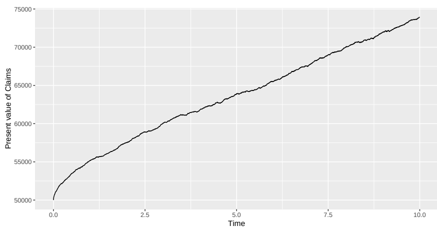
$R = 0$



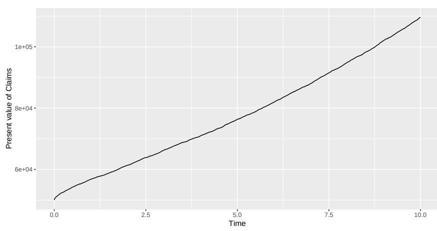
$R = 0.01$



$R = 0.025$



$R = 0.05$



$R = 0.1$

Figure 6.4: Effect of different values of R on the present value of claims

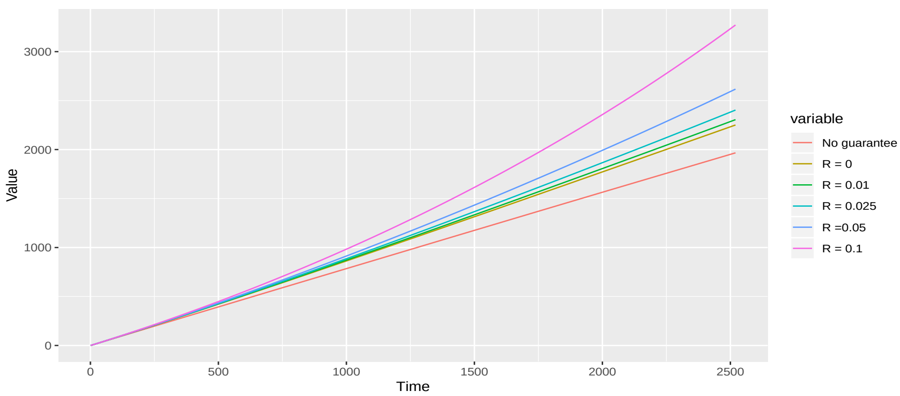
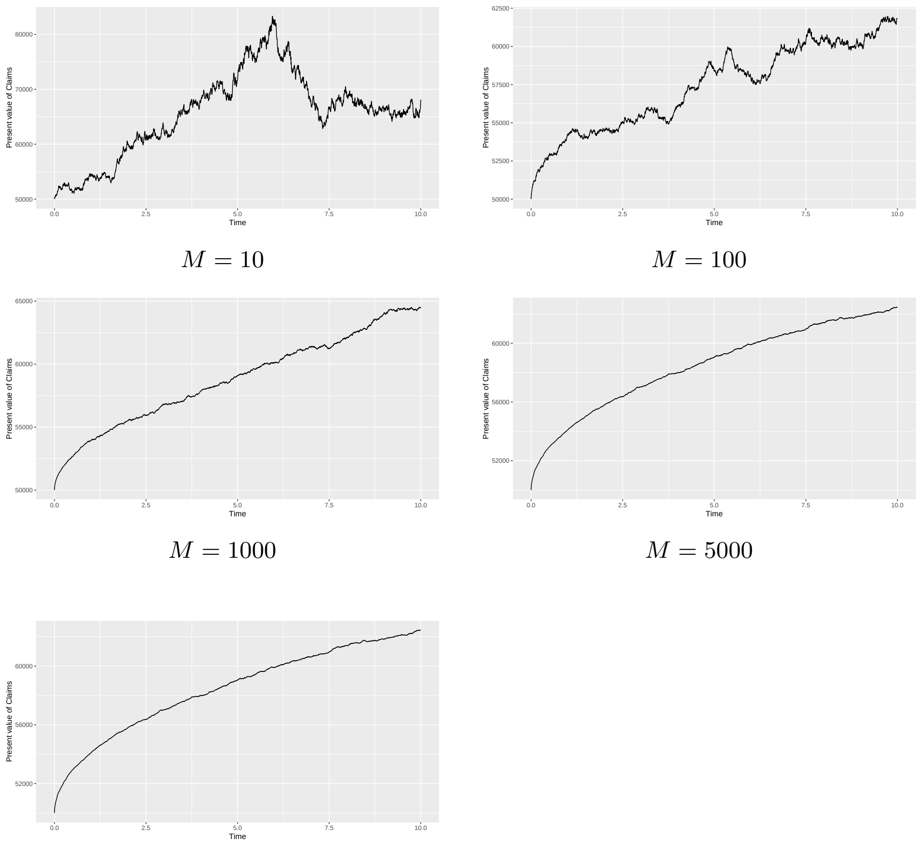


Figure 6.5: The premium reserve under different values of R



$M = 10$ $M = 100$

$M = 1000$ $M = 5000$

$M = 10000$

Figure 6.6: The paths of present values with various numbers of simulations

We proceed to check the convergence of the present value process with different numbers of simulations $M = 10, 100, 1000, 5000, 10000$, the results are shown in Figure 6.6. From our simulation scheme it is intuitive to see that because of the law of large numbers, the paths will converge almost surely as M increases. Figure 6.6 confirms the convergence.

Chapter 7

Extension and Future Work

7.1 Different Models for $\mathrm{X_t}$

We investigated the stochastic volatility model where the dynamics of the stochastic volatility is modeled by

$$\sigma_t = g(X_t)$$

for a non-negative Borel-measurable function g and $\{X_t\}_{t\in\mathbb{T}}$ is modeled by the weak solution to the SDE,

$$X_t = y + \int_0^t b(X_u)\,\mathrm{d}u + \beta B_t^H. \tag{7.1}$$

In our model, we considered $b(X_t) = 0$ as a simplification, the function $b(X_t)$ warrants further study, especially for the case where $b(X_t)$ is discontinuous.

Consider b to be of the form:

$$b(z) = a_1 \mathbb{1}_{(a,\infty)}(z) + a_2 \mathbb{1}_{(-\infty,a]} \tag{7.2}$$

where a_1, a_2 and a are constants, and a is called the threshold.

- For $a_1 = a_2 = a$, we have

  $$X_t = y + at + \sigma_t B_t^H.$$

  This introduces a linear term to X_t, which describes a constant "trend" of the market. Note if $a \equiv 0$, we recover the model discussed before.

- For $a_1 \neq a_2$, the resulting model could describe the so-called regime switching effect on the financial market due to e.g. natural catastrophes, financial crises and drastic regulatory changes. a_1 corresponds to the "level" the market is at before the shock and a_2 corresponds to the situation the market is in after the crisis.

By capturing the regime switching effects, the extended model is able to "react" to sudden changes in the market. In fact by Girsanov's theorem with respect to the fractional Brownian motion, it is shown in [19] that for b as in equation (7.2), there exists a weak solution to SDE (7.1). As for strong solutions, we refer readers to [1].

Another possibility is to consider $X_t,\ t > 0$ to be a fractional Lévy process given by

$$X_t = B^h_{\Gamma_t}$$

where $\Gamma_t,\ t \geq 0$ is a Gamma process. As for the definition and application of Lévy process, the reader may consult [21]

7.2 Optimizing $\mathrm{G}(\mathrm{t})$ and Stochastic Interest Rate

As shown in Chapter 6, the choice of parameters of the Guarantee $G(t)$ i.e. the return R and the initial value G_0 affects the present value of claims significantly. In practice, it is of great interest to the insurance company to determine the optimal choice of such parameters. This raises a optimization problem to be further studied.

In our model, we choose $r = 0.01$ in 5.1 i.e.

$$S_t = S_0 \exp\left(\int_0^t \left(r - \frac{1}{2}\sigma_s^2\right) \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s\right)$$

and the return R is also set to be constant. It is more realistic to invest both parameters in a stochastic setting.

For example, we could consider:

(i) The model where r_t involves a Brownian motion with drift:

$$r_t = r + \delta B_t.$$

(ii) The Vasicek-model where r_T is the solution to the SDE:

$$r_t = r_0 + \int_0^t a(b - r_s)\,\mathrm{d}s + \delta B_t, \qquad t \leq 0.$$

(iii) The Norberg-model with r_t defined by

$$r_t = \sum_{j \in \mathbb{S}} \mathbb{1}_{\{X_t = j\}} r_j(t)$$

where X_t is a Markov chain with state space $\mathbb{S}$ and $r_j(t)$ is a deterministic function.

Appendix A

Code

The code used to produce various plots are as follows

```
pkg <- c("somebm","stats","pracma","ggplot2",
         "reshape2","scales")
lapply(pkg,require, character.only = TRUE)

#Number of iterations
M <- 10000

# Simulations
#Hurst Parameter
H = 0.14

#time Span
T = 10

#Step of the Paths
N = 2520

#time squence
t <- seq(0,T,length.out = N)

#market interest rate
r = 0.01

#return rate promised by the insurance company
R = 0.01

#initiavalue of stock price
s0 <- 50000

#Values of the guarantee

G0 <- 50000
G <- G0*exp(R*t)
```

```
#paremeters of the model
eta <- 0.2
beta <- 0.3


#initiate matrices
PI <- matrix(ncol = M, nrow = N )
S <- matrix(ncol =M,nrow =N)

for (i in 1:M){
  #generating Brownian motion and fractional
  #Brownian motion
  x <- rnorm(n = N-1, sd = sqrt(10/N))
  Bm <- c(0, cumsum(x))
  fBm <- fbm(hurst = H, n = N - 1)
  fBm[1] <- 0

  #the form eta * exp(beta * fbm)
  sigma <- eta*exp(beta*fBm)
  sigma <- as.numeric(sigma)

  #case 3 constant volatility
  #sigma <-rep(0.2,N)

  #calculate the integral of r - 1/2 * sigma^2 from 0 to t
  #here used trapezoidal integration with pracma package
  s_1 <- cumtrapz(t,r - 1/2 * sigma^2)

  #calculate the stochastic integral of sigma form 0 to t
  s_2 <- cumsum(sigma * c(0,diff(Bm)))

  #calculate the stock price
  s <- s0 * exp(s_1 + s_2)
  S[,i] <- s

  #calculate the single premiums
  pi <- exp(-r*t)*pmax(s,G)
  PI[,i] <- pi
}


PI <- data.frame(PI)
S <- data.frame(S)

PV <- rowMeans(PI)
ST <-rowMeans(S)

#plot for the stock prices and present values
qplot(t,ST,geom = 'line', xlab = "Time",
```

```
      ylab = "Price" )
qplot(t,PV,geom = 'line',xlab =  'Time',
      ylab = 'Present␣value␣of␣Claims')

#id variable for position in matrix
S$id <- 1:nrow(S)

#reshape to long format
data_st <- melt(S,id.var="id")

#plot the generated stock prices
ggplot(data_st, aes(x=id,y=value,group=variable)) +
  geom_line(linetype = "solid") +
  stat_summary(aes(y = value,group=1), fun.y=mean,
               colour="red", geom="line",group=1)+
  xlab("Number␣of␣days") + ylab("Prices")


#plot the premium reserve for different R

v = matrix(ncol= 6 , nrow = N)
for (i in 1:M){
    pi <- exp(-r*t)*S[,i]
    PI[,i] <- pi
  }
PV <- rowMeans(PI)
v[,1] <- cumtrapz(t,PV*exp(0.004*-t)*0.004)


j = 2
for (R in c(0,0.01,0.025,0.05,0.1)){
  G <- G0*exp(R*t)

  for (i in 1:M){
    pi <- exp(-r*t)*pmax(S[,i],G)
    PI[,i] <- pi
  }
  PV <- rowMeans(PI)
  qplot(t,PV,geom = 'line',xlab =  'Time',
        ylab = 'Present␣value␣of␣Claims')
 v[,j] <- cumtrapz(t,PV*exp(0.004*-t)*0.004)
 j <- j+1
}

v <- as.data.frame(v)
v$id <- 1:nrow(v)
data_v <- melt(v,id.var="id")
plot_1 <-ggplot(data_v, aes(x=id,y=value,group=variable,
                            color = variable)) +
          geom_line(linetype = "solid") +
```

```
          xlab("Time") + ylab("Value")

p2 <-plot_1 +
  scale_color_discrete(labels = c("No␣guarantee","R␣=␣0",
                                  "R␣=␣0.01","R␣=␣0.025",
                                  "R␣=0.05","R␣=␣0.1"))
```

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