# ANNALS OF K-THEORY

**Algebraic *K*-theory of quotient stacks**

Amalendu Krishna and Charanya Ravi

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# Algebraic *K*-theory of quotient stacks

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We prove some fundamental results like localization, excision, Nisnevich descent, and the regular blow-up formula for the algebraic *K*-theory of certain stack quotients of schemes with affine group scheme actions. We show that the homotopy *K*-theory of such stacks is homotopy invariant. This implies a similar homotopy invariance property of the algebraic *K*-theory with coefficients.

## 1. Introduction

The higher algebraic *K*-theories of Quillen and Thomason–Trobaugh are among the most celebrated discoveries in mathematics. Fundamental results like localization, excision, Nisnevich descent, and the blow-up formula have played pivotal roles in almost every recent breakthrough in the *K*-theory of schemes; see, e.g., [Cortiñas 2006; Cortiñas et al. 2008; Schlichting 2010]. The generalization of these results to equivariant *K*-theory is the theme of this paper.

The significance of equivariant *K*-theory [Thomason 1987a] in the study of the ordinary (nonequivariant) *K*-theory is essentially based on two principles. First, it often turns out that the presence of a group action allows one to exploit representation-theoretic tools to study equivariant *K*-theory. Second, there are results (see, for instance, [Merkurjev 2005, Theorem 32]) which directly connect equivariant algebraic *K*-theory with the ordinary *K*-theory of schemes with group action. These principles have been effectively used in the past to study both equivariant and ordinary algebraic *K*-theory; see, for instance, [Joshua and Krishna 2015; Vezzosi and Vistoli 2003]. In addition, equivariant *K*-theory often allows one to understand various other cohomology theories of moduli stacks and moduli spaces from the *K*-theoretic point of view.

However, any serious progress towards the applicability of equivariant *K*-theory (of vector bundles) requires analogues for quotient stacks of the fundamental results of Thomason–Trobaugh. The goal of this paper is to establish these results, so that a very crucial gap in the study of the *K*-theory of quotient stacks can be filled. Special cases of these results were earlier proven in [Krishna 2009; Krishna and

Østvær 2012; Heller et al. 2015]. Here is a summary of our main results. The precise statements and the underlying notation can be found in the body of the text. We fix a field $k$.

**Theorem 1.1.** *Let $\mathcal{X}$ be a nice quotient stack over $k$ with the resolution property. Let $\mathbb{K}$ denote the (nonconnective) K-theory presheaf on the 2-category of nice quotient stacks. Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack with open complement $\mathcal{U}$. Then the following hold.*

(1) *There is a homotopy fibration sequence of $S^1$-spectra*

$$\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \rightarrow \mathbb{K}(\mathcal{X}) \rightarrow \mathbb{K}(\mathcal{U}).$$

(2) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies excision.*

(3) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies Nisnevich descent.*

(4) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies descent for regular blow-ups.*

**Theorem 1.2.** *The nonconnective homotopy K-theory presheaf KH on the 2-category of nice quotient stacks with resolution property satisfies the following.*

(1) *It is invariant under every vector bundle morphism (Thom isomorphism for stacks).*

(2) *It satisfies localization, excision, Nisnevich descent, and descent for regular blow-ups.*

(3) *If $\mathcal{X}$ is the stack quotient of a scheme by a finite nice group, then $KH(\mathcal{X})$ is invariant under infinitesimal extensions.*

The following result shows that $K$-theory with coefficients for quotient stacks is homotopy invariant, i.e., it satisfies the Thom isomorphism. No case of this result was yet known for stacks which are not schemes.

**Theorem 1.3.** *Let $\mathcal{X}$ be a nice quotient stack over $k$ with the resolution property and let $f : \mathcal{E} \rightarrow \mathcal{X}$ be a vector bundle. Then the following hold.*

(1) *For any integer $n$ invertible in $k$, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \rightarrow \mathbb{K}(\mathcal{E}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer $n$ nilpotent in $k$, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) \rightarrow \mathbb{K}(\mathcal{E}; \mathbb{Z}[1/n])$ is a homotopy equivalence.*

In the above results, a nice quotient stack means a stack of the form $[X/G]$, where $G$ is an affine group scheme over $k$ acting on a $k$-scheme $X$ such that $G$ is nice, i.e., it is either linearly reductive over $k$ or $\text{char}(k) = 0$. Group schemes of multiplicative type (e.g., diagonalizable group schemes) are notable examples of this in positive characteristic. We refer to Section 2B for more details.

*Applications.* Similar to the case of schemes, one expects the above results to be of central importance in the study of the *K*-theory of quotient stacks. Already by now, there have been two immediate major applications: (1) the *cdh*-descent and, (2) Weibel's conjecture for negative *KH*-theory of stacks. In a sense, these applications motivated the results of this paper.

Hoyois [2017] has constructed a variant of *KH*-theory for nice quotient stacks and has used the main results of this paper to prove the *cdh*-descent for this variant. The results of this paper (and their generalizations) have also been used recently by Hoyois and the first author [Hoyois and Krishna 2017] to prove *cdh*-descent for the *KH*-theory (as defined in Section 5) of nice stacks, and to prove Weibel's conjecture for the vanishing of negative *KH*-theory of such stacks.

Another application of the above results is related to a rigidity type theorem for the *K*-theory of semilocal rings. Let $A$ be a normal semilocal ring with isolated singularity with an action of a finite group $G$, and let $\widehat{A}$ denote its completion along the Jacobson radical. The rigidity question asks if the map $K'_*(G, A) \to K'_*(G, \widehat{A})$ is injective. If $G$ is trivial, this was proven for $K'_0(G, A)$ by Kamoi and Kurano [2002] for certain type of isolated singularities. They apply this result to characterize certain semilocal rings. The main tool of [Kamoi and Kurano 2002] is Theorem 1.1 for the ordinary *K*-theory of singular rings. We hope that the localization theorem for the *K*-theory of quotient stacks can now be used to prove the equivariant version of this rigidity theorem.

## 2. Perfect complexes on quotient stacks

Throughout this text, we work over a fixed base field $k$ of arbitrary characteristic. In this section, we fix notations, recall basic definitions and prove some preliminary results. We conclude the section with the proof of an excision property for the derived category of perfect complexes on stacks.

**2A.** *Notations and definitions.* Let $\mathbf{Sch}_k$ denote the category of separated schemes of finite type over $k$. A *scheme* in this paper will mean an object of $\mathbf{Sch}_k$. A group scheme $G$ will mean an affine group scheme over $k$. Recall that a stack $\mathcal{X}$ (of finite type) over the big fppf site of $k$ is said to be an algebraic stack over $k$ if the diagonal of $\mathcal{X}$ is representable by algebraic spaces and $\mathcal{X}$ admits a smooth, representable and surjective morphism $U \to \mathcal{X}$ from a scheme $U$. Throughout this text a "stack" will always refer to an algebraic stack. We shall say that $\mathcal{X}$ is a *quotient stack* if it is a stack of the form $[X/G]$ (see, for instance, [Laumon and Moret-Bailly 2000, §2.4.2]), where $G$ is an affine group scheme acting on a scheme $X$.

**2B.** *Nice stacks.* Given a group scheme $G$, let $\mathbf{Mod}^G(k)$ denote the category of $k$-modules with $G$-action. Recall that $G$ is said to be linearly reductive if the

"functor of $G$-invariants" $(-)^G : \mathbf{Mod}^G(k) \to \mathbf{Mod}(k)$, given by the submodule of $G$-invariant elements, is exact. If $\mathrm{char}(k) = 0$, it is well known that $G$ is linearly reductive if and only if it is reductive. In general, it follows from [Abramovich et al. 2008, Propositions 2.5, 2.7, Theorem 2.16] that $G$ is linearly reductive if there is an extension

$$1 \to G_1 \to G \to G_2 \to 1, \tag{2.1}$$

where each of $G_1$ and $G_2$ is either finite over $k$ of degree prime to the exponential characteristic of $k$, or is of multiplicative type (étale locally diagonalizable) over $k$. One knows that linearly reductive group schemes in positive characteristic are closed under the operations of taking closed subgroups and base change.

**Definition 2.2.** We shall say that a group scheme $G$ is *nice* if either it is linearly reductive or $\mathrm{char}(k) = 0$. If $G$ is nice and it acts on a scheme $X$, we shall say that the resulting quotient stack $[X/G]$ is nice.

**2C.** *Perfect complexes on stacks.* Given a stack $\mathcal{X}$, let $\mathrm{Sh}(\mathcal{X})$ denote the abelian category of sheaves of abelian groups, $\mathrm{Mod}(\mathcal{X})$ the abelian category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathrm{QC}(\mathcal{X})$ the abelian category of quasicoherent sheaves, each on the smooth-étale site Lis-Et$(\mathcal{X})$ of $\mathcal{X}$. Let $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ denote the category of all (possibly unbounded) chain complexes over $\mathrm{Mod}(\mathcal{X})$ whose cohomology lie in $\mathrm{QC}(\mathcal{X})$, and $\mathrm{Ch}(\mathrm{QC}(\mathcal{X}))$ the category of all chain complexes over $\mathrm{QC}(\mathcal{X})$. Let $D_{\mathrm{qc}}(\mathcal{X})$ and $D(\mathrm{QC}(\mathcal{X}))$ denote the corresponding derived categories. Let $D(\mathcal{X})$ denote the unbounded derived category of $\mathrm{Mod}(\mathcal{X})$. If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed substack with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$, we let

$$\mathrm{Ch}_{\mathrm{qc}, \mathcal{Z}}(\mathcal{X}) = \left\{ \mathcal{F} \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X}) \mid j^*(\mathcal{F}) \xrightarrow{\text{q. iso.}} 0 \right\}.$$

The derived category of $\mathrm{Ch}_{\mathrm{qc}, \mathcal{Z}}(\mathcal{X})$ will be denoted by $D_{\mathrm{qc}, \mathcal{Z}}(\mathcal{X})$. Recall that a stack $\mathcal{X}$ is said to have the resolution property if every coherent sheaf on $\mathcal{X}$ is a quotient of a vector bundle.

**Lemma 2.3.** *Let $\mathcal{X}$ be the stack quotient of a scheme $X$ with an action of a group scheme $G$. Then the following hold.*

(1) *Every quasicoherent sheaf on $\mathcal{X}$ is the direct limit of its coherent subsheaves.*

(2) *$\mathcal{X}$ has the resolution property if $X$ has an ample family of $G$-equivariant line bundles. In particular, $\mathcal{X}$ has the resolution property if $X$ is normal with an ample family of (nonequivariant) line bundles.*

(3) *$\mathcal{X}$ has the resolution property if $X$ is quasi-affine.*

*Proof.* Part (1) is [Thomason 1987b, Lemma 1.4]. For (2), note that $[\mathrm{Spec}(k)/G]$ has the resolution property [Thomason 1987b, Lemma 2.4]. Therefore, if $X$ has an ample family of $G$-equivariant line bundles, it follows from [Thomason 1987b, Lemma 2.6] that $\mathcal{X}$ has the resolution property. If $X$ is normal with an ample family

of (nonequivariant) line bundles, it follows from [Thomason 1987b, Lemmas 2.10, 2.14] that $\mathcal{X}$ has the resolution property. Part (3) is well known and follows, for example, from [Hall and Rydh 2017, Lemma 7.1]. □

Recall from [SGA 6 1971, Definition I.4.2] that a complex of $\mathcal{O}_X$-modules on a Noetherian scheme is perfect if it is Zariski locally quasi-isomorphic to a bounded complex of locally free sheaves.

**Definition 2.4.** Let $\mathcal{X}$ be a stack over $k$. A chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ is called *perfect* if for any affine scheme $U = \mathrm{Spec}(A)$ with a smooth morphism $s : U \to \mathcal{X}$, the complex of $A$-modules $s^*(P) \in \mathrm{Ch}(\mathrm{Mod}(A))$ is quasi-isomorphic to a bounded complex of finitely generated projective $A$-modules.

We shall denote the category of perfect complexes on $\mathcal{X}$ by $\mathrm{Perf}(\mathcal{X})$ and its derived category by $D^{\mathrm{perf}}(\mathcal{X})$. For a quotient stack with the resolution property, we can characterize perfect complexes in terms of their pull-backs to the total space of the quotient map.

**Lemma 2.5.** *Let* $f : X' \to X$ *be a faithfully flat map of Noetherian schemes. Let* $P$ *be a chain complex of quasicoherent sheaves on* $X$ *such that* $f^*(P)$ *is perfect on* $X'$. *Then* $P$ *is a perfect complex on* $X$.

*Proof.* By [Thomason and Trobaugh 1990, Proposition 2.2.12], a complex of quasicoherent sheaves is perfect if and only if it is cohomologically bounded above, its cohomology sheaves are coherent, and it has locally finite Tor-amplitude. But all these properties are known to descend from a faithfully flat cover. □

**Proposition 2.6.** *Let* $\mathcal{X}$ *be the stack quotient of a scheme* $X$ *with an action of a group scheme* $G$ *and let* $u : X \to \mathcal{X}$ *be the quotient map. Assume that* $\mathcal{X}$ *has the resolution property. Let* $P$ *be a chain complex of quasicoherent* $\mathcal{O}_X$-modules. *Then the following are equivalent.*

(1) *$P$ is perfect.*

(2) *$u^*(P)$ is perfect.*

(3) *$u^*(P)$ is quasi-isomorphic to a bounded complex of $G$-equivariant vector bundles in $\mathrm{Ch}(\mathrm{QC}^G(X))$, where $\mathrm{QC}^G(X)$ denotes the category of $G$-equivariant quasicoherent sheaves on $X$.*

*Proof.* (1) $\Rightarrow$ (2). We let $Q = u^*(P)$. Consider an open cover of $X$ by affine open subsets $\{\mathrm{Spec}(A_i)\}$. Let $s : U \to [X/G]$ be an atlas and $s_i : U_i \to \mathrm{Spec}(A_i)$ its base change to $\mathrm{Spec}(A_i)$, where $U_i$ are algebraic spaces. Take étale covers $t_i : V_i \to U_i$ of $U_i$, where the $V_i$ are schemes. Let $f_i : V_i \to U$ and $g_i : V_i \to \mathrm{Spec}(A_i)$ denote the obvious composite maps. It follows from (1) that $\boldsymbol{L}g_i^*(Q|_{\mathrm{Spec}(A_i)}) \simeq \boldsymbol{L}f_i^*(s^*(P))$ is a perfect complex on $V_i$. Therefore, by Lemma 2.5, $Q|_{\mathrm{Spec}(A_i)}$ is a perfect complex in $\mathrm{Ch}(\mathrm{Mod}(A_i))$. Equivalently, $Q$ is perfect.

$(2) \Rightarrow (3)$. We want to apply the inductive construction lemma [Thomason and Trobaugh 1990, Lemma 1.9.5] with $\mathcal{A}$ being $\mathrm{QC}^G(X)$, $\mathcal{D}$ the category of $G$-equivariant vector bundles on $X$ and $\mathcal{C}$ the category of complexes in $\mathrm{Ch}(\mathrm{QC}^G(X))$ satisfying (2). It is enough to verify that the hypothesis [loc. cit., 1.9.5.1] holds.

Suppose $C \in \mathcal{C}$ such that $H^i(C) = 0$ for $i \geq n$, and $q : \mathcal{F} \twoheadrightarrow H^{n-1}(C)$ in $\mathrm{QC}^G(X)$. By [Thomason and Trobaugh 1990, Proposition 2.2.3], $\mathcal{G} = H^{n-1}(C)$ is a coherent $\mathcal{O}_X$-module. Therefore, $\mathcal{G}$ is a coherent $G$-module. By Lemma 2.3(1), we can write $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ are coherent $G$-submodules of $\mathcal{F}$. Under the forgetful functor, this gives an epimorphism $q : \varinjlim \mathcal{F}_\alpha \twoheadrightarrow \mathcal{G}$ in $\mathrm{QC}(X)$, where $\mathcal{F}_\alpha$, $\mathcal{G}$ are coherent modules.

Now, as $\mathcal{G}$ is coherent and $X$ is Noetherian, we can find an $\alpha$ such that the composite map $\mathcal{F}_\alpha \hookrightarrow \mathcal{F} \xrightarrow{q} \mathcal{G}$ is surjective. By the resolution property, there exists $\mathcal{E} \in \mathcal{D}$ such that $\mathcal{E} \twoheadrightarrow \mathcal{F}_\alpha$. Hence the composite $\mathcal{E} \to \mathcal{F}_\alpha \hookrightarrow \mathcal{F} \xrightarrow{q} \mathcal{G}$ is also surjective. Applying the conclusion of [Thomason and Trobaugh 1990, Lemma 1.9.5] to $C^\bullet = P$ and $D^\bullet = 0$, we get a bounded above complex $E$ of $G$-vector bundles and a quasi-isomorphism $\phi : E \xrightarrow{\sim} P$ in $\mathrm{Ch}(\mathrm{QC}^G(X))$. Therefore, $E \in \mathcal{C}$.

Since $X$ is Noetherian, $E$ has globally finite Tor-amplitude. To show that $Q$ is quasi-isomorphic to a bounded complex over $\mathcal{D}$, it suffices to prove that the good truncation $\tau^{\geq a}(E)$ is a bounded complex of $G$-equivariant vector bundles and the map $E \to \tau^{\geq a}(E)$ is a quasi-isomorphism. It is enough to prove this claim by forgetting the $G$-action. But this follows exactly along the lines of the proof of [Thomason and Trobaugh 1990, Proposition 2.2.12].

$(3) \Longrightarrow (1)$ is clear.                                                □

**2D. *Perfect complexes and compact objects of $D_{\mathrm{qc}}(\mathcal{X})$.*** Recall that if $\mathcal{T}$ is a triangulated category which is closed under small coproducts, then an object $E \in \mathrm{Obj}(\mathcal{T})$ is called compact if the functor $\mathrm{Hom}_\mathcal{T}(E, -)$ on $\mathcal{T}$ commutes with small coproducts. The full triangulated subcategory of compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$. If $X$ is a scheme, one of the main results of [Thomason and Trobaugh 1990] is that a chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(X)$ is perfect if and only if it is a compact object of $D_{\mathrm{qc}}(X)$. For quotient stacks, this is a consequence of the results of [Neeman 1996; Hall and Rydh 2015]:

**Proposition 2.7.** *Let $\mathcal{X}$ be a nice quotient stack. Then a chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ is perfect if and only if it is a compact object of $D_{\mathrm{qc}}(\mathcal{X})$.*

*Proof.* Suppose $P$ is compact. We need to show that $s^*(P)$ is perfect on $U = \mathrm{Spec}(A)$ for every smooth map $s : U \to \mathcal{X}$. Since the compact objects of $D_{\mathrm{qc}}(U)$ are perfect, it suffices to show that $s^*(P)$ is compact. We deduce this using [Neeman 1996, Theorem 5.1].

The push-forward functor $\boldsymbol{R}s_* : D_{\mathrm{qc}}(U) \to D_{\mathrm{qc}}(\mathcal{X})$ is a right adjoint to the pullback $\boldsymbol{L}s^* : D_{\mathrm{qc}}(\mathcal{X}) \to D_{\mathrm{qc}}(U)$. As $\boldsymbol{R}s_*$ and $\boldsymbol{L}s^*$ both preserve small coproducts

(see the proof of Lemma 2.8 below), it follows from [Neeman 1996, Theorem 5.1] that $s^*(P)$ is compact.

If $P$ is perfect, then it is a compact object of $D_{qc}(\mathcal{X})$ by our assumption on $\mathcal{X}$ and [Hall and Rydh 2015, Theorem C]. $\qquad\square$

**Lemma 2.8.** *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack. Then the compact objects of $D_{qc,\mathcal{Z}}(\mathcal{X})$ are exactly those which are perfect in $\mathrm{Ch}_{qc}(\mathcal{X})$.*

*Proof.* It follows from Proposition 2.7 that $D_{\mathcal{Z}}^{\mathrm{perf}}(\mathcal{X}) \subseteq D_{qc,\mathcal{Z}}^c(\mathcal{X})$. To prove the other inclusion, let $K \in D_{qc,\mathcal{Z}}^c(\mathcal{X})$. We need to show that $K$ is a perfect complex in $D_{qc}(\mathcal{X})$. Let $s : V = \mathrm{Spec}(A) \to \mathcal{X}$ be any smooth morphism and set $T = s^{-1}(\mathcal{Z})$. Consider a set of objects $\{F_\alpha\}$ in $D_{qc,T}(V)$. Since $\mathcal{X}$ is a quotient stack, there exists a smooth atlas $u : X \to \mathcal{X}$, where $X \in \mathbf{Sch}_k$. This gives a 2-Cartesian square of stacks

$$\begin{array}{ccc} W & \xrightarrow{s'} & X \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ V & \xrightarrow{s} & \mathcal{X} \end{array} \qquad (2.9)$$

The maps $u$ and $s$ are Tor-independent because they are smooth. Since $\Delta_\mathcal{X}$ is representable and $V$ is affine, it follows that $s$ is representable. We conclude from [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] that $u^* \boldsymbol{R}s_*(F_\alpha) \xrightarrow{\simeq} \boldsymbol{R}s'_* u'^*(F_\alpha)$. It follows that $\boldsymbol{R}s_*(F_\alpha) \in D_{qc,\mathcal{Z}}(\mathcal{X})$. Using adjointness [Krishna 2009, Lemma 3.3], we get

$$\begin{aligned} \mathrm{Hom}_{D_{qc,T}(V)}\big(s^*(K), \textstyle\bigoplus_\alpha F_\alpha\big) &\simeq \mathrm{Hom}_{D_{qc,\mathcal{Z}}(\mathcal{X})}\big(K, \boldsymbol{R}s_*\big(\textstyle\bigoplus_\alpha F_\alpha\big)\big) \\ &\simeq^1 \mathrm{Hom}_{D_{qc,\mathcal{Z}}(\mathcal{X})}\big(K, \textstyle\bigoplus_\alpha \boldsymbol{R}s_*(F_\alpha)\big) \\ &\simeq^2 \textstyle\bigoplus_\alpha \mathrm{Hom}_{D_{qc,\mathcal{Z}}(\mathcal{X})}(K, \boldsymbol{R}s_*(F_\alpha)) \\ &\simeq \textstyle\bigoplus_\alpha \mathrm{Hom}_{D_{qc,T}(V)}(s^*(K), F_\alpha), \end{aligned}$$

where $\simeq^1$ follows from the fact that $\boldsymbol{R}s_*$ preserves small coproducts [Hall and Rydh 2017, Lemmas 2.5(3), 2.6(3)], and $\simeq^2$ follows since $K \in D_{qc,Z}^c(\mathcal{X})$. This shows that $s^*(K) \in D_{qc,T}^c(V)$. Since $V$ is affine, this implies that $s^*(K)$ is perfect. $\qquad\square$

**2E.** *Excision for derived category.* We now prove an excision property for the derived category of perfect complexes on stacks using the technique of Cartan–Eilenberg resolutions.

Let $\mathcal{A}$ be a Grothendieck category and let $D(\mathcal{A})$ denote the unbounded derived category of $\mathcal{A}$. Let $\mathrm{Ch}(\mathcal{A})$ denote the category of all chain complexes over $\mathcal{A}$. An object $A \in \mathrm{Ch}(\mathcal{A})$ is said to be *K-injective* if for every acyclic complex $J \in \mathrm{Ch}(\mathcal{A})$, the complex $\mathrm{Hom}^\bullet(J, A)$ is acyclic. Since $\mathcal{A}$ has enough injectives, a complex over $\mathcal{A}$ has a Cartan–Eilenberg resolution; see [EGA III$_1$ 1961, Chapitre 0, (11.4.2)].

It is known that a Cartan–Eilenberg resolution of an unbounded complex over $\mathcal{A}$ need not, in general, be a $K$-injective resolution. However, when $\mathcal{X}$ is a scheme or a Noetherian and separated Deligne–Mumford stack over a fixed Noetherian base scheme, it has been shown that for a complex $J$ of $\mathcal{O}_{\mathcal{X}}$-modules with quasicoherent cohomology, the total complex of a Cartan–Eilenberg resolution does give a $K$-injective resolution of $J$; see [Keller 1998; Krishna 2009, Proposition 2.2]. Our first objective is to extend these results to all algebraic stacks. We follow the techniques of [Krishna 2009] closely. Given a double complex $J^{\bullet,\bullet}$, let Tot$\hat{}(J)$ denote the (product) total complex.

**Proposition 2.10.** *Let $\mathcal{X}$ be a stack and let $K \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$. Let $E \xrightarrow{\epsilon} I^{\bullet,\bullet}$ be a Cartan–Eilenberg resolution of $E$ in $\mathrm{Ch}(\mathcal{X})$. Then $E \xrightarrow{\epsilon} \mathrm{Tot}\hat{}(I)$ is a $K$-injective resolution of $E$.*

*Proof.* Since $\mathrm{Mod}(\mathcal{X})$ is a Grothendieck category and $I^{\bullet,\bullet}$ is a Cartan–Eilenberg resolution, Tot$\hat{}(I)$ is a $K$-injective complex by [Weibel 1996, A.3]. We only need to show that $E \xrightarrow{\epsilon} \mathrm{Tot}\hat{}(I)$ is a quasi-isomorphism. Let

$$\tau^{\geq p}(E) := 0 \to E^p/B^p E \to E^{p+1} \to \cdots$$

denote the good truncation of $E$. Then $\{\tau^{\geq p}(E)\}_{p \in \mathbb{Z}}$ gives an inverse system of bounded below complexes with surjective maps such that $E \xrightarrow{\simeq} \varprojlim_p \tau^{\geq p}(E)$. Let $\tau^{\geq p}(I)$ denote the double complex whose $i$-th row is the good truncation of the $i$-th row of $I^{\bullet,\bullet}$ as above.

Let $L_p^{\bullet,\bullet} = \mathrm{Ker}(\tau^{\geq p}(I) \to \tau^{\geq p+1}(I))$. Then $I^{\bullet,\bullet} \twoheadrightarrow \tau^{\geq p}(I) \twoheadrightarrow \tau^{\geq p+1}(I)$ and $I^{\bullet,\bullet} \xrightarrow{\simeq} \varprojlim_p \tau^{\geq p}(I)$. Therefore, Tot$\hat{}(I) \xrightarrow{\simeq} \varprojlim_p \mathrm{Tot}\hat{}(\tau^{\geq p}(I))$. Moreover, since $\tau^{\geq p}(I)$ is a Cartan–Eilenberg resolution of the bounded below complex $\tau^{\geq p}(E)$, it is known that for each $p \in \mathbb{Z}$, $\tau^{\geq p}(E) \xrightarrow{\epsilon_p} \mathrm{Tot}\hat{}(\tau^{\geq p}(I))$ is a quasi-isomorphism.

Furthermore, the standard properties of Cartan–Eilenberg resolutions imply that $B^p E \to B^p I^{\bullet,\bullet}$ is an injective resolution and hence, the inclusions $B^p I^{\bullet,i} \hookrightarrow I^{\bullet,i}$ are all split. In particular, the maps $\tau^{\geq p}(I) \twoheadrightarrow \tau^{\geq p+1}(I)$ are termwise split surjective. Since $\tau^{\geq p}(I)$ are upper half plane complexes with bounded below rows, we conclude that the sequences

$$0 \to \mathrm{Tot}\hat{}(L_p) \to \mathrm{Tot}\hat{}(\tau^{\geq p}(I)) \to \mathrm{Tot}\hat{}(\tau^{\geq p+1}(I)) \to 0 \qquad (2.11)$$

are exact and are split in each degree.

Hence, we see that Tot$\hat{}(I) \xrightarrow{\simeq} \varprojlim_p \mathrm{Tot}\hat{}(\tau^{\geq p}(I))$, where each Tot$\hat{}(\tau^{\geq p}(I))$ is a bounded below complex of injective $\mathcal{O}_{\mathcal{X}}$-modules, and $\epsilon$ is induced by a compatible system of quasi-isomorphisms $\epsilon_p$. Furthermore, Tot$\hat{}(\tau^{\geq p}(I)) \to \mathrm{Tot}\hat{}(\tau^{\geq p+1}(I))$ is degreewise split surjective with kernel Tot$\hat{}(L_p)$, which is a bounded below complex of injective $\mathcal{O}_{\mathcal{X}}$-modules. Since $\mathcal{H}_i(E) \in \mathrm{QC}(\mathcal{X})$ and $\mathrm{QC}(\mathcal{X}) \subseteq \mathrm{Mod}(\mathcal{X})$

satisfies [Laszlo and Olsson 2008, Assumption 2.1.2], the proposition now follows from [Laszlo and Olsson 2008, Proposition 2.1.4]. $\square$

**Corollary 2.12.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks and let $E \in D_{qc}(\mathcal{Y})$. Then the natural map $\boldsymbol{R} f_*(E) \to \varprojlim_n \boldsymbol{R} f_*(\tau^{\geq n}(E))$ is an isomorphism in $D_{qc}(\mathcal{X})$.*

*Proof.* This is easily checked by replacing $E$ by a Cartan–Eilenberg resolution and using properties of Cartan–Eilenberg resolutions and good truncation. $\square$

Recall that a morphism $f : \mathcal{Y} \to \mathcal{X}$ of stacks is *representable* if for every algebraic space $T$ and a morphism $T \to \mathcal{X}$, the fiber product $T \times_{\mathcal{X}} \mathcal{Y}$ is represented by an algebraic space. If $T \times_{\mathcal{X}} \mathcal{Y}$ is represented by a scheme whenever $T$ is a scheme, we say that $f : \mathcal{Y} \to \mathcal{X}$ is *strongly representable*.

**Proposition 2.13.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a strongly representable étale morphism of stacks. Let $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{X}$ be a closed substack such that $f : \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Then $f^* : D_{qc,\mathcal{Z}}(\mathcal{X}) \to D_{qc,\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}}(\mathcal{Y})$ is an equivalence.*

*Proof.* We set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}$. Let us first assume that $E \in D_{qc,\mathcal{Z}}^+(\mathcal{X})$. We claim that the adjunction map $E \to \boldsymbol{R} f_* \circ f^*(E)$ is an isomorphism. The proof of this claim is identical to that of [Krishna and Østvær 2012, Proposition 3.4] which considers the case of schemes and Deligne–Mumford stacks. We take a smooth atlas $s : U \to \mathcal{X}$ with $U \in \mathbf{Sch}_k$ and note that $U \times_{\mathcal{X}} \mathcal{Y} \to U$ is an étale morphism in $\mathbf{Sch}_k$ because $f$ is strongly representable. As in the proof of [Krishna and Østvær 2012, Proposition 3.4], an application of [Hall and Rydh 2017, Corollary 4.13] now reduces the problem to the case of schemes. By similar arguments, if $F \in D_{qc,\mathcal{W}}^-(\mathcal{Y})$, the co-adjunction map $f^* \circ \boldsymbol{R} f_*(F) \to F$ is an isomorphism (see the proof of [Krishna and Østvær 2012, Theorem 3.5] for details).

To prove the proposition, we need to show that $f^*$ is fully faithful and essentially surjective on objects. To prove the first assertion, let $E \in D_{qc,\mathcal{Z}}(\mathcal{X})$. Since $f^*$ is exact, it commutes with good truncation. Applying this to the isomorphism $E \overset{\simeq}{\to} \varprojlim_n \tau^{\geq n}(E)$, we conclude from Corollary 2.12 and what we showed above for the bounded below complexes that the adjunction map $E \to \boldsymbol{R} f_* \circ f^*(E)$ is an isomorphism. If $E' \in D_{qc,\mathcal{Z}}(\mathcal{X})$ is now another object, then

$$
\begin{aligned}
\mathrm{Hom}_{D_{qc,\mathcal{Z}}(\mathcal{X})}(E, E') &\simeq \mathrm{Hom}_{D_{qc,\mathcal{Z}}(\mathcal{X})}(E, \boldsymbol{R} f_* \circ f^*(E')) \\
&\simeq \mathrm{Hom}_{D_{qc}(\mathcal{X})}(E, \boldsymbol{R} f_* \circ f^*(E')) \\
&\simeq^1 \mathrm{Hom}_{D_{qc}(\mathcal{Y})}(f^*(E), f^*(E')) \\
&\simeq \mathrm{Hom}_{D_{qc,\mathcal{W}}(\mathcal{Y})}(f^*(E), f^*(E')),
\end{aligned}
$$

where $\simeq^1$ follows from the adjointness of $(f^*, \boldsymbol{R} f_*)$ [Krishna 2009, Lemma 3.3].

To prove the essential surjectivity of $f^*$, let $F \in D_{qc, \mathcal{W}}(\mathcal{Y})$. If $F \in D_{qc, \mathcal{W}}^-(\mathcal{Y})$, then we have shown above that the map $f^* \circ \boldsymbol{R} f_*(F) \to F$ is an isomorphism. The general case follows from the bounded above case using the isomorphism $\varinjlim_n \tau^{\leq n}(F) \xrightarrow{\simeq} F$.                                                                    $\square$

## 3. Algebraic *K*-theory of nice quotient stacks

In this section, we prove Theorem 1.1. Let $\mathcal{X}$ be a stack. We begin with the definition and some preliminary results on the *K*-theory spectrum for stacks.

**3A.** *K-theory spectrum.* The algebraic *K*-theory spectrum $K(\mathcal{X})$ of $\mathcal{X}$ is defined to be the *K*-theory spectrum of the complicial bi-Waldhausen category of perfect complexes in $\mathrm{Ch}_{qc}(\mathcal{X})$ in the sense of [Thomason and Trobaugh 1990, §1.5.2]. Here, the complicial bi-Waldhausen category structure is given with respect to the degreewise split monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. The homotopy groups of the spectrum $K(\mathcal{X})$ are defined to be the *K*-groups of the stack $\mathcal{X}$ and are denoted by $K_n(\mathcal{X})$. Note that these groups are 0 if $n < 0$; see [Thomason and Trobaugh 1990, §1.5.3]. We shall extend this definition to negative integers later in this section. For a closed substack $\mathcal{Z}$ of $\mathcal{X}$, $K(\mathcal{X} \text{ on } \mathcal{Z})$ is the *K*-theory spectrum of the complicial bi-Waldhausen category of those perfect complexes on $\mathcal{X}$ which are acyclic on $\mathcal{X} \setminus \mathcal{Z}$.

**Lemma 3.1.** *For a stack $\mathcal{X}$ with affine diagonal, the inclusion of the complicial bi-Waldhausen category of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules into the category of perfect complexes in $\mathrm{Ch}_{qc}(\mathcal{X})$ induces a homotopy equivalence of their K-theory spectra.*

*Similarly, for a closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$, $K(\mathcal{X} \text{ on } \mathcal{Z})$ is homotopy equivalent to the K-theory spectra of the complicial bi-Waldhausen category of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules which are acyclic on $\mathcal{X} \setminus \mathcal{Z}$.*

*Proof.* For a stack $\mathcal{X}$ with affine diagonal, by [Lurie 2005, Theorem 3.8] the inclusion functors $\Phi : \mathrm{Ch}(\mathrm{QC}(\mathcal{X})) \to \mathrm{Ch}_{qc}(\mathcal{X})$ and $\Phi_{\mathcal{Z}} : \mathrm{Ch}_{\mathcal{Z}}(\mathrm{QC}(\mathcal{X})) \to \mathrm{Ch}_{qc, \mathcal{Z}}(\mathcal{X})$ induce equivalences of their left bounded derived categories. Therefore, they restrict to the equivalences of the derived homotopy categories of the bi-Waldhausen categories of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules and that of perfect complexes in $\mathrm{Ch}_{qc}(\mathcal{X})$, both with support in $\mathcal{Z}$ in the case of $\Phi_{\mathcal{Z}}$. By [Thomason and Trobaugh 1990, Theorem 1.9.8], these inclusions therefore induce homotopy equivalence of their *K*-theory spectra.                                                                    $\square$

**Lemma 3.2.** *Let $\mathcal{X}$ be a quotient stack with the resolution property. Consider the following list of complicial bi-Waldhausen categories:*

 (1) *bounded complexes of vector bundles on $\mathcal{X}$,*

(2) *perfect complexes in* $\mathrm{Ch}(\mathrm{QC}(\mathcal{X}))$,

(3) *perfect complexes in* $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$.

*Then the obvious inclusion functors induce homotopy equivalences of all their K-theory spectra. Furthermore, $K(\mathcal{X})$ is homotopy equivalent to the algebraic K-theory spectrum of the exact category of vector bundles on $\mathcal{X}$.*

*Proof.* The inclusion of (1) in (2) induces a homotopy equivalence of $K$-theory spectra by Proposition 2.6 and [Thomason and Trobaugh 1990, Theorem 1.9.8]. The inclusion of (2) in (3) induces homotopy equivalence of $K$-theory spectra by Lemma 3.1. The last assertion follows from [Thomason and Trobaugh 1990, Theorem 1.11.7]. □

**3B.** *The localization and excision for K-theory.* We now establish the localization sequence and excision for the $K$-theory of nice quotient stacks. We begin with the following localization at the level of $D_{\mathrm{qc}}(\mathcal{X})$.

**Proposition 3.3.** *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \hookrightarrow X$ be a closed substack with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$. Assume that $\mathcal{X}$ has the resolution property. Then the following hold.*

(1) $D_{\mathrm{qc}}(\mathcal{X})$, $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ *and* $D_{\mathrm{qc}}(\mathcal{U})$ *are compactly generated.*

(2) *The functor*

$$j^* : \frac{D_{\mathrm{qc}}(\mathcal{X})}{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})} \to D_{\mathrm{qc}}(\mathcal{U})$$

*is an equivalence of triangulated categories.*

*Proof.* The stack $\mathcal{U}$ has the resolution property by our assumption and [Gross 2017, Theorem A]. It follows from Proposition 2.7 that every perfect complex in $D_{\mathrm{qc}}(\mathcal{X})$ is compact, i.e., $\mathcal{X}$ is *concentrated*. Since $\mathcal{X}$ and $\mathcal{U}$ have affine diagonal with resolution property, it follows from [Hall and Rydh 2017, Proposition 8.4] that $D_{\mathrm{qc}}(\mathcal{X})$, $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ and $D_{\mathrm{qc}}(\mathcal{U})$ are compactly generated.

The second statement is an easy consequence of adjointness of the functors $(j^*, \boldsymbol{R}j_*)$ and works exactly like the case of schemes. One checks easily that $j^*$ is fully faithful and $j^* \circ \boldsymbol{R}j_*$ is the identity on $D_{\mathrm{qc}}(\mathcal{U})$. □

**Theorem 3.4** (localization sequence). *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \hookrightarrow X$ be a closed substack with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$. Assume that $\mathcal{X}$ has the resolution property. Then the morphism of spectra $K(\mathcal{X} \text{ on } \mathcal{Z}) \to K(\mathcal{X}) \to K(\mathcal{U})$ induce a long exact sequence*

$$\cdots \to K_i(\mathcal{X} \text{ on } \mathcal{Z}) \to K_i(\mathcal{X}) \to K_i(\mathcal{U}) \to K_{i-1}(\mathcal{X} \text{ on } \mathcal{Z})$$

$$\to \cdots \to K_0(\mathcal{X} \text{ on } \mathcal{Z}) \to K_0(\mathcal{X}) \to K_0(\mathcal{U}).$$

*Proof.* It follows from Proposition 2.7, Lemma 2.8 and Proposition 3.3 that there is a commutative diagram of triangulated categories

$$
\begin{array}{ccccc}
D_{\mathcal{Z}}^{\mathrm{perf}}(\mathcal{X}) & \longrightarrow & D^{\mathrm{perf}}(\mathcal{X}) & \longrightarrow & D^{\mathrm{perf}}(\mathcal{U}) \\
\downarrow & & \downarrow & & \downarrow \\
D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X}) & \longrightarrow & D_{\mathrm{qc}}(\mathcal{X}) & \longrightarrow & D_{\mathrm{qc}}(\mathcal{U})
\end{array}
\tag{3.5}
$$

where the bottom row is a localization sequence of triangulated categories and the top row is the sequence of full subcategories of compact objects of the corresponding categories in the bottom row. Moreover, each triangulated category in the bottom row is generated by its compact objects in the top row. We can thus apply [Neeman 1992, Theorem 2.1] to conclude that the functor

$$
\frac{D^{\mathrm{perf}}(\mathcal{X})}{D_{\mathcal{Z}}^{\mathrm{perf}}(\mathcal{X})} \to D^{\mathrm{perf}}(\mathcal{U})
\tag{3.6}
$$

is fully faithful, and an equivalence up to direct factors.

Let $\Sigma$ be the category whose objects are perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, and where a map $x \to y$ is a weak equivalence if the restriction $x|_{\mathcal{U}} \to y|_{\mathcal{U}}$ is a quasi-isomorphism in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{U})$. The cofibrations in $\Sigma$ are degreewise split monomorphisms. Then it is easy to see that $\Sigma$ is a complicial bi-Waldhausen model for the quotient category $D^{\mathrm{perf}}(\mathcal{X})/D_{\mathcal{Z}}^{\mathrm{perf}}(\mathcal{X})$. Thus, by the Waldhausen localization theorem [Thomason and Trobaugh 1990, Theorems 1.8.2, 1.9.8], there is a homotopy fibration of spectra $K(\mathcal{X} \text{ on } \mathcal{Z}) \to K(\mathcal{X}) \to K(\Sigma)$. It follows from (3.6) and [Neeman 1992, Lemma 0.6] that $K(\Sigma) \to K(\mathcal{U})$ is a covering map of spectra. In particular, $K_i(\Sigma) \xrightarrow{\simeq} K_i(\mathcal{U})$ for $i \geq 1$ and $K_0(\Sigma) \hookrightarrow K_0(\mathcal{U})$. $\qquad\square$

**Theorem 3.7** (excision). *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack. Let $f : \mathcal{Y} \to \mathcal{X}$ be a strongly representable étale morphism of stacks such that $f : \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Assume that $\mathcal{X}, \mathcal{Y}$ have the resolution property. Then $f^*$ induces a homotopy equivalence*

$$
f^* : K(\mathcal{X} \text{ on } \mathcal{Z}) \xrightarrow{\simeq} K(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}).
$$

*Proof.* We observe that since $f$ is strongly representable, $\mathcal{Y}$ is also a nice quotient stack. The theorem now follows directly from Lemma 2.8 and Proposition 2.13. $\quad\square$

**3C.** *Projective bundle formula.* In order to define the nonconnective $K$-theory of stacks, we need the projective bundle formula for their $K$-theory. This formula for the equivariant $K$-theory was proven in [Thomason 1993a, Theorem 3.1]. We adapt the argument of Thomason to extend it to the $K$-theory of all stacks. Though this formula is used in this text only for quotient stacks, its most general form plays

a crucial role in [Hoyois and Krishna 2017]. For details on the projective bundles over algebraic stacks, see [Laumon and Moret-Bailly 2000, Chapter 14].

**Theorem 3.8.** *Let $\mathcal{X}$ be a stack, $\mathcal{E}$ a vector bundle of rank $d$ and $p : \mathbb{P}\mathcal{E} \to \mathcal{X}$ the projective bundle associated to it. Let $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ be the fundamental invertible sheaf on $\mathbb{P}\mathcal{E}$ and $\mathcal{O}_{\mathbb{P}\mathcal{E}}(i)$ its $i$-th power in the group of invertible sheaves over $\mathcal{X}$.*

*Then the morphism of $K$-theory spectra induced by the exact functor that sends a sequence of $d$ perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, $(E_0, \ldots, E_{d-1})$ to the perfect complex*

$$p^* E_0 \oplus \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \otimes p^* E_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}\mathcal{E}}(1-d) \otimes p^* E_{d-1}$$

*induces a homotopy equivalence*

$$\Phi : \prod_d K(\mathcal{X}) \xrightarrow{\sim} K(\mathbb{P}\mathcal{E}).$$

*Similarly, for each closed substack $\mathcal{Z}$, the exact functor restricts to the subcategory of complexes acyclic on $\mathcal{X} \setminus \mathcal{Z}$ to give a homotopy equivalence*

$$\Phi : \prod_d K(\mathcal{X} \text{ on } \mathcal{Z}) \xrightarrow{\sim} K(\mathbb{P}\mathcal{E} \text{ on } \mathbb{P}(\mathcal{E}|\mathcal{Z})).$$

We need the following steps to prove this theorem.

**Lemma 3.9.** *Under the hypothesis of Theorem 3.8, let $F$ be a perfect complex in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ or in general a complex with quasicoherent and bounded cohomology. Then the canonical adjunction morphism (3.10) is a quasi-isomorphism:*

$$\eta : F \xrightarrow{\sim} \boldsymbol{R}p_* p^* F = \boldsymbol{R}p_*(\mathcal{O}_{\mathbb{P}\mathcal{E}} \otimes p^* F). \tag{3.10}$$

*In addition, for $j = 1, 2, \ldots, d - 1$, we have as a result of cancellation*

$$\boldsymbol{R}p_*(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-j) \otimes p^* F) \simeq 0. \tag{3.11}$$

*Proof.* The assertion of the lemma is fppf local on $\mathcal{X}$. Let $u : U \to \mathcal{X}$ be a smooth atlas for $\mathcal{X}$, where $U$ is a scheme. Since $p : \mathbb{P}\mathcal{E} \to \mathcal{X}$ is strongly representable, we can apply [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] to reduce to the case when $\mathcal{X} \in \mathbf{Sch}_k$. In this latter case, the lemma is proven in [Thomason 1993a, Lemma 3]. $\square$

**Lemma 3.12.** *Under the hypothesis of Theorem 3.8, if $E$ is a perfect complex in $\mathrm{Ch}_{\mathrm{qc}}(\mathbb{P}\mathcal{E})$, then the following hold.*

(1) $\boldsymbol{R}p_*(E)$ *is a perfect complex in* $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$.

(2) *If $\boldsymbol{R}p_*(E \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(i))$ is acyclic on $\mathcal{X}$ for $i = 0, 1, \ldots, d - 1$, then $E$ is acyclic on $\mathbb{P}\mathcal{E}$.*

*Proof.* Since the assertion is fppf local on $\mathcal{X}$ and the perfectness is checked by base change of $\mathcal{X}$ by smooth morphisms from affine schemes, we can use [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] again to replace $\mathcal{X}$ by a scheme. Part (1) then follows from [Thomason 1993a, Lemma 4] and (2) follows from [Thomason 1993a, Lemma 5]. □

*Proof of Theorem 3.8.* The proof follows exactly along the lines of the proof of [Thomason 1993a, Theorem 1], using Lemmas 3.9 and 3.12, which generalize [Thomason 1993a, Lemmas 3, 4, 5] to stacks. □

**3D. *K-theory of regular blow-ups of stacks.*** A closed immersion $\mathcal{Y} \to \mathcal{X}$ of stacks over $k$ is defined to be a *regular immersion* of codimension $d$ if there exists a smooth atlas $U \to \mathcal{X}$ of $\mathcal{X}$ such that $\mathcal{Y} \times_{\mathcal{X}} U \to U$ is a regular immersion of schemes of codimension $d$. This is well defined as $U$ is Noetherian and regular immersions behave well under flat base change and satisfy fpqc descent. For a closed immersion $i : \mathcal{Y} \to \mathcal{X}$, the blow-up of $\mathcal{X}$ along $\mathcal{Y}$ is defined to be $p : \widetilde{\mathcal{X}} = \mathrm{Proj}\big(\bigoplus_{n \geq 0} \mathcal{I}_{\mathcal{Y}}^n\big) \to \mathcal{X}$. See [Laumon and Moret-Bailly 2000, Chapter 14] for relative proj construction on stacks. Note that in the case of a regular immersion, $\widetilde{\mathcal{X}} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is a projective bundle over $\mathcal{Y}$, similar to schemes.

**Theorem 3.13.** *Let $i : \mathcal{Y} \to \mathcal{X}$ be a regular immersion of codimension $d$ of stacks. Let $p : \mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Y}$ and $j : \mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{X}', q : \mathcal{Y}' \to \mathcal{Y}$ be the maps obtained by base change. Then the square*

$$
\begin{array}{ccc}
K(\mathcal{X}) & \xrightarrow{\ i^* \ } & K(\mathcal{Y}) \\
{\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle q^*} \\
K(\mathcal{X}') & \xrightarrow[\ j^* \ ]{} & K(\mathcal{Y}')
\end{array}
\tag{3.14}
$$

*is homotopy Cartesian.*

*Proof.* This is proved in [Cortiñas et al. 2008, Proposition 1.5] in the case of schemes and an identical proof works for the case of stacks, in the presence of the results of Section 3C and Lemma 3.16. We give some details on the strategy of the proof. For $r = 0, \ldots, d-1$, let $D_r^{\mathrm{perf}}(\mathcal{X}') \subset D^{\mathrm{perf}}(\mathcal{X}')$ be the full triangulated subcategory generated by $Lp^* F$ and $Rj_* Lq^* G \otimes \mathcal{O}_{\mathcal{X}'}(-l)$ for $F \in D^{\mathrm{perf}}(\mathcal{X}), G \in D^{\mathrm{perf}}(\mathcal{Y})$ and $l = 1, \ldots, r$. Let $D_r^{\mathrm{perf}}(\mathcal{Y}') \subset D^{\mathrm{perf}}(\mathcal{Y}')$ be the full triangulated subcategory generated by $Lq^* G \otimes \mathcal{O}_{\mathcal{Y}'}(-l)$ for $G \in D^{\mathrm{perf}}(\mathcal{Y})$ and $l = 0, \ldots, r$. By Lemmas 3.9 and 3.16(1), $Lp^* : D^{\mathrm{perf}}(\mathcal{X}) \to D_0^{\mathrm{perf}}(\mathcal{X}')$ and $Lq^* : D^{\mathrm{perf}}(\mathcal{Y}) \to D_0^{\mathrm{perf}}(\mathcal{Y}')$ are equivalences. Exactly as in [Cortiñas et al. 2008, Lemma 1.2], one shows that $D_{d-1}^{\mathrm{perf}}(\mathcal{X}') = D^{\mathrm{perf}}(\mathcal{X}')$ and $D_{d-1}^{\mathrm{perf}}(\mathcal{Y}') = D^{\mathrm{perf}}(\mathcal{Y}')$ using Lemmas 3.12 and 3.16.

To prove the theorem, it is sufficient to show that $Lj^*$ is compatible with the filtrations on $D^{\mathrm{perf}}(\mathcal{X}')$ and $D^{\mathrm{perf}}(\mathcal{Y}')$:

$$D^{\text{perf}}(\mathcal{X}) \xrightarrow[\sim]{Lp^*} D_0^{\text{perf}}(\mathcal{X}') \hookrightarrow D_1^{\text{perf}}(\mathcal{X}') \hookrightarrow \cdots \hookrightarrow D_{d-1}^{\text{perf}}(\mathcal{X}') = D^{\text{perf}}(\mathcal{X}')$$

$$\left\downarrow{\scriptstyle Li^*} \qquad \left\downarrow{\scriptstyle Lj^*} \qquad \left\downarrow{\scriptstyle Lj^*} \qquad\qquad \left\downarrow{\scriptstyle Lj^*} \qquad (3.15)$$

$$D^{\text{perf}}(\mathcal{Y}) \xrightarrow[\sim]{Lq^*} D_0^{\text{perf}}(\mathcal{Y}') \hookrightarrow D_1^{\text{perf}}(\mathcal{Y}') \hookrightarrow \cdots \hookrightarrow D_{d-1}^{\text{perf}}(\mathcal{Y}') = D^{\text{perf}}(\mathcal{Y}')$$

and that for $r = 0, \ldots, d-2$, $Lj^*$ induces equivalences on quotient triangulated categories:

$$Lj^* : D_{r+1}^{\text{perf}}(\mathcal{X}')/D_r^{\text{perf}}(\mathcal{X}') \xrightarrow{\sim} D_{r+1}^{\text{perf}}(\mathcal{Y}')/D_r^{\text{perf}}(\mathcal{Y}').$$

Given this, it follows from [Thomason and Trobaugh 1990, Theorems 1.8.2, 1.9.8] that every square in (3.15) induces a homotopy Cartesian square of $K$-theory spectra.

To prove the compatibility of $Lj^*$, it is enough to check on generators and in this case, it can be reduced to the case of schemes using [Hall and Rydh 2017, Corollary 4.13]. To prove that $Lj^*$ induces equivalence on quotients, we first note that the composition

$$Lj^* \circ [\mathcal{O}_{\mathcal{X}'}(-r-1) \otimes Rj_*Lq^*] : D^{\text{perf}}(\mathcal{Y}) \to D_{r+1}^{\text{perf}}(\mathcal{X}')/D_r^{\text{perf}}(\mathcal{X}')$$
$$\to D_{r+1}^{\text{perf}}(\mathcal{Y}')/D_r^{\text{perf}}(\mathcal{Y}')$$

agrees with $\mathcal{O}_{\mathcal{Y}'}(-r-1) \otimes Lq^* : D^{\text{perf}}(\mathcal{Y}) \to D_{r+1}^{\text{perf}}(\mathcal{Y}')/D_r^{\text{perf}}(\mathcal{Y}')$, up to a natural equivalence. This follows as in the proof of [Cortiñas et al. 2008, Lemma 1.4] using [Hall and Rydh 2017, Corollary 4.13]. Therefore, it is enough to show that the functors $\mathcal{O}_{\mathcal{X}'}(-r-1) \otimes Rj_*Lq^*$ and $\mathcal{O}_{\mathcal{Y}'}(-r-1) \otimes Lq^*$ are equivalences. But the proof of this is exactly the same as the one in [Cortiñas et al. 2008, Proposition 1.5] for schemes. $\qquad\square$

**Lemma 3.16.** *Under the hypotheses of Theorem 3.13, the following hold.*

(1) *Let $F$ be a perfect complex on $\mathcal{X}$. Then the canonical adjunction morphism (3.17) is a quasi-isomorphism:*

$$\eta : F \xrightarrow{\sim} Rp_*Lp^*F = Rp_*(\mathcal{O}_{\mathcal{X}'} \otimes Lp^*F). \qquad (3.17)$$

(2) *Let $r$ be an integer such that $1 \le r \le d-1$. Let $\mathcal{A}_r'$ denote the full triangulated subcategory of $D^{\text{perf}}(\mathcal{X}')$ of those complexes $E$ for which $Rp_*(E \otimes \mathcal{O}_{\mathcal{X}'}(i)) \simeq 0$ for $0 \le i < r$. Then there exists a natural transformation $\partial$ of functors from $\mathcal{A}_r'$ to $D^{\text{perf}}(\mathcal{X}')$:*

$$\partial : (Rj_*Lq^*Rq_*(E \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{O}_{\mathcal{Y}'}(r-1)) \otimes \mathcal{O}_{\mathcal{X}'}(-r))[-1] \to E. \qquad (3.18)$$

*Moreover, $Rp_*(\partial \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is a quasi-isomorphism for $0 \le i < r+1$.*

(3) *Suppose $E \in D^{\text{perf}}(\mathcal{X}')$ is such that $Rp_*(E \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is acyclic on $\mathcal{X}$ for $i = 0, \ldots, d-1$. Then $E$ is acyclic on $\mathcal{X}'$.*

*Proof.* Statements (1) and (3) are proved in [Thomason 1993b] for schemes. The general case can be deduced from this exactly as in Lemmas 3.9 and 3.12. For (2), the existence of $\partial$ follows from [Thomason 1993b, Lemma 2.4(a)] as the construction of $\partial$ given there is natural in $\mathcal{X}$ for schemes. To check that $Rp_*(\partial \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is a quasi-isomorphism for $0 \le i < r+1$, we may again assume that $\mathcal{X}$ is a scheme, and this case follows from [loc. cit., Lemma 2.4(a)]. □

**3E.** *Negative K-theory of stacks.* Let $\mathcal{U} \hookrightarrow \mathcal{X}$ be an open immersion of stacks over $k$. As $K_0(\mathcal{X}) \to K_0(\mathcal{U})$ is not always surjective in the localization theorem, we want to introduce a nonconnective spectrum $\mathbb{K}(-)$ with $K(-)$ as its $(-1)$-connective cover, so that $\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{U})$ is a homotopy fiber sequence for any closed substack $\mathcal{Z}$ of $\mathcal{X}$ with complement $\mathcal{U}$. We define $\mathbb{K}$ only in the absolute case below. The construction of $\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z})$ follows similarly, as shown in [Thomason and Trobaugh 1990]. We shall use the following version of the Bass fundamental theorem for stacks to define $\mathbb{K}(\mathcal{X})$. The homotopy groups of $\mathbb{K}(\mathcal{X})$ will be denoted by $\mathbb{K}_i(\mathcal{X})$.

**Theorem 3.19** (Bass fundamental theorem). *Let $\mathcal{X}$ be a nice quotient stack with the resolution property and let $\mathcal{X}[T] = \mathcal{X} \times \mathrm{Spec}(k[T])$. Then the following hold.*

(1) *For $n \ge 1$, there is an exact sequence*

$$0 \to K_n(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_n(\mathcal{X}[T]) \oplus K_n(\mathcal{X}[T^{-1}])$$
$$\xrightarrow{(j_1^*, j_2^*)} K_n(\mathcal{X}[T, T^{-1}]) \xrightarrow{\partial_T} K_{n-1}(\mathcal{X}) \to 0.$$

*Here $p_1^*$, $p_2^*$ are induced by the projections $\mathcal{X}[T] \to \mathcal{X}$, etc., and $j_1^*$, $j_2^*$ are induced by the open immersions $\mathcal{X}[T^{\pm 1}] = \mathcal{X}[T, T^{-1}] \to \mathcal{X}[T]$, etc. The sum of these exact sequences for $n = 1, 2, \ldots$ is an exact sequence of graded $K_*(\mathcal{X})$-modules.*

(2) *For $n \ge 0$, $\partial_T : K_{n+1}(\mathcal{X}[T^{\pm 1}]) \to K_n(\mathcal{X})$ is naturally split by a map $h_T$ of $K_*(\mathcal{X})$-modules. Indeed, the cup product with $T \in K_1(k[T^{\pm 1}])$ splits $\partial_T$ up to a natural automorphism of $K_n(\mathcal{X})$.*

(3) *There is an exact sequence for $n = 0$:*

$$0 \to K_n(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_n(\mathcal{X}[T]) \oplus K_n(\mathcal{X}[T^{-1}]) \xrightarrow{(j_1^*, j_2^*)} K_n(\mathcal{X}[T^{\pm 1}]).$$

*Proof.* It follows from [Thomason 1987b, Lemma 2.6] that $\mathbb{P}^1_{\mathcal{X}}$ and $\mathcal{X}[T]$ are nice quotient stacks with the resolution property. It follows from Theorem 3.8 that there is an isomorphism $K_*(\mathbb{P}^1_{\mathcal{X}}) \simeq K_*(\mathcal{X}) \oplus K_*(\mathcal{X})$, where the two summands are $K_*(\mathcal{X})[\mathcal{O}]$ and $K_*(\mathcal{X})[\mathcal{O}(-1)]$ with respect to the external product $K(\mathcal{X}) \wedge K(\mathbb{P}^1_k) \to K(\mathbb{P}^1_{\mathcal{X}})$ and with $[\mathcal{O}], [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1_k)$. As for schemes, (1) now

follows directly from Theorems 3.4 and 3.7; see also [Thomason and Trobaugh 1990, Theorem 6.1].

For (2), it suffices to show that the composite map

$$\partial_T (T \cup p^*(-)) : K_n(\mathcal{X}) \to K_{n+1}(\mathcal{X}[T^{\pm 1}]) \to K_n(\mathcal{X})$$

is an automorphism of $K_n(\mathcal{X})$ for $n \geq 0$. By naturality and the fact that $\partial_T$ is a map of $K_*(\mathcal{X})$-modules, this reduces to showing that $\partial_T : K_1(k[T^{\pm 1}]) \to K_0(k)$ sends $T$ to $\pm 1$. But this is well known and (3) follows from (2) using the analogue of [Thomason and Trobaugh 1990, (6.1.5)] for stacks. $\square$

**Theorem 3.20.** *For a nice quotient stack $\mathcal{X}$ with the resolution property, there is a spectrum $\mathbb{K}(\mathcal{X})$ together with a natural map of spectra $K(\mathcal{X}) \to \mathbb{K}(\mathcal{X})$ which induces isomorphism $K_i(\mathcal{X}) \xrightarrow{\simeq} \mathbb{K}_i(\mathcal{X})$ for $i \geq 0$.*

*Let $\mathcal{Y}$ be a nice quotient stack with the resolution property and let $f : \mathcal{Y} \to \mathcal{X}$ be a strongly representable étale map. Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack such that $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ be the projective bundle associated to a vector bundle $\mathcal{E}$ on $\mathcal{X}$ of rank $r$. Then the following hold.*

(1) *There is a homotopy fiber sequence of spectra*

$$\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{X} \setminus \mathcal{Z}).$$

(2) *The map $f^* : \mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})$ is a homotopy equivalence.*

(3) *The map $\prod_0^{r-1} \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathbb{P}(\mathcal{E})), (a_0, \ldots, a_{r-1}) \mapsto \sum_i \mathcal{O}[-i] \otimes \pi^*(a_i)$, is a homotopy equivalence.*

*Proof.* The construction of the spectrum $\mathbb{K}(\mathcal{X})$ follows directly from Theorem 3.19 by the formalism given in (6.2)–(6.4) of [Thomason and Trobaugh 1990]. Like for schemes, the proof of (1), (2) and (3) is a standard deduction from Theorems 3.4, 3.7 and 3.8, using the inductive construction of $\mathbb{K}(\mathcal{X})$. $\square$

**3F.** *Schlichting's negative $K$-theory.* Schlichting [2006] defined negative $K$-theory of complicial bi-Waldhausen categories. Let $\mathcal{X}$ be a nice quotient stack. Schlichting's negative $K$-theory spectrum $K^{\mathrm{Scl}}(\mathcal{X})$ is the $K$-theory spectrum of the Frobenius pair associated to the category $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$. It follows from [Schlichting 2006, Theorem 8] that $K_i^{\mathrm{Scl}}(\mathcal{X}) = K_i(\mathcal{X})$ for $i \geq 0$. The following result shows that $K_i^{\mathrm{Scl}}(\mathcal{X})$ agrees with $\mathbb{K}_i(\mathcal{X})$ for $i < 0$.

**Theorem 3.21.** *Let $\mathcal{X}$ be a nice quotient stack with the resolution property. Then there are natural isomorphisms between $K_i^{\mathrm{Scl}}(\mathcal{X})$ and $\mathbb{K}_i(\mathcal{X})$ for $i \leq 0$.*

*Proof.* Let $p : \mathbb{P}^1_{\mathcal{X}} \to \mathcal{X}$ be the projection map. Then we can prove as in Theorem 3.8 that the functors $p^* : D^{\mathrm{perf}}(\mathcal{X}) \to D^{\mathrm{perf}}(\mathbb{P}^1_{\mathcal{X}})$ and $\mathcal{O}(-1) \otimes p^* : D^{\mathrm{perf}}(\mathcal{X}) \to D^{\mathrm{perf}}(\mathbb{P}^1_{\mathcal{X}})$,

which are induced by maps of their Frobenius models, induce isomorphisms

$$(p^*, \mathcal{O}(-1) \otimes p^*) : K_i^{\mathrm{Scl}}(\mathcal{X}) \oplus K_i^{\mathrm{Scl}}(\mathcal{X}) \xrightarrow{\simeq} K_i^{\mathrm{Scl}}(\mathbb{P}_{\mathcal{X}}^1)$$

for $i \leq 0$. It follows from the proof of Bass' fundamental theorem in [Thomason and Trobaugh 1990, Theorem 6.6(b)] that there is an exact sequence of abelian groups

$$0 \to K_i^{\mathrm{Scl}}(\mathcal{X}) \to K_i^{\mathrm{Scl}}(\mathcal{X}[T]) \oplus K_i^{\mathrm{Scl}}(\mathcal{X}[T^{-1}]) \to K_i^{\mathrm{Scl}}(\mathcal{X}[T, T^{-1}]) \to K_{i-1}^{\mathrm{Scl}}(\mathcal{X}) \to 0$$

for $i \leq 0$. As $K_0^{\mathrm{Scl}}(\mathcal{Y}) = \mathbb{K}_0(\mathcal{Y})$ for any stack $\mathcal{Y}$, the negative $K$-groups coincide. $\square$

## 4. Nisnevich descent for $K$-theory of quotient stacks

In this section, we prove Nisnevich descent in a 2-category of stacks whose objects are all quotients of schemes by action of a fixed group scheme. So let $G$ be a group scheme over $k$. Let $\mathbf{Sch}_k^G$ denote the category of separated schemes of finite type over $k$ with $G$-action. The equivariant Nisnevich topology on $\mathbf{Sch}_k^G$ and the homotopy theory of simplicial sheaves in this topology was defined and studied in detail in [Heller et al. 2015]. As an application of Theorem 3.20, we shall show in this section that the $K$-theory of quotient stacks for $G$-actions satisfies descent in the equivariant Nisnevich topology on $\mathbf{Sch}_k^G$.

**Definition 4.1** [Heller et al. 2015, Definition 2.1]. A *distinguished equivariant Nisnevich square* is a Cartesian square

$$
\begin{array}{ccc}
B & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle p} \\
A & \underset{j}{\hookrightarrow} & X
\end{array}
\tag{4.2}
$$

in $\mathbf{Sch}_k^G$ such that

(1) $j$ is an open immersion,

(2) $p$ is étale, and

(3) the induced map $(Y \setminus B)_{\mathrm{red}} \to (X \setminus A)_{\mathrm{red}}$ of schemes (without reference to the $G$-action) is an isomorphism.

**Remark 4.3.** We remark here that given a Cartesian square of the type (4.2) in $\mathbf{Sch}_k^G$, the closed subscheme $(X \setminus A)_{\mathrm{red}}$ (or $(Y \setminus B)_{\mathrm{red}}$) may not in general be $G$-invariant, unless $G$ is smooth. However, it follows from [Thomason 1987a, Lemma 2.5] that we can always find a $G$-invariant closed subscheme $Z \subset X$ such that $Z_{\mathrm{red}} = X \setminus A$. This closed subscheme can be assumed to be reduced if $G$ is smooth. Using the elementary fact that a morphism of schemes is étale if and only if the induced map of the associated reduced schemes is étale, it follows immediately that the condition (3) in Definition 4.1 is equivalent to

(3′) there is a $G$-invariant closed subscheme $Z \subset X$ with support $X \setminus A$ such that the map $Z \times_X Y \to Z$ in $\mathbf{Sch}_k^G$ is an isomorphism.

The collection of distinguished equivariant Nisnevich squares forms a *cd*-structure in the sense of [Voevodsky 2010]. The associated Grothendieck topology is called the equivariant *Nisnevich* topology. It is also called the *eN*-topology. It follows from [Heller et al. 2015, Theorem 2.3] that the equivariant Nisnevich *cd*-structure on $\mathbf{Sch}_k^G$ is complete, regular, and bounded. We refer to [Voevodsky 2010, §2] for the definition of a complete, regular, and bounded *cd*-structure.

Let $\mathbf{Sch}_{k/\mathrm{Nis}}^G$ denote the category of $G$-schemes $X$, such that $X$ admits a family of $G$-equivariant ample line bundles, equipped with the equivariant Nisnevich topology. Note that all objects of $\mathbf{Sch}_{k/\mathrm{Nis}}^G$ have the resolution property by Lemma 2.3. It follows from [Heller et al. 2015, Corollary 2.11] that for a sheaf $\mathcal{F}$ of abelian groups on $\mathbf{Sch}_{k/\mathrm{Nis}}^G$, we have $H_{G/\mathrm{Nis}}^i(X, \mathcal{F}) = 0$ for $i > \dim(X)$.

**Definition 4.4.** An equivariant morphism $Y \to X$ in $\mathbf{Sch}_k^G$ *splits* if there is a filtration of $X$ by $G$-invariant closed subschemes

$$\varnothing = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_0 = X, \tag{4.5}$$

such that for each $j$, the map $(X_j \setminus X_{j+1}) \times_X Y \to X_j \setminus X_{j+1}$ has a $G$-equivariant section. If $f$ is étale and surjective, the morphism is called an equivariant *split étale cover of X*.

**4A.** *Equivariant Nisnevich covers.* In [Heller et al. 2015, Proposition 2.15], it is shown that an equivariant étale morphism $Y \to X$ in $\mathbf{Sch}_k^G$ is an equivariant Nisnevich cover if and only if it splits. Further, when $G$ is a finite constant group scheme, it is shown that an equivariant étale map $f : Y \to X$ in $\mathbf{Sch}_k^G$ is an equivariant Nisnevich cover if and only if for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and $f$ induces isomorphisms $k(x) \simeq k(y)$ and $S_y \simeq S_x$. Here, for a point $x \in X$, the set-theoretic stabilizer $S_x \subseteq G$ is defined by $S_x = \{g \in G \mid g.x = x\}$ [Heller et al. 2015, Proposition 2.17].

Let $G^0$ denote the connected component of the identity element in $G$. Suppose that $G$ is of the form $G = \coprod_{i=0}^r g_i G^0$, where $\{e = g_0, g_1, \ldots, g_r\}$ are points in $G(k)$ which represent the left cosets of $G^0$. In the next proposition, we give an explicit description of the equivariant Nisnevich covers of reduced schemes $X \in \mathbf{Sch}_k^G$. For $x \in X$, let $\widetilde{S}_x := \{g_i \mid 0 \leq i \leq r, \ g_i.x = x\}$.

**Proposition 4.6.** *Let $G$ be a smooth affine group scheme over $k$ as above. A morphism $f : Y \to X$ in $\mathbf{Sch}_k^G$ is an equivariant split étale cover of a reduced scheme $X$ if and only if for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and $f$ induces isomorphisms $k(x) \simeq k(y)$ and $\widetilde{S}_y \simeq \widetilde{S}_x$.*

*Proof.* It is clear that a split étale $G$-equivariant family of morphisms satisfies the given conditions. The heart of the proof is to show the converse.

Suppose $Y \xrightarrow{f} X$ is such that for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and $f$ induces isomorphisms $k(x) \simeq k(y)$ and $\widetilde{S}_y \simeq \widetilde{S}_x$. Let $W$ be the regular locus of $X$. Then $W$ is a $G$-invariant dense open subscheme of $X$. Set $U = Y \times_X W$. Notice that $W$ is a disjoint union of its irreducible components, and each $f_U$ being étale, it follows that $U$ is regular and hence a disjoint union of its irreducible components.

Let $x \in W$ be a generic point of $W$. Then the closure $W_x = \overline{\{x\}}$ in $W$ is an irreducible component of $W$. By our assumption, there is a point $y \in U$ such that

$$f(y) = x, \qquad k_x \xrightarrow{\simeq} k_y, \quad \text{and} \quad \widetilde{S}_y \xrightarrow{\simeq} \widetilde{S}_x. \tag{4.7}$$

Then the closure $U_y = \overline{\{y\}}$ in $U$ is an irreducible component of $U$. Since $U_y \to W_x$ is étale and generically an isomorphism, it must be an open immersion. Thus $f$ maps $U_y$ isomorphically onto an open subset of $W_x$. We replace $W_x$ by this open subset $f(U_y)$ and call it our new $W_x$.

Let $GU_y$ be the image of the action morphism $\mu : G \times U_y \to U$. Notice that $\mu$ is a smooth map and hence open. This in particular implies that $GU_y$ is a $G$-invariant open subscheme of $U$ as $U_y$ is one of the disjoint irreducible components of $U$ and hence open. By the same reason, $GW_x$ is a $G$-invariant open subscheme of $W$.

Since the identity component $G^0$ is connected, it keeps $U_y$ invariant. Therefore, $y \in U$ is fixed by $G^0$ and hence $G$ acts on this point via its quotient $\overline{G} = G/G^0$. Since each $g_j G^0$ takes $U_y$ onto an irreducible component of $U$ and since $U$ has only finitely many irreducible components which are all disjoint, we see that $GU_y = U_0 \amalg U_1 \amalg \cdots \amalg U_n$ is a disjoint union of some irreducible components of $U$ with $U_0 = U_y$. In particular, for each $U_j$, we have $U_j = g_{j_i} G^0 U_y = g_{j_i} U_y$ for some $j_i$.

Since $f$ maps $U_y$ isomorphically onto $W_x$, we conclude from the above that $f$ maps each $U_j$ isomorphically onto one and only one $W_j$ such that $GW_x = f(GU_y) = W_0 \amalg W_1 \amalg \cdots \amalg W_m$ (with $m \leq n$) is a disjoint union of open subsets of some irreducible components of $W$ with $W_0 = W_x$. The morphism $f$ will map the open subscheme $GU_y$ isomorphically onto the open subscheme $GW_x$ if and only if no two components of $GU_y$ are mapped onto one component of $GW_x$. This is ensured by using the condition (4.7).

If two distinct components of $GU_y$ are mapped onto one component of $GW_x$, we can (using the equivariance of $f$) apply automorphisms by the $g_{j_i}$ and assume that one of these components is $U_y$. In particular, we can find $j \geq 1$ such that

$$W_x = f(U_y) = f(U_j) = f(g_{j_i} U_y) = g_{j_i} f(U_y) = g_{j_i} W_x. \tag{4.8}$$

But this implies that $g_{j_i} \in \widetilde{S}_x$ and $g_{j_i} \notin \widetilde{S}_y$. This violates the condition in (4.7) that $\widetilde{S}_y$ and $\widetilde{S}_x$ are isomorphic. We have thus shown that the morphism $f$ has a

$G$-equivariant splitting over a nonempty $G$-invariant open subset $GW_x$. Letting $X_1$ be the complement of this open subset in $X$ with reduced scheme structure, we see that $X_1$ is a proper $G$-invariant closed subscheme of $X$, and by restricting our cover to $X_1$, we get a cover for $X_1$ satisfying the given conditions. The proof of the proposition is now completed by the Noetherian induction. $\qquad\square$

**4B. *Equivariant Nisnevich descent.*** It is shown in [Heller et al. 2015, §3] that the category of presheaves of $S^1$-spectra on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ (denoted by $\mathrm{Pres}(\mathbf{Sch}^G_{k/\mathrm{Nis}})$) is equipped with the global and local injective model structures. A morphism $f : \mathcal{E} \to \mathcal{E}'$ of presheaves of spectra is called a *global weak equivalence* if the map $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a weak equivalence of $S^1$-spectra for every object $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. It is a global injective cofibration if $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a cofibration of $S^1$-spectra for every object $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. The map $f$ is called a *local weak equivalence* if it induces an isomorphism on the sheaves of stable homotopy groups of the presheaves of spectra in the *eN*-topology. A local (injective) cofibration is the same as a global injective cofibration.

A presheaf of spectra $\mathcal{E}$ on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ is said to satisfy the *equivariant Nisnevich descent* (*eN*-descent) if the fibrant replacement map $\mathcal{E} \to \mathcal{E}'$ in the local injective model structure of $\mathrm{Pres}(\mathbf{Sch}^G_{k/\mathrm{Nis}})$ is a global weak equivalence. Let $\mathcal{K}^G$ denote the presheaf of spectra on $\mathbf{Sch}^G_k$ which associates the spectrum $\mathbb{K}([X/G])$ to any $X \in \mathbf{Sch}^G_k$. As a consequence of Theorem 3.20, we obtain the following.

**Theorem 4.9.** *Let $G$ be a nice group scheme over $k$. Then the presheaf of spectra $\mathcal{K}^G$ on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ satisfies the equivariant Nisnevich descent.*

*Proof.* Since the *eN*-topology is regular, complete and bounded by [Heller et al. 2015, Theorem 2.3], it suffices to show using [Voevodsky 2010, Proposition 3.8] that $\mathcal{K}^G$ takes a square of the type (4.2) to a homotopy Cartesian square of spectra. In other words, we need to show that the square

$$
\begin{CD}
\mathbb{K}([X/G]) @>j^*>> \mathbb{K}([A/G]) \\
@Vp^*VV @VVp'^*V \\
\mathbb{K}([Y/G]) @>>j'^*> \mathbb{K}([B/G])
\end{CD}
\tag{4.10}
$$

is homotopy Cartesian. But this is an immediate consequence of Theorem 3.20. $\quad\square$

**Corollary 4.11.** *Let $G$ be a nice group scheme over $k$ and let $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. Then there is a strongly convergent spectral sequence*

$$
E_2^{p,q} = H^p_{eN}(X, \mathcal{K}^G_q) \Rightarrow \mathbb{K}_{q-p}([X/G]).
$$

*Proof.* This is immediate from Theorem 4.9 and [Heller et al. 2015, Theorem 2.3, Corollary 2.11]. $\qquad\square$

## 5. Homotopy invariance of *K*-theory with coefficients for quotient stacks

It is known that with finite coefficients, the ordinary algebraic *K*-theory of schemes satisfies the homotopy invariance property (see [Weibel 1989, Theorem 1.2, Proposition 1.6] for affine schemes and [Thomason and Trobaugh 1990, Theorem 9.5] for the general case). This is a hard result which was achieved by first defining a homotopy invariant version of algebraic *K*-theory [Weibel 1989] and then showing that with finite coefficients, this homotopy (invariant) *K*-theory coincides with the algebraic *K*-theory.

However, the proof of the agreement between algebraic *K*-theory and homotopy *K*-theory with finite coefficients requires the knowledge of a spectral sequence relating *NK*-theory and homotopy *K*-theory; see [Weibel 1989, Remark 1.3.1]. Recall here that $NK(X)$ denotes the homotopy fiber of the pull-back map $\iota^*$, where $\iota : X \hookrightarrow \mathbb{A}^1_k \times X$ denotes the 0-section embedding into the trivial line bundle over a scheme $X$. The existence of homotopy *K*-theory for quotient stacks is not yet known and one does not know if the above spectral sequence would exist for the homotopy *K*-theory of quotient stacks. In this section, we adopt a different strategy to extend the results of Weibel and Thomason–Trobaugh to the *K*-theory of nice quotient stacks (see Theorem 5.5).

**5A.** *Homotopy K-theory of stacks.* For $n \in \mathbb{N}$, let

$$\Delta_n = \operatorname{Spec}\big(k[t_0, \ldots, t_n]/(\textstyle\sum_i t_i - 1)\big).$$

Recall that $\Delta_\bullet = \{\Delta_n\}_{n \geq 0}$ forms a simplicial scheme whose face and degeneracy maps are given by the formulas

$$\partial_r(t_j) = \begin{cases} t_j & \text{if } j < r, \\ 0 & \text{if } j = r, \\ t_{j-1} & \text{if } j > r, \end{cases} \qquad \delta_r(t_j) = \begin{cases} t_j & \text{if } j < r, \\ t_j + t_{j+1} & \text{if } j = r, \\ t_{j+1} & \text{if } j > r. \end{cases}$$

**Definition 5.1.** For a nice quotient stack $\mathcal{X}$ with the resolution property, the *homotopy K-theory* is defined to be the spectrum

$$KH(\mathcal{X}) = \operatorname{hocolim}_n \mathbb{K}(\mathcal{X} \times \Delta_n).$$

It is clear from the definition that $KH(\mathcal{X})$ is contravariant with respect to morphisms of stacks. Furthermore, there is a natural map of spectra $\mathbb{K}(\mathcal{X}) \to KH(\mathcal{X})$. It is well known that $\mathbb{K}(\mathcal{X})$ is not a homotopy invariant functor. Our first result on $KH(\mathcal{X})$ is the following.

**Theorem 5.2.** *Let $\mathcal{X}$ be a nice quotient stack with the resolution property, and let $f : \mathcal{E} \to \mathcal{X}$ be a vector bundle morphism. Then the associated pull-back map $f^* : KH(\mathcal{X}) \to KH(\mathcal{E})$ is a homotopy equivalence.*

*Proof.* We first show that the map $KH(\mathcal{X}) \to KH(\mathcal{X} \times \Delta_n)$ is a homotopy equivalence for every $n \geq 0$. But this is essentially a direct consequence of the definition of $KH$-theory. By identifying $\Delta_n$ with $\mathbb{A}_k^n$ and using induction, one needs to show that the map $KH(\mathcal{X}) \to KH(\mathcal{X}[T])$ is a homotopy equivalence. Proof of this is identical to the case of the $KH$-theory of schemes [Weibel 1989, Theorem 1.2].

To prove the general case, we write $\mathcal{X} = [X/G]$, where $G$ is a group scheme over $k$ acting on a $k$-scheme $X$. We let $E = u^*(\mathcal{E})$, where $u : X \to \mathcal{X}$ is the quotient map. Then $E$ is a $G$-equivariant vector bundle on $X$ such that $[E/G] \simeq \mathcal{E}$.

We consider the standard fiberwise contraction map $H : \mathbb{A}_k^1 \times E \to E$. Explicitly, for an open affine $U = \mathrm{Spec}(A) \subseteq X$ over which $f$ is trivial (without $G$-action), $H|_U$ is induced by the $k$-algebra homomorphism $A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n, T]$ given by $X_j \mapsto TX_j$. It is then clear that this defines a unique map $H$ as above which is $G$-equivariant for the trivial $G$-action on $\mathbb{A}_k^1$. We have the commutative diagram

$$
\begin{array}{ccc}
& \{1\} \times E & \\
{\scriptstyle \mathrm{id}} \swarrow & \Big\downarrow {\scriptstyle i_1} & \searrow {\scriptstyle h_1} \\
E \xleftarrow{\; p \;} \mathbb{A}_k^1 \times E & \xrightarrow{\; H \;} & E \\
{\scriptstyle \mathrm{id}} \nwarrow & \Big\uparrow {\scriptstyle i_0} & \nearrow {\scriptstyle h_0} \\
& \{0\} \times E &
\end{array}
\tag{5.3}
$$

where $h_j = H \circ i_j$ for $j = 0, 1$ and $p$ is the projection map.

Let $\iota : X \hookrightarrow E$ denote the 0-section embedding, so that $f \circ \iota = \mathrm{id}_X$. So we only need to show that $f^* \circ \iota^*$ is the identity on $KH([E/G])$. Since $h_0 = \iota \circ f$, it suffices to show that $h_0^*$ is the identity.

It follows from the weaker version of homotopy invariance shown above (applied to $E$) that $p^*$ is an isomorphism on the $KH$-theory of the stack quotients. In particular, $i_0^* = (p^*)^{-1} = i_1^*$. Since $h_1 = \mathrm{id}_E$, we get $i_1^* \circ H^* = \mathrm{id}$, which in turn yields $H^* = (i_1^*)^{-1} = p^*$ and hence $h_0^* = i_0^* \circ H^* = i_0^* \circ p^* = \mathrm{id}$. This finishes the proof. $\qquad \square$

## 5B. *Proof of Theorem 1.2.*

The proof of Theorem 1.2 is a direct consequence of the definition of $KH(\mathcal{X})$ and similar results for the $\mathbb{K}$-theory. Part (1) of the theorem is Theorem 5.2. Part (2) follows directly from Theorems 3.20 and 3.13 because the homotopy colimit preserves homotopy fiber sequences.

We now prove (3). Let $G$ be a finite group acting on a scheme $X$ such that $X$ admits an ample family of line bundles. Then $X$ is covered by $G$-invariant affine open subschemes. By Theorem 4.9, it suffices to prove the theorem when

$X = \mathrm{Spec}(A)$ is affine. In this case, $\mathbb{K}([X/G])$ is homotopy equivalent to the $K$-theory of the exact category $\mathcal{P}^G(A)$ of finitely generated $G$-equivariant projective $A$-modules (see Lemma 3.2).

Since $G$ is also assumed to be nice, it follows from [Levine and Serpé 2008, Lemma 1.3] that $\mathcal{P}^G(A)$ is equivalent to the exact category $\mathcal{P}(A^{\mathrm{tw}}[G])$ of finitely generated projective $A^{\mathrm{tw}}[G]$-modules. Recall here that $A^{\mathrm{tw}}[G] = \bigoplus_{g \in G} A e_g$ and the product is defined by $(r_g \cdot e_g)(r_h \cdot e_h) = r_g \cdot (r_h \star g^{-1}) e_{gh}$, where $\star$ indicates the $G$-action on $A$.

If $I$ is a nilpotent ideal of $A$ with quotient $B = A/I$, it follows from Lemma 5.4 that the map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is surjective and its kernel is a nilpotent ideal of $A^{\mathrm{tw}}[G]$. We now apply [Weibel 1989, Theorem 2.3] to conclude that the map $KH(A^{\mathrm{tw}}[G]) \to KH((A/I)^{\mathrm{tw}}[G])$ is a homotopy equivalence. Since $G$ acts trivially on $\Delta_\bullet$, there is a canonical isomorphism $(A[\Delta_\bullet])^{\mathrm{tw}}[G] \simeq (A^{\mathrm{tw}}[G])[\Delta_\bullet]$. We conclude that the map $KH([\mathrm{Spec}(A)/G]) \to KH([\mathrm{Spec}(B)/G])$ is a homotopy equivalence. This finishes the proof. $\qquad\square$

**Lemma 5.4.** *Let $G$ be a finite group acting on commutative unital rings $A$ and $B$. Let $A \twoheadrightarrow B$ be a $G$-equivariant surjective ring homomorphism whose kernel is nilpotent. Then the induced map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is surjective and its kernel is nilpotent.*

*Proof.* Let $I$ denote the kernel of $f : A \twoheadrightarrow B$. By hypothesis, there exists an integer $n$ such that $I^n = 0$. Since the induced map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is a $G$-graded homomorphism induced by $f$ on each graded piece, it is a surjection and its kernel is given by $I^{\mathrm{tw}}[G] = \bigoplus_{g \in G} I e_g$. Since $I$ is a $G$-invariant ideal of $A$, each element of $(I^{\mathrm{tw}}[G])^n$ is of the form $(a_1.e_{g_1} + \cdots + a_m.e_{g_m})$, where $g_i \in G$ and $a_i \in I^n$. Therefore, $(I^{\mathrm{tw}}[G])^n = 0$. $\qquad\square$

**5C. $\mathbb{K}$-*theory of stacks with coefficients.*** For an integer $n \in \mathbb{N}$, let

$$\mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) := \mathrm{hocolim}(\mathbb{K}(\mathcal{X}) \xrightarrow{\cdot n} \mathbb{K}(\mathcal{X}) \xrightarrow{\cdot n} \cdots),$$

$$\mathbb{K}(\mathcal{X}; \mathbb{Z}/n) := \mathbb{K}(\mathcal{X}) \wedge \mathbb{S}/n,$$

where $\mathbb{S}/n$ is the mod-$n$ Moore spectrum. Our final result is the homotopy invariance property of $\mathbb{K}$-theory with coefficients.

The proof of Theorem 5.5 uses the notion of $K$-theory of dg-categories. We briefly recall its definition and refer to [Keller 2006, §5.2] for further details. Let $\mathcal{A}$ be a small dg-category. The category $D(\mathcal{A})$ is the localization of the category of dg $\mathcal{A}$-modules with respect to quasi-isomorphisms. The category of perfect objects $\mathrm{Per}(\mathcal{A})$ is the smallest triangulated subcategory of $D(\mathcal{A})$ containing the representable objects and closed under shifts, extensions and direct factors. The algebraic $K$-theory of $\mathcal{A}$ is defined to be the $K$-theory spectrum of the Waldhausen

category Per($\mathcal{A}$), where the cofibrations are the degreewise split monomorphisms and the weak equivalences are the quasi-isomorphisms.

**Theorem 5.5.** *Let $\mathcal{X}$ be a nice quotient stack over $k$ with the resolution property and let $f : \mathcal{E} \to \mathcal{X}$ be a vector bundle. Then the following hold.*

(1) *For any integer $n$ invertible in $k$, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer $n$ nilpotent in $k$, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}[1/n])$ is a homotopy equivalence.*

*Proof.* The category Perf($\mathcal{X}$) has a natural dg enhancement [Cisinski and Tabuada 2012, Example 5.5] whose algebraic $K$-theory (in the sense of $K$-theory of dg-categories) coincides with $\mathbb{K}(\mathcal{X})$ by [Keller 2006, Theorem 5.1]. It follows from Proposition 2.7 and [Hall and Rydh 2017, Proposition 8.4] that $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated and every perfect complex on $\mathcal{X}$ is compact. We conclude from [Tabuada 2017, Theorem 1.2] that the theorem holds when $f$ is the projection map $\mathcal{X}[T] \to \mathcal{X}$. To prove the general case, we use (5.3) and repeat the argument of Theorem 5.2 verbatim. $\square$

**Corollary 5.6.** *Let $\mathcal{X}$ be as in Theorem 5.5. Then the following hold.*

(1) *For any integer $n$ invertible in $k$, the natural map $\mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \to KH(\mathcal{X}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer $n$ nilpotent in $k$, the natural map $\mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) \to KH(\mathcal{X}; \mathbb{Z}[1/n])$ is a homotopy equivalence.*

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