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Universal base spaces and smoothability of face schemes of polyhedral manifolds

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Jarle Stavnes
Oslo, January 2019

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Chapter 1

Introduction

My thesis is devoted to the study of the algebraic deformation theory and the combinatorial structure of pairs (X, L), where X is a projective scheme formed as a disjoint union of toric varieties glued together equivariantly along toric prime divisors, and L is an ample line bundle. Specifically we ask if the pair is smoothable. Examples of such schemes are the stable toric varieties of Alexeev [Ale02; Ale15], and the central fibers of toric degenerations of Calabi Yau varieties from the Gross-Siebert program for mirror symmetry [GS06; GS10]. With our objective in mind, it will be useful to study the deformation functor of pairs $\mathrm{Def}_{(X,L)}$. The structure of the universal base space of $\mathrm{Def}_{(X,L)}$ can potentially shed light upon moduli considerations in this context. Thus the existence and structure of its smoothing components is particularly relevant.

The type of schemes we shall consider has an abstract classification, and a combinatorial counterpart. Abstractly, X is a projective, seminormal scheme equipped with an ample line bundle L over an algebraically closed field k of characteristic 0, such that the normalization $\overline{X}$ comes equipped with the action of an algebraic torus T with finitely many orbits and connected stabilizers. We also assume that the conductor locus is T-invariant. Let $\nu : \overline{X} \to X$ denote the normalization map. If $C \subseteq \overline{X}$ is the closure of a T-orbit $\mathcal{O}$, we furthermore assume that $C \to \nu(C)$ is the normalization of $\nu(C)$, and that $\mathcal{O} \to \nu(C)$ is injective. In addition we require that the induced automorphisms of orbit closures over $\nu(C)$ are T-equivariant.

Our definition above is a slight generalization of the notion of stable toric varieties; the difference is that we do not require that the action of T on $\overline{X}$ descends to an action on X. There is a hierarchy of similar but increasingly general notions, the most basic of which are Stanley-Reisner schemes associated to simplicial complexes. The irreducible components of such schemes are projective spaces, and the question of smoothability in this case was addressed in [Chr10]. As already mentioned, these generalized stable toric varieties we are considering also appears as the central fiber of the toric degenerations $\mathcal{X} \to T$ of Calabi Yau varieties, initially defined by Gross and Siebert in [GS03]. We will be preoccupied exclusively with the combinatorics and the deformation theory of the central fiber $\mathcal{X}_0$, and apart from this relation we will not pursue their program.

We will associate to the pair (X, L) a combinatorial object called a *polyhedral set*, and the study of these will be our primary consideration. The natural situation is when the polyhedral set realizes to an orientable manifold, in which case the dualizing sheaf ω_X is trivial. The combinatorial classification of pairs (X, L) is the following. First, let $\mathcal{P}$ denote the category of lattice polytopes defined as follows: an object of $\mathcal{P}$ is a full-dimensional lattice polytope $P \subseteq \mathbb{R}^n$, denoted (P, n), and a morphism $(P, n) \to (Q, m)$ in $\mathcal{P}$ is an injective affine transformation $f : \mathbb{Z}^n \to \mathbb{Z}^m$ identifying P with a face of Q and an associated character $\lambda_f : \mathbb{Z}^{n+1} \to \mathbf{G}_m$. One can associate to a pair (X, L) a category M with finite skeleton, and a functor $p : M \to \mathcal{P}$ such that X is isomorphic to the colimit $\varinjlim_{s\in M} X_{p(s)}$ of toric varieties, where X_P denotes the toric variety associated to the polytope P. This gives an explicit description of X as a union of toric varieties with torus orbits glued together along torus equivariant morphisms $X_P \to X_Q$ (induced by morphisms in $\mathcal{P}$). The polytopes $p(s)$ are determined up to affine equivalence by the ample invertible sheaf L. If we furthermore require that the functor p is a discrete fibration in the categorical sense, then M is uniquely determined up to isomorphism. In fact, there is a bijective correspondence between equivalence classes of pairs (X, L) and equivalence classes of discrete fibrations $p : M \to \mathcal{P}$ satisfying the following property: $(*)$ for any object $s \in M$, $\mathrm{Aut}_M(s) = \{\mathrm{id}_s\}$. This equivalence is expressed as an equivalence of groupoids in Corollary 3.3.8. Although not pursued, that may have implications for the moduli problem of parametrizing equivalence classes of pairs (X, Θ), where Θ is an ample Cartier divisor subject to certain mild assumptions (analogous to the moduli situation of stable toric pairs [Ale15]).

There is a categorical equivalence between discrete fibrations $p : M \to \mathcal{P}$ and presheaves $M : \mathcal{P}^{\mathrm{op}} \to \mathrm{Set}$. This will be exploited since presheaves are easier to work with. We call a presheaf M satisfying the property $(*)$ a *polyhedral set*, and we let $X = \mathbf{P}(M)$ denote its face scheme. For each $P \in \mathcal{P}$, let $\Gamma(P)$ denote the homogeneous coordinate ring of the toric variety X_P. Then we may alternatively write $\mathbf{P}(M)$ as $\mathrm{Proj}\,\Gamma(M)$, where $\Gamma(M) = \varprojlim_{s\in M} \Gamma(p(s))$. This is called the *face ring* of M. The category $\mathcal{P}$ comes equipped with the monoidal operation $\star$ of join of polytopes (technically we need to add the empty-polytope for $\star$ to be a monoidal product). The induced Day convolution product extends this operation to the subcategory $\mathcal{C} \subseteq \mathrm{Pre}(\mathcal{P})$ of polyhedral sets. The face ring and face scheme constructions can be defined categorically as the monoidal Yoneda extensions of the functors $P \mapsto \Gamma(P)$ and $P \mapsto X_P$, inducing functors $\Gamma : \mathcal{C} \to \mathrm{Ring}$ and $\mathbf{P} : \mathcal{C} \to \mathrm{Sch}_k$, such that $\Gamma(M \star N) \cong \Gamma(M) \otimes_k \Gamma(N)$. The functor $\mathbf{P}$ is particularly well-behaved, in the sense that it preserves intersections, unions and finite colimits (in particular group quotients). Moreover, it takes injections to closed immersions, and surjections to surjections. A polyhedral set M also has an associated topological realization $|M| = \varinjlim_{s\in M} p(s)$, and a number of properties of $\mathbf{P}(M)$ and $\Gamma(M)$ are determined by the topological properties of $|M|$. The most important basic properties we show are the following:

1) There are natural isomorphisms $H^i(|M|, k) \cong H^i(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)})$ for each integer i (Theorem 3.4.1).

2) There is a natural isomorphism
$$\mathrm{Pic}(\mathbf{P}(M)) \cong H^1(|M|, \mathbf{G}_m) \times \varprojlim_{s\in M} \mathrm{Pic}(X_{p(s)}),$$
where $\varprojlim_{s\in M} \mathrm{Pic}(X_{p(s)})$ is a finitely generated free abelian group (Theorem 3.5.2).

3) If $|M|$ is a homology manifold, then $\mathbf{P}(M)$ is locally Gorenstein, and the dualizing sheaf satisfies $\omega_{\mathbf{P}(M)}^{\otimes 2} \cong \mathcal{O}_{\mathbf{P}(M)}$. Moreover, $\omega_{\mathbf{P}(M)} \cong \mathcal{O}_{\mathbf{P}(M)}$ if and only if $|M|$ is an orientable homology manifold (Theorem 6.5.4).

4) If $|M|$ is a homology sphere, then $\Gamma(M)$ is Gorenstein.

Polyhedral sets generalizes the notion of polyhedral complexes, and the face ring $\Gamma(M)$ coincides with the ordinary face ring of a polyhedral complex [BG09]. In particular, if K is a simplicial complex, then $\Gamma(K)$ coincides with the Stanley-Reisner ring of K. Thus 3) is a generalization of the corresponding statement for simplicial complexes[BE91, Theorem 6.1]. It is a classical result[Rei76] by Reisner that the Cohen-Macaulayness of the Stanley-Reisner ring $\Gamma(K)$ is characterized by certain combinatorial conditions on K, which can be translated into topological conditions on its geometric realization $|K|$. Namely that for all $i < \dim K$, $\widetilde{H}^i(|K|; k) = 0$ and $H^i(|K|, |K| \backslash p; k) = 0$ for all points $p \in |K|$. In Theorem 6.3.3, we show that the same criteria also characterizes the *unimodular* polyhedral sets M for which its face ring $\Gamma(M)$ is Cohen-Macaulay (M is called unimodular if $p(s)$ is a unimodular simplex for each $s \in M$). We also give necessary and sufficient conditions for the unimodular M for which $\Gamma(M)$ is Gorenstein in Theorem 6.4.6, which coincides with the conditions provided for simplicial complexes[Sta96] by Hochster, namely that $M \cong S \star \Delta^n$, where S is a unimodular homology sphere and Δ^n a unimodular simplex. These two theorems are logically separate from our main objective, and should be viewed as independent results generalizing classical ones.

Local properties of the face scheme $\mathbf{P}(M)$ for a general polyhedral set M can be deduced by deformation to face schemes associated to unimodular sets. Using the existence of unimodular subdivisions of multiples of polytopes([BG09, Theorem 3.17]), we show that there exists an integer $d > 0$ such that the pair $(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(d))$ is deformation equivalent to a pair $(\mathbf{P}(N), \mathcal{O}_{\mathbf{P}(N)}(1))$ as the central fiber, where N is a simplicial complex (Theorem 2.9.9). Here N is a subdivision of a scaling of M, implying that $|M| \cong |N|$. Thus properties stable under deformation, such as being locally Cohen Macaulay or locally Gorenstein, can be verified via the topology of $|M|$, since we have deduced suitable topological criteria in the case of unimodular sets.

In Chapter 4, we extend the definition of a face scheme from polyhedral sets to more general combinatorial structures, called *open categories over* $\mathcal{P}$. An open category over $\mathcal{P}$ is a category U equipped with a *discrete Conduché fibration* $p : U \to \mathcal{P}$ [Joh99]. Discrete Conduché fibrations generalizes the notion of discrete fibrations, by relaxing the requirement on p. The rationale for the term open category is the following: any open category U can be universally completed into a polyhedral set M, inducing an open immersion $\mathbf{A}(U) \to \mathbf{P}(M)$. A *local isomorphism* $\phi : U \to V$ between open categories is a morphism which induces an isomorphism on comma categories $(s \downarrow U) \to (\phi(s) \downarrow V)$. The relevance of these definitions is that in this situation, we can prove that $\mathbf{A}(\phi) : \mathbf{A}(U) \to \mathbf{A}(V)$ is an étale morphism of schemes. This result will allow us to refine the face scheme $\mathbf{P}(M)$ of a polyhedral set M with an étale cover $\{\mathbf{A}(M_s) \to \mathbf{P}(M)\}$, where

$M_s = (s \downarrow M)$ is an open category, called the local category at $s \in M$. Another interesting fact is that if M is a polyhedral set, then the covering spaces of $|M|$ are in 1-to-1-correspondence with local isomorphisms $S \to M$ of polyhedral sets (Proposition 5.4.3).

In Chapter 5, we define the *link* $\mathrm{lk}_M(s)$ of a polyhedral set M at a face s (we require that the polytopes of M are smooth). It generalizes the ordinary link construction[MS05] for simplicial complexes K at a face s, which is defined as

$$\mathrm{lk}_K(s) = \{t \in K : t \cup s \in K \text{ and } t \cap s = \varnothing\}.$$

The link $\mathrm{lk}_M(s)$ is generally unimodular, and is defined as a certain left Kan extension. The topology of $|\mathrm{lk}_M(s)|$ determines the topology of $|M|$ locally, which allows us to translate certain combinatorial conditions on M into topological ones. Specifically, there is a basis of open sets for the topology of $|M|$, each of which is homeomorphic to an open subset of $|\mathrm{lk}_M(s) \star \Delta^{\dim(s)}|$ for some face $s \in M$. The relation between the link construction and open categories is that there exists a non-canonical isomorphism $\mathbf{A}(M_s) \cong \operatorname{Spec}\Gamma(\mathrm{lk}_M(s)) \times \mathbf{G}_m^{\dim s}$. The face rings $\Gamma(\mathrm{lk}_M(s))$ thus reflects the local properties of $\mathbf{P}(M)$, concretely in the sense that there exists morphisms $\operatorname{Spec}\Gamma(\mathrm{lk}_M(v)) \to \mathbf{P}(M)$ forming an étale cover $\mathcal{U}$ for v ranging over the vertices of M. This étale cover will be used to compute the local cohomology groups $\underline{H}^i_m(\Gamma(M))$ of $\Gamma(M)$ when M is unimodular, thus giving a criterion for when the face ring is Cohen-Macaulay. The proof draws its main idea from [Sta96, II, Theorem 4.1]. We substitute the ordinary Čech complex of $\mathbf{P}(M)$ with the alternating étale Čech complex $\check{\mathrm{C}}_{\mathrm{alt}}(\mathcal{U}, \mathcal{O}_M)$, from which we obtain a complex $\widetilde{C}^\bullet_\Delta(M, \Gamma(M))$ that computes the local cohomology groups of $\Gamma(M)$. This complex decomposes into a direct sum of tractable parts that are governed by the links of M.

Smoothability

In [AC10], the study of the deformation functor $\mathrm{Def}_{(X, \mathcal{O}_X(1))}$ was initiated for $X = \mathbf{P}(K)$, where K is a simplicial manifold. In the case where $|K|$ is a quotient of a $\{3,6\}$-tesselation of $\mathbb{R}^2$, a 3-dimensional smoothing component in the universal base space was identified in [Chr10]. This builds upon earlier computations of the T^1 and T^2-modules of Stanley-Reisner rings in [AC04]. The face scheme $\mathbf{P}(K)$ has a Zariski-cover of affine schemes $\{\operatorname{Spec}\Gamma(\mathrm{lk}_K(v)) \to \mathbf{P}(K)\}_{v \in K_0}$. Thus $T^1_{\Gamma(\mathrm{lk}_K(v))}$ and $T^2_{\Gamma(\mathrm{lk}_K(v))}$ determines the local first-order deformations and local obstruction of $\mathbf{P}(K)$ respectively.

In Chapter 7, the same general idea will be used for polyhedral sets. We will assume that $|M|$ is a 2-dimensional polyhedral manifold, and that each polytope of M is smooth (i.e. we assume that the normalization of $\mathbf{P}(M)$ is smooth). In this situation there exists an étale-cover on the form $\{\operatorname{Spec}\Gamma(\mathrm{lk}_M(v)) \to \mathbf{P}(M)\}_{v \in M_0}$. We may proceed in a similar fashion as in [AC10], now glueing local deformations in the étale-topology. As mentioned earler, our goal is to compute the universal deformation space of $\mathrm{Def}_{(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1)}$. However, a major obstacle for constructing deformation spaces is the obstruction to glueing local deformations to a global one. Section 7.1 is devoted to showing that that this obstruction vanishes in dimension 2, in addition to the vanishing of the obstruction to the lifting of a line bundle (Theorem 7.1.7). As a consequence, in Section 7.6 we are able to produce examples of 1-parameter smoothings of face schemes of various polyhedral spheres to K3 surfaces by constructing compatible infinitesimal local deformations. By standard results of [Art69], effective deformations with an ample line bundle are algebraizable, so this gives rise to proper, algebraic smoothings.

Theorem 7.3.2 is our main result. Here we find conditions under which it is possible to compute the universal deformation base space of $\mathrm{Def}_{(\mathbf{P}(M),\mathcal{O}_{\mathbf{P}(M)}(1))}$. Specifically, this works when each vertex link $\mathrm{lk}_M(v)$ contains between 3 and 6 components, and under a certain regularity condition on the facet polygons of M. We obtain a presentation of the universal base space as the spectrum of a quotient $\mathscr{R}_{M,\mathscr{O}} = \mathscr{P}_M/\mathfrak{a}_{\mathscr{O}}$ of a complete local regular ring $\mathscr{P}_M$, by an ideal of binomial equations $\mathfrak{a}_{\mathscr{O}}$ given as the minors of certain 2×3-matrices. This is a generalization of the analogous statement [AC10, Theorem 6.4] for simplicial complexes. Given some additional assumptions on M, Theorem 7.4.7 now answers our original question regarding smoothability of (X, L), where we have identified the smoothing components as the closure of a torus inside $\mathrm{Spec}(\mathscr{R}_M)$. Examples of this situation are finite index quotients of certain "admissible" periodic tesselations of the plane (i.e. torus tesselations), some of which we examine in Section 7.5. Smoothings of $(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1))$ are in this case polarized abelian surfaces. We compute some explicit examples of smoothing components using Macaulay2. One can also prove that the dimension $h^0(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1))$ is equal to the number of interior lattice points of the unique polytopes appearing in M, so a particularly interesting case is when the polytopes of M have empty interior, and are glued in such a way that M consists of only a single vertex. In this situation $h^0(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1)) = 1$, so in the torus case $(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1))$ is smoothable to principally polarized Abelian surfaces. This example is discussed in [AN99], and it appears on the boundary of Alexeev's compactification $\overline{AP_2}$ of the moduli space of principally polarized Abelian surfaces[Ale02] (see also [Ols08] for an outline of the construction).

Structure of thesis

The text is divided into seven chapters, including this introduction, and three appendices. What follows is a general description of the content and goals of each chapter.

Chapter 2: We give the preliminary definitions of the category of full-dimensional lattice polytopes $\mathcal{P}$ and the category of polyhedral sets $\mathcal{C}$. Most of our constructions will be given as Yoneda extensions. For example, this means that the definition of the geometric realization $|M|$ of a polyhedral set M is dictated by the forgetful functor $\mathcal{P} \to \mathrm{Top}$. We define the face ring $\Gamma(M)$ in Section 2.3, and in Section 2.7 and Section 2.8 we determine some of its properties. In Section 2.4, we define the *canonical pushout square* of a polyhedral set. This will be useful for inductive arguments in this and later chapters. The CW complex structure of $|M|$ is discussed in Section 2.5. In Section 2.9, we develop the formalism of subdividing polyhedral sets, and relate it to deformations of the face ring. An important result here is the existence of triangulations of polyhedral sets to simplicial complexes (Theorem 2.9.9).

Chapter 3: The face scheme is defined, and the categorical properties of the face scheme functor $\mathbf{P}$ as described above are proved, such as being finite colimit preserving (Proposition 3.0.3). Next, in Section 3.3, we show the 1-to-1-correspondence between pairs (X, L) and polyhedral sets. In Section 3.4 and Section 3.5, we show the existence of the natural isomorphisms $H^i(|M|; k) \cong H^i(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)})$ and $\operatorname{Pic}(\mathbf{P}(M)) \cong H^1(|M|, \mathbf{G}_m) \times \varprojlim_{s \in M} \operatorname{Pic}(X_{p(s)})$ respectively. In Section 3.6 we give an expression of the Hilbert polynomial of the face ring $\Gamma(M)$.

Chapter 4: In Section 4.1 we give the formal preliminaries for Conduché fibrations, and in Section 4.2 we apply it to $\mathcal{P}$ in particular. Section 4.3 and Section 4.4 are devoted to proving the result regarding étale morphisms as previously mentioned (Theorem 4.4.1).

Chapter 5: We define the notion of the link of a unimodular set, and in Section 5.3 we extend this definition to polyhedral sets consisting of smooth polytopes. We have separated these two situations, since more can be said in the former situation which will be crucial to the proofs of the theorems of Chapter 6. The content of Section 5.4 regarding the local topological properties of $|M|$ is particularly relevant.

Chapter 6: Our main goal is proving the classification theorem for Cohen Macaulay and Gorenstein unimodular sets, as described in the introduction above. In Section 6.1, we outline the construction of the alternating étale Čech complex, which will allow us to compute the local cohomology modules $H^i_m(\Gamma(M))$. In Section 6.2, we apply this to the étale cover $\{\mathbf{A}(X_v) \to \mathbf{P}(M)\}$, giving a more combinatorial representation of the Čech complex. In Section 6.3 and Section 6.4 we prove our result regarding Cohen Macaulay and Gorenstein unimodular sets respectively. In Section 6.5 we prove the result stated earlier regarding the the dualizing sheaf of $\mathbf{P}(M)$ in the situation where $|M|$ is a homology manifold.

Chapter 7: Here we answer our original question. In Section 7.1 we show in dimension 2 the vanishing of the cohomology group $H^2(\mathbf{P}(M), \mathcal{E}_{\mathcal{O}_{\mathbf{P}(M)}(1)})$, containing the obstructions to glueing local infinitesimal deformations. Next, in Section 7.2 we give a description of a basis for $H^0(\mathbf{P}(M), \mathcal{T}^1_{\mathbf{P}(M)/k})$. In Section 7.3 we prove our main theorem (Theorem 7.3.2) regarding the structure of the universal base space $\operatorname{Spec}(\mathscr{R}_{M,\mathscr{O}})$ of $(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1))$. In Section 7.4 we prove Theorem 7.4.7, identifying the smoothing components in $\operatorname{Spec}(\mathscr{R}_{M,\mathscr{O}})$. Section 7.5 contains our example computations of the smoothing components for quotients of admissible periodic plane tesselations. In Section 7.6 we give the idea behind smoothing polyhedral spheres to K3 surfaces. 10 explicit examples are found using Macaulay2.

Appendix A: Here the required background material is provided on the categorical notions we shall require. In particular, Lemma A.0.9 is required in order to give an explicit description of the Day convolution product $M \star N$ of polyhedral sets.

Appendix B: We outline the concept of Milnor patching for projective (and flat) modules and vector bundles. In particular, this will be used in the proof of Theorem 4.4.1 to prove flatness of $\mathbf{A}(U) \to \mathbf{A}(V)$ by induction on dimension. Proposition B.3.5 is instrumental for the classification of the Picard group of $\mathbf{P}(M)$ (Theorem 3.5.2).

Appendix C contains the Macaulay2 functions we have used to compute the examples of Section 7.5 and Section 7.6.

Chapter 2

Polyhedral sets

2.1 The category of lattice polytopes

A *lattice polytope* is a subset $P \subseteq \mathbb{R}^n$ for some $n \geq 0$, realized as the convex hull of integral lattice points. We will denote it as a pair by (P, n), but usually just by P. If P is full-dimensional, there exists a unique set of integers a_F and vectors $u_F \in \mathbb{Z}^{\dim(P)}$ of minimal length for each facet F such that P can be written as $\{m \in \mathbb{R}^n \mid \langle m, u_F \rangle \geq -a_F\}$. A *character* is a homomorphism $\lambda : \mathbb{Z}^{n+1} \to \mathbf{G}_m$, where denotes the multiplicative group k^*. An *affine transformation* is a function $f : \mathbb{Z}^m \to \mathbb{Z}^n$ on the form $x \mapsto u + Ax$, where $u \in \mathbb{Z}^n$ and $A : \mathbb{Z}^m \to \mathbb{Z}^n$ is a matrix whose columns can be extended to a basis for $\mathbb{Z}^n$. We denote the induced affine transformation on real vector spaces by $f_{\mathbb{R}} : \mathbb{R}^m \to \mathbb{R}^n$, which will also be called an affine transformation. The image $f_{\mathbb{R}}(P) \subseteq \mathbb{R}^n$ of a lattice polytope is again a lattice polytope. We define the category $\mathcal{P}$ as follows: The objects are full-dimensional lattice polytopes (P, n). An arrow $(Q, m) \to (P, n)$ is an affine transformation $f : \mathbb{Z}^m \to \mathbb{Z}^n$ such that $f_{\mathbb{R}}(P)$ is a face of Q, equipped with a character $\lambda_f : \mathbb{Z}^{m+1} \to \mathbf{G}_m$. We will sometimes denote an arrow by (f, λ_f). The character of a composition $(P, n) \xrightarrow{f} (Q, m) \xrightarrow{g} (R, r)$ is defined by $\lambda_{gf}(m, d) = \lambda_f(m, d)\lambda_g(f(m) - f(0) + df(0), d)$ for all $(m, d) \in \mathbb{Z}^n \times \mathbb{Z}$. If we include the empty-polytope $\varnothing$, we obtain the category $\mathcal{P}_+$. This object is initial in $\mathcal{P}_+$, and we will occasionally denote it by $(\varnothing, -1)$ for consistent notation. The motivation behind this definition is the following fact:

Proposition (Proposition 3.3.6)**.** *There is a natural bijective correspondence between isomorphisms $(f, \lambda_f) : P \to Q$ in $\mathcal{P}$ and pairs (ψ, ι), where $\psi : X_P \xrightarrow{\cong} X_Q$ is a torus-equivariant isomorphism, and $\iota : \psi^*\mathcal{O}_{X_Q}(D_Q) \xrightarrow{\cong} \mathcal{O}_{X_P}(D_P)$ is an isomorphism of line bundles. Here X_P denotes the projective toric variety associated to the polytope P, and D_P is the ample torus-invariant divisor corresponding to P.*

Let Top denote the category of compactly generated Hausdorff topological spaces. It is convenient to work in this subcategory of topological spaces by default as it is cartesian closed, in particular monoidally cocomplete. Let $\Re : \mathcal{P} \to \text{Top}$ and $\Re_+ : \mathcal{P}_+ \to \text{Top}$ be the functors given by $(P, n) \mapsto P$, i.e. forgetting the embedding into $\mathbb{R}^n$. We will briefly review the *join* operation $\star$ on Top. The join of a pair of topological spaces X, Y is defined as the quotient $X \star Y = (X \times I \times Y)/\sim$, where $(x, 0, y_1) \sim (x, 0, y_2)$ and $(x_1, 1, y) \sim (x_2, 1, y)$.

The join $X \star Y$ can alternatively be described as the pushout of the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "i_0"] & & X \times Y \arrow[dl, "i_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ X & & X \times I \times Y & & Y, \end{tikzcd}$$

where i_0 is the inclusion $(x, y) \mapsto (x, 0, y)$ and i_1 is the inclusion $(x, y) \mapsto (x, 1, y)$. In this generality the join operation $(X, Y) \mapsto X \star Y$ defines a monoidal product on Top, and the resulting monoidal category $(\mathrm{Top}, \star, \varnothing)$ is monoidally cocomplete.

The category $\mathcal{P}_+$ inherits from Top the join operation via $\Re_+$. If $(P, n), (Q, m) \in \mathcal{P}_+$, their join is $(P \star Q, n+m+1)$, where $P \star Q = \{((1-t)m, t, tn) \mid m \in P, n \in Q, t \in [0, 1]\} \subseteq \mathbb{R}^{n+m+1}$ (when P and Q are non-empty). If $f_1 : P_1 \to Q_1$ and $f_2 : P_2 \to Q_2$ are arrows, then $\lambda_{f_1 \star f_2}$ is given by $((m_1, s, m_2), d) \mapsto \lambda_{f_1}(m_1, d-s)\lambda_{f_2}(m_2, s)$. We can also write $P \star Q$ as the convex hull of the subset $P \times \{0\} \times \{0\} \cup \{0\} \times \{1\} \times Q \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$. The vertices of $P \star Q$ are on the form $(v, 0, 0)$ or $(0, 1, w)$, where v and w are vertices of P and Q respectively. If P (resp. Q) is the empty-polytope $\varnothing$, we have $P \star Q = Q$ (resp. $P \star Q = P$). The empty polytope $\varnothing$ is a unit for the join operation.

Lemma 2.1.1. *For any pair of objects $P_1, P_2 \in \mathcal{P}_+$ and arrow $f : Q \to P_1 \times P_2$, there exists a unique pair of arrows $f_1 : Q_1 \to P_1$, $f_2 : Q_2 \to P_2$ such that $f = f_1 \star f_2$. In the context of Definition A.0.2, this means that the functor $\star : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is a discrete fibration.*

Proof. By the elementary fact that the faces of $P_1 \star P_2$ are uniquely on the form $Q_1 \star Q_2$ for faces Q_1 and Q_2 of P_1 and P_2 respectively, one immediately obtains a unique corresponding decomposition of Q on the required form. Uniqueness of λ_{f_1} and λ_{f_2} is easily verified. $\square$

2.1.1 Unimodular simplices

Let Fin_+ denote the category of finite ordinals $[n] = \{0, \ldots, n\}$ (where $[-1] = \varnothing$), where an arrow $f : [n] \to [m]$ is an injective function, equipped with a character $\lambda_f : \mathbb{Z}^{[n]} \to \mathbf{G}_m$. Here $\mathbb{Z}^{[n]}$ denotes the free abelian group on basis vectors e_i for $i \in [n]$. The character of a composition is given by $\lambda_{gf}(e_i) = \lambda_g(e_{f(i)})\lambda_f(e_i)$. This category will serve as our model for unimodular simplices. The arrows are generated by three types of functions:

1) simplicial face maps $d_i : \{0, \ldots, n\} \to \{0, \ldots, n+1\}$ defined by $j \mapsto j$ if $j < i$, and $j \mapsto j+1$ if $j \geq i$ (for $i = 0, \ldots, n+1$), and with trivial character;

2) permuations $\sigma : \{0, \ldots, n\} \to \{0, \ldots, n\}$, with trivial character.

3) arrows $(\mathrm{id}_{[n]}, \lambda) : \{0, \ldots, n\} \to \{0, \ldots, n\}$ which are the identity as functions (but with arbitrary character λ).

We let Fin denote the full subcategory of Fin_+ of non-empty ordinals. We define $I_{\mathrm{Fin}} : \mathrm{Fin} \to \mathcal{P}$ by

$$[n] \mapsto |\Delta^n| := \{(x_1, \dots, x_n) \mid x_i \geq 0, \sum_{i=1}^{n} x_i \leq 1\} \subseteq \mathbb{R}^n$$

Functoriality is given as follows. First, identify the affine span of $\mathrm{Conv}(e_0, \dots, e_n) \subseteq \mathbb{R}^{[n]}$ with $\mathbb{R}^n$ by choosing basis vectors $e_1 - e_0, \dots, e_n - e_0$ centered at e_0. This identifies $\mathrm{Conv}(e_0, \dots, e_n)$ with $|\Delta^n|$. The corresponding affine transformation $\mathbb{R}^n \to \mathbb{R}^{[n]}$ is given by $0 \mapsto e_0$, and $e_i \mapsto e_i$ for $i > 0$. If $f : n \to m$ is a function, the affine transformation $\mathbb{Z}^{[n]} \to \mathbb{Z}^{[m]}$ defined by $e_i \mapsto e_{f(i)}$ identifies $\mathrm{Conv}(e_0, \dots, e_n)$ with a face of $\mathrm{Conv}(e_0, \dots, e_m)$, hence the affine transformation $I_{\mathrm{Fin}}(f) : \mathbb{Z}^n \to \mathbb{Z}^m$ induced by the same base change identifies $|\Delta^n|$ with a face of $|\Delta^m|$. It is given by $e_i \mapsto e_{f(i)}$, where we put $e_0 = 0$. On characters, we define $\lambda_{I_{\mathrm{Fin}}(f)}(e_i, 1) = \lambda_f(e_i)$, and $\lambda_{I_{\mathrm{Fin}}(f)}(0, 1) = \lambda_f(e_0)$. This is clearly functorial, and I_{Fin} extends to a functor I_{Fin_+} by mapping $[-1]$ to $\varnothing$.

Fin_+ can be equipped with an operation $+ : \mathrm{Fin}_+ \times \mathrm{Fin}_+ \to \mathrm{Fin}_+$, defined by $[n] + [m] = [n + m + 1]$ on objects, and for any pair of arrows $f : n \to n'$ and $g : m \to m'$, $h = f + g : [n] + [m] \to [n'] + [m']$ is given by

$$\begin{aligned} h(i) &= f(i) && \text{for } i = 0, \dots, n, \\ h(i) &= n' + 1 + g(i - n - 1) && \text{for } i = n + 1, \dots, n + m + 1, \\ \lambda_h(m_1, s, m_2) &= \lambda_f(m_1)\lambda_g(m_2) && \text{for } (m_1, s, m_2) \in \mathbb{Z}^{n+m+1}. \end{aligned}$$

This gives Fin_+ the structure of a symmetric monoidal category with unit $[-1]$.

Lemma 2.1.2. *The functor $I_{\mathrm{Fin}_+} : \mathrm{Fin}_+ \to \mathcal{P}_+$ is a fully faithful strong monoidal functor.*

Proof. Let $[n], [m] \in \mathrm{Fin}$, and note that any morphism $f : |\Delta^n| \to |\Delta^m|$ in $\mathcal{P}_+$ is uniquely determined by the character λ_f and the induced inclusion of vertices. In fact, the arrow $[n] \to [m]$ in Fin corresponding to this inclusion of vertices induces $f_{\mathbb{R}}$. This shows fully faithfulness. To see that it is strong monoidal, consider the isomorphism $\mathrm{Conv}(e_0, \dots, e_n) \star \mathrm{Conv}(e_0, \dots, e_m) \to \mathrm{Conv}(e_0, \dots, e_{n+m+1})$ with trivial character given by

$$((1-t)x_0, \dots, (1-t)x_n, t, ty_0, \dots, ty_m) \mapsto ((1-t)x_0, \dots, (1-t)x_n, ty_0, \dots, ty_m).$$

This clearly satisfies the coherence conditions of a monoidal functor, and via the canonical identifications, this induces a coherent isomorphism $|\Delta^n| \star |\Delta^m| \to |\Delta^{n+m+1}|$. □

2.2 Polyhedral sets

Definition 2.2.1. Let C be a category, and let $M : C^{\mathrm{op}} \to \mathrm{Set}$ be some presheaf. Let $(*)$ denote the following condition on M: For each $c \in C$, the group $\mathrm{Aut}_C(c)$ acts freely on M_c. We let $\mathrm{Pre}^{(*)}(C)$ denote the category of presheaves on C satisfying $(*)$.

By Proposition A.0.3, there is an equivalence of categories between the category of presheaves $\mathrm{Pre}(C)$ and the category of discrete fibration $\mathrm{Fib}(C)$. The condition $(*)$ translates to the following condition on a discrete fibrations $p : U \to C$: for all $s \in U$, $\mathrm{Aut}_U(s) = \{\mathrm{id}_s\}$.

Lemma 2.2.2. *The pushout of a diagram $M \xleftarrow{\phi} Z \xrightarrow{\psi} N$ in $\mathrm{Pre}(C)$ satisfies $(*)$ whenever $M, N, Z \in \mathrm{Pre}^{(*)}(C)$ and ϕ is a monomorphism.*

Proof. One easily verifies that $\mathrm{Aut}_C(c)$ acts freely on $M_c \times_{Z_c} N_c$ for each $c \in C$. □

Definition 2.2.3. We define the category *polyhedral sets* as $\mathcal{C} = \mathrm{Pre}^{(*)}(\mathcal{P})$. A *morphism* $M \to N$ of polyhedral sets $M, N \in \mathcal{C}$ is a natural transformation $\phi : M \to N$. A presheaf $M \in \mathrm{Pre}^{(*)}(\mathcal{P}_+)$ such that $M_\varnothing$ is a one-point set is called an *augmented polyhedral set.* The category $\mathcal{C}_+ = \mathrm{Pre}^{(*)}(\mathcal{P}_+)$ of augmented polyhedral sets will mainly serve as a notational convenience.

A *polyhedral subset* of M is an equivalence class of pairs (N, ϕ), where N is a polyhedral set and $\phi : N \to M$ is an injective morphism. Two such pairs (N_1, ϕ_1), (N_2, ϕ_2) are equivalent if there exists an isomorphism $\psi : N_1 \xrightarrow{\cong} N_2$ satisfying $\phi_2 = \phi_1\psi$. For a polyhedral set M, the *P-faces* of M are the elements of the set M_P. For each arrow $f : P \to Q$, $M(f) : M_Q \to M_P$ is the corresponding *face map*, also denoted f^M. For each P, we define *the P-polytope* Δ^P to be the presheaf $\mathcal{P}(-, P) : \mathcal{P}^{\mathrm{op}} \to \mathrm{Set}$ represented by P. Since all arrows of $\mathcal{P}$ are monomorphisms, each Δ^P is a polyhedral set. By the Yoneda lemma, there is a natural bijection $\mathrm{Hom}(\Delta^P, M) \to M_P$ between the P-faces of M and morphisms $\Delta^P \to M$ from the P-polytope to M. If $s \in M_P$ is a face, then we define $|s| = P$. We will denote the corresponding morphism $\Delta^{|s|} \to M$ by ζ^s_M. We will also denote $\Delta^{|s|}$ by Δ^s. A *facet* is a face $s \in M_P$ which is not in the image of any face map $f^M : M_Q \to M_P$ for any non-isomorphism $f : P \to Q$. Two faces $s \in M_P$ and $t \in M_Q$ are called *equivalent* if there exists an isomorphism $f : P \xrightarrow{\cong} Q$ such that $f^M(t) = s$. A *finite polyhedral set* is a polyhedral set with finitely many faces of each equivalence class.

Definition 2.2.4. Presheaves $M \in \mathrm{Pre}^{(*)}(\mathrm{Fin})$ are called *unimodular sets.* Analogously, one has the augmented version of presheaves $M \in \mathrm{Pre}^{(*)}(\mathrm{Fin}_+)$ such that $M_{[-1]}$ is a one-point set.

Consider the fully faithful functor $\widehat{I_{\mathrm{Fin}}} : \mathrm{Pre}(\mathrm{Fin}) \to \mathrm{Pre}(\mathcal{P})$ defined as the Yoneda extension (see Appendix A) of the embedding $I_{\mathrm{Fin}} : \mathrm{Fin} \to \mathcal{P}$. If M is a unimodular set, then it is clear that $\widehat{I_{\mathrm{Fin}}}(M)$ is given by by $\widehat{I_{\mathrm{Fin}}}(M)_P = M_n$ for each $n \in \mathrm{Fin}$ and $P \in \mathcal{P}$ with $P \cong I_{\mathrm{Fin}}(n)$, and $\varnothing$ otherwise. We define the *dimension* of an element $P \in \mathcal{P}$ to be its dimension $\dim(P)$ as a lattice polytope. The *dimension of a polyhedral set* M, denoted $\dim(M)$, is the largest integer N for which there exists a P with $\dim(P) = N$ and M_P non-empty. If no such N exist and M is not the empty-presheaf, we call M infinite-dimensional. We say that the empty-presheaf, denoted $\underline{\varnothing}$, is (-1)-dimensional. The *dimension* *of a face* $s \in M_P$ is $\dim(P)$, and is denoted $\dim(s)$. M_0 is the set of *vertices* of M. It is clear that the P-polytope Δ^P has dimension $\dim(P)$. Let M be a finite-dimensional polyhedral set. Then M is called *irreducible* if it cannot be written as a proper union of non-empty polyhedral subsets. Let s be a face of M, and consider the corresponding morphism $\zeta^s_M : \Delta^s \to M$. Then the *polyhedral subset corresponding to* s is the image $M^s = \zeta^s_M(\Delta^s) \subseteq M$. It is the minimal polyhedral subset of M containing s. M is irreducible if and only if it has (at most) a single equivalence class of facets, and can in general always be written uniquely as a union of its irreducible components.

The monoidal structure on $\mathrm{Pre}(\mathcal{P})$ is given by the Day convolution product (see Definition A.0.7 and [IK86, Section 4]). Per Lemma A.0.9, given $M, N \in \mathrm{Pre}(\mathcal{P}_+)$, the convolution product $M \star N : \mathcal{P}_+^{\mathrm{op}} \to \mathrm{Set}$ is defined as the left Kan extension of the functor $M \times N : \mathcal{P}_+^{\mathrm{op}} \times \mathcal{P}_+^{\mathrm{op}} \to \mathrm{Set} \times \mathrm{Set} \to^{\times} \mathrm{Set}$ along the induced comultiplication functor $\star^{\mathrm{op}} : \mathcal{P}_+^{\mathrm{op}} \times \mathcal{P}_+^{\mathrm{op}} \to \mathcal{P}_+^{\mathrm{op}}$. The characterization of $\star$ as a convolution product - coupled with the fact that it is induced by a discrete fibration - allows us to describe $M \star N$ explicitly. In particular, $\star$ defines a monoidal operation on $\mathcal{C}$.

Proposition 2.2.5. *If M and N are augmented polyhedral sets, then their* join *is an augmented polyhedral set, and is given by*

$$(M \star N)_R = \coprod_{\{(P,Q) | P \star Q = R\}} M_P \times N_Q. \tag{2.1}$$

for each $R \in \mathcal{P}_+$. The functorial structure is given as follows. For any arrow $h : R \to R'$, and for each pair (P_1, Q_1) with $P_1 \star Q_1 = R'$, let $h_1 : P_0 \to P_1$ and $h_2 : Q_0 \to Q_1$ be the unique pair of arrows such that $h = h_1 \star h_2$. Then the map $h^{M \star N} : (M \star N)_{R'} \to (M \star N)_R$ is given by $h_1^M \star h_2^N : M_{P_1} \times N_{Q_1} \to M_{P_0} \times N_{Q_0}$ on components.

Proof. The formulas follow from Lemma A.0.6 a) and b), which we may apply here since $\star$ is a discrete fibration by Lemma 2.1.1. It follows immediately from this that $M \star N$ also satisfies $(*)$. □

Remark 2.2.6. The inclusion $i : \mathcal{C} \to \mathrm{Pre}^{(*)}(\mathcal{P}_+)$ is given by adjoining a singleton set at $\varnothing$. By applying i, the join operation formula (2.1) makes sense for polyhedral sets $M, N \in \mathcal{C}$ as well, with the empty-presheaf $\underline{\varnothing}$ as a unit. Hence the category $\mathcal{C}$ of polyhedral sets is a monoidal category with respect to $\star$. The inclusion functor i preserves $\star$ and all connected colimits.

Remark 2.2.7. One analogously defines the join of two unimodular set via the monoidal operation $+ : \mathrm{Fin}_+ \times \mathrm{Fin}_+ \to \mathrm{Fin}_+$. Since $I_{\mathrm{Fin}_+} : \mathrm{Fin}_+ \to \mathcal{P}_+$ is a strong monoidal functor by Lemma 2.1.2, the inclusion $\mathrm{Pre}^{(*)}(\Delta) \to \mathcal{C}$ is a strong monoidal functor. As in Proposition 2.2.5, one has

$$(M \star N)_n = \coprod_{[m]+[r]=[n]} M_m \times N_r$$

for any pair of unimodular sets M, N.

The following proposition is a consequence of Proposition A.0.8.

Proposition 2.2.8 (**Geometric realization).** *The strong monoidal functor* $\Re_+ : \mathcal{P}_+ \to \mathrm{Top}$ *induces a cocontinuous strong monoidal geometric realization functor* $|-| : \mathrm{Pre}(\mathcal{P}_+) \to \mathrm{Top}$ *such that* $|\Delta^P| \cong P$ *for each* $P \in \mathcal{P}_+$*. In particular, there is a natural isomorphism*

$$|M \star N| \xrightarrow{\cong} |M| \star |N|$$

for each $M, N \in \mathrm{Pre}(\mathcal{P}_+)$*. Note that since the inclusion* $i : \mathrm{Pre}(\mathcal{P}) \to \mathrm{Pre}(\mathcal{P}_+)$ *preserves* $\star$*, the geometric realization functor* $|-| : \mathcal{C} \to \mathrm{Top}$ *also preserves* $\star$*.*

2.3 The face ring of a polyhedral set

Let $\mathscr{D}$ denote the category of positively graded k-algebras. $\mathscr{D}$ is a monoidal category, with the ordinary tensor product operation $\otimes_k$ and unit k. We recall some definitions from [CLS11]. The affine semigroup associated to a polytope P is defined as $A_P = \mathbb{N}[(P \cap \mathbb{Z}^n) \times \{1\}]$, and consists of all linear combinations of lattice points in $P \times \{1\} \subseteq \mathbb{Z}^{n+1}$ with non-negative integral coefficients. This notation is consistent if P is the empty-polytope, in which case we have $A_\varnothing = 0$. If $f : P \to Q$ is an arrow in $\mathcal{P}_+$ represented by an affine transformation $x \mapsto u + Ax$ with character λ_f, then one obtains an inclusion of affine semi-groups $A_f : A_P \to A_Q$ given by $(m, d) \mapsto (df_{\mathbb{R}}(\frac{m}{d}), d) = (ud + A(m), d)$. This is easily seen to define a functor $A : \mathcal{P}_+ \to \mathrm{SemiGrp}$. We will now define the face ring of a polyhedral set. Consider the contravariant functor $\Gamma' : \mathcal{P}_+ \to \mathscr{D}$ given on objects by $\Gamma'(P) = k[A_P]$, where

$$k[A_P] = \bigoplus_{d=0}^{\infty} \bigoplus_{m \in dP} k \cdot \chi^m$$

is the homogeneous coordinate ring (generated in degree 1) associated to the affine semigroup A_P. For any arrow $f : P \to Q$, we let $\Gamma'(f) : \Gamma'(Q) \to \Gamma'(P)$ be given by $\chi^{A_f(m)} \mapsto \lambda_f(m)\chi^m$ (and $\chi^m \mapsto 0$ otherwise).

Lemma 2.3.1. *The functor* $\Gamma' : \mathcal{P}_+ \to \mathscr{D}$ *is strong monoidal, where* $\mathscr{D}$ *is equipped with the tensor product operation.*

Proof. Note that $\Gamma'(\varnothing) = k$ is the unit of $\mathscr{D}$. Let $P, Q \in \mathcal{P}_+$, and consider the inclusions $i_1 : P \to P \star Q$, $i_2 : Q \to P \star Q$ given by $x \mapsto (x, 0, 0)$ and $y \mapsto (0, 1, y)$ respectively. A lattice point $m \in d(P \star Q)$ can always be written uniquely as (m_1, s, m_2) for some integer $0 \leq s \leq d$, where m_1 and m_2 are lattice points such that $m_1 \in (d-s)P$ and $m_2 \in sQ$. The decomposition $(m, d) = A_{i_1}(m_1, d-s) + A_{i_2}(m_2, s)$ is unique, so the (natural) homomorphism $\Gamma'(P \star Q) \to \Gamma'(P) \otimes_k \Gamma'(Q)$ given by $\chi^{(m,d)} \mapsto \chi^{(m_1,s)} \otimes \chi^{(m_2,d-s)}$ is an isomorphism. □

By Proposition A.0.8, Γ' induces a strong monoidal monoidally cocontinuous functor $\widehat{\Gamma} : \mathrm{Pre}(\mathcal{P}_+) \to \mathscr{D}$. Recall that the inclusion $i : \mathrm{Pre}(\mathcal{P}) \to \mathrm{Pre}(\mathcal{P}_+)$ given by adjoining a singleton at $\varnothing$ preserves $\star$ and all connected colimits. Hence the restricted functor $\Gamma := \widehat{\Gamma} \circ i : \mathrm{Pre}(\mathcal{P}) \to \mathscr{D}$ preserves all connected colimits, and is strong monoidal with respect to $\star$.

Definition 2.3.2. For a polyhedral set M, $\Gamma(M)$ is called the *face ring* of M.

From the preceding we have the following proposition.

Proposition 2.3.3. *The face ring functor $\Gamma : \mathcal{C} \to \mathscr{D}$ is a strong monoidal, contravariant functor preserving all connected colimits, such that $\Gamma(\Delta^P) = k[A_P]$ for all $P \in \mathcal{P}$. Thus there are natural isomorphisms*

$$\Gamma(M \star N) \cong \Gamma(M) \otimes_k \Gamma(N)$$

for any pair of polyhedral sets M, N. We remark that while $\Gamma(\Delta^P) = k[A_P]$ is a standard graded ring for each P, this is not necessarily the case for $\Gamma(M)$ in general.

Remark 2.3.4. Let M be a polyhedral set. It can be written as the colimit of a diagram $\theta_M : F_M \to \mathcal{C}$, where F_M is the category of elements of M. Endow F_M with an initial object $*$, and let $\mathcal{F}_M$ denote the endowed category. Consider the extended diagram $\Theta_M : \mathcal{F}_M \to \mathcal{C}$ given by $\Theta_M(*) = \underline{\varnothing}$, where $\underline{\varnothing} : \mathcal{P}^{\mathrm{op}} \to \mathrm{Set}$ is the empty-presheaf, also the initial presheaf. Then M is the colimit of the diagram $\Theta_M : \mathcal{F}_M \to \mathcal{C}$. Since $\mathcal{F}_M$ is connected, we have

$$\Gamma(M) = \varprojlim \Gamma \circ \Theta_M.$$

Note that this procedure may be carried out for any (possibly disconnected) diagram $H : I \to \mathcal{C}$ with $\varinjlim H = M$. We highlight this fact in the following remark.

Remark 2.3.5. Let $H : I \to \mathrm{Pre}(\mathcal{P})$ be a diagram of polyhedral sets, and assume that $M = \varinjlim H \in \mathcal{C}$. Endow I with an initial object $*$, and extend H to a functor $\mathcal{H} : \mathcal{I} \to \mathrm{Pre}(\mathcal{P})$, where $\mathcal{I}$ denotes the endowed category, and $\mathcal{H}(*) = \underline{\varnothing}$. Then $\varinjlim \mathcal{H} = M$, and $\Gamma(M) = \varprojlim \Gamma \circ \mathcal{H}$. In particular, for a coproduct $M = \coprod_i M_i$, the face ring $\Gamma(M)$ is the wide pullback of $\{\Gamma(M_i)\}_i$ over k. Explicitly, $\Gamma(M)_0 = k$, and $\Gamma(M)_d = \prod_i \Gamma(M_i)_d$ for each integer $d > 0$.

2.4 The canonical pushout square

Let M be a polyhedral set. For each face s of M, consider the corresponding morphism $\zeta_M^s : \Delta^s \to M$. These morphisms are natural in the following sense. Let $\phi : M \to N$ be a morphism, $s \in M_P$ and $t = \phi(s)$. Then the diagram

$$\begin{tikzcd} \Delta^s \arrow[r, "\zeta_M^s"] \arrow[d, equal] & M \arrow[d, "\phi"] \\ \Delta^t \arrow[r, "\zeta_N^t"] & N \end{tikzcd} \tag{2.2}$$

commutes. For each integer $\mathbf{n} \geq -1$, we define the $\mathbf{n}$-*truncation* $M^{\mathbf{n}}$ as the polyhedral subset of M given by $M^{\mathbf{n}}_P = M_P$ for all P with $\dim(P) \leq \mathbf{n}$, and $\varnothing$ otherwise. Let $\iota_M : M^{\mathbf{n}-1} \to M^{\mathbf{n}}$ denote the natural inclusion.

Definition 2.4.1. Consider the equivalence relation on $\mathcal{P}$ given by $P \sim Q$ whenever $P \cong Q$. For each equivalence class $[P]$, choose (once and for all) some *canonical representative* $P^{\text{can}} \in [P]$. Recall that two faces s, t of M are called equivalent if there exists an isomorphism $f : |s| \to |t|$ such that $f^M(t) = s$. For each isomorphism class $[s]$, choose some *canonical representative* $s^{\text{can}} \in M_{P^{\text{can}}}$. We define $J^{\mathbf{n}}_M \subseteq \coprod_{\dim(P)=\mathbf{n}} M_P$ as the subset of $\mathbf{n}$-dimensional faces s such that $s = s^{\text{can}}$. Let $M^{\text{can}} = \bigsqcup_{\mathbf{n}\geq 0} J^{\mathbf{n}}_M$ denote the set of canonical representatives.

Consider the indexed set of polytopes $\{\Delta^s\}_{s\in J^{\mathbf{n}}_M}$. For each $\mathbf{n}$-dimensional polytope Δ^P, let $\partial\Delta^P$ denote the $(\mathbf{n}-1)$-truncation $(\Delta^P)^{\mathbf{n}-1}$. This induces natural morphisms $\zeta^{s\prime}_M : \partial\Delta^s \to M^{\mathbf{n}-1}$ for each $s \in J^{\mathbf{n}}_M$. By the condition $(*)$, there is exactly one arrow $f : |s^{\text{can}}| \to |s|$ such that $f^M(s) = s^{\text{can}}$ for each $s \in M$. Hence it is easy to see that $M^{\mathbf{n}}$ is the colimit in the following natural diagram

$$\begin{tikzcd}
\{\Delta^s\}_{s\in J^{\mathbf{n}}_M} \arrow[r, "\{\zeta^s_M\}_{s\in J^{\mathbf{n}}_M}"] & M^{\mathbf{n}} \\
\{\partial\Delta^s\}_{s\in J^{\mathbf{n}}_M} \arrow[u, "\{\iota_{\Delta^s}\}_{s\in J^{\mathbf{n}}_M}"] \arrow[r, "\{\zeta^{s\prime}_M\}_{s\in J^{\mathbf{n}}_M}"] & M^{\mathbf{n}-1}, \arrow[u, "\iota_M"']
\end{tikzcd} \tag{2.3}$$

or for easier notation,

$$\begin{tikzcd}
\{\Delta^s\} \arrow[r, "\zeta^s_M"] & M^{\mathbf{n}} \\
\{\partial\Delta^s\} \arrow[u, "\iota_{\Delta^s}"] \arrow[r, "\zeta^{s\prime}_M"] & M^{\mathbf{n}-1}. \arrow[u, "\iota_M"']
\end{tikzcd} \tag{2.4}$$

Definition 2.4.2 (**The canonical pushout square**)**.** For each integer $\mathbf{n} \geq -1$, define $\Delta^{\mathbf{n}}_M = \coprod_{s\in J^{\mathbf{n}}_M} \Delta^s$, and let $\partial\Delta^{\mathbf{n}}_M$ denote the $(\mathbf{n}-1)$-truncation $(\Delta^{\mathbf{n}}_M)^{\mathbf{n}-1} = \coprod_{s\in J^{\mathbf{n}}_M} \partial\Delta^s$. The natural morphisms $\alpha_M : \Delta^{\mathbf{n}}_M \to M^{\mathbf{n}}$ and $\alpha'_M : \partial\Delta^{\mathbf{n}}_M \to M^{\mathbf{n}-1}$ are induced via the morphism $\zeta^s_M : \Delta^s \to M$. $M^{\mathbf{n}}$ fits into the natural pushout square

$$\begin{tikzcd}
\Delta^{\mathbf{n}}_M \arrow[r, "\alpha_M"] & M^{\mathbf{n}} \\
\partial\Delta^{\mathbf{n}}_M \arrow[u, "\iota_{\Delta_M}"] \arrow[r, "\alpha'_M"] & M^{\mathbf{n}-1}, \arrow[u, "\iota_M"']
\end{tikzcd} \tag{2.5}$$

called *the canonical pushout square for* M *of level* $\mathbf{n}$. If $\phi : N \to M$ is a morphism polyhedral sets, we denote each individual morphism in the morphism of diagrams by $\phi : M^{\mathbf{n}} \to N^{\mathbf{n}}, \phi^{\Delta} : \Delta_N^{\mathbf{n}} \to \Delta_M^{\mathbf{n}}$, $\phi' : N^{\mathbf{n}-1} \to M^{\mathbf{n}-1}$ and $\phi^{\partial\Delta} : \partial\Delta_N^{\mathbf{n}} \to \partial\Delta_M^{\mathbf{n}}$.

Definition 2.4.3. For each $P \in \mathcal{P}$, let $L(P)$ (resp. $L^*(P)$) denote the set of lattice points (resp. interior lattice points) of P. For any arrow $f : Q \to P$, we define $L(f) : L(Q) \to L(P)$ as the restriction of $f_{\mathbb{R}}$. This defines a functor $L : \mathcal{P} \to \mathrm{Set}$. Consider the Yoneda extension $L : \mathcal{C} \to \mathrm{Set}$. For each polyhedral set M, we have $L(M) = \varinjlim_{\Delta^P \to M} L(P)$. There is clearly a bijective correspondence $L(M) \cong \bigsqcup_{s \in M^{\mathrm{can}}} L^*(|s|)$.

2.5 CW structure

We will now see that the geometric realization of a polyhedral set M is a CW complex. Specifically, the realization of the diagram (2.4) determines the cell structure on $|M|$ and its characteristic maps. See [GJ09, Proposition 2.3] for the parallell situation in the case of simplicial sets.

Proposition 2.5.1. *Let M be a polyhedral set. Then the geometric realization $|M|$ is a CW complex with an $\mathbf{n}$-cell for each face $s \in J_M^{\mathbf{n}}$. Moreover, if $\phi : M \to N$ is a morphism, then $|\phi| : |M| \to |N|$ maps $\mathbf{n}$-cells of $|M|$ homeomorphically to $\mathbf{n}$-cells of $|N|$.*

Proof. Let $\mathbf{n} \geq 1$ be an integer. Then for each face $s \in M_P \subseteq J_M^{\mathbf{n}}$, the geometric realization of Δ^s is $|P| \cong \mathbb{D}^{\mathbf{n}}$, the $\mathbf{n}$-disc. Consider the polyhedral facets $F \subseteq P$, and note that

$$\partial\Delta^s = \bigcup_{F \text{ facet of } P} \Delta^F. \tag{2.6}$$

Thus we may write

$$|\partial\Delta^s| = \bigcup_{F \text{ facet of } P} F.$$

But this is the topological boundary of $|P|$, hence homeomorphic to the $(\mathbf{n}-1)$-sphere $\mathbb{S}^{\mathbf{n}-1}$. It follows that the induced map $|\iota_{\Delta^s}| : \mathbb{S}^{\mathbf{n}-1} \to \mathbb{D}^{\mathbf{n}}$ is the standard inclusion. The colimit diagram (2.4) defines a CW complex structure on $|M|$ in the following way. The space $|M^0|$ is a discrete set of points, and defines the 0-skeleton of $|M|$. For $\mathbf{n} \geq 1$, $|M^{\mathbf{n}}|$ is the $\mathbf{n}$-skeleton of $|M|$, and $|M^{\mathbf{n}}|$ is determined by $|M^{\mathbf{n}-1}|$ and the characteristic maps $|\zeta_M^s| : \mathbb{D}^{\mathbf{n}} \to |M^{\mathbf{n}}|$ and $|\zeta_M^{s\prime}| : \mathbb{S}^{\mathbf{n}} \to |M^{\mathbf{n}-1}|$ in the colimit diagram

$$\begin{tikzcd} \{\mathbb{D}^{\mathbf{n}}\} \arrow[r, "|\zeta_M^s|"] & |M^{\mathbf{n}}| \\ \{\mathbb{S}^{\mathbf{n}}\} \arrow[u, "|\iota_{\Delta^s}|"] \arrow[r, "|\zeta_M^{s\prime}|"] & |M^{\mathbf{n}-1}|. \arrow[u, "|\iota_M|"'] \end{tikzcd} \tag{2.7}$$

Since M is the colimit of the sequence of inclusions $M^0 \to M^1 \to \cdots \to M^{\mathbf{n}} \to \cdots$ as well, $|M|$ is the colimit of $|M^0| \to |M^1| \to \cdots \to |M^{\mathbf{n}}| \to \cdots$, which shows that the topology on $|M|$ agrees with the standard CW complex (weak) topology. If $\phi : M \to N$ is a morphism, naturality of (2.2) shows that that each $\mathbf{n}$-cell of $|M|$ is mapped homeomorphically to an $\mathbf{n}$-cell of $|N|$. $\square$

For a polyhedral set M of dimension $\mathbf{n}$, we may form the CW chain complex $C_\bullet^M$ arising from the CW structure of the geometric realization $|M|$. It consists of the abelian groups $C_i^M = \bigoplus_{s \in J_M^i} \mathbb{Z}s$ for each $i \geq 0$, freely generated by i-dimensional faces s which are canonical representatives. The canonical pushout square (2.5) for M of level $\mathbf{n}$ gives rise to an exact sequence of chain complexes

$$0 \to C_\bullet^{\partial \Delta_M} \xrightarrow{(\iota_{\Delta_M *}, -\alpha'_{M*})} C_\bullet^{\Delta_M^{\mathbf{n}}} \oplus C_\bullet^{M^{\mathbf{n}-1}} \xrightarrow{\alpha_{M*} \oplus \iota_{M*}} C_\bullet^{M^{\mathbf{n}}} \to 0. \tag{2.8}$$

To see this, note that

$$\begin{tikzcd} (\Delta_M^{\mathbf{n}})_P \arrow[r, "\alpha_M"] & M_P^{\mathbf{n}} \\ (\partial \Delta_M^{\mathbf{n}})_P \arrow[u, "\iota_{\Delta_M}"] \arrow[r, "\alpha'_M"] & M_P^{\mathbf{n}-1}, \arrow[u, "\iota_M"'] \end{tikzcd} \tag{2.9}$$

are pushout squares for each $P \in \mathcal{P}$, with $(\iota_M)_P$ and $(\iota_{\Delta_M})_P$ both bijections for $\dim(P) < \mathbf{n}$, and inclusions of the empty set for $\dim(P) = \mathbf{n}$. Furthermore, $(\alpha_M)_P$ is a bijection when $\dim(P) = \mathbf{n}$. It easily follows that (2.8) is exact.

2.6 Polyhedral complexes

We will now review two types of combinatorial structures: *simplicial complexes*[Sta96] and *polyhedral complexes*[BG02; BG09; Sta87]. See [BR05], [OY09] and [BKR08] for the related concept of *monoidal complexes* and its associated *toric face ring*. First, we need a simpler version of polyhedral sets, called *rigid polyhedral sets*.

Definition 2.6.1. Let $\mathcal{P}'$ denote the monoidal subcategory of $\mathcal{P}$ consisting of the same objects, but where $\mathrm{Hom}_{\mathcal{P}'}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{P}}(P, Q)$ consists of the arrows with trivial character $1 : \mathbb{Z}^n \to \mathbf{G}_m$. A presheaf $M \in \mathrm{Pre}^{(*)}(\mathcal{P}')$ is called a *rigid* polyhedral set. One can make the analogous definitions for $\mathrm{Fin}' \subseteq \mathrm{Fin}$.

There is an obvious pair of strict monoidal functors $U : \mathcal{P} \to \mathcal{P}'$ and $V : \mathcal{P}' \to \mathcal{P}$, defined as follows. Both U and V are the identity on objects. On arrows, $U(f, \lambda_f) = f$, and $V(f) = (f, 1)$. Taking Yoneda extensions, we obtain cocontinuous strong monoidal functors $\mathbb{U} : \mathrm{Pre}(\mathcal{P}) \to \mathrm{Pre}(\mathcal{P}')$ and $\mathbb{V} : \mathrm{Pre}(\mathcal{P}') \to \mathrm{Pre}(\mathcal{P})$. By considering canonical pushout squares (2.5), it follows from Lemma 2.2.2 that $\mathbb{U}$ and $\mathbb{V}$ restricts to functors $\mathbb{U} : \mathrm{Pre}^{(*)}(\mathcal{P}) \to \mathrm{Pre}^{(*)}(\mathcal{P}')$ and $\mathbb{V} : \mathrm{Pre}^{(*)}(\mathcal{P}') \to \mathrm{Pre}^{(*)}(\mathcal{P})$. By abuse of notation, we let $U : \mathrm{Fin} \to \mathrm{Fin}'$ and $V : \mathrm{Fin}' \to \mathrm{Fin}$ denote the respective restrictions of U and V, and similarly for $\mathbb{U} : \mathrm{Pre}^{(*)}(\mathrm{Fin}) \to \mathrm{Pre}^{(*)}(\mathrm{Fin}')$ and $\mathbb{V} : \mathrm{Pre}^{(*)}(\mathrm{Fin}') \to \mathrm{Pre}^{(*)}(\mathrm{Fin})$. Note that since $UV = \mathrm{id}_{\mathrm{Fin}'}$, there is a natural equivalence $\mathbb{U}\mathbb{V} \cong \mathrm{id}_{\mathrm{Pre}^{(*)}(\mathcal{P}')}$.

Definition 2.6.2. A *simplicial complex* K consists of a set of of vertices $V(K)$, and a set of faces $S(K)$. $S(K)$ consists of finite subsets of $V(K)$, such that $\{v\} \in S(K)$ for all $v \in V(K)$, and if $t \subseteq s \in S(K)$, then $t \in S(K)$. The dimension of a face $s = \{v_0, \dots, v_n\}$ is n.

More generally, there is the notion of polyhedral complexes. The following definition is a version of [BG02, Definition 2.1], which is also equivalent with [BG09, Definition 1.74].

Definition 2.6.3. A *(lattice) polyhedral complex* Π consists of a set of vertices $V(\Pi)$, and a set $S(\Pi)$ of subsets of $V(\Pi)$ such that $\{v\} \in S(\Pi)$ for all $v \in V(\Pi)$, and if $s, t \in S(\Pi)$, then $s \cap t \in S(\Pi)$. It furthermore comes equipped with the following data:

1) an embedding $s \to \mathbb{R}^{n_s}$ for each $s \in S(\Pi)$, such that the image of s constitutes the vertex set of an n_s-dimensional lattice polytope $s^* \subseteq \mathbb{R}^{n_s}$;

2) an embedding $\iota_{st} : s^* \to t^*$ for each inclusion $s \subset t$ in $S(\Pi)$, such that ι_{st} is an isomorphism of s^* with a face of t^* as lattice polytopes.

These embeddings are subject to the following compatibility conditions:

3) $\iota_{tu}\iota_{st} = \iota_{su}$ for each $s \subset t \subset u$ in $S(\Pi)$;

4) For each $t \in S(\Pi)$, and each face F of t^*, there exists an element $s \in S(\Pi)$ such that $s \subset t$ and $\iota(s^*) = F$.

We may consider a polyhedral complex Π as a rigid polyhedral set as follows. For each inclusion $s \subseteq t$, the embedding ι_{st} from 3) corresponds to a unique affine transformation $\mathbb{Z}^{n_s} \to \mathbb{Z}^{n_t}$. Via condition 4), this gives the operation $\Pi \to \mathrm{pre}(\mathcal{P}')$ defined by $s \mapsto \Delta^{s^*}$ the structure of a functor. We let $M(\Pi) = \varinjlim_{s \in \Pi} \Delta^{s^*}$. One can similarly defines the rigid unimodular set $M(K)$ corresponding to a simplicial complex K. If M is a polyhedral set and s is a face of M, recall that M^s is the image of $\zeta^s_M : \Delta^s \to M$. If a rigid polyhedral set M comes from a polyhedral complex, then it satisfies the following two conditions:

1) $\zeta^s_M : \Delta^s \to M$ is injective for all faces s;

2) For every pair of faces s and t, $M^s \cap M^t = M^u$ for some face u;

In fact, these three conditions characterizes the finite rigid polyhedral sets which are polyhedral complexes, and analogously the finite rigid unimodular polyhedral sets which are simplicial complexes. Indeed, one recovers the polyhedral complex Π up to isomorphism from $M(\Pi)$ by defining $V(\Pi) = M_0$, and defining $S(\Pi)$ to be the set of subsets $M^s_0 \subseteq M_0$ for faces s of M which are canonical representatives. The embeddings $M^s_0 \to \mathbb{R}^{\dim(s)}$ are given by the inclusions $|s| \subseteq \mathbb{R}^{\dim(s)}$. If M is unimodular, then condition 1) characterizes what is called *simplicial posets* (see [Sta91] or [LP11]).

Lemma 2.6.4. *If $\psi : \partial\Delta^n \to \partial\Delta^n$ is an automorphism such that $\mathbb{U}(\psi) = \mathrm{id}_{\partial\Delta^n}$, then there exists a unique automorphism $\widetilde{\psi} : \Delta^n \to \Delta^n$ satisfying $\widetilde{\psi}|_{\partial\Delta^n} = \psi$.*

Proof. We may write $\partial\Delta^n$ as a coequalizer

$$\coprod_{0\leq i<j<=n-1} \Delta^{n-2} \rightrightarrows \coprod_{0\leq i\leq n-1} \Delta^{n-1} \to \partial\Delta^n,$$

where the arrows $\Delta^{n-2} \rightrightarrows \Delta^{n-1}$ for the index (i,j) are d_i and d_{j-1}. An automorphism ψ satisfying $\mathbb{U}(\psi) = \mathrm{id}_{\partial\Delta^{\mathbf{n}}}$ corresponds to automorphisms ψ_i of Δ^{n-1} satisfying $\mathbb{U}(\psi_i) = \mathrm{id}_{\Delta^{n-1}}$ for each i, and $\psi_i d_{j-1} = \psi_j d_i$ for $i<j$. Each ψ_i corresponds to a character $\lambda_i : \mathbb{Z}^{n-1} \to \mathbf{G}_m$, and the above requirement translates to $\lambda_i(e_{d_{j-1}(k)}) = \lambda_j(e_{d_i(k)})$. Clearly, there is a unique character $\lambda : \mathbb{Z}^n \to \mathbf{G}_m$ restricting to λ_i via $d_i : [n-1] \to [n]$ for each i, and the result follows. $\square$

Proposition 2.6.5. *For every finite-dimensional unimodular set $N : \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Set}$, there exists a rigid unimodular set $M : \mathrm{Fin}'^{\mathrm{op}} \to \mathrm{Set}$ such that $\mathbb{V}(M) \cong N$.*

Proof. We will show the following by induction on $\mathbf{n} \geq -1$:

1) For any $\mathbf{n}$-dimensional $N, Z \in \mathrm{Pre}^{(*)}(\mathrm{Fin}')$ and any morphism $\phi : \mathbb{V}(N) \to \mathbb{V}(Z)$, there exists a unique automorphism ψ of $\mathbb{V}(N)$ satisfying $\mathbb{U}(\psi) = \mathrm{id}_N$ and $\phi = \mathbb{V}\mathbb{U}(\phi)\psi$.

2) For any $\mathbf{n}$-dimensional $N \in \mathrm{Pre}^{(*)}(\mathrm{Fin})$, there exists $M \in \mathrm{Pre}^{(*)}(\mathrm{Fin}')$ such that $\mathbb{V}(M) \cong N$.

Each case is trivial for $\mathbf{n} = -1$, so assume that $\mathbf{n} \geq 0$. Let $N \in \mathrm{Pre}^{(*)}(\mathrm{Fin})$ be $\mathbf{n}$-dimensional. By 2), there exists an $(\mathbf{n}-1)$-dimensional $M' \in \mathrm{Pre}^{(*)}(\mathrm{Fin}')$ and an isomorphism $\phi' : N^{\mathbf{n}-1} \to \mathbb{V}(M')$. Consider the morphism $\alpha'_N\phi' : \partial\Delta_N^{\mathbf{n}-1} \to \mathbb{V}(M')$. By 1), there exists a unique automorphism ψ of $\partial\Delta_N^{\mathbf{n}-1}$ such that $\mathbb{U}(\psi) = \mathrm{id}_{\partial\Delta_N^{\mathbf{n}-1}}$ and $\phi'\alpha'_N = \mathbb{V}\mathbb{U}(\phi')\alpha'_N\psi$. By Lemma 2.6.4, there is a unique morphism $\widetilde{\psi} : \Delta_N^{\mathbf{n}} \to \Delta_N^{\mathbf{n}}$ satisfying $\widetilde{\psi}|_{\partial\Delta_N^{\mathbf{n}}} = \psi$. Now, consider the pushout diagram

$$\begin{tikzcd}
\mathbb{U}(\Delta_N^{\mathbf{n}}) \arrow[r] & M \\
\mathbb{U}(\partial\Delta_N^{\mathbf{n}}) \arrow[u] \arrow[r, "\mathbb{U}(\phi'\alpha'_N)"] & M'. \arrow[u]
\end{tikzcd}$$

Since $\mathbb{V}$ preserves colimits, this induces a pushout diagram

$$\begin{tikzcd}
\Delta_N^{\mathbf{n}} \arrow[r] & \mathbb{V}(M') \\
\partial\Delta_N^{\mathbf{n}} \arrow[u] \arrow[r, "\mathbb{V}\mathbb{U}(\phi')\alpha'_N"] & \mathbb{V}(M). \arrow[u]
\end{tikzcd}$$

But the isomorphism of diagrams

$$\begin{tikzcd} \Delta^{\mathbf{n}}_N \arrow[d, "\widetilde{\psi}"] & \partial\Delta^{\mathbf{n}}_N \arrow[l] \arrow[r, "\alpha'_N"] \arrow[d, "\psi"] & N^{\mathbf{n}-1} \arrow[d, "\phi'"] \\ \Delta^{\mathbf{n}}_N & \partial\Delta^{\mathbf{n}}_N \arrow[l] \arrow[r, "\mathbb{V}\mathbb{U}(\phi')\alpha'_N"] & \mathbb{V}(M') \end{tikzcd}$$

induces a unique isomorphism $\phi : N \to \mathbb{V}(M)$. This shows 2). Next, let $\phi : \mathbb{V}(N) \to \mathbb{V}(Z)$ be any morphism as in 1). Consider the induced morphism $\phi' : \mathbb{V}(N^{\mathbf{n}-1}) \to \mathbb{V}(Z^{\mathbf{n}-1})$. By the inductive hypothesis, there is a unique automorphism ψ' of $\mathbb{V}(N^{\mathbf{n}-1})$ satisfying $\mathbb{U}(\psi') = \mathrm{id}_{N^{\mathbf{n}-1}}$ and $\phi' = \mathbb{V}\mathbb{U}(\phi')\psi'$. For the induced morphism $\phi^\Delta : \Delta^{\mathbf{n}}_{\mathbb{V}(N)} \to \Delta^{\mathbf{n}}_{\mathbb{V}(Z)}$, there clearly exists a unique automorphism ψ^Δ of $\Delta^{\mathbf{n}}_{\mathbb{V}(N)}$ satisfying $\mathbb{U}(\psi^\Delta) = \mathrm{id}_{\Delta^{\mathbf{n}}_{\mathbb{V}(N)}}$ and $\phi^\Delta = \mathbb{V}\mathbb{U}(\phi^\Delta)\psi^\Delta$, which necessarily restricts to an automorphism $\psi^{\Delta'}$ of $\partial\Delta^{\mathbf{n}}_{\mathbb{V}(N)}$ satisfying $\mathbb{U}(\psi^{\Delta'}) = \mathrm{id}_{\partial\Delta^{\mathbf{n}}_{\mathbb{V}(N)}}$ and $\phi^{\Delta'} = \mathbb{V}\mathbb{U}(\phi^\Delta)\psi^{\Delta'}$. Next we will show that $\alpha'_{\mathbb{V}(N)}\psi^{\Delta'} = \psi'\alpha'_{\mathbb{V}(N)}$. If so, we have an isomorphism of diagrams

$$\begin{tikzcd} \Delta^{\mathbf{n}}_{\mathbb{V}(N)} \arrow[d, "\psi^\Delta"] & \partial\Delta^{\mathbf{n}}_{\mathbb{V}(N)} \arrow[l] \arrow[r] \arrow[d, "\psi^{\Delta'}"] & \mathbb{V}(N^{\mathbf{n}-1}) \arrow[d, "\psi'"] \\ \Delta^{\mathbf{n}}_{\mathbb{V}(N)} & \partial\Delta^{\mathbf{n}}_{\mathbb{V}(N)} \arrow[l] \arrow[r] & \mathbb{V}(N^{\mathbf{n}-1}) \end{tikzcd}$$

inducing an isomorphism $\psi : \mathbb{V}(N) \to \mathbb{V}(N)$. So, by the inductive hypothesis 1), there is a unique automorphism π of $\partial\Delta^{\mathbf{n}}_N$ satisfying $\mathbb{U}(\pi) = \mathrm{id}_{\partial\Delta^{\mathbf{n}}_N}$ and $\psi'\alpha'_{\mathbb{V}(N)} = \alpha'_{\mathbb{V}(N)}\pi$. Thus we have

$$\alpha'_{\mathbb{V}(Z)}\phi^{\Delta'} = \phi'\alpha'_{\mathbb{V}(N)} = \mathbb{V}\mathbb{U}(\phi')\psi'\alpha'_{\mathbb{V}(N)} = \mathbb{V}\mathbb{U}(\phi')\alpha'_{\mathbb{V}(N)}\pi = \alpha'_{\mathbb{V}(Z)}\mathbb{V}\mathbb{U}(\phi^{\Delta'})\pi.$$

However, $\psi^{\Delta'}$ is the unique morphism satisfying $\alpha'_{\mathbb{V}(Z)}\phi^{\Delta'} = \alpha'_{\mathbb{V}(Z)}\mathbb{V}\mathbb{U}(\phi^{\Delta'})\psi^{\Delta'}$, which means that $\pi = \psi^{\Delta'}$. It follows that $\alpha'_{\mathbb{V}(N)}\psi^{\Delta'} = \psi'\alpha'_{\mathbb{V}(N)}$. Next, we consider the morphism $\mathbb{V}\mathbb{U}(\phi)\psi$. By the universal property of pushouts, we have $\mathbb{U}(\psi) = \mathrm{id}_{\mathbb{V}(N)}$ and $\phi = \mathbb{V}\mathbb{U}(\phi)\psi$. To show uniqueness of ψ, suppose that ψ_0 is any other automorphism of $\mathbb{V}(N)$ satisfying $\mathbb{U}(\psi_0) = \mathrm{id}_{\mathbb{V}(N)}$ and $\mathbb{V}\mathbb{U}(\phi)\psi_0 = \phi$. By the inductive hypothesis, we have $\psi'_0 = \psi'$ and $\psi_0^{\Delta'} = \psi^{\Delta'}$. It follows immediately from Lemma 2.6.4 that $\psi_0^\Delta = \psi^\Delta$. $\square$

2.7 The face ring of a polyhedral complex

See [Sta96] and [MS05] for a reference on Stanley-Reisner rings of simplicial complexes. Note that the Stanley-Reisner of the n-simplex $\mathrm{SR}(\Delta^n)$ is equal to the free polynomial algebra $k[x_0, \ldots, x_n]$ on the vertices of Δ^n. In [PRV04] it is noted that $\mathrm{SR}(K) \cong \varprojlim_{s\in K} k[x_0, \ldots, x_{|s|}]$, which in our notation means that

$\mathrm{SR}(K) \cong \Gamma(M(K))$. In [LP11] it is showed that the face ring of a simplicial poset (originally defined in [Sta91]) satisfies the corresponding limit formula. The polyhedral algebra $k[\Pi]$ of a polyhedral complex Π was defined in [BG02], which satisfies the limit formula $\varprojlim_{s \in S(\Pi)} k[A_{|s|}]$. So more generally, we have $k[\Pi] \cong \Gamma(M(\Pi))$ for polyhedral complexes Π.

Definition 2.7.1 ([BG02, p.4])**.** Let Π be a polyhedral complex. The *polyhedral algebra* associated to Π is defined as the k-algebra $k[\Pi]$ specified by the following conditions:

1) $k[\Pi]$ is generated by indeterminates x_m for $m \in L(\Pi)$ (here $L(\Pi)$ denotes the set of lattice points of Π, as defined in Definition 2.4.3);

2) for any element $s \in S(\Pi)$, the subalgebra of $k[\Pi]$ generated by $L(s^*)$ is naturally isomorphic to the k-algebra $k[A_{s^*}]$ associated to the affine semigroup A_{s^*};

3) if $x_{m_1}, \ldots, x_{m_n} \in L(\Pi)$, and there is no $t \in S(\Pi)$ such that each x_{m_i} belong to $L(t^*)$, then $x_{m_1} \ldots x_{m_n} = 0$.

One can alternatively describe the polyhedral algebra as follows:

$$k[\Pi] = k[x_m | m \in L(\Pi)]/(I^{\mathrm{bin}} + I^{\mathrm{mon}}), \tag{2.10}$$

where $I^{\mathrm{bin}} = \sum_{s \in S(\Pi)} \ker(\phi_s)$, where $\phi_s : k[\chi^m | m \in L(\Pi)] \to k[A_{s^*}]$ is the homomorphism given by $x_m \mapsto \chi^{(m,1)}$, and I^{mon} is the ideal of monomials $x_{m_1} \cdots x_{m_n}$ for which there exists no $t \in S(\Pi)$ such that $m_1, \ldots, m_n \in L(t)$. We summarize the preceding in the following proposition.

Proposition 2.7.2. *Let $M \in \mathcal{C}$ be a polyhedral complex. Then its face ring $\Gamma(M)$ is isomorphic to the polyhedral algebra given by the formula* (2.10)*. Hence if M is a simplicial complex, then $\Gamma(M)$ is the Stanley-Reisner ring of M.*

Corollary 2.7.3. *Let $P \in \mathcal{P}$. Then the homomorphism $\Gamma(\Delta^P) \to \Gamma(\partial\Delta^P)$ is surjective.*

Proof. Both Δ^P and $\partial\Delta^P$ are polyhedral complexes. Note that the induced map of lattice points $L(\partial\Delta^P) \to L(\Delta^P)$ is injective. Via the formula (2.10), the homomorphism $\Gamma(\Delta^P) \to \Gamma(\partial\Delta^P)$ is surjective onto the generators of $\Gamma(\partial\Delta^P)$. $\square$

2.8 Properties of the face ring

In contrast with polyhedral and simplicial complexes, the face ring $\Gamma(M)$ for general polyhedral sets M does not have a neat representation. This section will be devoted to examining the algebraic properties of $\Gamma(M)$ from a more abstract perspective.

Definition 2.8.1 (**The Milnor square for $\Gamma(M)$**)**.** Let M be a polyhedral set, and $\mathbf{n} \geq -1$ an integer. Consider the canonical pushout square of M of level $\mathbf{n}$ from Definition 2.4.2. By Proposition 2.3.3, Γ is preserves connected colimits, so applying Γ to (2.5) yields a natural pullback square

$$\begin{tikzcd} \Gamma(M^{\mathbf{n}}) \arrow[r, "\Gamma(\iota_M)"] \arrow[d, "\Gamma(\alpha_M)"'] & \Gamma(M^{\mathbf{n}-1}) \arrow[d, "\Gamma(\alpha'_M)"] \\ \Gamma(\Delta^{\mathbf{n}}_M) \arrow[r, "\Gamma(\iota_{\Delta_M})"] & \Gamma(\partial\Delta^{\mathbf{n}}_M), \end{tikzcd} \tag{2.11}$$

called *the Milnor square for* $\Gamma(M)$ *of level* $\mathbf{n}$. To justify this definition, we must verify that it indeed is a Milnor square (see Theorem B.1.1). Write $\Delta^{\mathbf{n}}_M = \coprod_{s \in J^{\mathbf{n}}_M} \Delta^s$ and $\partial\Delta^{\mathbf{n}}_M = \coprod_{s \in J^{\mathbf{n}}_M} \partial\Delta^s$, and note that each homomorphism $\Gamma(\iota_{\Delta^s}) : \Gamma(\Delta^s) \to \Gamma(\partial\Delta^s)$ is surjective by Corollary 2.7.3. By the description in Remark 2.3.5, $\Gamma(\Delta^{\mathbf{n}}_M)$ and $\Gamma(\partial\Delta^{\mathbf{n}}_M)$ are wide pullbacks over k, so the homomorphism $\Gamma(\iota_{\Delta_M}) : \Gamma(\Delta^{\mathbf{n}}_M) \to \Gamma(\partial\Delta^{\mathbf{n}}_M)$ is surjective. It follows that (2.11) is a Milnor square.

That (2.11) is a Milnor square is a central observation, as it permits use of the machinery of Appendix B. It will be useful for inductive arguments in this and future chapters, in particular via Definition 3.0.2. The following proposition will have extensive applications.

Proposition 2.8.2. *Let $\phi : M \to N$ be a morphism of finite-dimensional polyhedral sets. If ϕ is injective (resp. surjective), then $\Gamma(\phi) : \Gamma(N) \to \Gamma(M)$ is surjective (resp. injective).*

Proof. Let $\mathbf{n} \geq -1$ be an integer, and let N be of dimension $\mathbf{n}$. We proceed by induction on $\mathbf{n}$, noting that when $\mathbf{n} = -1$, the result follows trivially. Note that $\phi^{\Delta}_M : \Delta^{\mathbf{n}}_M \to \Delta^{\mathbf{n}}_N$ is injective (resp. surjective). Consider the Milnor squares for $\Gamma(M)$ and $\Gamma(N)$ from Definition 2.8.1. $\Gamma(\Delta^{\mathbf{n}}_M)$ and $\Gamma(\Delta^{\mathbf{n}}_N)$ are the wide pullbacks of $\{\Gamma(\Delta^s)\}_{s \in J^{\mathbf{n}}_M}$ and $\{\Gamma(\Delta^t)\}_{t \in J^{\mathbf{n}}_N}$ over k respectively, and similarly for $\Gamma(\partial\Delta^{\mathbf{n}}_M)$ and $\Gamma(\partial\Delta^{\mathbf{n}}_N)$. It follows that each individual vertical arrow of the homomorphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker\Gamma(\iota_{\Delta_N}) \arrow[r] \arrow[d] & \Gamma(\Delta^{\mathbf{n}}_N) \arrow[r] \arrow[d] & \Gamma(\partial\Delta^{\mathbf{n}}_N) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker\Gamma(\iota_{\Delta_M}) \arrow[r] & \Gamma(\Delta^{\mathbf{n}}_M) \arrow[r] & \Gamma(\partial\Delta^{\mathbf{n}}_M) \arrow[r] & 0 \end{tikzcd}$$

is surjective (resp. injective). Consider the homomorphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker\Gamma(\iota_{\Delta_N}) \arrow[r] \arrow[d] & \Gamma(N^{\mathbf{n}}) \arrow[r] \arrow[d] & \Gamma(N^{\mathbf{n}-1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker\Gamma(\iota_{\Delta_M}) \arrow[r] & \Gamma(M^{\mathbf{n}}) \arrow[r] & \Gamma(M^{\mathbf{n}-1}) \arrow[r] & 0 \end{tikzcd}$$

coming from (B.3). By the inductive hypothesis we may assume that $\Gamma(N^{\mathbf{n}-1}) \to \Gamma(M^{\mathbf{n}-1})$ is surjective (resp. injective). Since $\ker\Gamma(\iota_{\Delta_N}) \to \ker\Gamma(\iota_{\Delta_M})$ is surjective (resp. injective) as well, it follows immediately that $\Gamma(N^{\mathbf{n}}) \to \Gamma(M^{\mathbf{n}})$ is surjective (resp. injective). □

Lemma 2.8.3. *For any polyhedral set M, $\Gamma(M)$ is reduced.*

Proof. By Remark 2.3.4, $\Gamma(M)$ is a limit of reduced rings, hence reduced. □

Proposition 2.8.4. *Let M be a polyhedral set, and M_1, M_2 polyhedral subsets of M with corresponding injections $i_1 : M_1 \to M$, $i_2 : M_2 \to M$. Consider the polyhedral subsets $M_1 \cup M_2$ and $M_1 \cap M_2$ of M, with corresponding injections $\phi : M_1 \cup M_2 \to M$, $\psi : M_1 \cap M_2 \to M$. Then* $\ker\Gamma(i_1) \cap \ker\Gamma(i_2) = \ker\Gamma(\phi)$, *and* $\ker\Gamma(i_1) + \ker\Gamma(i_2) = \ker\Gamma(\psi)$.

Proof. By Proposition 2.8.2, the homomorphisms $\Gamma(i_1)$, $\Gamma(i_2)$, $\Gamma(\phi)$ and $\Gamma(\psi)$ are surjective, so we may assume that $M = M_1 \cup M_2$. Consider the pushout square

$$\begin{tikzcd}
M_1 \arrow[r, "i_1"] & M \\
M_1 \cap M_2 \arrow[u, "j_1"] \arrow[r, "j_2"] & M_2, \arrow[u, "i_2"']
\end{tikzcd} \tag{2.12}$$

where $j_1 : M_1 \cap M_2 \to M_1$ and $j_2 : M_1 \cap M_2 \to M_1$ are the induced injections. We must show that $\ker\Gamma(i_1) \cap \ker\Gamma(i_2) = 0$, and $\ker\Gamma(i_1) + \ker\Gamma(i_2) = \ker\Gamma(\psi)$. Now, since Γ preserves connected colimits, (2.12) gives a pullback square

$$\begin{tikzcd}
\Gamma(M) \arrow[r, "\Gamma(i_1)"] \arrow[d, "\Gamma(i_2)"'] & \Gamma(M_1) \arrow[d, "\Gamma(j_1)"] \\
\Gamma(M_2) \arrow[r, "\Gamma(j_2)"] & \Gamma(M_1 \cap M_2),
\end{tikzcd} \tag{2.13}$$

hence an exact sequence

$$0 \to \Gamma(M) \to \Gamma(M_1) \times \Gamma(M_2) \to \Gamma(M_1 \cap M_2) \to 0, \tag{2.14}$$

which we may write as

$$0 \to \Gamma(M) \to (\Gamma(M)/\ker\Gamma(i_1)) \times (\Gamma(M)/\ker\Gamma(i_2)) \to \Gamma(M)/\ker\Gamma(\psi) \to 0.$$

The result follows immediately. □

2.9 Deforming to a subdivison

We recall the definition of a *polyhedral subdivision* of a polytope[GKZ08, Chapter 7, Section 2] (we restrict to the case of lattice polytopes):

Definition 2.9.1. A *marked (lattice) polytope* is a pair (P, A), where $P \subseteq \mathbb{R}^n$ is a full-dimensional polytope, and A is a subset of $L(P) = P \cap \mathbb{Z}^n$ containing all the vertices of P. A *polyhedral subdivision* of P is a family $\{(P_i, A_i)\}_{i \in I}$ of marked n-dimensional lattice polytopes $P_i \subseteq P$ such that $A_i \subseteq A$, $A_i \cap Q_j = A_j \cap P_i$ and such that $\bigcup_{i \in I} P_i = P$. Such a subdivision is called *regular* (or *coherent*) if it is induced by a function $\psi : A \to \mathbb{R}$ in the following manner: Define $G_\psi = \mathrm{Conv}((w, y) \mid w \in A, y \in \mathbb{R}, y \geq \psi(w)) \subseteq \mathbb{R}^n \times \mathbb{R}$. The lower boundary of G_ψ is the graph of the piecewise linear function $g_\psi : P \to \mathbb{R}$ defined by $g_\psi(x) = \min\{y \mid (x, y) \in G_\psi\}$. Now let the polytopes $P_i \subseteq P$ be given as the set of domains of linearity of g_ψ (i.e. the convex subsets of P where g_ψ is an affine linear function), and let $A_i = \{w \in P_i \cap A \mid g_\psi(w) = \psi(w)\}$.

Remark 2.9.2. For any given regular subdivision of P, we may assume that ψ takes integral values. Indeed, if $\psi : A \to \mathbb{R}$ is any function, then a sufficiently approximate function $A \to \mathbb{Q}$ produces the same subdivision. Clearing denominators, the induced function $A \to \mathbb{Z}$ still produces the same subdivision.

Let $S(\psi) = \{(P_i, A_i)\}_{i \in I}$ denote the subdivision associated to $\psi : A \to \mathbb{Z}$. We can think of a subdivision $S(\psi)$ as the polyhedral complex $PS(P, \psi) = \bigcup_{i \in I} \Delta^{P_i}$ it gives rise to. This construction is functorial in the following sense. Let $f : Q \to P$ be a some arrow in $\mathcal{P}$. Consider the marked polytope $(Q, f_{\mathbb{R}}^{-1}(A))$ and the function $\psi f : f_{\mathbb{R}}^{-1}(A) \to \mathbb{Z}$. Then the subdivision $S(\psi f)$ is given as $\{(Q \cap f_{\mathbb{R}}^{-1}(P_i), A_i)\}_{i \in J}$ for the subset $J \subseteq I$ of indices such that $Q \cap f_{\mathbb{R}}^{-1}(P_i) \neq \varnothing$. Moreover, the arrow $f : Q \to P$ restricts to an arrow $f_i : Q \cap P_i \to P_i$ (with $\lambda_{f_i} = \lambda_f$) for each $i \in J$. These arrows induces a morphism $PS(\psi f) \to PS(\psi)$. To formalize this, let $\overline{\mathcal{P}}$ denote the category of pairs $(P, \psi : A \to \mathbb{Z})$, where (P, A) is a marked polyotope, and where the function $A_P \to \mathbb{Q}$ given by $(m, d) \mapsto d g_\psi(\frac{m}{d})$ takes integral values. An arrow $(Q, \phi : B \to \mathbb{Z}) \to (P, \psi : A \to \mathbb{Z})$ is an arrow $f : Q \to P$ in $\mathcal{P}$ such that $B = f_{\mathbb{R}}^{-1}(A)$ and $\phi = \psi f$. Then PS defines a functor $PS : \overline{\mathcal{P}} \to \mathcal{C}$. By taking the Yoneda extension one obtains the subdivision functor $PS : \mathrm{Pre}(\overline{\mathcal{P}}) \to \mathrm{Pre}(\mathcal{P})$ which preserves all colimits. By imposing the condition $(*)$ on presheaves $\overline{\mathcal{P}}^{\mathrm{op}} \to \mathrm{Set}$, one obtains the subdivision functor $PS : \mathrm{Pre}^{(*)}(\overline{\mathcal{P}}) \to \mathcal{C}$. We also have the forgetful functor $U : \overline{\mathcal{P}} \to \mathcal{P}$ given by $(P, \psi) \mapsto P$. Let $\mathbb{U} : \mathrm{Pre}^{(*)}(\overline{\mathcal{P}}) \to \mathcal{C}$ denote the Yoneda extension of U.

One can think of an element of $\mathrm{Pre}^{(*)}(\overline{\mathcal{P}})$ in the following way. Let M be a polyhedral set. Then a *marked pair structure* on M consists of a marked lattice polytope on the form $(A_s, |s|)$ and a function $\psi_s : A_s \to \mathbb{Z}$ for each face s of M, satisfying the condition of Remark 2.9.2, and also the following condition: for each arrow $f : P \to Q$ in $\mathcal{P}$ and each $s \in M_Q$, we have $A_{f^M(s)} = f_{\mathbb{R}}^{-1}(A_s)$ and $\psi_s f = \psi_{f^M(s)}$. In other words, a marked pair structure on M is a presheaf $\widehat{M} \in \mathrm{Pre}^{(*)}(\overline{\mathcal{P}})$ such that $\mathbb{U}(\widehat{M}) \cong M$. Note that the data consisting of the marked pairs $\{|s|, \psi_s : A_S \to \mathbb{Z}\}$ is equivalent to a function

$\Psi : A \to \mathbb{Z}$, where $A = \varprojlim_{s \in M} A_s$. Hence the category $\mathrm{Pre}^{(*)}(\overline{\mathcal{P}})$ is equivalent with the category of marked pairs $(M, \Psi : A \to \mathbb{Z})$, where $M \in \mathcal{C}$, $A \subseteq L(M) = \varinjlim_{\Delta^P \to M} L(P) = \coprod_{s \in M^{\mathrm{can}}} L^*(|s|)$, and each restriction $\Psi|_{A_s}$ satisfies the conditions of Remark 2.9.2. A morphism $(N, \Phi : B \to \mathbb{Z}) \to (M, \Psi : A \to \mathbb{Z})$ is simply a morphism $\phi : N \to M$ such that $L(\phi)^{-1}(A) = B$ and $\Psi \circ L(\phi)|_B = \Phi$.

Definition 2.9.3. Let M be a polyhedral set, and let (M, Ψ) be a marked pair structure on M. Then the polyhedral set $PS(M, \Psi)$ is called a *polyhedral subdivision* of M.

The functor $\mathbb{U}$ is given by $(M, \Psi) \mapsto M$. There is an isomorphism $|U(P, \psi)| \cong |PS(P, \psi)|$ of topological spaces given by the inclusions $Q_i \subseteq P$. This is natural in $(P, \psi) \in \overline{\mathcal{P}}$, so by the universal property of the Yoneda extension, there is an induced isomorphism $|\mathbb{U}(M, \Psi)| \cong |PS(M, \Psi)|$, natural in $(M, \Psi) \in \mathrm{Pre}^{(*)}(\overline{\mathcal{P}})$.

2.9.1 Unimodular triangulations

Definition 2.9.4. Let P be a polytope. For each integer $n \geq 1$, consider the *scaling functor* $n : \mathcal{P} \to \mathcal{P}$ defined by $P \mapsto nP$, where nP denotes the Minkowski multiple of P. On arrows $f : Q \to P$, we let $(nf)_{\mathbb{R}} = f_{\mathbb{R}}$ and $\lambda_{nf} = \lambda_f$. Let $n : \mathcal{C} \to \mathcal{C}$ denote the Yoneda extension. Then for any polyhedral set M, we have $nM = \varinjlim_{\Delta^P \to M} \Delta^{nP}$. This is called the *scaling of M by n*.

The scaling functor $n : \mathcal{P} \to \mathcal{P}$ extends to a scaling functor $n : \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ as follows. For a marked pair $(P, \psi : A \to \mathbb{Z}) \in \overline{\mathcal{P}}$, we define $n(P, \psi) = (nP, \psi)$, where ψ now denotes the function $nA = \{nm \mid m \in A\} \to \mathbb{Z}$ given by $m \mapsto n\psi(\frac{m}{n})$ for each $m \in A$. The Yoneda extension $n : \mathrm{Pre}^{(*)}(\overline{\mathcal{P}}) \to \mathrm{Pre}^{(*)}(\overline{\mathcal{P}})$ is given by $(M, \Psi) \mapsto (nM, \Psi)$ (where Ψ here is given by the functions $\psi_s : nA_s \to \mathbb{Z}$ for each face s of M).

Lemma 2.9.5. *For each integer $n \geq 1$, there is an isomorphism $\Gamma_k(nP) \xrightarrow{\cong} \Gamma(P)^{[n]} := \bigoplus_{d=0} \Gamma(P)_{dn}$ natural in $P \in \mathcal{P}$. Consequently, there is an isomorphism $\Gamma_k(nM) \xrightarrow{\cong} \Gamma_k(M)^{[n]}$ natural in $M \in \mathcal{C}$. Analogously, the natural homeomorphism $P \to nP$ induces a natural homeomorphism $|M| \xrightarrow{\cong} |nM|$.*

Proof. The first statement follows from the fact that $A_{nP} = \{(m, d) \in A_P \mid n \text{ divides } d\}$. Since the functor $(-)^{[n]}$ preserves limits, the rest is a consequence of the universal property of Yoneda extensions. □

Definition 2.9.6. Let M be a polyhedral set, and assume that $L^*(|s|)$ is non-empty for every $s \in M^{\mathrm{can}}$. Choose a lattice point $m_s \in L^*(|s|)$ for each s, and let $A = \{m_s \in L(M) \mid s \in M^{\mathrm{can}}\}$. Let $\Psi : A \to \mathbb{Z}$ be a function such that $\Psi(m_s) > \Psi(m_t)$ whenever $\dim(s) > \dim(t)$ (and suitably scaled so that the condition of Remark 2.9.2 is satisfied). Then (M, Ψ) is called a *barycentric* marked pair, and the resulting subdivision $PS(M, \Psi)$ is called a *barycentric* subdivision of M.

If $\phi : N \to M$ is a morphism, then a marked pair $(M, \Psi : A \to \mathbb{Z})$ of M pulls back to a marked pair $(N, \Psi|_N : (\phi^{\mathrm{can}})^{-1})(A) \to \mathbb{Z}$ of N, where $\Psi_N(m) := \Psi(\phi^{\mathrm{can}}(m))$.

Proposition 2.9.7. *Let M be a finite polyhedral set such that $L^*(|s|)$ is non-empty for every $s \in M^{\mathrm{can}}$, and let (M, Ψ) be a barycentric marked pair. Then the barycentric subdivision $PS(M, \Psi)$ satisfies condition 1) of Section 2.6. If M already satisfies 1), then $PS(M, \Psi)$ satisfies 2) as well (i.e. $\mathbb{U}PS(M, \Psi)$ is a polyhedral complex).*

Proof. We proceed by induction on the dimension $\mathbf{n}$ of M. Both statements are trivial for $\mathbf{n} = -1$, so we may assume that $\mathbf{n} \geq 0$. Let t be any face of $PS(M, \Psi)$. We will show that $\zeta^t_{PS(M,\Psi)} : \Delta^t \to PS(M, \Psi)$ is injective. Since PS preserves colimits, $PS(M, \Psi)$ can be written as the pushout of $PS(M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}}) \leftarrow PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M}) \to PS(\Delta^{\mathbf{n}}_M, \Psi_{\Delta^{\mathbf{n}}_M})$. Thus either $\zeta^t_{PS(M,\Psi)}$ factors through $PS(M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}})$ or Δ^s for an $\mathbf{n}$-dimensional face of $PS(M, \Psi)$. By the inductive hypothesis, we reduce to the latter case, and we may furthermore assume that t is a facet. Since (M, Ψ) is a barycentric marked pair, $\Delta^t \cap PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M}) = \Delta^u$ for some face u of Δ^t (we allow $|u| = \varnothing$ here). But $\Delta^t \backslash \Delta^u \to PS(M, \Psi)$ factors through $PS(\Delta^{\mathbf{n}}_M, \Psi|_{\Delta^{\mathbf{n}}_M}) \backslash PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M}) \subseteq PS(M, \Psi)$, and Δ^u (or $\underline{\varnothing}$) factors (injectively) through $PS(M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}})$, hence we are done.

Next, assume that M satisfies 1). It is clear that PS preserves injections, so $PS(\zeta^u_M) : PS(\Delta^u, \Psi|_{\Delta^u}) \to PS(M, \Psi)$ is injective for each face u. Let s, t be a pair of faces of $PS(M, \Psi)$. Now, $PS(\Delta^u, \Psi|_{\Delta^u})$ is a polyhedral complex, so if both M^s and M^t are contained in the image of $PS(\zeta^u_M)$, we have that $M^s \cap M^t = M^{u'}$ for some face u'. So assume otherwise. Again, we have that $\Delta^s \cap PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M}) = \Delta^{u_1}$ and $\Delta^s \cap PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M}) = \Delta^{u_2}$ for some pair of faces u_1 and u_2 of Δ^s and Δ^t respectively. Since M^s and M^t are not contained in the image of any $PS(\zeta^u_M)$, the intersection $M^s \cap M^t$ must be contained in the image of $PS(\partial\Delta^{\mathbf{n}}_M, \Psi|_{\partial\Delta^{\mathbf{n}}_M})$, and is therefore contained in $PS(M^{\mathbf{n}-1}, \Psi_{M^{\mathbf{n}-1}})$. By the inductive hypothesis, $M^s \cap M^t = M^u$ for some face u, verifying 2). □

Theorem 2.9.8 ([BG09, Theorem 3.17])**.** *Let M be a polyhedral complex. Then there exists an integer $n \geq 1$ and a marked pair (nM, Ψ) such that the subdivision $PS(nM, \Psi)$ of nM is unimodular (i.e. consists of unimodular simplices).*

Theorem 2.9.9. *Let M be a finite polyhedral set. Via repeated application of scaling ($M \mapsto nM$) and subdivisions ($M \mapsto PS(M, \Psi)$), one obtains a simplicial complex (i.e. a unimodular set which is isomorphic to $\mathbb{V}(Z)$ for some simplicial complex Z).*

Proof. Apply Proposition 2.9.7 twice, scaling each time if necessary to ensure that $L^*(|s|)$ is non-empty for all faces s, to produce a polyhedral set N such that $\mathbb{U}(N)$ is a polyhedral complex. By Theorem 2.9.8, there is a subdivision $(\mathbb{U}(N), \Psi)$ such that $PS(\mathbb{U}(N), \Psi) = \mathbb{U}PS(N, \Psi)$ is a simplicial complex. In particular, $PS(N,\Psi)$ is unimodular. So by Proposition 2.6.5, there exists a simplicial complex Z such that $\mathbb{V}(Z)\cong PS(N,\Psi)$. $\square$

2.9.2 A generalized face ring construction

We will now generalize the face ring construction to elements of $\overline{\mathcal{C}}$. Let R be a k-algebra DVR where $t\in R$ is a uniformizing parameter. Let K denote its field of fractions. Then we may define a contravariant functor $\Gamma'_R:\overline{\mathcal{P}_+}\to\mathscr{D}$ as follows. For each $(P,\psi:A\to\mathbb{Z})\in\overline{\mathcal{P}_+}$, let $\Gamma'_R(P,\psi)$ be the R-subalgebra of $K[\mathbb{Z}^{n+1}]=K[\chi^{(m,d)}\mid(m,d)\in\mathbb{Z}^{n+1}]$ generated by $t^{dg_\psi(\frac{m}{d})}\chi^{(m,d)}$, where $(m,d)\in A_{P_i}$ for some i. This is a modified version of [Ale02, Definition 2.8.3]. If $f:(Q,\phi:B\to\mathbb{Z})\to(P,\psi:A\to\mathbb{Z})$ is an arrow in $\overline{\mathcal{P}}$, then we define the R-algebra homomorphism $\Gamma'_R(f):\Gamma'_R(P,\psi)\to\Gamma'_R(Q,\phi)$ by $t^h\chi^{A_f(m,d)}\mapsto t^h\chi^{(m,d)}$, where $A_f:A_Q\to A_P$ denotes the usual homomorphism of affine semigroups (this homomorphism is induced by $t^{dg_\psi(\frac{m}{d})}\chi^{A_f(m,d)}\mapsto t^{dg_{\psi f}(\frac{m}{d})}\chi^{(m,d)}$ on generators). Γ'_R extends as a Yoneda extension to a functor $\widehat{\Gamma}_R:\mathrm{Pre}(\overline{\mathcal{P}_+})\to\mathscr{D}$, and as in Section 2.3 restricts to a functor $\Gamma_R:\mathcal{C}\to\mathscr{D}$, preserving all connected colimits. Note that each $\Gamma'_R(P,\psi)$ is torsion-free as an R-module, so the same is true for the limit $\Gamma_R(M,\Psi)=\varprojlim_{Y_{\overline{\mathcal{P}}}(P,\psi)\to(M,\Psi)}\Gamma_R(P,\psi)$. It follows that $\Gamma_R(M,\Psi)$ is a flat R-module. To avoid confusion, we will write $\Gamma_k(M)$ for the face ring of M over a field k, and similarly $\mathbf{P}_k(M)$ for the face scheme of M.

Proposition 2.9.10. *There is a natural isomorphism $\Gamma_R(P,\psi)\otimes_R K\xrightarrow{\cong}\Gamma_K(P)$, where $\Gamma_K(P)$ denotes the usual face-ring associated to P over the field K. Let $k=R/(t)$ be the residue field of R. Then there is a natural isomorphism $\Gamma_R(P,\psi)\otimes_R k\xrightarrow{\cong}\Gamma_k(PS(P,\psi))$.*

Proof. The first part is clear, as $\Gamma_R(P,\psi)\otimes_R K$ is given as the K-subalgebra of $K[\mathbb{Z}^{n+1}]$ generated by $\chi^{(m,d)}$ for $(m,d)\in\bigcup_i A_{P_i}$, in particular $\chi^{(m,1)}$, for every $m\in L(P)$. The isomorphism is given by $\chi^{(m,d)}\otimes r\mapsto r\chi^{(m,d)}$. For the other part, we follow along the lines of [Ale02, Lemma 2.8.4.]. Note first that $\Gamma_R(P,\psi)\otimes_R k\cong\Gamma_R(P,\psi)/t\Gamma_R(P,\psi)$ by flatness of $\Gamma_R(P,\psi)$. Consider now a product of generators in $\Gamma_R(P,\psi)/t\Gamma_R(P,\psi)$:

$$\prod_j t^{d_jg_\psi\left(\frac{m_j}{d_j}\right)}\chi^{(m_j,d_j)}=t^{\sum_j d_jg_\psi\left(\frac{m_j}{d_j}\right)}\chi^{\left(\sum_j m_j,\sum_j d_j\right)}.$$

Here

$$\sum_j d_jg_\psi\left(\frac{m_j}{d_j}\right)\geq\sum_j d_jg_\psi\left(\frac{\sum_j m_j}{\sum_j d_j}\right),$$

with equality if and only if every (m_j,d_j) is contained in the same A_{P_i} (i.e. $\frac{m_j}{d_j}\in P_i$). This follows from the fact that g_ψ is convex and that the P_i's are its domains of linearity. Thus any element of $\Gamma_R(P,\psi)\otimes_R k$ can be written as a sum of elements on the form $t^{dg_\psi(\frac{m}{d})}\chi^{(m,d)}\otimes r$ for $r\in k$ and $(m,d)\in A_{P_i}$ for some

i. Thus we may define a homomorphism $\Gamma_R(P,\psi)/t\Gamma_R(P,\psi) \to \Gamma_k(PS(P,\psi))$ by $t^{dg_\psi(\frac{m}{d})}\chi^{(m,d)} \otimes r \mapsto r\chi^{(m,d)}$ (see Definition 2.7.1 for the presentation of the face ring of a polyhedral complex). This is evidently a natural isomorphism. $\square$

Theorem 2.9.11. *There are natural isomorphisms* $\Gamma_R(M,\Psi) \otimes_R K \xrightarrow{\cong} \Gamma_K(M)$ *and* $\Gamma_R(M,\Psi) \otimes_R k \xrightarrow{\cong} \Gamma_k(PS(M,\Psi))$ *for finite polyhedral sets* M*.*

Proof. The natural homomorphisms in question are well-defined by Proposition 2.9.10. It remains to show that they are isomorphisms. We proceed by induction on the dimension $\mathbf{n}$ of M. The initial case of $\mathbf{n} = -1$ is trivial. It is easy to verify that the colimit diagram (2.4) for M induces a colimit diagram

$$\begin{tikzcd}
\{(\Delta^s, \Psi|_{\Delta^s})\} \arrow[r, "\zeta_M^s"] & (M, \Psi) \\
\{(\partial\Delta^s, \Psi|_{\partial\Delta^s})\} \arrow[u, "\iota_{\Delta^s}"] \arrow[r, "\zeta_M^{s\prime}"] & (M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}}). \arrow[u, "\iota_M"']
\end{tikzcd} \tag{2.15}$$

Applying $\Gamma_R : \overline{\mathcal{C}} \to \mathscr{D}$ to (2.15) yields an exact sequence

$$0 \to \Gamma_R(M,\Psi) \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_R(\Delta^s, \Psi|_{\Delta^s}) \times \Gamma_R(M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}}) \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_R(\partial\Delta^s, \Psi|_{\partial\Delta^s}). \tag{2.16}$$

Now, consider the homomorphism $\Gamma_R(\iota_{\Delta^s}) : \Gamma_R(\Delta^s, \Psi|_{\Delta^s}) \to \Gamma_R(\partial\Delta^s, \Psi|_{\partial\Delta^s})$. By Nakayama's lemma, $\Gamma_R(\iota_{\Delta^s})$ is surjective if and only if $\Gamma_R(\iota_{\Delta^s}) \otimes_R k$ is. By Proposition 2.9.10 and the inductive hypothesis, this is equivalent with the statement that $\Gamma_k(PS(\iota_{\Delta^s})) : \Gamma_k(PS(\Delta^s, \Psi|_{\Delta^s})) \to \Gamma_k(PS(\partial\Delta^s, \Psi|_{\partial\Delta^s}))$ is surjective. But this follows from Proposition 2.8.2. We conclude that (2.16) is right exact. Now, tensor (2.16) with K and compare with the exact sequence

$$0 \to \Gamma_K(M) \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_K(\Delta^s) \times \Gamma_K(M^{\mathbf{n}-1}) \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_K(\partial\Delta^s) \to 0. \tag{2.17}$$

By the inductive hypothesis and Proposition 2.9.10, it follows that $\Gamma_R(M,\Psi) \otimes_R K \to \Gamma_K(M)$ is an isomorphism. Next, applying the functor PS to (2.15) yields a new colimit diagram. It follows that the sequence

$$\begin{aligned}0 \to \Gamma_k(PS(M,\Psi)) \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_k(PS(\Delta^s, \Psi|_{\Delta^s})) \times \Gamma_k(PS(M^{\mathbf{n}-1}, \Psi|_{M^{\mathbf{n}-1}})) \\ \to \prod_{s\in J_M^{\mathbf{n}}} \Gamma_k(PS(\partial\Delta^s, \Psi|_{\partial\Delta^s}))\end{aligned} \tag{2.18}$$

is exact. Since the functor $\operatorname{Tor}_1^R(\Gamma_R(\partial\Delta^s, \Psi|_{\partial\Delta^s}), -)$ is trivial, tensoring (2.16) with k yields another exact sequence. Comparing this with (2.18) shows that $\Gamma_R(M,\Psi) \otimes_R k \to \Gamma_k(PS(M,\Psi))$ is an isomorphism. $\square$

Lemma 2.9.12. *Let* M *be a finite polyhedral set of dimension* $\mathbf{n}$*. Then,*

1) *if M is irreducible, then $\Gamma(M)$ is an integral domain;*

2) *there exists an integer $n > 0$ such that $\Gamma(M)^{[n]}$ is a finitely generated k-algebra of dimension $\mathbf{n} + 1$;*

3) *if $\phi : M \to N$ is a surjective morphism of polyhedral sets, then the homomorphism $\Gamma(\phi)^{[n]} : \Gamma(N)^{[n]} \to \Gamma(M)^{[n]}$ is finite for some $n > 0$.*

Proof. 1) Let $s \in M^{\mathrm{can}}$ be a facet of M, and let $P = |s|$. Since $\Delta^s \to M$ is surjective, $\Gamma(\zeta^s_M) : \Gamma(M) \to \Gamma(\Delta^P)$ is an injective by Proposition 2.8.2, where $\Gamma(\Delta^P) = k[A_P]$ is an integral domain. 2) Assume first that M is a simplicial complex. Then $\Gamma(M)$ is the Stanley-Reisner ring of M, and the result is well-known. By Lemma 2.9.5, the property in question is invariant under scaling $M \mapsto nM$. By Theorem 2.9.11, Nakayama's lemma implies that it is invariant under subdivisions $M \mapsto PS(M, \Psi)$ as well. Finally, by Theorem 2.9.9, one may iteratively scale and subdivide to end up with a simplicial complex of the same dimension. Hence we are done.

3) As in the proof of 2), this property is invariant under scaling and subdivision of N. Indeed, if (N, Ψ) is a marked pair structure on N, one obtains a marked pair structure $(M, \Psi|_M)$ on M, and a morphism $\phi : (M, \Psi|_M) \to (N, \Psi)$. If $PS(N, \Psi)$ is a simplicial complex, then $PS(M, \Psi|_M)$ is one as well, hence we have reduced to the case where both M and N are simplicial complexes. In this case, $\Gamma(\phi)$ is given by $x_w \mapsto \sum_{\phi(v)=w} x_v$. We will show that $\Gamma(M)$ is generated over $\Gamma(N)$ by its square-free monomials, of which there are finitely many. Choose some total order on the vertices of M, and assume for contradiction that $\mathbf{m} = x_{v_0}^{a_0} \cdots x_{v_n}^{a_n}$ is some monomial of $\Gamma(M)$ which cannot be generated by square-free monomials over $\Gamma(N)$. Here $v_0 < \cdots < v_n$, n is assumed to be maximal, and $(a_0, \ldots, a_n)$ is assumed to be minimal in lexicographical order. Let i be an integer such that $a_i > 1$, and define $w = \phi(v_i)$. Note that if $v \in M_0$ satisfies $\phi(v) = w$, then either $v = v_i$ or $v \notin \{v_0, \ldots, v_n\}$. This is because $\phi|_{\Delta^s} : \Delta^s \to N$ is injective for every face s of M. It follows that

$$\Gamma(\phi)(x_w) \cdot x_{v_0}^{a_0} \cdots x_{v_i}^{a_i - 1} \cdots x_{v_n}^{a_n} = \mathbf{m} \; + \; \text{monomials with } > n + 1 \text{ variables.}$$

The left hand side is generated by monomials since $(a_0, \ldots, a_i - 1 \ldots, a_n) < (a_0, \ldots, a_n)$. Since n was chosen maximal, we may rearrange the equation and write $\mathbf{m}$ as a linear combination of square-free monomials, which is a contradiction. $\square$

Chapter 3

The face scheme

Let $\mathcal{C}_f$ denote the full subcategory of $\mathcal{C}$ consisting of finite polyhedral sets. By Proposition 2.2.5, joins of finite polyhedral sets are finite, so $\mathcal{C}_f$ inherits a monoidal structure from $\mathcal{C}$. If $\phi : M \to N$ is a morphism of finite polyhedral sets, then $\Gamma(\phi)^{[n]} : \Gamma(M)^{[n]} \to \Gamma(N)^{[n]}$ is finite for some $n > 0$ by Lemma 2.9.12 3), inducing a well-defined morphism of schemes $\operatorname{Proj}\Gamma(M) \to \operatorname{Proj}\Gamma(N)$. This permits the following definition.

Definition 3.0.1. For a finite polyhedral set M, define $\mathbf{P}(M) = \operatorname{Proj}\Gamma(M)$. We will call this the *face scheme of* M. This defines a functor $\mathbf{P} : \mathcal{C}_f \to \mathrm{Sch}_k$ from the category of finite polyhedral sets to the category of schemes over $\operatorname{Spec} k$. Note that $\mathbf{P}(\Delta^P)$ is the ordinary toric variety associated with the polytope P (denoted X_P in [CLS11]). If $\phi : M \to N$ is a morphism in $\mathcal{C}_f$, we will let ϕ denote the morphism $\mathbf{P}(\phi) : \mathbf{P}(M) \to \mathbf{P}(N)$ whenever it is unambiguous. We will also denote the structure sheaf $\mathcal{O}_{\mathbf{P}(M)}$ by $\mathcal{O}_M$, and cohomology groups $H^i(\mathbf{P}(M), \mathcal{F})$ by $H^i(M, \mathcal{F})$ for sheaves $\mathcal{F}$ on $\mathbf{P}(M)$.

The main goals of this chapter is first to establish some basic properties of the face scheme of polyhedral sets, and then prove the classification theorem of Section 3.3. Next, in Section 3.4 we will prove that there is a natural isomorphism of k-vector spaces $H^i(M;k) \xrightarrow{\cong} H^i(M, \mathcal{O}_M)$ for each integer $i \geq 0$ (Theorem 3.4.1). Here $H^i(M;k)$ denotes the cellular cohomology groups of M with coefficients in k, which are naturally isomorphic to the singular cohomology groups $H^i(|M|;k)$. In Section 3.5 we will prove that the Picard group $\operatorname{Pic}(\mathbf{P}(M))$ is naturally isomorphic to $\operatorname{Deg}_M \times H^1(M;k^\times)$ (Theorem 3.5.2). Here Deg_M is a finitely generated free abelian group specifying the degree of the line bundle (Definition 3.5.1). Finally, we will consider the cohomology of the twisting sheaves $\mathcal{O}_M(d)$ in order to compute the Hilbert polynomial of $\Gamma(M)$.

Now, let A be a finitely generated and positively graded ring, and let $X = \operatorname{Proj}(A)$. Then for each integer n, the sheaf $\mathcal{O}_X(n)$ is defined as $\widetilde{A(n)}$. The graded k-algebras we will consider will be finitely generated, but not in general standard graded. This complicates things; for example, $\mathcal{O}_X(1)$ may potentially fail to be ample, or even locally free. See [Dol82, Section 1.5] for examples of these pathologies and more. However, X is always projective over k, and there exists an integer $n > 0$ such that $\mathcal{O}_X(n)$ is very ample. In Section 3.2 we will see that none of the pathologies from [Dol82] are present for $\mathbf{P}(M)$. For now, we will focus on the categorical properties of $\mathbf{P}$. From here on, all polyhedral sets will be assumed to be finite.

Definition 3.0.2 (The Milnor square for $\mathbf{P}(M)$). Let M be a polyhedral set, and $\mathbf{n} \geq -1$ an integer. Consider the Milnor square for $\Gamma(M)$ of level $\mathbf{n}$ from

Definition 2.8.1. Applying $\mathbf{P}$ to this pullback square yields the diagram

$$\begin{tikzcd} \mathbf{P}(\Delta_M^{\mathbf{n}}) \arrow[r, "\alpha_M"] & \mathbf{P}(M^{\mathbf{n}}) \\ \mathbf{P}(\partial\Delta_M^{\mathbf{n}}) \arrow[u, "\iota_{\Delta_M}"] \arrow[r, "\alpha'_M"] & \mathbf{P}(M^{\mathbf{n}-1}), \arrow[u, "\iota_M"'] \end{tikzcd} \tag{3.1}$$

which by Proposition B.2.1 b) is a pushout square of schemes. We call this *the Milnor square for* $\mathbf{P}(M)$ *of level* $\mathbf{n}$.

Proposition 3.0.3. *The functor* $\mathbf{P} : \mathcal{C}_f \to \mathrm{Sch}_k$ *preserves all finite colimits. In other words,* $\mathbf{P} : \mathcal{C}_f \to \mathrm{Sch}_k$ *is the pointwise Yoneda extension of the functor* $\mathcal{P} \to \mathrm{Sch}_k$ *given by* $P \mapsto \mathbf{P}(\Delta^P)$*. As a consequence, if* G *is a finite group acting on a polyhedral set* M*, then* $\mathbf{P}(M) \to \mathbf{P}(M/G)$ *is a group quotient for the induced action on* $\mathbf{P}(M)$ *in the category of schemes.*

Proof. We first observe that any finite colimit in $\mathcal{C}_f$ (resp. colimit in $\mathcal{C}$) can be computed pointwise. Indeed, this follows immediately from the fact that there does not exist any morphism $\phi : M \to N$ in $\mathrm{Pre}(\mathcal{P})$, where N is in $\mathcal{C}_f$ (resp. $\mathcal{C}$) and M is not. Thus we may always assume that a colimit in $\mathcal{C}_f$ (resp. $\mathcal{C}$) is a colimit in $\mathrm{Pre}(\mathcal{P})$ of the same diagram. Let $\mathbf{n} \geq -1$, and consider the canonical pushout square for $M \in \mathcal{C}_f$ of level $\mathbf{n}$ from Definition 2.4.2. Since the diagram is natural, it forms a pointwise pushout square of functors $\mathcal{C}_f \to \mathcal{C}_f$:

$$\begin{tikzcd} \Delta_-^{\mathbf{n}} \arrow[r] & (-)^{\mathbf{n}} \\ \partial\Delta_-^{\mathbf{n}} \arrow[u] \arrow[r] & (-)^{\mathbf{n}-1}. \arrow[u] \end{tikzcd} \tag{3.2}$$

The functors $(-)^{\mathbf{n}}, \Delta_-^{\mathbf{n}}, (-)^{\mathbf{n}}, \partial\Delta_-^{\mathbf{n}} : \mathcal{C} \to \mathcal{C}$ are respectively given by $M \mapsto M^{\mathbf{n}}$, $M \mapsto \Delta_M^{\mathbf{n}}$, $M \mapsto M^{\mathbf{n}-1}$ and $M \mapsto \partial\Delta_M^{\mathbf{n}}$. We will show that each of these four functors are cocontinuous. The functors $(-)^{\mathbf{n}}$ and $(-)^{\mathbf{n}-1}$ are clearly cocontinuous, and $\partial\Delta_-^{\mathbf{n}}$ is the composition of $(-)^{\mathbf{n}-1}$ with $\Delta_-^{\mathbf{n}}$, so we only need to verify that $\Delta_-^{\mathbf{n}}$ is cocontinuous. Restricting these functors to $\mathcal{C}_f$ and composing with $\mathbf{P}$ yields the four functors $\mathbf{P} \circ (-)^{\mathbf{n}}, \mathbf{P} \circ \Delta_-^{\mathbf{n}}, \mathbf{P} \circ (-), \mathbf{P} \circ \partial\Delta_- : \mathcal{C}_f \to \mathrm{Sch}_k$. By induction on $\mathbf{n}$ we will show that $\mathbf{P} \circ (-)^{\mathbf{n}}$ preserves all finite colimits. By the inductive hypothesis and the fact that $\partial\Delta_-$ is cocontinuous, we may assume that $\mathbf{P} \circ (-)^{\mathbf{n}-1}$ and $\mathbf{P} \circ \partial\Delta_-^{\mathbf{n}}$ preserves all finite colimits. We will show by an explicit computation that $\mathbf{P} \circ \Delta_-^{\mathbf{n}}$ preserves all finite colimits. The Milnor square for $\mathbf{P}(M)$ of level $\mathbf{n}$ from Definition 3.0.2 is (pointwise in $M \in \mathcal{C}_f$) the pushout square (3.2) composed with $\mathbf{P}$:

$$\begin{tikzcd} \mathbf{P} \circ \Delta_-^{\mathbf{n}} \arrow[r] & \mathbf{P} \circ (-)^{\mathbf{n}} \\ \mathbf{P} \circ \partial\Delta_-^{\mathbf{n}} \arrow[u] \arrow[r] & \mathbf{P} \circ (-)^{\mathbf{n}-1}. \arrow[u] \end{tikzcd} \tag{3.3}$$

We are now in a purely categorical situation. If $G, G_1, G_2, G_3 : C \to A$ are functors in a pointwise pushout square of functors

$$\begin{tikzcd} G_1 \arrow[r] & G \\ G_3 \arrow[u] \arrow[r] & G_2 \arrow[u] \end{tikzcd} \tag{3.4}$$

where each G_j preserves all finite colimits, then I claim that G preserves all finite colimits (even if we don't know that $\varinjlim G \circ F$ exists for finite diagrams $F : I \to A$). Note that since it is a pointwise pushout, it is a pushout in the functor category $[C, A]$. The dual statement in a more general setting is noted in [FK72, p.1]. It is a routine verification. In conclusion, $\mathbf{P} \circ (-1)^{\mathbf{n}}$ preserves all finite colimits. Now, let $F : I \to \mathcal{C}_f$ be a finite diagram with $\varinjlim F = M$. Let $\mathbf{n}$ be the maximal dimension of $F(i)$ for each $i \in I$. Then $\dim M = \mathbf{n}$, and $\mathbf{P} \circ (-)^{\mathbf{n}} \circ F = \mathbf{P} \circ F$. Since $\mathbf{P} \circ (-)^{\mathbf{n}}$ preserves all colimits, it follows that $\mathbf{P}(M)$ is the colimit of the diagram $\mathbf{P} \circ F : I \to \mathrm{Sch}_k$. Hence $\mathbf{P}$ preserves all finite colimits. It remains to show the following: 1) $\Delta^{\mathbf{n}}_{-} : \mathcal{C}_f \to \mathcal{C}_f$ is cocontinuous; 2) $\mathbf{P} \circ \Delta^{\mathbf{n}}_{-} : \mathcal{C}_f \to \mathrm{Sch}_k$ preserves all finite colimits.

For 1) we may equivalently show that that $\Delta^{\mathbf{n}}_{-}$ preserves finite coproducts and coequalizers. First, let $\{M_i\}_{i \in I}$ be a finite family of polyhedral sets, and let $M = \coprod_{i \in I} M_i$. Then it is clear that the induced function $J_M \to \coprod_{i \in I} J^{\mathbf{n}}_{M_i}$ is a bijection. It follows that $\Delta_M \to \coprod_{i \in I} \Delta^{\mathbf{n}}_{M_i}$ is an isomorphism. Second, let $\phi_1, \phi_2 : M \to N$ be a pair of morphisms of polyhedral sets, and let $M \rightrightarrows N \xrightarrow{\psi} Z$ be corresponding coequalizer diagram. Then $J^{\mathbf{n}}_M \rightrightarrows J^{\mathbf{n}}_N \to J^{\mathbf{n}}_Z$ is a coequalizer diagram of sets, and it follows easily that $\Delta^{\mathbf{n}}_M \rightrightarrows \Delta^{\mathbf{n}}_N \to \Delta^{\mathbf{n}}_Z$ is a coequalizer diagram.

We follows the same strategy for 2). In the same notation as above, $\Gamma(\coprod_{i \in I} M_i)$ is given as the wide pullback of $\{\Gamma(M_i)\}_{i \in I}$ over k via the description in Remark 2.3.5. Since the unique graded k-algebra homomorphism $\Gamma(\coprod_{i \in I} M_i) \to \prod_{i \in I} \Gamma(M_i)$ is an isomorphism in all degrees $d > 0$, it follows that $\coprod_{i \in I} \operatorname{Proj} \Gamma(M_i) \xrightarrow{\cong} \operatorname{Proj}(\Gamma(\coprod_{i \in I} M_i))$. Since $\Delta^{\mathbf{n}}_{-}$ preserves finite coproducts from 1), it follows now that $\mathbf{P} \circ \Delta^{\mathbf{n}}_{-}$ does as well. Next, let $\phi_1, \phi_2 : M \to N$ be a pair of morphisms of polyhedral sets, which from 1) induces a coequalizer diagram $\Delta^{\mathbf{n}}_M \rightrightarrows \Delta^{\mathbf{n}}_N \to \Delta^{\mathbf{n}}_Z$. We may write the induced diagram $\mathbf{P}(\Delta^{\mathbf{n}}_M) \rightrightarrows \mathbf{P}(\Delta^{\mathbf{n}}_N) \to \mathbf{P}(\Delta^{\mathbf{n}}_Z)$ as $\coprod_{u \in J^{\mathbf{n}}_M} \mathbf{P}(\Delta^u) \rightrightarrows \coprod_{t \in J^{\mathbf{n}}_N} \mathbf{P}(\Delta^t) \to \coprod_{s \in J^{\mathbf{n}}_Z} \mathbf{P}(\Delta^s)$. This is easily seen to be a coequalizer diagram of schemes. □

3.1 The category of vector bundles

We will use Appendix B.3 as a general reference for this section. Let M be a polyhedral set, and fix an integer $\mathbf{n} \geq -1$. Consider the Milnor square (3.1) for $\mathbf{P}(M)$ of level $\mathbf{n}$, arising from the Milnor square (2.11) of $\Gamma(M)$ of level $\mathbf{n}$. By Proposition B.3.3, there is a an equivalence of categories

$$\beta_M : \mathrm{VB}(M^{\mathbf{n}}) \to \mathrm{VB}(\Delta^{\mathbf{n}}_M) \times_{\mathrm{VB}(\partial \Delta^{\mathbf{n}}_M)} \mathrm{VB}(M^{\mathbf{n}-1}) \tag{3.5}$$

given by $\mathcal{E} \mapsto (\alpha_M^*\mathcal{E}, \iota_M^*\mathcal{E})$, where this pair is equipped with the canonical isomorphism $h : (\alpha'_M)^*\iota_M^*\mathcal{E} \to \iota_{\partial\Delta_M}^*\alpha_M^*\mathcal{E}$. We also have the inverse

$$\theta_M : \mathrm{VB}(\Delta_M^{\mathbf{n}}) \times_{\mathrm{VB}(\partial\Delta_M^{\mathbf{n}})} \mathrm{VB}(M^{\mathbf{n}-1}) \to \mathrm{VB}(M^{\mathbf{n}}), \tag{3.6}$$

where θ_M maps a pair $(\mathcal{E}_1, \mathcal{E}_2)$ equipped with an isomorphism $h : (\alpha'_M)^*\mathcal{E}_2 \to \iota_{\partial\Delta_M}^*\mathcal{E}_1$ to the vector bundle $\mathcal{E}$ determined by the pullback square

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & (\iota_M)_*\mathcal{E}_2 \arrow[d] \\
(\alpha_M)_*\mathcal{E}_1 \arrow[r] & (\alpha_M\iota_{\partial\Delta_M})_*\iota_{\partial\Delta_M}^*\mathcal{E}_1.
\end{tikzcd}$$

Let $\phi : N \to M$ be a morphism of polyhedral sets. Then the morphism $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ induces a pullback functor $\phi^* : \mathrm{VB}(M) \to \mathrm{VB}(N)$, defining the functor $\mathrm{VB}(-) : \mathcal{C}_f \to \mathrm{Cat}$. Since Cat is a 2-category, this is really a 2-functor. Recall the morphisms $\phi : M^{\mathbf{n}} \to N^{\mathbf{n}}$, $\phi^\Delta : \Delta_N^{\mathbf{n}} \to \Delta_M^{\mathbf{n}}$, $\phi' : N^{\mathbf{n}-1} \to M^{\mathbf{n}-1}$ and $\phi^{\partial\Delta} : \partial\Delta_N^{\mathbf{n}} \to \partial\Delta_M^{\mathbf{n}}$ from Definition 2.4.2. These induces pullback functors $\phi^* : \mathrm{VB}(M^{\mathbf{n}}) \to \mathrm{VB}(N^{\mathbf{n}})$, $\phi'^* : \mathrm{VB}(M^{\mathbf{n}-1}) \to \mathrm{VB}(N^{\mathbf{n}-1})$, $\phi^{\Delta *} : \mathrm{VB}(\Delta_M^{\mathbf{n}}) \to \mathrm{VB}(\Delta_N^{\mathbf{n}})$ and $\phi^{\Delta' *} : \mathrm{VB}(\partial\Delta_M^{\mathbf{n}}) \to \mathrm{VB}(\partial\Delta_N^{\mathbf{n}})$, and defines a functor

$$(\phi^{\Delta *}, \phi^{'*}) : \mathrm{VB}(\Delta_M^{\mathbf{n}}) \times_{\mathrm{VB}(\partial\Delta_M^{\mathbf{n}})} \mathrm{VB}(M^{\mathbf{n}-1}) \to \mathrm{VB}(\Delta_N^{\mathbf{n}}) \times_{\mathrm{VB}(\partial\Delta_N)} \mathrm{VB}(N^{\mathbf{n}-1}) \tag{3.7}$$

given by the following. On objects, a pair $(\mathcal{E}_1, \mathcal{E}_2)$ equipped with an isomorphism $h_M : \alpha'^*_M\mathcal{E}_2 \to \iota_{\Delta_M}^*\mathcal{E}_1$ is mapped to the pair $(\phi^{\Delta *}\mathcal{E}_1, \phi'^*\mathcal{E}_2)$ equipped with the isomorphism

$$h_N : \alpha'^*_N\phi'^*\mathcal{E}_2 \to \iota_{\Delta_N}^*\phi^{\Delta *}\mathcal{E}_1. \tag{3.8}$$

induced by $\phi^{\Delta' *}h_M : \phi^{\Delta' *}\alpha'^*_M\mathcal{E}_2 \to \phi^{\Delta' *}\iota_{\Delta_M}^*\mathcal{E}_1$. A morphism of pairs $(g_1, g_2) : (\mathcal{E}_1, \mathcal{E}_2) \to (\mathcal{F}_1, \mathcal{F}_2)$ is mapped to the morphism $(\phi^{\Delta *}g_1, \phi'^*g_2) : (\phi^{\Delta *}\mathcal{E}_1, \phi'^*\mathcal{E}_2) \to (\phi^{\Delta *}\mathcal{F}_1, \phi'^*\mathcal{F}_2)$. It is easily seen that the pair of morphisms $(\phi^{\Delta *}g_1, \phi'^*g_2)$ satisfies the condition (B.13), and is therefore well-defined. This defines the functor

$$\mathrm{VB}(\Delta_-^{\mathbf{n}}) \times_{\mathrm{VB}(\partial\Delta_-^{\mathbf{n}})} \mathrm{VB}((-)^{\mathbf{n}-1}) : \mathcal{C}_f \to \mathrm{Cat}\,.$$

Proposition 3.1.1. *The equivalences of categories β_M and θ_M define natural equivalences of functors*

$$\beta : \mathrm{VB}((-)^{\mathbf{n}}) \overset{\cong}{\leftrightarrows} \mathrm{VB}(\Delta_-^{\mathbf{n}}) \times_{\mathrm{VB}(\partial\Delta_-^{\mathbf{n}})} \mathrm{VB}((-)^{\mathbf{n}-1}) : \theta,$$

Proof. To show that β_M defines a natural equivalence β, we need to verify that for any morphism of polyhedral sets $\phi : M \to N$, there are natural equivalences of functors

$$\beta_N \circ \phi^* \cong (\phi^{\Delta *}, \phi^{'*}) \circ \beta_M \text{ and } \phi^* \circ \theta_M \cong \theta_N \circ (\phi^{\Delta *}, \phi'^*). \tag{3.9}$$

Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}(M^{\mathbf{n}})$, and consider the canonical isomorphisms $\alpha_N^*\phi^*\mathcal{E} \cong \phi^{\Delta*}\alpha_M^*\mathcal{E}$ and $\iota_N^*\phi^*\mathcal{E} \cong \phi'^*\iota_M^*\mathcal{E}$. Then it is easily seen that the induced natural isomorphism

$$\beta_M(\phi^*\mathcal{E}) = (\alpha_N^*\phi^*\mathcal{E}, \iota_N^*\phi^*\mathcal{E}) \xrightarrow{\cong} (\phi^{\Delta*}\alpha_M^*\mathcal{E}, \phi'^*\iota_M^*\mathcal{E}) = (\phi^{\Delta*}, \phi'^*)(\beta_M(\mathcal{E})) \tag{3.10}$$

satisfies the condition (B.13), and is therefore a morphism of pairs. This shows that β_M defines a natural equivalence β, and it follows formally that θ_M defines a natural equivalence θ as well. □

3.2 Properties of the face scheme

The results of Section 2.8 and Section 2.9 regarding face rings immediately translates to facts about face schemes, which we summarize now.

Proposition 3.2.1. *Let M and N be polyhedral sets. Then,*

a) $\mathbf{P}(M)$ *is a reduced projective scheme over k of dimension $\dim(M)$;*

b) *if $\phi : M \to N$ is an injective morphism, then $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ is a closed immersion;*

c) *if M_1, M_2 are polyhedral subsets of M, then $\mathbf{P}(M_1) \cup \mathbf{P}(M_2) = \mathbf{P}(M_1 \cup M_2)$ and $\mathbf{P}(M_1) \cap \mathbf{P}(M_2) = \mathbf{P}(M_1 \cap M_2)$ as closed subschemes of $\mathbf{P}(M)$;*

d) *if M is irreducible, then $\mathbf{P}(M)$ is an irreducible scheme;*

e) *if $\{M_i\}_{i=1}^n$ are the irreducible components of M, then $\{\mathbf{P}(M_i)\}_{i=1}^n$ are the irreducible components of $\mathbf{P}(M)$;*

f) *if $\phi : M \to N$ is any morphism, then $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ is a finite morphism;*

g) *if $\phi : M \to N$ is a surjective morphism, then $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ is surjective, and $\phi^\sharp : \mathcal{O}_N \to \phi_*\mathcal{O}_M$ is injective;*

Proof. a) Being reduced is stable under localization, so $\mathbf{P}(M)$ is reduced by Lemma 2.8.3. There is an integer $n > 0$ such that $\Gamma^{[n]}(M)$ is a finitely generated $(\mathbf{n}+1)$-dimensional k-algebra by Lemma 2.9.12 2). Since $\mathbf{P}(M) \cong \operatorname{Proj}\Gamma^{[n]}(M)$, it is projective over k of dimension $\mathbf{n}$. b) follows from Proposition 2.8.2. c) follows from Proposition 2.8.4. d) follows from Lemma 2.9.12 1). e) follows from c) and d). f) $\Gamma^{[n]}(\phi)$ is finite for some $n > 0$ by Lemma 2.9.12 3), so $\phi^\sharp$ is finite as well. Since $\mathbf{P}(\phi)$ is induced by a morphism of graded rings, it is affine — and therefore finite. g) $\Gamma(\phi)$ is injective by Proposition 2.8.2, and localization preserves injections, hence $\phi^\sharp$ is injective. Since ϕ is finite, it must be surjective. □

Proposition 3.2.2. *Let M be a polyhedral set, and n, m integers.*

a) *The canonical morphism*

$$\mathcal{O}_M(n) \otimes \mathcal{O}_M(m) \to \mathcal{O}_M(n+m) \tag{3.11}$$

is an isomorphism. Hence $\mathcal{O}_M(d)$ is an invertible sheaf for every integer d.

b) *The line bundle $\mathcal{O}_M(1)$ is ample.*

c) *Let $\phi : N \to M$ be a morphism of polyhedral sets, and let d be an integer. Then the canonical morphism*

$$\phi^*(\mathcal{O}_M(d)) \to \mathcal{O}_N(d) \tag{3.12}$$

is an isomorphism.

Proof. We will prove a), b) and c) simultaneously by induction. Let $\mathbf{n} \geq -1$ be an integer, and suppose that the assertions holds for all maximally $(\mathbf{n}-1)$-dimensional polyhedral sets. Assume that M is $\mathbf{n}$-dimensional, and consider the Milnor square (2.11) for $\Gamma(M)$ of level $\mathbf{n}$. Choosing $P = \Gamma(M)(d)$, Proposition B.2.1 a) yields a pullback square

$$\begin{tikzcd}
\mathcal{O}_M(d) \arrow[r] \arrow[d] & (\iota_M)_*(\mathcal{O}_{M^{\mathbf{n}-1}}(d)) \arrow[d] \\
(\alpha_M)_*(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d)) \arrow[r] & (\alpha_M \iota_{\partial\Delta^{\mathbf{n}}_M})_*(\mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}(d)).
\end{tikzcd} \tag{3.13}$$

Since $\Gamma(\Delta^{\mathbf{n}}_M)$ and $\Gamma(\partial\Delta^{\mathbf{n}}_M)$ are standard graded, the canonical morphism $\iota^*_{\Delta_M} : \mathcal{O}_{\Delta^{\mathbf{n}}_M}(d) \to \mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}(d)$ is an isomorphism. By the inductive hypothesis, the canonical morphism $(\alpha'_M)^*\mathcal{O}_{M^{\mathbf{n}-1}}(d) \to \mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}(d)$ is an isomorphism. Hence by Proposition B.3.1, $\mathcal{O}_M(d)$ is a vector bundle. Now, consider the pair $(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d), \mathcal{O}_{M^{\mathbf{n}-1}}(d))$ equipped with the canonical isomorphism $(\alpha'_M)^*\mathcal{O}_{M^{\mathbf{n}-1}}(d) \to \iota^*_{\Delta_M}\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d)$. Since (3.13) is a pullback square and $\mathcal{O}_M(d)$ is a vector bundle, it follows by definition that $\theta_M(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d), \mathcal{O}_{M^{\mathbf{n}-1}}(d)) \cong \mathcal{O}_M(d)$. Again, since $\Gamma(\Delta_M)$ is standard graded, the canonical morphism $\mathcal{O}_{\Delta^{\mathbf{n}}_M}(n) \otimes \mathcal{O}_{\Delta^{\mathbf{n}}_M}(m) \to \mathcal{O}_{\Delta^{\mathbf{n}}_M}(n+m)$ is an isomorphism. By the inductive hypothesis, we may assume that the canonical morphism $\mathcal{O}_{M^{\mathbf{n}-1}}(n) \otimes \mathcal{O}_{M^{\mathbf{n}-1}}(m) \to \mathcal{O}_{M^{\mathbf{n}-1}}(n+m)$ is an isomorphism. By Proposition B.3.4 b), it follows that the canonical morphism $\mathcal{O}_M(n) \otimes \mathcal{O}_M(m) \to \mathcal{O}_M(n+m)$ is an isomorphism, showing a). Now, there exists an integer n such that $\mathcal{O}_M(n)$ is very ample. Since $\mathcal{O}_M(1)^{\otimes n} \cong \mathcal{O}_M(n)$, it follows that $\mathcal{O}_M(1)$ is ample, showing b). Let now $\phi : N \to M$ be a morphism. Since $\Gamma(\Delta^{\mathbf{n}}_M) \to \Gamma(\Delta^{\mathbf{n}}_N)$ is a homomorphism of standard graded rings, the canonical morphism $\phi^{\Delta *}(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d)) \to \mathcal{O}_{\Delta^{\mathbf{n}}_N}(d)$ is an isomorphism. By the inductive hypothesis, the canonical morphism $\phi'^*(\mathcal{O}_{M^{\mathbf{n}-1}}(d)) \to \mathcal{O}_{N^{\mathbf{n}-1}}(d)$ is an isomorphism. The isomorphism $\phi^*\theta_M(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d), \mathcal{O}_{M^{\mathbf{n}-1}}(d)) \cong \theta_N(\phi^{\Delta *}(\mathcal{O}_{\Delta^{\mathbf{n}}_M}(d)), \phi'^*(\mathcal{O}_{M^{\mathbf{n}-1}}(d)))$ is provided by naturality of θ_M from Proposition 3.1.1. This shows that the canonical morphism $\phi^*(\mathcal{O}_M(d)) \to \mathcal{O}_N(d)$ is an isomorphism. This shows c). By induction, we are done. $\square$

Definition 3.2.3. Let M be a polyhedral set, and $\mathbf{n} \geq -1$ an integer. Then we define $T^{\mathbf{n}}_M = \mathbf{P}(M^{\mathbf{n}}) \backslash \mathbf{P}(M^{\mathbf{n}-1})$. With Definition 2.8.1 in mind, Proposition B.2.1 implies that $\alpha_M : \Delta^{\mathbf{n}}_M \to M^{\mathbf{n}}$ induces an isomorphism $\alpha_M^{-1}(T^{\mathbf{n}}_M) \xrightarrow{\cong} T^{\mathbf{n}}_M$. But the ideal sheaf of $\mathbf{P}(\Delta^{\mathbf{n}}_M) \backslash \alpha_M^{-1}(T^{\mathbf{n}}_M)$ in $\mathbf{P}(\Delta^{\mathbf{n}}_M)$ is equal to the ideal sheaf of $\mathbf{P}(\partial \Delta^{\mathbf{n}}_M)$ by Lemma B.3.2 a). Thus α_M restricts to an isomorphism $\alpha_M : T^{\mathbf{n}}_{\Delta^{\mathbf{n}}_M} \to T^{\mathbf{n}}_M$, which furthermore is natural in M.

Lemma 3.2.4. *Let $\phi : M \to N$ be a morphism of polyhedral sets. Then,*

a) *the scheme-theoretic image $\phi(\mathbf{P}(M))$ in $\mathbf{P}(N)$ is equal to $\mathbf{P}(\phi(M))$;*

b) *if $Z \subseteq N$ is a polyhedral subset, then the preimage $\phi^{-1}(\mathbf{P}(Z))$ is equal to $\mathbf{P}(\phi^{-1}(Z))$ as closed subsets of $\mathbf{P}(M)$.*

Proof. a) The morphism $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ factors as a surjective morphism $\mathbf{P}(M) \to \mathbf{P}(\phi(M))$ followed by a closed immersion $\mathbf{P}(\phi(M)) \to \mathbf{P}(N)$. Since the scheme-theoretic image of a reduced scheme is reduced, it follows that $\phi(\mathbf{P}(M))$ and $\mathbf{P}(\phi(M))$ are equal as closed subschemes of $\mathbf{P}(N)$. b) We will show that $\mathbf{P}(\phi^{-1}(Z)) = \phi^{-1}(\mathbf{P}(Z))$ set-theoretically. We proceed by induction on the dimension of N, noting that equality is immediate for the base case of $N = \underline{\varnothing}$. Let N be $\mathbf{n}$-dimensional, and consider the restricted morphism $\phi : T^{\mathbf{n}}_M \to T^{\mathbf{n}}_N$. But note that $\mathbf{P}(Z) \cap T^{\mathbf{n}}_N = T^{\mathbf{n}}_Z$, and that $\mathbf{P}(N^{\mathbf{n}-1}) \cap \mathbf{P}(Z) = \mathbf{P}(Z^{\mathbf{n}-1})$ by Proposition 3.2.1 c). Hence $\phi^{-1}(\mathbf{P}(Z)) = \phi^{-1}(T^{\mathbf{n}}_Z) \cup \phi^{-1}(\mathbf{P}(Z^{\mathbf{n}-1}))$. By the inductive hypothesis, $\phi^{-1}(\mathbf{P}(Z^{\mathbf{n}-1})) = \mathbf{P}(\phi^{-1}(Z^{\mathbf{n}-1}))$ as a subset of $\mathbf{P}(M^{\mathbf{n}-1})$, hence of $\mathbf{P}(M)$ as well. Via the isomorphisms $\alpha_M : T^{\mathbf{n}}_{\Delta^{\mathbf{n}}_M} \to T^{\mathbf{n}}_M$ and $\alpha_Z : T^{\mathbf{n}}_{\Delta^{\mathbf{n}}_Z} \to T^{\mathbf{n}}_Z$, $\phi^{-1}(T^{\mathbf{n}}_Z)$ is easily seen to be equal to $T^{\mathbf{n}}_{\phi^{-1}(Z)}$. We conclude that $\mathbf{P}(\phi^{-1}(Z)) = \phi^{-1}(\mathbf{P}(Z))$ as closed subsets of $\mathbf{P}(M)$. □

3.3 An equivalence between groupoids of pairs

Lemma 3.3.1. *Let M be a polyhedral set, and let S be the set of canonical representatives of facets of M. Define $\overline{M} = \coprod_{s \in S} \Delta^s$. Then the induced morphism $\nu : \mathbf{P}(\overline{M}) \to \mathbf{P}(M)$ is the normalization of $\mathbf{P}(M)$. In particular, if M is* pure *of dimension $\mathbf{n}$, meaning that the facets are all $\mathbf{n}$-dimensional, then $\alpha_M : \mathbf{P}(\Delta^{\mathbf{n}}_M) \to \mathbf{P}(M)$ is the normalization of $\mathbf{P}(M)$.*

Proof. We will first show that if M is an irreducible polyhedral set with canonical facet s, then $\zeta^s_M : \mathbf{P}(\Delta^s) \to \mathbf{P}(M)$ is the normalization of M. Since $\mathbf{P}(\Delta^s)$ is normal, ζ^s_M is finite and both $\mathbf{P}(\Delta^s)$ and $\mathbf{P}(M)$ contain a copy of $T^{\dim(s)}_M$ (see Definition 3.2.3) as an open subset, the claim immediately follows. For a general polyhedral set M, the irreducible components of $\mathbf{P}(M)$ are on the form $\mathbf{P}(M^s)$ for canonical facets s of M, and the result follows. □

If A is a reduced Noetherian ring such that its normalization $A \to \overline{A}$ is finite (i.e. A is a Noetherian *Mori ring*), then the *seminormalization* of A in $\overline{A}$ is the largest subring $A' \subseteq \overline{A}$ containing A such that the following conditions hold.

1) for all $p \in \operatorname{Spec} A$, there exists at most one $q \in \operatorname{Spec} A'$ over p (i.e. such that $q \cap A = p$),

2) the canonical homomorphism $k(p) \to k(q)$ is an isomorphism.

A is a *seminormal* ring it it is equal to its seminormalization. A Noetherian normal ring is automatically seminormal. See [Tra70], [Swa80] and [GT80] for further details. For our purposes, the following fact will be useful:

Lemma 3.3.2 ([Tra70, Lemma 1.3])**.** *Let A be a seminormal ring, and let $\overline{A}$ be its normalization. Then the conductor ideal $\operatorname{Ann}_A(\overline{A}/A)$ of A in $\overline{A}$ is radical.*

A locally Noetherian scheme is called called seminormal if any affine open subscheme is the spectrum of a seminormal ring.

Lemma 3.3.3. *Let M be a polyhedral set. Then $\mathbf{P}(M)$ a seminormal scheme.*

Proof. By [Swa80, Corollary 3.3], an arbitrary limit of seminormal rings is seminormal. Now, $\Gamma(M)$ can be written as a limit of rings on the form $\Gamma(\Delta^P)$ as in the proof of Lemma 2.8.3. Since each $\mathbf{P}(\Delta^P)$ is a normal scheme, its local affine pieces $\Gamma(\Delta^P)_{(f)}$ are normal as well. Thus the local affine pieces of $\mathbf{P}(M)$ can be written as a limit of normal rings, and is therefore a seminormal scheme. □

Definition 3.3.4. Fix an $\mathbf{n}$-dimensional torus $T \cong \mathbf{G}_m^{\mathbf{n}}$ over a field k. Let $\operatorname{Pairs}(T)$ be the category of pairs (X, L), where

1) X is a projective, seminormal scheme over a field k of dimension $\mathbf{n}$;

2) L is an ample invertible sheaf on X;

3) T acts on the normalization $\overline{X}$ with finitely many orbits (then $\overline{X}$ is necessarily a disjoint union $\coprod_j Z_j$ of toric varieties, where the torus acting on Z_j is the quotient of T with the stabilizer of its generic point);

4) If $\mathcal{O} \subseteq \overline{X}$ is an orbit, then $\nu|_{\mathcal{O}} : \mathcal{O} \to X$ is injective, where $\nu : \overline{X} \to X$ denotes the normalization morphism;

5) If $C \subseteq \overline{X}$ is an orbit closure, then $C \to \nu(C)$ is the normalization of $\nu(C)$. We moreover assume that the automorphisms between components over $\nu(C)$ are torus-equivariant.

A morphism $(X, L) \to (X', L')$ of pairs is a pair (f, g) where $f : X \to X'$ is a finite surjection such that $\overline{f} : \overline{X} \to \overline{X'}$ is T-equivariant and an isomorphism restricted componentwise, and g is an isomorphism $f^*L' \to L$.

Let $C(X) = \overline{X} \times_X X_{nn}$ denote the conductor locus. Then X fits into a pushout diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\nu"] & X \\ C(X) \arrow[u] \arrow[r] & X_{nn} \arrow[u] \end{tikzcd} \tag{3.14}$$

This can be seen locally: Let A be a seminormal ring. Then ideal of the conductor locus is the largest common ideal I of A and $\overline{A}$. Hence

$$0 \to A \to \overline{A} \times A/I \to \overline{A}/I$$

is exact, meaning that 3.14 is a pushout diagram for $X = \mathrm{Spec}(A)$. The projective case easily follows from Proposition B.2.1.

Lemma 3.3.5. *If X is a scheme satisfying the above conditions, then $C(X)$ is torus-invariant.*

Proof. It follows from Item 5) that $\nu^{-1}\nu(C)$ is torus-invariant whenever C is an orbit closure. Indeed, any component C' of it must satisfy $\nu(C') = \nu(C)$. The uniquely induced isomorphism $C \xrightarrow{\cong} C'$ is per assumption torus-equivariant, which implies torus-invariance of C'.

Let S be the set of orbit closures in $\overline{X}$, and let Z be the union of those $C \in S$ for which there exists a different $C' \in S$ such that $\nu(C') = \nu(C)$. We will show that $C(X) = Z$. To see that $Z \subseteq C(X)$, suppose that $x \in \overline{X} \backslash C(X)$. Then there is an open subset $U \subseteq X$ containing $\nu(x)$ such that $\nu|_{\nu^{-1}(U)} : \nu^{-1}(U) \to U$ is an isomorphism. Assume that $x \in Z$. Then there exists a pair of distinct orbit closures $C, C' \in S$ such that $x \in C$ and $\nu(C) = \nu(C')$. But then $U \cap \nu(C) = U \cap \nu(C')$ is a dense open subset of $\nu(C) = \nu(C')$. Thus we may choose points $y \in \mathcal{O} \subseteq C$ and $y' \in \mathcal{O}' \subseteq C'$ in their respective open dense orbits such that $\nu(y) = \nu(y') \in U$. But since C and C' are different, $\mathcal{O}$ and $\mathcal{O}'$ must be disjoint. This violates Item 4).

Conversely, suppose that $x \in C(X)$. We will use the following fact, which will be proved later: if A is a seminormal k-algebra of finite type such that $\mathrm{Spec}(\overline{A}) \to \mathrm{Spec}(A)$ is bijective and induces an isomorphism of residue fields of closed points, then A is normal. Note that the residue fields of the closed points of $\overline{X}$ are isomorphic to k. Hence $\overline{X} \to X$induces an isomorphism of residue fields of closed points. Let x be such that $\nu(x) \in C(X)$. By the fact above, there exists $y \neq x$ such that $\nu(y) = \nu(x)$. Let C_x denote the closure of the orbit of x, and similarly for y. Now, $y \in \nu^{-1}\nu(C_x)$, which implies that $C_y \subseteq \nu^{-1}\nu(C_x)$ by torus-invariance. Hence $\nu(C_x) = \nu(C_y)$. If $C_x = C_y$, then $x = y$ by Item 4), which is a contradiction. Hence $C_x \neq C_y$, which implies that $x \in Z$. To prove the fact we mentioned, let m be a maximal ideal of A, and consider the homomorphism $A_m \to \overline{A}_m$. In [Tra70], seminormality of A means that $A = A + m'\overline{A}_m$, where m' is the unique maximal ideal of $\overline{A}$ lying over m. Hence $m = m'$, which by Nakayama's lemma implies that $A_m \cong \overline{A}_m$. This works for all maximal ideals, so $A = \overline{A}$. $\square$

Proposition 3.3.6. *There is a natural bijective correspondence between isomorphisms $(f, \lambda_f) : P \to Q$ in $\mathcal{P}$ and pairs (ψ, ι), where $\psi : X_P \xrightarrow{\cong} X_Q$ is a torus-equivariant isomorphism of projective toric varieties, and $\iota : \psi^*\mathcal{O}_{X_Q}(D_Q) \xrightarrow{\cong} \mathcal{O}_{X_P}(D_P)$ is an isomorphism of line bundles.*

Proof. First, let $\psi : X_P \to X_Q$ be a toric isomorphism such that $\psi^*\mathcal{O}_{X_Q}(D_Q) \cong \mathcal{O}_{X_P}(D_P)$. In the notation of [CLS11, Theorem 3.3.4], ψ corresponds to an isomorphism of lattices $\overline{\psi} : N_1 \to N_2$, where for each cone $\sigma_R \subseteq \Sigma_P$, $\overline{\psi}_{\mathbb{R}}(\sigma_R) = \sigma_{R'}$ for some cone $\sigma_{R'} \subseteq \Sigma_Q$. Consider the dual isomorphism $\overline{\psi}^\vee : M_2 \to M_1$. In the notation of [CLS11, Theorem 6.2.7], there exists a unique torus-invariant cartier divisor D' on X_P such that $\psi_{D'} = \psi_{D_Q}\overline{\psi}$ and $\mathcal{O}_{X_P}(D') \cong \psi^*\mathcal{O}_{X_Q}(D_Q)$. The cartier data for D_P is $\{v\}_{v\in P}$ (ranging over the vertices $v \in P$), which will allow us to compute the cartier data for D'. Let $v \in P$ be a vertex, and let $\sigma_{v'}$ be the image of σ_v under $\overline{\psi}_{\mathbb{R}}$. Then we know that for all $u \in \sigma_v$, $\psi_{D'}(u) = \psi_{D_Q}(\overline{\psi})$, i.e. $\langle m_{\sigma_v}, u\rangle = \langle v', \overline{\psi}(u)\rangle = \langle \overline{\psi}^\vee(v'), u\rangle$. It follows that $m_{\sigma_v} = \overline{\psi}^\vee(v')$. Next, since $D' \sim D_P$, we know that $D' - D_P = \operatorname{div}(\chi^m)$. Hence $\langle \overline{\psi}^\vee(v') - v, u\rangle = \langle m, u\rangle$ for all $u \in \sigma(v)$ (here we're using that $\{m\}_{v\in P}$ is cartier data for χ^m). It follows that $\overline{\psi}^\vee(v') = v + m$. Hence $\overline{\psi}^\vee : M_2 \to M_1$ maps Q isomorphically to the translated polytope $P + m$. Thus $f = \overline{\psi}^\vee - m$ defines a affine transformation $P \to Q$.

Next, note that a torus-equivariant isomorphism $\psi : X_P \to X_Q$ is exactly the same as a toric isomorphism $\psi' : X_P \to X_Q$ composed with multiplication from the torus $\lambda : X_P \to X_P$ by some element $\lambda \in T$, which corresponds to some homomorphism $\lambda_0 : \mathbb{Z}^n \to \mathbf{G}_m$. Moreover, the isomorphism $\iota : \psi^*\mathcal{O}_{X_Q}(D_Q) \to \mathcal{O}_{X_P}(D_P)$ induces an isomorphism $\mathcal{O}_{X_Q}(D_Q) \to \psi_*\mathcal{O}_{X_P}(D_P)$, which is just $\psi^\sharp$ composed with multiplication $t : \mathcal{O}_{X_Q}(D_Q) \to \mathcal{O}_{X_Q}(D_Q)$ by some from $t \in \mathbf{G}_m$. Let $f : P \to Q$ be the affine transformation induced by ψ', and define $\lambda_f : \mathbb{Z}^{\dim(P)+1} \to \mathbf{G}_m$ by $\lambda_f(m, d) = t^d\lambda_0(m)$. The isomorphism $(f, \lambda_f) : P \to Q$ in $\mathcal{P}$ induces an isomorphism $\Gamma(\Delta^Q) \to \Gamma(\Delta^P)$, which in turn induces the isomorphism $\psi : X_P \to X_Q$ and the isomorphism $\iota : \psi^*\mathcal{O}_{X_Q}(D_Q) \to \mathcal{O}_{X_P}(D_P)$. This provides the bijective correspondence. □

Let $\mathcal{C}^{\mathbf{n}}_{fs}$ denote the subcategory of $\mathcal{C}_f$ consisting of $\mathbf{n}$-dimensional finite polyhedral sets, where morphisms $\phi : M \to N$ are required to be surjective. If $\phi : M \to N$ is a surjection, then $\phi : \mathbf{P}(M) \to \mathbf{P}(N)$ is finite, surjective, and induces isomorphisms over components on normalizations. Furthermore, the homomorphism $\Gamma(N) \to \Gamma(M)$ induces an isomorphism $\iota_\phi : \phi^*\mathcal{O}_N(1) \xrightarrow{\cong} \mathcal{O}_M(1)$. Hence there is a well-defined functor $\mathbf{P}_{\mathrm{p}} : \mathcal{C}^{\mathbf{n}}_{fs} \to \mathrm{Pairs}(T)$, given by $M \mapsto (\mathbf{P}(M), \mathcal{O}_M(1))$, and $\phi \mapsto (\mathbf{P}(\phi), \iota_\phi)$.

Theorem 3.3.7. *The functor* $\mathbf{P}_{\mathrm{p}} : \mathcal{C}^{\mathbf{n}}_{fs} \to \mathrm{Pairs}(T)$ *is an equivalence of categories.*

Proof. We will show essential surjectivity, then fullness and faithfulness by induction on the dimension $\mathbf{n}$ of T (the base case of $\mathbf{n} = -1$ is trivial). So let $(X, L) \in \mathrm{Pairs}(T)$. Then $\overline{X}$ is a disjoint union of toric varieties $\coprod_j Z_j$, and via the ample line bundles $L|_{Z_j}$, we may assume that $Z_j = \mathbf{P}(\Delta^{P_j})$ and that $L|_{\mathbf{P}(\Delta^{P_j})} \cong \mathcal{O}_{\Delta^{P_j}}(1)$. So choose isomorphisms $\psi_j : L|_{Z_j} \to \mathcal{O}_{\Delta^{P_j}}(1)$. Since $C(X)$ is torus-invariant, it is necessarily on the form $\coprod_j \mathbf{P}(N_j)$ for polyhedral subsets $N_j \subseteq \Delta^{P_j}$. Moreover, the isomorphisms ψ_j induces isomorphisms $\psi'_j : L_{\mathbf{P}(N_j)} \to \mathcal{O}_{N_j}(1)$. Now, any orbit closure of $\overline{X}$ is on the form $\mathbf{P}(\Delta^Q)$ for some face $Q \subseteq P_j$ and some j. Consider the set S of maximal orbit closures in $\coprod \mathbf{P}(N_j)$. These cover X_{nn}, and the disjoint union of the orbit closures of a subset of S forms the normalization $\overline{X_{nn}}$ of X_{nn} by Item 5). There is a naturally induced action of T on $\overline{X_{nn}}$. This induces a commutative square

$$\begin{tikzcd} \overline{X} \arrow[r] & X \\ \overline{X_{nn}} \arrow[u, "\pi"] \arrow[r] & X_{nn}, \arrow[u] \end{tikzcd}$$

where $\pi : \overline{X_{nn}} \to \overline{X}$ is torus-equivariant. We may write $\overline{X_{nn}} = \coprod_i \mathbf{P}(\Delta^{Q_i})$, and the torus acting on $\mathbf{P}(\Delta^{Q_i})$ is on the form $T/\operatorname{Stab}(\eta_j)$, where η_j is the generic point of $\mathbf{P}(\Delta^{Q_i})$. Let T' be an $(\mathbf{n}-1)$-dimensional torus, and choose some homomorphism $\mu : T' \to T$ inducing a surjection $T' \to T/\operatorname{Stab}(\eta_j)$ for each j. Via the action of T' on $\overline{X_{nn}}$, the pair $(X_{nn}, L|_{X_{nn}})$ now satisfies all conditions above, except possibly seminormality. That is seen as follows: By [Kol96, Theorem 7.2.5], the morphism $C(X) \to X_{nn}$ is seminormal. Since $C(X)$ is seminormal, the morphism $\overline{C(X)} \to X_{nn}$ is seminormal as well. But the normalization $\overline{X_{nn}} \to X_{nn}$ clearly factors through $\overline{C(X)} \to X_{nn}$ via a seminormal morphism $\overline{X_{nn}} \to \overline{C(X)}$, so we conclude that $\overline{X_{nn}} \to X_{nn}$ is seminormal. Hence X_{nn} is seminormal as well. By the inductive hypothesis, there exists a finite $(\mathbf{n}-1)$-dimensional polyhedral set M' such that the pair $(X_{nn}, L|_{X_{nn}})$ is isomorphic to $(\mathbf{P}(M'), \mathcal{O}_{M'}(1))$. Recall from Lemma 3.3.1 that the normalization of $\mathbf{P}(M')$ is on the form $\mathbf{P}(\overline{M'})$.

Consider the morphism $C(X) \to \mathbf{P}(M')$. Note that the action of T' on $\overline{X_{nn}}$ can be extended to an action on $\overline{C(X)}$ by Item 5) (which is compatible with the action of T). Thus the induced morphism $\overline{\phi} : \overline{C(X)} \to \mathbf{P}(\overline{M'})$ is T'-equivariant, and also an isomorphism over components. There is also a canonical isomorphism $\psi' : \phi^* \mathcal{O}_{M'}(1) \to L|_{C(X)}$. By the inductive hypothesis, the morphism $(\phi, \psi') : (C(X), L|_{C(X)}) \to (\mathbf{P}(M'), \mathcal{O}_{M'}(1))$ corresponds to a surjection $\phi : \coprod_j N_j \to M'$. We define M as the following pushout:

$$\begin{tikzcd} \coprod_j \Delta^{P_j} \arrow[r] & M \\ \coprod_j N_j \arrow[u] \arrow[r, "\phi"] & M' \arrow[u] \end{tikzcd}$$

This forms a pushout square

$$\begin{tikzcd} \coprod_j \mathbf{P}(\Delta^{P_j}) \arrow[r] & \mathbf{P}(M) \\ \coprod_j \mathbf{P}(N_j) \arrow[u] \arrow[r, "\phi"] & \mathbf{P}(M') \arrow[u] \end{tikzcd}$$

and by comparing with 3.14, there is a unique induced isomorphism $f : X \to \mathbf{P}(M)$. By the choices involved, the diagram $L|_{\overline{X}} \to L|_{C(X)} \leftarrow L|_{X_{nn}}$ of line bundles on X is isomorphic to the diagram $\mathcal{O}_{\coprod \Delta^{P_j}}(1) \to \mathcal{O}_{\coprod N_j}(1) \leftarrow \mathcal{O}_{M'}(1)$, hence by taking pullbacks there is a unique induced isomorphism $L \to \mathcal{O}_M(1)$. We conclude that the pairs (X, L) and $(\mathbf{P}(M), \mathcal{O}_M(1))$ are isomorphic.

For fullness, let $(f, \iota) : (\mathbf{P}(M_1), \mathcal{O}_{M_1}(1)) \to (\mathbf{P}(M_2), \mathcal{O}_{M_2}(1))$ be a morphism of pairs. Then $\overline{f} : \mathbf{P}(\overline{M_1}) \to \mathbf{P}(\overline{M_1})$ is surjective, T-equivariant, and induces an isomorphism on components. Let $\nu_i : \overline{M_i} \to M_i$ for $i = 1, 2$ denote the morphisms from Lemma 3.3.1. Via the induced isomorphism $\overline{\iota} = \mathbf{P}(\nu_2)^* \iota : \overline{f}^* \mathcal{O}_{\overline{M_2}}(1) \to \mathcal{O}_{\overline{M_1}}(1)$, the pair $(\overline{f}, \overline{\iota})$ is induced by some morphism $\overline{\phi} : \overline{M_1} \to \overline{M_2}$ by Proposition 3.3.6. Let s be any face of M_1, and let t be a face of $\overline{M_1}$ mapping to s. Then we may write $\zeta^s_M = \nu_1 \zeta^t_{\overline{M_1}}$. Hence

$$
\begin{aligned}
(f, \iota)\mathbf{P}_{\mathrm{p}}(\zeta^s_{M_1}) &= (f, \iota)\mathbf{P}_{\mathrm{p}}(\nu_1)\mathbf{P}_{\mathrm{p}}(\zeta^t_{\overline{M_1}}) = \mathbf{P}_{\mathrm{p}}(\nu_2)(\overline{f}, \overline{\iota})\mathbf{P}_{\mathrm{p}}(\zeta^t_{\overline{M_1}}) \\
&= \mathbf{P}_{\mathrm{p}}(\nu_2)\mathbf{P}_{\mathrm{p}}(\overline{\phi})\mathbf{P}_{\mathrm{p}}(\zeta^t_{\overline{M_1}}) = \mathbf{P}_{\mathrm{p}}(\zeta^{\nu_2\overline{\phi}(t)}_{M_2}).
\end{aligned}
$$

So define $\phi : M_1 \to M_2$ by $\phi(s) = \nu_2\overline{\phi}(t)$, where t is some face of $\overline{M_1}$ such that $\nu_1(t) = s$. This is well-defined, since $\mathbf{P}_{\mathrm{p}}(\zeta^{t_1}_{M_2}) = \mathbf{P}_{\mathrm{p}}(\zeta^{t_2}_{M_2})$ if and only if $t_1 = t_2$ (by Proposition 3.3.6). It is easily seen that ϕ defines a morphism of polyhedral sets. Now, we have that $(f, \iota)\mathbf{P}_{\mathrm{p}}(\zeta^s_{M_1}) = \mathbf{P}_{\mathrm{p}}(\phi)\mathbf{P}_{\mathrm{p}}(\zeta^s_{M_1})$ for all faces s. Since $\mathbf{P}(M_1) = \varinjlim \mathbf{P}(\Delta^s)$, it follows that $\mathbf{P}(\phi) = f$ and $\iota = \iota_\phi$, hence that $(f, \iota) = \mathbf{P}_{\mathrm{p}}(\phi)$.

For faithfulness, suppose that $\phi, \phi' : M_1 \to M_2$ is a pair of morphisms such that $\mathbf{P}_{\mathrm{p}}(\phi) = \mathbf{P}_{\mathrm{p}}(\phi')$. Let s be a face of M_1. Then $\mathbf{P}_{\mathrm{p}}(\phi)\mathbf{P}_{\mathrm{p}}(\zeta^s_{M_1}) = \mathbf{P}_{\mathrm{p}}(\phi')\mathbf{P}_{\mathrm{p}}(\zeta^s_{M_1})$, so that $\mathbf{P}_{\mathrm{p}}(\zeta^{\phi(s)}_{M_2}) = \mathbf{P}_{\mathrm{p}}(\zeta^{\phi'(s)}_{M_2})$. By Proposition 3.3.6 again, we must have $\phi(s) = \phi'(s)$. We conclude that $\phi = \phi'$. $\square$

Corollary 3.3.8. *There is an equivalence of groupoids*

$$\mathbf{P}_{\mathrm{p}} : \mathrm{Isom}(\mathcal{C}_f) \to \mathrm{Isom}(\mathrm{Pairs}(T)). \tag{3.15}$$

I.e., the groupoid of pairs isomorphic to (X, L) is equivalent to the groupoid of finite polyhedral sets isomorphic to M, where $(\mathbf{P}(M), \mathcal{O}_M(1))$ corresponds to the pair (X, L).

3.4 Structure sheaf cohomology

We will use Section 2.5 as a general reference for the notation which follows. Consider the function $\deg : C^M_0(k) \to k$, mapping each generator of $C^M_0(k)$ to 1. This induces a natural homomorphism $\deg_M : k \to H^0(M; k)$, which is an isomorphism whenever M is connected and non-trivial. In particular, $\deg_{\Delta^P}$ is an isomorphism for each $P \in \mathcal{P}$. Note further that $H^0(\Delta^P; k)$ is the only non-zero cohomology module of Δ^P, which follows from the fact that $|\Delta^P| \cong \mathbb{D}^{\dim(P)}$ is contractible. Let $j_M : k \to H^0(M, \mathcal{O}_M)$ denote the natural homomorphism induced by the morphism from the constant presheaf to the structure sheaf: $\underline{k} \to \mathcal{O}_M$. Similarly, j_M is an isomorphism whenever M is connected and non-trivial, so in particular j_{Δ^P} is an isomorphism for each $P \in \mathcal{P}$. Moreover, $H^0(\Delta^P, \mathcal{O}_{\Delta^P})$ is the only non-zero cohomology module of $\mathcal{O}_{\Delta^P}$ by [CLS11, Theorem 9.13 b)].

Theorem 3.4.1. *There are natural isomorphisms* $\eta^i_M : H^i(M; k) \to H^i(M, \mathcal{O}_M)$ *of contravariant functors* $\mathcal{C}_f \to \mathrm{Vec}_k$ *for each integer* $i \geq 0$*.*

Proof. We will define η^i_M inductively on the dimension of M. If $M = \underline{\varnothing}$, then η^i_M is given as the trivial isomorphism for all integers $i \geq 0$. Let $\mathbf{n} \geq 0$, and assume that each η^i_M is defined for and natural in $(\mathbf{n}-1)$-dimensional M, and such that the diagram

$$\begin{tikzcd} & k \arrow[dl, "\deg_M"'] \arrow[dr, "j_M"] & \\ H^0(M; k) \arrow[rr, "\eta^0_M"] & & H^0(M, \mathcal{O}_M) \end{tikzcd} \tag{3.16}$$

commutes. Note that (3.16) uniquely determines isomorphisms $\eta^i_{\Delta^P}$ for all $P \in \mathcal{P}$ and $i \geq 0$. Let M be $\mathbf{n}$-dimensional, and consider the canonical pushout square (2.5) for M of level $\mathbf{n}$:

$$\begin{tikzcd} \Delta^{\mathbf{n}}_M \arrow[r, "\alpha_M"] & M \\ \partial\Delta^{\mathbf{n}}_M \arrow[u, "\iota_{\Delta_M}"] \arrow[r, "\alpha'_M"] & M^{\mathbf{n}-1}. \arrow[u, "\iota_M"'] \end{tikzcd}$$

Since $\Delta^{\mathbf{n}}_M = \coprod_{s \in J^{\mathbf{n}}_M} \Delta^s$, the morphisms $\zeta^s_{\Delta^{\mathbf{n}}_M} : \Delta^s \to \Delta^{\mathbf{n}}_M$ splits the cohomology groups $H^0(\Delta^{\mathbf{n}}_M; k)$ and $H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M})$ into direct sums $\bigoplus_{s \in J^{\mathbf{n}}_M} H^0(\Delta^s; k)$ and $\bigoplus_{s \in J^{\mathbf{n}}_M} H^0(\Delta^s, \mathcal{O}_{\Delta^s})$ respectively. We define $\eta^0_{\Delta^{\mathbf{n}}_M} : H^0(\Delta^{\mathbf{n}}_M; k) \to H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M})$ as the unique k-linear homomorphism making the diagram

$$\begin{tikzcd} H^0(\Delta^{\mathbf{n}}_M; k) \arrow[r, "\cong"] \arrow[d, "\eta^0_{\Delta^{\mathbf{n}}_M}"] & \bigoplus_{s \in J^{\mathbf{n}}_M} H^0(\Delta^s; k) \arrow[d, "\bigoplus_{s \in J^{\mathbf{n}}_M} \eta^0_{\Delta^s}"] \\ H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M}) \arrow[r, "\cong"] & \bigoplus_{s \in J^{\mathbf{n}}_M} H^0(\Delta^s, \mathcal{O}_{\Delta^s}) \end{tikzcd} \tag{3.17}$$

commute. It is clear that $\eta^0_{\Delta^{\mathbf{n}}_M}$ satisfies commutativity of (3.16). Since $\partial\Delta^{\mathbf{n}}_M$ similarly splits as $\coprod_{s \in J^{\mathbf{n}}_M} \partial\Delta^s$, and (3.16) commutes for $\eta^0_{\partial\Delta^{\mathbf{n}}_M}$ by the inductive hypothesis, it follows immediately that the diagram

$$\begin{tikzcd} H^0(\Delta^{\mathbf{n}}_M; k) \arrow[r, "\eta^0_{\Delta^{\mathbf{n}}_M}"] \arrow[d, "\iota^*_{\Delta_M}"] & H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M}) \arrow[d, "\iota^\sharp_{\Delta_M}"] \\ H^0(\partial\Delta^{\mathbf{n}}_M; k) \arrow[r, "\eta^0_{\partial\Delta^{\mathbf{n}}_M}"] & H^0(\partial\Delta^{\mathbf{n}}_M, \mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}) \end{tikzcd} \tag{3.18}$$

commutes. Via the Milnor square (2.11) of $\Gamma(M)$ of level $\mathbf{n}$, (B.12) yields the exact sequence

$$0 \to \mathcal{O}_M \to \alpha_{M*}\mathcal{O}_{\Delta^{\mathbf{n}}_M} \oplus \iota_{M*}\mathcal{O}_{M^{\mathbf{n}-1}} \to \gamma_{M*}\mathcal{O}_{\partial\Delta^{\mathbf{n}}_M} \to 0, \tag{3.19}$$

where $\gamma_M = \iota_M \alpha'_M$. Consider the partial isomorphism between the long exact sequence induced by (2.8) and the long exact sequences in cohomology for (3.19):

$$\begin{tikzcd}
H^{i-1}(\Delta^{\mathbf{n}}_M;k) \oplus H^{i-1}(M^{\mathbf{n}-1};k) \arrow[r, "(\eta^{i-1}_{\Delta^{\mathbf{n}}_M}{,}\eta^{i-1}_{M^{\mathbf{n}-1}})"] \arrow[d, "\iota^*_{\Delta_M} \oplus -\alpha'^*_M"] & H^{i-1}(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M}) \oplus H^{i-1}(M^{\mathbf{n}-1}, \mathcal{O}_{M^{\mathbf{n}-1}}) \arrow[d, "\iota^\sharp_{\Delta_M} \oplus -\alpha'^\sharp_M"] \\
H^{i-1}(\partial\Delta^{\mathbf{n}}_M;k) \arrow[r, "\eta^{i-1}_{\partial\Delta^{\mathbf{n}}_M}"] \arrow[d] & H^{i-1}(\partial\Delta^{\mathbf{n}}_M, \mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}) \arrow[d] \\
H^{i}(M;k) \arrow[d] & H^{i}(M, \mathcal{O}_M) \arrow[d] \\
H^{i}(\Delta^{\mathbf{n}}_M;k) \oplus H^{i}(M^{\mathbf{n}-1};k) \arrow[r, "(\eta^{i}_{\Delta^{\mathbf{n}}_M}{,}\eta^{i}_{M^{\mathbf{n}-1}})"] \arrow[d, "\iota^*_{\Delta_M} \oplus -\alpha'^*_M"] & H^{i}(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M}) \oplus H^{i}(M^{\mathbf{n}-1}, \mathcal{O}_{M^{\mathbf{n}-1}}) \arrow[d, "\iota^\sharp_{\Delta_M} \oplus -\alpha'^\sharp_M"] \\
H^{i}(\partial\Delta^{\mathbf{n}}_M;k) \arrow[r, "\eta^{i}_{\partial\Delta^{\mathbf{n}}_M}"] & H^{i}(\partial\Delta^{\mathbf{n}}_M, \mathcal{O}_{\partial\Delta^{\mathbf{n}}_M})
\end{tikzcd} \tag{3.20}$$

Commutativity of (3.20) is ensured by the inductive hypothesis and commutativity of (3.18). By exactness of both columns, there is a uniquely determined isomorphism $\eta^i_M : H^i(M;k) \to H^i(M, \mathcal{O}_M)$ for each $i \geq 0$, making the entire diagram commute. This defines η^i_M for all polyhedral sets M of dimension $\mathbf{n}$, and does not conflict with the original definition of $\eta^i_{\Delta^{\mathbf{n}}_M}$. Indeed, since $\alpha_{\Delta^{\mathbf{n}}_M} : \Delta^{\mathbf{n}}_{\Delta^{\mathbf{n}}_M} \to \Delta^{\mathbf{n}}_M$ is an isomorphism, both definitions coincide. To verify that η^0_M fits into (3.16), it is sufficient to note that

$$\begin{tikzcd}
& k \arrow[dl, "\deg_{\Delta^{\mathbf{n}}_M}"'] \arrow[dr, "j_{\Delta^{\mathbf{n}}_M}"] & \\
H^0(\Delta^{\mathbf{n}}_M;k) \arrow[rr, "\eta^0_{\Delta^{\mathbf{n}}_M}"] \arrow[d, "\alpha^*_M"] & & H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M}) \arrow[d, "\alpha^\sharp_M"] \\
H^0(M;k) \arrow[rr, "\eta^0_M"] & & H^0(M, \mathcal{O}_M)
\end{tikzcd} \tag{3.21}$$

commutes. Now, let $\phi : N \to M$ be any morphism. We must show that

$$\begin{tikzcd}
H^i(M;k) \arrow[r, "\eta^i_M"] \arrow[d, "\phi^*"] & H^i(M,\mathcal{O}_M) \arrow[d, "\phi^\sharp"] \\
H^i(N;k) \arrow[r, "\eta^m_N"] & H^i(N,\mathcal{O}_N)
\end{tikzcd} \tag{3.22}$$

commutes for each $i \geq 0$. It is clear that

$$\begin{tikzcd}
H^0(\Delta^{\mathbf{n}}_M;k) \arrow[r, "\eta^0_{\Delta^{\mathbf{n}}_M}"] \arrow[d, "\phi^{\Delta *}"] & H^0(\Delta_M,\mathcal{O}_{\Delta^{\mathbf{n}}_M}) \arrow[d, "\phi^{\Delta\sharp}"] \\
H^0(\Delta^{\mathbf{n}}_N;k) \arrow[r, "\eta^0_{\Delta^{\mathbf{n}}_N}"] & H^0(\Delta_N,\mathcal{O}_{\Delta^{\mathbf{n}}_N})
\end{tikzcd} \tag{3.23}$$

commutes via the defining diagrams (3.17) for $\eta^0_{\Delta^{\mathbf{n}}_M}$ and $\eta^0_{\Delta^{\mathbf{n}}_N}$. By the inductive hypothesis, this yields a pair of homomorphisms of long exact sequences (in simplified notation)

$$\begin{tikzcd}
H^{i-1}(\partial\Delta^{\mathbf{n}}_M) \arrow[r] \arrow[d] & H^{i-1}(\mathcal{O}_{\partial\Delta^{\mathbf{n}}_M}) \arrow[r] \arrow[d] & H^{i-1}(\mathcal{O}_{\partial\Delta^{\mathbf{n}}_N}) \arrow[d] \\
H^i(M) \arrow[r, "\eta^i_M"] \arrow[d] & H^i(\mathcal{O}_M) \arrow[r, "\phi^\sharp"] \arrow[d] & H^i(\mathcal{O}_N) \arrow[d] \\
H^i(\Delta^{\mathbf{n}}_M) \oplus H^i(M^{\mathbf{n}-1}) \arrow[r] & H^i(\mathcal{O}_{\Delta^{\mathbf{n}}_M}) \oplus H^i(\mathcal{O}_{M^{\mathbf{n}-1}}) \arrow[r] & H^i(\mathcal{O}_{\Delta^{\mathbf{n}}_N}) \oplus H^i(\mathcal{O}_{N^{\mathbf{n}-1}})
\end{tikzcd} \tag{3.24}$$

and

$$\begin{tikzcd}
H^{i-1}(\partial\Delta^{\mathbf{n}}_M) \arrow[r] \arrow[d] & H^{i-1}(\partial\Delta^{\mathbf{n}}_N) \arrow[r] \arrow[d] & H^{i-1}(\mathcal{O}_{\partial\Delta^{\mathbf{n}}_N}) \arrow[d] \\
H^i(M) \arrow[r, "\phi^*"] \arrow[d] & H^i(N) \arrow[r, "\eta^i_N"] \arrow[d] & H^i(\mathcal{O}_N) \arrow[d] \\
H^i(\Delta^{\mathbf{n}}_M) \oplus H^i(M^{\mathbf{n}-1}) \arrow[r] & H^i(\Delta^{\mathbf{n}}_N) \oplus H^i(N^{\mathbf{n}-1}) \arrow[r] & H^i(\mathcal{O}_{\Delta^{\mathbf{n}}_N}) \oplus H^i(\mathcal{O}_{N^{\mathbf{n}-1}})
\end{tikzcd} \tag{3.25}$$

Both (3.24) and (3.25) commutes by construction (even if $\dim N < \mathbf{n}$). By the inductive hypothesis and commutativity of (3.23), (3.22) commutes as well. By induction, we are done. $\square$

3.5 Classification of the Picard group

If M is a polyhedral set, we let $\mathrm{Pic}(M)$ denote the Picard group of $\mathbf{P}(M)$.

Definition 3.5.1. Consider the functor $\mathrm{Pic} : \mathcal{P}^{\mathrm{op}} \to \mathrm{Ab}$ given by $P \mapsto \mathrm{Pic}(\Delta^P)$. Let $\mathrm{Deg} : \mathcal{C}_f \to \mathrm{Ab}$ be the left Kan-extension of Pic. For each polyhedral set M, we have $\mathrm{Deg}_M = \varprojlim_{s \in F_M} \mathrm{Pic}(\Delta^s)$, called *the group of degree functions*. Each morphism $\zeta_M^s : \mathbf{P}(\Delta^s) \to \mathbf{P}(M)$ induces a homomorphism $\zeta_M^{s*} : \mathrm{Pic}(M) \to \mathrm{Pic}(\Delta^s)$. This makes $\mathrm{Pic}(M)$ a cone to the diagram, which induces a homomorphism $\omega : \mathrm{Pic}(M) \to \mathrm{Deg}_M$ which is natural in M. For each line bundle $\mathcal{E}$ on $\mathbf{P}(M)$, the associated degree function $\omega_{\mathcal{E}} \in \mathrm{Deg}_M$ is given by $\omega_{\mathcal{E}}(s) = \zeta_M^{s*}\mathcal{E} \in \mathrm{Pic}(\mathbf{P}(\Delta^s))$.

Since Deg is defined as left Kan-extension, it preserves finite colimits. In particular, the sequence

$$1 \to \mathrm{Deg}_{M^{\mathbf{n}}} \to \mathrm{Deg}_{\Delta_M^{\mathbf{n}}} \times \mathrm{Deg}_{M^{\mathbf{n}-1}} \to \mathrm{Deg}_{\partial\Delta_M^{\mathbf{n}}} \tag{3.26}$$

induced by (2.5) is exact for each integer $\mathbf{n} \geq 0$.

Theorem 3.5.2. *Let M be a polyhedral set. Then there exists a natural isomorphism of groups*

$$S_M : \mathrm{Pic}(M) \to \mathrm{Deg}_M \times H^1(M; \mathbf{G}_m) \tag{3.27}$$

such that

$$S_M(\mathcal{O}_M(m)) = (\omega_m, 1) \tag{3.28}$$

for each $m \in \mathbb{Z}$, where $\omega_m \in \mathrm{Deg}_M$ is given by $\omega_m(s) = m[D_{|s|}]$.

Proof. We will define S_M inductively on the dimension $\mathbf{n}$ of M. If M is 0-dimensional, then S_M is given as the trivial isomorphism. So we may assume that $\mathbf{n} \geq 1$. Consider the natural exact sequence

$$\begin{gathered} H^0(\Delta_M^{\mathbf{n}}, \mathcal{O}^*_{\Delta_M^{\mathbf{n}}}) \times H^0(M^{\mathbf{n}-1}, \mathcal{O}^*_{M^{\mathbf{n}-1}}) \to H^0(\partial\Delta_M^{\mathbf{n}}, \mathcal{O}^*_{\partial\Delta_M^{\mathbf{n}}}) \to \mathrm{Pic}(M) \to \\ \mathrm{Pic}(\Delta_M^{\mathbf{n}}) \times \mathrm{Pic}(M^{\mathbf{n}-1}) \to \mathrm{Pic}(\partial\Delta_M^{\mathbf{n}}). \end{gathered} \tag{3.29}$$

from Proposition B.3.5. Via (2.8), $H^1(M; \mathbf{G}_m)$ similarly and naturally fits inside the exact sequence

$$\begin{gathered} H^0(\Delta_M^{\mathbf{n}}, \mathbf{G}_m) \times H^0(M^{\mathbf{n}-1}, \mathbf{G}_m) \to H^0(\partial\Delta_M^{\mathbf{n}}, \mathbf{G}_m) \to H^1(M; \mathbf{G}_m) \to \\ H^1(\Delta_M^{\mathbf{n}}; \mathbf{G}_m) \times H^1(M^{\mathbf{n}-1}; \mathbf{G}_m) \to H^1(\partial\Delta_M^{\mathbf{n}}; \mathbf{G}_m). \end{gathered} \tag{3.30}$$

Since $H^1(\Delta_M^{\mathbf{n}}; \mathbf{G}_m) = 0$, adjoining (3.30) and (3.26) yields a natural exact sequence

$$\begin{gathered} H^0(\Delta_M^{\mathbf{n}}, \mathbf{G}_m) \times H^0(M^{\mathbf{n}-1}, \mathbf{G}_m) \to H^0(\partial\Delta_M^{\mathbf{n}}, \mathbf{G}_m) \to \mathrm{Deg}_M \times H^1(M; \mathbf{G}_m) \to \\ \mathrm{Deg}_{\Delta_M^{\mathbf{n}}} \times \mathrm{Deg}_{M^{\mathbf{n}-1}} \times H^1(M^{\mathbf{n}-1}; \mathbf{G}_m) \to \mathrm{Deg}_{\partial\Delta_M^{\mathbf{n}}} \times H^1(\partial\Delta_M^{\mathbf{n}}; \mathbf{G}_m). \end{gathered} \tag{3.31}$$

Analogously with the proof of Theorem 3.4.1, one obtains a natural isomorphism $H^0(M, \mathcal{O}_M^*) \to H^0(M, \mathbf{G}_m)$. By the inductive hypothesis, S is well-defined and natural in $(\mathbf{n} - 1)$-dimensional polyhedral sets. Via the canonical isomorphism $\mathrm{Pic}(\Delta^P) \xrightarrow{\cong} \mathrm{Deg}_{\Delta^P}$ for $P \in \mathcal{P}$ (which satisfies (3.28)), one may extend S to an isomorphism $S_{\Delta^{\mathbf{n}}_M} : \mathrm{Pic}(\Delta^{\mathbf{n}}_M) \to \mathrm{Deg}_{\Delta^{\mathbf{n}}_M}$ (cf. (3.17)). Consider the partially defined isomorphism between (3.29) and (3.31). If $\mathbf{n} = 1$, then $\mathrm{Pic}(M^{\mathbf{n}-1})$ and $\mathrm{Deg}_{M^{\mathbf{n}-1}} \times H^1(M^{\mathbf{n}-1}; \mathbf{G}_m)$ are trivial, so that the image of $\mathrm{Pic}(M)$ (resp. $\mathrm{Deg}_M \times H^1(M; \mathbf{G}_m)$) is a subgroup of the free abelian group $\mathrm{Pic}(\Delta^{\mathbf{n}}_M)$ (resp. $\mathrm{Deg}_{\Delta^{\mathbf{n}}_M}$). By split-exactness, there is a uniquely induced isomorphism $S_M : \mathrm{Pic}(M) \to \mathrm{Deg}_M \times H^1(M; \mathbf{G}_m)$ making the entire diagram commute. On the other hand, if $\mathbf{n} \geq 2$, then $H^0(\Delta^{\mathbf{n}}_M, \mathbf{G}_m) \to H^0(\partial\Delta^{\mathbf{n}}_M, \mathbf{G}_m)$ (resp. $H^0(\Delta^{\mathbf{n}}_M, \mathbf{G}_m) \to H^0(\partial\Delta^{\mathbf{n}}_M, \mathbf{G}_m)$) is surjective, so that $\mathrm{Pic}(M)$ (resp. $\mathrm{Deg}_M \times H^1(M; \mathbf{G}_m)$) injects into $\mathrm{Pic}(\Delta^{\mathbf{n}}_M) \times \mathrm{Pic}(M^{\mathbf{n}-1})$ (resp. $\mathrm{Deg}_{\Delta^{\mathbf{n}}_M} \times \mathrm{Deg}_{M^{\mathbf{n}-1}} \times H^1(M^{\mathbf{n}-1}; \mathbf{G}_m)$). Thus there is a uniquely induced isomorphism S_M in this case as well. One checks naturality of S by the same method as in the proof of Theorem 3.4.1. Naturality applied to the morphisms $\zeta^s_M : \Delta^s \to M$ moreover verifies (3.28). By induction, we are done. $\square$

We finish this section with the following observation particular to unimodular sets.

Proposition 3.5.3. *If M is a unimodular set, then Deg_M is naturally isomorphic to $H^0(|M|\backslash|M^0|; \mathbb{Z})$. Hence Theorem 3.5.2 takes the form of a natural isomorphism*

$$S_M : \mathrm{Pic}(M) \to H^0(|M|\backslash|M^0|; \mathbb{Z}) \times H^1(M; \mathbf{G}_m).$$

Proof. We will show that there is a natural isomorphism of abelian groups $\mathrm{Deg}_M \to H^0(|M|\backslash|M^0|)$. Consider the category F_M of faces of M, and let $F_M^{\geq 1}$ denote the full subcategory consisting of faces of dimension ≥ 1. Note that since $\mathrm{Pic}(\Delta^n) = \mathbb{Z}$ for each $n \in \Delta$, $n \neq 0$, the group of degrees Deg_M is a direct sum of copies of $\mathbb{Z}$ for each connected component of $F_M^{\geq 1}$. Thus we may write $F_M^{\geq 1} = \coprod_{i \in I} F_M^i$, and $\mathrm{Deg}_M = \bigoplus_{i \in I} \mathbb{Z}$. Similarly, $|M|\backslash|M^0|$ splits into a disjoint union $\coprod_{i \in I} Z_i$, so that $H^0(|M|\backslash|M^0|) \cong \bigoplus_{i \in I} \mathbb{Z}$. One easily observes that the obvious isomorphism $\mathrm{Deg}_M \to H^0(|M|\backslash|M^0|)$ is natural in M. $\square$

3.6 Hilbert polynomial

In this section we will give an expression for the Hilbert polynomial $P_M(d)$ for the face ring $\Gamma(M)$.

Lemma 3.6.1. *Let $V(\sigma_Q)$ denote the orbit-closure $\overline{O(\sigma_Q)}$ corresponding to a face Q of a polytope P (see [CLS11, p. 121]). Then $V(\sigma_Q) = \mathbf{P}(\Delta^Q)$ as closed subschemes of $\mathbf{P}(\Delta^P)$.*

Proof. We have $V(\sigma_Q) = V(I^Q)$ where $I^Q = (\chi^{(m,d)} | (m,d) \notin A_Q)$ by [GKZ08, Chapter 5, Prop 1.9]. Here A_Q is identified with the subgroup of A_P under $A_f : A_Q \to A_P$. Thus, I^Q is equal to the kernel of $\Gamma(\Delta^P) \to \Gamma(\Delta^Q)$, so $V(\sigma_Q)$ is therefore identified with the reduced subscheme $\mathbf{P}(\Delta^Q)$. $\square$

The orbit closure $V(\sigma_F)$ for each facet F of a polytope P defines a torus-invariant Weil-divisor D_F on $\mathbf{P}(\Delta^P)$, and $K_P = -\sum_F D_F$ is the canonical divisor of $\mathbf{P}(\Delta^P)$ [CLS11, Theorem 8.2.3]. If P is given by inequalities $\langle u_F, m\rangle \geq -a_F$, then the divisor $D_P = \sum_F a_F D_F$ is a very ample divisor corresponding to $\mathcal{O}_{\Delta^P}(1)$.

Lemma 3.6.2. *Let M be a finite polyhedral set. Then for each integer $\mathbf{n} \geq 1$, there is an exact sequence*

$$0 \to \bigoplus_{s\in J_M^{\mathbf{n}}} \zeta^s_{M*}\mathcal{O}_{\Delta^s}(K_{|s|}) \to \mathcal{O}_{M^{\mathbf{n}}} \to \iota_{M*}\mathcal{O}_{M^{\mathbf{n}-1}} \to 0. \tag{3.32}$$

Proof. Via the Milnor square (2.11) of $\Gamma(M)$ of level $\mathbf{n}$, (B.11) yields the exact sequence

$$0 \to \alpha_{M*}\mathcal{I} \to \mathcal{O}_{M^{\mathbf{n}}} \to \iota_{M*}\mathcal{O}_{M^{\mathbf{n}-1}} \to 0, \tag{3.33}$$

where $\mathcal{I}$ is the ideal sheaf of $\mathbf{P}(\partial\Delta^{\mathbf{n}}_M)$ in $\mathbf{P}(\Delta^{\mathbf{n}}_M)$. Since $\mathbf{P}(\Delta^{\mathbf{n}}_M) = \coprod_{s\in J^{\mathbf{n}}_M}\mathbf{P}(\Delta^s)$, and since $\mathbf{P}(\partial\Delta^s)$ is a reduced closed subscheme of $\mathbf{P}(\Delta^s)$ corresponding to the union of $V(\sigma_F)$, we have by [CLS11, Proposition 4.0.28] that the ideal $\mathcal{I}$ is isomorphic to the sheaf $\mathcal{O}_{\Delta^s}(K_{|s|})$. Thus (3.33) translates into the exact sequence (3.32). □

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_M$-modules, we define $\Gamma_*(\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}}\Gamma(M, \mathcal{F}(d))$, which is a graded $\Gamma_*(\mathcal{O}_M)$-module.

Proposition 3.6.3. *Let M be an $\mathbf{n}$-dimensional polyhedral set. Then for each integer $d \geq 1$, the natural homomorphism*

$$\Gamma(M)_d \to \Gamma_*(\mathcal{O}_M)_d \tag{3.34}$$

is an isomorphism. For $d = 0$, it is injective and an isomorphism if and only if M is connected. Moreover,

$$\dim_k(\Gamma(M)_d) = \sum_{n=0}^{\mathbf{n}}\sum_{s\in J^n_M} L^*(d|s|) \tag{3.35}$$

for each each integer $d \geq 1$ (recall that $L^*(P)$ denotes the number of interior lattice points of P). For $d = 0$, the dimension is 1.

Proof. Note that $\Gamma(M)$ is positively graded in general, so we have always have $\dim_k(\Gamma(M)_0) = 1$ as required. The dimension of the vector space $\Gamma_*(\mathcal{O}_M)_0 = H^0(M, \mathcal{O}_M)$ is equal to the number of components of $\mathbf{P}(M)$, hence by Proposition 3.2.1 c) equal to the number of components of M. This verifies the statements for $d = 0$.

If $\mathbf{n} = 0$, then M is a simplicial complex. By Proposition 2.7.2, $\Gamma(M) = k[x_v : v \in J^0_M]/(x_v x_w : v \neq w)$, and the formula (3.35) clearly holds. On the other hand, $\mathbf{P}(M) = \coprod_{s\in J^0_M}\operatorname{Spec} k$, so (3.34) is clearly an isomorphism in positive degrees. We proceed by induction on $\mathbf{n}$, and we may assume that $\mathbf{n} \geq 1$. Fix an integer $d \geq 1$. Since $\Gamma(M) \to \Gamma(M^{\mathbf{n}-1})$ is surjective, it follows from the inductive hypothesis that $\Gamma_*(\mathcal{O}_M)_d \to \Gamma_*(\mathcal{O}_{M^{\mathbf{n}-1}})_d$ is surjective. This means that (3.32) induces an exact sequence

$$0 \to \Gamma_*(\mathcal{I})_d \to \Gamma_*(\mathcal{O}_M)_d \to \Gamma_*(\mathcal{O}_{M^{\mathbf{n}-1}})_d \to 0.$$

Here $\mathcal{I} = \widetilde{I}$, where $I = \ker \Gamma(\iota_{\Delta_M})$. Consider the induced homomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I_d \arrow[r] \arrow[d] & \Gamma(M)_d \arrow[r] \arrow[d] & \Gamma(M^{\mathbf{n}-1})_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_*(\mathcal{I})_d \arrow[r] & \Gamma_*(\mathcal{O}_M)_d \arrow[r] & \Gamma_*(\mathcal{O}_{M^{\mathbf{n}-1}})_d \arrow[r] & 0.
\end{tikzcd} \tag{3.36}$$

By the inductive hypothesis, the vertical arrow to the right is an isomorphism. It remains to show that the vertical arrow to the left is an isomorphism. By (B.3) and (B.11), this arrow also appears in the following homomorphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & I_d \arrow[r] \arrow[d] & \Gamma(\Delta_M^{\mathbf{n}})_d \arrow[r] \arrow[d] & \Gamma(\partial\Delta_M^{\mathbf{n}})_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_*(\mathcal{I})_d \arrow[r] & \Gamma_*(\mathcal{O}_{\Delta_M^{\mathbf{n}}})_d \arrow[r] & \Gamma_*(\mathcal{O}_{\partial\Delta_M^{\mathbf{n}}})_d \arrow[r] & 0.
\end{tikzcd} \tag{3.37}$$

By the inductive hypothesis, $\Gamma(\partial\Delta^s)_d \to H^0(\partial\Delta^s, \mathcal{O}_{\partial\Delta^s}(d))$ is an isomorphism for all $s \in J_M^{\mathbf{n}}$. Hence it will suffice to show that $\Gamma(\Delta^s)_d \to H^0(\Delta^s, \mathcal{O}_{\Delta^s}(d))$ is an isomorphism for all $s \in J_M^{\mathbf{n}}$. It is injective since $\Gamma(\Delta^s)$ is reduced, and by [CLS11, Theorem 5.4.8] the dimensions agree. Hence it is an isomorphism, and we conclude that $\Gamma(M)_d \to \Gamma_*(\mathcal{O}_M)_d$ is an isomorphism.

Now, by the inductive hypothesis,

$$\dim_k \Gamma(M^{\mathbf{n}-1})_d = \sum_{n=0}^{\mathbf{n}-1} \sum_{s \in J_M^n} L^*(d|s|).$$

Since $\dim_k \Gamma(M)_d = \dim_k \Gamma(M^{\mathbf{n}-1})_d + \dim_k I_d$, the formula (3.35) follows if we can show that $\dim_k I_d = \sum_{s \in J_M^{\mathbf{n}}} L^*(d|s|)$. For this, it will suffice to show the formula $\dim_k I_d^s = L^*(d|s|)$ for each face $s \in J_M^{\mathbf{n}}$, where $I^s = \ker(\Gamma(\iota_{\Delta^s}))$. Since $\coprod_{F \text{ facet of } P} \Delta^F \to \partial\Delta^P$ is surjective, the homomorphism $\Gamma(\partial\Delta^P) \subseteq \prod \Gamma(\Delta^F)$ is injective. The kernel of $\Gamma(\Delta^P)_d \to \prod_F \Gamma(\Delta^F)_d$ is clearly $\bigoplus_{m \in L^*(dP)} k \cdot \chi^{(m,d)}$, which proves the formula. By induction, we are done. $\square$

Proposition 3.6.3 yields the formula $h^0(\mathcal{O}_M(1)) = \sum_{n=0}^{\mathbf{n}} \sum_{s \in J_M^n} L^*(|s|)$ for the ample line bundle $\mathcal{O}_M(1)$. In particular, if the ≥ 1-dimensional faces of M correspond to polytopes without interior points (e.g. if M is a unimodular set), then $h^0(\mathcal{O}_M(1)) = |J_M^0|$.

Lemma 3.6.4. *Let M be a polyhedral set of dimension $\mathbf{n} \geq 1$ with no isolated vertices. Then $H^0(M, \mathcal{O}_M(-d)) = 0$ for all $d > 0$.*

Proof. First, let $\mathbf{n} = 1$. Since M has no isolated vertices, the morphism $\coprod_{n \geq 1} \coprod_{s \in J_M^n} \Delta^s \to M$ is surjective. By Proposition 3.2.1 f), the induced morphism $\mathcal{O}_M \to \bigoplus \zeta^s_{M*}\mathcal{O}_{\Delta^s}$ is injective. Twisting with $-d$ for an integer $d > 0$ and taking global sections yields an injection $H^0(M, \mathcal{O}_M(-d)) \subseteq \bigoplus H^0(M, \mathcal{O}_{\Delta^s}(-d)) = 0$, and the result follows from this. □

Lemma 3.6.5. *Let M be a polyhedral set. Then $H^i(M, \mathcal{O}_M(d)) = 0$ for all integers $i > 0$ and $d > 0$.*

Proof. Let M be $\mathbf{n}$-dimensional. If $\mathbf{n} = 0$, then the dimension of $\mathbf{P}(M)$ is 0 and the result follows immediately. We proceed by induction on $\mathbf{n}$, and we may assume that $\mathbf{n} \geq 1$. Consider the exact sequence (3.32). Twisting with an integer $d > 0$ yields an exact sequence

$$0 \to \bigoplus_{s \in J_M^{\mathbf{n}}} \zeta^s_{M*}\mathcal{O}_{\Delta^s}(dD_{|s|} + K_{|s|}) \to \mathcal{O}_M(d) \to \iota_{M*}\mathcal{O}_{M^{\mathbf{n}-1}}(d) \to 0. \qquad (3.38)$$

If $\mathbf{n} \geq 1$, then by Serre duality $H^i(\mathcal{O}_{\Delta^s}(dD_{|s|} + K_{|s|}) = H^{\mathbf{n}-i}(\mathcal{O}_{\Delta^s}(-dD_{|s|})$ for all $s \in J_M^{\mathbf{n}}$. By application of [CLS11, Theorem 9.2.7], these cohomology groups are trivial for $i > 0$. By the inductive hypothesis, the induced long exact sequence in cohomology of (3.38) now immediately yields the result. □

Fix an $\mathbf{n}$-dimensional polyhedral set M. Then Proposition 3.6.3 gives a formula for the Hilbert polynomial of $\Gamma(M)$:

Theorem 3.6.6. *The Hilbert polynomial of the face ring $\Gamma(M)$ is*

$$H_M(d) = \sum_{s \in M^{\mathrm{can}}} (-1)^{\dim(s)} \operatorname{Ehr}_{|s|}(-d),$$

where $\operatorname{Ehr}_P(d)$ denotes the Ehrhart polynomial of $P \in \mathcal{P}$ (see [CLS11, Theorem 9.4.2]).

Chapter 4

Open categories

An open category over $\mathcal{P}$ is a category U equipped with a *discrete Conduché fibration* $p : U \to \mathcal{P}$. Such functors generalizes the notion of discrete fibrations (Appendix A.0.2) by relaxing the requirements on p. Conduché fibrations also go by the name of *unique factorization lifting functors*, see [Joh99] or [BN00] for a reference. An open category U can be universally completed into a polyhedral set $\mathbb{L}(U)$, and we will define the associated a face scheme $\mathbf{A}(U)$ as an open subscheme of $\mathbf{P}(\mathbb{L}(U))$. If U is an open category and $s \in U$, then then comma category $U_s = (s \downarrow U)$ is also an open category. If a morphism $\phi : U \to V$ induces an isomorphism $U_s \to V_{\phi(s)}$ for each $s \in U$, then ϕ is called a local isomorphism. The goal of this chapter is proving Theorem 4.4.1, where we will see that a local isomorphism induces an étale morphism $\mathbf{A}(U) \to \mathbf{A}(V)$ of schemes.

4.1 Discrete Conduché fibrations

We will use Appendix A.0.2 as a general reference for the notation in this section. Fix a small category C. Then one may extend the category of discrete fibrations $\mathrm{Fib}(C)$ to the larger category $\mathrm{CFib}(C)$ of discrete Conduché fibrations as follows:

Definition 4.1.1 ([Joh99]). A *discrete Conduché fibration over C* (abbreviated *Conduché fibration*) is a small category (U, p) over C satisfying the following axiom.

$(*)$ *Unique factorization lift*: For every arrow $h : t \to s$ in U and factorization $p(t) \xrightarrow{f_1} c \xrightarrow{f_2} p(s)$ of $p(h)$, there exists a unique factorization $t \xrightarrow{h_1} u \xrightarrow{h_2} s$ of h such that $p(h_1) = f_1$ and $p(h_2) = f_2$.

In particular, for every object $s \in U$ and arrow $f : c \to p(s)$, there exists at most one lifting $h : t \to s$ of f along s. We define the category $\mathrm{CFib}(C)$ as the full subcategory of $(\mathrm{Cat} \downarrow C)$ consisting of Conduché fibrations over C.

It follows from uniqueness of lifts that $\mathrm{Fib}(C)$ is a full subcategory of $\mathrm{CFib}(C)$ via the forgetful functor $T : \mathrm{Fib}(C) \to \mathrm{CFib}(C)$. This defines a functor $\mathbb{T} = T \circ I : \mathrm{Pre}(C) \to \mathrm{CFib}(C)$. For each Conduché fibration (U, p), consider the diagram $H_U = Y_C \circ p : U \to \mathrm{Pre}(C)$. For ease of notation, C^s will denote the representable functor $C(-, p(s))$ for each $s \in U$. We define the functor $\mathbb{L} : \mathrm{CFib}(C) \to \mathrm{Pre}(C)$ by $\mathbb{L}(U) = \varinjlim H_U$, which as a colimit comes equipped with associated morphisms $\zeta_U^s : C^s \to \mathbb{L}(U)$ for each $s \in U$. A morphism $\phi : U \to V$ of Conduché fibrations induces a natural transformation of diagrams $H_\phi : H_U \to H_V \phi$, hence a unique natural transformation $\mathbb{L}(\phi) : \mathbb{L}(U) \to \mathbb{L}(V)$ satisfying $\mathbb{L}(\phi) \circ \zeta_U^s = \zeta_V^{\phi(s)}$ for all $s \in U$. This fully defines the functor $\mathbb{L} : \mathrm{CFib}(C) \to \mathrm{Pre}(C)$.

Proposition 4.1.2. *The pair* $(\mathbb{L}, \mathbb{T})$ *forms an adjunction* $\mathbb{L} : \mathrm{CFib}(C) \leftrightarrows \mathrm{Pre}(C) : \mathbb{T}$*. Thus* $\mathbb{L}$ *preserves all colimits, and* $\mathbb{T}$ *preserves all limits.*

Proof. Let (U, p) be a Conduché fibration, and M a presheaf. We define the unit $\eta_U : U \to \mathbb{T}\mathbb{L}(U)$ of the adjunction by $\eta_U(s) = (p(s), \zeta^s_U(\mathrm{id}_{p(s)}))$. The morphisms $\zeta^s_M : C^s \to M$ for each element $s \in M(c)$ induces a unique isomorphism $\epsilon_M : \mathbb{L}\mathbb{T}(M) \to M$ such that $\epsilon_M \circ \zeta^{(c,s)}_{\mathbb{L}\mathbb{T}(M)} = \zeta^s_M$, which defines the counit. It is straight-forward to check that these natural transformations creates an adjoint pair $(\mathbb{L}, \mathbb{T})$. □

Let (U, p) be a Conduché fibration, and let $s \in U$. Consider the comma category $U_s = (s \downarrow U)$, and the forgetful functor $\gamma(s) : U_s \to U$ defined on objects by $(t, f : s \to t) \mapsto t$. The composition $p_s : U_s \xrightarrow{\gamma(s)} U \xrightarrow{p} C$ makes (U_s, p_s) a category over C. Any morphism $\phi : U \to V$ of Conduché fibrations induces a functor $\phi_s : U_s \to V_{\phi(s)}$ defined by $(t, f) \mapsto (\phi(t), \phi(f))$. We say that ϕ is a *local isomorphism* if ϕ_s is an isomorphism for all $s \in U$. If M is a presheaf, we will denote $\mathbb{T}(M)_{(c,s)}$ by M_s for any $c \in C$ and $s \in M(c)$. We will occasionally denote objects (t, f) of U_s by $s \xrightarrow{f} t$.

Proposition 4.1.3. *For every* $s \in U$*,* (U_s, p_s) *is a Conduché fibration. Moreover, for each* $s \in U$*,* $\gamma(s) : U_s \to U$ *is a local isomorphism.*

Proof. Let $h : (t_1, f_1) \to (t_2, f_2)$ be an arrow in U_s, and suppose that $p(t_1) \xrightarrow{g_1} c \xrightarrow{g_2} p(t_2)$ is a factorization of $p(h)$. Since $h : t_1 \to t_2$ is an arrow in U, there is a factorization $t_1 \xrightarrow{h_1} t \xrightarrow{h_2} t_2$ of h in U such that $p(h_1) = g_1$ and $p(h_2) = g_2$. But then $(t_1, f_1) \xrightarrow{h_1} (t, h_1 f_2) \xrightarrow{h_2} (t_2, f_2)$ is a factorization of $h : (t_1, f_1) \to (t_2, f_2)$ with $p_s(h_1) = g_1$ and $p_s(h_2) = g_2$. Thus U_s satisfies the unique factorization lift axiom.

Next we will now show that $\gamma(s) : U_s \to U$ is a local isomorphism. So pick an object $(t, f : s \to t)$ in U_s, and consider the induced morphism $(\gamma(s))_{(t,f)} : (U_s)_{(t,f)} \to U_t$. This morphism sends an object $((u, g : s \to u), h : (t, f) \to (u, g))$ to $(u, h : t \to s)$. We will define an inverse morphism $\gamma' : U_t \to (U_s)_{(t,f)}$ as follows. On objects, let $(u, h) \mapsto ((u, hf), h : (t, f) \to (u, hf))$. An arrow $g : (u, h) \to (u', h')$ in U_t may also be considered as an arrow $g : (u, hf) \to (u', h'f)$ in U_s, and hence defines an arrow $g : \gamma'((u, h)) \to \gamma'((u', h'))$. This clearly defines a functor, and is an inverse of $(\gamma(s))_{(t,f)}$. □

If U is a Conduché fibration, then any arrow $f : s \to t$ in U induces a natural morphism $U_f : U_t \to U_s$, given on objects by $(t, g) \mapsto (s, gf)$. By Proposition 4.1.3, such morphisms are always local isomorphisms. Note that a morphism $\phi : U \to V$ is a local isomorphism if and only if $\phi_s : U_s \to V_{\phi(s)}$ is bijective on objects for every $s \in U$; if ϕ_s is bijective, an inverse functor is uniquely determined by the unique factorization lift axiom.

Definition 4.1.4. Let $\phi : U \to V$ and $\psi : W \to V$ be a pair of morphisms of Conduché fibrations. We define the fibered product $U \times_V W$ as a category over C

as the strict pullback of categories $U \times_V W$ equipped with the induced projection map $p = p_U \times p_V : U \times_V W \to C$. The objects of $U \times_V W$ are pairs (s, t) where $s \in U$, $t \in W$ and $\phi(s) = \psi(t)$. An arrow $(s', t') \to (s, t)$ is a pair of arrows $f : s' \to s$, $g : t' \to t$ such that $\phi(f) = \psi(g)$, or equivalently $p_U(f) = p_W(g)$ by uniqueness of lifts.

The following lemma is easily verified.

Lemma 4.1.5. *$U \times_V W$ is a Conduché fibration over C, and equipped with the projection morphisms $\pi_1 : U \times_V W \to U$ and $\pi_2 : U \times_V W \to W$ it is a fibered product in* $\mathrm{CFib}(C)$.

Lemma 4.1.6. *Let $\phi : U \to V$ be a morphism of Conduché fibrations. Then,*

a) *The functor ϕ is faithful;*

b) *If $W \subseteq V$ is a full Conduché subfibration, the inverse image $\phi^{-1}(W)$ consisting of objects $\phi^{-1}(\mathrm{ob}\, W)$ and arrows $\phi^{-1}(\mathrm{Hom}(W))$ is a full Conduché subfibration of U;*

c) *If $W \subseteq V$ is a full Conduché subfibration, there is an isomorphism $\phi^{-1}(W) \to W \times_V U$, such that the diagram*

$$\begin{tikzcd} \phi^{-1}(W) \arrow[r] \arrow[dr] & U \times_V W \arrow[d, "\pi_1"] \\ & U \end{tikzcd} \tag{4.1}$$

commutes;

d) *If ϕ is a local isomorphism, and $\psi : W \to V$ is any morphism, then $\pi_2 : U \times_V W \to W$ is a local isomorphism;*

e) *If $\phi : U \to V$ is a local isomorphism, then ϕ is a full and faithful functor.*

Proof. a) Suppose that $h_1, h_2 : t \to s$ are two arrows of U such that $\phi(h_1) = \phi(h_2)$. Then in particular $p_U(h_1) = p_U(h_2)$, which implies that $h_1 = h_2$ by uniqueness of lifts.

b) Clearly, $\phi^{-1}(W)$ forms a subcategory of U. So we only need to verify the existence of factorization lifts. Let $h : t \to s$ be an arrow in $\phi^{-1}(W)$, and suppose that $p(s) \xrightarrow{f} c \xrightarrow{g} p(t)$ is a factorization of $p(h)$. This factorization lifts to a factorization $t \to u \to s$ in U, which maps to a factorization lift $\phi(s) \to \phi(u) \to \phi(t)$ in V. Since W is a full Conduché subfibration, this unique lifting exists in W, hence the factorization $t \to u \to s$ exists in $\phi^{-1}(W)$.

c) We define the morphism $\phi^{-1}(W) \to W \times_V U$ by $s \mapsto (\phi(s), s)$ on objects, and $f \mapsto (\phi(f), f)$ on arrows. This is clearly a morphism of Conduché fibrations making (4.1) commute, and $\pi_2 : W \times_V U \to U$ produces an inverse morphism by restricting the codomain to $\phi^{-1}(W)$.

d) Let $\psi : W \to V$ be any morphism, and consider the projection $\pi_2 : U \times_V W \to W$. Let (s,t) be an object of $U \times_V W$, and consider the induced morphism $(\pi_2)_{(s,t)} : (U \times_V W)_{(s,t)} \to W_t$. Since $\phi_s : U_s \to V_{\phi(s)}$ is an isomorphism, we may define an inverse $\tau : W_t \to (U \times_V W)_{(s,t)}$ as follows. For any object (t', f) of W_t, the object $(\psi(t'), \psi(f))$ in $V_{\psi(t)}$ lifts to a unique object (s', g) in U_s via ψ_s. Thus we define $\tau((t', f))$ to be $((s', t'), (g, f))$. Let $h : (t', f) \to (t'', f')$ be an arrow in W_t, i.e. an arrow $h : t' \to t''$ such that $hf = f'$. Then $(\psi(t''), \psi(h))$ in $V_{\psi(t')}$ lifts to a unique object (s'', h') in $U_{s'}$ via $\psi_{s'}$. Let (s'', g') be the unique lifting of $(\psi(t''), f')$ in U_s, and define $\tau(h)$ to be the arrow $(h', h) : ((s', t'), (g, f)) \to ((s'', t''), (g', f'))$. Uniqueness of lifts in U makes τ functorial, and it is clearly an inverse to $(\pi_2)_{(s,t)}$.

e) ϕ is full by the definition of a local isomorphism, and faithful by a). $\square$

4.1.1 Perimeter

Lemma 4.1.7. *Let (U, p) be a Conduché fibration. Then the unit transformation $\eta_U : U \to \mathbb{TL}(U)$ is injective and a local isomorphism. In particular, η_U makes U a full subcategory of $\mathbb{TL}(U)$ by Lemma 4.1.6 e).*

Proof. Note first that for any object $s \in \mathbb{TL}(U)$, there exists by construction of the colimit an arrow $s \xrightarrow{f} t$ in $\mathbb{TL}(U)$ for some t in the image of η_U. Suppose that $s_0, s_1 \in U$ are two objects with $\eta_U(s_0) = \eta_U(s_1)$. We will show that $s_0 = s_1$. Define $c = p(s_0) = p(s_1)$. Then by construction of the colimit, there exists a zig-zag diagram $s_0 = t_0 \xrightarrow{f_0} t_1 \xleftarrow{f_1} \dots \xleftarrow{f_{r-1}} t_r = s_1$ for some collection of objects $t_i \in U$, arrows f_i in U, and elements $h_i \in C^{t_i}(c)$ such that $h_0 = h_r = \mathrm{id}_c$ and the induced zig-zag diagram of representable functors $C^{s_0} \to C^{t_1} \leftarrow C^{t_2} \to \dots \to C^{t_{r-1}} \leftarrow C^{s_1}$ maps the elements h_i compatibly to each other. If $r = 1$, then there exists an arrow $f_0 : s_0 \to s_1$, which must be the identity arrow since it is the unique lifting of id_c. Proceeding inductively, assume that $r \geq 2$. Consider the diagram $C^{s_0} \to C^{t_1} \leftarrow C^{t_2}$. Here $\mathrm{id}_c \in C^{s_0}(c)$ maps to $p(f_0) = h_1$, and $h_2 \in C^{t_2}(c)$ maps to $p(f_1)h_2 = h_1$. Hence $p(f_1)h_2 = p(f_0)$, which is a factorization of $p(f_0)$. Thus there exists a unique factorization $s_0 \xrightarrow{g_1} t_2 \xrightarrow{g_2} t_1$ of f_0 such that $p(g_1) = h_2$ and $p(g_2) = p(f_1)$. In particular, the arrow $g_1 : s_0 \to t_2$ induces a morphism $C^{s_0} \to C^{t_2}$ mapping id_c to h_2. We may now shorten the zig-zag diagram of length r to one of length $r-1$. By induction, we conclude that $s_0 = s_1$ and that η_U is injective on objects. By Lemma 4.1.6 a), η_U is faithful, so we may consider U as a subcategory of $\mathbb{TL}(U)$.

Next, suppose that $h : s_0 \to s_1$ is any arrow in $\mathbb{TL}(U)$ for some pair of objects $s_0, s_1 \in U$. Write $f = p(h) : c' \to c$, $t_0 = \zeta_U^{s_0}(\mathrm{id}_{c'})$ and $t_1 = \zeta_U^{s_1}(\mathrm{id}_c)$. Then $\mathbb{L}(U)(f)(t_1) = t_0$, which means that $\zeta_U^{s_1}(f) = t_0$, where f is considered an object of $C^{s_1}(c')$. Like before, this means that there exists a zig-zag diagram $s_0 = t_0 \xrightarrow{f_0} t_1 \xleftarrow{f_1} \dots \xleftarrow{f_{r-1}} t_r = s_1$ and elements $h_i \in C^{t_i}(c')$ such that $h_0 = \mathrm{id}_{c'}$, $h_r = f$ and the elements h_i are compatibly sent to each other in the induced zig-zag diagram $C^{s_0} \to C^{t_1} \leftarrow C^{t_2} \to \dots \to C^{t_{r-1}} \leftarrow C^{s_1}$. By the previous argument, we may iteratively reduce the length of such a zig-zag diagram to one on the form $s_0 \xrightarrow{g} s_1$ such that the morphism $C^{s_0} \to C^{s_1}$ maps $\mathrm{id}_{c'}$ to f. This means in particular that $p(g) = f$, which implies that $g = h$ by uniquenss of lifts. Thus η_U is a full functor, and U is a full subcategory of $\mathbb{TL}(U)$.

Let now $s \in U$, and consider the induced morphism $(\eta_U)_s : U_s \to \mathbb{TL}(U)_s$. Suppose that (t, f) and (t', g) is a pair of objects of U_s mapping to the same object in $\mathbb{TL}(U)_s$. Then in particular, $t = t'$ since η_U is injective on objects. But $p(f) = p(g)$, so by uniqueness of lifts we must have $f = g$. Hence $(\eta_U)_s$ is injective. Conversely, let (t, f) be any object of $\mathbb{TL}(U)_s$. Then there exists an object s' in U and an arrow $t \xrightarrow{g} s'$ in $\mathbb{TL}(U)$. The composition $s \xrightarrow{gf} s'$ exists in U since it is a full subcategory. In $\mathbb{TL}(U)$, the factorization $s \xrightarrow{f} t \xrightarrow{g} s'$ is the unique lifting of the factorization $p(s) \xrightarrow{p(f)} p(t) \xrightarrow{p(g)} p(s')$. But there is also such a factorization lifting in U, which must be identical to $s \xrightarrow{f} t \xrightarrow{g} s'$ by uniqueness. So the arrow $s \xrightarrow{f} t$ exists in U, and therefore as an object in U_s. Hence $(\eta_U)_s$ is surjective, and therefore bijective. It follows that $(\eta_U)_s$ is an isomorphism. ☐

Definition 4.1.8. Let U be a full Conduché subfibration of a discrete fibration M such that the inclusion $U \to M$ is a local isomorphism. Then we define *the complement of U in M*, denoted $M \backslash U$, to be the full subcategory M consisting of objects $\mathrm{ob}\, M \backslash \mathrm{ob}\, U$, equipped with the projection morphism $M \backslash U \to C$ induced by that of M.

Lemma 4.1.9. *The the complement $M \backslash U$ is a discrete fibration.*

Proof. We must show that for every arrow $f : c' \to c$ in C and object $s \in M \backslash U$, there exists a lifting $h : t \to s$ of f along s with $t \in M \backslash U$. Given such an arrow f and object s over c, there certainly exists such a lifting $h : t \to s$ in M. Assume for contradiction that $t \in U$. Since the induced morphism $U_t \to M_t$ is an isomorphism, there exist an object $t \xrightarrow{g} s'$ in U_t mapping to the object $t \xrightarrow{h} s$ in M_t. This implies that $s' = s$, a contradiction, so we conclude that $t \in M \backslash U$. ☐

Lemma 4.1.7 and Lemma 4.1.9 allows us to make the following definition.

Definition 4.1.10. We define the *perimeter of U*, denoted U^∂, as the discrete subfibration $\mathbb{TL}(U) \backslash U$ of $\mathbb{TL}(U)$.

Complementary to Lemma 4.1.9, the lemma below shows that the set of injective local isomorphism $U \to M$ from a Conduché fibration U to a discrete fibration M is in one-to-one correspondence with complements $M \backslash N$ of discrete subfibrations N.

Lemma 4.1.11. *Let M be a discrete fibration, and $N \subseteq M$ a discrete subfibration. Then the full subcategory of M with objects $M \backslash N$ is a Conduché subfibration of M such that the inclusion $M \backslash N \to M$ is a local isomorphism.*

Proof. First we show that $M\backslash N$ is a Conduché fibration. Let $f : s \to t$ be an arrow of $M\backslash N$, and suppose that $p(s) \to c \to p(t)$ is a factorization of $p(f)$. Since M is a discrete fibration, there exists a unique lifting $s \xrightarrow{g} s' \xrightarrow{h} t$ of this factorization in M. Assume that $s' \in N$. Then $s \xrightarrow{g} s'$ is the unique lifting of $p(g)$ in N. This means that $s \in N$, which is impossible. Hence $s' \in M\backslash N$, which shows that $M\backslash N$ is a Conduché fibration. Next we show that the inclusion $\phi : M\backslash N \to M$ is a local isomorphism. Let $s \in M\backslash N$, and consider the local morphism $\phi_s : (M\backslash N)_s \to M_s$. This is an isomorphism if and only if $t \in M\backslash N$ for every object $(s, f : s \to t)$ in M_s. But if $t \in N$ for any such object, then $f : s \to t$ is the unique lifting of $p(f)$ in N. This means that $s \in N$, which is impossible. We conclude that ϕ is a local isomorphism. □

4.2 Open categories over $\mathcal{P}$

Definition 4.2.1. We will now consider Conduché fibrations $p : U \to \mathcal{P}$ satisfying $(*)$: for all $s \in U$, $\mathrm{Aut}_U(s) = \{\mathrm{id}_s\}$. We will call *open categories over* $\mathcal{P}$ (abbreviated *open categories*). Recall that the category of discrete fibrations satisfying $(*)$ is equivalent to $\mathrm{Pre}^{(*)}(\mathcal{P})$ (Definition 2.2.1). We note that if $U \to V$ is a morphism of Conduché fibrations where V satisfies $(*)$, then U also satisfies $(*)$.

Definition 4.2.2. Let U be an open category. For any *face* $s \in U$, we let $|s|$ denote the object $p(s) \in \mathcal{P}$. We define the dimension $\dim(s)$ of a face $s \in U$ to be $\dim(|s|)$, and the dimension of U to be the supremum of the dimensions of its faces, and -1 if U is empty. For any non-negative integer $\mathbf{n}$, let the $\mathbf{n}$-*truncation* $U^{\mathbf{n}}$ denote the full subcategory of U consisting of all maximally $\mathbf{n}$-dimensional faces of U. It is clear that this is an open category, and that $\mathbb{TL}(U)^{\mathbf{n}} \cap U = U^{\mathbf{n}}$ as full open subcategories of $\mathbb{TL}(U)$. If s,t is a pair of faces in U, and $f : |s| \to |t|$ is an arrow in $\mathcal{P}$ which lifts to an arrow $s \to t$ in U, we will denote this arrow by f as well.

Definition 4.2.3. Let U be an open category, and consider the morphisms $\mathbb{T}(\zeta^s_U) : \mathbb{T}(\Delta^s) \to \mathbb{TL}(U)$ for faces $s \in U$. We define the *open polytope* Δ^s_U as the preimage $\mathbb{T}(\zeta^s_U)^{-1}(U) \subseteq \mathbb{T}(\Delta^s)$. Then $\mathbb{T}(\zeta^s_U)$ restricts to a morphism $\xi^s_U : \Delta^s_U \to U$ which is natural in U. If $s \in U$ is of dimension $\mathbf{n}$, we will denote the $(\mathbf{n}-1)$-truncation $(\Delta^s_U)^{\mathbf{n}-1}$ by $\partial\Delta^s_U$.

We note the following. The induced morphism $\mathbb{L}(\Delta^s_U) \to \mathbb{L}(\Delta^s)$ is an isomorphism, which follows from the fact that ξ^s_U preserves terminal objects. The morphisms $\zeta^s_{\mathbb{L}(U)} : \Delta^s \to \mathbb{L}(U)$ for faces $s \in U^\partial$ factors through $\mathbb{L}(U^\partial)$, so $\mathbb{T}(\zeta^s_{\mathbb{L}(U)})^{-1}(U)$ is the empty-category.

Lemma 4.2.4. *Let U be an open category, and s a face of U corresponding to the morphism $\zeta^s_U : \Delta^s \to \mathbb{L}(U)$. Then $(\Delta^s_U)^\partial = \mathbb{T}(\zeta^s_U)^{-1}(U^\partial)$ as full open subcategories of $\mathbb{T}(\Delta^s)$.*

Proof. By our observation above, $\mathbb{L}(\Delta^s_U)$ is isomorphic to Δ^s, so we may consider

$(\Delta_U^s)^\partial$ as the full open subcategory $\mathbb{T}(\zeta_U^s)^{-1}(U)^\partial$ of $\mathbb{T}(\Delta^s)$. For faces $t \in \mathbb{T}(\Delta^s)$, we have

$$t \in \mathbb{T}(\zeta_U^s)^{-1}(U)^\partial \Leftrightarrow \mathbb{T}(\zeta_U^s)(t) \notin U \Leftrightarrow \mathbb{T}(\zeta_U^s)(t) \in U^\partial \Leftrightarrow t \in \mathbb{T}(\zeta_U^s)^{-1}(U^\partial).$$

Hence $(\Delta_U^s)^\partial$ and $\mathbb{T}(\zeta_U^s)^{-1}(U^\partial)$ consists of the same objects. It remains to show that $\mathbb{T}(\zeta_U^s)^{-1}(U^\partial)$ is a full subcategory of $\mathbb{T}(\Delta^s)$. However, this immediately follows from the fact that U^∂ is a full subcategory of $\mathbb{TL}(U)$. $\square$

Definition 4.2.5. Let U an open category. For each $\mathbf{n}$, we have a natural inclusion $\iota_U : U^{\mathbf{n}-1} \to U$. If $\phi : U \to V$ is a morphism of open categories, let $\phi' : U^{\mathbf{n}-1} \to V^{\mathbf{n}-1}$ denote the induced morphism of $(\mathbf{n}-1)$-truncations. Define $\Delta_U^{\mathbf{n}}$ as the coproduct $\coprod_{s \in J_U^{\mathbf{n}}} \Delta_U^s$, where $J_U^{\mathbf{n}} := U \cap J_{\mathbb{L}(U)}^{\mathbf{n}}$. This is clearly an open category, and let $\alpha_U : \Delta_U \to U$ denote the induced morphism. For each $s \in U$, the morphisms $\Delta_U^s \to \Delta_V^{\phi(s)}$ induces a morphism $\phi^\Delta : \Delta_U \to \Delta_V$, and we have that $\alpha_V \circ \phi^\Delta = \phi \circ \alpha_U$. This makes α_U natural in U. We also define $\partial\Delta_U^{\mathbf{n}}$ as the $(\mathbf{n}-1)$-truncation $(\Delta_U^{\mathbf{n}})^{\mathbf{n}-1} = \coprod_{s \in J_U^{\mathbf{n}}} \partial\Delta_U^s$.

Remark 4.2.6. While $U^{\mathbf{n}-1} = U \cap \mathbb{TL}(U)^{\mathbf{n}-1}$ as full open subcategories of U, it is generally not the case that the induced morphism $\mathbb{L}(U^{\mathbf{n}-1}) \to \mathbb{L}(U)^{\mathbf{n}-1}$ is an isomorphism. Informally speaking, $U^{\mathbf{n}-1}$ loses track of the perimeter of the facets of U. In general, this morphism is neither injective nor surjective.

Lemma 4.2.7. *If $\phi : U \to V$ is a local isomorphism, then the morphism $\tau_{\mathbf{n}} = \alpha_U \times_{V^{\mathbf{n}}} \phi^\Delta : \Delta_U^{\mathbf{n}} \to U^{\mathbf{n}} \times_{V^{\mathbf{n}}} \Delta_V^{\mathbf{n}}$ is an isomorphism for each integer $\mathbf{n} \geq -1$.*

Proof. We will define an inverse map $\mu : U^{\mathbf{n}} \times_{V^{\mathbf{n}}} \Delta_V^{\mathbf{n}} \to \Delta_U^{\mathbf{n}}$. Let (s, f) be a face of $U^{\mathbf{n}} \times_{V^{\mathbf{n}}} \Delta_V^{\mathbf{n}}$. Then f is contained in a unique open polytope Δ_V^v for some $v \in J_V^{\mathbf{n}}$. Let $u = \xi_V^v(f)$, and consider the arrow $f : u \to v$. Since $\phi(s) = u$, the isomorphism $\phi_s : U_s \to V_u$ produces a unique arrow $f : s \to t$ in U such that $\phi(t) = v$. Thus t is $\mathbf{n}$-dimensional. Consider the corresponding morphism $\xi_U^t : \Delta_U^t \to U$. We may consider f as a face of Δ_U^t, which satisfies $\xi_U^t(f) = s$. On objects, we define $\mu(s, f) = f$. Uniqueness of t ensures that this is well-defined. Now, let $g : (s_0, f_0) \to (s_1, f_1)$ be an arrow in $U^{\mathbf{n}} \times_{V^{\mathbf{n}}} \Delta_V^{\mathbf{n}}$. Since $g : f_0 \to f_1$ is an arrow connecting f_0 and f_1, they are contained in the same open polytope Δ_V^v. Define $u_i = \xi_V^v(f_i)$ for $i = 0, 1$. Again, since ϕ is a local isomorphism, there exists unique liftings $f_0 : s_0 \to t_0$ and $f_1 : s_1 \to t_1$ in U of $f_0 : u_0 \to v$ and $f_1 : u_1 \to v$ respectively. Since $g : s_0 \to s_1$ is an arrow, and $f_0 = f_1 g$, there is an arrow $f_0 : s_0 \to t_1$. By uniqueness, we must have $t_0 = t_1$. Note finally that the arrow $g : f_0 \to f_1$ in Δ^{t_0} exists in $\Delta_U^{t_0}$, since as an arrow in $\mathbb{T}(\Delta^{t_0})$ it maps to the arrow $g : s_0 \to s_1$ via $\zeta_U^{t_0}$. Thus on arrows, we define $\mu(g) = g$ as an arrow $f_1 \to f_0$ in $\Delta_U^{t_0}$. This is clearly functorial, and defines an inverse of $\tau_{\mathbf{n}}$. $\square$

4.3 Associated scheme

We say that an open category U is *finite* if $\mathbb{L}(U)$ is. From here on, all open categories and polyhedral sets will be assumed to be finite.

Definition 4.3.1. Let (U, p) be an open category, and consider the inclusion of the perimeter $U^\partial \subseteq \mathbb{TL}(U)$. This induces an inclusion of polyhedral sets $j_U : \mathbb{L}(U^\partial) \to \mathbb{L}(U)$, hence a closed immersion $j_U : \mathbf{P}(\mathbb{L}(U^\partial)) \to \mathbf{P}(\mathbb{L}(U))$. We define the *face scheme* $\mathbf{A}(U)$ of U as the complement $\mathbf{P}(\mathbb{L}(U))\backslash\mathbf{P}(\mathbb{L}(U^\partial))$.

Let $\phi : U \to V$ be a morphism. Then ϕ induces a morphism of schemes $\mathbf{P}\mathbb{L}(\phi) : \mathbf{P}\mathbb{L}(U) \to \mathbf{P}\mathbb{L}(V)$. Since $\mathbb{TL}(\phi)(U) \subseteq V$, we have $\mathbb{TL}(\phi)^{-1}(V^\partial) \subseteq U^\partial$. This means that $\mathbb{L}(\phi)^{-1}(\mathbb{L}(V^\partial)) \subseteq \mathbb{L}(U^\partial)$. Thus by Lemma 3.2.4,

$$\mathbf{P}\mathbb{L}(\phi)^{-1}(\mathbf{P}(\mathbb{L}(V^\partial))) = \mathbf{P}(\mathbb{L}(\phi)^{-1}\mathbb{L}(V^\partial)) \subseteq \mathbf{P}(\mathbb{L}(U^\partial)),$$

so $\mathbf{P}\mathbb{L}(\phi)$ restricts to a morphism $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$. This defines a functor $\mathbf{A} : \mathrm{CFib}(\mathcal{P})_f \to \mathrm{Sch}_k$, associating a scheme to each finite open category. Furthermore, the open immersions $\mathbf{A}(U) \subseteq \mathbf{P}(\mathbb{L}(U))$ defines a natural transformation $\kappa : \mathbf{A} \to \mathbf{P} \circ \mathbb{L}$. The morphism κ_U is an isomorphism if and only if U is a discrete fibration, or equivalently, on the form $\mathbb{T}(M)$ for some polyhedral set M.

Lemma 4.3.2. *Let U be an open category, and let s be a face of U. Consider the induced morphism $\zeta_U^s : \mathbf{P}(\Delta^s) \to \mathbf{P}(\mathbb{L}(U))$. Then $\mathbf{A}(\Delta_U^s) = (\zeta_U^s)^{-1}(\mathbf{A}(U))$ as open subschemes of $\mathbf{P}(\Delta^s)$.*

Proof. As an open subscheme of $\mathbf{P}(\Delta^s)$, we have that $\mathbf{A}(\Delta_U^s) = \mathbf{P}(\Delta^s)\backslash\mathbf{P}(\mathbb{L}((\Delta_U^s)^\partial))$. By Lemma 4.2.4, $\mathbf{P}(\mathbb{L}((\Delta_U^s)^\partial)) = \mathbf{P}(\mathbb{L}(\mathbb{T}(\zeta_U^s)^{-1}(U^\partial)))$, which is equal to the preimage $\mathbf{P}((\zeta_U^s)^{-1}(\mathbb{L}(U^\partial)))$. By Lemma 3.2.4, this is equal to $(\zeta_U^s)^{-1}(\mathbf{P}(\mathbb{L}(U^\partial)))$ as closed subsets of $\mathbf{P}(\Delta^s)$. Hence $\mathbf{A}(\Delta_U^s)$ is equal to

$$\mathbf{P}(\Delta^s)\backslash(\zeta_U^s)^{-1}(\mathbf{P}(\mathbb{L}(U^\partial))) = (\zeta_U^s)^{-1}(\mathbf{P}(\mathbb{L}(U))\backslash\mathbf{P}(\mathbb{L}(U^\partial))) = (\zeta_U^s)^{-1}(\mathbf{A}(U)).$$

□

Proposition 4.3.3. *Let U be an open category, and let $M \subseteq \mathbb{L}(U)$ be an inclusion of polyhedral sets. Then the embedding $U \cap \mathbb{T}(M) \to \mathbb{T}(M)$ induces an isomorphism $\mathbf{A}(\mathbb{T}(M) \cap U) \to \mathbf{A}(U) \cap \mathbf{P}(M)$.*

Proof. Note first that the induced morphism $\mathbb{L}(U \cap \mathbb{T}(M)) \to \mathbb{L}(U)$ factors uniquely through M via some morphism $\phi : \mathbb{L}(U \cap \mathbb{T}(M)) \to M$, so the induced morphism $\mathbf{A}(U \cap \mathbb{T}(M)) \to \mathbf{A}(U)$ restricts to a morphism $\mathbf{A}(\mathbb{T}(M) \cap U) \to \mathbf{A}(U) \cap \mathbf{P}(M)$, where $\mathbf{A}(U)$ and $\mathbf{P}(M)$ are considered subschemes of $\mathbf{P}(\mathbb{L}(U))$. We proceed by induction on the dimension $\mathbf{n}$ of M, noting that for $M = \underline{\varnothing}$, the statement is trivial. Consider the Milnor squares of schemes

$$\begin{tikzcd}
\coprod_{s \in J^{\mathbf{n}}_{\mathbb{T}(M)}} \mathbf{P}(\Delta^s) \arrow[r] & \mathbf{P}(M) \\
\coprod_{s \in J^{\mathbf{n}}_{\mathbb{T}(M)}} \mathbf{P}(\partial\Delta^s) \arrow[r] \arrow[u] & \mathbf{P}(M^{\mathbf{n}-1}) \arrow[u]
\end{tikzcd},$$

$$
\begin{tikzcd}
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{P}(\Delta^s) \arrow[r] & \mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M))) \\
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{P}(\partial\Delta^s) \arrow[r] \arrow[u] & \mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1}). \arrow[u]
\end{tikzcd}
\tag{4.2}
$$

$\mathbf{A}(U)\cap\mathbf{P}(M)$ and $\mathbf{A}(U\cap\mathbb{T}(M))$ are open subsets of $\mathbf{P}(M)$ and $\mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M)))$ respectively. Note that for any $s\in U^{\partial}\cap\mathbb{T}(M)$, the preimages $(\zeta^s_{\mathbb{T}(M)})^{-1}(U\cap M)$ are empty. It is also clear that $\Delta^s_{U\cap\mathbb{T}(M)}=\Delta^s_U$ as full open subcategories of $\mathbb{T}(\Delta^s)$ for any $s\in U\cap\mathbb{T}(M)$. Moreover, since $\partial\Delta^s_U=\partial\Delta^s\cap\Delta^s_U$, the inductive hypothesis ensures that the induced morphism $\mathbf{A}(\partial\Delta^s_U)\to\mathbf{A}(\Delta^s_U)\cap\mathbf{P}(\partial\Delta^s)$ is an isomorphism compatible with the induced morphisms to $\mathbf{P}(\partial\Delta^s)$. By Lemma 4.3.2, we have $\mathbf{A}(\Delta^s_U)=(\zeta^s_U)^{-1}(\mathbf{A}(U))$. Hence by Proposition B.2.1, there are induced pushout squares

$$
\begin{tikzcd}
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{A}(\Delta^s_U) \arrow[r] & \mathbf{A}(U)\cap\mathbf{P}(M) \\
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{A}(\partial\Delta^s_U) \arrow[r] \arrow[u] & \mathbf{A}(U)\cap\mathbf{P}(M^{\mathbf{n}-1}), \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{A}(\Delta^s_U) \arrow[r] & \mathbf{A}(U\cap\mathbb{T}(M)) \\
\coprod_{s\in J^{\mathbf{n}}_{U\cap\mathbb{T}(M)}} \mathbf{A}(\partial\Delta^s_U) \arrow[r] \arrow[u] & \mathbf{A}(U\cap\mathbb{T}(M))\cap\mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1}). \arrow[u]
\end{tikzcd}
$$

The inductive hypothesis yields isomorphisms

$$\mathbf{A}(U)\cap\mathbf{P}(M^{\mathbf{n}-1})\xrightarrow{\cong}\mathbf{A}(U\cap\mathbb{T}(M^{\mathbf{n}-1})), \tag{4.3}$$

and

$$\mathbf{A}(U\cap\mathbb{T}(M))\cap\mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1})\xrightarrow{\cong}\mathbf{A}(U\cap\mathbb{T}(M)\cap\mathbb{T}(\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1})). \tag{4.4}$$

These isomorphisms are compatible with the morphisms into $\mathbf{P}(M^{\mathbf{n}-1})$, hence into $\mathbf{P}(M)$. Recall from Remark 4.2.6 that $U\cap\mathbb{T}(M)\cap\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1}=(U\cap\mathbb{T}(M))^{\mathbf{n}-1}$, which is isomorphic to $U\cap\mathbb{T}(M^{\mathbf{n}-1})$ compatibly into $\mathbb{T}(M)$. This means that the isomorphisms (4.3) and (4.4) composes to an isomorphism

$$\mathbf{A}(U\cap\mathbb{T}(M))\cap\mathbf{P}(\mathbb{L}(U\cap\mathbb{T}(M))^{\mathbf{n}-1})\xrightarrow{\cong}\mathbf{A}(U)\cap\mathbf{P}(M^{\mathbf{n}-1})$$

compatibly into $\mathbf{P}(M)$. Hence $\mathbf{A}(U)\cap\mathbf{P}(M)$ and $\mathbf{A}(U\cap\mathbb{T}(M))$ are pushouts of isomorphic diagrams, inducing a unique isomorphism between them compatible with the morphisms into $\mathbf{P}(M)$. $\square$

Proposition 4.3.3 and Proposition B.2.1 permits the following definition.

Definition 4.3.4 (**The Milnor square for $\mathbf{A}(U)$).** Let U be an open category, and $\mathbf{n} \geq -1$ an integer. Then $\mathbf{A}(U^{\mathbf{n}-1}) = \mathbf{A}(U^{\mathbf{n}}) \cap \mathbf{P}(\mathbb{L}(U)^{\mathbf{n}-1})$ and $\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) = \mathbf{A}(\Delta^{\mathbf{n}}_U) \cap \mathbf{A}(\partial\Delta^{\mathbf{n}}_{\mathbb{L}(U)})$ as open subschemes of $\mathbf{P}(\mathbb{L}(U)^{\mathbf{n}-1})$ and $\mathbf{P}(\partial\Delta^{\mathbf{n}}_{\mathbb{L}(U)})$ respectively. Restricting to $\mathbf{A}(U^{\mathbf{n}}) \subseteq \mathbf{P}(M^{\mathbf{n}})$ in the Milnor square (3.1) for $\mathbf{P}(M)$ of level $\mathbf{n}$ yields a natural pushout square

$$\begin{tikzcd}
\mathbf{A}(\Delta^{\mathbf{n}}_U) \arrow[r] & \mathbf{A}(U^{\mathbf{n}}) \\
\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \arrow[r] \arrow[u] & \mathbf{A}(U^{\mathbf{n}-1}), \arrow[u]
\end{tikzcd} \tag{4.5}$$

called the *Milnor square for* $\mathbf{A}(U)$ *of level* $\mathbf{n}$.

Lemma 4.3.5. *A morphism of open categories $\phi : U \to V$ is surjective (resp. injective) if and only if $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is surjective (resp. injective).*

Proof. We proceed by induction on the dimension $\mathbf{n}$ of V. Note that the base case of $V = \varnothing$ is trivial. Consider the Milnor square (4.5) for $\mathbf{A}(V)$ of level $\mathbf{n}$. We may assume that $\mathbf{n} \geq 0$. Assume first that $\phi : U \to V$ is surjective (resp. injective). Since $\phi' : U^{\mathbf{n}-1} \to V^{\mathbf{n}-1}$ is surjective (resp. injective), the inductive hypothesis implies that $\phi' : \mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{A}(V^{\mathbf{n}-1})$ is surjective (resp. injective). In the notation of Definition 3.2.3, it remains to show that the restricted morphism $T^{\mathbf{n}}_{\mathbb{L}(U)} \cap \mathbf{A}(U) \to T^{\mathbf{n}}_{\mathbb{L}(V)} \cap \mathbf{A}(V)$ is surjective (resp. injective). But note that $\mathbb{L}(\phi) : \mathbb{L}(U) \to \mathbb{L}(V)$ is surjective (resp. injective on $\mathbf{n}$-dimensional faces). This means that the induced morphism $T^{\mathbf{n}}_{\mathbb{L}(U)} \to T^{\mathbf{n}}_{\mathbb{L}(V)}$ is surjective (resp. injective) by Lemma 3.2.4. But $T^{\mathbf{n}}_{\mathbb{L}(U)} = \mathbf{P}(\mathbb{L}(U)) \backslash \mathbf{P}(\mathbb{L}(U)^{\mathbf{n}-1})$, and since $U^{\partial} \subseteq \mathbb{L}(U)^{\mathbf{n}-1}$, we have that $T^{\mathbf{n}}_{\mathbb{L}(U)} \subseteq \mathbf{A}(U)$. This shows that $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is surjective (resp. injective). Conversely, assume that $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is surjective (resp. injective). As a function of sets, ϕ decomposes as a disjoint union of the pair of surjective (resp. injective) morphisms $\phi' : \mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{A}(V^{\mathbf{n}-1})$ and $T^{\mathbf{n}}_{\mathbb{L}(U)} \to T^{\mathbf{n}}_{\mathbb{L}(V)}$. By the inductive hypothesis, $\phi' : U^{\mathbf{n}-1} \to V^{\mathbf{n}-1}$ is surjective (resp. injective). Since $T^{\mathbf{n}}_{\mathbb{L}(U)}$ and $T^{\mathbf{n}}_{\mathbb{L}(V)}$ are disjoint unions of schemes on the form $\mathbf{P}(\Delta^P) \backslash \mathbf{P}(\partial\Delta^P)$, one for each face in $U^{\mathbf{n}}$ and $V^{\mathbf{n}}$ respectively, it follows that the entire morphism $\phi : U \to V$ is surjective (resp. injective). By induction, we are done. □

4.4 Étale morphisms

The aim now is to prove the following theorem, which will be done through a series of lemmas.

Theorem 4.4.1. *Let $\phi : U \to V$ be a local isomorphism of open categories. Then $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is étale.*

Lemma 4.4.2. *Let $\phi : U \to V$ be a local isomorphism of open categories. Then,*

a) $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ *is flat;*

b) *for any other morphism* $\psi : W \to V$*, the induced morphism*

$$\pi_1 \times \pi_2 : \mathbf{A}(U \times_V W) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W) \tag{4.6}$$

is an isomorphism.

Proof. Let V be $\mathbf{n}$-dimensional. We will prove a) and b) simultaneously, proceeding by induction on $\mathbf{n}$. Note that the base case of $V = \varnothing$ is trivial, so we may assume that $\mathbf{n} \geq 0$. The Milnor square (4.5) for $\mathbf{A}(V)$ of level $\mathbf{n}$ is induced by the Milnor square (3.1) for $\mathbf{P}(\mathbb{L}(V))$ by restriction to $\mathbf{A}(V)$, and similarly for $\mathbf{A}(U)$. By Proposition B.2.1, (4.5) is locally a Milnor square of rings, hence we may apply Theorem B.1.1 a) to verify flatness of $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$. Interpreting the conditions of the proposition for schemes, we are required to show the following:

1) the morphisms $\phi^\Delta : \mathbf{A}(\Delta^{\mathbf{n}}_U) \to \mathbf{A}(\Delta^{\mathbf{n}}_V)$ and $\phi' : \mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{A}(V^{\mathbf{n}-1})$ are flat;

2) the canonical morphisms

$$\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \to \mathbf{A}(U^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(\partial\Delta^{\mathbf{n}}_V) \tag{4.7}$$

and

$$\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \to \mathbf{A}(\partial\Delta^{\mathbf{n}}_V) \times_{\mathbf{A}(\Delta^{\mathbf{n}}_V)} \mathbf{A}(\Delta^{\mathbf{n}}_U) \tag{4.8}$$

are isomorphisms.

1) $\phi^\Delta : \mathbf{A}(\Delta^{\mathbf{n}}_U) \to \mathbf{A}(\Delta^{\mathbf{n}}_V)$ is locally on the form $\mathbf{A}(\Delta^s_U) \to \mathbf{A}(\Delta^{\phi(s)}_V)$ for faces $s \in U$. By Lemma 4.3.2, each of these morphisms are open immersions — hence flat. The truncated morphism $\phi' : U^{\mathbf{n}-1} \to V^{\mathbf{n}-1}$ is a local isomorphism since $\phi : U \to V$ is, so $\phi' : \mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{A}(V^{\mathbf{n}-1})$ flat by the inductive hypothesis. 2) By the inductive hypothesis, the canonical morphism $\mathbf{A}(U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} \partial\Delta^{\mathbf{n}}_V) \to \mathbf{A}(U^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(\partial\Delta^{\mathbf{n}}_V)$ is an isomorphism. But since $\phi : U^{\mathbf{n}} \to V^{\mathbf{n}}$ is a local isomorphism, Lemma 4.2.7 implies that the induced morphism $\tau_{\mathbf{n}} : \Delta^{\mathbf{n}}_U \to U \times_V \Delta^{\mathbf{n}}_V$ is an isomorphism. Truncating yields an isomorphism $\tau'_{\mathbf{n}} : \partial\Delta^{\mathbf{n}}_U \to U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} \partial\Delta^{\mathbf{n}}_V$, thus composing to an isomorphism

$$\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \xrightarrow{\mathbf{A}(\tau'_{\mathbf{n}})} \mathbf{A}(U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} \partial\Delta^{\mathbf{n}}_V) \xrightarrow{\cong} \mathbf{A}(U^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(\partial\Delta^{\mathbf{n}}_V)$$

which is just (4.7). Locally, (4.8) is on the form $\mathbf{A}(\partial\Delta^s_U) \to \mathbf{A}(\partial\Delta^{\phi(s)}_V) \times_{\mathbf{A}(\Delta^{\phi(s)}_V)} \mathbf{A}(\Delta^s_U)$. Since $\mathbf{A}(\Delta^s_U) \to \mathbf{A}(\Delta^{\phi(s)}_V)$ is an open immersion, the fibered product is just the intersection $\mathbf{A}(\partial\Delta^{\phi(s)}_V) \cap \mathbf{A}(\Delta^s_U)$ in $\mathbf{A}(\Delta^{\phi(s)}_V)$. By Proposition 4.3.3,

$$\mathbf{A}(\partial\Delta^{\phi(s)}_V) = \mathbf{A}(\Delta^{\phi(s)}_V \cap \partial\Delta^{\phi(s)}) = \mathbf{A}(\Delta^{\phi(s)}_V) \cap \mathbf{P}(\partial\Delta^{\phi(s)}).$$

Hence

$$\mathbf{A}(\partial\Delta^{\phi(s)}_V) \cap \mathbf{A}(\Delta^s_U) = \mathbf{P}(\partial\Delta^{\phi(s)}) \cap \mathbf{A}(\Delta^{\phi(s)}_V) \cap \mathbf{A}(\Delta^s_U) = \mathbf{P}(\partial\Delta^s) \cap \mathbf{A}(\Delta^s_U) = \mathbf{A}(\partial\Delta^s_U).$$

This means that (4.8) is an isomorphism. We conclude that ϕ is flat.

Applying Theorem B.1.1 b) here shows that the canonical morphisms

$$\mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(V^{\mathbf{n}-1}), \tag{4.9}$$

$$\mathbf{A}(\Delta^{\mathbf{n}}_U) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\Delta^{\mathbf{n}}_V), \tag{4.10}$$

$$\mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_V) \tag{4.11}$$

are isomorphisms. Next, let $\psi : W \to V$ be any morphism. We will show that (4.6) is an isomorphism by induction on the dimension of W. However, if $\dim W < \mathbf{n}$, then $\mathbf{A}(W) \to \mathbf{A}(V)$ factors as $\mathbf{A}(W) \to \mathbf{A}(V^{\mathbf{n}-1}) \to \mathbf{A}(V)$, so by (4.9) and the inductive hypothesis,

$$\begin{aligned}\mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W) &\cong \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(V^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(W)\\ &\cong A(U^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(W) \cong A(U \times_V W).\end{aligned}$$

The case where W is $\mathbf{n}$-dimensional remains. Applying $\mathbf{A}(U) \times_{\mathbf{A}(V)} (-)$ to the the Milnor square (4.5) for $\mathbf{A}(W)$ yields a diagram

$$\begin{tikzcd} \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\Delta^{\mathbf{n}}_W) \arrow[r] & \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W) \\ \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W) \arrow[r] \arrow[u] & \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W^{\mathbf{n}-1}). \arrow[u] \end{tikzcd} \tag{4.12}$$

Since $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is flat, this is locally a Milnor square. To show that (4.6) is an isomorphism, we will compare (4.12) with the Milnor square for $\mathbf{A}(W \times_V U)$ of level $\mathbf{n}$:

$$\begin{tikzcd} \mathbf{A}(\Delta^{\mathbf{n}}_{U\times_V W}) \arrow[r] & \mathbf{A}(U \times_V W) \\ \mathbf{A}(\partial\Delta^{\mathbf{n}}_{U\times_V W}) \arrow[r] \arrow[u] & \mathbf{A}(U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} W^{\mathbf{n}-1}). \arrow[u] \end{tikzcd} \tag{4.13}$$

We require that the following induced morphisms are isomorphisms:

$$\begin{aligned}(*)\ &\mathbf{A}(\Delta^{\mathbf{n}}_{U\times_V W}) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\Delta^{\mathbf{n}}_W),\\ (**)\ &\mathbf{A}(U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} W^{\mathbf{n}-1}) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W^{\mathbf{n}-1}),\\ (***)\ &\mathbf{A}(\partial\Delta^{\mathbf{n}}_{U\times_V W}) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W).\end{aligned}$$

First, observe that for each face $(s,t) \in U \times_V W$, $\Delta^{(s,t)}_{U\times_V W} = \Delta^s_U \cap \Delta^t_W$ as subpolytopes of $\Delta^{(s,t)}$. This implies that the canonical morphism $\Delta^{(s,t)}_{U\times_V W} \to \Delta^s_U \times_{\Delta^{\psi t}_V} \Delta^t_W$ is an isomorphism. Since $\Delta^s \times_{\Delta^{\mathbf{n}}_V} \Delta^t_U = \varnothing$ whenever $\phi(s) \neq \psi(t)$, it follows that the canonical morphism $\Delta^{\mathbf{n}}_{U\times_V W} \to \Delta^{\mathbf{n}}_U \times_{\Delta^{\mathbf{n}}_V} \Delta^{\mathbf{n}}_W$ is an isomorphism. By truncation, this yields an isomorphism $\partial\Delta^{\mathbf{n}}_U \times_{\partial\Delta^{\mathbf{n}}_V} \partial\Delta^{\mathbf{n}}_W \xrightarrow{\cong} \partial\Delta^{\mathbf{n}}_{W\times_V U}$. The inductive hypothesis and the isomorphisms (4.9) and (4.11) shows that $(**)$ and $(***)$ are isomorphisms:

$$\begin{aligned}
(**)\ \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(W^{\mathbf{n}-1}) &\xrightarrow{\cong} \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(V^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(W^{\mathbf{n}-1}) \\
&\xrightarrow{\cong} \mathbf{A}(U^{\mathbf{n}-1}) \times_{\mathbf{A}(V^{\mathbf{n}-1})} \mathbf{A}(W^{\mathbf{n}-1}) \\
&\xrightarrow{\cong} \mathbf{A}(U^{\mathbf{n}-1} \times_{V^{\mathbf{n}-1}} W^{\mathbf{n}-1}), \\
(***)\ \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W) &\xrightarrow{\cong} \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_V) \times_{\mathbf{A}(\partial\Delta^{\mathbf{n}}_V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W) \\
&\xrightarrow{\cong} \mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \times_{\mathbf{A}(\partial\Delta^{\mathbf{n}}_V)} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W) \\
&\xrightarrow{\cong} \mathbf{A}(\partial\Delta^{\mathbf{n}}_U) \times_{\partial\Delta^{\mathbf{n}}_V} \mathbf{A}(\partial\Delta^{\mathbf{n}}_W) \\
&\xrightarrow{\cong} \mathbf{A}(\partial\Delta^{\mathbf{n}}_{U\times_V W}).
\end{aligned}$$

Finally, we will show that $(*)$ is an isomorphism. By (4.10), we have

$$\begin{aligned}
\mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\Delta^{\mathbf{n}}_W) &\xrightarrow{\cong} \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(\Delta^{\mathbf{n}}_V) \times_{\mathbf{A}(\Delta^{\mathbf{n}}_V)} \mathbf{A}(\Delta^{\mathbf{n}}_W) \\
&\xrightarrow{\cong} \mathbf{A}(\Delta^{\mathbf{n}}_U) \times_{\mathbf{A}(\Delta^{\mathbf{n}}_V)} \mathbf{A}(\Delta^{\mathbf{n}}_W).
\end{aligned}$$

Thus we are require that the canonical morphism $\mathbf{A}(\Delta^{\mathbf{n}}_{U\times_V W}) \to \mathbf{A}(\Delta^{\mathbf{n}}_U) \times_{\mathbf{A}(\Delta^{\mathbf{n}}_V)} \mathbf{A}(\Delta^{\mathbf{n}}_W)$ is an isomorphism. Since $\mathbf{A}(\Delta^s_U) \times_{\mathbf{A}(\Delta^{\mathbf{n}}_V)} \mathbf{A}(\Delta^t_W) = \varnothing$ whenever $\phi(s) \neq \psi(t)$, this locally amounts to showing that the morphisms $\mathbf{A}(\Delta^{(s,t)}_{U\times_V W}) \to \mathbf{A}(\Delta^s_U) \times_{\mathbf{A}(\Delta^{\psi t}_V)} \mathbf{A}(\Delta^t_W) = \mathbf{A}(\Delta^s_U) \cap \mathbf{A}(\Delta^t_W)$ are isomorphisms for each face $(s,t) \in U \times_V W$, or in other words, that $\mathbf{A}(\Delta^{(s,t)}_{U\times_V W}) = \mathbf{A}(\Delta^s_U) \cap \mathbf{A}(\Delta^t_W)$ as open subsets of $\mathbf{P}(\Delta^{(s,t)})$. This is equivalent to showing that $\mathbf{P}((\Delta^{(s,t)}_{U\times_V W})^\partial) = \mathbf{P}((\Delta^s_U)^\partial) \cup \mathbf{P}((\Delta^t_W)^\partial)$. First write $\mathbf{P}((\Delta^s_U)^\partial) \cup \mathbf{P}((\Delta^t_W)^\partial) = \mathbf{P}((\Delta^s_U)^\partial \cup (\Delta^t_W)^\partial)$. Then observe that $(\Delta^s_U)^\partial \cup (\Delta^t_W)^\partial = \Delta^{(s,t)} \backslash (\Delta^s_U \cap \Delta^t_W)$. But $\Delta^s_U \cap \Delta^t_W = \Delta^{(s,t)}_{U\times_V W}$, so $(\Delta^s_U)^\partial \cup (\Delta^t_W)^\partial = (\Delta^{(s,t)}_{U\times_V W})^\partial$. We conclude that $\mathbf{P}((\Delta^{(s,t)}_{U\times_V W})^\partial) = \mathbf{P}((\Delta^s_U)^\partial) \cup \mathbf{P}((\Delta^t_W)^\partial)$, and that $(*)$ is an isomorphism. This shows that the diagrams (4.12) and (4.13) are isomorphic. We conclude that the canonical morphism $\mathbf{A}(U \times_V W) \to \mathbf{A}(U \times_V W)$ is an isomorphism. By induction, we are done. $\square$

Lemma 4.4.3. *Let $\phi : U \to V$ be a local isomorphism of open categories. Then the diagonal morphism $D : U \to U \times_V U$ is an injective local isomorphism.*

Proof. The identity map $U \to U$ factors as $U \xrightarrow{D} U \times_V U \xrightarrow{\mathrm{pr}_2} U$, and by Lemma 4.1.6 d), the projection map pr_2 is a local isomorphism. It follows that D is a local isomorphism, and it is clearly injective. $\square$

Lemma 4.4.4. *Let M be a polyhedral set, and let $N \subseteq M$ be a polyhedral subset. Then the complement $\mathbb{T}(M)\backslash\mathbb{T}(N) \subseteq \mathbb{T}(M)$ induces an open immersion $\mathbf{A}(\mathbb{T}(M)\backslash\mathbb{T}(N)) \to \mathbf{P}(M)$, identifying $\mathbf{A}(\mathbb{T}(M)\backslash\mathbb{T}(N))$ with $\mathbf{P}(M)\backslash\mathbf{P}(N)$.*

Proof. Let M be $\mathbf{n}$-dimensional. We proceed by induction on $\mathbf{n}$, noting that the base case of $M = \varnothing$ is trivial. Define $W = \mathbf{P}(M)\backslash\mathbf{P}(N)$. Let $U = \mathbb{T}(M)\backslash\mathbb{T}(N)$, and let $\phi : U \to \mathbb{T}(M)$ denote the inclusion. Consider the induced morphism $\mathbb{TL}(\phi) : \mathbb{TL}(U) \to \mathbb{T}(M)$. Clearly, $\mathbb{TL}(\phi)^{-1}(\mathbb{T}(N)) \subseteq U^{\partial}$, so the morphism $\mathbf{P}(\mathbb{L}(U)) \to \mathbf{P}(M)$ restricts to a morphism $\mathbf{A}(U) \to W$. Consider the Milnor square (3.1) of $\mathbf{P}(M)$ of level $\mathbf{n}$, and note that $\mathbf{P}(M^{\mathbf{n}-1}) \cap W = \mathbf{P}(M^{\mathbf{n}-1})\backslash\mathbf{P}(N^{\mathbf{n}-1})$. By Proposition B.2.1 c), the following diagram

$$\begin{tikzcd}
\alpha_M^{-1}(W) \arrow[r] & W \\
\alpha_M^{-1}(W) \cap \mathbf{P}(\partial\Delta_M^{\mathbf{n}}) \arrow[r] \arrow[u] & \mathbf{P}(M^{\mathbf{n}-1})\backslash\mathbf{P}(N^{\mathbf{n}-1}) \arrow[u]
\end{tikzcd}$$

is a pushout square. We will compare this with the Milnor square (4.5) of $\mathbf{A}(U)$ of level $\mathbf{n}$. We require that the canonical morphisms

$$(*)\ \mathbf{A}(\Delta_U) \to \alpha_M^{-1}(W)$$
$$(**)\ \mathbf{A}(U^{\mathbf{n}-1}) \to \mathbf{P}(M^{\mathbf{n}-1})\backslash\mathbf{P}(N^{\mathbf{n}-1})$$
$$(***)\ \mathbf{A}(\partial\Delta_U^{\mathbf{n}}) \to \alpha_M^{-1}(W) \cap \mathbf{P}(\partial\Delta_M^{\mathbf{n}})$$

are isomorphisms. First we note that $(**)$ is an isomorphism by the inductive hypothesis. Next, observe that $\alpha_M^{-1}(W) = \mathbf{P}(\Delta_M^{\mathbf{n}})\backslash\mathbf{P}(\alpha_M^{-1}(N))$. This means that $\alpha_M^{-1}(W) \cap \mathbf{P}(\partial\Delta_M^{\mathbf{n}}) = \mathbf{P}(\partial\Delta_M^{\mathbf{n}})\backslash\mathbf{P}(\alpha_M^{-1}(N)^{\mathbf{n}-1})$. Since M is $\mathbf{n}$-dimensional, U is maximally $\mathbf{n}$-dimensional, so the induced morphism $\Delta_{\mathbb{L}(U)}^{\mathbf{n}} \to \Delta_M^{\mathbf{n}}$ is an injection. Clearly, $\Delta_M^{\mathbf{n}}\backslash\Delta_{\mathbb{L}(U)}^{\mathbf{n}} \subseteq \alpha_M^{-1}(N)$, and it follows immediately that $\Delta_U^{\mathbf{n}} = \alpha_{\mathbb{L}(U)}^{-1}(U) \to \Delta_M^{\mathbf{n}}\backslash\alpha_M^{-1}(N)$ is an isomorphism. Truncating yields an isomorphism $\partial\Delta_U^{\mathbf{n}} \to \partial\Delta_M^{\mathbf{n}}\backslash\alpha_M^{-1}(N)^{\mathbf{n}-1}$, so $(***)$ is an isomorphism by the inductive hypothesis. It remains to verify that $(*)$ is an isomorphism. This amounts to showing that for every $\mathbf{n}$-dimensional face $s \in U$, the morphism $\mathbf{A}(\Delta_U^s) \to \mathbf{P}(\Delta^s)\backslash\mathbf{P}((\zeta_M^s)^{-1}(N))$ is an isomorphism. Since $\mathbf{A}(\Delta_U^s) = \mathbf{P}(\Delta^s)\backslash\mathbf{P}((\Delta_U^s)^{\partial})$, we reduce to showing that $(\Delta_U^s)^{\partial} \xrightarrow{\cong} (\zeta_M^s)^{-1}(N)$. But note that $(\Delta_U^s)^{\partial} = \Delta^s\backslash(\zeta_{\mathbb{L}(U)}^s)^{-1}(U)$ and $(\zeta_M^s)^{-1}(N) = \Delta^s\backslash(\zeta_M^s)^{-1}(U)$, and finally observe that $(\zeta_{\mathbb{L}(U)}^s)^{-1}(U) = (\zeta_M^s)^{-1}(U)$. We conclude that $(*)$ is an isomorphism. These isomorphisms induces an isomorphism of pushout squares, showing that $\mathbf{A}(U) \to W$ is an isomorphism. By induction, we are done. $\square$

Lemma 4.4.5. *Let $\phi : U \to V$ be an injective local isomorphism. Then $\phi : \mathbf{A}(U) \to \mathbf{A}(V)$ is an open immersion.*

Proof. First note that the composition $U \xrightarrow{\phi} V \xrightarrow{\eta_V} \mathbb{TL}(V)$ is an injective local isomorphism by Lemma 4.1.7. By definition, $\eta_V : \mathbf{A}(V) \to \mathbf{P}(\mathbb{L}(V))$ is an open immersion, so we may assume that V is on the form $\mathbb{T}(M)$ for some polyhedral set M. By Lemma 4.1.9, the open subcategory $\mathbb{T}(M)\backslash U$ of $\mathbb{T}(M)$ may be identified with $\mathbb{T}(N)$ for some polyhedral subset N of M. But $U \cong \mathbb{T}(M)\backslash\mathbb{T}(N)$, so Lemma 4.4.4 implies that $\mathbf{A}(U) \to \mathbf{P}(M)$ is an open immersion. $\square$

Remark 4.4.6. Let M be a polyhedral complex as in Section 2.6 (or more generally a polyhedral poset), and let s be a face of M. Then the morphism $M_s \to M$ is injective. Indeed, suppose that (t, f_1),(t, f_2) is a pair of faces of M_s. Since $\zeta^t_M : \Delta^t \to M$ is injective, f_1 must be equal to f_2. By Proposition 4.1.3 and Lemma 4.4.5, the morphism $\mathbf{A}(M_s) \to \mathbf{P}(M)$ is an open immersion.

We are now in a position to prove Theorem 4.4.1:

Proof. By Lemma 4.4.2 a), ϕ is flat. By Lemma 4.4.3 and Lemma 4.4.5, the morphism $\mathbf{A}(D) : \mathbf{A}(U) \to \mathbf{A}(U \times_V U)$ is an open immersion. By Lemma 4.4.2 b), this is the same as the diagonal morphism $\mathbf{A}(U) \to \mathbf{A}(U) \times_{\mathbf{A}(V)} \mathbf{A}(U)$. This shows that ϕ is unramified as well, hence étale. □

Proposition 4.4.7. *Let G be a group acting freely on a polyhedral set M via a homomorphism $\rho : G \to \mathrm{Aut}(M)$, and let $\pi : M \to M/G$ be the categorical group quotient. Then π is a local isomorphism.*

Proof. Let s be a face of M, and consider the local morphism $\pi_s : M_s \to (M/G)_{\overline{s}}$, where $\overline{s}$ is the image of s. Suppose that $s \xrightarrow{f} t$ and $s \xrightarrow{f} t'$ are two objects of M_s mapping to the same object $\overline{s} \xrightarrow{f} \overline{t}$ of $(M/G)_{\overline{s}}$. Then $\overline{t'} = \overline{t}$, so there exists an element $g \in G$ such that $\rho(g)(t) = t'$. But then $f : s \to t$ and $f : \rho(g)(s) \to t$ are two liftings of the same arrow f in $\mathcal{P}$. By uniqueness of lifts, $s = \rho(g)(s)$. Since G acts freely, g is equal to the identity element e. This means that π_s is injective. Next, let $\overline{s} \xrightarrow{f} \overline{t}$ be an object of $(M/G)_{\overline{s}}$, where $\overline{t}$ is the image of some face t of M. Let $f : s' \to t$ be the unique lifting of f in $\mathcal{P}$. Then by uniqueness of lifts in (M/G), $\overline{s'} = \overline{s}$, so there exists an element $g \in G$ such that $\rho(g)(s') = s$. Thus the object $s \xrightarrow{f} \rho(g^{-1})(t)$ of M_s is mapped to $\overline{s} \xrightarrow{f} \overline{t}$ in $(M/G)_{\overline{s}}$, and π_s is therefore surjective. Thus π_s is bijective, and therefore an isomorphism. □

Propositions 4.1.3 and 4.4.7 and Theorem 4.4.1 immediately yields the following corollaries.

Corollary 4.4.8. *Let U be an open category, and $s \in U$ as face of U. Then the induced morphism $\mathbf{A}(U_s) \to \mathbf{A}(U)$ is étale.*

Corollary 4.4.9. *Let G be a group acting freely on a polyhedral set M, and let $\pi : M \to M/G$ be the categorical group quotient. Then the categorical group quotient $\pi : \mathbf{P}(M) \to \mathbf{P}(M/G)$ of schemes is étale.*

Chapter 5

The link of a polyhedral set

In this chapter we will define the *link* of a unimodular open category, and later more generally of an open category over the category $\mathcal{P}^{\mathrm{sm}}$ of smooth polytopes. The link of an open category U at a face s is denoted $\mathrm{lk}_U(s)$, and is generally a unimodular set. This generalizes the ordinary link construction[MS05, Definition 1.38] for simplicial complexes. We will further show that there are isomorphisms

$$\mathbf{A}(M_s) \cong \operatorname{Spec}\Gamma(\mathrm{lk}_M(s)) \times \mathbf{G}_m^{\dim s},$$

which will become useful in Chapter 6 and Chapter 7. In particular, the relation between links $\mathrm{lk}_M(s)$ and the local topology of $|M|$ becomes crucial when it comes to the topological characterization of Cohen-Macaulay and Gorenstein properties we shall see in Theorem 6.3.3 and Theorem 6.4.6.

5.1 The link construction

Let s be a face of an open category U over Fin. Let $n = |s|$, and consider the open category U_s. Then U_s defines a functor $U_s : (n \downarrow \mathrm{Fin})^{op} \to \mathrm{Set}$ as follows: On objects, $(m, f) \mapsto \{(t, f) \in U_s : |t| = n\}$. Let $h : (m, f) \to (r, g)$ be an arrow in $(n \downarrow \mathrm{Fin})$. Then $g = h \circ f$ in Fin. For any object (t, g) in $U_s(r, g)$, the factorization $n \xrightarrow{f} m \xrightarrow{h} r$ lifts to a unique factorization $s \xrightarrow{f} t' \xrightarrow{h} t$ of g, and we define $U_s(h)$ by $(t, g) \mapsto (t', f)$. The assignment $U \mapsto U_s$ is natural in pairs (U, s). We define a functor $M_n : (n \downarrow \mathrm{Fin}) \to \mathrm{Fin}_+$ as follows: On objects, we let $M_n(m, f) = [m] \setminus \operatorname{im} f$, which is (uniquely) identified with the ordinal $[m - n - 1] \in \mathrm{Fin}_+$. If $h : (m, f) \to (r, g)$ is an arrow, then $M_n(h) : [m] \setminus \operatorname{im} f \to [r] \setminus \operatorname{im} g$ is given by the restriction of $h : [m] \to [r]$, and $\lambda_{M_n(h)} : \mathbb{Z}^{[m-n-1]} \to \mathbf{G}_m$ is given by $e_i \mapsto \lambda_h(e_j)$, where $j \mapsto i$ via $[m] \setminus \operatorname{im} f \to [m - n - 1]$.

Definition 5.1.1. Let n be an object of Fin. We define the *link functor* $\mathrm{lk} : \mathrm{Pre}((n \downarrow \mathrm{Fin})) \to \mathrm{Pre}(\mathrm{Fin}_+)$ as the Yoneda extension of $Y_{\mathrm{Fin}_+} \circ M_n : (n \downarrow \mathrm{Fin}) \to \mathrm{Pre}(\mathrm{Fin}_+)$ along the Yoneda embedding $Y_{(n\downarrow\mathrm{Fin})}$.

$$\begin{tikzcd}
(n \downarrow \mathrm{Fin}) \arrow[r, "Y_{\mathrm{Fin}_+} \circ M_n"] \arrow[d, "Y_{(n\downarrow\mathrm{Fin})}"'] & \mathrm{Pre}(\mathrm{Fin}_+) \\
\mathrm{Pre}((n \downarrow \mathrm{Fin})) \arrow[ur, "\mathrm{lk}"'] &
\end{tikzcd}$$

Let $j : \mathrm{Fin} \to \mathrm{Fin}_+$ denote the embedding of categories. For any pair (U, s), we define the *link of s in U* as the unimodular set $\mathrm{lk}(U_s) \circ j^{\mathrm{op}} : \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Set}$, denoted $\mathrm{lk}_U(s)$. This is functorial in pairs (U, s).

For any pair (U, s), we have $\mathrm{lk}_U(s) = \varinjlim_{s \xrightarrow{f} t} \Delta^{M_{|s|}((|t|,f))}$. Note the similarity with the description of $\mathbb{L}(U_s)$ as $\varinjlim_{s \xrightarrow{f} t} \Delta^t$. In fact — as we will see — there is a close connection between the face rings of $\mathrm{lk}_U(s)$ and $\mathbb{L}(U_s)$.

Definition 5.1.2. A functor $F : A \to B$ is fibered in groupoids if

- for every arrow $f : b' \to b$ in B and object $a \in A$ with $F(a) = b$, there exists a lifting $g : a' \to a$ of f in A, and
- for every pair of arrows $g : a' \to a$, $g' : a'' \to a$ and any arrow $f : F(a'') \to F(a')$ such that $F(g)f = F(g')$, there exist a unique lifting $h : a'' \to a'$ of f such that $gh = g'$.

If F is fibered in groupoids, then the fiber category $F^{-1}(b)$ (consisting of objects $a \in A$ such that $F(a) = b$, and arrows $g : a \to a'$ such that $F(g) = \mathrm{id}_b$) is a groupoid for each $b \in B$.

Lemma 5.1.3. *For each $n \in \mathrm{Fin}$, the functor $M_n : (n \downarrow \mathrm{Fin}) \to \mathrm{Fin}_+$ is fibered in groupoids.*

Proof. Let $(m, f) \in (n \downarrow \mathrm{Fin})$, and let $h : [r] \to M_n(m, f) = [m - n - 1]$ be an arrow. Let $g : [n] \to [r+n+1]$ be any arrow, and $h' : [r+n+1] \to [m]$ the unique arrow such that $h'g = f$ and such that when restricted to $[r + n + 1] \setminus \mathrm{im}(g)$, it agrees with the arrow $[r + n + 1] \setminus \mathrm{im}(g) \to [m] \setminus \mathrm{im}(f)$ induced by h. Then h defines an arrow $(r + n + 1, g) \to (m, f)$ such that $M_n(h') = h$. Next, let $g : (m', f') \to (m, f)$ and $g' : (m'', f'') \to (m, f)$ be a pair arrows, and let $h : M_n(m'', f'') \to M_n(m', f')$ be an arrow such that $M_n(g)h = M_n(g')$. Via the construction above, there exists a lifting $h' : (m'', f'') \to (m', f')$ of h. Moreover, the fact that $(gh')f'' = gf' = g'f''$ and $M_n(gh') = M_n(g')$ implies that $gh' = g$. This further implies uniqueness, since every arrow in Fin is a monomorphism. □

Lemma 5.1.4. *Let U be a unimodular open category, and let s be a face of U. Then,*

a) $\mathrm{lk}_U(s)_n = \{(t, f) \in U_s : \dim t = \dim s + n + 1$ *and* $t \in U^{\mathrm{can}}\}$ *for all* $n \in \mathrm{Fin}$*, and*

b) *every arrow in $\mathbb{T}(\mathrm{lk}_U(s))$ is on the form $(t, f) \xrightarrow{M_n(g)} (u, gf)$. Hence $\mathrm{lk}_U(s)$ satisfies $(*)$.*

Proof. Let $[r] = |s|$. By Lemma 5.1.3, M_r is fibered in groupoids. Hence by a similar argument as in Lemma A.0.6 a), $\mathrm{lk}_U(s)_n = \varinjlim_{M_r(m,f)=[n]} U_s(m, f)$, where the colimit is taken over the fiber of $[n]$, i.e. the groupoid of objects $(m, f) \in (r \downarrow \mathrm{Fin})$ such that $M_r(m, f) = [n]$. It follows from the condition $(*)$ that objects on the form (t, f) where $t \in U^{\mathrm{can}}$ is a choice of representatives for each isomorphism class in this groupoid. Hence a) gives the desired representation of the colimit.

b) Suppose that $(u, g) \in U_s(m', g)$, and let $h : [n] \to M_r(m', g)$ be any arrow. Then there exists a lifting of h to an arrow $h' : (m, f) \to (m', g)$ such that $M_r(h') = h$, unique if f is fixated. Here $g = h'f$. By definition, $h^{\mathrm{lk}_U(s)}(u, g) = (t, f)$, where $t \xrightarrow{h'} u$ is the unique lifting of $m \xrightarrow{h'} m'$ in U. The requirement that t is a canonical representative fixes f, and therefore h'. This yields the arrow $(t, f) \xrightarrow{M_n(h')} (u, h'f)$ in $\mathbb{T}(\mathrm{lk}_U(s))$, and clearly every arrow arises in this way. $\square$

5.2 Properties and relations with the face scheme

Lemma 5.2.1. *Let $m \in \mathrm{Fin}$, and let $f \in \Delta^m_n$ be any face. Then $\mathrm{lk}_{\Delta^m}(f) \cong \Delta^{M_n(m,f)}$.*

Proof. This follows immediately from the observation that the open category Δ^m_f defines the representable functor $(n \downarrow \mathrm{Fin})(-, (m, f)) : (n \downarrow \mathrm{Fin})^{\mathrm{op}} \to \mathrm{Set}$. $\square$

Remark 5.2.2. Let K be a simplicial complex. The ordinary link construction for K at a face s is defined as $\{t \in K : t \cup s \in M \text{ and } t \cap s = \varnothing\}$. Let M be the polyhedral complex corresponding to K. By Remark 4.4.6, the morphism $M_s \to M$ is injective. Moreover, by Lemma 5.1.4, we have $\mathrm{lk}_M(s)_n = \{t \in \mathbb{T}(M)^{\mathrm{can}} : s \subseteq t \text{ and } |t| = |s| + n + 1\}$. Thus the assignment $t \mapsto t \backslash s$ defines an isomorphism between the two versions of links.

Lemma 5.2.3. *Let U be an open category over* Fin*, s a face of U, and (t, f) a face of $\mathrm{lk}_U(s)$. Then $\mathrm{lk}_{\mathrm{lk}_U(s)}(t, f) \cong \mathrm{lk}_U(t)$.*

Proof. Let $|s| = n$ and $|t| = m$. Consider the functor $F : (m \downarrow \mathrm{Fin}) \to (M_n(m, f) \downarrow \mathrm{Fin})$ defined as follows. Any object (r, g) of $(m \downarrow \mathrm{Fin})$ can be considered as an arrow $g : (m, f) \to (r, gf)$ in $(n \downarrow \mathrm{Fin})$, thus inducing an arrow $M_n(g) : M_n(m, f) \to M_n(r, gf)$. We define $F(r, g) = (M_n(r, gf), M_n(g))$. Any arrow $h : (r_1, g_1) \to (r_2, g_2)$ in $(m \downarrow \mathrm{Fin})$ can be considered as an arrow $h : (r_1, g_1 f) \to (r_2, g_2 f)$ in $(n \downarrow \mathrm{Fin})$, thus inducing and arrow $M_n(h) : M_n(r_1, g_1 f) \to M_n(r_2, g_2 f)$. Since $M_n(h) \circ M_n(g_2) = M_n(g_1)$, this defines an arrow $F(h) : F(\overline{r_1}, g_1) \to F(\overline{r_2}, g_2)$. Functoriality of F is clear.

We will now define a natural transformation of functors $\mu : U_t \to \mathrm{lk}_U(s)_{(t,f)} \circ F$. But first we must unwind some of the definitions. Let (r, g) be an object of $(m \downarrow \mathrm{Fin})$, and let $(u, g) \in U_t(r, g)$. Then $(u, gf) \in U_s(r, gf)$, and we have an arrow $g : (t, f) \to (u, gf)$ in U_s as an open category. Consider the arrow $g : (m, f) \to (r, gf)$ in $(m \downarrow \mathrm{Fin})$. By definition, $U_s(g)(u, gf) = (t, f)$. Consider the universal transformation $U_s \to \mathrm{lk}(U_s) \circ M_n$. By naturality, the diagram

$$\begin{tikzcd}
U_s(r, gf) \arrow[r] \arrow[d, "U_s(g)"] & \mathrm{lk}(U_s)(M_n(r, gf)) \arrow[d, "\mathrm{lk}(U_s)(M_n(g))"] \\
U_s(m, f) \arrow[r] & \mathrm{lk}(U_s)(M_n(m, f))
\end{tikzcd}$$

commutes. It follows that $\mathrm{lk}_U(s)^{M_n(g)}(u, gf) = (t, f)$ as faces of $\mathrm{lk}_U(s)$. Hence $(t, f) \xrightarrow{M_n(g)} (u, gf)$ is a valid face of $\mathrm{lk}_U(s)_{(t,f)}$. In other words, $((u, gf), M_n(g)) \in \mathrm{lk}_U(s)_{(t,f)}(F(r, g))$. We define $\mu_{(r,g)} : U_t(r, g) \to \mathrm{lk}_U(s)_{(t,f)}(F(r, g))$ by $(u, g) \mapsto ((u, gf), M_n(g))$. To show that μ is natural, consider an arrow $h : (r_1, g) \to (r_2, hg)$ in $(m \downarrow \mathrm{Fin})$, and let $(u, hg) \in U_t(r_2, hg)$. By definition, $U_t(h)(u, hg)$ is equal to a face on the form $(v, g) \in U_t(r_1, g)$ such that $t \xrightarrow{g} v \xrightarrow{h} u$ is the unique lifting $m \xrightarrow{g} r_1 \xrightarrow{h} r_2$. It is clear that $F(h) = M_n(h) : ((v, gf), M_n(g)) \to ((u, hgf), M_n(hg))$ is a valid arrow in $\mathrm{lk}_U(s)_{(t,f)}$, and naturality of μ follows.

Next we show that there is an equality of functors $M_m = M_q \circ F$, where $q = M_n(m, f)$. It is clear that for each object (r, g), $M_m(r, g)$ and $M_q(M_n(r, gf), M_n(g))$ are equal as ordinals. If $h : (r, g) \to (r', hg)$ be an arrow, then $F(h) = M_n(h)$, and it is easily seen that $M_m(h) = M_q(M_n(h))$.

Consider the universal transformations $\eta_1 : U_t \to \mathrm{lk}(U_t) \circ M_m$ and $\eta_2 : \mathrm{lk}_U(s)_{(t,f)} \to \mathrm{lk}(\mathrm{lk}_U(s)_{(t,f)}) \circ M_q$. By the universal property of η_1, there exists a unique natural transformation $\delta : \mathrm{lk}(U_t) \to \mathrm{lk}(\mathrm{lk}_U(s)_{(t,f)})$ such that the induced diagram

$$\begin{tikzcd}
U_t \arrow[r, "\mu"] \arrow[dr, "\eta_1"'] & \mathrm{lk}_U(s)_{(t,f)} \circ F \arrow[r, "\eta_2 F"] & \mathrm{lk}(\mathrm{lk}_U(s)_{(t,f)}) \circ M_m \\
& \mathrm{lk}(U_t) \circ M_m \arrow[ur, "\delta M_m"'] &
\end{tikzcd}$$

commutes. This induces a morphism $\phi : \mathrm{lk}_U(t) \to \mathrm{lk}_{\mathrm{lk}_U(s)}(t, f)$ such that $\phi(u, g) = ((u, gf), M_n(g))$ for all faces (u, g) of $\mathrm{lk}_U(t)$. Now, every face of $\mathrm{lk}_{\mathrm{lk}_U(s)}(t, f)$ is on the form $((u, gf), M_n(g))$. Indeed, suppose that $((u, g), h)$ is a face, for some arrow $g : n \to r$. Then $h : (t, f) \to (u, g)$ is an arrow in $\mathbb{T}(\mathrm{lk}_U(s))$, which by Lemma 5.1.4 b) is on the form $M_n(g') : (t, f) \to (u, g'f)$. Hence $((u, g), h) = ((u, g'f), M_n(g'))$. It follows that ϕ is surjective. Next, suppose that $\phi(u, g) = \phi(v, h)$. Then $((u, gf), M_n(g)) = ((v, hf), M_n(h))$, so $u = v$, $gf = hf$ and $M_n(g) = M_n(h)$. Define $f' := gf = hf : n \to r$. Then $g, h : (m, f) \to (r, f')$ are both liftings of $M_n(g) = M_n(h)$, hence equal by the proof of Lemma 5.1.3. It follows that ϕ is injective, and therefore an isomorphism. □

Lemma 5.2.4. *Let M, N be a pair of unimodular sets, $s \in M$ and $t \in N$. Then $\mathrm{lk}_{M \star N}((s, t)) \cong \mathrm{lk}_M(s) \star \mathrm{lk}_N(t)$.*

Proof. Consider the representation of the links from Lemma 5.1.4 a), and the representation of join from Remark 2.2.7. This allows us to define a morphism $\phi : \mathrm{lk}_M(s) \star \mathrm{lk}_N(t) \to \mathrm{lk}_{M \star N}((s, t))$ by $((u, f), (v, g)) \mapsto ((u, v), f + g)$, which is well-defined and clearly an isomorphism by Lemma 5.1.3. □

Lemma 5.2.5. *Let P be a polytope, $v \in P$ a vertex, and $f : 0 \to P$ the associated arrow. Then the closed subsets $V(\chi^{(v,1)})$ and $\mathbf{P}((\Delta^P_f)^\partial)$ of $\mathbf{P}(\Delta^P)$ are equal.*

Proof. By Lemma 3.6.1, we have $\mathbf{P}(\Delta^Q) = V(\sigma_Q) = V(\chi^{(m,d)} | (m, d) \notin \sigma_Q) \subseteq V(\chi^{(v,1)})$ for all $v \notin Q$, whenever $Q \to P$ is a face. Hence $\mathbf{P}((\Delta^P_f)^\partial) =$

$\bigcup_{v\notin F}\mathbf{P}(\Delta^F) \subseteq V(\chi^{(v,1)})$. Now suppose that $\chi^{(v,1)}(p) = 0$ for some point $p \in \mathbf{P}(\Delta^P)$. To show the reverse inclusion, we require that $\chi^{(m,d)}(p) = 0$ for all $(m,d) \notin \sigma_F$ whenever $v \notin F$. Suppose that $(m,d) \in C(P)$ is such a lattice point. Then for each integer k, we have $\chi^{(m,d)}(p)^k = \chi^{(km,kd)}(p)$. If it is possible to write $\chi^{(km,kd)} = \chi^{(v,1)}\chi^{(m',d')}$ for some $k > 0$ and $(m',d') \in C(P)$, we are done. In other words, we require that $km-v \in (kd-1)P \Leftrightarrow \langle km-v, u_F\rangle \geq -(kd-1)a_F$ for all facets F. Rewriting the inequality as $dk(\langle \frac{m}{d}, u_F\rangle + a_F) \geq \langle v, u_F\rangle + a_F$, one observes that it is obtained for all F such that $\frac{m}{d} \notin F$ for k sufficiently large. On the other hand, if $\frac{m}{d} \in F$, then $v \in F$ as well, so the inequality is automatically satisfied. □

For each arrow $g : Q \to P$, define $\theta_g = \prod_{f:0\to Q}\chi^{(gf(0),1)} \in \Gamma(\Delta^P)$. If U is an open category, and s is a face of U, then we define $\theta_s \in \Gamma(\mathbb{L}(U_s))$ via its restriction to $\theta_f \in \Gamma(\Delta^{(t,f)})$ for each $(t,f) \in U_s$.

Corollary 5.2.6. *Let $f : Q \to P$ be any arrow in $\mathcal{P}$. Then the closed subsets $V(\theta_f)$ and $\mathbf{P}((\Delta^P_f)^\partial)$ of $\mathbf{P}(\Delta^P)$ are equal.*

Proof. By Lemma 5.2.5, we have to show that $(\Delta^P_f)^\partial = \bigcup_{g:0\to Q}(\Delta^{(P,f)}_{fg})^\partial$, i.e. $\Delta^P_f = \bigcap_{g:0\to Q}\Delta^P_{fg}$. The open subcategory $\Delta^P_f \subseteq \Delta^P$ consists of the arrows $h : R \to P$ which factors $f : Q \to P$. For any arrow $g : 0 \to Q$, an arrow $h : R \to P$ factoring f also factors fg, so $\Delta^P_f \subseteq \Delta^P_{fg}$. For the other direction, suppose that $h : R \to P$ factors every arrow on the form $fg : 0 \to P$. Then R must contain every vertex of Q, and thus contains Q. One can easily produce a factorization $Q \to R \xrightarrow{h} P$ of f. □

Proposition 5.2.7. *Let U be an open category, and s a face of U. Then the open subschemes $\mathbf{A}(U_s)$ and $D_+(\theta_s)$ of $\mathbf{P}(\mathbb{L}(U_s))$ are equal.*

Proof. By Corollary 5.2.6, there is an exact sequence

$$0 \to \sqrt{(\theta_f)} \to \Gamma(\Delta^{(t,f)}) \to \Gamma((\Delta^{(t,f)}_{U_s})^\partial) \to 0 \tag{5.1}$$

for each face (t,f) of U_s. This sequence is moreover natural in (t,f). Now, observe that $\Delta^{(t,f)}_{U_s} = \Delta^t_f$ as open subcategories of Δ^t. Indeed, both Δ^t_f and $\Delta^{(t,f)}_{U_s}$ can be viewed as the subset of Δ^t consisting of arrows $g : R \to |t|$ which factors $f : |s| \to |t|$. It follows that $(\zeta^{(t,f)}_{U_s})^{-1}(U^\partial_s) = (\Delta^t_f)^\partial$. Since colimits in the category of sets commutes with fibered products, there are isomorphisms $\varinjlim_{s\xrightarrow{f}t}(\Delta^t \times_{\mathbb{L}(U_s)} U^\partial_s) \xrightarrow{\cong} (\varinjlim_{s\xrightarrow{f}t}\Delta^t) \times_{\mathbb{L}(U_s)} U^\partial_s$. In other words, $\varinjlim_{s\xrightarrow{f}t}(\Delta^t_f)^\partial \xrightarrow{\cong} U^\partial_s$. This is a connected colimit, so by Proposition 2.3.3 we have that $\Gamma(U^\partial_s) \xrightarrow{\cong} \varprojlim_{s\xrightarrow{f}t}\Gamma((\Delta^t_f)^\partial)$. Taking the limit of (5.1), one obtains an exact sequence

$$0 \to \varprojlim_{s\xrightarrow{f}t}\sqrt{(\theta_f)} \to \Gamma(\mathbb{L}(U_s)) \to \Gamma(U^\partial_s) \to 0.$$

It is easy to show that $\varprojlim_{s \xrightarrow{f} t} \sqrt{(\theta_f)} = \sqrt{(\theta_s)}$ as ideals of $\Gamma(\mathbb{L}(U_s))$, and the result follows from this. $\square$

Lemma 5.2.8. *Let U be an open category over* Fin*, and let s be a face of U. Then there exists an isomorphism $\mathrm{lk}_U(s) \star \Delta^s \xrightarrow{\cong} \mathbb{L}(U_s)$, which restricts to an isomorphism $\mathrm{lk}_U(s) \star \partial\Delta^s \xrightarrow{\cong} U_s^\partial$.*

Proof. Define $n = |s|$. For each face (t, f) of U_s, let $a_f : [M_n(|t|, f)] \to [t]$ denote the arrow in Fin_+ corresponding to the inclusion of ordinals $[t] \setminus \mathrm{im}\, f \to [t]$. Consider the homeomorphism $\phi_{(t,f)} : I_{\mathrm{Fin}_+}(M_n(|t|, f)) \star |s| \to |t|$ given by

$$(x_0, \dots, x_{M_n(|t|,f)}, t, y_0, \dots, y_n) \mapsto (z_0, \dots, z_{|t|}),$$

where $z_{f(i)} = t y_i$ for each i, and $z_{a_f(j)} = (1-t)x_j$ for each j (we put $\lambda_{\phi_{(t,f)}} = 1$). Here we have identified $I_{\mathrm{Fin}_+}(M_n(|t|, f))$ with $\mathrm{Conv}(e_0, \dots, e_n) \subseteq \mathbb{R}^{[n]}$ for simplicity of notation. These maps are evidently natural in faces (t, f), and the corresponding isomorphisms $\psi_{(t,f)} : \Delta^{M_n(|t|,f)} \star \Delta^s \to \Delta^t$ induces an isomorphism $\mathrm{lk}_U(s) \star \Delta^s \to \mathbb{L}(U_s)$ upon taking colimits. By the proof of Proposition 5.2.7, there is an isomorphism $\varinjlim_{s \xrightarrow{f} t} (\Delta^t_f)^\partial \xrightarrow{\cong} U_s^\partial$, and — as observed — $(\Delta^t_f)^\partial$ can be written as the union $\bigcup_{j=0}^n \Delta^{d_{f(j)}}$. Note that the inclusions $|\Delta^{d_{f(j)}}| \to |\Delta^t|$ are identified with the restriction of $\phi_{(t,f)}$ to $|\Delta^{M_n(|t|,f)} \star \Delta^{d_j}|$. Since $|\partial\Delta^s| = \bigcup_{j=0}^n |\Delta^{d_j}|$, this identifies the restriction of $\psi_{(t,f)}$ to $\Delta^{M_n(|t|,f)} \star \partial\Delta^s$ with the inclusion $(\Delta^t_f)^\partial \to \Delta^t$. Taking colimits we obtain an isomorphism $\mathrm{lk}_U(s) \star \partial\Delta^s \to U_s^\partial$, which is the restriction of ϕ. $\square$

Corollary 5.2.9. *Let U be a unimodular open category, and let s a face of U. Then there is a natural isomorphism*

$$\Gamma(\mathrm{lk}_U(s)) \otimes \Gamma(\Delta^s)_{(\theta_{\mathrm{id}_{|s|}})} \xrightarrow{\cong} \Gamma(\mathbb{L}(U_s))_{(\theta_s)} \tag{5.2}$$

inducing an isomorphism

$$\mathbf{A}(U_s) \to \mathrm{Spec}(\Gamma(\mathrm{lk}_U(s))) \times_k (\mathbf{G}_m)^{\dim s}. \tag{5.3}$$

5.3 Extension to smooth polytopes

Let P be a polytope, and let v be a vertex of P. For each edge E containing v, let $w_{E,v}$ denote the first lattice point along E other than v. Consider the set of vectors $\{w_{E,v} - v\}_{v \in E} \in \mathbb{Z}^{\dim(P)}$. If this set forms a $\mathbb{Z}$-basis for $\mathbb{Z}^{\dim(P)}$ for all vertices v, then P is called *smooth*. Let $\mathcal{P}^{\mathrm{sm}}$ denote the full subcategory of $\mathcal{P}$ consisting of smooth polytopes respectively.

Definition 5.3.1. A *smooth* open category is an open category U over $\mathcal{P}^{\mathrm{sm}}$. Similarly, a polyhedral set M is called smooth if M_P is non-empty only for smooth polytopes. Note that Δ^P is smooth if and only if P is smooth. Also note that unimodular sets are smooth.

Remark 5.3.2. We continue with the notation above. Following [CLS11], $C_v = \mathrm{Cone}(P \cap \mathbb{Z}^{\dim(P)} - v)$ is the dual of the maximal cone $\sigma_v \in \Sigma_P$, and the corresponding affine toric variety U_{σ_v} is isomorphic to $\operatorname{Spec} k[C_v \cap \mathbb{Z}^{\dim(P)}]$. Since $\{w_{E,v} - v\}_{v \subset E}$ is a $\mathbb{Z}$-basis, the cone C_v is generated by the vectors $w_{E,v} - v$, so the assignment $(P, v) \mapsto C_v$ is functorial. Moreover, there is a natural isomorphism $k[C_v \cap \mathbb{Z}^{\dim(P)}] \to \Gamma(\Delta^P)_{(\chi^{(v,1)})}$ given on generators by $\chi^m \mapsto \frac{\chi^{(m,1)}}{\chi^{(v,1)}}$. Again, since $\{w_{E,v} - v\}_{v \subset E}$ is a $\mathbb{Z}$-basis, the ring $\Gamma(\Delta^P)_{(\chi^{(v,1)})}$ is a polynomial ring on generators $\frac{\chi^{(w_{E,v},1)}}{\chi^{(v,1)}}$.

We define a functor $M_{(P,v)} : (P \downarrow \mathcal{P}^{\mathrm{sm}}) \to \mathrm{Fin}_+$ as follows: On objects, we let

$$M_{(P,v)}(Q, f) = \{E \subseteq Q | E \text{ is an edge such that } f_{\mathbb{R}}(v) \subseteq Q \text{ and } E \not\subseteq f_{\mathbb{R}}(P)\}.$$

Since Q is smooth, $M_{(P,v)}(Q, f)$ has cardinality $\dim(Q)$. We order this set compatibly with the order on $\{w_{E,f(v)} - f(v)\}_{E \in M_{(P,v)}(Q,f)} \subseteq \mathbb{Z}^{\dim(P)}$ under the the lexicographical order on $\mathbb{Z}^{\dim(Q)}$. Hence $M_{(P,v)}(Q, f)$ can be identified with the ordinal $[\dim(Q) - \dim(P) - 1] \in \mathrm{Fin}_+$. Any arrow $h : (Q, f) \to (R, g)$ induces an inclusion of sets $M_{(P,v)}(Q, f) \subseteq M_{(P,v)}(R, g)$, and we define $M_{(P,v)}(h) : [\dim(Q) - \dim(P) - 1] \to [\dim(R) - \dim(P) - 1]$ to be the corresponding arrow in Fin_+, and the character $\lambda_{M_{(P,v)}(h)} : \mathbb{Z}^{M_{(P,v)}(Q,f)} \to \mathbf{G}_m$ is given by $e_E \mapsto \lambda_h(w_{E,f(v)}, 1)$. If P is a vertex itself, we will write M_0 for $M_{(0,0)}$.

Let s be a face of a smooth open category U, and let $P = |s|$. Analogous to before, U_s defines a functor $U_s : (P \downarrow \mathcal{P}^{\mathrm{sm}})^{op} \to \mathrm{Set}$: On objects $(Q, f) \mapsto \{(t, f) \in U_s : |t| = Q\}$. Let $h : (Q, f) \to (R, g)$ be an arrow in $(P \downarrow \mathcal{P}^{\mathrm{sm}})$. Then $g = h \circ f$ in $\mathcal{P}^{\mathrm{sm}}$. For any object (t, g) in $U_s(R, g)$, the factorization $P \xrightarrow{f} Q \xrightarrow{h} R$ lifts to a unique factorization $s \xrightarrow{f} t' \xrightarrow{h} t$ of g, and we define $U_s(h)$ by $(t, g) \mapsto (t', f)$.

Definition 5.3.3. We define the *link functor* $\mathrm{lk}^v : \mathrm{Pre}((P \downarrow \mathcal{P}^{\mathrm{sm}})) \to \mathrm{Pre}(\mathrm{Fin}_+)$ as the Yoneda extension of $Y_{\mathrm{Fin}_+} \circ M_{(P,v)} : (P \downarrow \mathcal{P}^{\mathrm{sm}}) \to \mathrm{Pre}(\mathrm{Fin}_+)$ along the Yoneda embedding $Y_{(P \downarrow \mathcal{P}^{\mathrm{sm}})}$. Let $j : \mathrm{Fin} \to \mathrm{Fin}_+$ denote the embedding of categories. For any pair (U, s) and vertex $v \in |s|$, we define the *link of s in U* as the unimodular set $\mathrm{lk}^v(U_s) \circ j^{\mathrm{op}} : \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Set}$, denoted $\mathrm{lk}^v_U(s)$. This is functorial in triples (U, s, v).

Continuing with the notation above, consider the homomorphism

$$\Gamma(\Delta^{M_{(P,v)}(Q,f)}) = k[x_E | E \in M_{(P,v)}(Q, f)] \to \Gamma(\Delta^Q)$$

given by $x_E \mapsto \chi^{(w_{E,f(v)},1)}$, and the homomorphism $\Gamma(\Delta^P) \to \Gamma(\Delta^Q)$ given by $\chi^{(m,d)} \mapsto \lambda_f(m, d)^{-1} \chi^{A_f(m,d)}$. They are both evidently natural in $(Q, f) \in (0 \downarrow \mathcal{P}^{\mathrm{sm}})$, and induce a natural homomorphism

$$\Big(\Gamma(\Delta^{M_{(P,v)}(Q,f)}) \otimes_k \Gamma(\Delta^P)\Big)_{(1 \otimes \theta_{\mathrm{id}_P})} \to \Gamma(\Delta^Q)_{(\theta_f)}. \tag{5.4}$$

This is in fact an isomorphism, which follows from the observation in Remark 5.3.2 (both sides can be viewed as polynomial rings localized at a monomial).

Proposition 5.3.4. *Let U be a smooth open category, let s be a face of U, and let $v \in |s|$ be a vertex. Then there is a natural isomorphism*

$$\Gamma(\mathrm{lk}_U^v(s)) \otimes_k \Gamma(\Delta^s)_{(\theta_{\mathrm{id}_{|s|}})} \xrightarrow{\cong} \Gamma(\mathbb{L}(U_s))_{(\theta_s)} \tag{5.5}$$

inducing an isomorphism

$$\mathbf{A}(U_s) \to \operatorname{Spec}(\Gamma(\mathrm{lk}_U^v(s))) \times_k \mathbf{G}_m^{\dim(s)}. \tag{5.6}$$

Proof. Let (t, g) be a face of U_s. Via (5.4), we have

$$\Gamma(\Delta^{M_{(|s|,v)}(|t|,g)}) \otimes_k \Gamma(\Delta^s)_{(\theta_{\mathrm{id}_{|s|}})} \to \Gamma(\Delta^t)_{(\theta_g)}, \tag{5.7}$$

naturally in $(t, g) \in U_s$. Taking the limit on both sides yields the desired injection (5.5). $\square$

5.4 Links and local topology

Let M be a polyhedral set, and let $p \in |M|$ be any point. Then there is a unique face $s(p)$ of M (such that $|s(p)|$ is a canonical representative) of minimal dimension such that $p \in |M^{s(p)}|$. We will also write p for the unique point in $|\Delta^{s(p)}|\backslash|\partial\Delta^{s(p)}|$ which maps to $p \in |M|$ via $|\zeta_M^{s(p)}|$. For each arrow $f : s(p) \to t$ in $\mathbb{T}(M)$, we define

$$B_f = \operatorname{int}(\epsilon(|\Delta^t| - f_{\mathbb{R}}(p)) + f_{\mathbb{R}}(p)) \subseteq |\Delta^t| \text{ (for some fixed } \epsilon < 1).$$

We also define $B_t = \bigcup_{f:s(p)\to t} B_f$. If ϵ is sufficiently small, the open subsets B_f are pairwise disjoint. Note that for each factorization $s \xrightarrow{h} u \xrightarrow{g} t$ of $f : s \to t$, we have

$$|\zeta_{\Delta^t}^g|(B_h) = \operatorname{im}|\zeta_{\Delta^t}^g| \cap B_f. \tag{5.8}$$

Proposition 5.4.1. *There exists a contractible open neighbourhood $B_\epsilon^M(p)$ of p such that*

(∗) $|\zeta_M^t|^{-1}(B_\epsilon^M(p)) = B_t$ *for each face t of M.*

Proof. Suppose first that $s(p)$ is a facet. Then $|\Delta^{s(p)}|\backslash|\partial\Delta^{s(p)}| \to |M|$ is the inclusion of a cell, and $B_{s(p)}$ defines the required contractible open neighbourhood satisfying (∗). We proceed by induction on the dimension $\mathbf{n}$ of M. If $\mathbf{n} = 0$, then $s(p)$ is necessarily a facet, and we are done. So assume that $\mathbf{n} \geq 1$, and that $s(p)$ is not a facet. By the inductive hypothesis, there exists a neighbourhood $B_\epsilon^{M^{\mathbf{n}-1}}(p) \subseteq |M^{\mathbf{n}-1}|$ of p satisfying (∗). Then $|\zeta_M^t|^{-1}(B_\epsilon^{M^{\mathbf{n}-1}}(p)) = B_t$ for each face t of $M^{\mathbf{n}-1}$. However, if t is an $\mathbf{n}$-dimensional face of M, then

$$|\zeta_M^t|^{-1}(B_\epsilon^{M^{\mathbf{n}-1}}(p)) = \bigcup_{\substack{u \xrightarrow{g} t \\ \dim u \leq \mathbf{n}-1}} |\zeta_{\Delta^t}^g|(B_u) = \bigcup_{\substack{u \xrightarrow{g} t \\ \dim u \leq \mathbf{n}-1}} \bigcup_{s \xrightarrow{h} u} \operatorname{im}|\zeta_{\Delta^t}^g| \cap B_\epsilon(|\zeta_{\Delta^t}^{gh}|(p)) \tag{5.9}$$

by (5.8). Any arrow $g : u \to t$ factors through some $(\mathbf{n}-1)$-dimensional face, so the union (5.9) remains unaffected if we restrict ourselves to those faces u with $\dim u = \mathbf{n}-1$. Since $B_\epsilon(|\zeta^{gh}_{\Delta^t}|(p))$ does not intersect any $(\mathbf{n}-1)$-dimensional subpolytope of $|\Delta^t|$ other than $\operatorname{im}|\zeta^g_{\Delta^t}|$, we have for each $(\mathbf{n}-1)$-dimensional face u that $\operatorname{im}|\zeta^g_{\Delta^t}| \cap B_\epsilon(|\zeta^{gh}_{\Delta^t}|(p)) = |\partial\Delta^t| \cap B_\epsilon(|\zeta^{gh}_{\Delta^t}|(p))$. The arrows $gh : s \to t$ ranges over all arrows $f : s \to t$ as g and h varies. In conclusion,

$$\begin{aligned} |\zeta^t_M|^{-1}(B^{M^{\mathbf{n}-1}}_\epsilon(p)) &= \bigcup_{\substack{u \xrightarrow{g} t \\ \dim u = \mathbf{n}-1}} \bigcup_{s \xrightarrow{h} u} |\partial\Delta^t| \cap B_\epsilon(|\zeta^{gh}_{\Delta^t}|(p)) \\ &= \bigcup_{s \xrightarrow{f} t} |\partial\Delta^t| \cap B_\epsilon(|\zeta^{gh}_{\Delta^t}|(p)) = |\partial\Delta^t| \cap B_t. \end{aligned}$$

Thus we may define $B^{M^{\mathbf{n}}}_\epsilon(p) = B^{M^{\mathbf{n}-1}}_\epsilon(p) \cup \bigcup_{\dim t = \mathbf{n}} B_t \backslash |\partial\Delta^t|$, which satisfies $(*)$. It is an open subset of $|M|$ since each $|\zeta^t_M|^{-1}(B^{M^{\mathbf{n}}}_\epsilon(p))$ is open. A homotopy from $B^{M^{\mathbf{n}-1}}_\epsilon(p)$ to p is obtained by letting $\epsilon \to 0$, so it is also contractible. By induction, we are done. $\square$

Lemma 5.4.2. *Let $\phi : M \to N$ be a morphism of polyhedral sets, and let $p \in |M|$. If $\phi_{s(p)} : M_{s(p)} \to N_{\phi(s(p))}$ is an isomorphism, then $|\phi| : |M| \to |N|$ restricts to a homeomorphism $B^M_\epsilon(p) \to B^N_\epsilon(|\phi|(p))$.*

Proof. Assume first that $s(p)$ is a facet. Since $\phi_{s(p)}$ is an isomorphism, $\phi(s(p))$ is a facet as well. Then it is clear that $B_{s(p)} \to B_{\phi(s(p))}$ is an isomorphism. It will suffice to show that for each integer $\mathbf{n}$, $|\phi| : |M^{\mathbf{n}}| \to |N^{\mathbf{n}}|$ restricts to such a homeomorphism. We proceed by induction on $\mathbf{n}$, and we may assume that $s(p)$ is not a facet. The case $\mathbf{n} = 0$ is therefore already proved, so we may assume that $\mathbf{n} \geq 1$. Since $s(p)$ is a face of $M^{\mathbf{n}-1}$, p is contained in $|M^{\mathbf{n}-1}|$. By the inductive hypothesis, $|\phi'| : |M^{\mathbf{n}-1}| \to |N^{\mathbf{n}-1}|$ restricts to a homeomorphism $B^{M^{\mathbf{n}-1}}_\epsilon(p) \to B^{N^{\mathbf{n}-1}}_\epsilon(|\phi|(p))$. By construction, we need to show that $|\phi|$ restricts to a bijection

$$\bigcup_{u \in J^{\mathbf{n}}_M} B_u \backslash |\partial\Delta^u| \to \bigcup_{t \in J^{\mathbf{n}}_N} B_t \backslash |\partial\Delta^t|.$$

We reduce to showing that $|\phi^\Delta| : \Delta^{\mathbf{n}}_M \to \Delta^{\mathbf{n}}_N$ restricts to a bijection $\bigcup_{\phi(u)=t} B_u \to B_t$ for each fixed $t \in J^{\mathbf{n}}_N$. For each arrow $f : s(p) \to u$ in $\mathbb{T}(M)$, the open neighbourhood $B_\epsilon(|\zeta^f_{\Delta^u}|(p)) \subseteq B_u$ maps bijectively to $B_\epsilon(|\zeta^f_{\Delta^t}|(|\phi|(p))) \subseteq B_t$, where $f : \phi(s(p)) \to t$ is the induced arrow in $\mathbb{T}(M)$. So we simply require that for each arrow $f : \phi(s(p)) \to t$ in $\mathbb{T}(N)$, there exists a unique arrow $g : s(p) \to u$ in $\mathbb{T}(M)$ mapping to g. But this is exactly the content of $\phi_{s(p)}$ being a bijection. We conclude that $|\phi|$ restricts to a bijection $B^M_\epsilon(p) \to B^N_\epsilon(|\phi|(p))$. Next, the open sets $B^M_\epsilon(p)$ for varying p and ϵ clearly forms a basis for the topology of $|M|$. This implies that $|\phi|$ is an open map, so the restriction of $|\phi|$ is a homeomorphism. By induction, we are done. $\square$

Proposition 5.4.3. *Let M be a polyhedral set, and let $p : S \to |M|$ be a covering space. Then there exists a local isomorphism $\pi : N \to M$ such that $|\pi| = p$.*

Proof. By [Bre93, Theorem 8.10], S inherits a unique CW complex structure from $|M|$: For each integer $\mathbf{n} \geq 0$, the characteristic maps $f_\alpha : \mathbb{D}^{\mathbf{n}} \to S$ are exactly those for which $p \circ f_\alpha : \mathbb{D}^{\mathbf{n}} \to |M|$ is a characteristic map of $|M|$. In other words, the characteristic maps $f_\alpha : \mathbb{D}^{\mathbf{n}} \to S$ are the liftings of the characteristic maps $\mathbb{D}^{\mathbf{n}} \to |M|$, unique upon specification of the image of any point by f_α. We only have to show that the CW complex structure on S is realized by a polyhedral set N, and that p is induced by a morphism $\pi : N \to M$. For each $P \in \mathcal{P}^{\mathrm{simp}}$, define

$$N_P = \{h : |\Delta^P| \to S : p \circ h = |\zeta_M^s| : |\Delta^s| \to |M| \text{ for some } s \in M_P\}.$$

For each arrow $f : P \to Q$, $f^N : N_Q \to N_P$ is given by $h \mapsto h \circ |\zeta_{\Delta^Q}^f|$. This clearly defines a polyhedral set N, and the morphism $\pi : N \to M$ is given by $h \mapsto s$, where s is the face of M such that $p \circ h = |\zeta_M^s|$. Consider the CW complex structure on $|N|$ given by the diagram (2.7). By definition, the characteristic maps $f_\alpha : \mathbb{D}^{\mathbf{n}} \to |N|$ are such that $p \circ f_\alpha$ is a characteristic map of $|M|$. By construction of N, these are all the liftings of the characteristic maps of $|M|$. Thus $|N|$ and S are canonically identified in such a way that for each face h of N, we have $|\zeta_N^h| = h$. Now, $\pi \circ \zeta_N^h = \zeta_M^{\pi(h)}$ for every face h of N, so $|\pi| \circ h = |\zeta_M^{\pi(h)}| = p \circ h$. Since the characteristic maps are jointly surjective, it follows that $|\pi| = p$. Next, let $h \in N_P$, and consider the local morphism $\pi_h : N_h \to M_{\pi(h)}$. We will show that π_h is bijective. Let $f : P \to Q$ be any arrow in $\mathcal{P}^{\mathrm{simp}}$ and suppose that (h_1, f) and (h_2, f) is a pair of faces of N_h mapping to the same face (t, f) of $M_{\pi(h)}$. We need to show that $h_1 = h_2$. Since $t = \pi(h_1) = \pi(h_2)$, we have $|\pi| \circ h_1 = |\zeta_M^t| = |\pi| \circ h_2$. This means that h_1 and h_2 are both liftings of $|\zeta_M^t|$, hence uniquely specified by a point in their image. But $h_1 \circ |\zeta_{\Delta^Q}^f| = h = h_2 \circ |\zeta_{\Delta^Q}^f|$, and thus share a point of specification. We conclude that $h_1 = h_2$, and that π_h is injective. Suppose now that (t, f) is any face of $M_{\pi(h)}$ for some arrow $f : P \to Q$, and consider the lifting $h' : |\Delta^Q| \to S$ of $|\zeta_M^t|$ specified by the image of a point in the image of $|\zeta_{\Delta^Q}^f| : |\Delta^P| \to |\Delta^Q|$. Then h and $h' \circ |\zeta_{\Delta^Q}^f|$ are both liftings of $|\zeta_M^{\pi(h)}|$ sharing a point of specification, hence equal. This means that the induced face (h', f) of N_h maps to (h, f), which implies that that π_h is surjective. ☐

Lemma 5.4.2 and Proposition 5.4.3 yields the following result.

Corollary 5.4.4. *A morphism $\phi : M \to N$ of polyhedral sets is a local isomorphism if and only if $|\phi| : |M| \to |N|$ is a covering space, and all covering spaces of $|N|$ are obtained in this way.*

Lemma 5.4.5. *Let M be a unimodular set, let $p \in |M|$ be a point, and define $n = \dim(s(p))$. Then $H_i(|M|, |M| \backslash p) \cong \widetilde{H}_{i-n-1}(\mathrm{lk}_M(s(p)))$ for each integer i.*

Proof. Since $H_i(|M|, |M| \backslash p) \cong H_i(B_\epsilon^M(p), B_\epsilon^M(p) \backslash p)$, we may replace M with $\mathbb{L}(M_{s(p)})$ by Lemma 5.4.2, and hence with $\mathrm{lk}_M(s) \star \Delta^{s(p)}$ by Lemma 5.2.8. Thus we have reduced to showing the following claim: If M is on the form $N \star \Delta^{s(p)}$, and $s(p) = (\varnothing, \mathrm{id}_n)$, then $H_i(|M|, |M| \backslash p) \cong \widetilde{H}_{i-n-1}(|N|)$ for all

i. Since $|M| \cong |N| \star |\Delta^{s(p)}|$ is contractible, we may equivalently prove that $\widetilde{H}_i(|M|\backslash p) \cong \widetilde{H}_{i-n}(|N|)$ for all i. Here $p \in |N| \star |\Delta^{s(p)}|$ corresponds to an interior point q of $|\Delta^{s(p)}| \cong \mathbb{D}^n$, where $\mathbb{D}^n$ is the n-disc. It is easy to see that there is a homotopy equivalence between $|N| \star |\Delta^n|\backslash p$ and $|N| \star \mathbb{S}^{n-1}$. But $|N| \star \mathbb{S}^{n-1}$ is isomorphic to the n-fold suspension $\Sigma^n|N|$, so

$$\widetilde{H}_i(|M|\backslash p) \cong \widetilde{H}_i(\Sigma^n(|N|)) = \widetilde{H}_{i-n}(|N|).$$

$\square$

Chapter 6

Cohen-Macaulay and Gorenstein properties

This chapter will be concerned with the following objectives: First, for unimodular sets M, we will give a complete classification of those M for which $\Gamma(M)$ is Cohen-Macaulay(Theorem 6.3.3) and Gorenstein(Theorem 6.4.6) respectively. Second, we will use the degenerations via subdivisions from Section 2.9 to show that the dualizing sheaf of $\mathbf{P}(M)$ (for a polyhedral set M) satisfies $\omega_M^{\otimes 2} \cong \mathcal{O}_M$ (resp. $\omega_M \cong \mathcal{O}_M$) if and only if $|M|$ is a homology manifold (resp. orientable homology manifold). The general argument is that these properties are stable in proper families, hence it suffices to show it for the degenerate case of simplicial manifolds. That uses the general idea of [BE91, Theorem 6.1], where the corresponding result regarding simplicial complexes is shown.

A Noetherian graded ring A of dimension n is Cohen-Macaulay if and only if $\underline{H}_m^i(A) = 0$ for all integers $i < n$, where $m = A_+$. See [GW78] or [Eis05] for a reference on graded local cohomology. Naturally, the first step will be to compute the graded local cohomology groups of the face ring $\Gamma(M)$. Second, we will generalize a theorem of Eisenbud[BE91, Theorem 6.1], and show that the dualizing sheaf of $\mathbf{P}(M)$ (for a smooth polyhedral set M) satisfies $\omega_M^{\otimes 2} \cong \mathcal{O}_M$ (resp. $\omega_M \cong \mathcal{O}_M$) whenever $|M|$ is a homology manifold (resp. orientable homology manifold).

Let M be a polyhedral set, and let $m = \Gamma(M)_+$. Recall that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_M$-modules, then $\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(M, \mathcal{F}(n))$. Let R be a graded $\Gamma(M)$-module. Then there is an exact sequence

$$0 \to \underline{H}_m^0(R) \to M \to \Gamma_*(\widetilde{R}) \to \underline{H}_m^1(R) \to 0 \tag{6.1}$$

and graded isomorphisms

$$\bigoplus_{d\in\mathbb{Z}} H^i(M, \widetilde{R}(d)) \cong \underline{H}_m^{i+1}(R) \tag{6.2}$$

relating the local cohomology groups of the module M with the cohomology of the twisting sheaves $\widetilde{R}(d)$. The requirement given in [GW78, Chapter 5] (called $\sharp$) is that $\Gamma(M)_0 = k$ and that there exists an integer $n_0 > 0$ such that the Veronesian subring $\Gamma(M)^{[n]}$ is generated in degree 1 for all $n \geq n_0$. This is the case here since the homomorphism $\Gamma(M) \to \Gamma_*(\mathcal{O}_M)$ is an isomorphism in positive degrees (Proposition 3.6.3), and $\mathcal{O}_M(1)$ is an ample invertible sheaf (Proposition 3.2.2). The computation of $\underline{H}_m^i(\Gamma(M))$ when M is unimodular therefore consists of computing the cohomology groups $H^i(M, \mathcal{O}_M(d))$ for integers d and $i \geq 0$. We have already computed some cases: $H^i(M, \mathcal{O}_M) \cong H^i(M; k)$ for $d = 0$ (Theorem 3.4.1), and $H^i(M, \mathcal{O}_M(d)) = 0$ for $i > 0$ and $d > 0$ (Lemma 3.6.5).

The remaining case is $d < 0$. This is harder, and will involve a more technical approach using étale cohomology. The computation involves a variant of the étale Čech complex called the *alternating* étale Čech complex.

6.1 The alternating Étale Cech complex

We will use [Tam94] as a general reference for étale cohomology and Čech cohomology. We will briefly outline the notation and definitions involved (with minor notational deviations). For each étale morphism $f : V \to X$, the global sections functor $\Gamma_V : \mathrm{Pre}(X_{\text{ét}}) \to \mathrm{Ab}$ is defined by $F \mapsto F(V)$. Γ_V has a left adjoint $f_! : \mathrm{Ab} \to \mathrm{Pre}(X_{\text{ét}})$, defined by $f_! A(W) = \bigoplus_{\phi \in \mathrm{Hom}(W,V)} A$. Since $(f_!, \Gamma_U)$ is an adjoint pair, there are natural isomorphisms $\mathrm{Hom}(f_!\mathbb{Z}, F) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{Z}, \Gamma_V(F)) = F(V)$ for presheaves F on $X_{\text{ét}}$. Let $\underline{\mathbb{Z}}$ denote the constant presheaf of $\mathbb{Z}$. Then the *trace map* $\mathrm{Tr}_f : f_!\mathbb{Z} \to \underline{\mathbb{Z}}$ is defined as the map adjoint to $\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}$. For each $W \to X$ in $X_{\text{ét}}$, $\mathrm{Tr}_f(W) : \bigoplus_{\phi \in \mathrm{Hom}(W,V)} \mathbb{Z} \to \mathbb{Z}$ is given by summing each element of the direct sum. Let $\mathrm{Shv}(X_{\text{ét}})$ denote the category of sheaves on the étale site $X_{\text{ét}}$. By sheafification, the induced functor $f_! : \mathrm{Ab} \to \mathrm{Shv}(X_{\text{ét}})$ is left adjoint to the global sections functor $\Gamma_V : \mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Ab}$. The right derived functors $R^p\Gamma_X : \mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Ab}$ defines the *étale cohomology groups* $R^p\Gamma_X(F) = H^p_{\text{ét}}(X, F)$ of F for each integer $p \geq 0$. The *Leray spectral sequence* associated with the inclusion of sites $\epsilon : X_{\text{Zar}} \to X_{\text{ét}}$ relates the étale cohomology of F with the Zariski-cohomology of F ([Tam94, p.86]):

$$E_2^{pq} = H^p_{\text{Zar}}(X, R^q\epsilon^s(F)) \Rightarrow H^{p+q}_{\text{ét}}(X, F). \tag{6.3}$$

Here $\epsilon^s : \mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Shv}(X_{\text{Zar}})$ is given by $(\epsilon^s F)(U) = F(U)$. If $\mathcal{F}$ is a quasi-coherent sheaf on X, then there is an induced étale sheaf $\mathcal{F}_{\text{ét}}$ on $X_{\text{ét}}$ given by $V \mapsto \Gamma(V, f^*\mathcal{F})$. Note that $\epsilon^s(\mathcal{F}_{\text{ét}}) = \mathcal{F}$. The spectral sequence (6.3) collapses for $F = \mathcal{F}_{\text{ét}}$, inducing isomorphisms

$$H^p_{\text{Zar}}(X, \mathcal{F}) \xrightarrow{\cong} H^p_{\text{ét}}(X, \mathcal{F}_{\text{ét}}) \tag{6.4}$$

for every integer $p \geq 0$ ([Tam94, p.103]).

et $\mathcal{U} = \{U_i\}_{i \in I}$ be an étale cover of X, and let F be a presheaf of abelian groups on $X_{\text{ét}}$. Define $U = \coprod_{i \in I} U_i$, and let $U^{\times p} = U \times_X \ldots \times_X U$ denote the p-fold fibered product over X. Let $\mathrm{pr}_j : U^{\times(p+1)} \to U^{\times p}$ denote the projection from the j'th factor. We denote the fibered product $U_{i_0} \times_X \ldots \times_X U_{i_p}$ by $U_{i_0 \ldots i_p}$. Then $U^{\times(p+1)}$ can be written as $\coprod_{(i_0,\ldots,i_p) \in I^{p+1}} U_{i_0 \ldots i_p}$, and the projection morphisms $\mathrm{pr}_j : U^{\times(p+1)} \to U^{\times p}$ restricts to projection morphisms $\mathrm{pr}_j : U_{i_0 \ldots i_p} \to U_{i_0 \ldots \widehat{i_j} \ldots i_p}$. Consider the presheaf $\mathbb{Z}_U := \bigoplus_{i \in I} f^i_! \mathbb{Z}$, where $f^i : U_i \to X$ are the morphisms associated to the cover $\mathcal{U}$. We define the trace map $\mathrm{Tr} : \mathbb{Z}_U \to \underline{\mathbb{Z}}$ as the sum of the trace maps $\mathrm{Tr}_{f_i} : f^i_! \mathbb{Z} \to \underline{\mathbb{Z}}$. Then Tr induces a chain complex

$$C_\bullet : \mathbb{Z}_U \leftarrow \mathbb{Z}_U^{\otimes 2} \leftarrow \cdots \leftarrow \mathbb{Z}_U^{\otimes(p+1)} \leftarrow \cdots \tag{6.5}$$

which is acyclic in positive degrees ($C_\bullet$ can be viewed as the *Hochschild complex* of $\mathbb{Z}_U$, see [Wei94]). Here $C_p = \mathbb{Z}_U^{\otimes(p+1)}$, and the differentials $d : C_{p+1} \to C_p$ are given on each $V \to X$ by

$$e_{i_0} \otimes \ldots \otimes e_{i_p} \mapsto \sum_{j=0}^{p} (-1)^j \operatorname{Tr}(V)(e_j) e_{i_0} \otimes \ldots \otimes \widehat{e_{i_j}} \otimes \ldots \otimes e_{i_p}$$

locally on basis elements e_i of $\mathbb{Z}_U(V)$. The natural bijections $\operatorname{Hom}(V, U_{i_0 \ldots i_p}) \cong \prod_{j=0}^{p} \operatorname{Hom}(V, U_{i_j})$ induces isomorphisms $\mathbb{Z}_U^{\otimes(p+1)} \cong \bigoplus_{(i_0,\ldots,i_p)\in I^{p+1}} f_!^{i_0\ldots i_p}\mathbb{Z}$ for each p, where each $f^{i_0\cdots i_p}$ denotes the induced morphism $U_{i_0\ldots i_p} \to X$. Hence (6.5) takes the form

$$C_\bullet : \bigoplus_{i\in I} f_!^i \mathbb{Z} \leftarrow \cdots \leftarrow \bigoplus_{(i_0,\ldots,i_p)\in I^{p+1}} f_!^{i_0\ldots i_p}\mathbb{Z} \leftarrow \cdots$$

By adjointness, there are natural isomorphisms $\operatorname{Hom}(f_!^{i_0\cdots i_p}\mathbb{Z}, F) \xrightarrow{\cong} F(U_{i_0\ldots i_p})$ for every presheaf F. Thus $\operatorname{Hom}(C^\bullet, F)$ is the étale Čech complex associated to F.

We refer to [Stacks, Tag 0721] for the more refined version of the Čech complex, called the *alternating* Čech complex. We will outline the construction here. Consider now the *Koszul complex* associated with the trace map $\operatorname{Tr} : \mathbb{Z}_U \to \underline{\mathbb{Z}}$:

$$K_\bullet : \Lambda^1\mathbb{Z}_U \leftarrow \Lambda^2\mathbb{Z}_U \leftarrow \cdots \leftarrow \Lambda^{p+1}\mathbb{Z}_U \leftarrow \cdots \tag{6.6}$$

Here the exterior product $K_p = \Lambda^{p+1}\mathbb{Z}_U$ is the quotient of $C_p = \mathbb{Z}_U^{\otimes(p+1)}$ by the subgroup of elements on the form

$$e_{i_0} \otimes \ldots \otimes e_{i_p} - \operatorname{sgn}(\sigma) e_{i_{\sigma(0)}} \otimes \ldots \otimes e_{i_{\sigma(p)}}$$

for permutations $\sigma \in S_{p+1}$. The differential $d^p : K_p \to K_{p-1}$ is induced by that of $C_\bullet$, and is given on each $V \to X$ by

$$e_{i_0} \wedge \ldots \wedge e_{i_p} \mapsto \sum_{j=0}^{p} (-1)^j \operatorname{Tr}(V)(e_j) e_{i_0} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_p}.$$

For each presheaf F, we define $\check{C}^\bullet_{\text{alt}}(\mathcal{U}, F) = \operatorname{Hom}(K_\bullet, F)$. Define $\check{H}^0_{\text{alt}}(\mathcal{U}, -) = H^0(\check{C}^\bullet_{\text{alt}}(\mathcal{U}, -))$, and note that this is a left-exact functor. Let $\check{H}^p_{\text{alt}}(\mathcal{U}, -)$ denote the right-derived functors $R^p\check{H}^0_{\text{alt}}(\mathcal{U}, -)$. These are called the *alternating* Čech cohomology groups associated with $\mathcal{U}$. Let I be an injective object of $\operatorname{Pre}(X_{\text{ét}})$. Since $\operatorname{Hom}(-, I)$ is an exact functor and $K_\bullet$ is acyclic in positive degrees, the alternating Čech complex $\check{C}^\bullet_{\text{alt}}(\mathcal{U}, I)$ is as well. This means that the functors $H^p(\check{C}^\bullet_{\text{alt}}(\mathcal{U}, -))$ are *effaceable* for each $p > 0$, so that $(H^p(\check{C}^\bullet(\mathcal{U}, -)))_{p\geq 0}$ defines a *universal* ∂-functor. This means that there are canonical isomorphisms

$$\check{H}^p_{\text{alt}}(\mathcal{U}, F) \xrightarrow{\cong} H^p(\check{C}^\bullet_{\text{alt}}(\mathcal{U}, F)). \tag{6.7}$$

Assume that $\mathcal{U}$ is a separated cover of X. Then the induced morphism $f : U \to X$ is separated, and the diagonal morphism $\Delta : U \to U \times_X U$ is a closed immersion. Since f is étale, it is also an open immersion. We define $W_1 = U \times_X U \backslash \Delta(U)$, which is a closed and open subset of $U \times_X U$. More generally, consider the diagonals $\Delta_p^{j,r} \subseteq U^{\times(p+1)}$ for $j \neq r$ (on closed points, $\Delta_p^{j,r}$ is given by $\{(u_0, \dots, u_p) \in U^{\times(p+1)} : u_j = u_r\}$). Let $\Delta_p = \bigcup_{j \neq r} \Delta_p^{j,r}$, and define $W_p = U^{\times(p+1)} \backslash \Delta_p$. Then W_p is a closed and open subset of $U^{\times(p+1)}$. We also define $W_0 = U$. Consider the action of the symmetric group S_{p+1} on $U^{\times(p+1)}$ given by permuting the factors. The fixed points of any non-trivial group elements are contained in the subset $\Delta_p \subseteq U^{\times(p+1)}$, which means that the action of S_{p+1} restricts to a free action on W_p. Assume now that F is a sheaf. Then the induced action of S_{p+1} on $F(U^{\times(p+1)})$ restricts to a right action on $F(W_p) \subseteq F(U^{\times(p+1)})$. Let K_p^+ denote the sheafification of K_p. Note that the sheafification of $\mathbb{Z}_U$ is $f_!\mathbb{Z}$. Thus $K_\bullet^+$ is the Koszul complex of the **surjective** trace map $\mathrm{Tr}_f : f_!\mathbb{Z} \to \underline{\mathbb{Z}}$ of sheaves, where $\underline{\mathbb{Z}}$ here denotes the constant sheaf of $\mathbb{Z}$. Since $f_!\mathbb{Z}$ is a locally free abelian group, this means that $K_\bullet^+ \to \underline{\mathbb{Z}}$ is a **resolution** of $\underline{\mathbb{Z}}$. The surjection of complexes $C_\bullet \to K_\bullet$ induces a surjection of sheafifications $C_\bullet^+ \to K_\bullet^+$.

Lemma 6.1.1 ([Stacks, Tag 0726])**.** *The inclusion*

$$\mathrm{Hom}(K_p^+, F) \subseteq \mathrm{Hom}(C_p^+, F) \cong F(U^{\times(p+1)})$$

identifies $\mathrm{Hom}(K_p^+, F)$ *with the* S_{p+1}*-anti-invariant sections of* $F(W_p)$*. In other words,*

$$\mathrm{Hom}(K_p, F) = \mathrm{Hom}(K_p^+, F) = \{x \in F(W_p) : x = \mathrm{sgn}(\sigma) x \cdot \sigma\}.$$

Proof. Heuristically, the sections of $\mathrm{Hom}(K_p, F) \subseteq \mathrm{Hom}(C_p^+, F)$ vanishes on each subgroup $F(\Delta_p^{j,r}) \subseteq F(U^{\times(p+1)})$ because basis elements on the form $e_{i_0} \otimes \dots \otimes e_{i_p}$ where $i_j = i_r$ vanishes in K_p. These sections are moreover the S_{p+1}-anti-invariant sections of $F(W_p)$ since K_p is a group quotient of C_p under the signed action of S_{p+1}. □

Since $K_\bullet^+ \to \underline{\mathbb{Z}}$ is a resolution, left-exactness of the functor $\mathrm{Hom}(-, F)$ induces an isomorphism $H^0(\mathrm{Hom}(K_\bullet^0, F)) \xrightarrow{\cong} \mathrm{Hom}(\underline{\mathbb{Z}}, F)$. In other words, $\check{H}^0_{\mathrm{alt}}(\mathcal{U}, F) \xrightarrow{\cong} F(X)$. The global sections functor $\Gamma_X : \mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Ab}$ is therefore identified with the composition

$$\mathrm{Shv}(X_{\text{ét}}) \xrightarrow{i} \mathrm{Pre}(X_{\text{ét}}) \xrightarrow{\check{H}^0_{\mathrm{alt}}(\mathcal{U}, -)} \mathrm{Ab},$$

and the Leray spectral sequence associated with i relates the étale cohomology of F with the alternating Čech cohomology of F:

$$E_2^{pq} = \check{H}^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}_{\text{ét}}(X, F),$$

where $\mathcal{H}^q(-) : \mathrm{Shv}(X_{\text{ét}}) \to \mathrm{Pre}(X_{\text{ét}})$ denotes the derived functors $R^q i$ of i. There are canonical isomorphisms $\mathcal{H}^q(F)(V) \cong H^q_{\text{ét}}(V, F)$ for every $V \in X_{\text{ét}}$ ([Tam94, p.57]). We conclude with the following lemma:

Lemma 6.1.2. *Let $\mathcal{U}$ be an affine étale cover of a separated scheme X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for each integer $p \geq 0$, there is a natural isomorphism*

$$H^p(\check{C}^\bullet_{\text{alt}}(\mathcal{U}, \mathcal{F}_{\text{ét}})) \xrightarrow{\cong} H^p_{\text{Zar}}(X, \mathcal{F}).$$

Proof. Since U is affine and X is separated, the p-fold fibered product $U^{\times p}$ is affine. This means that each $U_{i_0 \ldots i_p}$ is affine. Hence by (6.4), $\mathcal{H}^q(\mathcal{F}_{\text{ét}})(U_{i_0 \ldots i_p}) \cong H^q_{\text{ét}}(\mathcal{F}_{\text{ét}})(U_{i_0 \ldots i_p}) \cong H^q_{\text{Zar}}(\mathcal{F}_{\text{ét}})(U_{i_0 \ldots i_p}) = 0$ for $q > 0$. This means that we have $\check{C}^\bullet_{\text{alt}}(\mathcal{U}, \mathcal{H}^q(\mathcal{F}_{\text{ét}})) = 0$ for all $q > 0$, and (6.1) collapses. This induces natural isomorphisms $\check{H}^p(\mathcal{U}, \mathcal{F}_{\text{ét}}) \xrightarrow{\cong} H^p_{\text{ét}}(X, \mathcal{F}_{\text{ét}})$ for each integer $p \geq 0$, and composing with (6.7) yields the desired isomorphism. $\square$

6.2 The alternating Čech complex associated to a unimodular set

As a standing assumption for this chapter, all polyhedral sets considered are assumed to be finite. Let X be a polyhedral set. By Corollary 4.4.8, the morphisms $\psi_v : \mathbf{A}(M_v) \to \mathbf{P}(M)$ are étale for vertices $v \in M_0$. For every face s of M, there exists a vertex $v \in M_0$ and an arrow $v \to s$ in $\mathbb{T}(M)$. This means that the morphism $U := \coprod_{v \in J^0_M} M_v \to M$ is surjective, so by Lemma 4.3.5, $\mathcal{U} = \{\mathbf{A}(M_v) \to \mathbf{P}(M)\}_{v \in J^0_M}$ is an étale cover of $\mathbf{P}(M)$. By Proposition 5.2.7, $\mathcal{U}$ is an affine cover. We note that by Remark 4.4.6, in the particular situation where M is a polyhedral poset, the cover $\mathcal{U}$ is a Zariski open cover of M. Consider the induced surjective étale morphism $\mathbf{A}(U) \to \mathbf{P}(M)$. Let $U^{\times p} = U \times_M \cdots \times_M U$ denote the p-fold fibered product over M. Then by Lemma 4.4.2, the canonical morphism $\mathbf{A}(U^{\times(p+1)}) \to \mathbf{A}(U)^{\times(p+1)}$ is an isomorphism, and we may write

$$\mathbf{A}(U)^{\times(p+1)} = \coprod_{(v_0, \ldots, v_p) \in (J^0_M)^{p+1}} \mathbf{A}(M_{v_0} \times_M \cdots \times_M M_{v_p}).$$

The S_{p+1}-action on $\mathbf{A}(U)^{\times(p+1)}$ is induced by the corresponding S_{p+1}-action on $U^{\times(p+1)}$ given by permuting the factors. The union of diagonals $\Delta_p \subseteq \mathbf{A}(U)^{p+1}$ corresponds to the union of diagonals in $U^{\times(p+1)}$, which we will denote by D_p. We define $W_p = U^{\times(p+1)} \backslash D_p$, so that $\mathbf{A}(W_p) = \mathbf{A}(U^{\times(p+1)}) \backslash \Delta_p$ by Lemma 4.4.4.

Let $s_0, \ldots, s_p$ be a sequence of faces in M which are canonical representatives. Let $\text{Cat}(s_0, \ldots, s_p)$ denote the category of objects $(s, \{f_i : s_i \to s\}^p_{i=0})$, where $s \in M^{\text{can}}$, and where $f_i : s_i \to s$ are arrows in $\mathbb{T}(M)$ for each i. An arrow $h : (s, \{f_i\}) \to (t, \{g_i\})$ is an arrow $h : s \to t$ in $\mathbb{T}(M)$ satisfying $g_i = hf_i$ for each i. The partial order $\leq$ on $\text{Cat}(s_0, \ldots, s_p)$ is defined by $(s, \{f_i\}) \leq (t, \{g_i\})$ whenever there exists an arrow $(s, \{f_i\}) \to (t, \{g_i\})$. Let $\text{Min}(s_0, \ldots, s_p)$ denote the set of minimal objects.

Lemma 6.2.1. *Let $s_0, \ldots, s_n$ be a sequence of faces in M. Then,*

a) *there is an isomorphism*

$$M_{s_0\dots s_p} := M_{s_0} \times_M \cdots \times_M M_{s_p} \xrightarrow{\cong} \coprod_{(s,\{f_i\})\in \mathrm{Min}(s_0,\dots,s_p)} M_s;$$

b) *the projection morphisms* $\mathrm{pr}_j : M_{s_0\dots s_p} \to M_{s_0\dots\widehat{s_j}\dots s_p}$ *are given on each component by* $M_{h_j} : M_s \to M_{s(j)}$.

Proof. a) For each $(s,\{f_i\}) \in \mathrm{Min}(s_0,\dots,s_p)$, the morphisms $M_{f_i} : M_s \to M_{s_i}$ induces a morphism $M_s \to M_{s_0\dots s_p}$. Here a face $(t, f : s \to t)$ of M_s is mapped to the tuple of faces $((t, f\circ f_0),\dots,(t, f\circ f_p))$. Consider the induced morphism $\gamma_p : \coprod_{(s,\{f_i\})\in\mathrm{Min}(s_0,\dots,s_p)} M_s \to M_{s_0\dots s_p}$. We will define an inverse ψ_p of γ_p. Any tuple in $M_{s_0\dots s_p}$ is on the form $((t,g_0),\dots,(t,g_p))$, and defines an element $(t,\{g_i\})$ in $\mathrm{Cat}(s_0,\dots,s_p)$. Let $(s,\{f_i\}) \in \mathrm{Min}(s_0,\dots,s_p)$ be a minimal object such that $(s,\{f_i\}) \le (t,\{g_i\})$. Then there exists an arrow $f : (s,\{f_i\}) \to (t,\{g_i\})$, and the face (t,f) of M_s maps to $((t,g_0),\dots,(t,g_p))$. Suppose that $f' : (s',\{f'_i\}) \to (t,\{g_i\})$ is any other arrow in $\mathrm{Cat}(s_0,\dots,s_p)$ such that $(s',\{f'_i\})$ is minimal. Then $f\circ f_i = g_i = f'\circ f'_i$ for each i. Consider the arrows $f_{\mathbb{R}} : |s| \to |t|$ and $f'_{\mathbb{R}} : |s'| \to |t|$ in Fin, and note that $f_{\mathbb{R}}(|s|) \cap f'_{\mathbb{R}}(|s'|)$ both contains $\bigcup_i (g_i)_{\mathbb{R}}(|s_i|) \subseteq |t|$. But the sub-polytope $f_{\mathbb{R}}(|s|) \cap f'_{\mathbb{R}}(|s'|) \subseteq |t|$ corresponds to some arrow $f'' : s'' \to t$ in $\mathbb{T}(M)^{\mathrm{can}}$. One easily observes that this induces an element $(s'',\{f''_i\})$ in $\mathrm{Cat}(s_0\dots,s_p)$ which is less than or equal to both $(s,\{f_i\})$ and $(s',\{f'_i\})$. This is a contradiction unless $(s,\{f_i\})$ is the unique such minimal object, and f is the unique arrow to $(t,\{g_i\})$. Hence we may define ψ_p by $((t,g_0),\dots,(t,g_p)) \mapsto (t,f)$, where (t,f) is located in the term M_s corresponding to the index $(s,\{f_i\})$. This clearly determines a morphism, which is an inverse of γ_p.

b) We have to show that the diagram

$$\begin{tikzcd}
M_s \arrow[r, "\gamma_p"] \arrow[d, "M_{h_j}"'] & M_{s_0\dots s_p} \arrow[d, "\mathrm{pr}_j"] \\
M_{s(j)} \arrow[r, "\gamma_{p-1}"] & M_{s_0\dots\widehat{s_j}\dots s_p}
\end{tikzcd}$$

commutes for each j and $(s,\{f_i\}) \in \mathrm{Min}(s_0,\dots,s_p)$. Let (t,f) be a face of M_s. Then $\mathrm{pr}_j(\gamma_p(t,f)) = ((t,f\circ f_0),\dots,\widehat{(t,f\circ f_j)},\dots(t,f\circ f_p))$. On the other hand, the arrows of the object $(s(j),\{g_i\})$ satisfies $h_j g_i = f_i$ for each $i \ne j$. Since $M_{h_j}(t,f) = (t, fh_j)$, it immediately follows that $\gamma_{p-1}(M_{h_j}(t,f)) = \mathrm{pr}_j(\gamma_p(t,f))$. □

Lemma 6.2.2. *The union of diagonals* $D_p \subseteq U^{\times(p+1)}$ *is*

$$\coprod_{\substack{(s,\{f_i\})\in\mathrm{Min}(v_0,\dots,v_p)\\ \exists i\ne j: f_i=f_j}} M_s, \quad \Rightarrow \quad W_p = \coprod_{\substack{(s,\{f_i\})\in\mathrm{Min}(v_0,\dots,v_p)\\ f_i\ne f_j \forall i\ne j}} M_s.$$

Proof. The action of S_{p+1} on $U^{\times(p+1)}$ permutes the factors, so the union of diagonals D_p can be identified with the fixed points of non-trivial group elements. For every non-trivial $\sigma \in S_{p+1}$, the induced morphism $\sigma : U^{\times(p+1)} \to U^{\times(p+1)}$ restricts on components to isomorphisms $M_{v_0 \dots v_p} \to M_{v_{\sigma(0)} \dots v_{\sigma(p)}}$. By the representation of $M_{v_0 \dots v_p}$ from Lemma 6.2.1, σ further restricts to an isomorphism from M_s corresponding to the index $(s, \{f_i\}) \in \mathrm{Min}(v_0, \dots, v_p)$ to M_s corresponding to the index $(s, \{f_{\sigma(i)}\}) \in \mathrm{Min}(v_{\sigma(0)}, \dots, v_{\sigma(p)})$. Hence σ has no fixed points unless the arrows $f_i : v_i \to s$ and $f_{\sigma(i)} : v_{\sigma(i)} \to s$ are equal for all i. So the non-trivial fixed points of $M_{v_0 \dots v_p}$ are exactly the terms M_s corresponding to an index $(s, \{f_i\})$ such that there exists a non-trivial permutation σ such that $f_{\sigma(i)} = f_i$ for all i. This does not happen exactly when the f_i's are all different, and the result follows. $\square$

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}(M)$. By Lemma 6.2.1, the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F}_{\text{ét}})$ is given by

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}_{\text{ét}}) = \bigoplus_{(s,\{f_i\}) \in \mathrm{Min}(v_0,\dots,v_p)} \Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F}),$$

and the differentials $d^p : \check{C}^{p-1}(\mathcal{U}, \mathcal{F}_{\text{ét}}) \to \check{C}^p(\mathcal{U}, \mathcal{F}_{\text{ét}})$ are given by the alternating sums of the canonical homomorphisms $M_{h_j}^* : \Gamma(\mathbf{A}(M_{s(j)}), \psi_{s(j)}^* \mathcal{F}) \to \Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F})$ for $j = 0, \dots, p$. Thus by Lemma 6.2.2, the alternating Čech complex groups $\check{C}^p_{\text{alt}}(\mathcal{U}, \mathcal{F}_{\text{ét}})$ are the S_{p+1}-anti-invariant sections of

$$\mathcal{F}_{\text{ét}}(\mathbf{A}(W_p)) = \bigoplus_{\substack{(s,\{f_i\}) \in \mathrm{Min}(v_0,\dots,v_p) \\ f_i \neq f_j \forall i \neq j}} \Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F}). \tag{6.8}$$

The action of a permutation $\sigma \in S_{p+1}$ on $\mathcal{F}_{\text{ét}}(\mathbf{A}(W_p))$ is given by mapping each term $\Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F})$ corresponding to the index $(s, \{f_i\}) \in \mathrm{Min}(v_0, \dots, v_p)$ identically into $\Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F})$ corresponding to the index $(s, \{f_{\sigma^{-1}(i)}\}) \in \mathrm{Min}(v_{\sigma^{-1}(0)}, \dots, v_{\sigma^{-1}(p)})$. In other words, it is given by $(x \cdot \sigma)_{(s,\{f_i\})} = x_{(s,\{f_{\sigma(i)}\})}$. Hence $x_{(s,\{f_i\})} = \mathrm{sgn}(\sigma) x_{(s,\{f_{\sigma(i)}\})}$ in $\check{C}^p_{\text{alt}}(\mathcal{U}, \mathcal{F}_{\text{ét}})$ for all $\sigma \in S_{p+1}$.

For the rest of this section we shall assume that M is a unimodular set. In this situation, an object $(s, \{f_i\})$ is minimal if and only if the functions $f_i : |s_i| \to |s|$ in Fin are jointly surjective. If $(s, \{f_i\})$ is minimal, let T_j denote the subset $\cup_{i \neq j} \operatorname{im} f_i \subseteq \{0, \dots, p\}$ for each $j \in \{0, \dots, p\}$. Then the inclusion $T_j \subseteq \{0, \dots, p\}$ corresponds to an arrow $h_j : s(j) \to s$ in Fin where $s(j) \in M^{\text{can}}$, and the inclusion $\operatorname{im} f_i \subseteq T_j$ determines an arrow $g_i : s_i \to s(j)$ in Fin satisfying $h_j g_i = f_i$ for each $i \neq j$. Observe that $(s(j), \{g_i\}_{i \neq j})$ is in $\mathrm{Min}(s_0, \dots, \widehat{s_j}, \dots, s_n)$. If $(s, \{f_i\})$ is an object of $\mathrm{Min}(v_0, \dots, v_p)$ and $f_0, \dots, f_p$ are all different, then the arrows $f_i : |v_i| \to |s|$ in Fin form a permutation of the inclusions $\{i\} \subseteq \{0, \dots, p\}$ for $i = 0, \dots, p$. So the integral order on $\{0, \dots, p\}$ induces a unique associated permutation $\sigma \in S_{p+1}$ such that $f_{\sigma(i)}(0) = i$ for each i.

Definition 6.2.3. We define the subcomplex $\check{C}^\bullet_\Delta(\mathcal{U},\mathcal{F})$ of $\check{C}^\bullet(\mathcal{U},\mathcal{F}_{\text{ét}})$ as

$$\check{C}^p_\Delta(\mathcal{U},\mathcal{F}) = \bigoplus_{s\in J^p_M} \Gamma(\mathbf{A}(M_s),\psi_s^*\mathcal{F}),$$

where each index $s \in J^p_M$ corresponds to the unique minimal element $(s,\{f_i\}) \in \mathrm{Min}(v_0,\dots,v_p)$ such that each $f_i : |v_i| \to |s|$ satisfies $f_i(0) = i$. The induced differentials $d^p : \check{C}^{p-1}_\Delta(\mathcal{U},\mathcal{F}) \to \check{C}^p_\Delta(\mathcal{U},\mathcal{F})$ are given by

$$(d^p(x))_{s\in J^p_M} = \sum_{j=0}^{p}(-1)^j M^*_{h_j}(x_{s(j)}).$$

For each integer $p \geq 0$, consider the homomorphism $\phi^p : \check{C}^p_\Delta(\mathcal{U},\mathcal{F}) \to \check{C}^p_{\text{alt}}(\mathcal{U},\mathcal{F}_{\text{ét}})$ given by $(\phi^p(x))_{(s,\{f_i\})} = \operatorname{sgn}(\sigma)x_s$, where σ is the permutation such that $f_{\sigma(i)}(0) = i$ for each i. To see that this is well-defined, we must show that $\phi^p(x)$ is S_{p+1}-anti-invariant. So let $\tau \in S_{p+1}$. Then we require that $(\phi^p(x))_{(s,\{f_i\})} = \operatorname{sgn}(\tau)(\phi^p(x))_{(s,\{f_{\tau(i)}\})}$. But $(\phi^p(x))_{(s,\{f_{\tau(i)}\})} = \operatorname{sgn}(\pi)x_s$, where π is the permutation such that $f_{\tau\pi(i)}(0) = i$ for each i. Thus $\tau\pi = \sigma$, and $\operatorname{sgn}(\sigma)x_s = \operatorname{sgn}(\tau)\operatorname{sgn}(\pi)x_s$ as required.

Lemma 6.2.4. *The homomorphisms ϕ^p induces an isomorphism of complexes*

$$\phi : \check{C}^\bullet_\Delta(\mathcal{U},\mathcal{F}) \xrightarrow{\cong} \check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}_{\text{ét}}).$$

Proof. First we must show that ϕ respects the differentials. Let $x = (x_s) \in \check{C}^{p-1}_\Delta(\mathcal{U},\mathcal{F})$ be a section. Then

$$\phi^p(d^p(x))_{(s,\{f_i\})} = \operatorname{sgn}(\sigma)(d^p(x))_s = \sum_{j=0}^{p}\operatorname{sgn}(\sigma)(-1)^j M^*_{h_j}(x_{s(j)}),$$

where $\sigma \in S_{p+1}$ is the unique permutation such that $f_{\sigma(i)}(0) = i$ for each i. On the other hand,

$$\begin{aligned}
(d^p(\phi^{p-1}(x)))_{(s,\{f_i\})} &= \sum_{j=0}^{p}(-1)^j M^*_{h_j}((\phi^{p-1}(x))_{(s(j),\{g_i\}_{i\neq j})})\\
&= \sum_{j=0}^{p}(-1)^j M^*_{h_j}(\operatorname{sgn}(\tau_j)x_{s(j)})\\
&= \sum_{j=0}^{p}\operatorname{sgn}(\tau_j)(-1)^j M^*_{h_j}(x_{s(j)})\\
&= \sum_{j=0}^{p}\operatorname{sgn}(\tau_{\sigma(j)})(-1)^{\sigma(j)} M^*_{h_j}(x_{s(j)}),
\end{aligned}$$

where each arrow $g_i : v_i \to s(j)$ corresponds to the inclusion $\operatorname{im} f_i \subseteq T_j = \{0,\dots,\widehat{f_j(0)},\dots,p\}$, and τ_j is the permutation associated with the set of arrows

$\{g_i\}_{i\neq j}$ for each j. To verify that $(\phi^p(d^p(x)))_{(s,\{f_i\})} = (d^p(\phi^{p-1}(x)))_{(s,\{f_i\})}$, we must show that $\operatorname{sgn}(\sigma)(-1)^j = \operatorname{sgn}(\tau_{\sigma(j)})(-1)^{\sigma(j)}$ for each j. Substituting j with $\sigma^{-1}(j)$, this can be rewritten as

$$\operatorname{sgn}(\sigma)(-1)^{\sigma^{-1}(j)} = \operatorname{sgn}(\tau_j)(-1)^j. \tag{6.9}$$

Since $f_i(0) = \sigma^{-1}(i)$ for each i, σ^{-1} is represented by the permutation

$$\begin{pmatrix} 0 & 1 & \cdots & p \\ f_0(0) & f_1(0) & \cdots & f_p(0) \end{pmatrix}.$$

Similarly, τ_j^{-1} is represented by the permutation

$$\begin{pmatrix} 0 & \cdots & \widehat{j} & \cdots & p \\ f_0(0) & \cdots & \widehat{f_j(0)} & \cdots & f_p(0) \end{pmatrix}.$$

Let $P(\sigma^{-1}) = (e_{\sigma^{-1}(0)}, \ldots, e_{\sigma^{-1}(p)})$ be the $p \times p$ permutation matrix associated with σ^{-1}. Then $P(\tau_j^{-1})$ is obtained from $P(\sigma^{-1})$ by eliminating the j'th column and the $\sigma^{-1}(j)$'th row. Hence $\det P(\tau_j^{-1})$ is equal to the $(p-1) \times (p-1)$ minor $P(\sigma^{-1})_{(\sigma^{-1}(j),j)}$. The cofactor expansion of $P(\sigma^{-1})$ along the j'th column computes the determinant of $P(\sigma^{-1})$ as follows:

$$\det P(\sigma^{-1}) = (-1)^j \sum_{i=0}^{p} (-1)^i \det P(\sigma^{-1})_{i,j} = (-1)^j (-1)^{\sigma^{-1}(j)} \det P(\tau_j^{-1}).$$

The determinant of a permutation matrix is equal to the sign of the permutation, and it follows that $\operatorname{sgn}(\sigma^{-1}) = (-1)^j(-1)^{\sigma^{-1}(j)} \operatorname{sgn}(\tau_j^{-1})$. Thus we obtain the equation (6.9).

Next we must show that ϕ is a bijection. It is clearly injective, and by S_{p+1}-anti-invariance any section $x \in \check{C}^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{F}_{\text{ét}})$ satisfies $x_{(s,\{f_i\})} = \operatorname{sgn}(\sigma) x_{(s,\{f_{\sigma(i)}\})}$ for all $\sigma \in S_{p+1}$. So each section is determined by its restriction to $\Gamma(\mathbf{A}(M_s), \psi_s^* \mathcal{F})$ for any choice of index $(s, \{f_i\})$. Thus ϕ is surjective, and therefore an isomorphism of complexes. $\square$

Lemma 6.2.5. *For each integer $p \geq 0$, there is an isomorphism*

$$H^p_{\mathrm{Zar}}(M, \mathcal{F}) \cong H^p(\check{C}^\bullet_\Delta(\mathcal{U}, \mathcal{F})).$$

Proof. By Proposition 5.2.7, the étale cover $\mathcal{U} = \{\mathbf{A}(M_v) \to \mathbf{P}(M)\}_{v \in J^0_M}$ is affine. Now the statement follows from Lemma 6.2.4 and Lemma 6.1.2 $\square$

6.3 The Cohen-Macaulay property for unimodular sets

Definition 6.3.1. Let M be a unimodular set, and let R be a graded $\Gamma(M)$-module. Then we define the *reduced complex* of R as

$$\widetilde{C}^\bullet_\Delta(M, R) : 0 \to R \to \check{C}^\bullet_\Delta(M, \bigoplus_{d\in\mathbb{Z}} \widetilde{R}(d)),$$

where $R \to \check{C}^0_\Delta(M, \bigoplus_{d\in\mathbb{Z}} \widetilde{R}(d))$ is given by the canonical homomorphisms

$$R_d \to H^0(M, \widetilde{R}(d)) \to \bigoplus_{v\in J^0_M} \Gamma(\mathbf{A}(M_v), \psi_v^* \widetilde{R}(d)).$$

The grading of the complex is given by $\widetilde{C}^{-1}_\Delta(M,R) = R$, and $\widetilde{C}^p_\Delta(M,R) = \check{C}^p_\Delta(M,\widetilde{R})$ for each $p \geq 0$.

By the exact sequence (6.1) and the isomorphisms (6.2), we have $H^p(\widetilde{C}^\bullet_\Delta(M,R)) = \underline{H}^{p+1}_m(R)$ for all $p \geq 0$. Of particular interest is the case where $R = \Gamma(M)$. From Proposition 5.2.7, $\mathbf{A}(M_s)$ is equal to the distinguished open subset $D_+(\theta_s)$ of $\mathbf{P}(\mathbb{L}(M_s))$. So for each face s of M, we have

$$\begin{aligned}\Gamma(\mathbf{A}(M_s), \psi_s^*(\bigoplus_{d\in\mathbb{Z}} \mathcal{O}_M(d))) &\cong \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_{\mathbb{L}(M_s)}(d)(D_+(\theta_s)) \\ &\cong \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathbb{L}(M_s))(d)_{(\theta_s)} = \Gamma(\mathbb{L}(M_s))_{\theta_s}.\end{aligned}$$

This means that we may describe the reduced complex $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))$ as

$$0 \to \Gamma(M) \to \bigoplus_{v\in J^0_M} \Gamma(\mathbb{L}(M_v))_{\theta_v} \to \cdots \to \bigoplus_{s\in J^p_M} \Gamma(\mathbb{L}(M_s))_{\theta_s} \to \cdots \tag{6.10}$$

The induced differentials are given by the alternating sum of the homomorphisms $\Gamma(M_{h_j}) : \Gamma(\mathbb{L}(M_{s(j)}))_{\theta_{s(j)}} \to \Gamma(\mathbb{L}(M_s))_{\theta_s}$.

Definition 6.3.2. Let M be a unimodular set. Then we define the complex $K^\bullet(M)$ as the subcomplex of $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))$ given by $K^{-1}(M) = \Gamma(M)$, and $K^p(M) = \bigoplus_{s\in J^p_M} \Gamma(\mathbb{L}(M_s))$.

In the following theorem we will utilize our constructions thus far. The proof draws its main idea from the proof of [Sta96, II, Theorem 4.1], by essentially decomposing the chain complex $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))$ into a direct sum of complexes on the form $K^\bullet(M)$ and $K^\bullet(\mathrm{lk}_M(s))$.

Theorem 6.3.3. *Let M be a non-trivial unimodular set. Then $\underline{H}^0_m(\Gamma(M)) = 0$, and for each $i \geq 0$, the Hilbert series of the graded $\Gamma(M)$-module $\underline{H}^{i+1}_m(\Gamma(M))$ is*

$$H_{\underline{H}^{i+1}_m(\Gamma(M))}(t) = \widetilde{H}^i(M;k) + \sum_{p\in M^{\mathrm{can}}} \dim_k \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p);k)\frac{t^{-|p|-1}}{(1-t^{-1})^{|p|+1}}.$$

In particular, $\Gamma(M)$ is Cohen-Macaulay if and only if

1) $\widetilde{H}^i(M;k) = 0$ *for all* $i < \dim M$, *and*

2) $\widetilde{H}^i(\mathrm{lk}_M(p);k) = 0$ *for all* $p \in \mathbb{T}(M)$ *and* $i < \dim M - |p| - 1$.

By Lemma 5.4.5 and Theorem 3.4.1, this can be equivalently rephrased as

1') $\widetilde{H}^i(|M|;k) = 0$ *for all* $i < \dim M$*, and*

2') $H^i(|M| \mid p) = 0$ *for all points* $p \in |M|$ *and* $i < \dim M$*.*

Proof. We consider the reduced complex $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))$. Define $x_f^\alpha = x_{f(0)}^{\alpha_0}\cdots x_{f(n)}^{\alpha_n} \in \Gamma(\Delta^m)_{x_f}$ for arrows $f : n \to m$ in Fin and $\alpha \in \mathbb{Z}^{n+1}$. Let $s \in M^{\mathrm{can}}$, and let (t,f) be a face of M_s. Consider the basis for the k-vector space $\Gamma(\Delta^t)_{x_f}$ consisting of 1) monomials $\mathbf{m} \in \Gamma(\Delta^t)$ and 2) elements on the form $\frac{\mathbf{m}}{x_{fg}^\alpha}$, where $g : p \to s$ is some arrow in M^{can}, $\alpha \in \mathbb{Z}_{>0}^{|p|+1}$, and $\mathbf{m}$ is a monomial in $\Gamma(\Delta^t)$ which is not divisible by any generator on the form $x_{fg(j)}$. For each arrow $g : p \to s$ in M^{can}, we may write $\Gamma(\Delta^t) = \Gamma(\Delta^{M_{|p|}(|t|,fg)}) \otimes_k \Gamma(\Delta^p)$, where the generators of $\Gamma(\Delta^p)$ correspond to $x_{g(0)},\dots,x_{g(|p|)}$. In this notation, a basis element of the first type corresponds to $\mathbf{m}\otimes 1$, and a basis element of the second type corresponds to $\mathbf{m}\otimes \frac{1}{x_g^\alpha}$. By this we get the following vector space decomposition:

$$\Gamma(\Delta^t)_{x_f} = \Gamma(\Delta^t) \oplus \bigoplus_{\substack{p\in M^{\mathrm{can}}\\ \alpha\in\mathbb{Z}_{>0}^{|p|+1}}} \bigoplus_{p\xrightarrow{g}s} \Gamma(\Delta^{M_{|p|}(|t|,fg)}) \otimes_k k\{\frac{1}{x_g^\alpha}\}.$$

Alternatively, we may write

$$\Gamma(\Delta^t)_{x_f} = \Gamma(\Delta^t) \oplus \bigoplus_{\substack{p\in M^{\mathrm{can}}\\ \alpha\in\mathbb{Z}_{>0}^{|p|+1}}} \bigoplus_{p\xrightarrow{g}s} \Gamma(\Delta^{M_{|p|}(|t|,fg)})(|\alpha|),$$

where $|\alpha| = \alpha_0 + \cdots + \alpha_{|p|}$, and where $\Gamma(\Delta^{M_{|p|}(|t|,fg)})(|\alpha|)$ denotes the graded vector space with degrees shifted by $|\alpha|$. Observe that for any arrow $h : (t,f) \to (t',f')$ in M_s, $\Gamma(\zeta^h_{\Delta^{t'}}) : \Gamma(\Delta^{t'})_{x_{f'}} \to \Gamma(\Delta^t)_{x_f}$ maps any element on the form $1\otimes\frac{1}{x_g^\alpha}$ to some non-zero scalar multiple of $1\otimes\frac{1}{x_g^\alpha}$. Indeed, since these elements can be written as $\frac{1}{x_{f'g}^\alpha}$ and $\frac{1}{x_{fg}^\alpha}$ respectively, the equality $hf = f'$ means that $x_{f'g(j)} = x_{hfg(j)} \mapsto \lambda_h(fg(j))x_{fg(j)}$. It is clear that any monomial $\mathbf{m} \in \Gamma(\Delta^{t'})$ not divisible by any generator on the form $x_{f'g(j)}$ is mapped to a non-zero scalar multiple of a monomial $\mathbf{m'} \in \Gamma(\Delta^t)$ which is not divisible by any generator on the form $x_{fg(j)}$. This means that $\Gamma(\zeta^h_{\Delta^{t'}})$ splits into homomorphisms $\Gamma(\Delta^{t'}) \to \Gamma(\Delta^t)$ and $\Gamma(\Delta^{M_{|p|}(|t'|,f'g)})(|\alpha|) \to \Gamma(\Delta^{M_{|p|}(|t|,fg)})(|\alpha|)$ via their respective decompositions. This justifies the following computation:

$$\begin{aligned}\Gamma(\mathbb{L}(M_s))_{\theta_s} &= \varprojlim_{s\xrightarrow{f}t} \Gamma(\Delta^t)_{x_f} = \varprojlim_{s\xrightarrow{f}t}\left(\Gamma(\Delta^t) \oplus \bigoplus_{\substack{p\in M^{\mathrm{can}}\\ \alpha\in\mathbb{Z}_{>0}^{|p|+1}}} \bigoplus_{p\xrightarrow{g}s} \Gamma(\Delta^{M_{|p|}(|t|,fg)})(|\alpha|)\right)\\ &= \left(\varprojlim_{s\xrightarrow{f}t} \Gamma(\Delta^t)\right) \oplus \bigoplus_{\substack{p\in M^{\mathrm{can}}\\ \alpha\in\mathbb{Z}_{>0}^{|p|+1}}} \bigoplus_{p\xrightarrow{g}s} \left(\varprojlim_{s\xrightarrow{f}t} \Gamma(\Delta^{M_{|p|}(|t|,fg)})(|\alpha|)\right).\end{aligned}$$

Notice that $\varprojlim_{s \xrightarrow{f} t} \Gamma(\Delta^t) = \Gamma(\mathbb{L}(M_s))$. Moreover, for each non-identity arrow $g : p \to s$, we have $\varprojlim_{s \xrightarrow{f} t} \Gamma(\Delta^{M_{|p|}(|t|, fg)}) = \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,g)})$. If $g : p \to s$ is the identity arrow, then $\mathrm{lk}_M(p) = \varinjlim_{s \xrightarrow{f} t} \Delta^{M_{|p|}(|t|, f)}$. By abuse of notation, we shall write $\Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(p,\mathrm{id})})$ for $\Gamma(\mathbb{L}(\mathrm{lk}_M(p)))$. Taken together, we may write

$$\Gamma(\mathbb{L}(M_s))_{\theta_s} = \Gamma(\mathbb{L}(M_s)) \oplus \bigoplus_{\substack{p \in M^{\mathrm{can}} \\ \alpha \in \mathbb{Z}^{|p|+1}_{>0}}} \left(\bigoplus_{p \xrightarrow{g} s} \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,g)})(|\alpha|) \right).$$

Next we observe that the differentials of $\widetilde{C}^{\bullet}_{\Delta}(M, \Gamma(M))$ respect this decomposition: The homomorphism $\Gamma(M_{h_j}) : \Gamma(\mathbb{L}(M_{s(j)}))_{\theta_{s(j)}} \to \Gamma(\mathbb{L}(M_s))_{\theta_s}$ is given by the localization homomorphisms $\Gamma(\Delta^t)_{x_{fh_j}} \to \Gamma(\Delta^t)_{x_f}$ for each face (t, f) of M_s. Thus any basis element on the form $\frac{y}{x^{\alpha}_{fh_j \circ g}}$ maps to a non-zero scalar multiple of $\frac{y}{x^{\alpha}_{f \circ h_j g}}$. Hence $\Gamma(\Delta^t)_{x_{fh_j}}$ restricts to a homomorphism $\Gamma(\mathbb{L}(\mathrm{lk}_M(p)_{(s(j),g)}))(|\alpha|) \to \Gamma(\mathbb{L}(\mathrm{lk}_M(p)_{(s,h_j g)}))(|\alpha|)$. This means that the reduced complex splits as a direct sum

$$\widetilde{C}^{\bullet}_{\Delta}(M, \Gamma(M)) = K^{\bullet}(M) \oplus \bigoplus_{\substack{p \in M^{\mathrm{can}} \\ \alpha \in \mathbb{Z}^{|p|+1}_{>0}}} T^{\bullet}_p[-|p| - 1](|\alpha|),$$

where $T^{-1}_p = \Gamma(\mathrm{lk}_M(p))$, and

$$T^i_p = \bigoplus_{\substack{p \xrightarrow{g} s \text{ in } M^{\mathrm{can}} \\ |s| = i + |p| + 1}} \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,g)}) = \bigoplus_{(s,g) \in J^i_{\mathrm{lk}_M(p)}} \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,g)})$$

for each $i \geq 0$. Here $T^{\bullet}_p[-|p| - 1]$ denotes the chain complex shifted by $-|p| - 1$. The induced differentials of $T^{\bullet}_p$ are given by the sum of the homomorphisms $(-1)^j \Gamma(\mathrm{lk}_M(p)_{M_{|p|}(h_j)}) : \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s(j),g)}) \to \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,h_j g)})$. We will now define an isomorphism of complexes $K^{\bullet}(\mathrm{lk}_M(p)) \to T^{\bullet}_p$. The terms of these two complexes are the same, but the differentials of $K^{\bullet}(\mathrm{lk}_M(p))$ are given as the alternate sum of the homomorphisms

$$(-1)^k \Gamma(\mathrm{lk}_M(p)_{h_k}) : \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{h_k(s,g)}) \to \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,g)}).$$

By slight abuse of notation, we define $h_0(s, g) = (p, \mathrm{id})$ for vertices $(s, g) \in \mathrm{lk}_M(p)_0$.

For each arrow $g : n \to m$ in Fin and $k = 0, \ldots, m - n - 1$, let $j = E(g, k)$ denote the integer satisfying $M_n(d_j) = d^k$. Observe that for each $s \in M^{\mathrm{can}}$, $h_j = d_j \sigma_j$ for some automorphism σ_j of $|s(j)|$. Since $M_{|p|}(h_j)$ is equal to some $h_{k'}$, it therefore follows that $M_{|p|}(h_j) = h_k$. Thus $h_k(s, h_{E(g,k)} g) = (h_{E(g,k)}(s), g)$ for each arrow $g : p \to h^M_{E(g,k)}(s)$, and the differentials of $T^{\bullet}_p$ can written as the sum of the homomorphisms $(-1)^{E(g,k)}\Gamma(\mathrm{lk}_M(p)_{h_k}) : \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(h^M_{E(g,k)}(s),g)}) \to \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,h_{E(g,k)}g)})$. To find our isomorphism $K^\bullet(\mathrm{lk}_M(p)) \to T^\bullet_p$, we are therefore required to find signs $\mathrm{sgn}(s,g) \in \{1,-1\}$ such that the diagram

$$\begin{tikzcd}
\Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(h^M_{E(g,k)}(s),g)}) \arrow[r, "(-1)^k\Gamma(\mathrm{lk}_M(p)_{h_k})"] \arrow[d, "\mathrm{sgn}(h^M_{E(g,k)}(s),g)\Gamma(\mathrm{id})"] & \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,h_{E(g,k)}g)}) \arrow[d, "\mathrm{sgn}(s,h_{E(g,k)}g)\Gamma(\mathrm{id})"] \\
\Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(h^M_{E(g,k)}(s),g)}) \arrow[r, "(-1)^{E(g,k)}\Gamma(\mathrm{lk}_M(p)_{h_k})"] & \Gamma(\mathbb{L}(\mathrm{lk}_M(p))_{(s,h_{E(g,k)}g)})
\end{tikzcd}$$

commutes for all faces (s,g) in $\mathrm{lk}_M(p)^{\mathrm{can}}$ and $k = 0,\ldots,|s|-|p|-1$. Thus we must find a function $\mathrm{sgn} : \mathrm{lk}_M(p)^{\mathrm{can}} \to \{1,-1\}$ satisfying

$$\mathrm{sgn}(s,h_{E(g,k)}g)(-1)^k = \mathrm{sgn}(h^M_{E(g,k)}(s),g)(-1)^{E(g,k)}. \tag{6.11}$$

It follows immediately by induction that $g : p \to s$ can be written as a composition of arrows on the form $h_j : t(j) \to t$. Moreover, by construction we have $h_j h_i = h_i h_{j-1}$ whenever $j > 1$, so by ([Mac98, Proposition 2, p.174]) g can be written uniquely as $h_{j_0}\cdots h_{j_r}$, with $j_0 > \cdots > j_r$, and thus we may define $\mathrm{sgn}(s,g) = (-1)^{r+\Sigma^r_{i=0}j_i}$. Note that $E(g,k)$ is the integer in the k'th position of $[m]\setminus \mathrm{im}\, g$, where $[m] = |s|$. If we write $[m]\setminus \mathrm{im}\, g = \{j_0,\ldots,j_r\}$, then $E(g,k) = j_k$. Now, $h_{j_k}g = h_{j_0+1}\cdots h_{j_k+1}h_{j_k}h_{j_{k+1}}\cdots h_{j_r}$ via the relations $h_j h_i = h_i h_{j-1}$ for $j > i$. This means that $\mathrm{sgn}(s,h_{j_k}g) = (-1)^{E(g,k)+(k+1)+(r+1)+\Sigma^r_{i=0}j_i} = (-1)^{E(g,k)+k}\,\mathrm{sgn}(h^M_{E(g,k)}(s),g)$, immediately verifying (6.11). This yields our desired isomorphism $K^\bullet(\mathrm{lk}_M(p)) \to T^\bullet_p$. Thus we can write the reduced complex as

$$\widetilde{C}^\bullet_\Delta(M,\Gamma(M)) = K^\bullet(M) \oplus \bigoplus_{\substack{p\in M^{\mathrm{can}}\\ \alpha\in\mathbb{Z}^{|p|+1}_{>0}}} K^\bullet(\mathrm{lk}_M(p))[-|p|-1](|\alpha|).$$

By Proposition 3.6.3 and Lemma 3.6.5, the degree d part $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))_d$ is acyclic for $d > 0$. Since $K^\bullet(M)$ is a direct summand of $\widetilde{C}^\bullet_\Delta(M,\Gamma(M))$, this means that $K^\bullet(M)_d$ is acyclic for $d > 0$ as well. But $K^\bullet(M)_0$ is simply the augmented cellular cochain complex $0 \to k \to C^\bullet_M(k)$, and it follows that $H^i(K^\bullet(M))$ is equal to the reduced cohomology group $\widetilde{H}^i(M;k)$. The same argument applies to the complexes $K^\bullet(\mathrm{lk}_M(p))$ as well, so that $H^i(K^\bullet(\mathrm{lk}_M(p))(|\alpha|)) = \widetilde{H}^i(\mathrm{lk}_M(p);k)(|\alpha|)$. Hence

$$H^i(\widetilde{C}^\bullet_\Delta(M,\Gamma(M))) = \widetilde{H}^i(M;k) \oplus \bigoplus_{p\in M^{\mathrm{can}}}\ \bigoplus_{\alpha\in\mathbb{Z}^{|p|+1}_{>0}} \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p))(|\alpha|).$$

In other words, we have $H^i(\widetilde{C}^\bullet_\Delta(M,\Gamma(M)))_0 = \widetilde{H}^i(M;k)$, and for $d < 0$ fixed,

$$H^i(\widetilde{C}^\bullet_\Delta(M,\Gamma(M)))_d = \bigoplus_{p\in M^{\mathrm{can}}}\ \bigoplus_{\substack{\alpha\in\mathbb{Z}^{|p|+1}_{>0}\\ |\alpha|=-d}} \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p);k).$$

Now, there are $\binom{-d-1}{|p|}$ elements of the set $\{\alpha \in \mathbb{Z}^{|p|+1}_{>0} : |\alpha| = -d\}$, so

$$\dim_k H^i(\widetilde{C}^\bullet_\Delta(M, \Gamma(M)))_d = \sum_{p \in M^{\mathrm{can}}} \binom{-d-1}{|p|} \dim_k \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p); k).$$

Let $i \geq 0$. Using that $H^i(\widetilde{C}^\bullet_\Delta(M, \Gamma(M))) \cong \underline{H}^{i+1}_m(\Gamma(M))$, the Hilbert series of $\underline{H}^{i+1}_m(\Gamma(M))$ becomes

$$\begin{aligned} H_{H^{i+1}_m(\Gamma(M))}(t) &= \widetilde{H}^i(M;k) + \sum_{d=-\infty}^{-1} t^d \sum_{p \in M^{\mathrm{can}}} \binom{-d-1}{|p|} \dim_k \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p);k) \\ &= \widetilde{H}^i(M;k) + \sum_{p \in M^{\mathrm{can}}} \dim_k \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p);k) t^{-1} \sum_{d=0}^{\infty} (t^{-1})^d \binom{d}{|p|} \\ &= \widetilde{H}^i(M;k) + \sum_{p \in M^{\mathrm{can}}} \dim_k \widetilde{H}^{i-|p|-1}(\mathrm{lk}_M(p);k) \frac{t^{-|p|-1}}{(1-t^{-1})^{|p|+1}}. \end{aligned}$$

(here we have used the formula $\sum_{d=0}^{\infty} t^{-d}\binom{d+n}{n} = \frac{1}{(1-t^{-1})^{n+1}}$). Finally, the homomorphism $\Gamma(M) \to \Gamma_*(\mathcal{O}_M)$ is injective by Proposition 3.6.3, so $\underline{H}^0_m(\Gamma(M))$ is trivial, and we are done. □

6.4 The Gorenstein property for unimodular sets

Before proving Theorem 6.4.6, we need some preliminary facts regarding the non-normal locus of $\mathbf{P}(M)$, and its dualizing sheaf.

Lemma 6.4.1. *Let M be a unimodular set. Then the non-normal locus $\mathbf{P}(M)_{\mathbf{n}}$ of $\mathbf{P}(M)$ is equal to $\mathbf{P}(N(M))$, where $N(M)$ is the polyhedral subset of M consisting of faces t such that $\mathrm{lk}_M(t) \ncong \Delta^{\dim \mathrm{lk}_M(t)}$.*

Proof. By Lemma 5.2.3, $N(M)$ defines a polyhedral subset of M. Indeed, if $s \in M_n$, and $f : m \to n$ is any arrow, then $\mathrm{lk}_M(s) = \mathrm{lk}_{\mathrm{lk}_M(f^M(s))}(s, f)$. If $\mathrm{lk}_M(s)$ is not isomorphic to a simplex, then by Lemma 5.2.1, $\mathrm{lk}_M(f^M(s))$ cannot be either. Hence $f^M(s) \in N(M)$. First we will show that $\mathbf{P}(N(M)) \subseteq \mathbf{P}(M)_{\mathbf{n}}$. Consider the normalization $\nu : \mathbf{P}(\overline{M}) \to \mathbf{P}(M)$ from Lemma 3.3.1, where $\overline{M} = \coprod_{s \in S} \Delta^s$. Then the restriction $\mathbf{P}(\overline{M}) \backslash \nu^{-1}(\mathbf{P}(M)_{\mathbf{n}}) \to \mathbf{P}(M) \backslash \mathbf{P}(M)_{\mathbf{n}}$ is an isomorphism. Let p be a point of $\mathbf{P}(N(M))$, and let s be the minimal face of $N(M)$ for which $p \in \mathbf{P}(M^s)$. Since $\mathrm{lk}_M(s)$ is not a simplex, it is either not irreducible, or the morphism $\Delta^n \to \mathrm{lk}_M(s)$ associated to its unique facet is not injective. In the first case, suppose that (t_1, f_1) and (t_2, f_2) is a pair of facets of $\mathrm{lk}_M(s)^{\mathrm{can}}$. If $t_1 \neq t_2$, then $\mathbf{P}(M^s) \subseteq \mathbf{P}(M^{t_1}) \cap \mathbf{P}(M^{t_2})$, so that p lies in the intersection of two irreducible components of $\mathbf{P}(M)$. Since $\mathbf{P}(\overline{M})$ is locally integral, ν cannot be an isomorphism around p. On the other hand, if $t := t_1 = t_2$, consider the morphism $\zeta^t_M : \Delta^t \to M$ and the subsimplices Δ^{f_1} and Δ^{f_2} of Δ^t. Then the restricted morphisms $\zeta^t_M|_{\Delta^{f_1}} : \Delta^{f_1} \to M$ and $\zeta^t_M|_{\Delta^{f_2}} : \Delta^{f_2} \to M$ are both identical to

$\zeta^s_M : \Delta^s \to M$. Suppose that there exists an open neighbourhood U of p such that the restricted morphism $\nu^{-1}(U) \to U$ is an isomorphism. Then the induced morphism $V := \mathbf{P}(\Delta^t) \cap \nu^{-1}(U) \to \mathbf{P}(M)$ is an open immersion; in particular it is injective. Since $\mathbf{P}(\Delta^{f_1}) \cup \mathbf{P}(\Delta^{f_2}) \subseteq (\zeta^t_M)^{-1}(\mathbf{P}(M^s))$, the restricted morphism

$$\big(V \cap \mathbf{P}(\Delta^{f_1})\big) \cup \big(V \cap \mathbf{P}(\Delta^{f_2})\big) \to \mathbf{P}(M^s) \tag{6.12}$$

is injective. However, the intersections $V \cap \mathbf{P}(\Delta^{f_i})$ are non-empty, and therefore dense in $\mathbf{P}(\Delta^{f_i})$. Define $V_1 = V \backslash \mathbf{P}(\Delta^{f_2})$ and $V_2 = V \backslash \mathbf{P}(\Delta^{f_1})$. Since both morphisms $\mathbf{P}(\Delta^{f_i}) \to \mathbf{P}(M^s)$ are surjective morphisms of integral schemes, the induced morphisms $V_i \cap \mathbf{P}(\Delta^{f_i}) \to \mathbf{P}(M^s)$ are dominant, and therefore their images have non-empty intersection. But then (6.12) cannot possibly be injective. This is a contradiction, showing that ν is not an isomorphism around p. In the second case, where $\mathrm{lk}_M(s)$ is irreducible, let (t, f) be its unique (canonical) facet, and suppose that $f_1 : m_1 \to n$ and $f_2 : m_2 \to n$ is a pair of faces of Δ^n mapping to the same face (u, g) of $\mathrm{lk}_M(s)$. Then $f_1 g = f = f_2 g$. This means that $\mathrm{lk}_M(u)$ is not irreducible, as it contains the pair of facets (t, f_1),(t, f_2). But since $p \in \mathbf{P}(M^u)$, the same argument as above shows that $p \in \mathbf{P}(M)_{\mathbf{n}}$. This shows that $\mathbf{P}(N(M)) \subseteq \mathbf{P}(M)_{\mathbf{n}}$.

Conversely, consider the restriction $\mathbf{P}(\overline{M}) \backslash \nu^{-1}(\mathbf{P}(N(M))) \to \mathbf{P}(M) \backslash \mathbf{P}(N(M))$. By Lemma 3.2.4 and Lemma 4.4.4, this morphism is induced by the morphism $\phi : \overline{M} \backslash \nu^{-1}(N(M)) \to M \backslash N(M)$ of open categories. Since $\nu : \overline{M} \to M$ is surjective, ϕ is as well. This means that the induced local morphisms $\phi_t : \overline{M}_t \to M_{\phi(t)}$ are surjective for each $t \in \overline{M} \backslash \nu^{-1}(N(M))$. The induced morphisms of links $\mathrm{lk}_{\overline{M}}(t) \to \mathrm{lk}_M(t)$ are surjective morphisms of simplices, hence isomorphisms. This means that ϕ is a local isomorphism. However, suppose that ϕ maps a pair of faces s_1,s_2 to the same face s. Let (t, f) be a facet of M_s. Since ϕ_{s_1} and ϕ_{s_2} are isomorphisms, there exists facets (t_1, f) and (t_2, f) of $\overline{M}_{s_1}$ and $\overline{M}_{s_2}$ respectively mapping to (t, f). However, there is a bijective correspondence between the facets of $\overline{M}$ and the facets of M, which means that $t_1 = t_2$. Hence $s_1 = s_2$, which shows that ϕ is injective. In conclusion, the morphism $\mathbf{P}(\overline{M}) \backslash \nu^{-1}(\mathbf{P}(N(M))) \to \mathbf{P}(M) \backslash \mathbf{P}(N(M))$ is an isomorphism, and by definition of $\mathbf{P}(M)_{\mathbf{n}}$, this shows that $\mathbf{P}(M)_{\mathbf{n}} \subseteq \mathbf{P}(N(M))$. $\square$

Lemma 6.4.2. *Let M, N be a pair of unimodular sets. Then there is an isomorphism $N(M \star N) \cong N(M) \star N \cup M \star N(N)$.*

Proof. By definition, $N(M \star N)$ consists of the faces (s, t) for which $\mathrm{lk}_{M \star N}((s, t))$ is not a simplex. By Lemma 5.2.4, $\mathrm{lk}_{M \star N}((s, t)) \cong \mathrm{lk}_M(s) \star \mathrm{lk}_N(t)$. This unimodular set is a simplex if and only if both $\mathrm{lk}_M(s)$ and $\mathrm{lk}_N(t)$ are, and the result follows from Lemma 6.4.1 $\square$

Definition 6.4.3. Let M be an $\mathbf{n}$-dimensional polyhedral set. Then M is called a *homology manifold* (with respect to the field k) if $|M|$ is, i.e. if the reduced homology groups $\widetilde{H}_i(|M| \mid p; k) = 0$ and $\widetilde{H}_{\mathbf{n}-|s|-1}(|M| \mid p; k) \cong k$ for all $i < \mathbf{n} - |s| - 1$ and points $p \in |M|$. If $H_{\mathbf{n}}(M; k) \cong k$, then M is called an *orientable* homology manifold. If moreover $\widetilde{H}_i(|M|; k) = 0$ for all $i < \mathbf{n}$ then M

is called a *homology sphere* (with respect to k). Note that by Lemma 5.4.5, a unimodular set M is a homology manifold if and only if $\widetilde{H}_{\mathbf{n}-|s|-1}(\mathrm{lk}_M(s); k) \cong k$ and $\widetilde{H}_i(\mathrm{lk}_M(s); k) = 0$ for all faces s and $i < \mathbf{n} - |s| - 1$.

Lemma 6.4.4. *Let M be an $\mathbf{n}$-dimensional polyhedral homology manifold with respect to k. Then $N(M) = M^{\mathbf{n}-1}$.*

Proof. The only polyhedral set which is simultaneously a simplex and a homology sphere is $\underline{\varnothing}$. Hence $N(M)$ contains all non-facets. Since M is pure, the statement follows. $\square$

For an affine morphism $f : X \to Y$ of schemes, the quasi-coherent $\mathcal{O}_X$-module $f^!_{\mathrm{sh}}\mathcal{G}$ is defined by the formula $f_* f^!_{\mathrm{sh}}\mathcal{G} = \mathscr{H}om_Y(f_*\mathcal{O}_X, \mathcal{G})$, for any quasi-coherent sheaf $\mathcal{G}$ on Y ([Vak17, Section 30.3]). The functor $f^!_{\mathrm{sh}}$ is a right adjoint to the functor f_*. If f is finite and flat, the functor $f^!_{\mathrm{sh}}$ is denoted $f^!$, and is part of the six functor formalism. If f is finite and X and Y are projective schemes of the same dimension, then $f^!_{\mathrm{sh}}\omega_Y \cong \omega_X$. If Y is also locally Gorenstein, then the dualizing sheaf ω_Y is invertible. By the projection formula and via the adjunction $(f_*, f^!_{\mathrm{sh}})$, we have

$$\begin{aligned} f_* f^!_{\mathrm{sh}}\omega_Y &= \mathscr{H}om_Y(f_*\mathcal{O}_X, \omega_Y) \cong \mathscr{H}om_Y(f_*\mathcal{O}_X \otimes \omega_Y^\vee, \mathcal{O}_Y) \\ &\cong \mathscr{H}om_Y(f_* f^*\omega_Y^\vee, \mathcal{O}_Y) \cong f_* \mathscr{H}om_X(f^*\omega_Y^\vee, f^!_{\mathrm{sh}}\mathcal{O}_Y) \\ &\cong f_* \mathscr{H}om_X(\mathcal{O}_X, f^!_{\mathrm{sh}}\mathcal{O}_Y \otimes f^*\omega_Y) \cong f_*(f^!_{\mathrm{sh}}\mathcal{O}_Y \otimes f^*\omega_Y). \end{aligned}$$

Thus we obtain the formula

$$\omega_X \cong f^!_{\mathrm{sh}}\mathcal{O}_Y \otimes f^*\omega_Y. \tag{6.13}$$

Lemma 6.4.5. *Let $f : X \to Y$ be an affine étale morphism of schemes. Then $f^!_{\mathrm{sh}}\mathcal{G} \cong f^*\mathcal{G}$ for each quasi-coherent sheaf $\mathcal{G}$ on Y. In particluar, if f is also finite and X and Y are projective schemes of the same dimension, then $f^*\omega_Y \cong \omega_X$.*

Proof. Consider the fiber square

$$\begin{tikzcd} X \times_Y X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

The projections are affine and f is flat, so by the observation in [Vak06, Section 8], there is a natural isomorphism $\mathrm{pr}_1^* f^!_{\mathrm{sh}}\mathcal{G} \cong (\mathrm{pr}_2)^!_{\mathrm{sh}} f^*\mathcal{G}$. Next, the diagonal morphism $\Delta : X \to X \times_Y X$ is an open immersion since f is unramified, and a section to each projection. Applying $\Delta^* = \Delta^!_{\mathrm{sh}}$ on both sides yields the result. $\square$

Let X be a reduced projective scheme, and let $\nu : \overline{X} \to X$ be the normalization of X. Consider the injection $\nu^\sharp : \mathcal{O}_X \to \nu_*\mathcal{O}_{\overline{X}}$. Then the conductor ideal sheaf is defined as $\mathcal{I} = \mathrm{Ann}_X(\nu_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X)$. Locally on affine $U \subseteq X$, note that $\mathcal{I}(U) = \mathcal{O}_X(U)$ if and only if $\nu|_{\nu^{-1}(U)} : \nu^{-1}(U) \to U$ is an isomorphism. Thus the support of $\mathcal{O}_X/\mathcal{I}$ is the locus of where ν is not an isomorphism, i.e. the non-normal locus of X. Assume that X is seminormal. Then the ideal $\mathcal{I}$ is radical by Lemma 3.3.2, and therefore cuts out the reduced subscheme $X_{\mathbf{n}}$ of X. It follows that the ideal sheaves $\mathcal{I}$ and $\mathcal{I}_{X_{\mathbf{n}}}$ are equal. If $A \subseteq \overline{A}$ is the integral closure of a reduced ring A, then $\mathrm{Ann}_A(\overline{A}/A) \cong \mathrm{Hom}_A(\overline{A}, A)$ ([HS06, Lemma 2.4.2]). The conductor ideal $\mathrm{Ann}_A(\overline{A}/A)$ is moreover characterized by being the largest common ideal of A and $\overline{A}$. Globally, this translates to $\mathscr{H}om_X(\nu_*\mathcal{O}_{\overline{X}}, \mathcal{O}_X) \cong \mathcal{I}_{X_{\mathbf{n}}} = \nu_*\mathcal{I}_{\nu^{-1}(X_{\mathbf{n}})}$. Hence

$$\mathcal{I}_{\nu^{-1}(X_{\mathbf{n}})} \cong \nu^!_{\mathrm{sh}}\mathcal{O}_X. \tag{6.14}$$

6.4.1 Classifying Gorenstein unimodular sets

Theorem 6.4.6. *Let M be a unimodular set. Then the following are equivalent:*

a) $\Gamma(M)$ *is Gorenstein.*

b) *There exists a unimodular homology sphere S (with respect to k) such that $M \cong S \star \Delta^n$ for some integer $n \geq -1$.*

c) *There exists a unimodular homology sphere S such that $\Gamma(M) \cong \Gamma(S) \otimes_k \Gamma(\Delta^n)$ for some integer $n \geq -1$.*

Moreover, $\omega_M \cong \mathcal{O}_M(-n-1)$. In particular, $|M|$ is a homology sphere if and only if $\Gamma(M)$ is Gorenstein and $\omega_M \cong \mathcal{O}_M$.

Proof. We proceed by induction on the dimension $\mathbf{n}$ of M. For any $\mathbf{n}$, c) immediately follows from b), so we can limit ourself to proving a) $\Rightarrow$ b) and c) $\Rightarrow$ a). First we will deal with the cases $\mathbf{n} = -1, 0$ separately. If $\mathbf{n} = -1$, then $M = \underline{\varnothing}$ and $\Gamma(M) = k$. A field is a Gorenstein ring and $\underline{\varnothing}$ is a homology sphere, so the implications hold. If $\mathbf{n} = 0$, then M is a disjoint union of points. By Proposition 2.7.2, $\Gamma(M)$ is on the form $k[x_v : v \in J^0_M]/(x_v x_w : v \neq w)$. $\Gamma(M)$ is clearly Gorenstein if $|J^0_M| = 1$, and in this case $\mathbb{L}(M_v) \to M$ is an isomorphism, where v is the vertex of M (satisfying $\mathrm{lk}_M(v) \cong \underline{\varnothing}$). To determine whether $\Gamma(M)$ is Gorenstein when $|J^0_M| > 1$, we divide out by the non-zero divisor $\sum_{v \in J^0_M} x_v$. Pick a vertex $v_0 \in J^0_M$. Then $k \oplus k\{x_v : v \neq v_0\}$ is a graded decomposition of the quotient ring. Observe that the socle is $(|J^0_M| - 1)$-dimensional. To be a Gorenstein ring, we therefore require $|J^0_M| = 2$. In this case, M is a homology sphere. We conclude that a) $\Rightarrow$ b). Conversely, if $\Gamma(M)$ is on the form $\Gamma(N) \otimes_k \Gamma(\Delta^n)$ for a homology sphere N, then M is either the disjoint union of two points or on the form $\Gamma(\Delta^0)$. As we have seen, $\Gamma(M)$ is Gorenstein in both cases. In conclusion, c) $\Rightarrow$ a).

We may now assume that $\mathbf{n} \geq 1$. We begin with the direction c) $\Rightarrow$ a). Suppose that M is a homology sphere. Then by Theorem 6.3.3, $\Gamma(M)$ is Cohen-Macaulay. By Lemma 5.4.5, each link is a homology sphere. Thus by the inductive hypothesis, $\Gamma(\mathrm{lk}_M(v))$ is a Gorenstein ring for each vertex $v \in M_0$. By Corollary 5.2.9, there is an isomorphism $\mathbf{A}(M_v) \cong \operatorname{Spec}\Gamma(\mathrm{lk}_M(v))$. This means that the étale cover $\{\mathbf{A}(M_v) \to \mathbf{P}(M)\}_{v \in J_M^0}$ consists of locally Gorenstein schemes. However, an étale morphism of schemes (where every closed point is a k-point) induces an isomorphism on stalks after completion [Stacks, Tag 039M], and by [Mat89, Theorem 18.3] a local finite-dimensional Noetherian ring is Gorenstein if and only its completion is. It follows that $\mathbf{P}(M)$ is locally Gorenstein. This means that the dualizing sheaf ω_M is invertible. We will show that ω_M is trivial. Assume first that $\mathbf{n} \geq 2$. By Proposition 3.5.3, ω_M is represented by an element of $H^0(|M|\backslash|M^0|;\mathbb{Z}) \times \mathrm{Pic}_0(M)$. But $H^0(|M|\backslash|M^0|;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank equal to the number of connected components of $|M|\backslash|M^0|$. Since M is a homology sphere of dimension ≥ 2, the long exact sequence of relative homology reduces to an exact sequence

$$0 \to H_1(|M| \mid |M^0|;k) \to H_0(|M|\backslash|M^0|;k) \to H_0(|M|;k) \to H_0(|M| \mid |M^0|;k) \to 0.$$

For each i, $H_i(|M| \mid |M^0|;k) \cong \bigoplus_{v \in J_M^0} \widetilde{H}_{i-1}(\mathrm{lk}_M(v);k)$ by Lemma 5.4.5. But each $\mathrm{lk}_M(v)$ is a ≥ 1-dimensional homology sphere, so $H_0(|M| \mid |M^0|;k) = 0$ and $H_1(|M| \mid |M^0|;k) = 0$. Thus $H_0(|M|\backslash|M^0|;k) \cong k$, and therefore $H^0(|M|\backslash|M^0|;k) \cong k$. Hence ω_M is represented by a pair $(d,\mathcal{E})$, where d is an integer, and $\mathcal{E}$ has trivial degree function. In other words, $\zeta_M^{s*}\omega_M \cong \mathcal{O}_{\Delta^s}(d)$ for all faces s of M. In particular, $\alpha_M^*\omega_M \cong \mathcal{O}_{\Delta_M^{\mathbf{n}}}(d)$. By Serre duality and Theorem 3.4.1, we have $\dim_k H^0(M,\omega_M) = \dim_k H^{\mathbf{n}}(M,\mathcal{O}_M) = \dim_k H^{\mathbf{n}}(M;k) = 1$, and $\dim_k H^{\mathbf{n}}(M,\omega_M) = \dim_k H^0(M,\mathcal{O}_M) = 1$. Consider the exact sequence

$$0 \to \omega_M \to \mathcal{O}_{\Delta_M^{\mathbf{n}}}(d) \oplus \mathcal{O}_{M^{\mathbf{n}-1}}(\iota_M^*\omega_M) \to \mathcal{O}_{\partial\Delta_M^{\mathbf{n}}}(d) \to 0,$$

and the induced long exact sequence in cohomology. Since $H^0(M,\omega_M) \subseteq H^0(\Delta_M^{\mathbf{n}},\mathcal{O}_{\Delta_M^{\mathbf{n}}}(d))$, we must have $d \geq 0$. Assume that $d > 0$. The end of the long exact sequence is on the form $H^{\mathbf{n}-1}(\partial\Delta_M^{\mathbf{n}},\mathcal{O}_{\partial\Delta_M^{\mathbf{n}}}(d)) \to H^{\mathbf{n}}(M,\omega_M) \to H^{\mathbf{n}}(\Delta_M^{\mathbf{n}},\mathcal{O}_{\Delta_M^{\mathbf{n}}}(d)) \to 0$. Since $d > 0$, $H^{\mathbf{n}}(\Delta_M^{\mathbf{n}},\mathcal{O}_{\Delta_M^{\mathbf{n}}}(d))$ vanishes. But the cohomology group $H^{\mathbf{n}-1}(\partial\Delta_M^{\mathbf{n}},\mathcal{O}_{\partial\Delta_M^{\mathbf{n}}}(d))$ vanishes as well, which is seen by the long exact sequence in cohomology associated to the exact sequence

$$0 \to \mathcal{O}_{\Delta_M^{\mathbf{n}}}(-\mathbf{n}-1+d) \to \mathcal{O}_{\Delta_M^{\mathbf{n}}}(d) \to \mathcal{O}_{\partial\Delta_M^{\mathbf{n}}} \to 0.$$

Hence $H^{\mathbf{n}}(M,\omega_M) = 0$, which is a contradiction. We conclude that $d = 0$. Next we may follow in parallel with the proof of [BE91, Theorem 6.1]. Let x be a nontrivial global section of ω_M, and suppose that x vanishes at a point $P \in \mathbf{P}(M)$. Then P is in the image of a morphism $\zeta_M^s : \mathbf{P}(\Delta^s) \to \mathbf{P}(M)$ for some facet s. Since $\zeta_M^{s*}\omega_M \cong \mathcal{O}_{\Delta^s}$, we must have $\zeta_M^{s*}(x) = 0$. Since $|M|\backslash|M^0|$ is connected, the same argument used in Proposition 3.5.3 shows that any facet t can be connected with s by a chain $s \leftarrow u_1 \to \cdots \leftarrow u_m \to t$ of ≥ 1-dimensional faces $u_1,\ldots,u_m$ of M. If u connects s with a facet t via a simple chain $t \xleftarrow{f} u \xrightarrow{g} s$, then $\zeta_{\Delta^t}^{f*}\zeta_M^{t*}(x) = \zeta_{\Delta^s}^{g*}\zeta_M^{s*}(x) = 0$. This implies that $\zeta_M^{t*}(x) = 0$. Thus $\zeta_M^{t*}(x) = 0$ for all facets t by induction on the length of a chain connecting s with t. Since

$H^0(M, \omega_M) \subseteq H^0(\Delta^{\mathbf{n}}_M, \mathcal{O}_{\Delta^{\mathbf{n}}_M})$, this is a contradiction. So x vanishes nowhere, and therefore trivializes ω_M. We conclude that $\omega_M \cong \mathcal{O}_M$. Next we consider the case where $\mathbf{n} = 1$. Since $H_0(M; k) \cong k$, M is a connected graph. It is moreover a 2-regular graph, since $\mathrm{lk}_M(v)$ is the disjoint union of two points for each vertex v. It follows that M is a cycle. Consider the unique morphism $\phi : M \to \mathcal{L}$, where $\mathcal{L}$ is the loop (a cycle with one edge). Since $|\mathcal{L}|\backslash|\mathcal{L}^0|$ is connected, applying the same argument as above shows that $\omega_{\mathcal{L}} \cong \mathcal{O}_{\mathcal{L}}$. Observe that ϕ is a local isomorphism, so that $\phi : \mathbf{P}(M) \to \mathbf{P}(\mathcal{L})$ is étale by Theorem 4.4.1. By Lemma 6.4.5, we have $\omega_M \cong \phi^*\omega_{\mathcal{L}}$, so $\omega_M \cong \mathcal{O}_M$. Thus for general $\mathbf{n}$, $\Gamma(M)$ is Cohen-Macaulay and $\omega_M \cong \mathcal{O}_M$. It follows from [GW78, (5.1.9)] that $\Gamma(M)$ is a graded Gorenstein ring (with canonical module $\Gamma(M)$).

Assume now that $\Gamma(M) \cong \Gamma(\Delta^n) \otimes \Gamma(N)$ for some homology sphere N and some integer $n \geq 0$. The graded polynomial ring $\Gamma(\Delta^n)$ is Gorenstein with canonical module $\Gamma(\Delta^n)[-n-1]$, and by the inductive hypothesis, $\Gamma(N)$ is a graded Gorenstein ring with canonical module $\Gamma(N)$. It follows immediately that $\Gamma(M)$ is Gorenstein with canonical module $\Gamma(M)[-n-1]$ (see [Eis95, Section 21.11]). By [GW78, (5.1.8)], we have that $\omega_M \cong \mathcal{O}_M(-n-1)$.

For the other direction $a) \Rightarrow b)$, assume that $\Gamma(M)$ is a Gorenstein ring. In particular, $\Gamma(M)$ is Cohen-Macaulay. Hence by Theorem 6.3.3, we have that $\widetilde{H}^i(M; k) = 0$ for all $i < \mathbf{n}$. Let s be a face of M, and consider the étale morphism $\mathbf{A}(M_s) \to \mathbf{P}(M)$. By [GW78, Lemma 5.1.10], $\mathbf{P}(M)$ is locally Gorenstein, so each $\mathbf{A}(M_s)$ is Gorenstein as well. But $\mathbf{A}(M_s) \cong \operatorname{Spec}\Gamma(\mathrm{lk}_M(s)) \times (\mathbf{G}_m)^{\dim s}$ by Corollary 5.2.9, and it follows that $\Gamma(\mathrm{lk}_M(s))$ is a Gorenstein ring. By the inductive hypothesis, $\mathrm{lk}_M(s)$ is on the form $N \star \Delta^n$ for some homology sphere N and some integer $n \geq -1$. Thus $|\,\mathrm{lk}_M(s)|$ which is of one of the following two types: Either it is a homology sphere ($n = -1$), or it is the cone over some other topological space ($n \geq 0$). In the latter case it is contractible. So in either case, we have by Lemma 5.4.5 that $\widetilde{H}_{i-|s|-1}(\mathrm{lk}_M(s); k) = 0$ for all $i < \mathbf{n}$, and $\widetilde{H}_{\mathbf{n}-|s|-1}(\mathrm{lk}_M(s); k) \cong k$ or $\widetilde{H}_{\mathbf{n}-|s|-1}(\mathrm{lk}_M(s); k) \cong 0$. By [GW78, (5.1.9)] we have $\omega_M \cong \mathcal{O}_M(-n)$ for some integer n. Assume that $n < 0$. Then by Serre duality, $H^0(M, \mathcal{O}_M) = H^{\mathbf{n}}(M, \mathcal{O}_M(-n))$. But Lemma 3.6.5 implies that $H^{\mathbf{n}}(\mathcal{O}_M(-n)) = 0$. Since $\dim_k H^0(M, \mathcal{O}_M) > 0$, this is a contradiction, so $n \geq 0$. By Serre duality again, $\dim_k H^0(M, \mathcal{O}_M(d+1-n)) = \dim_k H^{\mathbf{n}}(M, \mathcal{O}_M(-d-1))$ for all integers d. Thus from Theorem 6.3.3 and Proposition 3.6.3, we obtain the formula

$$\sum_{m=0}^{\mathbf{n}} \sum_{s \in J_M^m} \dim_k \widetilde{H}^{\mathbf{n}-m-1}(\mathrm{lk}_M(s); k) \binom{d}{m} = \sum_{m=0}^{\mathbf{n}} |M_m| \binom{d-n}{m} \tag{6.15}$$

for integers $d \geq 0$. Assume first that $n = 0$. Then (6.15) forces the equalities $\dim_k \widetilde{H}^{\mathbf{n}-|s|-1}(\mathrm{lk}_M(s); k) = 1$ for each face s of M. By Serre duality, $\dim_k H^{\mathbf{n}}(M, \mathcal{O}_M) = \dim_k H^0(M, \mathcal{O}_M) = 1$. Hence by Theorem 3.4.1, we have $\dim_k \widetilde{H}^{\mathbf{n}}(M; k) = 1$. Thus M is a homology sphere, as the homological conditions are now obtained from the universal coefficient theorem. Next, assume that $n > 0$. Since $\widetilde{H}^{\mathbf{n}}(M, \mathcal{O}_M(-n)) = H^0(M, \mathcal{O}_M) = 1$, by Theorem 6.3.3 we obtain the formula

$$\sum_{m=0}^{\mathbf{n}} \sum_{s \in J_M^m} \dim_k \widetilde{H}^{\mathbf{n}-m-1}(\mathrm{lk}_M(s);k) \binom{n-1}{m} = 1. \tag{6.16}$$

This implies that there exists a unique face $s \in M^{\mathrm{can}}$ of dimension $\dim s \le n-1$ such that $\dim_k \widetilde{H}^{\mathbf{n}-|s|-1}(\mathrm{lk}_M(s);k) = 1$. Thus $\mathrm{lk}_M(s)$ is a homology sphere. Since $\dim s = n-1$, we have $n \le \mathbf{n}+1$.

By Lemma 5.2.8, we have $\mathbb{L}(M_s) \cong \mathrm{lk}_M(s) \star \Delta^s$, so that $\Gamma(\mathbb{L}(M_s)) \cong \Gamma(\mathrm{lk}_M(s)) \otimes_k \Gamma(\Delta^s)$. The inductive hypothesis implies that $\Gamma(\mathbb{L}(M_s))$ is Gorenstein with canonical module $\Gamma(\mathbb{L}(M_s))[-n]$. Thus $\omega_{\mathbb{L}(M_s)} \cong \mathcal{O}_{\mathbb{L}(M_s)}(-n)$. Consider now the morphism $\phi : \mathbf{P}(\mathbb{L}(M_s)) \to \mathbf{P}(M)$ induced by $\phi : \mathbb{L}(M_s) \to M$, and observe that $\phi^*\omega_M \cong \omega_{\mathbb{L}(M_s)}$. By (6.13), we have that $\phi^!_{\mathrm{sh}}\mathcal{O}_M \cong \mathcal{O}_{\mathbb{L}(M_s)}$. This means that

$$\phi^{\Delta !}_{\mathrm{sh}}(\alpha_M)^!_{\mathrm{sh}}\mathcal{O}_M \cong (\alpha_{\mathbb{L}(M_s)})^!_{\mathrm{sh}}\phi^!_{\mathrm{sh}}\mathcal{O}_M \cong (\alpha_{\mathbb{L}(M_s)})^!_{\mathrm{sh}}\mathcal{O}_{\mathbb{L}(M_s)}.$$

By Lemma 6.4.1, $\mathbf{P}(M)_{\mathbf{n}} = \mathbf{P}(N(M))$ and $\mathbf{P}(\mathbb{L}(M_s))_{\mathbf{n}} = \mathbf{P}(N(\mathbb{L}(M_s)))$. By (6.14), we have $\phi^{\Delta !}\mathcal{I}_{\alpha_M^{-1}(N(M))} = \mathcal{I}_{\alpha^{-1}_{\mathbb{L}(M_s)}(N(\mathbb{L}(M_s)))}$. However, $\phi^{\Delta} : \mathbf{P}(\Delta^{\mathbf{n}}_{\mathbb{L}(M_s)}) \to \mathbf{P}(\Delta^{\mathbf{n}}_M)$ is locally on $\mathbf{P}(\Delta^{\mathbf{n}}_{\mathbb{L}(M_s)})$ an isomorphism, so the functors $\phi^{\Delta !}$ and ϕ^* are easily seen to be naturally equivalent. Hence $\phi^{\Delta *}\mathcal{I}_{\alpha_M^{-1}(N(M))} = \mathcal{I}_{\alpha^{-1}_{\mathbb{L}(M_s)}(N(\mathbb{L}(M_s)))}$, which implies that $\phi^{-1}(\mathbf{P}(\alpha_M^{-1}(N(M)))) = \mathbf{P}(\alpha^{-1}_{\mathbb{L}(M_s)}(N(\mathbb{L}(M_s))))$. Thus by Lemma 3.2.4, $\mathbf{P}(\phi^{-1}(N(M))) = \mathbf{P}(N(\mathbb{L}(M_s)))$. This implies that $\phi^{-1}(N(M)) = N(\mathbb{L}(M_s))$.

By Lemma 5.2.8, the isomorphism $\mathbf{P}(\mathbb{L}(M_s)) \to \mathbf{P}(\mathrm{lk}_M(s) \star \Delta^s)$ restricts to an isomorphism $\mathbf{P}(M_s^P) \to \mathbf{P}(\mathrm{lk}_M(s) \star \partial\Delta^s)$. By Lemma 6.4.2, we have $N(\mathrm{lk}_M(s) \star \Delta^s) = N(\mathrm{lk}_M(s)) \star \Delta^s \cup \mathrm{lk}_M(s) \star N(\Delta^s)$. However, $\mathbf{P}(\Delta^s)$ is normal, so $N(\Delta^s) = \varnothing$. Also, by Lemma 6.4.4, we have $N(\mathrm{lk}_M(s)) = \mathrm{lk}_M(s)^{\mathbf{n}-|s|-2}$. Hence $N(\mathrm{lk}_M(s) \star \Delta^s) = \mathrm{lk}_M(s)^{\mathbf{n}-|s|-2} \star \Delta^s$. It follows that

$$\begin{aligned}\mathbf{P}(M_s^P \cap N(\mathbb{L}(M_s))) &= \mathbf{P}(M_s^P) \cap \mathbf{P}(N(\mathbb{L}(M_s))) \\ &\cong \mathbf{P}(\mathrm{lk}_M(s) \star \partial\Delta^s) \cap \mathbf{P}(\mathrm{lk}_M(s)^{\mathbf{n}-|s|-2} \star \Delta^s) \\ &= \mathbf{P}(\mathrm{lk}_M(s)^{\mathbf{n}-|s|-2} \star \partial\Delta^s).\end{aligned}$$

Since $\mathrm{lk}_M(s)^{\mathbf{n}-|s|-2} \star \partial\Delta^s \subseteq (\mathrm{lk}_M(s) \star \Delta^s)^{\mathbf{n}-2}$, the closed subscheme $\mathbf{P}(M_s^P \cap N(\mathbb{L}(M_s))$ does not have any components of dimension $\ge (\mathbf{n}-1)$. This implies that $M_s^P \cap N(\mathbb{L}(M_s)) \subseteq M^{\mathbf{n}-2}$. Since $\Delta^{\mathbf{n}}_{\mathbb{L}(M_s)} \to \Delta^{\mathbf{n}}_M$ is a local isomorphism, it follows that $\mathbb{L}(M_s)\backslash N(\mathbb{L}(M_s)) \to M\backslash N(M)$ is as well. Since $M_s \to M$ is also a local isomorphism, so is the morphism $(\mathbb{L}(M_s)\backslash N(\mathbb{L}(M_s))) \cup M_s \to M$ of open categories. In conclusion, the morphism $\mathbb{L}(M_s)\backslash\mathbb{L}(M_s)^{\mathbf{n}-2} \to M\backslash M^{\mathbf{n}-2}$ is a local isomorphism.

We will show that both of these open categories are connected. If $\mathbf{n}=1$ this is clear. Assume that $\mathbf{n}>1$, and suppose that $M = N \cup Z$ with $N \cap Z \subseteq M^{\mathbf{n}-2}$, where N and Z are both nontrivial pure $\mathbf{n}$-dimensional. Let $t \in M_{\mathbf{n}-2}$ be a facet of this intersection. Then $\mathrm{lk}_M(t)$ is 1-dimensional. However, its 1-dimensional faces are on the form $t \to y$ for $y \in N$, or $t \to z$ for $z \in Z$. But such facets cannot share a vertex $t \to u$, since $u \in M_{\mathbf{n}-1}$. It follows that $\mathrm{lk}_M(t)$ is not connected, so that $\widetilde{H}^0(\mathrm{lk}_M(t); k) \neq 0$. This is a contradiction. The same argument applies to $\mathbb{L}(M_s)$.

We will now show that ϕ is surjective. Pick $s \in M_{\mathbf{n}}$ not in the image of ϕ. Then s can be connected via elements in $M \backslash M^{\mathbf{n}-2}$ to a face $s' \in M_{\mathbf{n}}$ which is in the image of ϕ. Replacing s and s' with appropriately chosen other faces of the chain if necessary, we may assume that there exists a pair of arrows $f : t \to s$ and $g : t \to s'$ in $\mathbb{T}(M)$ for some $t \in M \backslash M^{\mathbf{n}-2}$. Let $u \in \mathbb{L}(M_s)_{\mathbf{n}}$ be a face with $\phi(u) = s'$, and let $r = g^M(u)$. Then the local morphism $\phi_r : \mathbb{L}(M_s)_r \to M_t$ cannot be an isomorphism since $t \to s$ is not in the image. This is a contradiction, and we conclude that ϕ is surjective.

Next we will show that for each facet t of M, there is maximally one face on the form $s \to t$ in M_s. Suppose on the contrary that there exists two different arrows $f_1, f_2 : s \to t$, and let t_1, t_2 denote the corresponding facets in $\mathbb{L}(M_s)$. Let $i \in \operatorname{im} f_1 \backslash \operatorname{im} f_2$, and consider the arrow $d_i : [|t| - 1] \to [|t|]$. Then there exists a factorization $s \xrightarrow{h} d_i^M(t) \xrightarrow{d_i} t$ of $f_2 : s \to t$, but no factorization $s \to d_i^M(t) \xrightarrow{d_i} t$ of $f_1 : s \to t$. Hence $d_i \in \Delta^t_{f_2} \cap (\Delta^t_{f_1})^P$. This implies that $u_2 = d_i^{\mathbb{L}(M_s)}(t_2) \in M_s$, and $u_1 = d_i^{\mathbb{L}(M_s)}(t_1) \in M_s^P$. However, both faces maps to $u = d_i^M(t)$ via the morphism $\mathbb{L}(M_s) \to M$. Since $M_s^P \cap N(\mathbb{L}(M_s)) \subseteq M^{\mathbf{n}-2}$, and $u_1 \in M_{\mathbf{n}-1}$, we have $u_1 \in \mathbb{L}(M_s) \backslash N(\mathbb{L}(M_s))$. By Lemma 6.4.1, $\mathrm{lk}_{\mathbb{L}(M_s)}(u_1)$ is a simplex. But $\mathbb{L}(M_s) \backslash \mathbb{L}(M_s)^{\mathbf{n}-2} \to M \backslash M^{\mathbf{n}-2}$ is a local isomorphism, which implies that $\mathrm{lk}_{\mathbb{L}(M_s)}(u_1) \cong \mathrm{lk}_{\mathbb{L}(M_s)}(u_2) \cong \mathrm{lk}_M(u)$. By Lemma 5.2.3, $\mathrm{lk}_M(u) \cong \mathrm{lk}_{\mathrm{lk}_M(s)}(u, h)$, which is a homology sphere. This is a contradiction, since u is not a facet.

In conclusion, the map $\mathbb{L}(M_s)_{\mathbf{n}} \to M_{\mathbf{n}}$ is bijective. Hence $\mathbf{P}(M)$ and $\mathbf{P}(\mathbb{L}(M_s))$ have the same normalization, so $\phi^{\Delta} : \Delta^{\mathbf{n}}_{\mathbb{L}(M_s)} \to \Delta^{\mathbf{n}}_M$ is an isomorphism. Since $\mathbb{L}(M_s) \to M$ is surjective, $\phi^{\sharp} : \mathcal{O}_M \to \phi_* \mathcal{O}_{\mathbb{L}(M_s)}$ is injective. By [HS06, Lemma 2.4.2], there is an isomorphism

$$\mathrm{Ann}_M(\phi_* \mathcal{O}_{\mathbb{L}(M_s)} / \mathcal{O}_M) \cong \mathscr{H}\!om_M(\phi_* \mathcal{O}_{\mathbb{L}(M_s)}, \mathcal{O}_M) = \phi_* \phi^!_{\mathrm{sh}} \mathcal{O}_M.$$

Since $\phi^!_{\mathrm{sh}} \mathcal{O}_M \cong \mathcal{O}_{\mathbb{L}(M_s)}$, we obtain an isomorphism $\phi_* \mathcal{O}_{\mathbb{L}(M_s)} \cong \mathrm{Ann}_M(\phi_* \mathcal{O}_{\mathbb{L}(M_s)} / \mathcal{O}_M)$, which is an ideal sheaf of $\mathcal{O}_M$. It follows that $\phi^{\sharp} : \mathcal{O}_M \to \phi_* \mathcal{O}_{\mathbb{L}(M_s)}$ is an isomorphism. Since ϕ is affine, $\mathbf{P}(\mathbb{L}(M_s)) \to \mathbf{P}(M)$ is an isomorphism. By Lemma 4.3.5, this implies that $\mathbb{L}(M_s) \to M$ is an isomorphism, and thus $M \cong \mathrm{lk}_M(s) \star \Delta^s$, showing b). $\square$

6.5 Regular subdivisions of face schemes

We continue with the notation from Section 2.9. We define $\mathbf{P}_R(M, \Psi) = \mathrm{Proj}(\Gamma_R(M, \Psi))$. This yields a functor $\mathbf{P}_R : \overline{\mathcal{C}} \to \mathrm{Sch}_R$, since any homomorphism on the form $\Gamma_R(\phi) : \Gamma_R(N, \Phi) \to \Gamma_R(M, \Psi)$ is finite. The natural grading on $\Gamma_R(M, \Psi)$ defines a sheaf $\mathcal{O}_{\mathbf{P}_R(M, \Psi)}(1)$.

Lemma 6.5.1. *Let M be a finite polyhedral set, and let R be a k-algebra DVR with residue field k and field of fractions K. Then the morphism $\mathbf{P}_R(M,\Psi) \to \operatorname{Spec}(R)$ is flat with central fiber $\mathbf{P}_k(PS(M,\Psi))$, and generic fiber $\mathbf{P}_K(M)$. Moreover, the sheaf $\mathcal{O}_{\mathbf{P}_R(M,\Psi)}(1)$ is a relatively ample line bundle with central fiber $\mathcal{O}_{\mathbf{P}_k(PS(M,\Psi))}(1)$, and generic fiber $\mathcal{O}_{\mathbf{P}_K(M)}(1)$. Hence the pairs $(\mathbf{P}_K(M), \mathcal{O}_M(1))$ and $(\mathbf{P}_k(PS(M,\Psi)), \mathcal{O}_{\mathbf{P}_k(PS(M,\Psi))}(1))$ are deformation equivalent.*

Proof. The both statements follows from Theorem 2.9.11 (and the fact that relative ampleness is a local property on the base). □

Lemma 6.5.2. *Let M be a polyhedral set. Then there is a natural isomorphism of pairs $(\mathbf{P}(nM), \mathcal{O}_{nM}(1)) \cong (\mathbf{P}(M), \mathcal{O}_M(n))$.*

Proof. This follows immediately from Lemma 2.9.5. □

Theorem 6.5.3. *If M is a finite polyhedral set, there exists an integer $n \geq 1$ such that pair $(\mathbf{P}_k(M), \mathcal{O}_M(n))$ is deformation equivalent to $(\mathbf{P}_k(Z), \mathcal{O}_Z(1))$ for some simplicial complex Z such that $|Z| \cong |M|$ (and any field k).*

Proof. This follows from a straight-forward application of Theorem 2.9.9, Lemma 6.5.2 and Lemma 6.5.1. □

Theorem 6.5.4. *Let M be a polyhedral homology manifold. Then $\mathbf{P}(M)$ is locally Gorenstein, and $\omega_M^{\otimes 2} \cong \mathcal{O}_M$. Moreover, M is orientable if and only if $\omega_M \cong \mathcal{O}_M$.*

Proof. Assume first that M is unimodular. Since M is a homology manifold, each link $\operatorname{lk}_M(s)$ is a homology sphere for each face s of M. Consider the étale cover $\{\mathbf{A}(M_s) \to \mathbf{P}(M)\}_{s \in \mathbb{T}(M)}$. By Corollary 5.2.9, the morphisms $\mathbf{A}(M_s) \to \operatorname{Spec}\Gamma(\operatorname{lk}_M(s)) \times_k (\mathbf{G}_m)^{\dim s}$ are isomorphisms. Since the Gorenstein property is preserved under étale morphisms, each $\mathbf{A}(M_s)$ is Gorenstein by Theorem 6.4.6. It follows that $\mathbf{P}(M)$ is locally Gorenstein. Assume that M is an orientable homology manifold, and let $\mathbf{n} = \dim M$. By Lemma 3.3.1, $\alpha_M : \mathbf{P}(\Delta_M^{\mathbf{n}}) \to \mathbf{P}(M)$ is the normalization of $\mathbf{P}(M)$. By Lemma 6.4.4 and Lemma 6.4.1, we have $\mathbf{P}(M)_{\mathbf{n}} = \mathbf{P}(M^{\mathbf{n}-1})$. Hence $(\alpha_M)^!_{\mathrm{sh}}\mathcal{O}_M \cong \mathcal{I}_{\partial\Delta_M^{\mathbf{n}}}$ by (6.14). By (3.33), we have $\alpha_* \mathcal{I}_{\partial\Delta_M^{\mathbf{n}}} \cong \mathcal{I}_{M^{\mathbf{n}-1}}$, so $\alpha_{M*}(\alpha_M)^!_{\mathrm{sh}}\mathcal{O}_M \cong \mathcal{I}_{M^{\mathbf{n}-1}}$. However, we also have $\mathcal{I}_{\partial\Delta_M^{\mathbf{n}}} \cong \omega_{\Delta_M^{\mathbf{n}}}$ by Lemma 3.6.2. Thus we may rewrite (6.13) as $\omega_{\Delta_M^{\mathbf{n}}} \cong \omega_{\Delta_M^{\mathbf{n}}} \otimes \alpha_M^* \omega_M$, which implies that $\alpha_M^* \omega_M \cong \mathcal{O}_{\Delta_M^{\mathbf{n}}}$. The same argument as in the proof of Theorem 6.4.6 now shows that ω_M has a non-trivial global section, which trivializes ω_M. This shows that $\omega_M \cong \mathcal{O}_M$ whenever M is orientable.

Next we assume that M is non-orientable. By Proposition 5.4.3, the orientable double cover $\pi : \widetilde{M} \to M$ is a local isomorphism. By Theorem 4.4.1, $\pi : \mathbf{P}(\widetilde{M}) \to \mathbf{P}(M)$ is étale. We know that $\pi_1(|\widetilde{M}|) \subseteq \pi_1(|M|)$ is a subgroup of index two, hence by abelianizing we get an exact sequence

$$0 \to H_1(\widetilde{M}) \to H_1(M) \to \mathbb{Z}/2 \to 0.$$

By the universal coefficient theorem, dualizing with respect to $\mathbf{G}_m$ yields an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/2, \mathbf{G}_m) \to H^1(M; \mathbf{G}_m) \to H^1(\widetilde{M}; \mathbf{G}_m).$$

Now, since π is étale, we have $\pi^*\omega_M \cong \omega_{\widetilde{M}}$ by Lemma 6.4.5. But $\widetilde{M}$ is orientable, so $\pi^*\omega_M \cong \mathcal{O}_{\widetilde{M}}$. It follows that the degree function of ω_M is trivial, hence by Theorem 3.5.2, ω_M is represented by an element of $H^1(|M|, \mathbf{G}_m)$. Via the natural isomorphism $\mathrm{Pic}_0(\omega_M) \xrightarrow{\cong} H^1(M; \mathbf{G}_m)$, ω_M is identified with an element in the kernel of $H^1(M; \mathbf{G}_m) \to H^1(\widetilde{M}; \mathbf{G}_m)$. But $\mathrm{Hom}(\mathbb{Z}/2, \mathbf{G}_m)$ is isomorphic to $\mathbb{Z}/2$ (or trivial if $\operatorname{char} k = 2$), so $\omega_M^{\otimes 2}$ is trivial in $H^1(M; \mathbf{G}_m)$. We conclude that $\omega_M^{\otimes 2} \cong \mathcal{O}_M$. Since M is not orientable, we have $H^{\mathbf{n}}(M; k) = 0$. By Serre duality, $H^0(M, \omega_M) \cong H^{\mathbf{n}}(M, \mathcal{O}_M) \cong H^{\mathbf{n}}(M; k) = 0$. Hence ω_M does not have a global section, and is therefore not trivial.

Finally, let M be a general polyhedral homology manifold. By Theorem 6.5.3, there exists a marked pair (M, Ψ) such that $Z = PS(M, \Psi)$ is unimodular and $|Z| \cong |M|$. By the above, $\mathbf{P}_k(Z)$ is locally Gorenstein. This property is preserved under deformation equivalence, so $\mathbf{P}_k(M)$ is locally Gorenstein as well by Theorem 6.5.3. For the rest of the statements, it will suffice to show that for projective schemes X, the properties $\omega_X \cong \mathcal{O}_X$, $\omega_X^{\otimes 2} \cong \mathcal{O}_X$ and $H^0(X, \omega_X) = 0$ are open stable under generization in proper families of locally Gorenstein schemes. Indeed, suppose that $\mathcal{X} \to \mathrm{Spec}(R)$ is such a family, where R is a DVR. Then its relative dualizing complex is invertible ([Stacks, Tag 0DW9]), hence it has an invertible relative dualizing sheaf $\omega_{\mathcal{X}/R}$. Assume that either of the properties above apply for its central fiber $\mathcal{X}_0$. We will show that the same applies to its general fiber $\mathcal{X}_\eta$. Since $\omega_{\mathcal{X}/R}|_{\mathcal{X}_t} \cong \omega_{\mathcal{X}_t}$ in general ([Stacks, Tag 0BZW]), we can lift a trivializing section of $\omega_{\mathcal{X}_0}$) (resp. $\omega_{\mathcal{X}_0}^{\otimes 2}$) to a trivializing section of $\omega_{\mathcal{X}/R}$ (resp. $\omega_{\mathcal{X}/R}^{\otimes 2}$). This induces a trivializing section of $\omega_{\mathcal{X}_\eta}$ (resp. $\omega_{\mathcal{X}_\eta}^{\otimes 2}$). Finally, since $\omega_{\mathcal{X}/R}$ is invertible, $H^0(X, \omega_{X/R}) = 0 \Rightarrow H^0(\mathcal{X}, \omega_{\mathcal{X}/R}) = 0 \Rightarrow H^0(\mathcal{X}_\eta, \omega_{\mathcal{X}_\eta/R}) = 0$. □

Corollary 6.5.5. *If M is a polyhedral homology sphere (with respect to k), then $\Gamma(M)$ is Gorenstein. In particular, $\Gamma(\partial\Delta^P)$ is Gorenstein for all $P \in \mathcal{P}$.*

Proof. By Theorem 6.5.4, we have $\omega_M \cong \mathcal{O}_M$. Thus by Serre duality and Lemma 3.6.5, $H^i(M, \mathcal{O}_M(-d)) = H^{\mathbf{n}-i}(M, \mathcal{O}_M(d)) = 0$ for all $d > 0$ and $i < \mathbf{n}$. Since M is a homology sphere, $H^i(M, \mathcal{O}_M) = 0$ for all $0 < i < \mathbf{n}$ as well. This means that $\Gamma(M)$ is Cohen-Macaulay, and by [GW78, (5.1.9)], also Gorenstein. □

Chapter 7

Deformations of face schemes of polyhedral manifolds

This chapter will be concerned with the following objectives: First, we will prove the vanishing of the obstructions to glueing local infinitesimal deformations of $(\mathbf{P}(M), \mathcal{O}_M(1))$ in dimension 2. This is essentially done via reduction to the simplicial case. Second, we compute the universal base space for the deformation functor of pairs under some restrictions on M. Then we find further conditions under which the smoothing component is identifiable, and apply it to two classes of examples in the final two sections.

We refer to [Har10] for the basic terminology on deformation theory. For additional results, [Ser06] and [Art69] will be referenced. We consider the functor $F : k\text{-alg} \to \mathrm{Set}$ given by $F(B) = \{(f : \mathcal{X} \to \mathrm{Spec}(B), L)\}/\cong$, where f is proper and flat, and L is an ample line bundle. Functoriality is given by base change. This functor is locally of finite presentation (i.e. it preserves inductive limits), which follows from standard techniques. Let X be a reduced and projective scheme over k, and let L be an ample line bundle on X. Consider the over-category $k\text{-alg}/k$, consisting of k-algebras B equipped with a homomorphism $\phi : B \to k$. The local functor $F_{(X,L)} : k\text{-alg}/k \to \mathrm{Set}$ is given by $F_{(X,L)}(B \xrightarrow{\phi} k) = \{(X', L') \in F(B) : F(\phi)(X', L') = (X, L)\}$. We will focus on the local deformation functor $\mathrm{Def}_{(X,L)}$, which is equal to $F_{(X,L)}$ restricted to the category of local Artinian k-algebras. It maps A to the set of equivalence classes of pairs $(X' \to \mathrm{Spec}(A), L')$, where X' is a deformation of X over $\mathrm{Spec}(A')$, and $L' \otimes_{A'} k \cong L$ (note that deformations $X' \to \mathrm{Spec}(A')$ are automatically proper since A' is finite over k). When X is smooth, its basic properties are well known ([Ser06, Section 3.3.3]). In particular, $\mathrm{Def}_{(X,L)}$ has a miniversal family. However, the smoothness assumption is redundant, as is shown in [AC10]. In fact, since L is ample, $\mathrm{Def}_{(X,L)}$ is pro-representable and has unique effectivizations of formal families. This essentially follows from representability of the Hilbert functor and the Picard functor. As a consequence of that, all effective formal families are algebraizable:

Proposition 7.0.1. *Any formal family $\{(X_n, L_n)\}$ for $\mathrm{Def}_{(X,L)}$ is algebraizable (including the universal formal family). I.e. there exists an algebraic deformation $(\mathcal{X} \to \mathrm{Spec}(B), \mathcal{L})$ and compatible isomorphisms $(\mathcal{X} \times_B B/m^n, \mathcal{L} \otimes_B B/m^n) \xrightarrow{\cong} (X_n, L_n)$. Moreover, the base space $\mathrm{Spec}(B)$ for the universal deformation is unique in the étale-topology locally around $0 \in \mathrm{Spec}(B)$.*

Proof. Since $\mathrm{Def}_{(X,L)}$ has unique effectivizations, the above statement holds for the universal formal family by [Art69, Theorem 1.6] and [Art69, Theorem 1.7].

Since any formal family $\{(X_n, L_n)\}$ is effective, it is automatically algebraizable as a base change of an algebraic universal deformation. $\square$

We will now summarize from [AC10] what we shall require here. Consider the natural morphism dlog : $\mathcal{O}_X^* \to \Omega_X$ given locally by $u \mapsto \frac{du}{u}$, and let $c : H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X)$ denote the induced homomorphism in cohomology. Since $H^1(X, \Omega_X) \cong \mathrm{Ext}_X^1(\mathcal{O}_X, \Omega_X)$, the element $c(L)$ induces an extension

$$0 \to \Omega_X \to Q_L \to \mathcal{O}_X \to 0. \tag{7.1}$$

This sequence is locally split-exact, hence by dualizing we obtain an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to \mathcal{T}_X \to 0,$$

where $\mathcal{E}_L = Q_L^\vee$.

Theorem 7.0.2 ([AC10, Theorem 3.1])**.**

1) *The functor* $\mathrm{Def}_{(X,L)}$ *has a miniversal family (i.e.* $\mathrm{Def}_{(X,L)}$ *satisfies* $(H0)$–$(H3)$ *of Schlessinger's criterion [Har10, Theorem 16.2]).*

2) *There is an isomorphism* $\mathrm{Def}_{(X,L)}(k[\epsilon]) \xrightarrow{\cong} \mathrm{Ext}_X^1(Q_L, \mathcal{O}_X)$*.*

3) *There is an exact sequence*

$$0 \to H^1(X, \mathcal{E}_L) \to \mathrm{Ext}_X^1(Q_L, \mathcal{O}_X) \to H^0(X, \mathcal{T}_X^1) \to H^2(X, \mathcal{E}_L),$$

where $H^1(X, \mathcal{E}_L)$ *parametrizes the locally trivial first-order deformations of* X*.*

4) *The obstructions for* $\mathrm{Def}_{(X,L)}$ *lie in* $H^0(X, \mathcal{T}_X^2)$, $H^1(X, \mathcal{T}_X^1)$ *and* $H^2(X, \mathcal{E}_L)$*.*

7.1 Vanishing of obstructions

For the rest of this chapter, we shall assume that M is 2-dimensional polyhedral manifold. We will prove that the obstruction module $H^2(M, \mathcal{E}_L)$ vanishes, where $L = \mathcal{O}_M(1)$. If M is a simplicial manifold, then the vanishing of $H^2(M, \mathcal{E}_L)$ is known [AC10, Theorem 6.1]. We will prove the general result in two steps. First, we show that the property that $H^2(M, \mathcal{E}_L) = 0$ is preserved under scaling of M, and then under certain types of subdivisions called *simple* subdivisions. Second, we show that it is possible to iteratively scale and subdivide M via simple subdivisions until one ends up with a simplicial manifold.

Definition 7.1.1. Let (M, Ψ) be a marked pair structure on a 2-dimensional polyhedral manifold M. Then the *inner skeleton* $S \subseteq PS(M, \Psi)$ of the subdivision is the closure of $PS(M, \Psi)^1 \backslash PS(M^1, \Psi|_{M^1})$. Hence S is 1-dimensional, and its edges are those which are not contained in $PS(M^1, \Psi|_{M^1})$. A subdivision of M is called *simple* if $H_1(|S|) = 0$.

First, we shall require a more general version of (7.1). Let $\pi : X \to \mathrm{Spec}(R)$ be a projective, flat morphism, where R is some finitely generated k-algebra. Let L be a line bundle on X. Consider the morphism $\mathrm{dlog}_R : \mathcal{O}_X^* \to \Omega_{X/R}$ given by $u \mapsto \frac{d(u)}{u}$. It is easy to check that this is a morphism of abelian groups. This induces a homomorphism $c_R : \mathrm{Pic}(X) \xrightarrow{\cong} H^1(X, \mathcal{O}_X^*) \to H^1(\Omega_{X/R})$. Since $H^1(X, \Omega_{X/R}) \cong \mathrm{Ext}^1_X(\mathcal{O}_X, \Omega_{X/R})$, the element $c_R(L)$ induces an extension

$$c_R(L) : 0 \to \Omega_{X/R} \to Q_L \to \mathcal{O}_X \to 0.$$

As before, this sequence is locally split-exact, hence by dualizing we obtain an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to \mathcal{T}_X \to 0,$$

where $\mathcal{E}_L = Q_L^\vee$.

Proposition 7.1.2. *Let n be a positive integer. Then $Q_{nL} \cong Q_L$, so that $\mathcal{E}_{nL} \cong \mathcal{E}_L$.*

Proof. We shall require the following fact, which follows immediately from the construction in [Wei94, Theorem 3.4.3]: Let R be a ring, and A, B R-modules. Let $r \in R$ be an invertible element, and $e \in \mathrm{Ext}^1_R(A, B)$ an extension class. Then e and re are isomorphic as R-modules. This easily globalizes to the corresponding statement for quasi-coherent sheaves on X, so we can apply it to the extension $c(L) \in \mathrm{Ext}^1_X(\mathcal{O}_X, \Omega_X)$. First, since the homomorphism c is additive, we have $c(L^{\otimes n}) = nc(L)$. Now, n is invertible in $H^0(X, \mathcal{O}_X)$, and thus the middle terms of the extensions corresponding to $nc(L)$ and $c(L)$ are isomorphic. In other words, $Q_{nL} \cong Q_L$. □

Proposition 7.1.3. *Let $x \in \mathrm{Spec}(R)$ be a point, and consider the fiber $X_x \to \mathrm{Spec}(k(x))$. Let $L_x := L|_{X_x} = L \otimes_R k(x)$. Then $(Q_L)|_{X_x} \cong Q_{L_x}$.*

Proof. Let $f : X_x \to X$ denote the closed immersion of the fiber over x. Observe that $\Omega_{X/R} \otimes_R k(x) \cong f_* \Omega_{X_x/k(x)}$. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}_X^* \arrow[r, "\mathrm{dlog}_R"] \arrow[d] & \Omega_{X/R} \arrow[d] \\ f_*\mathcal{O}_{X_x}^* \arrow[r, "f_* \mathrm{dlog}_{k(x)}"] & f_*\Omega_{X_x/k(x)}. \end{tikzcd}$$

This induces a commutative diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}_X^*) \arrow[r, "c_R"] \arrow[d] & H^1(X, \Omega_{X/R}) \arrow[d] \\ H^1(X, f_*\mathcal{O}_{X_x}^*) \arrow[r] \arrow[d] & H^1(X, f_*\Omega_{X_x/k(x)}) \arrow[d] \\ H^1(X_x, \mathcal{O}_{X_x}^*) \arrow[r, "c_{k(x)}"] & H^1(X_x, \Omega_{X_x/k(x)}). \end{tikzcd}$$

The vertical arrow on the right is just the composition $H^1(X, \Omega_{X/R}) \to H^1(X, \Omega_{X/R}) \otimes_R k(x) \xrightarrow{\cong} H^1(X_x, \Omega_{X_x/k(x)})$. Thus $c_{k(x)}(L|_{X_x}) = c_R(L) \otimes 1$. If we view $c_R(L) \otimes 1$ as an element of $\mathrm{Ext}^1_{X_x}(\mathcal{O}_{X_x}, \Omega_{X_x/k(x)}) \cong H^1(X_x, \Omega_{X_x/k(x)})$, it corresponds to

$$c_R(L) \otimes 1 : 0 \to \Omega_{X_x/k(x)} \to (Q_L)|_{X_x} \to \mathcal{O}_{X_x} \to 0.$$

Since this also corresponds to the extension $c_{k(x)}(L|_{X_x})$, we conclude that $(Q_L)|_{X_x} \cong Q_{L_x}$. □

We will now apply the above in the situation where $\mathcal{X} := \mathbf{P}_R(M, \Psi) \to \mathrm{Spec}(R)$ is a deformation arising from a marked pair structure (M, Ψ), where $R = k[t]_p$ and $p = (t)$. In this situation, the central fiber $(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}}(1)|_{\mathcal{X}_0})$ is isomorphic to a pair $(\mathbf{P}(Z), \mathcal{O}_Z(1))$ for $Z = PS(M, \Psi)$. Define $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$ and $L = \mathcal{O}_Z(1)$, and consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{t} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_Z \to 0. \tag{7.2}$$

This yields an exact sequence (see [Har10, Theorem 3.4])

$$0 \to \mathcal{T}_{\mathcal{X}/R} \xrightarrow{t} \mathcal{T}_{\mathcal{X}/R} \to \mathcal{T}_{Z/k} \to \mathcal{T}^1_{\mathcal{X}/R} \xrightarrow{t} \mathcal{T}^1_{\mathcal{X}/R}.$$

If we let C denote the sheaf of t-torsion sections of $\mathcal{T}^1_{\mathcal{X}/R}$, then we get the exact sequence

$$0 \to \mathcal{T}_{\mathcal{X}/R} \otimes_R k \to \mathcal{T}_{Z/k} \to C \to 0. \tag{7.3}$$

Now, by Proposition 7.1.3, there is an isomorphism $(Q_{\mathcal{L}})|_{\mathcal{X}_0} \cong Q_L$ (compatible with the isomorphism $\Omega_{\mathcal{X}/R}|_{\mathcal{X}_0} \to \Omega_{Z/k}$). So if we apply $\mathscr{H}om\,(Q_{\mathcal{L}}, -)$ to (7.2), one obtains an injective morphism $\mathcal{E}_{\mathcal{L}} \otimes_R k \to \mathcal{E}_L$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathcal{X}} \otimes_R k \arrow[r] \arrow[d] & \mathcal{E}_{\mathcal{L}} \otimes_R k \arrow[r] \arrow[d] & \mathcal{T}_{\mathcal{X}/R} \otimes_R k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_Z \arrow[r] & \mathcal{E}_L \arrow[r] & \mathcal{T}_{Z/k} \arrow[r] & 0,
\end{tikzcd}$$

By the snake lemma and (7.3) there is an induced exact sequence on the form

$$0 \to \mathcal{E}_{\mathcal{L}} \otimes_R k \to \mathcal{E}_L \to C \to 0. \tag{7.4}$$

Our next goal is to find conditions under which $H^1(\mathcal{X}, C) = 0$. To do this, we must consider a new type of face ring construction:

Remark 7.1.4. There is an alternative way of interpreting the face ring construction $(M, \Psi) \mapsto \Gamma_R(M, \Psi)$ from Section 2.9, and it involves considering each ring $\Gamma_R(P, \psi)$ as the coordinate ring of a polyhedron Q, which essentially forms a homotopy between P and the subdivision $PS(P, \psi)$. The general setup is as follows. Let $\mathcal{Q}_+$ denote the category of full-dimensional *lattice* *polyhedrons* $Q \subseteq \mathbb{R}^n$ [CLS11, Definition 7.1.3](including the empty-polyhedron $\varnothing$). If $Q = \{m \in \mathbb{Z}^{\dim(Q)} \mid \langle m, u_F\rangle \geq -a_F\}$ is a lattice polyhedron, let $C_Q = \{m \in \mathbb{Z}^{\dim(Q)} \mid \langle m, u_F\rangle \geq 0\}$ denote its recession cone. The category $\mathcal{Q}_+$ may be defined analogously to $\mathcal{P}_+$, where arrows $(f, \lambda_f) : Q_1 \to Q_2$ are affine transformations $f : \mathbb{Z}^{\dim(Q_1)} \to \mathbb{Z}^{\dim(Q_2)}$, $f_{\mathbb{R}} : \mathbb{R}^{\dim(Q_1)} \to \mathbb{R}^{\dim(Q_2)}$ identifies Q_1 with a face of Q_2, and where $\lambda_f : \mathbb{Z}^{\dim(Q_1)} \to \mathbf{G}_m^{\times}$ is a character. The affine semigroup associated to Q is defined as $A_Q = \mathbb{N}[(Q \times \{1\} \cup C_Q \times \{0\}) \cap \mathbb{Z}^{\dim(Q)+1}]$. For each arrow $f : Q_1 \to Q_2$ represented by an affine transformation $x \mapsto u + Ax$, there is an induced homomorphism $A_{Q_1} \to A_{Q_2}$ given by $(m, d) \mapsto (df_{\mathbb{R}}(\frac{m}{d}), d) = (ud + A(m), d)$. This is analogous to and a generalization of the definition of the functor $A : \mathcal{P}_+ \to \mathcal{Q}$ for $P \in \mathcal{P}$.

We define the face ring of Q as $\Gamma'(Q) = k[A_Q]$. Functoriality is given as before: $\chi^{A_f(m)} \mapsto \lambda_f(m)\chi^m$ (and $\chi^m \mapsto 0$ otherwise). The grading is given by $k[A_P] = \bigoplus_{d=0}^{\infty} \bigoplus_{(m,d)\in A_Q} k \cdot \chi^{(m,d)}$. This defines a functor $\Gamma' : \mathcal{Q}_+ \to \mathscr{D}_{\geq 0}$, where $\mathscr{D}_{\geq 0}$ denotes the category of non-negatively graded k-algebras. Now, $k[C_Q \cap \mathbb{Z}^{\dim(Q)}]$ is the degree 0 part of $\Gamma'(Q)$, and the graded inclusion $k[C_Q \cap \mathbb{Z}^{\dim(Q)}] \to \Gamma'(Q)$ given by $\chi^m \mapsto \chi^{(m,0)}$ is natural. As in Section 2.3, this defines a functor $\Gamma : \mathrm{Pre}^{(*)}(\mathcal{Q}) \to \mathscr{D}_{\geq 0}$, where $\mathcal{Q} \subseteq \mathcal{Q}_+$ is the subcategory of non-empty polyhedra. A presheaf $M \in \mathrm{Pre}^{(*)}(\mathcal{Q})$ will be called a *generalized polyhedral set*, and the definitions analogous to those prior apply. As before, there is an induced functor $\mathbf{P} : \mathrm{Pre}^{(*)}(\mathcal{Q})_f \to \mathrm{Sch}_k$, which associates to a finite generalized polyhedral set M the scheme $\mathbf{P}(M) = \mathrm{Proj}(\Gamma(M))$. There is nothing essentially different about this construction other than for the fact that $\Gamma(M)$ is not necessarily positively graded. Thus, all results of Chapter 4, and also Lemma 5.2.5, Corollary 5.2.6 and Proposition 5.2.7 have analogies in this situation. Hence there is an étale cover $\{\mathbf{A}(M_s) \to \mathbf{P}(M)\}_s$ for faces s of M, and also $\mathbf{A}(M_s) \cong \operatorname{Spec} \Gamma(\mathbb{L}(M_s))_{(\theta_s)}$. These facts will be used later.

To tie this up with subdivisions, let (P, ψ) be a marked pair, let $n = \dim(P)$, and consider the polyhedron Q defined as the union of vertical lines intersecting the lower boundary G_ψ. I.e. $Q = \{(m, s) \in \mathbb{R}^{\dim(P)+1} \mid m \in P, s \geq g_\psi(m)\}$. The recession cone $C_Q \subseteq \mathbb{R}^n$ is 1-dimensional and generated by the unit vector e_n. Thus $t := \chi^{(e_n,0)} \in \Gamma'(Q)$ is a well-defined element of degree zero, and $\Gamma'(Q)$ is a positively graded $k[t]$-algebra. One immediately observes that $\Gamma'(Q)$ is generated as a $k[t]$-algebra by the elements $t^{dg_\psi(\frac{m}{d})}\chi^{(m,d)}$, where $(m, d) \in A_{P_i}$ for some i. If $k[t] \to R$ is the homomorphism given by $t \mapsto t$, then this observation yields a natural isomorphism of R-algebras

$$\Gamma(Q) \otimes_{k[t]} R \cong \Gamma_R(P, \psi). \tag{7.5}$$

The assignment $(P, \psi) \mapsto Q$ defines a functor $\overline{\mathcal{P}} \to \mathcal{Q}$, and the Yoneda extension induces a functor $\mathrm{Pre}^{(*)}(\overline{\mathcal{P}}) \to \mathrm{Pre}^{(*)}(\mathcal{Q})$. Hence to a given marked pair (M, Ψ), there is an associated generalized polyhedral set $\widetilde{M}$ such that $\Gamma(\widetilde{M})$ is a positively graded $k[t]$-algebra, $\Gamma(\widetilde{M}) \otimes_{k[t]} R \cong \Gamma_R(M, \Psi)$ (since (7.5) is natural), and $\mathbf{P}(\widetilde{M}) \times_{k[t]} R \cong \mathbf{P}_R(M, \Psi)$. Note that we also have natural inclusions $PS(P, \psi) \subseteq \Delta^Q$, and that $\Gamma'(Q) \otimes_{k[t]} k \cong \Gamma(PS(P, \psi))$ by (7.5). Thus for the induced inclusion $PS(M,\Psi) \subseteq \widetilde{M}$ of generalized polyhedral sets, we have $\Gamma(PS(M,\Psi)) \cong \Gamma(\widetilde{M}) \otimes_{k[t]} k$ (the last part follows from a similar argument as in the proof of Theorem 2.9.11). We also have $\mathbf{A}(\widetilde{M}_s) \times_{k[t]} k \cong \mathbf{A}(PS(M,\Psi)_s)$ for every face s of $PS(M,\Psi)$.

Proposition 7.1.5. *If the subdivision associated to a marked pair (M,Ψ) is simple, i.e. if $H_1(|S|) = 0$ where $S \subseteq PS(M,\Psi)$ is the inner skeleton of the subdivision, then $H^1(\mathcal{X}, C) = 0$.*

Proof. Consider the homomorphism $\Omega_{\Gamma_R(P,\psi))} \to \Gamma_R(P,\psi)$ given by $d(t^s\chi^m) \mapsto s \otimes t^s\chi^m$. This is well-defined due to (7.5), since all relations between generators in $\Gamma_R(P,\psi)$ are binomials scaling the powers of t equally on each side. It is also natural in (P,ψ), hence by taking limits one obtains a homomorphism $\Omega_{\Gamma_R(M,\Psi)} \to \Gamma_R(M,\Psi)$ satisfying $d(t) \mapsto t$. Its degree is 0, hence it corresponds to a global section $D \in H^0(\mathcal{X}, \mathcal{T}_{\mathcal{X}/k})$. Consider now the exact sequence $\mathcal{T}_{\mathcal{X}/k} \to \mathcal{T}_{R/k}(\mathcal{O}_{\mathcal{X}}) \to \mathcal{T}^1_{\mathcal{X}/R}$ ([Har10, Theorem 3.5]). Let $\overline{D}$ be the image of D in $H^0(\mathcal{X}, \mathcal{T}_{R/k}(\mathcal{O}_{\mathcal{X}}))$. It corresponds to the R-module homomorphism $\Omega_{R/k} \to \mathcal{O}_{\mathcal{X}}$ given by $d(t) \mapsto t$. Now, let D_0 denote the homomorphism $\Omega_{R/k} \to \mathcal{O}_{\mathcal{X}}$ given by $d(t) \mapsto 1$, and note that $\overline{D} = tD_0$. Let $\overline{D_0}$ denote the image of D_0 in $H^0(\mathcal{X}, \mathcal{T}^1_{\mathcal{X}/R})$. Then $t\overline{D_0}$ is equal to the image of $\overline{D}$, so by exactness we have $t\overline{D_0} = 0$. It follows that $\overline{D_0} \in H^0(\mathcal{X}, C)$, and this global section corresponds to a homomorphism $\lambda : \mathcal{O}_{\mathcal{X}} \to C$.

As in Remark 7.1.4, let $\widetilde{M}$ denote the generalized polyhedral set associated with the subdivision. Then $\mathcal{X} \cong \mathbf{P}(\widetilde{M}) \times_{k[t]} R$. In order to determine the support of C, which is contained in $\mathbf{P}(Z) = \mathrm{Sing}(\mathcal{X}_0)$, where $Z := PS(M,\Psi) \subseteq \widetilde{M}$, we must study the structure of the polyhedron Q associated to (P,ψ) further. For every lattice point $m \in P$, vertical line through $(m,0)$ intersects the lower hull G_ψ in a lattice point. Hence if E is an edge of Q contained in some vertical facet (of infinite area), it follows by a routine computation on facet normal vectors that the cone $\sigma_E \subseteq \Sigma_Q$ is smooth (given that P is smooth). If $\{(v_1,s_1),(v_2,s_2)\}$ are the vertices of E, this implies that $U_{\sigma_E} \cong \mathrm{Spec}(\Gamma(\Delta^Q)_{(\chi^{(v_1,s_1,1)}\chi^{(v_2,s_2,1)})})$ is smooth, and therefore

$$\Gamma(\Delta^Q)_{(t^{s_1}\chi^{(v_1,1)}t^{s_2}\chi^{(v_2,1)})} \cong k[t] \otimes_k \Gamma(\Delta^P)_{(\chi^{(v_1,1)}\chi^{(v_2,1)})}.$$

If $E' \subseteq P$ (with vertices $\{w_1,w_2\}$) is the edge which contains $\{v_1,v_2\}$, then clearly $\Gamma(\Delta^P)_{(\chi^{(v_1,1)}\chi^{(v_2,1)})} \cong \Gamma(\Delta^P)_{(\chi^{(w_1,1)}\chi^{(w_2,1)})}$. Hence if $f : E \to Q$ and $f' : E' \to P$ denote the corresponding arrows of the inclusions, we have

$$\Gamma(\Delta^Q)_{(\theta_f)} \cong k[t] \otimes_k \Gamma(\Delta^P)_{(\theta_{f'})}. \tag{7.6}$$

Now, let e be an edge of Z which does not belong to S. Let $e \to t_1$, $e \to t_2$ be the two canonical faces of Z_e. Since e is not contained in S, the edge $(f_i)_{\mathbb{R}}(|e|) \subseteq |t_i|$ is contained in a vertical facet. Via the natural isomorphism (7.6), we obtain that $\mathbf{A}(\widetilde{M}_e) \cong \mathbf{A}^1 \times_k \mathbf{A}(M_{e'})$ for some edge e' of M. Since $\mathbf{A}(Z_e) \cong \mathbf{A}(\widetilde{M}_e) \times_{k[t]} R$, this means that $\mathcal{X} \to \mathrm{Spec}(R)$ induces a trivial deformation over the étale neighbourhood $\mathbf{A}(Z_e)$. It follows immediately that $C|_{\mathbf{A}(Z_e)} = 0$, hence C is supported on $\mathbf{P}(S) \subseteq \mathbf{P}(Z)$, and λ induces an injective homomorphism $\lambda' : \mathcal{O}_Y \to C$, where $Y \subseteq \mathbf{P}(Z)$ is a subscheme satisfying $Y_0 \subseteq \mathbf{P}(S)$.

Conversely, let e be an edge of S. Let s be the facet of M such that e is in the image of $PS(\Delta^s, \psi_s) \to \widetilde{M}$. Let $P = |s|$, and let Q be the polyhedron associated to the pair (P, ψ_s). Let $f : E \to Q$ denote the associated arrow. Then $\mathbf{A}(\widetilde{M}_e) \cong \mathbf{A}(\Delta^Q_f) \cong \operatorname{Spec}(\Gamma(\Delta^Q)_{(\theta_f)})$. This is an affine toric variety, hence its coordinate ring is generated by some set of characters, including t. Now, the general fiber of $\mathbf{A}(\widetilde{M}_e) \to \operatorname{Spec}(k[t])$ is smooth, with central fiber $\mathbf{A}(Z_e) \cong \operatorname{Spec} k[x,z,y]_y/(xz)$. Hence we must have $\Gamma(\Delta^Q)_{(\theta_f)} \cong k[t,x,z,y]_y/(xz - t^N y^n)$ for some integers $N \geq 0$, $n \in \mathbb{Z}$. It follows immediately that $C|_{\mathbf{A}(Z_e)}$ is generated by $\overline{D_0}|_{\mathbf{A}(Z_e)}$ (which corresponds to the homomorphism given by $xz - t^N y^n \mapsto N t^{N-1} y^n$). Thus $\mathcal{O}_Y \to C$ induces an isomorphism over $\mathbf{A}(Z_e)$. Since Y is closed, we have that $Y_0 = \mathbf{P}(S)$. Consider the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_Y \to \mathcal{O}_S \to 0$. Now, $\mathcal{I}$ is supported at a closed subscheme of $\mathcal{X}$ lying over the zero-dimensional scheme $\mathbf{P}(S^0) \subseteq \mathbf{P}(S)$, which must be affine. Since $H^1(S, \mathcal{O}_S) = H_1(S; k) = 0$, the long exact sequence in cohomology yields that $H^1(Y, \mathcal{O}_Y) = 0$. However, $\operatorname{coker}(\lambda')$ is also supported at an affine closed subscheme of $\mathcal{X}$ lying over the zero-dimensional scheme $\mathbf{P}(S^0) \subseteq \mathbf{P}(S)$. From the long exact sequence in cohomology of $0 \to \mathcal{O}_Y \to C \to \operatorname{coker}(\lambda') \to 0$ we get that $H^1(\mathcal{X}, C) = 0$. In conclusion, $H_1(|S|) = 0$ implies that $H^1(\mathcal{X}, C) = 0$. □

Proposition 7.1.6. *Let M be a 2-dimensional polyhedral manifold. Then it is possible to iteratively scale and subdivide M (i.e. $M \mapsto nM$ or $M \mapsto PS(M, \Psi)$) into a simplicial manifold, such that at each step the inner skeleton S of the subdivision satisfies $H_1(|S|) = 0$.*

Proof. First scale M so that every facet has an interior point. Then choose an interior lattice point m of a facet s, and consider the subdivision of $|s|$ corresponding to some $\psi_s : A_s = \operatorname{Vert}(|s|) \cup \{m\} \to \mathbb{Z}$ satisfying $\psi_s(v) = 0$ for $v \in \operatorname{Vert}(|s|)$ and $\psi_s(m) < 0$. The subdivision consists of a line drawn from m to each vertex of $|s|$. It also defines a subdivision of M, where every other facet is subdivided trivially. Moreover, the inner skeleton S is clearly contractible, so $H_1(|S|) = 0$ in this case. Do the same process for every other facet of M. After these subdivisions, M now contains no edges which are contained in the same facet in two distinct ways. Next, without scaling, we perform the same iterative subdivisions of each facet whenever possible, in order to remove every interior lattice point of every facet of M. After that, choose any lattice point m interior to some edge e of M, and subdivide the two (distinct) facets t_1, t_2 containing e via the functions $\psi_{t_i} : \operatorname{Vert}(|t_i|) \cup \{m\} \to \mathbb{Z}$ satisfying $\psi_{t_i}(v) = 0$ for $v \in \operatorname{Vert}(|t_i|)$ and $\psi_{t_i}(m) < 0$. This subdivision consists of a line drawn from m to each vertex opposite of $|e| \subseteq |t_i|$. Again, S is clearly contractible. Via these subdivisions we end up with a polyhedral set without any facet or edges with interior lattice points. By Pick's theorem, the areas of all facet lattice polytopes must $\frac{1}{2}$, and it is easy to see that a polygon with area $\frac{1}{2}$ must be a triangle affinely isomorphic to the simplicial triangle Δ^2. Hence M has been subdivided into a simplicial manifold in the desired fashion. □

Theorem 7.1.7. *Let M be a 2-dimensional polyhedral set. Then $H^2(M, \mathcal{E}_L) = 0$.*

Proof. By Proposition 7.1.6 and Proposition 7.1.5, we can iteratively scale and subdivide M (via some sequence $M = M_0 \mapsto M_1 \mapsto \cdots \mapsto M_N$), such that at each subdivision step $M_r \mapsto M_{r+1}$ we have $H^1(\mathcal{X}, C) = 0$ (where $\mathcal{X} \to \mathrm{Spec}(R)$ is the deformation associated to the subdivision). At each subdivision step, in the situation of (7.4), $H^1(\mathcal{X}, C) = 0$ and $H^2(M_{r+1}, \mathcal{E}_{\mathcal{O}_{M_{r+1}}(1)}) = 0$ implies that $H^2(\mathcal{X}, \mathcal{E}_{\mathcal{L}}) = 0$. Since $\mathcal{X} \to \mathrm{Spec}(R)$ is flat, the sheaf $\mathcal{E}_{\mathcal{L}} = Q_{\mathcal{L}}^{\vee}$ is flat as well as a sheaf of R-modules. Hence $H^2(\mathcal{X}_\eta, (\mathcal{E}_{\mathcal{L}})_\eta) = 0$, where $(\mathcal{X}_\eta, \mathcal{L}_\eta) \cong (\mathbf{P}_K(M_r), \mathcal{O}_{M_r}(1))$ is the generic fiber. But the homomorphism $R \to K$ is flat, so it easily follows that $(\mathcal{E}_{\mathcal{L}})_\eta \cong ((Q_{\mathcal{L}})_\eta)^{\vee} \cong \mathcal{E}_{\mathcal{O}_{M_r}(1)}$, where the last isomorphism comes from Proposition 7.1.3. In conclusion, $H^2(M_{r+1}, \mathcal{E}_{\mathcal{O}_{M_{r+1}(1)}}) = 0$ implies that $H^2(M_r, \mathcal{E}_{\mathcal{O}_{M_r(1)}}) = 0$. At each scaling step ($M_r \mapsto M_{r+1} = nM_r$), we have $\mathcal{O}_{M_{r+1}}(1) \cong \mathcal{O}_{M_r}(n)$. Hence by Proposition 7.1.2, $H^2(M_{r+1}, \mathcal{E}_{\mathcal{O}_{M_{r+1}}(1)}) = 0$ implies $H^2(M_r, \mathcal{E}_{\mathcal{O}_{M_r}(1)}) = 0$ in this situation as well. Finally, by [AC10, Theorem 6.1], we have $H^2(M_N, \mathcal{E}_{\mathcal{O}_{M_N}(1)}) = 0$, since M_N is a simplicial complex. By the above, we conclude that $H^2(M, \mathcal{E}_{\mathcal{O}_M(1)}) = 0$. □

7.2 First-order deformations

Definition 7.2.1. The *valency* $\mathrm{val}(v)$ of a vertex $v \in M_0$ is defined as the number of edges of the n-cycle $\mathrm{lk}_M(v)$.

In this section we will restrict our attention to the situation where $\mathrm{val}(v) \geq 3$ for all $v \in M_0$. In this case, $\mathrm{lk}_M(v)$ is a simplicial complex on the form C_n, where $n = \mathrm{val}(v)$ and C_n denotes the n-cycle (i.e. the simplicial complex with n vertices and n edges forming a circle). Since $\mathbf{A}(M_v) = \mathrm{Spec}(\Gamma(C_n))$, the methods of [AC04] can be used to compute the first-order deformations of $\mathbf{P}(M)$. Our present goal is to find a suitable characterization of a basis for the cohomology group $H^0(M, \mathcal{T}^1_{M/k})$ via the étale Čech cohomology complex. Recall the étale cover $\{\mathbf{A}(M_v) \to \mathbf{P}(M)\}_{v \in M_0^{\mathrm{can}}}$. By Lemma 6.2.1, we have $\mathbf{A}(M_v) \times_{\mathbf{P}(M)} \mathbf{A}(M_w) = \coprod_{v \to s \leftarrow w} \mathbf{A}(M_s)$, where s ranges over all minimal elements with arrows from v and w. But note that $\mathcal{T}^1_{M/k}|_{\mathbf{A}(M_s)} = 0$ whenever $\dim(s) = 2$, since $\mathbf{A}(M_s)$ is smooth. Since the restriction $\mathcal{T}^1_{M/k}|_{\mathbf{A}(M_s)}$ is only non-trivial when s is an edge or a vertex, Definition 6.2.3 and the proof of Lemma 6.2.4 can easily be adapted to the present situation, and as a consequence, $H^0(M, \mathcal{T}^1_{M/k})$ is equal to the kernel of the homomorphism

$$\delta : \bigoplus_{v \in M_0^{\mathrm{can}}} \mathcal{T}^1_{\mathbf{A}(M_v)/k} \to \bigoplus_{e \in M^{\mathrm{can}}, dim(e)=1} \mathcal{T}^1_{\mathbf{A}(M_e)/k}, \tag{7.7}$$

which is given by $\delta((\lambda_v)_v)_e = \lambda_{d_1(e)}|_{\mathbf{A}(M_e)} - \lambda_{d_0(e)}|_{\mathbf{A}(M_e)}$ for each edge $e \in M^{\mathrm{can}}$. Hence it becomes crucial to understand $T^1_{\Gamma(C_n)/k}$ for $n \geq 3$, and also the restriction homomorphisms $\mathcal{T}^1_{\mathbf{A}(M_{d_i(e)})/k} \to \mathcal{T}^1_{\mathbf{A}(M_e)/k}$ for $i = 0, 1$.

7.2.1 n-cycles and normal forms

As before, let C_n denote the n-cycle for $n \geq 3$. Its Stanley-Reisner ring can be described as

$$\Gamma(C_n) = \begin{cases} n = 3 : k[x_0, x_1, x_2]/(x_0 x_1 x_2), \\ n \geq 4 : k[x_0, \ldots, x_{n-1}]/(x_i x_j \mid |i-j| \geq 2). \end{cases}$$

For simplicity, the index of the variables x_i can be any integer, but is interpreted as reduced modulo n. By [AC04, Example 17], a k-basis for $T^1_{\Gamma(C_n)/k}$ is given as follows: For $n = 3$, we have generators $\phi_i^{(k)}$ for $k \geq -1$ and $i = 0, 1, 2$. Here $\phi_i^{(k)}$ corresponds to the homomorphism given by $x_0 x_1 x_2 \mapsto x_i^{k+1}$, so we have $\phi_0^{(-1)} = \phi_1^{(-1)} = \phi_2^{(-1)}$. For $n = 4$, we have generators $\phi_i^{(k)}$ for $k \geq 0$, and $i = 0, 1, 2, 3$. Here $\phi_i^{(k)}$ corresponds to $x_{i-1} x_{i+1} \mapsto x_i^k$, mapping the other relation to 0. Hence $\phi_0^{(0)} = \phi_2^{(0)}$, and $\phi_1^{(0)} = \phi_3^{(0)}$. For $n \geq 5$, we have generators $\phi_i^{(k)}$ for $k \geq 1$ and $i = 0, 1, \ldots, n-1$. Here $\phi_i^{(k)}$ corresponds to $x_{i-1} x_{i+1} \mapsto x_i^k$, mapping the other relations to zero.

In order to lift to higher-order deformations, we shall require the notion of *normal forms* (see [AC10]), which corresponds to certain deformations of $\Gamma(C_n)$ (for $3 \leq n \leq 6$) over Artinian local rings. The definition will involve the following data:

- A ring $\mathscr{R}_n$ for $3 \leq n \leq 6$, given as the quotient of the power-series ring $\mathscr{P}_n = k[[t_i^{(k)}]]$ by a finitely generated ideal $\mathfrak{a}_n$. In $\mathscr{P}_n$ we have one variable $t_i^{(k)}$ for each generator $\phi_i^{(k)} \in T^1_{\Gamma(C_n)/k}$, where by convention we put $t_i^{(k)} = t_j^{(k)}$ whenever $\phi_i^{(k)} = \phi_j^{(k)}$.
- A finite set of elements $\mathcal{I}_n \subseteq k[x_1, \ldots, x_n][[t_j^{(k)}]]$.

The ideal $\mathfrak{a}_n$ is defined as the zero-ideal for $n \leq 5$, but $\mathfrak{a}_6$ is defined as the 2×2-minors of the matrix

$$\begin{bmatrix} t_0^{(1)} & t_2^{(1)} & t_4^{(1)} \\ t_3^{(1)} & t_5^{(1)} & t_1^{(1)} \end{bmatrix} \tag{7.8}$$

The elements forming $\mathcal{I}_n$ for $n \geq 3$ are given in [AC10, p.19].

Proposition 7.2.2 ([AC10, Proposition 6.6])**.** *Let $f : \mathscr{R}_n \to A$ (for $3 \leq n \leq 6$) be a k-algebra homomorphism, where A is a local Artinian k-algebra, such that all but finitely many $t_i^{(k)}$ are mapped to* 0*. Then the quotient of $A[x_0, \ldots, x_{n-1}]$ by the ideal generated by the image of $\mathcal{I}_n$ under f defines an infinitesimal deformation $Z_f \to \operatorname{Spec}(A)$ of $\operatorname{Spec}(\Gamma(C_n))$.*

An infinitesimal deformation of $\mathrm{Spec}(\Gamma(C_n))$ over $\mathrm{Spec}(A)$ is said to be in *normal form* if it is on the form Z_f for some homomorphism $f : \mathscr{R}_n \to A$.

7.2.2 A basis for $H^0(M, \mathcal{T}^1_{M/k})$

We shall assume here that M is a 2-dimensional smooth polyhedral manifold (i.e. its faces are smooth polytopes). Let $e \in M$ be an edge, and let $v_i = d_i(e)$ for $i = 0, 1$ be the vertices of e (which may be equal). Define $E = |e|$, and let

$$y_i := \frac{\chi^{(w_{E,v_i},1)}}{\chi^{(v_i,1)}} = \chi^{(w_{E,v_i}-v_i,0)} \in \Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})}.$$

Then $y_0 = y_1^{-1}$, and $\Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})} \cong k[y_1]_{y_1} \cong k[y_0]_{y_0}$. Since M is a manifold, $\mathrm{lk}^{v_i}_M(e)$ is the disjoint union of two points, corresponding to a pair of arrows $e \xrightarrow{f_1} t_1$, $e \xrightarrow{f_2} t_2$ where t_1, t_2 are facets.

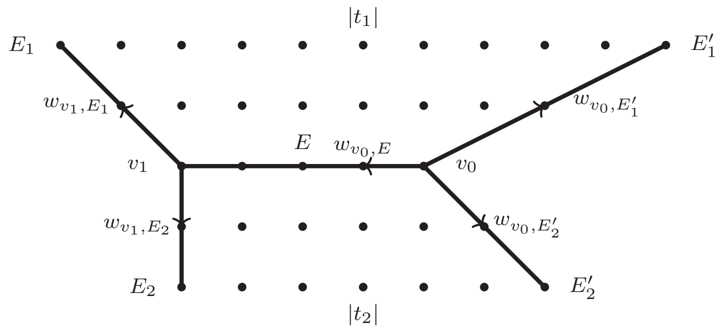

Figure 7.1: The local picture around $e \in M$.

Hence $\Gamma(\mathrm{lk}^{v_i}(e)) = k[x_i, z_i]/(x_i z_i)$. Consider the isomorphisms

$$\pi_i : \mathbf{A}(M_e) \to \mathrm{Spec}(\Gamma(\mathrm{lk}^{v_i}_U(s))) \times_k \mathrm{Spec}(\Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})}))$$

from Proposition 5.3.4. These induces an isomorphism

$$\pi_e = \pi_0 \pi_1^{-1} : k[y_1]_{y_1} \otimes_k k[x_1, z_1]/(x_1 z_1) \to k[y_0]_{y_0} \otimes_k k[x_0, z_0]/(x_0 z_0).$$

Clearly $\pi_e(y_1) = y_0^{-1}$. Moreover, $\boxed{\pi_e(x_1) = x_0 y_0^{n(e,t_1)}}$ and $\boxed{\pi_e(z_1) = z_0 y_0^{n(e,t_2)}}$ for some integers $n(e,t_1), n(e,t_2)$, which we will determine next. Note first that $\Gamma(\mathrm{lk}^{v_i}_U(s)) \otimes_k \Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})}$ is isomorphic to the pullback of the diagram

$$\Gamma(\Delta^{t_1})_{(\theta_{f_1})} \to \Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})} \leftarrow \Gamma(\Delta^{t_2})_{(\theta_{f_2})}.$$

In $\Gamma(\Delta^{t_1})_{(\chi^{(v_1,1)}\chi^{(v_0,1)})}$, x_1 and x_0 corresponds to $\chi^{(w_{E_1,v_1}-v_1,0)}$ and $\chi^{(w_{E'_1,v_0}-v_0,0)}$ respectively (see Figure 7.1), where $E_1, E'_1 \neq E$ are the edges in $|t_1|$ containing

v_1, v_0 respectively. Similarly, in $\Gamma(\Delta^{t_2})_{(\chi^{(v_1,1)}\chi^{(v_0,1)})}$, z_1 and z_0 corresponds to $\chi^{(w_{E_2,v_1}-v_1,0)}$ and $\chi^{(w_{E_2',v_0}-v_0,0)}$ respectively, where $E_2, E_2' \neq E$ are the edges in $|t_2|$ containing v_1, v_0 respectively. It follows that $n(e,t_1)$ is the integer satisfying $\chi^{(w_{E_1,v_1}-v_1,0)} = \chi^{(w_{E_1',v_0}-v_0,0)}(\chi^{(w_{E,v_0}-v_0,0)})^{n(e,t_1)}$. In other words,

$$w_{E_1,v_1} - v_1 = w_{E_1',v_0} - v_0 + n(e,t_1)(w_{E,v_0} - v_0).$$

Analogously, $n(e,t_2)$ is the integer satisfying

$$w_{E_2,v_1} - v_1 = w_{E_2',v_0} - v_0 + n(e,t_2)(w_{E,v_0} - v_0).$$

Such integers exist due to the fact that $|t_1|$ and $|t_2|$ are smooth polytopes. Hence $n(e,t_1)$ (resp. $n(e,t_2)$) is the number of lattice points of E minus the number of lattice points of the line segment between w_{E_1,v_1} and w_{E_1',v_0} (resp. w_{E_2,v_0} and w_{E_2',v_0}). See Figure 7.2 for a graphical depiction. Note that the sum $\boxed{n(e) := n(e,t_1) + n(e,t_2)}$ depends only on e.

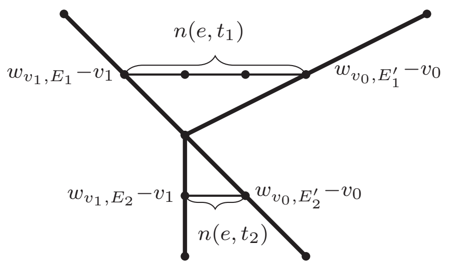

Figure 7.2:

Consider now the module $T^1_{k[x_0,z_0,y_0]_{y_0}/(x_0z_0)/k} \cong k[y_0]_{y_0}$. It is generated by the elements $\boxed{\phi_e^{(k)} := y_0^k}$ for each $k \in \mathbb{Z}$. The homomorphism corresponding to y_0^k is the one given by $x_0z_0 \mapsto y_0^k$. Thus the isomorphism

$$k[y_0]_{y_0} \cong T^1_{k[x_0,z_0,y_0]_{y_0}/(x_0z_0)/k} \to T^1_{k[x_1,z_1,y_1]_{y_1}/(x_1z_1)/k} \cong k[y_1]_{y_1} \tag{7.9}$$

induced by π_e is given by $\boxed{y_0^k \mapsto y_1^{-n(e)-k}.}$

Let $v \in M_0$ be any vertex. Then we relabel the generators $\phi_i^{(k)}$ of $T^1_{\Gamma(\mathrm{lk}_M(v))/k}$ as $\phi^{(k)}_{(e_i,f_i)}$, where (e_i,f_i) ranges over the vertices of $\mathrm{lk}_M(v)$ in some cyclic order (for $i = 0,\ldots,\mathrm{val}(v)-1$). We are now in a position to describe (7.7). For each edge $e \in M_1^{\mathrm{can}}$, let us identify $\mathbf{A}(M_e)$ with $\mathrm{Spec}(\Gamma(\mathrm{lk}_U^{d_1(e)}(e))) \times_k \mathrm{Spec}(\Gamma(\Delta^e)_{(\theta_{\mathrm{id}_E})})$. Then $\mathbf{A}(M_e) \to \mathbf{A}(M_{d_1(e)})$ corresponds to the localization homomorphism $\Gamma(C_n) \to k[x_1,z_1,y_1]_{y_1}/(x_1z_1)$. On the other hand, the morphism $\mathbf{A}(M_e) \to \mathbf{A}(M_{d_0(e)})$ corresponds to the localization homomorphism composed with the isomorphism π_e. This means that the restriction homomorphism $\mathcal{T}^1_{\mathbf{A}(M_{d_1(e)})/k} \to \mathcal{T}^1_{\mathbf{A}(M_e)/k}$

is given by $\phi^{(k)}_{(e,d_1)} \mapsto \phi^{(k)}_e$, while the restriction homomorphism $\mathcal{T}^1_{\mathbf{A}(M_{d_0(e)})/k} \to \mathcal{T}^1_{\mathbf{A}(M_e)/k}$ is given by $\boxed{\phi^{(k)}_{(e,d_0)} \mapsto \phi^{(-n(e)-k)}_e.}$

It follows that via (7.7), the element $\phi^{(k)}_{(e,f)} \in \mathcal{T}^1_{\mathbf{A}(M_v)/k}$ is mapped to $\phi^{(k)}_e$ whenever $v = d_1(e)$, and $\phi^{(-n(e)-k)}_e$ whenever $v = d_0(e)$. We will not give an explicit description of a basis for $H^0(M, \mathcal{T}^1_{M/k})$, because it will not be necessary for our purposes, but we will give a sufficiently descriptive implicit characterization:

Definition 7.2.3. Each basis element of $H^0(M, \mathcal{T}^1_{M/k})$ is on the form $\psi^{(k)}_{(e,f)}$, for $k \geq -1$ and $(e, f : v \to e) \in \mathrm{lk}_M(v)^{\mathrm{can}}$, which (if it exists) is uniquely determined by the condition that $\psi^{(k)}_{(e,f)}|_{\mathbf{A}(M_v)} = \phi^{(k)}_{(e,f)}$, and that its restriction to $\mathbf{A}(M_w)$ is zero for the maximal number of $w \in M_0^{\mathrm{can}}$.

Determining whether the global section $\psi^{(k)}_{(e,f)}$ exist is a purely combinatorial problem. For example, if $f' : v' \to e$ is the arrow in M^{can} distinct from $f : v \to e$, then $\psi^{(k)}_{(e,f)}$ and $\psi^{(-n(e)-k)}_{(e,f')}$ are equal as global sections, and exist contemporaneously. Of course, $\mathrm{val}(v) = 3$ is a necessary condition for that $\psi^{(-1)}_{(e,f)}$ can exist, and $\mathrm{val}(v) \leq 4$ is necessary for that $\psi^{(0)}_{(e,f)}$ can exist. In this latter situation, we also have $\psi^{(0)}_{(e,f)} = \psi^{(0)}_{(e',f')}$ when (e', f') as a vertex of $\mathrm{lk}_M(v)$ lies opposite of (e, f). Otherwise, if $\mathrm{val}(v) \geq 5$, then $\psi^{(k)}_{(e,f)}$ can only exist if $k \geq 1$.

7.3 The universal base space

Definition 7.3.1. Let $\mathscr{P}_M$ denote the k-algebra on the generators $\{w_j^*\}_j$ dual to some chosen basis $\{w_j\}_j$ of $H^1(M, \mathcal{E}_L)$, and generators $\{T^{(k)}_{(e,f)}\}_{(v \xrightarrow{f} e, k) \in I}$ (one for each basis element $\psi^{(k)}_{(e,f)}$ of $H^0(M, \mathcal{T}^1_{M/k})$). As a notational convention, we put $T^{(k)}_{(e,f)} = 0$ if $\psi^{(k)}_{(e,f)}$ does not exist.

For each vertex v with $\mathrm{val}(v) = 6$, and for each arrow $f : v \to e$ where e is an edge, let $g_{(e,f)}$ be some element of the completion $\widehat{\mathscr{P}}_M$. Let $\mathscr{O} = \{g_{(e,f)}\}_{(e,f)}$ denote the set of these elements, and consider the ideal $\mathfrak{a}_{\mathscr{O}} \subseteq \widehat{\mathscr{P}}_M$ given as the 2×2-minors of the matrices

$$\begin{bmatrix} g_{(e_0,f_0)} & g_{(e_2,f_2)} & g_{(e_4,f_4)} \\ g_{(e_3,f_3)} & g_{(e_5,f_5)} & g_{(e_1,f_1)} \end{bmatrix} \tag{7.10}$$

for vertices v with $\mathrm{val}(v) = 6$. In this situation we define $\mathscr{R}_{M,\mathscr{O}} := \widehat{\mathscr{P}}_M / \mathfrak{a}_{\mathscr{O}}$.

Theorem 7.3.2. *Let M be a 2-dimensional smooth polyhedral manifold such that $3 \leq \mathrm{val}(v) \leq 6$ for every vertex $v \in M_0$. Assume in addition that for every edge* *e, the inequality* $n(e) < \delta_{d_1(e)} + \delta_{d_0(e)}$ *holds, where*

$$\delta_v := \begin{cases} 0 & \mathrm{val}(v) \geq 5 \\ 1 & \mathrm{val}(v) \geq 4 \\ 2 & \mathrm{val}(v) \geq 3 \end{cases}$$

(this is always the case when, for example, M is unimodular). Then we may find $\mathscr{O}$ as above with $g_{(e,f)} = T^{(1)}_{(e,f)} +$ higher order terms, such that $\mathrm{Spec}(\mathscr{R}_{M,\mathscr{O}})$ *is the universal base space for the deformation functor* $\mathrm{Def}_{(\mathbf{P}(M),\mathcal{O}_M(1))}$. *In particular, if* $3 \leq \mathrm{val}(v) \leq 5$ *for all vertices v, then the universal base space is regular.*

Proof. The proof will be similar to that of [AC10, Theorem 6.7], except for the fact that we have to glue our local deformations in the étale topology, and that we have to take in consideration that there may be locally trivial deformations as well. We will recursively construct a sequence of deformations of pairs

$$\begin{array}{ccc} (X_n, L_n) & \longrightarrow & (X_{n+1}, L_{n+1}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R_n) & \longrightarrow & \mathrm{Spec}(R_{n+1}) \end{array} \tag{7.11}$$

of $(\mathbf{P}(M), \mathcal{O}_M(1))$ and liftings of line bundles L_n on X_n, where each R_n is a local Artinian quotient of $\widehat{\mathscr{P}}_M$ such that $R_n = R_{n+1}/m^{n+1}$, and where $m \subseteq \widehat{\mathscr{P}}_M$ is the maximal ideal. The inequalities are assumed in order to construct a compatible system of local deformations. The R_n's will be chosen in such a way that $\varprojlim_n R_n \cong \mathscr{R}_{M,\mathscr{O}}$ for some set $\mathscr{O} = \{g_{(e,f)}\} \subseteq \widehat{\mathscr{P}}_M$. Note that a deformation X_n corresponds to a coherent sheaf of flat R_n-algebras $\mathcal{O}_{X_n}$ on $X_0 = \mathbf{P}(M)$. We will show that there exists homomorphisms of k-algebras $h_v^{(n)} : \mathscr{R}_{\mathrm{val}(v)} \to R_n$ for each n, lifting $h_v^{(n-1)}$, such that the deformation $X_n|_{\mathbf{A}(M_v)} := \mathrm{Spec}(\mathcal{O}_{X_n|_{\mathbf{A}(M_v)}})$ of $\mathbf{A}(M_v)$ over $\mathrm{Spec}(R_n)$ corresponds to $h_v^{(n)}$ (per Proposition 7.2.2). The elements $g_{(e,f)}$ will be defined recursively via the sequence of compatible elements $g^{(n)}_{(e,f)} := h_v^{(n)}(t^{(1)}_{(e,f)}) \in R_n$.

To begin with, define $R_0 = \widehat{\mathscr{P}}_M/m$, and let $h_v^{(0)}$ map every generator to 0. For $n = 1$, define $R_1 = \widehat{\mathscr{P}}_M/m^2$, and let $h_v^{(1)} : \mathscr{R}_{\mathrm{val}(v)} \to R_1$ be given by $t_i^{(k)} \mapsto T^{(k)}_{(e_i,f_i)}$. Then the associated local deformations $U_v^{(1)}$ of $\mathbf{A}(M_v)$ corresponds to $\sum_{k\in\mathbb{Z},(e,f)\in \mathrm{lk}_M(v)} \phi^{(k)}_{(e,f)}$. Indeed, this can be seen by plugging in the corresponding variables in the particular equations which define $\mathcal{I}_n$ in Proposition 7.2.2. Moreover, these deformations agree on intersections. I.e. $U^{(1)}_{d_1(e)}|_{\mathbf{A}(M_e)} \cong U^{(1)}_{d_0(e)}|_{\mathbf{A}(M_e)}$ for each edge e. They can be found in [AC10, p.19]. Next, since $H^2(X_0, \mathcal{E}_{L_0}) = 0$ by Theorem 7.1.7, there is no obstruction to the existence of a compatible system of isomorphisms

$(\phi_{vw}, \pi_{vw}) : (U_v^{(1)}, \mathcal{O}_{U_v^{(1)}})|_{\mathbf{A}(M_v)\times_M \mathbf{A}(M_w)} \to (U_w^{(1)}, \mathcal{O}_{U_w^{(1)}})|_{\mathbf{A}(M_v)\times_M \mathbf{A}(M_w)}$ (reducing to the identity on $(\mathcal{O}_{\mathbf{A}(M_{vw})}|_{\mathbf{A}(M_{vw})}, L_0|_{\mathbf{A}(M_{vw})}))$ for each pair of vertices v, w. Perturb the system $(\phi_{vw}, \pi_{vw})_{vw}$ by an element on the form $\sum_j a_j w_j^* \otimes w_j \in H^1(X_0, \mathcal{E}_{L_0}) \otimes_k m/m^2$ for general coefficients a_j. We may glue now glue the pairs $(U_v^{(1)}, \mathcal{O}_{U^{(1)}})$ together via the perturbed system $(\phi'_{vw})_{vw}$ to a deformation (X_1, L_1) of (X_0, L_0) over $\mathrm{Spec}(R_1)$. Since the coefficients of $\sum_j a_j w_j^* \otimes w_j$ were chosen generally, the resulting Kodaira Spencer map $\mathrm{Hom}(R_1, k[\epsilon]) \to \mathrm{Def}_{(X_0,L_0)}(k[\epsilon])$ is bijective.

For the inductive step, assume that R_m and suitable deformations $X_m \to \mathrm{Spec}(R_m)$ exists for $m < n$, where $n > 1$. Then we will construct R_n and a suitable deformation $X_n \to \mathrm{Spec}(R_n)$. First, let $g'_{(e,f)}$ be an arbitrary lifting of $g_{(e,f)}^{(n-1)}$ to $\widehat{\mathscr{P}}_M$. Let $\mathscr{O}' = \{g'_{(e,f)}\}_{(e,f)} \subseteq \widehat{\mathscr{P}}_M$, and define $R_n = \widehat{\mathscr{P}}_M/(\mathfrak{a}_{\mathscr{O}'} + m^{n+1})$, where $\mathfrak{a}_{\mathscr{O}'}$ is the ideal defined as the minors of (7.10). Next, let h'_v be an arbitrary lifting of $h_v^{(n-1)}$. If $\mathrm{val}(v) = 6$, we impose the requirement that $h'_v(t_i^{(1)}) = g'_{(e_i,f_i)}$ (in which case the local obstruction equations $\mathfrak{a}_6$ are mapped to 0 as required). Each h'_v define deformations U'_v over $\mathrm{Spec}(R_n)$ which are liftings of $U_v^{(n-1)}$. However, the deformations $U_v^{(n-1)}|_{\mathbf{A}(M_{vw})}$ and $U_w^{(n-1)}|_{\mathbf{A}(M_{vw})}$ may fail to be isomorphic. Their differences define an element $D_e = \sum_{k\in\mathbb{Z}} b_k \phi_e^{(k)} \in H^0(\mathbf{A}(M_e), \mathcal{T}^1_{\mathbf{P}(M)/k}) \otimes m^n/m^{n+1}$ for each edge $e \in M_1^{\mathrm{can}}$, where each $b_k \in m^n$. To remedy this, we need to adjust the h_v's. Note that an adjustment of $h_{d_1(e)}^{(n)}(t_i^{(k)}) \mapsto h_{d_1(e)}^{(n)}(t_{(e_i,f_i)}^{(k)}) + b$ for some $b \in m^n$ has the effect of perturbing the difference D_e by $+b\phi_{e_i}^{(k)}$ ($D_{e'}$ for $e' \neq e_i$ is not affected). On the other hand, an adjustment of $h_{d_0(e)}^{(n)}(t_i^{(k)}) \mapsto h_{d_0(e)}^{(n)}(t_{(e_i,f_i)}^{(k)}) + b$ has the effect of perturbing the difference by $-b\phi_{e_i}^{(-n(e_i)-k)}$. This can be seen by inspecting the equations defining $\mathcal{I}_n$ (see [AC10, p.19]). Now, the inequality $n(e) < \delta_{d_1(e)} + \delta_{d_0(e)}$ ensures that each term $b_k\phi_e^{(k)}$ lies in the image of either $H^0(\mathbf{A}(M_{d_1(e)}), \mathcal{T}^1_{\mathbf{P}(M)/k})$ or $H^0(\mathbf{A}(M_{d_0(e)}), \mathcal{T}^1_{\mathbf{P}(M)/k})$, and this allows us to eliminate every term $b_k\phi_e^{(k)}$ by iterative adjustments. Indeed, the inequality implies that that any integer $k \in \mathbb{Z}$ can be written as k_0 or $-n(e) - k_1$, where k_i is an integer such that $\phi_{(e,d_i)}^{(k_i)}$ is a generator of $\mathcal{T}^1_{\mathbf{A}(M_{d_i(e)})/k}$ for $i = 0, 1$. So let $g_{(e,f)}^{(n)}$ be the new elements given by $h_v^{(n)}$, and let $\mathscr{O}^{(n)} = \{g_{(e,f)}^{(n)}\}_{(e,f)}$. Then R_n is still isomorphic to $\widehat{\mathscr{P}}_M/(\mathfrak{a}_{\mathscr{O}^{(n)}} + m^{n+1})$. Indeed, $\mathfrak{a}_{\mathscr{O}^{(n)}} = \mathfrak{a}_{\mathscr{O}'} (\mathrm{mod}\ m^{n+1})$ since the adjustment $g'_{(e,f)} \mapsto g_{(e,f)}^{(n)}$ in R_n is by an element of m^n. Let $U_v^{(n)}$ be the new local deformations after these adjustments. Then they will agree on intersections, and since $H^2(X_0, L_0) = 0$ there exists a compatible system of isomorphisms $(\phi_{vw}^{(n)}, \pi_{vw}^{(n)}) : (U_v^{(n)}, \mathcal{O}_{U_v^{(n)}})|_{\mathbf{A}(M_{vw})} \to (U_w^{(n)}, \mathcal{O}_{U_w^{(n)}})|_{\mathbf{A}(M_{vw})}$, reducing to the identity like above. We may glue the pairs to a global deformation (X_n, L_n) lifting (X_{n-1}, L_{n-1}).

In conclusion, the deformations of (7.11) have now been constructed, and satisfies the required conditions. The Kodaira Spencer map is bijective by construction, and thus $\operatorname{Spec}(\mathscr{R}_{M,\mathscr{O}})$ will be the the universal base space. $\square$

Proposition 7.3.3. *Under the conditions of Theorem 7.3.2, assume in addition that* $\operatorname{val}(v) \neq 5$ *for all vertices* v. *Furthermore, assume that* $T^{(k)}_{(e,f)} = 0$ *whenever* $k \geq 2$ *for* $\operatorname{val}(v) = 6$, $k \geq 1$ *for* $\operatorname{val}(v) = 4$, *and* $k \geq 0$ *for* $\operatorname{val}(v) = 3$. *Under these assumptions, each* $g_{(e,f)} = T^{(1)}_{(e,f)}$. *Hence the universal formal family* $\{(X_n, L_n)\}$ *is definable over the finitely generated* k*-algebra*

$$\mathcal{R}_{M,\mathscr{O}} := \mathscr{P}_M/\mathfrak{a}_{\mathscr{O}},$$

where $\mathfrak{a}_{\mathscr{O}}$ *is the* 2×2*-minors of the matrices*

$$\begin{bmatrix} T_{e_0} & T_{e_2} & T_{e_4} \\ T_{e_3} & T_{e_5} & T_{e_1} \end{bmatrix}.$$

associated to each vertex of valency 6.

Proof. The proof follows in the exact same manner as for Theorem 7.3.2, with the additional observation that in this situation no adjustment of the homomorphisms $h'_v : \mathscr{R}_{\operatorname{val}(v)} \to R_n$ is necessary in order for $U'_{d_1(e)}|_{\mathbf{A}(M_e)}$ and $U'_{d_0(e)}|_{\mathbf{A}(M_e)}$ to be isomorphic. Specifically, we define $h^{(n)}_v : \mathscr{R}_{\operatorname{val}(v)} \to R_n$ (for each n) by $t^{(k)}_i \mapsto T^{(k)}_{(e_i,f_i)}$. We will show that the equations defining the local deformations $U^{(n)}_v$, as defined by h_v, are linear in the variables of R_n. In such a situation, we will have $U^{(n)}_{d_1(e)}|_{\mathbf{A}(M_e)} \cong U^{(n)}_{d_0(e)}|_{\mathbf{A}(M_e)}$ in general. Indeed, the linear terms will in each case yield equality on the nose (for the same reason we have that for $n = 1$). Again, we refer to [AC10, p.19] for the particular equations, which via the normal forms h_v define the associated deformation. Note that for valency 3, linearity is automatic. **Valency 4:** The ideal defining $U^{(n)}_v \subseteq \operatorname{Spec}(R_n[x_0, x_1, x_2, x_3])$ is on the form

$$(x_0x_2 + T^{(0)}_{(e_1,f_1)}, x_1x_3 + T^{(0)}_{(e_0,f_0)}).$$

By inverting x_0 (which amounts to restricting to $\mathbf{A}(M_{e_0})$), we obtain a description of the resulting subscheme $U^{(n)}_v|_{\mathbf{A}(M_{e_1})} \subseteq \operatorname{Spec}(R_n[x_0, x_1, x_2]_{x_1})$ as given by the ideal

$$(x_0x_2 + T^{(0)}_{(e_0,f_0)}).$$

The choice of x_1 was without loss of generality, and the same conclusion holds for x_i for $i = 0, \ldots, 3$. **Valency 6:** The ideal defining $U^{(n)}_v \subseteq \operatorname{Spec}(R_n[x_i \mid i = 0, \ldots 5])$ is on the form

$$(x_{i+1}x_{i-1} + x_iT^{(1)}_{(e_i,f_i)}, x_jx_{j+3} - T^{(1)}_{(e_{j+1},f_{j+1})}T^{(1)}_{(e_{j+2},f_{j+2})} \mid i = 0, \ldots, 5, j = 0, 1, 2).$$

Again, by a straight-forward computation after inverting x_0, say, one obtains the ideal $(x_1x_5 + x_0T_{(e_0,f_0)})$, which has with linear terms. This computation relies on the vanishing of the minors of (7.8). $\square$

7.4 Smoothability

Next we come to the question of smoothability.

Lemma 7.4.1. *Let $f : X \to Y$ be a flat morphism of schemes of finite type. Then the set $U_f = \{y \in Y \mid X_y \to \operatorname{Spec}(k(y))$ is smooth$\}$ is open in Y.*

Proof. Let $y \in U_f$, and let $y' \in Y$ be a generization of y. Let R be some DVR, and let $\operatorname{Spec}(R) \to Y$ be some morphism covering the generization $y' \rightsquigarrow y$. Let $g : X' \to \operatorname{Spec}(R)$ denote the base change of f. Now, U_g contains the closed point of $\operatorname{Spec}(R)$. Since g is of finite type, every closed point of X' is also a closed point of the central fiber X'_0. Thus g is smooth at all closed points in the sense of [Stacks, Tag 01V9] (i.e. g is smooth at $x \in X'$ if $X'_{g(x)} \to \operatorname{Spec}(k(g(x)))$ is smooth around $x \in X'_{f(x)}$). But this is an open condition, so g is smooth at every point. Hence U_g contains the generic point of $\operatorname{Spec}(R)$ as well, and thus $y' \in U_f$. □

Definition 7.4.2. A morphism $f : X \to Y$ of schemes is called *generically smooth* if it is flat, proper, and $X_\eta \to \operatorname{Spec}(k(\eta))$ is smooth for the generic point η of each component of Y. By Lemma 7.4.1, the set U_f is then open and dense in Y.

Definition 7.4.3. Let (X_0, L_0) be a pair of a projective scheme over a field k, and an ample line bundle L_0. Then a *smoothing* of (X_0, L_0) is a proper, generically smooth deformation $(\mathcal{X} \to \operatorname{Spec}(k[[t]]), \mathcal{L})$ of the pair (X_0, L_0). If such a smoothing exists, then by Proposition 7.0.1 there exists an algebraization to a generic smoothing over a non-singular curve C of finite type over k. A *formal smoothing* of X_0 is a formal family $(X_n \to \operatorname{Spec}(A_n), L_n)$, where $A_n := k[t]/(t^{n+1})$, such that there exists an integer n_0 such that $t^{n_0}\mathcal{T}^1_{X_n/A_n}(\mathcal{F}) = 0$ for all n and for every coherent sheaf $\mathcal{F}$ on X_n.

Proposition 7.4.4. *Let X_0 be a projective scheme equipped with an ample line bundle L_0 which is formally smoothable. Then (X_0, L_0) is smoothable.*

Proof. Suppose that (X_n, L_n) is a formal smoothing of (X_0, L_0), and let $(\widehat{X}, \widehat{L})$ be the induced effectivization over $\widehat{A} := k[[t]]$. Let m be an integer such that $\widehat{L}^{\otimes m}$ is very ample. Then $(X_n, L_n^{\otimes m})$ is a formal smoothing of $(X_0, L_0^{\otimes m})$. Considering the associated embeddings, X_n is a formal smoothing of X_0 in the sense of [Har10, Chapter 29]. Thus by [Har10, Proposition 29.5], $\widehat{X} \to \operatorname{Spec}(\widehat{A})$ is generically smooth. □

Remark 7.4.5. There is also the well-defined notion of a universal generic smoothing of a pair (X_0, L_0), which can be realized as the restriction of the universal effective deformation to its generically smooth base locus: Let $(\mathscr{X} \to \operatorname{Spec}(\widehat{R}), \mathscr{L})$ be the (unique) universal effective deformation of (X_0, L_0). Then by Lemma 7.4.1, there is a maximal open set $U \subseteq \operatorname{Spec}(\widehat{R})$ over which $\mathscr{X}|_U \to U$ is smooth. Let S be the closure of U, and let $(\mathscr{X}', \mathscr{L}') = (\mathscr{X}|_S, \mathscr{L}|_S)$. Suppose that $(\mathscr{Y} \to T, \mathscr{H})$ is any generically smooth effective deformation of (X_0, L_0). Since $\operatorname{Def}_{(X_0,L_0)}$ has unique effectivizations, there is a unique morphism $f : T \to \operatorname{Spec}(\widehat{R})$ such that

$(\mathscr{Y}, \mathscr{H}) \cong (\mathscr{X} \times_{\widehat{R}} T, \mathscr{L} \otimes_{\widehat{R}} T)$. But since $\mathscr{Y}$ is generically smooth, f factors as $T \to S \to \operatorname{Spec}(\widehat{R})$. Indeed, if V is the maximal open subset of $\mathscr{Y}$ such that $\mathscr{Y}|_V \to V$ is smooth, then $f(V) \subseteq U$. Hence $f(T) \subseteq S$. Thus $\mathscr{Y} \cong \mathscr{X}' \times_S T$. The morphism $T \to S$ is obviously unique, hence $(\mathscr{X}' \to S, \mathscr{L}')$ can be called a universal generic smoothing. By Proposition 7.0.1, the universal smoothing $(\mathscr{X}' \to S, \mathscr{L}')$ is algebraizable.

Lemma 7.4.6. *Suppose that (X, L) is a smoothing of $(\mathbf{P}(M), \mathcal{O}_M(1))$, for some polyhedral manifold M. Let K_X be the canonical divisor of X. Then $2K_X = 0$, and $K_X = 0$ if and only if M is orientable. Hence by the Kodaira classification of surfaces, $|M| \cong \mathbb{S}^2$ implies that X is a K3 surface, $|M| \cong \mathbb{S}^1 \times \mathbb{S}^1$ implies that X is an abelian surface, $|M| \cong \mathbb{R}P^2$ implies that X is an Enriques surface.*

Proof. The statements about the canonical divisor on X follows from Theorem 6.5.4, and the last part of the proof. □

Theorem 7.4.7. *Under the conditions of Proposition 7.3.3, $(\mathbf{P}(M), \mathcal{O}_M(1))$ is smoothable if $T^{(1)}_{(e,f)}$ is non-zero for $\operatorname{val}(v) = 6$, $T^{(0)}_{(e,f)}$ is non-zero for $\operatorname{val}(v) = 4$, and $T^{(1)}_{(e,f)}$ is non-zero for $\operatorname{val}(v) = 3$. Hence, one may consistently define*

$$T_e := \begin{cases} T^{(1)}_{(e,f)} & \text{if } \operatorname{val}(f^M(e)) = 6 \\ T^{(0)}_{(e,f)} & \text{if } \operatorname{val}(f^M(e)) = 4 \\ T^{(-1)}_{(e,f)} & \text{if } \operatorname{val}(f^M(e)) = 3 \end{cases}$$

for each edge e of M. Equivalently, $n(e) = \delta_{d_1(e)} + \delta_{d_0(e)} - 2$ for all edges e. In this case, the base space for the universal smoothing of $\mathbf{P}(M)$ are the main components *of* $\operatorname{Spec}(\mathscr{R}_{M,\mathscr{O}})$*, i.e. the closure $\mathcal{B}_M := \overline{D(\prod_e T_e)}$ of the main torus $D(\prod T_e)$.*

Proof. We will utilize the following criteria for determining whether a deformation is generically smooth or not, which we highlight in a remark because it will become useful later on as well.

Remark 7.4.8. Let (X, L) be a pair where X is a reduced projective scheme, and where L is an ample line bundle satisfying $H^2(X, \mathcal{E}_L) = 0$. Let $\{U_i\}$ be a finite affine étale cover of X, and let $\mathcal{U}_i \to \operatorname{Spec}(k[t])$ be morphisms with central fiber U_i over $t = 0$. Define $\widehat{A} = k[[t]]$, and $A_n = k[t]/t^n$. Assume further that structure sheaves $\mathcal{O}_{(\mathcal{U}_i)_n}$ of $(\mathcal{U}_i)_n = \mathcal{U}_i \times_{k[t]} A_n$ are flat over A_n (as coherent sheaves of A_n-algebras on U_i), and satisfy

$$\mathcal{O}_{(\mathcal{U}_i)_n}|_{U_{ij}} \cong \mathcal{O}_{(\mathcal{U}_j)_n}|_{U_{ij}} \tag{7.12}$$

for each $n \geq 0$. Note that by the infinitesimal criterion for flatness, each $\mathcal{U}_i$ is flat over $\operatorname{Spec}(k[t])$ at $t = 0$. Since $H^2(X, \mathcal{E}_L) = 0$, these isomorphisms may be modified to satisfy the cocycle condition, glueing in the étale topology to a deformation $X_n \to \operatorname{Spec}(A_n)$ of X equipped with liftings L_n of L. Assume now that $\mathcal{U}_i \to \operatorname{Spec}(k[t])$ are generic smoothings for each i. Then there exists integers

n_i such that $t^{n_i}\mathcal{T}^1_{\mathcal{U}_i/k[t]}(\mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on $\mathcal{U}_i$. In particular, $t^{n_i}\mathcal{T}^1((\mathcal{U}_i)_n/A_n, \mathcal{F}|_{(\mathcal{U}_i)_n}) = 0$ for every coherent sheaf $\mathcal{F}$ on X_n for all n. Hence for $n_0 = \max(n_i)$, we have $t^{n_0}\mathcal{T}^1(X_n/A_n, \mathcal{F}) = 0$. Thus (X_n, L_n) is a formal smoothing, and therefore the induced effectivization $\widehat{X} \to \operatorname{Spec}(\widehat{A})$ is generically smooth by Proposition 7.4.4. Conversely, if some $\mathcal{U}_i$ is not a generic smoothing, then there exists no integer n_i as above. Hence $\widehat{X}$ cannot be generically smooth in this case.

By Proposition 7.3.3, the universal formal family $\{(X_n, L_n)\}$ is definable over $\operatorname{Spec}(\mathcal{R}_{M,\mathscr{O}}) = V(\mathfrak{a}_{\mathscr{O}}) \subseteq \operatorname{Spec}(\mathscr{P}_{M,\mathscr{O}})$. Here $\mathfrak{a}_{\mathscr{O}}$ is a homogeneous ideal, so there is a linear complete curve $C \subseteq \operatorname{Spec}(\widehat{\mathscr{P}_{M,\mathscr{O}}})$ through every dimension 1 point of $\operatorname{Spec}(\mathscr{R}_{M,\mathscr{O}})$. Such a curve is a complete DVR, so the closure $\mathcal{B}_M \subseteq \operatorname{Spec}(\widehat{\mathscr{P}_{M,\mathscr{O}}})$ of the smooth base locus is precisely the closure of the union of the curves C which induce generic smoothings $\mathscr{X}|_C \to C$. By Remark 7.4.5, $\mathcal{B}_M$ is the base locus of the universal generic smoothing. Any curve C is determined by a linear homomorphism $f_C : \widehat{\mathscr{P}_{X,\mathscr{O}}} \to \widehat{A}$, given on generators by $T_e \mapsto a_e t$ and $w_j^* \mapsto b_j t$ for some constants $a_e, b_j \in k$ satisfying $a_e = a_{e'}$ whenever $T_e = T_{e'}$. The a_e's must the satisfy equations given by the vanishing of the minors of (7.8). Thus f_C induces a homomorphism $R_n \to A_n$. Now, the restriction $\mathscr{X}|_C$ is the completion of the formal family $\{X_n \times_{R_n} A_n \to \operatorname{Spec}(A_n)\}$. Consider the affine étale cover $\{U_v^{(n)} \times_{R_n} A_n\}_v$ of $X_n \times_{R_n} A_n$. We observe that each $U_v^{(n)}$ is on the form $\mathcal{U}_v \times_{k[t]} A_n$, where $\mathcal{U}_v \subseteq \operatorname{Spec}(k[t][x_0, \ldots, x_{\operatorname{val} v - 1}])$ are defined in Table 7.1. By Remark 7.4.8, $\mathscr{X}|_C$ is generically smooth if and only if each

Table 7.1: The local deformations $\mathcal{U}_v$.

Valency	**Ideal of $\mathcal{U}_v$**
3	$(x_0x_1x_2 - a_{e_0}t)$
4	$(x_0x_2 + a_{e_1}t, x_1x_3 + a_0t)$
6	$(x_{i+1}x_{i-1} + a_{e_i}x_it, x_jx_{j+3} - a_{e_{j+1}}a_{e_{j-1}}t^2 \mid i = 0, \ldots, 5, j = 0, 1, 2)$

$\mathcal{U}_v \to \operatorname{Spec}(k[t])$ is generically smooth. Via Macaulay2 one can check that every fiber of $\mathcal{U}_v \to \operatorname{Spec}(k[t])$ will be singular if any $a_{e_i} = 0$. On the other hand, the fiber over $t = 1$ is smooth if $a_{e_i} \neq 0$ for all i. By Lemma 7.4.1, it is generically smooth in this case. In conclusion, $\mathcal{B}_M$ is the closure of the union of curves C which are determined by homomorphisms f_C satisfying $a_e \neq 0$ for every edge e, and this is precisely the closure of $D(\prod_e T_e)$. □

7.5 Degenerations of abelian surfaces

Our object of consideration here will be a class of torus tilings represented by polyhedral sets M arising as quotients of certain *admissible* (Definition 7.5.2) periodic tesselations of the plane $\mathbb{R}^2$ by smooth lattice polygons (with vertices in $\mathbb{Z}^2$). Thus M will be a quotient Λ/G, where Λ is a polyhedral set defining a periodic lattice tesselation of the plane, and G is a subgroup of the linear translation group of Λ, which we shall assume is of finite index in $\mathbb{Z}^2$. By Lemma 7.4.6, if $(\mathbf{P}(M), \mathcal{O}_M(1))$ is smoothable, then it is to abelian surfaces.

Any subgroup $G \subseteq \mathbb{Z}^2$ of finite index has unique generators on the form $(n, 0)$ and (r, m), for some integers $n, m \geq 1$ and $0 \leq r < n$. A fundamental domain of Λ relative to G is a minimal compact subtiling of Λ which is surjective onto Λ/G. There is a canonical fundamental domain $K_G \subseteq \Lambda$ defined as the tiling of the closed rectangle with corners $(0, 0), (n, 0), (0, m), (n, m)$. The basic object of our consideration is the equivelar tesselation $\{3, 6\}$, which partitions the plane into triangles. Let us call this Λ_0.

Definition 7.5.1. Let $G \subseteq \mathbb{Z}^2$ be some subgroup of finite index. Then we define $\mathcal{N}_G = \Lambda_0/G$.

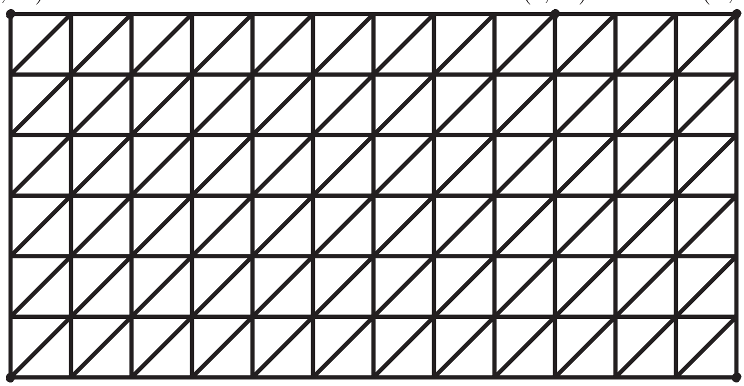

Figure 7.3: The fundamental domain K_G relative to G. In $\mathcal{N}_G = \Lambda_0/G$, the vertices $(0, 0)$, $(n, 0)$ and (r, m) are identified. If $r = 0$ then all four corners are identified.

Definition 7.5.2. A tesselation Λ of the plane is a 2-dimensional polyhedral set consisting of lattice polytopes which can be embedded into $\mathbb{R}^2$ (with vertices in $\mathbb{Z}^2$) without overlap. Λ is called *admissible* if it can be subdivided into the basic tesselation Λ_0, and moreover satisfies the following condition: for every vertex $v \in \Lambda$, either $\mathrm{val}(v) \neq 5$, or if $\mathrm{val}(v) = 4$, then the edges containing v split into two pairs of parallel lines.

Theorem 7.5.3. *Let M be a quotient Λ/G of an admissible tesselation Λ. Then M satisfies the conditions of Theorem 7.4.7. Hence in particular, $(\mathbf{P}(M), \mathcal{O}_{\mathbf{P}(M)}(1))$ is smoothable, and $\mathcal{B}_M = \overline{D(\prod T_e)}$ is the base space for the universal generic smoothing.*

Proof. We need to show that $n(e) = \delta_{d_1(e)} + \delta_{d_0(e)} - 2$ for each edge e. In Table 7.2 we list the various possibilities of triples $(\mathrm{val}(d_1(e)), \mathrm{val}(d_0(e)), n(e))$

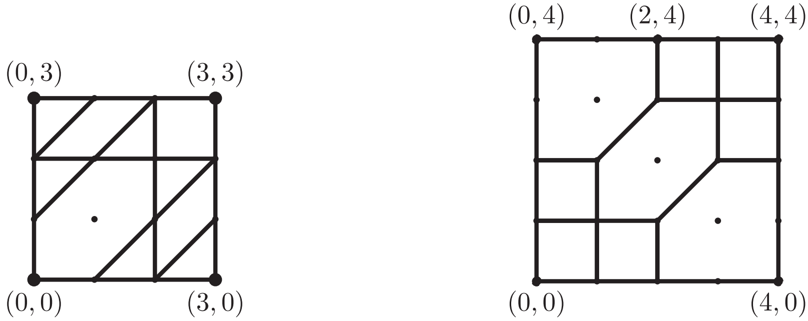

Figure 7.4: Examples of fundamental domains of admissible tesselations.

that may occur, and from one deduces which generators $T^{(k)}_{(e,f)}$ that can be possibly non-zero. We assume that $\mathrm{val}(d_1(e)) \geq \mathrm{val}(d_0(e))$. The remaining cases are obtained by inverting the table below. $\square$

$\big(\mathrm{val}(d_1(e)), \mathrm{val}(d_0(e)), n(e)\big)$	**Possible** $T^{(k)}_{(e,f)}$
$(6, 6, -2)$	$T^{(1)}_{(d_1(e),e)} = T^{(1)}_{(d_0(e),e)}$
$(6, 4, -1)$	$T^{(1)}_{(d_1(e),e)} = T^{(0)}_{(d_0(e),e)}$
$(6, 3, 0)$	$T^{(1)}_{(d_1(e),e)} = T^{(-1)}_{(d_0(e),e)}$
$(4, 4, 0)$	$T^{(0)}_{(d_1(e),e)} = T^{(0)}_{(d_0(e),e)}$
$(3, 3, 2)$	$T^{(-1)}_{(d_1(e),e)} = T^{(-1)}_{(d_0(e),e)}$

Table 7.2: We see that there are no situations that could potentially violate the conditions of Proposition 7.3.3.

See [Chr10] for an investigation of the universal base space $\mathrm{Spec}(\mathcal{R}_{M,\mathscr{O}})$ and the universal generic smoothing base space $\mathcal{B}_M$ in the case where M is a simplicial complex quotient of the basic tesselation Λ_0 by triangles. The minimal such situation is $(n, m, r) = (7, 1, 3)$.

Remark 7.5.4. Let M be a quotient Λ/G of an admissible tesselation Λ. We will provide a method for computing $\mathcal{B}_M$, given $\mathcal{B}_{\mathcal{N}_G}$. By Theorem 7.5.3, the generators of $\mathcal{P}_M$ are either on the form T_e for some edge e of M, or on the form w_j^* corresponding to a dual basis of the space $H^1(\mathbf{P}(M), \mathcal{E}_{\mathcal{O}_{\mathbf{P}(M)}(1)})$ of locally trivial first-order deformations. Note that if we let $\mathcal{R}'_M$ denote the subalgebra of $\mathcal{R}_M$ generated by the T_e's only, then $\mathcal{R}_M = \mathcal{R}'_M \otimes_k k[\{w_j^*\}_j]$. Thus, if we let $d_M := \dim H^1(M, \mathcal{E}_L)$, then $\mathcal{B}_M = \overline{D(T_e)} \times_k \mathbf{A}^{d_M}$, where $D(T_e) \subseteq \mathrm{Spec}(\mathcal{R}'_M)$. Now, via the description in Proposition 7.3.3, one observes that the homomorphism $\gamma : \mathcal{R}'_{\mathcal{N}_G} \to \mathcal{R}'_M$ given by

$$T_e \mapsto \begin{cases} T_{e'} & \text{if } e' \text{ is an edge containing } e, \\ 1 & \text{otherwise} \end{cases}$$

is well-defined and surjective (it does not however induce a well-defined map on completions).

Let v be a vertex of M. If $\mathrm{val}(v) = 4$, then we have $T_{e_i} = T_{e_{i+2}}$ for $i = 0, 1$. If $\mathrm{val}(v) = 3$, then $T_{e_0} = T_{e_1} = T_{e_2}$. One observes that these relations are equivalent with that the minors of the matrix

$$\begin{bmatrix} \gamma(T_{e_0}) & \gamma(T_{e_2}) & \gamma(T_{e_4}) \\ \gamma(T_{e_3}) & \gamma(T_{e_5}) & \gamma(T_{e_1}) \end{bmatrix} \tag{7.13}$$

vanishes for every vertex w of $\mathcal{N}_G$, where e_i are the edges appearing in $\mathrm{lk}_X(w)$. Hence we see that the universal generic smoothing base space factor $\mathcal{B}'_M$ is obtained by base changing $\mathcal{B}'_{\mathcal{N}_G}$ along $\mathrm{Spec}(\mathcal{R}'_M) \to \mathrm{Spec}(\mathcal{R}'_{\mathcal{N}_G})$. It other words, $\mathcal{B}'_M \subseteq \mathcal{B}'_{\mathcal{N}_G}$ is the subscheme cut out by the relations $T_e - 1$ for edges e is not contained in any edge of M, and $T_e - T_{e'}$ whenever e and e' are contained in the same edge of M.

7.5.1 Example computations

In what follows we will compute the universal generic smoothing in some special cases, using Macaulay2. The functions used are found in Appendix C.0.1. It computes the closure of the torus $\overline{D(\prod T_e)}$ inside the quotient ring $\mathcal{R}_{M,\mathscr{O}} = \mathcal{P}_M / \mathfrak{a}_{\mathscr{O}}$, where $\mathfrak{a}_{\mathscr{O}}$ is generated by the minors of the matrices

$$\begin{bmatrix} T_{e_0} & T_{e_2} & T_{e_4} \\ T_{e_3} & T_{e_5} & T_{e_1} \end{bmatrix}$$

for each vertex v. The output is in each case easily recognized to correspond to what we have listed. We have restricted the list to the cases where the resulting components can feasibly be written down. The degree is computed using the Riemann-Roch formula $L^2 = 2H^0(X, \mathcal{O}_X(1))$ for abelian surfaces (where $L = \mathcal{O}_X(1)$). Since $H^0(X, \mathcal{O}_X(1))$ is constant in families, then by Proposition 3.6.3 the degree L^2 is equal to $2L(M)$, where $L(M)$ is the number of lattice points of M.

Example 7.5.5. We consider quotients $M = \Lambda_0/G$ of the basic tesselation Figure 7.3. The Macaulay2 function used is called **basicTesselation**. The central fiber $(\mathbf{P}(M), \mathcal{O}_M(1))$ is a *stable quasiabelian pair* in the sense of [AN99], and this tesselation is one of the two Delaunay decompositions of the plane. The most degenerate example $\Lambda_0/\mathbb{Z}^2$ is also discussed there. It consists of two triangles, three edges and one vertex. It has a 3-parameter smoothing to principally polarized abelian surfaces.

$(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{r})$ $\dim \mathcal{B}'_M$	**Components of $\mathcal{B}'_M$ for Example 7.5.5** **Degree of polarization** $= 2nm$
$(1,1,0)$ 3	$\mathbf{A}^3$ 2 (a principal polarization)
$(2,1,0)$ 4	$\mathbf{A}^4$ 4
$(2,1,1)$ 4	$\mathbf{A}^4$ 4
$(2,2,0)$ 6	$\mathbf{A}^6 \cup \mathbf{A}^6$ which intersect at the origin 8
$(3,1,0)$ 5	$\mathbf{A}^5$ 6
$(3,1,1)$ 5	$\mathbf{A}^5$ 6
$(3,1,2)$ 5	$V(x_5x_7 - x_4x_8, x_9x_7 - x_2x_8, x_5x_6 - x_3x_8, x_4x_6 - x_3x_7, x_2x_6 - x_1x_7, x_9x_6 - x_1x_8, x_9x_4 - x_2x_5, x_2x_3 - x_1x_4, x_9x_3 - x_1x_5) \subseteq \mathbf{A}^9$ 6

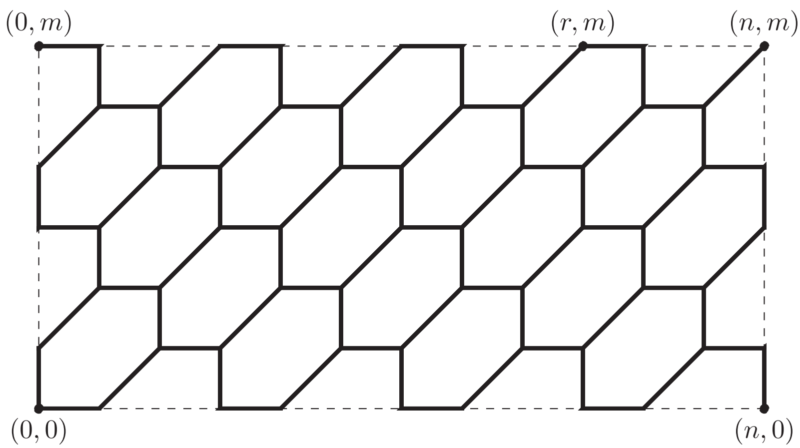

Figure 7.5: All deformations parameters T_e are equated since $h^0(\mathcal{T}^1_{\mathbf{P}(M)/k}) = 1$.

Example 7.5.6. We consider quotients of the tesselation by hexagons Figure 7.5. In this case, no matter what G is, all generators T_e will be equal. Hence $\mathcal{B}_M = \mathbf{A}^{d_M+1}$.

Example 7.5.7. We consider quotients of the tesselation $\{4,4\}$ by squares Figure 7.6. In this case, the number of generators $\{T_e\}_e$ depends on the tuple (n,m,r). Indeed, since each valency is 4, the generators of $\mathcal{P}_M$ corrsponds to the orbits of the actions by $\mathbb{Z}\times 0$ on the set of vertical edges of Λ/G, and $0\times\mathbb{Z}$ on the horizontal edges. Thus there are m horizontal equivalence classes, and $\gcd(n,r)$ vertical equvalence classes (where $\gcd(n,0) := n$). Thus $\mathcal{B}_M \cong \mathbf{A}^{d_M+m+\gcd(n,r)}$.

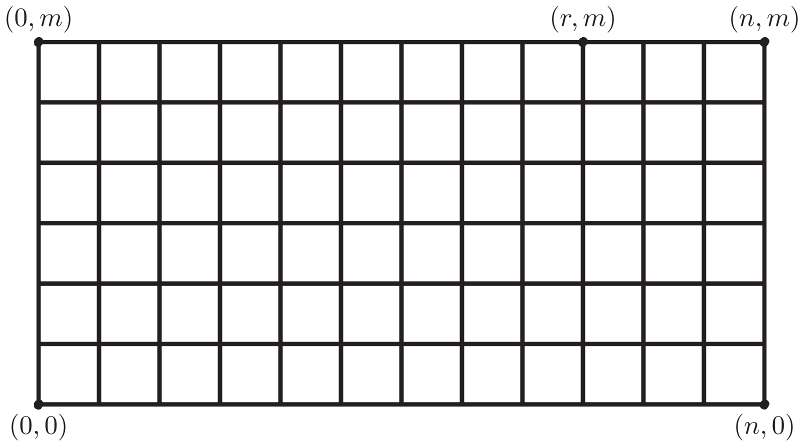

Figure 7.6: Deformation parameters T_e,$T_{e'}$ are equated if e and e' are adjacent and parallel. Thus $h^0(\mathbf{P}(M), \mathcal{T}^1_{\mathbf{P}(M)/k})$ is equal to the number of vertical and horizontal paths on the torus.

Example 7.5.8. We consider quotients of the following tesselation by squares Figure 7.7. Here the valencies are 3 and 6. The translation group is generated by the vectors $(2,1)$ and $(1,2)$. This gives the following requirements on (n,m,r): $3 \mid n$ and $3 \mid 2r-m$. By Remark 7.5.4, we obtain a description of $\mathcal{B}_M$ by dividing out by the ideal generated by $T_e - 1$ for every edge e of $\mathcal{N}_G$ which does not appear in M. Hence there is a single generator T_v for every vertex $v \in M_0$ of valency 3. The Macaulay2 function used is called **crossTesselation**.

7.6 Degenerations of K3 surfaces

Let M be a smooth polyhedral sphere. If $(\mathbf{P}(M), \mathcal{O}_M(1))$ has a smoothing, it is to K3 surfaces by Lemma 7.4.6. However, the conditions of Theorem 7.4.7 rarely holds in this case. But there is another way of produce examples of smoothable $\mathbf{P}(M)$, and hence proper families of K3 surfaces. Assume that all vertices of

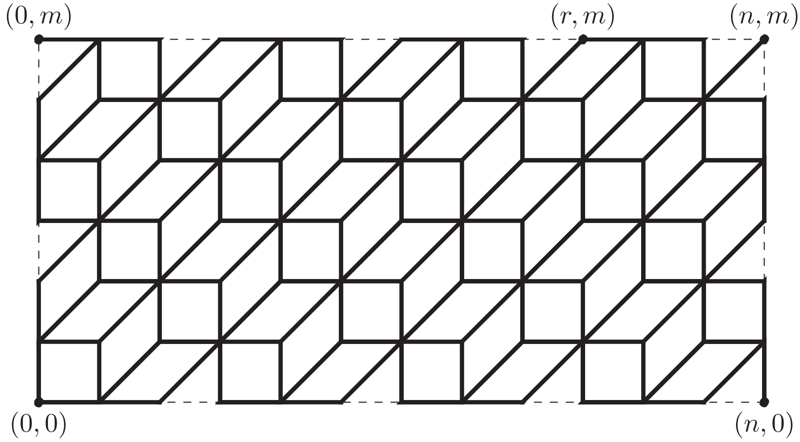

Figure 7.7: The universal base space is non-trivial, and as for Λ_0/G it is generally a union of varieties defined by binomial ideals.

(n,m,r) $\dim \mathcal{B}'_M$	**Components of $\mathcal{B}'_M$ for Example 7.5.8** **Degree of polarization**
$(3,1,2)$ 2	$\mathbf{A}^2$ 6
$(3,2,1)$ 3	$\mathrm{Cone}(\mathbf{P}^1 \times \mathbf{P}^1)$ 12
$(6,2,1)$ 5	$V(x_0y_0 - zw, x_1y_1 - zw, x_2y_2 - zw) \subseteq \mathbf{A}^8$ 12
$(6,2,4)$ 5	$V(x_4x_6 - x_5x_7, x_3x_6 - x_2x_7, x_8x_6 - x_1x_7, x_2x_4 - x_3x_5, x_1x_4 - x_8x_5, x_8x_2 - x_1x_3) \subseteq \mathbf{A}^8$ 12

M has valency 3. We will attempt to define a smoothing of $\mathbf{P}(M)$ by choosing local generic smoothings of $\mathbf{A}(M_v)$ on the form $\mathcal{U}_v = \mathrm{Spec}(k[t, x_0, x_1, x_1]/(x_0x_1x_1 + tF_v(x_0, x_1, x_1)))$, where $F_v(x_0, x_1, x_1)$ is a sum of monomials on the form $x_0^a x_1^b x_1^c$. Note that such schemes are automatically flat over $t = 0$. Then we will apply Remark 7.4.8 to glue these infinitesimally, which will induce a unique effective deformation, which may be a smoothing. Thus we will require that $F_{d_1(e)}$ maps to $F_{d_0(e)}$ under the transition functions π_e. This is computationally feasible for a given polyhedral sphere, but to reduce the complexity of the situation we will assume that F_v are equal for all v. Thus our problem becomes finding F which are invariant under π_e.

Definition 7.6.1. Let M be a smooth polyhedral sphere, such that every vertex has valency 3. Let t be any facet of M, and e any edge contained in t. Let e' and

e'' denote the edges of t adjacent to e. Then we assume that $n(e', t) = n(e'', t)$. We call such polygons $|t|$ *regular*. Moreover, if t' and t'' are the facets of M containing e' and e'' other than t, we furthermore assume that $n(e', t') = n(e'', t'')$. If M satisfies these conditions, we call it 3-*regular*. See Figure 7.8 for a picture describing the situation.

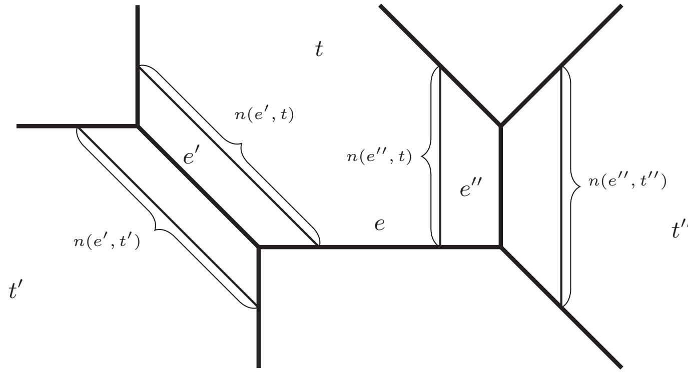

Figure 7.8: Note: the figure is slightly imprecise as the braced lines over each edge $\widetilde{e}$ has length $L(|\widetilde{e}|) + n(\widetilde{e}, t)$.

Examples of 3-regular polyhedral spheres are 3-regular Archimedean solids, where its regular polygons replaced by a suitable set of lattice polygons $\{P_1, P_2, P_3\}$, which are regular in the above sense. See Figure 7.9 for some basic examples of regular polygons. Let v be a vertex of M. Let e_0, e_1, e_2 be the edges

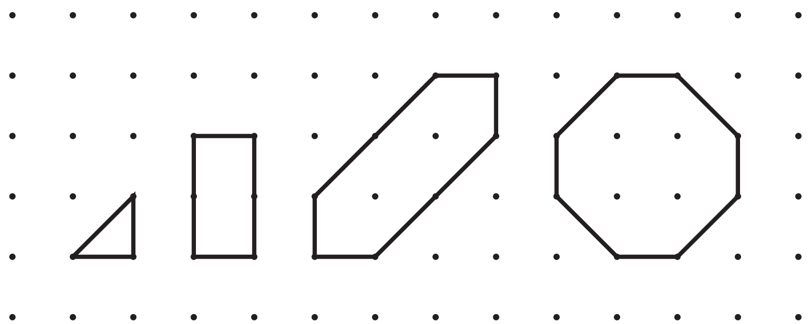

Figure 7.9: Regular polygons

and t_0, t_1, t_2 the facets of M which appear as the vertices and edges of $\mathrm{lk}_M(v)$, ordered cyclically in the sense of Figure 7.10. Consider the tuple

$$(n(e_0, t_0), n(e_1, t_0), n(e_1, t_1), n(e_2, t_1), n(e_2, t_2), n(e_0, t_2)). \tag{7.14}$$

By Definition 7.6.1, this is an invariant of M up to a permutation of the edges (and corresponding permutation of the facets). We will now describe a

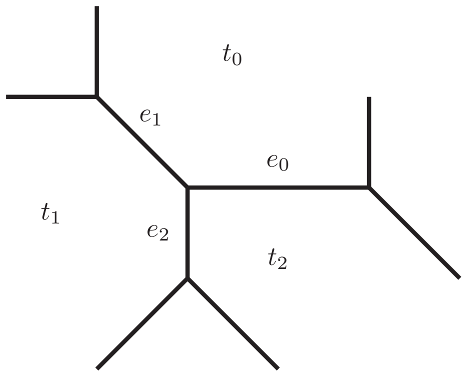

Figure 7.10: The local picture around a vertex of a 3-regular polyhedral sphere

procedure for finding invariant polynomials F. We will use the notation from Section 7.2.2. Let e be an edge of M. Using the isomorphism (7.9), we get that the ideal of $\mathcal{U}^{(n)}_{d_1(e)}|_{\mathbf{A}(M_e)}$ is on the form $(x_0x_1x_2 + tF(x_0, x_1, x_2))$, while the ideal of $\mathcal{U}^{(n)}_{d_0(e)}|_{\mathbf{A}(M_e)}$ is on the form $(x_0x_1x_2x_0^{n(e)-2} + tF(x_0^{-1}, x_1x_0^{n(e,s_1)}, x_2x_0^{n(e,s_2)})$, where $x_0 \in \Gamma(\mathrm{lk}_M(d_1(e)))$ is the variable corresponding to $(e, d_1(e)) \in \mathrm{lk}_M(d_1(e))$, and $e \to s_1$ and $e \to s_2$ are the facets of M_e corresponding to x_1 and x_2 respectively. A sufficient criterion for (7.12) is that these two ideals are equal for all edges e. Hence we get three requirements on F for each vertex v:

$$\begin{aligned} F(x_0, x_1, x_2) &= x_0^{2-n(e_0)} F(x_0^{-1}, x_1x_0^{n(e_0,t_0)}, x_2x_0^{n(e_0,t_2)}) \\ F(x_0, x_1, x_2) &= x_1^{2-n(e_1)} F(x_0x_1^{n(e_1,t_0)}, x_1^{-1}, x_2x_1^{n(e_1,t_1)}) \\ F(x_0, x_1, x_2) &= x_2^{2-n(e_2)} F(x_0x_2^{n(e_2,t_2)}, x_1x_2^{n(e_2,t_1)}, x_2^{-1}). \end{aligned} \tag{7.15}$$

Here e_i is the edge corresponding to x_i. We posit that F can be written as a sum of monomials $x_0^a x_1^b x_2^c$ for tuples (a, b, c) in some finite subset $S \subseteq \mathbb{N}^3$ containing $(0, 0, 0)$. The above transformations maps such a monomial to the following:

$$x_0^a x_1^b x_2^c \mapsto \begin{cases} x_0^{-a+n(e_0,t_0)b+n(e_0,t_2)c+2-n(e_0)} x_1^b x_2^c \\ x_0^a x_1^{-b+n(e_1,t_0)a+n(e_1,t_1)c+2-n(e_1)} x_2^c \\ x_0^a x_1^b x_2^{-c+n(e_2,t_2)a+n(e_2,t_1)b+2-n(e_2)} \end{cases}.$$

This can be translated to affine transformations $f_i : \mathbb{Z}^3 \to \mathbb{Z}^3$ given as follows:

$$\begin{aligned} f_1(a, b, c) &= (-a + n(e_0, t_0)b + n(e_0, t_2)c + 2 - n(e_0), b, c) \\ f_2(a, b, c) &= (a, -b + n(e_1, t_0)a + n(e_1, t_1)c + 2 - n(e_1), c) \\ f_3(a, b, c) &= (a, b, -c + n(e_2, t_2)a + n(e_2, t_1)b + 2 - n(e_2)) \end{aligned}$$

Of course, each affine transformation $f_i : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection through a hyperplane. We have to assume that $\boxed{n(e) \leq 2}$ for all edges e, so that f_i will map $(0, 0, 0)$ to a vector with non-negative entries. We will now generate a finite set $S \subseteq \mathbb{Z}^3$ of vectors with non-negative entries containing $(0, 0, 0)$, which is invariant under application of f_1, f_2, f_3. So let G be the subgroup of the group of isometries $\mathrm{Isom}(\mathbb{Z}^3)$ generated by $\{f_1, f_2, f_3\}$, and define $S = \{g(0, 0, 0)\}_{g \in G}$. If S consists of vectors with non-negative entries, then we may define $F = \sum_{(a,b,c) \in T} x_0^a x_1^b x_2^c$, and this F will satisfy the equations (7.15). Finally, suppose that $V(x_0x_1x_2 + tF(x_0, x_1, x_2))$ is generically smooth over $\mathrm{Spec}(k[t])$. Then we may apply Remark 7.4.8. This completes the procedure, and we may now investigate what configurations of 6-tuples on the form (7.14) can give rise to a suitable set $S \subseteq \mathbb{N}^3$ and polynomial F.

7.6.1 Resulting smoothings

The above procedure yields an invariant F for every possible 6-tuple on the form (7.14). We have listed in Table 7.3 each situation where the polyhedral complex M is realizable as an Archimedean solid using regular polygons. Note that M is not necessarily the boundary of a 3-dimensional lattice polytope; at least not a priori. The list is a selection of the output of the function **regularDeformation** from Appendix C.0.2. To see that each tuple is realizable, one simply compares the listed tuple with the tuple corresponding to the polyhedral complex obtained from that Archimedean solid with its facets replaced by regular polygons. Note that there are two non-equivalent listings of the truncated cuboctahedron. The degree is computed using the Riemann-Roch polarization degree formula $L^2 = 2(H^0(X, \mathcal{O}_X(1)) - 2)$ for K3 surfaces. Similarly to before, by Proposition 3.6.3 we have that the degree L^2 is equal to $2(L(M) - 2)$. The number of lattice points $L(M)$ is easily calculated. The faces present in these examples are triangles, squares, hexagons and octagons. It is probable that the Archimedean solids containing decagons are smoothable as well, although one would have to use different polynomials F_v for each vertex to construct such families.

Table 7.3: $(\mathbf{P}(M), \mathcal{O}_M(1))$ has a proper 1-parameter smoothing to polarized K3 surfaces

6-tuple **M is boundary of**	**Polynomial F** **Degree of polarization**
$(-1,-1,-1,-1,-1,-1)$ Tetrahedron	$x_0^4+x_1^4+x_2^4+1$ $L^2=4$
$(0,0,0,0,-1,-1)$ Triangular prism	$x_0^3x_1^2+x_1^2x_2^3+x_0^3+x_2^3+x_1^2+1$ $L^2=8$
$(0,0,0,0,0,0)$ Cube	$x_0^2x_1^2x_2^2+x_0^2x_1^2+x_0^2x_2^2+x_1^2x_2^2+x_0^2+x_1^2+x_2^2+1$ $L^2=12$
$(1,1,0,0,0,0)$ Hexagonal prism	$x_0^2x_1^2x_2^2+x_0^2x_1x_2^2+x_0x_1^2x_2^2+x_0^2x_1^2+x_0^2x_1$ $+x_0x_1^2+x_0x_2^2+x_1x_2^2+x_2^2+x_0+x_1+1$ $L^2=24$
$(1,1,1,1,-1,-1)$ Truncated tetrahedron	$x_0^2x_1^2x_2^2+x_0^2x_1^2+x_1^2x_2^2+x_0^2+x_2^2+1$ $L^2=28$
$(1,2,0,0,0,0)$ Octahedral prism	$x_0^2x_1^2x_2^2+x_0x_1^2x_2^2+x_0^2x_1^2+x_0x_1^2+x_0x_2^2+x_2^2$ $+x_0+1$ $L^2=44$
$(1,1,1,1,0,0)$ Truncated octahedron	$x_0^2x_1^2x_2^2+x_0^2x_1^2x_2+x_0x_1^2x_2^2+x_0^2x_1x_2+x_0x_1^2x_2$ $+x_0x_1x_2^2+x_0x_1+x_0x_2+x_1x_2+x_0+x_2+1$ $L^2=60$
$(2,1,1,2,0,0)$ Truncated cube	$x_0^2x_1^2x_2^2+x_0^2x_1^2x_2+x_0x_1^2x_2^2+x_0^2x_1x_2+x_0x_1x_2^2$ $+x_0^2x_1+x_1x_2^2+x_0x_1+x_1x_2+x_0+x_2+1$ $L^2=92$
$(1,2,0,0,1,1)$ Truncated cuboctahedron	$x_0^2x_1^2x_2^2+x_0^2x_1^2x_2+x_0x_1^2x_2+x_0x_2+x_2+1$ $L^2=152$
$(2,1,1,1,0,0)$ Truncated cuboctahedron	$x_0^2x_1^2x_2^2+x_0^2x_1^2x_2+x_0^2x_1x_2+x_1x_2+x_2+1$ $L^2=152$

Appendices

Appendix A

Kan extensions

We refer to [Mac98, Chapter X, Section 3] for a treatment on Kan extensions. We will briefly summarize the construction, in particular the case of left Kan extensions along Yoneda embeddings, and include some results from [IK86] on monoidal Kan extensions.

Definition A.0.1. Given categories A, B and C, and functors $F : A \to C$, $I : A \to B$, a *left Kan extension of F along I* is a functor $\mathrm{Lan}_I(F) : B \to C$ together with a natural transformation $\eta_F : F \to \mathrm{Lan}_I(F)I$ called the universal transformation, written

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "I"'] \arrow[dr, phantom, "\Rightarrow^{\eta_F}"] & C \\ B \arrow[ur, "\mathrm{Lan}_I(F)"'] & \end{tikzcd} \tag{A.1}$$

satisfying the following universal property: Given any functor $M : B \to C$ together with a natural transformation $\mu : F \to MI$, there exists a unique natural transformation $\delta : \mathrm{Lan}_I(F) \to M$ such that the diagram

$$\begin{tikzcd} & \mathrm{Lan}_I(F)I \arrow[dr, "\delta I"] & \\ F \arrow[ur, "\eta_F"] \arrow[rr, "\mu"] & & MI \end{tikzcd} \tag{A.2}$$

commutes.

The left Kan extension $\mathrm{Lan}_I(F)$ equipped with its universal transformation is unique up to natural equivalence. Whenever C is *cocomplete*, i.e. has all colimits, then every functor $F : A \to C$ has a left Kan extension. The universal transformation $\eta_F : F \to \mathrm{Lan}_I(F)I$ is natural in natural transformations $\phi : F \to G$, inducing a functor on functor categories $\mathrm{Lan}_I : [A, C] \to [B, C]$ which is left adjoint to the precomposition functor $I^* : [B, C] \to [A, C]$. It is a general fact that left adjoint functors are *cocontinuous*, i.e. preserves all colimits (see [Mac98, Chapter V, Theorem 2]). Thus left Kan extensions are always cocontinuous when C is cocomplete. If I is fully faithful, then η_F is a natural equivalence and Lan_I is fully faithful as well.

A.0.1 Construction of the left Kan extension

Assume that A and B are small categories. Given a functor $F : A \to C$, the left Kan extension functor $\mathrm{Lan}_I(F) : B \to C$ along with the universal transformation

$\eta_F : F \to \mathrm{Lan}_I(F)I$ can be constructed as follows. For each object $b \in B$, consider the comma category $(I \downarrow b)$ consisting of pairs $(a, f : I(a) \to b)$ for objects $a \in A$, and arrows $f : I(a) \to b$. An arrow $(a, f : I(a) \to b) \to (a', f' : I(a') \to b)$ consists of an arrow $g : a \to a'$ in A such that $f' \circ I(g) = f$.

For each $b \in B$, the functor $F : A \to C$ induces a diagram $F_b : (I \downarrow b) \to C$, defined by $F_b(a, f) = F(a)$, and $F_b\,(g : (a, f) \to (a', f')) = F(g)$. We define $\mathrm{Lan}_I(F)(b)$ as the colimit $\varinjlim F_b$ of this diagram. The colimit comes equipped with the universal natural transformation $\tau_b : F_b \to \mathrm{Lan}_I(F)(b)$, where $\mathrm{Lan}_I(F)(b)$ is regarded as the constant functor on $(I \downarrow b)$ with value $\mathrm{Lan}_I(F)(b)$. For each $a \in A$, consider the natural transformation $\tau_{I(a)} : F_{I(a)} \to \mathrm{Lan}_I(F)(I(a))$. We define $\eta_F : F \to \mathrm{Lan}_I(F)$ by $\eta_F(a) = \tau_{I(a)}(a, \mathrm{id}_{I(a)}) : F(a) \to \mathrm{Lan}_I(F)(I(a))$.

Consider the functor $(I \downarrow -) : B \to \mathrm{Cat}$, sending any $b \in B$ to the category $(I \downarrow b)$, and any arrow $h : b \to b'$ to the functor $(I \downarrow h) : (I \downarrow b) \to (I \downarrow b')$. The functor $(I \downarrow h)$ is defined by mapping any object (a, f) to $(a, h \circ f)$, and any arrow $g : (a, f) \to (a', f')$ to $g : (a, h \circ f) \to (a', h \circ f')$. Let $h : b \to b'$ be an arrow in B. Then the natural transformation $\tau_{b'} : F_{b'} \to \mathrm{Lan}_I(F)(b')$ yields a natural transformation $\tau_{b'}(I \downarrow h) : F_b \to \mathrm{Lan}_I(F)(b')$. The universal property of τ_b induces the arrow $\mathrm{Lan}_I(F)(h) : \mathrm{Lan}_I(F)(b) \to \mathrm{Lan}_I(F)(b')$.

A.0.2 Discrete fibrations

Definition A.0.2. Fix a small category C. Then a *category over* C is a small category U equipped with a functor $p : U \to C$, and is denoted (U, p). A *discrete fibration over* C is a category (U, p) over C such that for every object $s \in U$ and arrow $f : c \to p(s)$ in C, there exists a unique lifting $h : t \to s$ of f along s. This defines the category $\mathrm{Fib}(C)$ of discrete fibrations over C as a full subcategory of $(\mathrm{Cat} \downarrow C)$.

Let $X : C^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf on C. Consider the category of objects $I(X) = (Y_C \downarrow X)$ of X as a category over C via the projection functor $p_X : I(X) \to C$ defined by $(c, s \in X(c)) \mapsto c$ on objects, and $(f : (c', t) \to (c, s)) \mapsto (f : c' \to c)$ on arrows. The following fact is well-known.

Proposition A.0.3. *The assignment $X \mapsto (I(X), p_X)$ defines an equivalence of categories $I : \mathrm{Pre}(C) \xrightarrow{\cong} \mathrm{Fib}(C)$.*

A.0.3 Yoneda extensions

Definition A.0.4. Given a functor $F : A \to B$, a *Yoneda extension of* F is a left Kan extension of $F : A \to B$ along $Y_A : A \to \mathrm{Pre}(A)$.

Of particular importance is the Yoneda extension of a composition $A \xrightarrow{F} B \xrightarrow{Y_B} \mathrm{Pre}(B)$ along the Yoneda embedding $Y_A : A \to \mathrm{Pre}(A)$. In fact given a presheaf $X : A^{\mathrm{op}} \to \mathrm{Set}$, then using the general construction of the left Kan extension, we have the following simple formula for the Yoneda extension

$\widehat{X} = \mathrm{Lan}_{Y_A}(Y_B \circ F)(X)$:

$$\widehat{X} = \varinjlim_{A(-,a)\to X} B(-, F(a)). \tag{A.3}$$

Notably, for representable functors we have $\widehat{A(-,a)} = B(-, F(a))$. There is an alternative way of describing $\widehat{X} : B^{\mathrm{op}} \to \mathrm{Set}$; it is in fact a left Kan extension itself:

Lemma A.0.5. *In the situation above, the functor* $\widehat{X} = \mathrm{Lan}_{Y_A}(Y_B \circ F)(X)$ *is a left Kan extension of* $X : A^{\mathrm{op}} \to \mathrm{Set}$ *along* $F^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$*. In other words, there is a natural equivalence of functors* $\mathrm{Lan}_{Y_A}(Y_B \circ F) \cong \mathrm{Lan}_{F^{\mathrm{op}}}$*.*

Proof. Note first that since Y_A is fully faithful, the universal transformation $Y_B \circ F \to \mathrm{Lan}_{Y_A}(Y_B \circ F) \circ Y_A$ is a natural equivalence. The adjunction between $\mathrm{Lan}_{F^{\mathrm{op}}}$ and $(F^{\mathrm{op}})^*$ yields a natural bijection $\mathrm{Hom}(\mathrm{Lan}_{F^{\mathrm{op}}}(A(-,a)), Z) \cong \mathrm{Hom}(A(-,a), Z \circ F^{\mathrm{op}}))$ for every object $a \in A$ and preshaf $Z : B^{\mathrm{op}} \to \mathrm{Set}$. By the Yoneda lemma, there are natural bijections $\mathrm{Hom}(A(-,a), Z \circ F^{\mathrm{op}})) \cong Z_{F(a)} \cong \mathrm{Hom}(B(-, F(a)), Z)$. It follows that there is a natural bijection $\mathrm{Lan}_{F^{\mathrm{op}}}(A(-,a)) \cong B(-, F(a))$. I.e, there are natural equivalences $\mathrm{Lan}_{F^{\mathrm{op}}} \circ Y_A \cong Y_B \circ F \cong \mathrm{Lan}_{Y_A}(Y_B \circ F) \circ Y_A$. Since $\mathrm{Lan}_{F^{\mathrm{op}}}$ is cocontinuous, using the fact that every presheaf is a colimit of representable presheaves results in an equivalence $\mathrm{Lan}_{F^{\mathrm{op}}} \cong \mathrm{Lan}_{Y_A}(Y_B \circ F)$. $\square$

When $F : A \to B$ is a discrete fibration, the functor $\widehat{X}$ has a particularly nice description.

Lemma A.0.6. *In the situation of Lemma A.0.5, assume additionally that F is a discrete fibration. Then,*

a) *for every $b \in B$, $\widehat{X}(b) = \coprod_{b=F(a)} X(a)$;*

b) *the universal transformation $\eta_X : X \to \widehat{X} \circ F^{op}$ is given on each $a \in A$ by the inclusion of the term $X(a)$ into $\widehat{X}(F(a))$;*

c) *for each arrow $f : b' \to b$ in B, the map $\widehat{X}(f) : \widehat{X}(b) \to \widehat{X}(b')$ is given on components as $X(g) : X(a) \to X(a')$ for each $a \in A$ with $F(a) = b$ and lifting $g : a' \to a$ of f;*

d) *if $\phi : X \to Y$ is a natural transformation, the induced transformation $\widehat{\phi} : \widehat{X} \to \widehat{Y}$ is for each $b \in B$ given on components by $\phi_a : X_a \to Y_a$ for each $a \in A$ with $F(a) = b$.*

Proof. a) By the general construction we have that $\widehat{X}(b) = \varinjlim_{b \xrightarrow{f} F(c)} X(c)$, where the colimit is taken over the comma category $(F^{\mathrm{op}} \downarrow b) \cong (b \downarrow F)$. Every arrow $f : b \to F(c)$ lifts uniquely to an arrow $h : a \to c$, where $F(a) = b$. This induces maps $X(h) : X(c) \to X(a)$. Let $g : (c', f') \to (c, f)$ be any arrow in $(b \downarrow F)$, and consider the respective liftings $h : a \to c$ and $h' : a' \to c'$ of f and

f'. Since $F(g) \circ f' = f$, then $g \circ h' = h$ by uniqueness of lifts. Hence $a = a'$, and the composition $X(c) \xrightarrow{X(g)} X(c') \xrightarrow{X(h')} X(a)$ is equal to $X(h)$. It follows that $\coprod_{b=F(a)} X(a)$ is a cone to the diagram $(b \downarrow F) \to \mathrm{Set}$. It is clearly the initial one, since each $X(a)$ already appears in the colimit corresponding to the identity arrow $b \to F(a)$. b) follows immediately by the construction of $\tau_{F(a)}$. c) follows from b) and naturality of the universal transformation $\eta_X : X \to \widehat{X} \circ F^{op}$. d) follows immediately from b) and naturality of η_X in X. □

A.0.4 Monoidal Kan extensions

We refer to [Mac98, Chapter XI] for the full definition of a monoidal category, monoidal functors and monoidal natural transformations. We will refer to [IK86] for further results. A cocomplete monoidal category D is called *monoidally cocomplete* if for each $d \in D$, the functors $- \otimes d, d \otimes - : D \to D$ are both cocontinuous. A cartesian closed category is monoidally cocomplete with respect to the product operation, since these functors are left-adjoints.

Definition A.0.7 ([IK86, Section 4])**.** Let $(C, \otimes, 1_C)$ be a monoidal category. Then the *Day convolution product* $* : \mathrm{Pre}(C) \times \mathrm{Pre}(C) \to \mathrm{Pre}(C)$ is defined as the left Kan extension of the composite functor $C \times C \xrightarrow{\otimes} C \xrightarrow{Y_C} \mathrm{Pre}(C)$ along $Y_C \times Y_C : C \times C \to \mathrm{Pre}(C) \times \mathrm{Pre}(C)$.

The convolution product $X * Y : C^{\mathrm{op}} \to \mathrm{Set}$ can be written as

$$\varinjlim_{C(-,c_1)\to X, C(-,c_2)\to Y} C(-, c_1 \otimes c_2). \tag{A.4}$$

The Day convolution product gives the presheaf category $\mathrm{Pre}(C)$ the structure of a monoidal category with unit $Y_C(1_C) = C(-, 1_C)$.

Proposition A.0.8. *Let C be a monoidal category. Then,*

a) *the presheaf category* $\mathrm{Pre}(C)$ *is monoidally cocomplete under the Day convolution operation, and the Yoneda embedding* $Y_C : C \to \mathrm{Pre}(C)$ *is a strong monoidal functor of monoidal categories;*

b) *If a monoidal category D is monoidally cocomplete, and $F : C \to D$ is a strong monoidal functor, then the Yoneda extension* $\widehat{F} : \mathrm{Pre}(C) \to D$ *is the unique (up to monoidal equivalence) cocontinuous strong monoidal functor such that $\widehat{F} \circ Y_C$ and F are monoidally equivalent. The monoidal equivalence is given by the universal equivalence $\eta : F \to \widehat{F} \circ Y_C$ associated to $\widehat{F}$ as the left Kan extension of F.*

Proof. a) follows from [IK86, Proposition 4.1], and b) follows from [IK86, Theorem 5.1]. □

Note that Set is a symmetric monoidal category via the cartesian product operation $\mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$, with the unit being the one-point set. It is also cartesian closed, hence monoidally cocomplete. Using this, there is an alternative way of describing the convolution product more explicitly, analogously to Lemma A.0.5:

Lemma A.0.9. *Let* $(C, \otimes, 1_C)$ *be a monoidal category. Then for each pair of presheaves* $X, Y \in \mathrm{Pre}(C)$*, the Day convolution product* $X * Y : C^{\mathrm{op}} \to \mathrm{Set}$ *is the left Kan extension of the composition* $X \times Y : C^{\mathrm{op}} \times C^{\mathrm{op}} \xrightarrow{(X,Y)} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}$ *along* $\otimes^{\mathrm{op}} : C^{\mathrm{op}} \times C^{\mathrm{op}} \to C^{\mathrm{op}}$*. In other words, there is a natural equivalence of functors* $\mathrm{Lan}_{\otimes^{\mathrm{op}}} \cong \mathrm{Lan}_{Y_C \times Y_C}(Y_C \circ \otimes)$.

Proof. Note first that since $Y_C \times Y_C$ is fully faithful, the universal transformation $Y_C \circ \otimes \to \mathrm{Lan}_{Y_C \times Y_C}(Y_C \circ \otimes) \circ (Y_C \times Y_C)$ is a natural equivalence. The adjunction between $\mathrm{Lan}_{\otimes^{\mathrm{op}}}$ and $(\otimes^{\mathrm{op}})^*$ induces a natural bijection

$$\mathrm{Hom}(\mathrm{Lan}_{\otimes^{\mathrm{op}}}(C(-, c_1) \times C(-, c_2)), Z) \cong \mathrm{Hom}(C(-, c_1) \times C(-, c_2), Z \circ \otimes^{\mathrm{op}})) \tag{A.5}$$

for every pair of objects $c_1, c_2 \in C$ and presheaf $Z : C^{\mathrm{op}} \times C^{\mathrm{op}} \to \mathrm{Set}$. Via a monoidal version of the Yoneda lemma, there are natural bijections

$$\mathrm{Hom}(C(-, c_1) \times C(-, c_2), Z \circ \otimes^{\mathrm{op}})) \cong Z_{c_1 \otimes c_2} \cong \mathrm{Hom}(C(-, c_1 \otimes c_2), Z) \tag{A.6}$$

(the first bijection is given by $\phi \mapsto \phi_{(c_1, c_2)}((\mathrm{id}_{c_1}, \mathrm{id}_{c_2})))$. It follows that $\mathrm{Lan}_{\otimes^{\mathrm{op}}}(C(-, c_1) \times C(-, c_2)) \cong C(-, c_1 \otimes c_2)$. I.e., there are natural equivalences

$$\mathrm{Lan}_{\otimes^{\mathrm{op}}} \circ (Y_C \times Y_C) \cong Y_C \circ \otimes \cong \mathrm{Lan}_{Y_C \times Y_C}(Y_C \circ \otimes) \circ (Y_C \times Y_C). \tag{A.7}$$

Now, $\mathrm{Lan}_{\otimes^{\mathrm{op}}}$ is cocontinuous and both Set and $\mathrm{Pre}(C)$ are monoidally cocomplete (by Proposition A.0.8 a)), so writing the presheaves in both arguments as colimits of representables shows the equivalence $\mathrm{Lan}_{\otimes^{\mathrm{op}}} \cong \mathrm{Lan}_{Y_C \times Y_C}(Y_C \circ \otimes)$. □

Appendix B

Milnor patching

B.1 Milnor patching for flat and projective modules

In this section we will primarily summarize the procedure of patching projective and flat modules per [Fer03]. The statement below is a generalization of Milnor patching per [Mil71], extending the case of projective modules to that of flat modules as well.

Theorem B.1.1 ([Fer03, Theorem 2.2]). *Let*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_2 \arrow[d, "j_2"] \\ R_1 \arrow[r, "j_1"] & R_3 \end{tikzcd} \tag{B.1}$$

be a pullback square of rings, with j_1 surjective. This is called a Milnor square*. Let P_1, P_2 be projective (resp. flat) modules over R_1 and R_2 respectively, and let $h : R_3 \otimes_{R_2} P_2 \to R_3 \otimes_{R_1} P_1$ be an isomorphism. Consider the pullback square*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P_2 \arrow[d, "h\circ(1\otimes \mathrm{id})"] \\ P_1 \arrow[r, "1\otimes \mathrm{id}"] & R_3 \otimes_{R_1} P_1 \end{tikzcd} \tag{B.2}$$

of R-modules yielding an R-module P. Then,

a) *P is a projective (resp. flat) R-module. Moreover, if P_1 and P_2 are finitely generated over R_1 and R_2 respectively, then P is finitely generated over R;*

b) *the modules P_1 and P_2 are isomorphic to $R_1 \otimes_R P$ and $R_2 \otimes_R P$ respectively via the canonical maps;*

c) *every projective (resp. flat) R-module arise in this way for appropriately chosen P_1, P_2 and h.*

The following lemma is easily verified.

Lemma B.1.2. *Given a Milnor square like* (B.1)*, then the following sequences of R-modules are exact:*

$$0 \to \ker j_1 \to R \to R_2 \to 0, \tag{B.3}$$

$$0 \to R \to R_1 \times R_2 \to R_3 \to 0. \tag{B.4}$$

B.2 Milnor squares of schemes

In this section we will show that certain Milnor squares of graded rings induces pushouts of projective schemes. See [Sch05] and [Fer03] for similar results of this type, and also [Fer03, p. 6.2] for an example of how a pushout of projective schemes is not projective in general.

Proposition B.2.1. *Let*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_1 \arrow[r] & R_3 \end{tikzcd}$$

be a Milnor square of finitely generated positively graded k-algebras with $R_1 \to R_3$ surjective and $R_2 \to R_3$ finite.

a) *for any graded projective R-module P and homogeneous element $x \in R$, the induced diagram*

$$\begin{tikzcd} P_{(x)} \arrow[r] \arrow[d] & (P \otimes_R R_2)_{(x_2)} \arrow[d] \\ (P \otimes_R R_1)_{(x_1)} \arrow[r] & (P \otimes_R R_3)_{(x_3)} \end{tikzcd} \tag{B.5}$$

is a pullback square, where x_1, x_2, x_3 are the respective images of x;

b) *The induced diagram of schemes*

$$\begin{tikzcd} \mathrm{Proj}(R) & \mathrm{Proj}(R_2) \arrow[l] \\ \mathrm{Proj}(R_1) \arrow[u] & \mathrm{Proj}(R_3) \arrow[l] \arrow[u] \end{tikzcd} \tag{B.6}$$

is a pushout square of schemes.

c) *More generally, if $U \subseteq \mathrm{Proj}(R)$ is an open subscheme, then the induced diagram*

$$\begin{tikzcd} U & U_2 \arrow[l] \\ U_1 \arrow[u] & U_3 \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout square of schemes, where U_1, U_2, U_3 are the respective preimages of U.

Proof. a) Tensoring the exact sequence (B.4) with P and then localizing yields an exact sequence

$$0 \to P_{(x)} \to (P \otimes_R R_1)_{(x_1)} \times (P \otimes_R R_2)_{(x_2)} \to (P \otimes_R R_3)_{(x_3)} \to 0.$$

It follows that (B.5) is a pullback square.

b) Let X_1, X_2, X_3 be projective schemes, $X_3 \to X_1$ a closed immersion and $X_3 \to X_2$ a finite morphism. Then the conditions of [Fer03, Theorem 7.1] are satisfied, so that the amalgated sum $X' = X_1 \cup_{X_3} X_2$ of ringed spaces is a scheme, and the induced morphisms $X_i \to X'$ are morphisms of schemes. See [Fer03, Scolie 4.3] for a precise definition. As a topological space, X' is just the pushout $X_1 \cup_{X_3} X_2$. The structure sheaf on X' is defined as the pullback $\mathcal{O}_{X_1} \times_{\mathcal{O}_{X_3}} \mathcal{O}_{X_2}$. If $U \subseteq X'$ is an open subset, then one easily observes that U is the amalgated sum $U_1 \cup_{U_3} U_2$ of ringed spaces, where each U_i is the restriction of U to X_i. If each $X_i = \mathrm{Spec}(A_i)$ is affine, then $X' = \mathrm{Spec}(A_1 \times_{A_3} A_2)$. Crucially, the amalgated sum is a pushout in the category of ringed spaces. In the current situation, one can without much difficulty show that X' is a pushout in the category of locally ringed spaces, and therefore in the category of schemes as well.

Now, let $X = \mathrm{Proj}(R)$, and define $X_i = \mathrm{Proj}(R_i)$. Then via (B.6), X is a cone to the diagram. Thus there is a uniquely induced morphism $f : X' = X_1 \cup_{X_3} X_2 \to X$ of schemes. Let $x \in R$ be any homogeneous element, and consider the distinguished open affine $D_+(x) \subseteq X$. Let $V = f^{-1}(D_+(x))$. By a), we have $D_+(x) = D_+(x_1) \cup_{D_+(x_3)} D_+(x_2)$ (choose $P = R$). By the above discussion, this amalgated sum characterizes V as well. Hence $f|_V : V \to D_+(x)$ is an isomorphism, so it follows that $f : X' \to X$ is an isomorphism.

c) follows immediately from the proof of b). $\square$

B.3 Milnor patching for vector bundles

Patching of projective modules via Milnor diagrams of rings can be generalized to patching vector bundles via pushout squares of schemes. By a vector bundle we mean a locally free sheaf of finite rank. Let

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_2 \arrow[d] \\ R_1 \arrow[r] & R_3 \end{tikzcd} \tag{B.7}$$

be a Milnor square of graded rings as in Proposition B.2.1, and let

$$\begin{tikzcd} X & X_2 \arrow[l, "i_2"'] \\ X_1 \arrow[u, "i_1"] & X_3 \arrow[l, "j_1"] \arrow[u, "j_2"'] \end{tikzcd} \tag{B.8}$$

be the corresponding pushout square of schemes, where j_1 is a closed immersion. We call such a pushout square a *Milnor square of schemes*. Let i_3 denote the morphism $i_1 \circ j_1 = i_2 \circ j_2$. Since the diagrams (B.5) are pullback squares,

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "i_2^\sharp"] \arrow[d, "i_1^\sharp"] & i_{2*}\mathcal{O}_{X_2} \arrow[d] \\ i_{1*}\mathcal{O}_{X_1} \arrow[r] & i_{3*}\mathcal{O}_{X_3} \end{tikzcd} \tag{B.9}$$

is a pullback square of $\mathcal{O}_X$-modules. The following analogues of Theorem B.1.1 and Lemma B.1.2 can be verified locally.

Proposition B.3.1 (**Milnor patching for vector bundles**). *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles on X_1 and X_2 respectively, and let $h : j_2^*\mathcal{E}_2 \to j_1^*\mathcal{E}_1$ be an isomorphism. Let $\mathcal{E}_1 \to j_{1*}j_1^*\mathcal{E}_1$ and $\mathcal{E}_2 \to j_{2*}j_2^*\mathcal{E}_2$ be the canonical morphisms, and consider the pullback square of $\mathcal{O}_X$-modules*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "p_2"] \arrow[d, "p_1"] & i_{2*}\mathcal{E}_2 \arrow[d, "i_{2*}(j_{2*}h\circ\eta_2)"] \\ i_{1*}\mathcal{E}_1 \arrow[r, "i_{1*}\eta_1"] & i_{3*}j_1^*\mathcal{E}_1. \end{tikzcd} \tag{B.10}$$

Then,

a) *$\mathcal{E}$ is a vector bundle on X;*

b) *the canonical morphisms $i_1^*\mathcal{E} \to \mathcal{E}_1$ and $i_2^*\mathcal{E} \to \mathcal{E}_2$ are isomorphisms;*

c) *every vector bundle on X arise in this way for appropriately chosen $\mathcal{E}_1, \mathcal{E}_2$ and h.*

Lemma B.3.2. *Given a pullback square of $\mathcal{O}_X$-modules like* (B.9), *then the following sequences of $\mathcal{O}_X$-modules are exact:*

$$0 \to i_{1*} \ker j_1^\sharp \to \mathcal{O}_X \to i_{2*}\mathcal{O}_{X_2} \to 0, \tag{B.11}$$

$$0 \to \mathcal{O}_X \to i_{1*}\mathcal{O}_{X_1} \oplus i_{2*}\mathcal{O}_{X_2} \to i_{3*}\mathcal{O}_{X_3} \to 0. \tag{B.12}$$

In the situation of (B.8), let $\mathrm{VB}(X)$ denote the category of vector bundles on X. The fibered product of categories $\mathrm{VB}(X_1) \times_{\mathrm{VB}(X_3)} \mathrm{VB}(X_2)$ consists of pairs of finite vector bundles $(\mathcal{E}_1, \mathcal{E}_2)$ on X_1 and X_2 respectively, equipped with an isomorphism $h : j_2^*\mathcal{E}_2 \to j_1^*\mathcal{E}_1$ of vector bundles on X_3. A morphism of pairs $(\mathcal{E}_1, \mathcal{E}_2) \to (\mathcal{E}_1', \mathcal{E}_2')$ equipped with respective isomorphisms h and h', is a pair of morphisms $g_1 : \mathcal{E}_1 \to \mathcal{E}_1'$, $g_2 : \mathcal{E}_2 \to \mathcal{E}_2'$ such that the diagram

$$\begin{tikzcd} j_1^*\mathcal{E}_1 \arrow[d, "j_1^*g_1"] & j_2^*\mathcal{E}_2 \arrow[l, "h"'] \arrow[d, "j_2^*g_2"] \\ j_1^*\mathcal{E}_1' & j_2^*\mathcal{E}_2' \arrow[l, "h'"'] \end{tikzcd} \tag{B.13}$$

commutes. We define the functor

$$\beta : \mathrm{VB}(X) \to \mathrm{VB}(X_1) \times_{\mathrm{VB}(X_3)} \mathrm{VB}(X_2) \tag{B.14}$$

by $\mathcal{E} \mapsto (i_1^*\mathcal{E}, i_2^*\mathcal{E})$ equipped with the canonical isomorphism $h_{\mathcal{E}} : j_2^* i_2^* \mathcal{E} \to j_1^* i_1^* \mathcal{E}$, which is natural in $\mathcal{E}$. A morphism of vector bundles $f : \mathcal{E} \to \mathcal{F}$ on X maps to the morphism of pairs $(i_1^* f, i_2^* f) : (i_1^*\mathcal{E}, i_2^*\mathcal{E}) \to (i_1^*\mathcal{F}, i_2^*\mathcal{F})$, which satisfies the commutative diagram (B.13) by naturality of $h_{\mathcal{E}}$. We also define the functor

$$\theta : \mathrm{VB}(X_1) \times_{\mathrm{VB}(X_3)} \mathrm{VB}(X_2) \to \mathrm{VB}(X) \tag{B.15}$$

by mapping a pair $(\mathcal{E}_1, \mathcal{E}_2)$ equipped with an isomorphism h to the vector bundle $\mathcal{E}$ defined as the pullback of (B.10). $\mathcal{E}$ is characterized up to isomorphism, so θ is defined by making a choice of $\mathcal{E}$ for each pair $(\mathcal{E}_1, \mathcal{E}_2)$. A morphism of pairs $(f_1, f_2) : (\mathcal{E}_1, \mathcal{E}_2) \to (\mathcal{F}_1, \mathcal{F}_2)$ induces a morphism of their corresponding pullback squares on the form of (B.10). If θ maps each pair to $\mathcal{E}$ and $\mathcal{F}$ respectively, then the morphism of pullback squares induces a unique morphism $\theta(f_1, f_2) : \mathcal{E} \to \mathcal{F}$. Functoriality of β and θ is easily verified.

Proposition B.3.3. *There is an equivalence of categories*

$$\beta : \mathrm{VB}(X) \cong \mathrm{VB}(X_1) \times_{\mathrm{VB}(X_3)} \mathrm{VB}(X_2) : \theta.$$

Proof. We will establish natural equivalences $\eta : \mathrm{id} \to \theta \circ \beta$ and $\mu : \mathrm{id} \to \beta \circ \theta$ of the composites of β and θ with identity functors. Let $\mathcal{E}$ be a vector bundle on X. Then $\theta \circ \beta$ maps $\mathcal{E}$ to a vector bundle $\mathcal{E}'$ such that

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "p_2'"'] \arrow[d, "p_1'"'] & i_{2*}\mathcal{E}_2 \arrow[d] \\
i_{1*}\mathcal{E}_1 \arrow[r] & i_{3*}j_1^*\mathcal{E}_1
\end{tikzcd}$$

is a pullback square. However, $\mathcal{E}$ is the pullback of the same diagram (B.10), which means there is a unique isomorphism $\eta_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}'$ such that $p_1' \circ \eta_{\mathcal{E}} = p_1$ and $p_2' \circ \eta_{\mathcal{E}} = p_2$. Naturality of $\eta_{\mathcal{E}}$ is easily verified. Conversely, consider a pair $(\mathcal{E}_1, \mathcal{E}_2)$ equipped with an isomorphism $h : j_2^*\mathcal{E}_2 \to j_1^*\mathcal{E}_1$. Let $\mathcal{E} = \theta(\mathcal{E}_1, \mathcal{E}_2)$, and consider the pullback square (B.10). β maps $\mathcal{E}$ to the pair $(i_1^*\mathcal{E}, i_2^*\mathcal{E})$ equipped with the isomorphism $h_{\mathcal{E}} : j_2^* i_2^* \mathcal{E} \to j_1^* i_1^* \mathcal{E}$. Recall the induced isomorphisms $\psi_1 : i_1^*\mathcal{E} \to \mathcal{E}_1$ and $\psi_2 : i_2^*\mathcal{E} \to \mathcal{E}_2$ from Proposition B.3.1 b). To see that this defines an isomorphism $\mu_{(\mathcal{E}_1, \mathcal{E}_2)} = (\psi_1, \psi_2) : (i_1^*\mathcal{E}, i_2^*\mathcal{E}) \to (\mathcal{E}_1, \mathcal{E}_2)$, we have to verify that

$$\begin{tikzcd}
j_1^* i_1^* \mathcal{E} \arrow[d, "j_1^*\psi_1"] & j_2^* i_2^* \mathcal{E} \arrow[l, "h_{\mathcal{E}}"'] \arrow[d, "j_2^*\psi_2"] \\
j_1^*\mathcal{E}_1 & j_2^*\mathcal{E}_2 \arrow[l, "h"']
\end{tikzcd}$$

commutes. Locally, in the notation of Proposition B.2.1, this is the diagram

$$\begin{tikzcd}
P \otimes_{R_{(x)}} (R_3)_{(x_3)} \arrow[d] & P \otimes_{R_{(x)}} (R_3)_{(x_3)} \arrow[l] \arrow[d] \\
P_1 \otimes_{(R_1)_{(x_1)}} (R_3)_{(x_3)} & P_2 \otimes_{(R_2)_{(x_2)}} (R_3)_{(x_3)}. \arrow[l, "h"']
\end{tikzcd} \tag{B.16}$$

However, this is just the diagram (B.2) tensored with $(R_3)_{(x_3)}$. We conclude that (B.16) commutes. Next, let $(f_1, f_2) : (\mathcal{E}_1, \mathcal{E}_2) \to (\mathcal{F}_1, \mathcal{F}_2)$ be a morphism of pairs, and let $\mathcal{F} = \theta(\mathcal{F}_1, \mathcal{F}_2)$. By definition of $\theta(f_1, f_2)$, the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr] \arrow[dr, "\theta(f_1,f_2)"] \arrow[dd] & & i_{2*}\mathcal{E}_2 \arrow[d] \\
& \mathcal{F} \arrow[r] \arrow[d] & i_{2*}\mathcal{F}_2 \\
i_{1*}\mathcal{E}_1 \arrow[r] & i_{1*}\mathcal{F}_1 &
\end{tikzcd} \tag{B.17}$$

commutes. Naturality of $\mu_{(\mathcal{E}_1,\mathcal{E}_2)}$ follows easily. $\square$

We define the tensor product $(\mathcal{E}_1, \mathcal{E}_2) \otimes (\mathcal{F}_1, \mathcal{F}_2)$ of pairs of vector bundles (equipped with respective isomorphisms h_1,h_2) as $(\mathcal{E}_1 \otimes \mathcal{F}_1, \mathcal{E}_2 \otimes \mathcal{F}_2)$ equipped with the isomorphism h given by

$$j_2^*(\mathcal{E}_2 \otimes \mathcal{F}_2) \cong j_2^*\mathcal{E}_2 \otimes j_2^*\mathcal{F}_2 \xrightarrow{h_1 \otimes h_2} j_1^*\mathcal{E}_1 \otimes j_1^*\mathcal{E}_1 \otimes j_2^*\mathcal{F}_1 \cong j_1^*(\mathcal{E}_1 \otimes \mathcal{F}_1). \tag{B.18}$$

Proposition B.3.4. *There are natural isomorphisms*

a) $\beta(\mathcal{E} \otimes \mathcal{F}) \cong \beta(\mathcal{E}) \otimes \beta(\mathcal{F})$,

b) $\theta((\mathcal{E}_1, \mathcal{E}_2) \otimes (\mathcal{F}_1, \mathcal{F}_2)) \cong \theta(\mathcal{E}_1, \mathcal{E}_2) \otimes \theta(\mathcal{F}_1, \mathcal{F}_2)$.

Proof. a) Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X. Then there are isomorphisms $i_1^*(\mathcal{E} \otimes \mathcal{F}) \to i_1^*\mathcal{E} \otimes i_1^*\mathcal{F}$, and $i_2^*(\mathcal{E} \otimes \mathcal{F}) \to i_2^*\mathcal{E} \otimes i_2^*\mathcal{F}$, natural in $\mathcal{E}$ and $\mathcal{F}$. Clearly, the induced diagram

$$\begin{tikzcd}
j_2^* i_2^*(\mathcal{E} \otimes \mathcal{F}) \arrow[r, "h_{\mathcal{E}\otimes\mathcal{F}}"] \arrow[d] & j_1^* i_1^*(\mathcal{E} \otimes \mathcal{F}) \arrow[d] \\
j_2^* i_2^*\mathcal{E} \otimes j_2^* i_2^*\mathcal{F} \arrow[r, "h_{\mathcal{E}} \otimes h_{\mathcal{F}}"] & j_1^* i_1^*\mathcal{E} \otimes j_1^* i_1^*\mathcal{F}
\end{tikzcd} \tag{B.19}$$

commutes.

b) Let $(\mathcal{E}_1, \mathcal{E}_2)$ and $(\mathcal{F}_1, \mathcal{F}_2)$ be pairs of vector bundles with respective isomorphisms h_1, h_2, and let $\mathcal{H} = \theta(\mathcal{E}_1 \otimes \mathcal{F}_1, \mathcal{E}_2 \otimes \mathcal{F}_2)$. Then there exists a unique morphism $u : \mathcal{E} \otimes \mathcal{F} \to \mathcal{H}$ such that the diagram

$$
\begin{tikzcd}
\mathcal{E} \otimes \mathcal{F} \arrow[dr, "u"] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
& \mathcal{H} \arrow[r] \arrow[d] & i_{2*}(\mathcal{E}_2 \otimes \mathcal{F}_2) \arrow[r] \arrow[d] & i_{2*}\mathcal{E}_2 \otimes i_{2*}\mathcal{F}_2 \arrow[dd] \\
& i_{1*}(\mathcal{E}_1 \otimes \mathcal{F}_1) \arrow[r] \arrow[d] & i_{3*}j_1^*(\mathcal{E}_1 \otimes \mathcal{F}_1) \arrow[dr] & \\
& i_{1*}\mathcal{E}_1 \otimes i_{1*}\mathcal{F}_1 \arrow[rr] & & i_{3*}j_1^*\mathcal{E}_1 \otimes i_{3*}j_1^*\mathcal{F}_1
\end{tikzcd}
\tag{B.20}
$$

commutes. It remains to verify that $\mathcal{E} \otimes \mathcal{F}$ is the pullback of the large square of (B.20). This may be seen locally, and follows from Proposition B.2.1 a). Naturality is easily seen. □

The Picard group $\mathrm{Pic}(X)$ is the group of equivalence classes of the subcategory $\mathrm{VB}^1(X)$ of $\mathrm{VB}(X)$ consisting of vector bundles of rank 1. By Proposition B.3.3 and Proposition B.3.4, we may also consider $\mathrm{Pic}(X)$ as the group of equivalence classes of triples $(\mathcal{E}_1, \mathcal{E}_2, h) \in \mathrm{VB}^1(X_1) \times_{\mathrm{VB}^1(X_3)} \mathrm{VB}^1(X_2)$.

Proposition B.3.5. *There is an exact sequence of abelian groups*

$$
\begin{aligned}
1 \to H^0(X, \mathcal{O}_X^*) \to H^0(X_1, \mathcal{O}_{X_1}^*) \times H^0(X_2, \mathcal{O}_{X_2}^*) \to H^0(X_3, \mathcal{O}_{X_3}^*) \xrightarrow{\phi} \\
\mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2) \to \mathrm{Pic}(X_3),
\end{aligned}
$$

where ϕ is given by $h \mapsto$ isomorphism class of $(\mathcal{O}_{X_1}, \mathcal{O}_{X_2}, h)$, where $h \in H^0(X_3, \mathcal{O}_{X_3}^*)$ is considered as an isomorphism between $j_2^*\mathcal{O}_{X_2}$ and $j_1^*\mathcal{O}_{X_1}$.

Proof. The map ϕ is multiplicative by Proposition B.3.4. Exactness at $H^0(X, \mathcal{O}_X^*)$ and $H^0(X_1, \mathcal{O}_{X_1}^*) \times H^0(X_2, \mathcal{O}_{X_2}^*)$ follows from exactness of (B.12), and exactness at $\mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$ follows from Proposition B.3.3. It remains to show that $\ker \psi \subseteq \operatorname{im} \phi$, since the other inclusion is immediate. So let $\mathcal{E} \in \ker(\beta)$. Then $i_1^*\mathcal{E} \otimes (i_2^*\mathcal{E})^\vee$ is trivial, so there exists isomorphisms $g_1 : i_1^*\mathcal{E} \to \mathcal{O}_{X_1}$ and $g_2 : i_2^*\mathcal{E} \to \mathcal{O}_{X_2}$. Consider the isomorphism $h = j_2^* g_2 \circ \circ h_{\mathcal{E}}^{-1} \circ (j_1^* g_1)^{-1} : j_2^*\mathcal{O}_{X_2} \to j_1^*\mathcal{O}_{X_1}$, and note that (g_1, g_2) forms an isomorphism between $(i_1^*\mathcal{E}, i_2^*\mathcal{E}, h_{\mathcal{E}})$ and $(\mathcal{O}_{X_1}, \mathcal{O}_{X_2}, h)$. If we consider h as an element of $H^0(X_3, \mathcal{O}_{X_3}^*)$, then it is clear that $\phi(h)$ is equal to the isomorphism class $\mathcal{E}$. □

Appendix C

Code

C.0.1 Components of the base space

```
-- Adjusts a coordinate to fit into the fundamental
-- domain relative to the translation group G.
torus = (n,m,r) -> (
    f := (a,b) -> (
        if  b > m-1 then return f(a-r,b-m)
        else if b < 0 then return f(a+r,b+m)
        else if a > n-1 then return f(a-n,b)
        else if a < 0 then return f(a+n,b)
        else return {a,b}
        );
    return f;
    );

-- takes input the three integers (n,m,r) defining the fundamental domain
-- relative to G,  and outputs the ideal defining the closure of D(\prod T_e)
triangles = (n,m,r) -> (
    f := torus(n,m,r);
    A := QQ[t_{0,0}..t_{n-1,m-1},u_{0,0}..u_{n-1,m-1},v_{0,0}..v_{n-1,m-1}];
    B := A[s_{0,0}..s_{n-1,m-1},q_{0,0}..q_{n-1,m-1},w_{0,0}..w_{n-1,m-1}];
    I := ideal(0);
    for i from 0 to n-1 do (
        for j from 0 to m-1 do (
            I = I + ideal(t_{i,j}*v_(f(i,j-1))-v_{i,j}*t_(f(i-1,j)));
            I = I + ideal(t_{i,j}*u_{i,j}-u_(f(i-1,j-1))*t_(f(i-1,j)));
            I = I + ideal(v_{i,j}*u_{i,j}-u_(f(i-1,j-1))*v_(f(i,j-1)));
            I = I + ideal(s_{i,j}*t_{i,j}-1,q_{i,j}*u_{i,j}-1,w_{i,j}*v_{i,j}-1);
            );
        );
    g := map(B/I,A);
    return ker(g);
    );

-- Base changes the result from before by putting the appropriate generators
-- equal to 1.
squares = (n,m,r) -> (
    f := torus(n,m,r);
    A := QQ[t_{0,0}..t_{n-1,m-1},u_{0,0}..u_{n-1,m-1},v_{0,0}..v_{n-1,m-1}];
    B := A[s_{0,0}..s_{n-1,m-1},q_{0,0}..q_{n-1,m-1},w_{0,0}..w_{n-1,m-1}];
    I := ideal(0);
    for i from 0 to n-1 do (
        for j from 0 to m-1 do (
            I = I + ideal(t_(f(1+2*i+j,1+i+2*j))-1,
                v_(f(1+2*i+j,1+i+2*j))-1,u_(f(2*i+j,i+2*j))-1);
            I = I + ideal(t_{i,j}*v_(f(i,j-1))-v_{i,j}*t_(f(i-1,j)));
            I = I + ideal(t_{i,j}*u_{i,j}-u_(f(i-1,j-1))*t_(f(i-1,j)));
            I = I + ideal(v_{i,j}*u_{i,j}-u_(f(i-1,j-1))*v_(f(i,j-1)));
            I = I + ideal(s_{i,j}*t_{i,j}-1,q_{i,j}*u_{i,j}-1,w_{i,j}*v_{i,j}-1);
```

```
        );
    );
    g := map(B/I,A);
    return ker(g);
    );

-- Outputs the components of the closure of D(\prod T_e) in the
-- basic tesselation
basicTesselation = (n,m,r) -> (
    J := triangles(n,m,r);
    D := decompose(J);
    return D;
    );

-- Outputs the components of the closure of D(\prod T_e) in the
-- tesselation by squares
crossTesselation = (n,m,r) -> (
    J := squares(n,m,r);
    D := decompose(J);
    return D;
    );
```

C.0.2 Invariant polynomials

```
-- Returns the respective affine transformations f_i.
transform = (n2,n1,m0,m2,r1,r0) -> (
    affine = (i,l) -> (
        if (i == 1) then (
            a := 2-n2-n1-l_0+n1*l_1+n2*l_2;
            return {a,l_1,l_2};
            )
        else if (i == 2) then (
            b := 2-m0-m2-l_1+m0*l_0+m2*l_2;
            return {l_0,b,l_2};
            )
        else if (i == 3) then (
            c := 2-r1-r0-l_2+r1*l_1+r0*l_0;
            return {l_0,l_1,c};
            );
        );
    return affine;
    );

-- Checks whether a vector contains negative elements
neg = l -> (
    if (l_0 < 0) or (l_1 < 0) or (l_2 < 0) then return true else return false;
    );

-- Returns 0 or an invariant polynomial F such that V(x_0x_1x_2+tF)
-- is generically smooth
invariant = r -> (
    f := transform(r_5,r_0,r_1,r_2,r_3,r_4);
    L := {{0,0,0}};
    val := true;
    -- Loops through the list L containing {0,0,0} until it is invariant
    -- under the f_i's.
    while val do (
```

```
        L1 := L;
        val = false;
        for l in L do (
            l1 := f(1,l);
            l2 := f(2,l);
            l3 := f(3,l);
            if not member(l1,L1) then (L1 = L1 | {l1}; val = true);
            if not member(l2,L1) then (L1 = L1 | {l2}; val = true);
            if not member(l3,L1) then (L1 = L1 | {l3}; val = true);
            );
        if any(L1,neg) then val = false;
        L = L1;
        );
    -- Ensures that no negative coordinates appear.
    if not any(L,neg) then (
        R := QQ[x_0,x_1,x_2];
        F := 0;
        --  Defines the invariant F and checks that V(x_0x_1x_2+F) is smooth
        for l in L do F = F + x_0^(l_0)*x_1^(l_1)*x_2^(l_2);
        B := singularLocus(ideal(x_0*x_1*x_2+F));
        J := ker(map(B,R));
        if J == 1 then return F else return 0;
        )  else return 0;
    );

-- Outputs a list of 6-tuples and suitable invariant polynomials F.
regularDeformation = T -> (
    M := {};
    for s in T do (
        F := invariant(s);
        if F != 0 then M = M | {{s,F}};
        );
    return M;
    );

cycle = t -> {t_2,t_3,t_4,t_5,t_0,t_1};
pert = t -> {t_2,t_3,t_0,t_1,t_4,t_5};

-- Removes any elements from a list of 6-tuples which are duplicates in
-- the sense that they describe the same combinatorial situation.
ord = T -> (
    T1 := T;
    for t in T do (
        T1 = delete(t,T1);
        T1 = delete(pert(t),T1);
        T1 = delete(cycle(t),T1);
        T1 = T1 | {t};
        );
    return T1;
    );

-- Makes a list of all possible 6-tuples, and removes combinatorial duplicates
T = ord(toList({-1,-1,-1,-1,-1,-1}.. {2,2,2,2,2,2}));
-- Gives a list of all pairs of 6-tuples and corresponding suitable polynomials
M = regularDeformation(T);
for m in M do print m;
```

Bibliography

[AC04] Altmann, K. and Christophersen, J. A. "Cotangent cohomology of Stanley-Reisner rings". In: *Manuscripta Math.* 115.3 (2004), pp. 361–378.

[AC10] Altmann, K. and Christophersen, J. A. "Deforming Stanley-Reisner schemes". In: *Math. Ann.* 348.3 (2010), pp. 513–537.

[Ale02] Alexeev, V. "Complete moduli in the presence of semiabelian group action". In: *Ann. of Math. (2)* 155.3 (2002), pp. 611–708.

[Ale15] Alexeev, V. *Moduli of weighted hyperplane arrangements.* Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer, Basel, 2015, pp. vii+104.

[AN99] Alexeev, V. and Nakamura, I. "On Mumford's construction of degenerating abelian varieties". In: *Tohoku Math. J. (2)* 51.3 (1999), pp. 399–420.

[Art69] Artin, M. "Algebraization of formal moduli. I". In: *Global Analysis (Papers in Honor of K. Kodaira).* Univ. Tokyo Press, Tokyo, 1969, pp. 21–71.

[BE91] Bayer, D. and Eisenbud, D. "Graph curves". In: *Adv. Math.* 86.1 (1991). With an appendix by Sung Won Park, pp. 1–40.

[BG02] Bruns, W. and Gubeladze, J. "Polyhedral algebras, arrangements of toric varieties, and their groups". In: *Computational commutative algebra and combinatorics (Osaka, 1999).* Vol. 33. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2002, pp. 1–51.

[BG09] Bruns, W. and Gubeladze, J. *Polytopes, rings, and K-theory.* Springer Monographs in Mathematics. Springer, Dordrecht, 2009, pp. xiv+461.

[BKR08] Bruns, W., Koch, R., and Römer, T. "Gröbner bases and Betti numbers of monoidal complexes". In: *Michigan Math. J.* 57 (2008). Special volume in honor of Melvin Hochster, pp. 71–91.

[BN00] Bunge, M. and Niefield, S. "Exponentiability and single universes". In: *J. Pure Appl. Algebra* 148.3 (2000), pp. 217–250.

[BR05] Brun, M. and Römer, T. "Subdivisions of toric complexes". In: *J. Algebraic Combin.* 21.4 (2005), pp. 423–448.

[Bre93] Bredon, G. E. *Topology and geometry.* Vol. 139. Graduate Texts in Mathematics. Springer-Verlag, New York, 1993, pp. xiv+557.

[Chr10] Christophersen, J. A. *Deformations of equivelar Stanley-Reisner abelian surfaces.* 2010. eprint: arXiv:1003.4004.

[CLS11] Cox, D. A., Little, J. B., and Schenck, H. K. *Toric varieties.* Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841.

[Dol82] Dolgachev, I. "Weighted projective varieties". In: *Group actions and vector fields (Vancouver, B.C., 1981).* Vol. 956. Lecture Notes in Math. Springer, Berlin, 1982, pp. 34–71.

[Eis05] Eisenbud, D. *The geometry of syzygies.* Vol. 229. Graduate Texts in Mathematics. A second course in commutative algebra and algebraic geometry. Springer-Verlag, New York, 2005, pp. xvi+243.

[Eis95] Eisenbud, D. *Commutative algebra.* Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995, pp. xvi+785.

[Fer03] Ferrand, D. "Conducteur, descente et pincement". In: *Bull. Soc. Math. France* 131.4 (2003), pp. 553–585.

[FK72] Freyd, P. J. and Kelly, G. M. "Categories of continuous functors. I". In: *J. Pure Appl. Algebra* 2 (1972), pp. 169–191.

[GJ09] Goerss, P. G. and Jardine, J. F. *Simplicial homotopy theory.* Modern Birkhäuser Classics. Reprint of the 1999 edition [MR1711612]. Birkhäuser Verlag, Basel, 2009, pp. xvi+510.

[GKZ08] Gelfand, I. M., Kapranov, M. M., and Zelevinsky, A. V. *Discriminants, resultants and multidimensional determinants.* Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+523.

[GS03] Gross, M. and Siebert, B. "Affine manifolds, log structures, and mirror symmetry". In: *Turkish J. Math.* 27.1 (2003), pp. 33–60.

[GS06] Gross, M. and Siebert, B. "Mirror symmetry via logarithmic degeneration data. I". In: *J. Differential Geom.* 72.2 (2006), pp. 169–338.

[GS10] Gross, M. and Siebert, B. "Mirror symmetry via logarithmic degeneration data, II". In: *J. Algebraic Geom.* 19.4 (2010), pp. 679–780.

[GT80] Greco, S. and Traverso, C. "On seminormal schemes". In: *Compositio Math.* 40.3 (1980), pp. 325–365.

[GW78] Goto, S. and Watanabe, K. "On graded rings. I". In: *J. Math. Soc. Japan* 30.2 (1978), pp. 179–213.

[Har10] Hartshorne, R. *Deformation theory.* Vol. 257. Graduate Texts in Mathematics. Springer, New York, 2010, pp. viii+234.

[HS06] Huneke, C. and Swanson, I. *Integral closure of ideals, rings, and modules.* Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006, pp. xiv+431.

[IK86] Im, G. B. and Kelly, G. M. "A universal property of the convolution monoidal structure". In: *J. Pure Appl. Algebra* 43.1 (1986), pp. 75–88.

[Joh99] Johnstone, P. “A note on discrete Conduché fibrations”. In: *Theory Appl. Categ.* 5 (1999), No. 1, 1–11.

[Kol96] Kollár, J. *Rational curves on algebraic varieties.* Vol. 32. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996, pp. viii+320.

[LP11] Lü, Z. and Panov, T. “Moment-angle complexes from simplicial posets”. In: *Cent. Eur. J. Math.* 9.4 (2011), pp. 715–730.

[Mac98] Mac Lane, S. *Categories for the working mathematician.* Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314.

[Mat89] Matsumura, H. *Commutative ring theory.* Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1989, pp. xiv+320.

[Mil71] Milnor, J. *Introduction to algebraic K-theory.* Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971, pp. xiii+184.

[MS05] Miller, E. and Sturmfels, B. *Combinatorial commutative algebra.* Vol. 227. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xiv+417.

[Ols08] Olsson, M. C. *Compactifying moduli spaces for abelian varieties.* Vol. 1958. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2008, pp. viii+278.

[OY09] Okazaki, R. and Yanagawa, K. “Dualizing complex of a toric face ring”. In: *Nagoya Math. J.* 196 (2009), pp. 87–116.

[PRV04] Panov, T., Ray, N., and Vogt, R. “Colimits, Stanley-Reisner algebras, and loop spaces”. In: *Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001).* Vol. 215. Progr. Math. Birkhäuser, Basel, 2004, pp. 261–291.

[Rei76] Reisner, G. A. “Cohen-Macaulay quotients of polynomial rings”. In: *Advances in Math.* 21.1 (1976), pp. 30–49.

[Sch05] Schwede, K. “Gluing schemes and a scheme without closed points”. In: *Recent progress in arithmetic and algebraic geometry.* Vol. 386. Contemp. Math. Amer. Math. Soc., Providence, RI, 2005, pp. 157–172.

[Ser06] Sernesi, E. *Deformations of algebraic schemes.* Vol. 334. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, pp. xii+339.

[Sta87] Stanley, R. "Generalized H-vectors, intersection cohomology of toric varieties, and related results". In: *Commutative algebra and combinatorics (Kyoto, 1985)*. Vol. 11. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1987, pp. 187–213.

[Sta91] Stanley, R. P. "f-vectors and h-vectors of simplicial posets". In: *J. Pure Appl. Algebra* 71.2-3 (1991), pp. 319–331.

[Sta96] Stanley, R. P. *Combinatorics and commutative algebra*. Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. x+164.

[Stacks] Stacks Project Authors, T. *Stacks Project*. https://stacks.math.columbia.edu. 2018.

[Swa80] Swan, R. G. "On seminormality". In: *J. Algebra* 67.1 (1980), pp. 210–229.

[Tam94] Tamme, G. *Introduction to étale cohomology*. Universitext. Translated from the German by Manfred Kolster. Springer-Verlag, Berlin, 1994, pp. x+186.

[Tra70] Traverso, C. "Seminormality and Picard group". In: *Ann. Scuola Norm. Sup. Pisa (3)* 24 (1970), pp. 585–595.

[Vak06] Vakil, R. *Foundations of Algebraic Geometry Classes 53 and 54*. http://math.stanford.edu/~vakil/0506-216/216class5354.pdf. 2006.

[Vak17] Vakil, R. *Foundations of Algebraic Geometry*. http://math.stanford.edu/~vakil/216blog/FOAGjun0417public.pdf. 2017.

[Wei94] Weibel, C. A. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450.