Whitehead torsion of inertial h-cobordisms*

Bjørn Jahren and Sławomir Kwasik

Abstract

We study the Whitehead torsion of inertial h-cobordisms, continuing an investigation started in [12]. Of particular interest is a nested sequence of subsets of the Whitehead group, and a number of examples are given to show that these subsets are all different in general. The main new results are Theorems 2.5, 2.6 and 2.7. Proposition 5.2 is a partial correction to [12, Lemma 8.1].

1 Introduction

The h-cobordism theorem plays a crucial role in modern geometric topology, providing the essential link between homotopy and geometry. Indeed, comparing manifolds of the same homotopy type, one can often use surgery methods to produce h-cobordisms between them, and then hope to be able to show that the Whitehead torsion $\tau(W^{n+1}; M^n)$ in $\mathrm{Wh}(\pi_1(M^n))$ is trivial. By the s-cobordism theorem, the two manifolds will then be isomorphic (homeomorphic or diffeomorphic, according to which category we work in).

The last step, however, is in general very difficult, and what makes the problem even more complicated, but at the same time more interesting, is that there exist h-cobordisms with non-zero torsion, but where the ends still are isomorphic (cf. [8], [9], [18], [12]). Such h-cobordisms are called *inertial*. The central problem is then to determine the subset of elements of the Whitehead group $\mathrm{Wh}(\pi_1(M^n))$ which can be realized as Whitehead torsion of inertial h-cobordisms of the manifold M. This is in general very difficult, and only partial results in this direction are known ([8], [9], [18]).

The purpose of this note is to shed some light on this important problem.

2 Inertial h-cobordisms

In this section we recall basic notions and constructions concerning various types of h-cobordisms. We will follow the notation and terminology of [12].

For convenience we choose to formulate everything in the category of topological manifolds, but for most of what we are going to say, this does not make much difference. See Section 5 for more on the relations between the different categories.

An h-cobordism $(W; M, M')$ is a compact manifold W with two boundary components M and M', each of which is a deformation retract of W.

We will think of this as an h-cobordism from M to M', thus distinguishing it from the dual h-cobordism $(W; M', M)$. Since the pair $(W; M)$ determines M', we will often use the notation $(W; M)$ for $(W; M, M')$. We denote by $\mathcal{H}(M)$ the set of homeomorphism classes relative M of h-cobordisms from M.

If X is a path connected space, we denote by $\mathrm{Wh}(X)$ the Whitehead group $\mathrm{Wh}(\pi_1(X))$. Note that this is independent of choice of base point of X, up to unique isomorphism.

The s-cobordism theorem (cf. [20], [21]) says that if M is a closed connected manifold of dimension at least 5 there is a one-to-one correspondence between $\mathcal{H}(M)$ and $\mathrm{Wh}(M)$ associating to the h-cobordism $(W; M, M')$ its Whitehead torsion $\tau(W; M) \in \mathrm{Wh}(M)$. Given an element $(W; M, M') \in \mathcal{H}(M)$ the restriction of a retraction $r : W \to M$ to M' is a homotopy equivalence $h : M' \to M$, uniquely determined up to homotopy. By a slight abuse of language, any such h will be referred to as "the natural homotopy equivalence". It induces a unique isomorphism
$$h_* : \mathrm{Wh}(M') \to \mathrm{Wh}(M).$$

Recall also that there is an involution $\tau \to \bar{\tau}$ on $\mathrm{Wh}(M)$ induced by tranposition of matrices and inversion of group elements (cf. [21], [22]). If M is non-orientable, the involution is also twisted by the orientation character $\omega : \pi_1(M) \to \{\pm 1\}$, i.e. inversion of group elements is replaced by
$$\tau \mapsto \omega(\tau)\tau^{-1}. \tag{1}$$

Let $(W; M, M')$ and $(W; M', M)$ be dual h-cobordisms with M and M' of dimension n. Then $\tau(W; M)$ and $\tau(W; M')$ are related by the Milnor duality formula (cf. [21], [12])
$$h_*(\tau(W; M')) = (-1)^n \overline{\tau(W; M)}.$$

Definition 2.1. *The* inertial set *of a closed connected manifold M is defined as*
$$I(M) = \{(W; M, M') \in \mathcal{H}(M) | M \cong M'\},$$
or the corresponding subset of $\mathrm{Wh}(M)$.

There are many ways to construct inertial h-cobordisms. Here we recall three of these.

A. Let G be an arbitrary (finitely presented) group. Then there is a 2-dimensional finite simplicial complex K with $\pi_1(K) \cong G$. Let $\tau_0 \in \mathrm{Wh}(G)$ be an element with the property that $\tau_0 = \tau(f)$ for some homotopy self-equivalence

$f : K \to K$. Denote by $N(K)$ a regular neighborhood of K in a high-dimensional Euclidean space $\mathbb{R}^n (n \geqslant 5$ will do). By general position, we can approximate $f : K \to K \subseteq N(K)$ by an embedding whose image has neighborhood $N'(K) \subset \operatorname{int} N(K)$. By uniqueness up to PL isotopy in codimension $> \dim K$, we obtain $N'(K) \approx N(K)$. Then $W = N(K) - \operatorname{int} N'(K)$ is an inertial h-cobordism whose torsion $\tau(W; \partial N'(K))$ can be identified with τ_0 via the π_1-isomorphisms $\partial N'(K) \subset N(K) \supset K$ (cf. [8], [9]).

B. Let $f : M \to M$ be a homotopy self-equivalence of a closed manifold and let $\tau_0 = \tau(f) \in \operatorname{Wh}(M)$. Approximate $f : M \to M \subset M \times D^n$ by an embedding (cf. [26]), where D^n is the n-dimensional disk, $n > \dim M$. In the same way as in A, this will lead to an inertial h-cobordism between two copies of $M \times S^{n-1}$, with torsion τ_0 (cf. [12]).

C. Let $(W; M, M')$ be an h-cobordism with torsion $\tau_0 = \tau(W; M)$. Form the *double* (cf. [21], [12]):

$$(\widetilde{W}; M, M) := \left(W \underset{M'}{\cup} W; M, M \right)$$

Then $\tau(\widetilde{W}; M) = \tau_0 + (-1)^n \bar{\tau}_0$ and this again often leads to a nontrivial inertial h-cobordism; for example if n is odd and the involution $- : \operatorname{Wh}(M) \to \operatorname{Wh}(M)$ is nontrivial.

It will be convenient to introduce the notation $D(M)$ for the subgroup $\{\tau + (-1)^n \bar{\tau} | \tau \in \operatorname{Wh}(M)\}$ of $\operatorname{Wh}(M)$. Note that $D(M)$ depends only on $\pi_1(M)$, orientation and the dimension of M.

The construction in **C** leads to h-cobordisms that are particularly simple and have special properties: not only do they come with canonical identifications of the two ends, but they are also *strongly inertial*.

Definition 2.2 (Jahren–Kwasik [12]). *The h-cobordism $(W; M, M')$ is called* strongly inertial, *if the natural homotopy equivalence $h : M' \to M$ is homotopic to a homeomorphism.*

The set of Whitehead torsions of strongly inertial h-cobordisms will be denoted by $SI(M)$. It was observed in [12] that $SI(M) \subseteq \operatorname{Wh}(M)$ is a subgroup.

Obviously $SI(M) \subseteq I(M)$ and there are many examples of inertial but not strongly inertial h-cobordism, for example constructed using the methods in **A** or **B**. In fact, for any closed connected manifold M of dimension $n \geqslant 5$, we have

$$I(M \#_k (S^p \times S^{n-p})) = \operatorname{Wh}(M \#_k (S^p \times S^{n-p})),$$

for $2 \leq p \leq n - 2$ and k big enough [7]. (If $\pi_1(M)$ is finite, $k = 2$ suffices.)

However, for $SI(M)$ there are restrictions. For example, since the natural homopy equivalence h is homotopic to a homeomorphism, its Whitehead torsion $\tau(h)$ must vanish. But we have

$$\tau(h) = -\tau(W; M) + (-1)^n \overline{\tau(W; M)},$$

(see e.g. [12, formula (5.1)]) so $\tau(W; M)$ must satisfy the equation $\tau(W; M) = (-1)^n \overline{\tau(W; M)}$, i.e.

$$SI(M) \subseteq A(M) := \{\tau \in \mathrm{Wh}(M) | \tau = (-1)^n \bar{\tau}\}. \qquad (2)$$

In special cases we have even stronger restrictions, as in the following result (Theorem 1.3 in [12]):

Theorem 2.3 (Jahren–Kwasik [12])**.** *Suppose M is a closed oriented manifold of odd dimension with finite abelian fundamental group. Then every strongly inertial h-cobordism from M is trivial.*

This result motivated us to look more closely at strongly invertible h-cobordisms with finite fundamental groups. Our main interest is the following:

Problem 2.4. *Let M^n be a closed n-dimensional (oriented) manifold with $n \geqslant 5$ and with finite fundamental group $\pi_1(M^n)$. Determine the subset $SI(M^n)$ of $\mathrm{Wh}(M^n)$. In particular, is $SI(M^n) = D(M)$?*

Note that if G is a finite abelian group, then the involution $- : \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ is trivial (cf. [22]), and consequently $D(M^n) = \{0\}$ for n odd. Hence, in this case $SI(M) = D(M)$ by Theorem 2.3.

Our first new observation is that $SI(M^n) = \{0\}$ also for odd dimensional manifolds M^n with $\pi_1(M^n)$ finite periodic, namely:

Theorem 2.5. *Let $(W^{n+1}; M^n, N^n)$ be a strongly inertial h-cobordism with M orientable, n odd and $\pi = \pi_1(M^n)$ finite periodic. Then $W^{n+1} = M^n \times I$ for $n \geqslant 5$. Hence $SI(M^n) = \{0\}$.*

The class of finite periodic fundamental groups has attracted a lot of attention in topology of manifolds and transformation groups (cf. [19], [16]). The most extensive classification results for manifolds with finite fundamental groups involve this class of groups.

Let M^n be a closed, oriented manifold with $\pi_1(M^n)$ finite abelian. If n is odd, then, as we observed, $SI(M^n) = \{0\}$. In the even dimensional case the situation is quite different.

Theorem 2.6. *For every $n \geqslant 3$ there are oriented manifolds M^{2n} with $\pi_1(M^{2n})$ finite cyclic and with $\{0\} \neq D(M) \neq SI(M)$.*

The following result shows that orientability is essential in Theorem 2.3:

Theorem 2.7. *In every odd dimension $n \geqslant 5$ there are closed nonorientable manifolds with finite, cyclic fundamental groups such that $\{0\} \neq D(M) \neq SI(M)$.*

Remarks 2.8. (*i*). There are obvious inclusions $\{0\} \subset D(M) \subset SI(M) \subset I(M) \subset \mathrm{Wh}(M)$. In addition it is proved in [10] that $A(M) \subset I(M)$, such that combined with (2) we have a sequence of subsets

$$\{0\} \subset D(M) \subset SI(M) \subset A(M) \subset I(M) \subset \mathrm{Wh}(M). \qquad (3)$$

Clearly each of these inclusions can be an equality for some M, but for each pair of subsets we now have examples of manifolds where the inclusion is proper. In addition to those given by Theorems 2.6 and 2.7, we have:

- For $SI(M) \neq A(M)$, see e.g. [12, Example 6.4].)

- For $A(M) \neq I(M) \neq Wh(M)$, see [9, Remark 6.2], which implies that $I(M)$ is not always a group. The inequality $I(M) \neq Wh(M)$ just means that there are h-cobordant manifolds which are not homeomorphic. Numerous examples exist in the literature, see e.g. [21].

(*ii*) $D(M)$ and $A(M)$ depend only on the fundamental group (with orientation character), and Khan [14] has shown that $SI(M)$ is homotopy invariant (cf. Corollary 3.2 below). It would be interesting to know if $SI(M)$ also only depends on the fundamental group. If so, it is a functorial, algebraically defined subgroup of $Wh(M)$ between $D(M)$ and $A(M)$. What could it be?

Observe that the quotient $A(M)/D(M)$ is equal to the Tate cohomology group $\hat{H}^n(\mathbb{Z}_2; Wh(M))$, where $n = \dim M$, and therefore $SI(M)/D(M)$ is a subgroup. Another description of this subgroup is given in the beginning of Section 3. Here we only record the following trivial consequence of the fact that $\hat{H}^n(\mathbb{Z}_2; Wh(M))$ is 2-torsion:

Lemma 2.9. *$A(M)$, $SI(M)$ and $D(M)$ have the same ranks as abelian groups.*

Note that Hausmann has shown that $I(M)$ is *not* homotopy invariant [9, Theorem 6.6]. As mentioned already, it is not a subgroup of $Wh(M)$ in general, but it is preserved by the involution $\tau \mapsto (-1)^{n+1}\bar{\tau}$ [9, Lemma 5.6].

There is one more piece of structure that we should mention: the group $\pi_0(\mathrm{Top}(M))$ of isotopy classes of homeomorphisms of M acts on $Wh(M)$ via the isomorphisms induced on the fundamental group. (Recall that $Wh(M)$ is independent of choice of base point.) Geometrically, this corresponds to changing an h-cobordism $(W; M)$ by the way M is identified with part of the boundary of W. Hence the orbits represent equivalence classes under homeomorphisms preserving boundary components, but not necessary the identity on any of them. A simple example to illustrate this is the case where $M = P_1 \# P_2$, where P_1 and P_2 are copies of the same manifold. Since $Wh(M) \approx Wh(P_1) \oplus Wh(P_2)$ ([24]), this means that every h-cobordism from M is a band-connected sum $W_1 \#_{S^{n-1} \times I} W_2$ of h-cobordisms from P_1 and P_2, and the homeomorphisms interchanging P_1 and P_2 just interchanges W_1 and W_2.

The observation now is that the action of $\pi_0(\mathrm{Top}(M))$ preserves the filtration (3): geometrically on $D(M)$, $SI(M)$ and $I(M)$, and algebraically on $A(M)$ (as well as $D(M)$).

Note also that on $Wh(M)$ this action factors through an action of the group $\pi_0(\mathrm{Aut}(M))$ of homotopy classes of homotopy equivalences of M. Since the action of $\pi_0(\mathrm{Aut}(M))$ is defined algebraically, it must also preserve the functorial subgroups $D(M)$ and $A(M)$.

This action does not have an easy geometric interpretation, but $SI(M)$ is still preserved, by the more subtle functoriality of [14, Theorem 3.1], as explained

in Corollary 3.2 below. However, it is an easy consequence of [9, Theorem 6.1] that it does *not* in general preserve $I(M)$.

3 Proofs

In this section all manifolds have dimension at least five. The proofs are based on the following commutative diagram, henceforth refered to as *the main diagram*. The rows are the Sullivan-Wall exact sequences for topological surgery [26], which we identify with the algebraic surgery sequences of Ranicki (see [23, Theorem 18.5]), and the columns are part of the Rothenberg sequences for L-groups and structure sets. Then the diagram is a functorial diagram of groups.

$$\begin{array}{ccc}
\hat{H}^{n+3}(\mathbb{Z}_2; \mathrm{Wh}(M)) & = \!\!=\!\!= & \hat{H}^{n+3}(\mathbb{Z}_2; \mathrm{Wh}(M)) \\
\big\downarrow & & \big\downarrow \\
L^s_{n+2}(M) \xrightarrow{\ \gamma^s\ } S^s(M \times I) & \xrightarrow{\ \eta^s\ } N(M \times I) \xrightarrow{\ \theta^s\ } L^s_{n+1}(M) \\
\big\downarrow{\scriptstyle l_1} \qquad\qquad \big\downarrow{\scriptstyle t} & & \big\downarrow{\scriptstyle =} \qquad\qquad \big\downarrow{\scriptstyle l_0} \\
L^h_{n+2}(M) \xrightarrow{\ \gamma^h\ } S^h(M \times I) & \xrightarrow{\ \eta^h\ } N(M \times I) \xrightarrow{\ \theta^h\ } L^h_{n+1}(M) \\
\big\downarrow{\scriptstyle \delta_L} \qquad\qquad \big\downarrow{\scriptstyle \delta_S} & & \\
\hat{H}^{n+2}(\mathbb{Z}_2; \mathrm{Wh}(M)) & = \!\!=\!\!= & \hat{H}^{n+2}(\mathbb{Z}_2; \mathrm{Wh}(M))
\end{array}$$

We want to understand the quotient group $SI(M)/D(M)$, and the clue is the following observation:

Lemma 3.1 (Khan [14])**.** *Let M be a closed manifold of dimension $\geqslant 5$. Then*

$$SI(M)/D(M) \approx \mathrm{im}\, \delta_S \subset \hat{H}^n(\mathbb{Z}_2; \mathrm{Wh}(M)) \subset \mathrm{Wh}(M)/D(M).$$

Proof. (For a slightly different proof, see [14].) Recall that an element of $S^h(M \times I)$ is represented by a homotopy equivalence $f : W \to M \times I$ which is a homeomorphism on the boundary. Hence we can think of W as an h-cobordism from M, and as such it is clearly strongly inertial. Since the map δ_S is induced by $(f : W \to M \times I) \mapsto \tau(\mathrm{pr}_M \circ f) = \tau(W; M)$, the inclusion $\supseteq$ follows.

To prove the opposite inclusion, let $(W; M, N)$ be a strongly inertial h-cobordism representing an element z in $SI(M)/D(M)$, and let $H : N \times I \to M$ be a homotopy from the natural homotopy equivalence $h_W = r_M | N$ to a homeomorphism. Define a map $W \to M$ as the composite $W \xrightarrow{\approx} W \cup_N N \times I \to M$, where the last map is H on the collar $N \times I$ and the retraction r_M on W. Combined with any map $(W; M, N) \to (I; 0, 1)$ this defines an element of $S^h(M \times I)$ with image $z \in SI(M)/D(M)$. $\qquad\square$

We include the following corollary, which is our way of understanding Theorem 3.1 in [14] and its proof.

Corollary 3.2 (Khan [14])**.** *Let* $f : M \to M'$ *be a homotopy equivalence of closed manifolds of dimension* $\geqslant 5$. *Then the induced isomorphism* $f_* : \mathrm{Wh}(M) \to \mathrm{Wh}(M')$ *restricts to an isomorphism* $f_* : SI(M) \to SI(M')$.

Proof. We need to verify that $f_*(SI(M)) \subseteq SI(M')$.

Lemma 3.1 and functoriality of the surgery exact sequence imply that the induced homomorphism $f_* : \mathrm{Wh}(M)/D(M) \to \mathrm{Wh}(M')/D(M')$ restricts to a homomorphism $f_* : SI(M)/D(M) \to SI(M')/D(M')$. In other words, if $x \in SI(M)$, then $f_*(x) = y + d$, where $y \in SI(M')$ and $d \in D(M')$. But then obviously also $f_*(x) \in SI(M')$. $\qquad\square$

Our strategy to prove Theorems 2.6 and 2.7 will now be to show that in these cases the homomorphism l_1 in the diagram above is not onto. Then δ_S will be nontrivial, and consequently $SI(M) \neq D(M)$. The algebraically defined group $D(M)$ is often easy to compute explicitly.

Proof of Theorem 2.6. We need to study the map of *even* L-groups: $l_1 : L_{2m}^s(\pi) \to L_{2m}^h(\pi)$, where $\pi = \pi_1(M)$ and $2m = \dim M + 2$. Now assume that $\pi = \mathbb{Z}_k$ is a cyclic group of *odd* order $k \geqslant 5$. Then $Wh(M)$ is free abelian and the involution is trivial [22][1], hence $\hat{H}^{2m+1}(\mathbb{Z}_2; Wh(M)) = \{0\}$ and l_1 is injective. In fact, its image splits off as the free part, plus a $\mathbb{Z}_2$ (Arf invariant) if m is odd. Therefore any other torsion in $L_{2m}^h(\mathbb{Z}_k)$ must map nontrivially by δ_L.

The possible extra torsion can be computed from the Rothenberg sequence relating L_*^h and L_*^p-groups:

$$\longrightarrow L_{2m+1}^p(\mathbb{Z}_k) \longrightarrow H^2(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_k])) \longrightarrow L_{2m}^h(\mathbb{Z}_k) \longrightarrow L_{2m}^p(\mathbb{Z}_k),$$

where the groups $L_{2m+1}^p(\mathbb{Z}_k)$ vanish by [2, Corollary 4.3]. An example where $H^2(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_k]))$ is nontrivial is provided by [13, Theorem 7.1], where it is shown that $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}_{15}]) \approx \mathbb{Z}_2$. Hence, if we choose M to be any orientable, closed manifold of even dimension and fundamental group $\mathbb{Z}_{15}$, then $D(M) \neq SI(M)$ by commutativity of the main diagram and Lemma 3.1.

Recall now that $\mathrm{Wh}(\mathbb{Z}_{15}) \approx \mathbb{Z}^4$ (see e. g. [3, 11.5]), with trivial involution. Then $D(M) = 2 \mathrm{Wh}(\mathbb{Z}_{15}) \approx \mathbb{Z}^4$. (It follows that also $SI(M) \approx \mathbb{Z}^4$.) $\qquad\square$

Proof of Theorem 2.7. Consider now the cyclic 2-group $\mathbb{Z}_{2^k}$, $k \geqslant 4$, with the (unique) nontrivial orientation character $\omega : \mathbb{Z}_{2^k} \to \{\pm 1\}$. Computations in [5, Theorem B and formula p.44] give

$$L_{2m+1}^h(\mathbb{Z}_{2^k}, \omega) \xrightarrow[\approx]{\delta_L} H^1(\mathbb{Z}_2; Wh(\mathbb{Z}_{2^k})^\omega) \approx (\mathbb{Z}_2)^{2^{k-3}},$$

where the superscript ω indicates that the cohomology is with respect to the involution twisted by ω, as in formula (1). Hence, if we let M be a nonorientable manifold of odd dimension $2m-1 \geqslant 5$ and fundamental group $\mathbb{Z}_{2^k}$, then $\mathrm{im}\, \delta_S \neq \{0\}$. Hence $S(M) \neq D(M)$ by Lemma 3.1.

One way of constructing such M is as follows: choose a closed, orientable manifold P with fundamental group $\mathbb{Z}_{2^k}$, e. g. a lens space. Identifying $H^1(P; \mathbb{Z}_2)$

7

with $\mathrm{Hom}(\mathbb{Z}_{2^k}, \mathbb{Z}_2)$, we can find a vector bundle ξ over P of rank $r \geqslant 3$ such that $r + \dim P$ is even and with first Stiefel-Whitney class $w_1(\xi) = \omega$.

The total space M of the associated sphere bundle $S(\xi)$ is then of odd dimension $\geqslant 5$. Denoting stable equivalence of vector bundles by $\simeq$, the stable tangent bundles $T(M)$ of M and $T(P)$ of P are then related by

$$T(M) \simeq p^* T(P) \oplus p^* \xi.$$

The first Stiefel-Whitney class of M is then given by

$$w_1(M) = p^* w_1(P) + p^* w_1(\xi) = p^* w_1(\xi),$$

which corresponds to ω under the isomorphism $\pi_1(M) \approx \pi_1(P)$.

To finish the proof, observe that since $Wh(M)$ is free abelian, the non-trivial subgroup $SI(M)$ is free abelian. By Lemma 2.9, $D(M)$ must also be non-trivial (in fact, free abelian of the same rank as $SI(M)$). $\qquad\square$

Proof of Theorem 2.5. The proof is similar to the proof of Theorem 1.3 in [12] (Theorem 2.3 above), but now Lemma 3.1 simplifies part of the argument. We need that the following three statements are valid also for periodic groups:

(i) The involution $- : \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi)$ is trivial.

(ii) The homomorphism l_1 is surjective.

(iii) The homomorphism l_0 is injective on the image of θ^s.

(i) is Claim 3 and **(ii)** is Claim 1 in [17], so it remains to prove **(iii)**.

Since $\mathrm{im}\,\theta^s \subseteq L_{n+1}^s(\pi_2)$, where π_2 is the Sylow 2-subgroup of π [27], it is enough to show that the restriction $l_0 : L_{n+1}^s(\pi_2) \to L_{n+1}^h(\pi_2)$ to $\mathrm{im}\,\theta^s$ is injective. To this end note that $\mathrm{SK}_1(\pi_2) = 0$ (cf. [22]), where

$$\mathrm{SK}_1(\pi) := \mathrm{Ker}(K_1(\mathbb{Z}[\pi]) \longrightarrow K_1(\mathbb{Q}[\pi]))$$

is the torsion subgroup of $Wh(\pi)$. Indeed π_2 is either generalized quaternionic or cyclic [19].

As a consequence $L_{n+1}^s(\pi_2) \cong L_{n+1}'(\pi_2)$ where $L_*'(-)$ are the weakly simple L-groups of C.T.C. Wall from [28]. Now, there is an exact sequence (cf. [28, p. 78])

$$0 \to L_{2m}^s(\pi_2) \to L_{2m}^h(\pi_2) \to \mathrm{Wh}'(\pi_2) \otimes \mathbb{Z}_2 \to L_{2m-1}^s(\pi_2) \to L_{2m-1}^h(\pi_2) \to 0$$

for any m. Setting $n = 2m - 1$, we see that $l_0 | \mathrm{im}\,\theta^s$ is injective, as claimed.

Given this, diagram chase, as in the proof of half of the five lemma, proves that the map t in the main diagram is surjective. Then $SI(M) = D(M)$ by Lemma 3.1, and $D(M) = \{0\}$ by fact **(i)** above. $\qquad\square$

4 Further remarks

Let M be a closed manifold of *odd* dimension. Then $D(M) = \{\tau - \bar{\tau} | \tau \in \text{Wh}(M)\}$, which vanishes if and only if the involution is trivial. Therefore we get the following curious restatement of a special case of our problem:

Question 4.1. *Is $SI(M) = \{0\}$ if the involution* $- : \text{Wh}(M) \to \text{Wh}(M)$ *is the identity?*

'Only if' is trivial here, since $D(M) \subseteq S(M)$.

Theorems 2.3 and 2.5 give examples of such manifolds, and more examples are provided by

Example 4.2. Let M be a closed, orientable, odd dimensional manifold with finite fundamental group π, and assume that $\text{SK}_1(\pi) = 0$. Examples of such groups are dihedral groups and many nonabelian metacyclic groups; see [22, ch. 14] for more. Then $\text{Wh}(\pi)$ is torsion free, and the standard involution $- : \text{Wh}(\pi) \to \text{Wh}(\pi)$ is trivial [1]. It follows that

$$A(M) = \{\tau \in Wh(M) | \tau = -\bar{\tau}\} = \{0\},$$

hence also $SI(M) = \{0\}$.

Thus we have vanishing results for $SI(M)$ for odd-dimensional orientable manifolds M with finite fundamental groups which are abelian, periodic or such that $SK_1(\pi_1(M)) = \{0\}$. There are overlaps between these classes of groups, but no inclusions. For example, abelian groups containing $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ and the periodic groups containing $\mathbb{Z}_p \times Q(8)$, where $p \geqslant 3$ is prime and $Q(8)$ is the quaternionic group of order 8, all have nontrivial SK_1 (cf. [22]).

Example 4.3. We should point out that there are finite groups G such that the standard involution on $\text{Wh}(G)$ is non-trivial. An odd-dimensional, orientable manifold M with $\pi_1(M) \approx G$ will then have $SI(M) \neq \{0\}$, by the discussion above.

To give just one example of such a group, let p be an odd prime and let G be a p-group such that $SK_1(\hat{\mathbb{Z}}_p G)_{(p)}$ is non-trivial, for example the group given in Example 8.11 of [22]. Then the argument on page 323 of [22] shows that that the involution $- : \text{Wh}(G) \to \text{Wh}(G)$ is nontrivial.

Remark 4.4. In this section we have only considered manifolds of odd dimension. If M is an *even*-dimensional, orientable manifold with a non-trivial fundamental group as in Example 4.2, then $S(M) \neq \{0\}$, since it will contain $D(M) = \{2\tau | \tau \in Wh(M)\}$.

Remark 4.5. There exist 4-dimensional inertial s-cobordisms which are not topological products. (cf. [4], [16]). Note that in this dimension all h-cobordisms are inertial [25, Theorem 1.4].

9

5   Appendix: On topological invariance

It is a consequence of the s-cobordism theorem and smoothing theory that if M is a closed manifold and $\dim M \geqslant 5$, then the classification of h-cobordisms from M up to isomorphism relative to M is the same in the three categories TOP, PL and $DIFF$. For example, if M is smooth and $(W; M)$ is a topological h-cobordism, then W has a smooth structure, unique up to concordance, extending that of M, and if two such h-cobordisms are homeomorphic rel M, then they are also diffeomorphic rel M.

However, the following question is more subtle:

Question 5.1. *Suppose $(W; M, N)$ is a smooth h-cobordism which is inertial in* TOP, *does it follow that it is also inertial in* DIFF*?*

In other words: if M and N are homeomorphic, are they then also diffeomorphic? (Similar questions can of course be asked for the pairs of categories $(DIFF,PL)$ and $(PL,TOP$).)

Note that this indeed holds for the examples provided by the general results and constructions above; for example $D(M)$, $A(M)$ and those obtained by connected sum with products of spheres, and in Lemma 8.1 of [12] we claimed that the answer is always yes. However, this was based on an overly optimistic interpretation of the product structure theorem for smoothings, and it does not hold as it stands[1]. We have, unfortunately, not been able to correct this in general, but here is a proof in the case of *strongly inertial h-cobordisms.*

Proposition 5.2. *Let M be a smooth, closed manifold. If W is a* PL *h-cobordism from M, then W has a smooth structure compatible with the given structure on M, unique up to concordance. If W is strongly inertial in* PL*, then it is also strongly inertial in* DIFF*.*

Replacing the pair of categories $(\mathrm{DIFF}, \mathrm{PL})$ by $(\mathrm{DIFF}, \mathrm{TOP})$ or $(\mathrm{PL}, \mathrm{TOP})$, a similar result is true, provided M has dimension at least 5.

Proof. Denote by $\Gamma(M)$ the set of concordance classes of smoothings of the underlying PL manifold M. By smoothing theory [15, Essay IV, §10] this is a homotopy functor. In particular, if $(W; M, N)$ is an h-cobordism, the inclusions $M \subset_{j_M} W$ and $N \subset_{j_N} W$ induce restriction *isomorphisms*

$$\Gamma(M) \xleftarrow[\approx]{j_M^*} \Gamma(W) \xrightarrow[\approx]{j_N^*} \Gamma(N).$$

This proves the first part of the Proposition and also defines a unique concordance class of structures on N.

Now let M_α be the given structure on M, W_α a structure on W restricting to M_α and N_α the restriction of this again to N, such that $(W_\alpha; M_\alpha, N_\alpha)$ is a smooth h-cobordism. Observe that since j_M has a homotopy inverse r_M, the composite isomorphism $\Gamma(M) \to \Gamma(N)$ is induced by $r_M \circ j_N$, i.e. the natural homotopy equivalence h_W. But if the h-cobordism is PL strongly inertial, the

---

[1]We would like to thank Jean-Claude Hausmann for pointing out the error in [12].

isomorphism is also induced by a PL homeomorphism f. This means that N_α is concordant to the smoothing $N_{f^*\alpha}$ on N transported from M_α by f in such a way that f becomes a *diffeomorphism* between $N_{f^*\alpha}$ and M_α.

Let $(N \times I)_\beta$ be a concordance between N_α and $N_{f^*\alpha}$, i.e. a smooth structure restricting to N_α on $N \times \{0\}$ and $N_{f^*\alpha}$ on $N \times \{1\}$. By the product structure theorem ([11, part I]) there is a diffeomorphism $H : (N \times I)_\beta \to N_\alpha \times I$ restricting to the identity on $N \times \{0\}$. Then $F(x,t) = H(f(x),t)$ defines a homotopy (in fact PL isotopy) between f and a diffeomorphism between M_α and N_α. But f was homotopic to h_W.

The proofs in the other cases are analogous, but one now needs the triangulation theory of [15], which is only valid in dimensions $\geqslant 5$. □

Remark 5.3. *If* $\dim M = 4$, *Question 5.1 has a negative answer, even in the strongly inertial case. In fact, the first counterexamples to the h-cobordism theorem given by Donaldson in [6] are even* strongly *inertial, so even Proposition 5.2 in case* (DIFF,TOP) *fails in this dimension.*

References

[1] A. Bak, *The involution on Whitehead torsion*, General Topology and Appl. 7 (1977), no. 2, 201–206.

[2] A. Bak and M. Kolster, *The computation of odd dimensional projective surgery groups of finite groups*, Topology, Vol. 21(1982), 55–63.

[3] M. Cohen, *A course in Simple Homotopy Theory*, Graduate Texts in Mathematics, Springer Verlag, 1973.

[4] S. Cappell and J. Shaneson, *On 4-dimensional s-cobordisms*, J. Differential Geom, 22(1985), 97–115.

[5] ——, *Torsion in L-groups*, Algebraic and Geometric Tropology, LNM 1126, Springer 1985

[6] S. Donaldson, *Irrationality and the h-cobordism conjecture*, J. Differential Geom. 26 (1987), no. 1, 141–168.

[7] A. Hatcher and T. Lawson. *Stability theorems for "concordance implies isotopy" and "h-cobordism implies diffeomorphism"*, Duke Math. J., 43(3):555–560, 1976.

[8] J. C. Hausmann, *h-cobordismes entres variétés homeomorphes*, Comment. Math. Helv. 50(1970), 9–13.

[9] ——, *Open books and h-cobordisms*, Comm. Math. Helv. 55(1980), 330–346.

[10] J.-C. Hausmann and B. Jahren, *A simplification theorem in manifold topology*, L'Enseignement Mathématique, to appear. ArXiv:1802.03186.

[11] M. W. Hirsch and B. Mazur, *Smoothings of Piecewise Linear Manifolds*, Annals of Mathematics sudies, no. 80, Princeton University Press 1974.

[12] B. Jahren and S. Kwasik, *How different can h-cobordant manifolds be?*, Bull. London Math. Soc. 47(2015), 617–630.

[13] M. Kervaire and M. P. Murthy, *On projective class groups of cyclic groups of prime power order*, Comment. Math. Helv. 52(1977), 415–452.

[14] Q. Khan, *Free transformations of $S^1 \times S^n$ of odd square-free period.* Indiana Univ. Math. J. 66(5)(2017), 1453–1482.

[15] R. Kirby, L. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations.* Annals of Mathematics Study, No. 88(1977), Princeton Univ. Press.

[16] ———, *On s-cobordisms of metacyclic prism manifolds*, Invent. Math. 97(1987), 523–552.

[17] ———, *On h-cobordisms of spherical space forms*, Proc. Amer. Math. Soc. 127(1999) 1525-1532.

[18] T. Lawson, *Inertial h-cobordisms with finite cyclic fundamental groups*, Proc. Amer. Math. Soc. 441(1974), 492–496.

[19] J. Madsen, C. B. Thomas and C. T. C. Wall, *The topological spherical space form problem II,* Topology 15(1976), 375–382.

[20] B. Mazur, *Relative neighborhoods and the theorems of Smale*, Annals of Math. 77(1963), 232–249.

[21] J. Milnor, *Whitehead torsion,* Bull. Amer. Math. Soc. 721(1966), 358–426.

[22] R. Oliver, *Whitehead groups of finite groups,* London Math. Soc. Lect. Notes Series 132(1986), Cambridge University Press.

[23] A. Ranicki, *Algebraic and geometric Surgery*, Oxford math. Monographs, 2002.

[24] J. Stallings, *Whitehead torsion of free products*, Ann. of Math. (2) 82 1965 354–363.

[25] V. G. Turaev, *Towards the topological classification of geometric 3-manifolds*, In Topology and geometry—Rohlin Seminar, volume 1346 of Lecture Notes in Math., pages 291323. Springer, Berlin, 1988.

[26] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, New York, 1970.

[27] ——, *Formulae for surgery obstructions*, Topoogy 15(1976), 189–210.

[28] ——, *Classifications of hermitian forms. VI Group rings*, Annals of Math. 103(1976), 1–80.

Bjørn Jahren
Department of Mathematics
University of Oslo
0316 Oslo
Norway
bjoernj@math.uio.no

Slawomir Kwasik
Department of Mathematics
Tulane University
New Orleans, LA 70122
USA
kwasik@tulane.edu