Fragmentation at the Foundation

by

Hans Robin Solberg
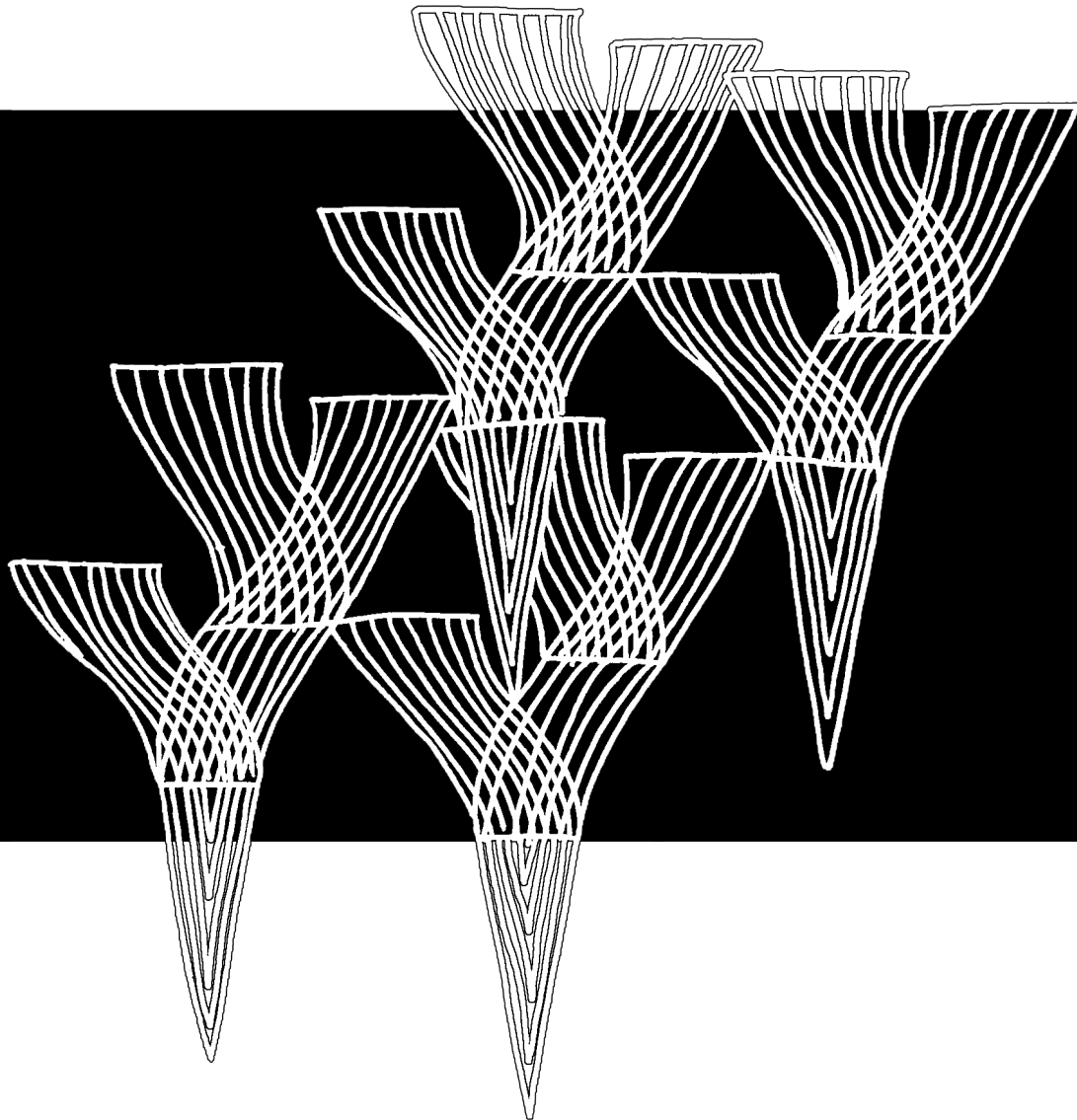
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Abstract

The standard axiomatization of set theory known as ZFC provides maybe the most widely accepted foundation for mathematics. But there are natural mathematical statements, such as Cantor's continuum hypothesis (CH), that can neither be proved nor disproved from ZFC. These statements are said to be independent of ZFC. The independence phenomenon in set theory can be seen as motivating two very different conceptions of mathematical reality.

Universism has it that ZFC is about a particular mathematical structure, namely the maximal and unique universe of all sets V. The universist claims that the independence phenomenon shows us that our description of V is incomplete but, claims the universist, the independent statements are either true or false in the unique universe. So, we ought to try to formulate and justify new axioms to strengthen our theory and uniquely decide the independent statements. This way we get a more complete description of V. *Multiversism*, on the other hand, has it that there are many different universes of sets where statements independent of ZFC, such as CH, hold in some of them and fail in others. Thus, there is no point in trying to decide these statements in a unique manner. Instead, we should be content to explore in more detail the different set theoretic universes, all of which, according to the multiversist, are equally real.

In this thesis I further explore multiversism. I assess some of the strengths and weaknesses of the view by focusing primarily on philosophical rather than technical issues. I also investigate in more detail in what way multiversism challenges the standard accounts of justification in set theory that the universists have traditionally hoped to avail themselves of.

Acknowledgements

Contents

Introduction

Whatever the universe of mathematics is, whether real or not, whether an independently existing domain or a product of our minds, it is surely a strange place. Examples abound: we have the one-sided nonorientable surface with boundary known as the Möbius strip, the geometric figure called Gabriel's horn which has finite volume but infinite surface area, the empty function from $\varnothing$ to $\varnothing$, or take, for example, the fact that there is a one-to-one correspondence between the unit interval $[0, 1]$ and all the points of $\mathbb{R}^n$, for any natural number n, a fact that prompted Cantor, who proved it, to write "I see it, but I don't believe it!". Yet another example is the "paradoxical" yet provable Banach-Tarski decomposition of a solid ball in 3-dimensional space into a finite number of pieces that can be rearranged so as to yield two balls of the same size as the original ball. And there is the transfinite hierarchy of modern-day set theory, with its unending stock of larger and larger infinities. The universe of mathematics is not only strange but also immensely rich in entities and structures.

But things may be stranger yet. So far we have referred to "the universe" of mathematics, assuming that there is sense to be made of a unique and coherent domain where all mathematical entities and structures belong. But can we really talk about *the* universe of mathematics at all? What if mathematical reality is so varied, so fractured, that it cannot be thought of in a single coherent framework? Might it be that there is not one mathematical universe but a plurality of universes?

In addition to the richness of mathematical objects and structures, there is also a seeming open-endedness to mathematical operations that suggests mathematical reality

cannot be a completely unified and delimited whole. This thought was nicely expressed by the mathematician Saunders Mac Lane:

> Understanding Mathematical operations leads repeatedly to the formation of totalities: The collection of all prime numbers, the set of all points on an ellipse, the manifold of all lines in 3-space, the manifold of all positions and velocities of a mechanical system, the set of all subsets of a set, the set of all power series expansions for a function (its Riemannian surface) or the category of all topological spaces. There are no upper limits; it is useful to consider the "universe" of all sets (a class) or the category *Cat* of all small categories as well as CAT, the category of all big categories. This is the idea of a *totality*, and these are some of its many formulations. After each careful delimitation, bigger totalities appear. No set theory and no category theory can encompass them all–and they are needed to grasp what Mathematics does. (Mac Lane 1986:390)

Despite this richness of entities and the open-endedness of mathematical operations, some still think that the idea of a definite universe wherein all of mathematics belongs makes sense and that there could be a theory capable of encompassing all of mathematics. That way we could provide some unity to all the varied riches of mathematics. Others think differently, and claim that the idea of a definite universe does not make sense in the case of mathematics, and that mathematical reality is somehow indefinite or that there must be a plurality of universes.

The aim of this thesis is to examine such views, especially views that posit many universes, from a philosophical stance while drawing on mathematical and metamathematical results. The focus will be on set theory. As we will see, this allows the questions about the unity or disunity of mathematics raised so far, together with potential answers, to be put in sharper terms.

Why set theory?

A response to the seeming disunity of mathematics characterized above is to avoid it by providing a *foundation* or framework for mathematics. One chief task for a foundational or framework theory, then, is to show that all the varied entities and structures we find in different branches of mathematics really can be thought of in a unified way.

Set theory developed, in part, around the aim of providing such a foundational theory. By considering only sets and how they are structured by the membership-relation, mathematicians have been able to give a language and axiomatic theory capable of unifying all (or at least almost all) of conventional mathematics. Today, the standard axiomatization

of set theory is called ZFC which is stated in the language $\mathscr{L}_\in$, which is a first-order language with $\in$ as the only non-logical symbol.[1]

The language of set theory is greatly expressive in that all mathematical language (developed so far) can be translated into it, and ZFC is a powerful mathematical theory able to prove most of conventional mathematics. One way to express this is to say that any acceptable informal mathematical proof should be reconstructable as a purely formal derivation from the axioms of ZFC. For example, within set theory one can develop the whole of classical arithmetic and analysis by representing their objects of study (natural numbers and real numbers), and the operations that apply to them (addition, multiplication and so on), as sets. Any theorems provable from the standard axioms of arithmetic and analysis suitably translated are thus provable in ZFC. This holds also of almost all other branches of mathematics, such as geometry, topology and algebra. In this way, set theory allows us to bring all the varied entities and structures throughout the different branches of mathematics into one vast arena. This unification of mathematics in set theory is a truly remarkable achievement of twentieth century mathematics.

Having said that, what exactly is required of a foundational or framework theory for mathematics, if a foundation is required at all, is up for discussion. Many take set theory to be the best candidate for a foundation around, while others have defended different foundations such as Category theory (see Lawvere 1964 and 1966, and for further discussion, Feferman 2013) and more recently the project known under the name of Univalent foundations closely related to the development of homotopy type theory. All the same, ZFC remains the most widely accepted foundational theory.

Maddy (2017) presents several roles a foundational theory, such as ZFC, can be asked to play. Two of them she deems spurious:

- **Epistemic Source:** Knowledge of the foundational axioms of set theory together with knowledge of the mathematically acceptable rules of inference is to ground our knowledge of the theorems of mathematics.
- **Metaphysical Insight:** The reduction of a mathematical object to a given set reveals the true metaphysical identity of that object.

However, there are also five legitimate roles, summarized by Maddy (2017:317) as:

- **Meta-mathematical Corral:** To allow for meta-mathematical consideration of the whole expanse of the vast subject of classical mathematics at once.
- **Elucidation:** To provide the conceptual resources and construction techniques to clarify old mathematical notions in order to take on new demands.
- **Risk Assessment:** To provide a scale of consistency strength (such as the large cardinal hierarchy in set theory arguably does).

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[1]The language and theory is given in Chapter 1 below.

- **Shared Standard:** To serve as a benchmark of mathematical proof. As such, ZFC is a theory of provability in mathematics.
- **Generous Arena:** To give a framework in which the various branches of mathematics appear side-by-side, so that the objects, results, methods and resources of classical mathematics can be pooled for fruitful interaction.

Several of these roles have been both defended and argued against by various philosophers and mathematicians. For example, Burgess (2015) stresses the importance of several of these roles, such as **Shared Standard** and **Generous Arena**, while Tanswell (2015) argues against **Shared Standard** by trying to motivate that not all informally legitimate proofs are formalizable (at least in a way that stays true to the original content of the informal proof). A last, non-foundational role for set theory is as our best mathematical theory of the infinite. For our purposes, it is in what way, if at all, set theory provides a **Generous Arena** that is the central issue.

Similarly to how number theorists think of themselves as describing and studying a specific class of objects and its structure, that is, the set of natural numbers $\mathbb{N}$, many set theorists think of themselves as describing what is called the cumulative hierarchy V, sometimes also called the universe of all sets. The upshot of the unification of mathematics in set theory is that we now have a candidate for the position of being the unique and coherent domain, the vast arena, where all mathematical entities and structures belong, namely the universe of set theory V.

But is there such a unique and definite universe of all sets? Recently, some set theorists and philosophers of mathematics have defended views that deny the existence of a unique and definite V, and not because of a general anti-realism about mathematical objects or structures. Instead, they have argued that we should believe that there is a plurality of distinct universes of sets with distinct set theoretic truths, so-called multiverse views in set theory. As such, they claim that even at the fundamental level there is no single definite universe where mathematics take place but many universes. The foundation itself is fragmented.

So, moving to our foundational or framework theory, we can again ask the question about the unity or disunity of mathematics in the following way: is there, then, a single universe of sets or are there many? Further questions also arise: What *is* a multiverse view in set theory? Why should we believe in such a view? Can it be stated in a philosophically satisfying way? How do more or less philosophical (or at least informal) views about the existence of a definite universe or a multiverse in set theory affect set theoretical research programs, practice and methodology, if at all? These are the central questions that drive this thesis.

Structure of the thesis

In chapter 1, "ZFC, Independence and Gödel's Program", I start by stating the language and axioms of ZFC together with the development of the idea of the cumulative hierarchy V as the structure described by ZFC. After that I review the most central developments in set theory from the construction of Gödel's minimal inner model L and onwards. I focus in particular on the model theory of set theory, the independence phenomenon and the study of possible extensions of ZFC as part of the research program known as Gödel's program.

In chapter 2, "Multiversism in Set Theory", I start by stating the view that set theorists are studying a definite and unique structure V, sometimes called the universe of all sets. I then look at an alternative view, which has gained in popularity recently, claiming that there are many distinct universes of sets. I try to clarify such a multiverse view philosophically primarily by paying attention to conceptual and metaphysical features of the view. The rest of the chapter is devoted to assessing the different strengths and weaknesses of multiversism often while comparing it to universism. The aim of the chapter is to establish multiversism as a philosophically interesting pluralist conception of the subject matter of set theory.

In chapter 3, "Multiversism and Mathematical Evidence", I look at the relationship between the multiverse view and methodology in mathematics. I focus in particular on what happens to the status and legitimacy of two kinds of evidence in mathematics, so-called intrinsic and extrinsic, conditional on the multiverse view being true. I conclude that the legitimacy of these methods in establishing new basic principles of sets is under serious doubt if multiversism is true. In the end I compare the idea of a set theoretic multiverse to the idea of a universe with regards to potential fruitfulness for further set theoretic practice. Although, I agree that for the most part these debates in the philosophy of mathematics probably have little real effect on the practice of set theory, I argue that such ideas understood as heuristic devices might still have some impact on the degree of perceived freedom among set theorists in formulating new research projects.

Chapter 1

ZFC, Independence and Gödel's Program

Before we explore the idea that there are many distinct set theoretic universes it will be useful to review today's most standard axiomatization of set theory, ZFC, together with results and further developments that are important to the later discussion. Particularly important are the independence results in set theory, showing that certain statements ϕ in the language of set theory are such that neither ϕ nor $\neg\phi$ are provable from ZFC. As we will see, these results are pivotal to the motivation for and formulation of both a one-and-definite universe view and a multiverse view.

In this chapter, I first present ZFC together with the idea of the cumulative hierarchy V. After that I turn to the independence phenomenon and its significance. I focus in particular on what is probably the most famous statement independent of ZFC, the so-called continuum hypothesis (abbreviated as CH). As part of this we will look briefly at the way an independence result is established by the use of inner models and forcing. Lastly, I discuss a central research program in set theory known as Gödel's program, formulated in part as a response to the independence phenomenon.

1.1   ZFC

Set theory is the study of sets. A set is a collection of distinct objects, its members, into another object, namely the set of those objects. Here we are mainly concerned with pure set theory. In pure set theory one studies and formulates theories concerning so-called pure sets: a set x is called pure if all the members of x are sets, as are all members of the members of x, and so on. So, in pure set theory, every member of a set is a set.

Today, the standard axiomatization of set theory is ZFC, short for Zermelo-Fraenkel set theory with Choice. In this section we first give the language and state the axioms of ZFC. After that we look at what kind of structure the axioms might be taken to describe.

1.1.1   The Language and Formulas of ZFC

The language of ZFC, $\mathscr{L}_\in$, is a one-sorted first order language with symbols for the standard logical connectives and the identity relation. In addition we add a sole non-logical symbol $\in$, which is a binary relation symbol intuitively interpreted as denoting a membership relation. So $x \in y$ is read as something along the lines of "x is a member of y" or "x is an element of y".

The formulas of $\mathscr{L}_\in$ are built up from the atomic formulas, $x \in y$ and $x = y$, using the standard connectives, $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$, and quantifiers $\forall$ and $\exists$. In addition, it is common to expand $\mathscr{L}_\in$ by adding symbols for defined constants, relations and operations, such as $\varnothing$ (empty set), $\subseteq$ (subset), $\cup$ (union), $\cap$ (intersection) and $\times$ (cartesian product). If care is taken with the definition of each added symbol, each formula in the expanded language can be written in a form that has only $\in$ as the sole non-logical symbol.

For instance, one set being a subset of another, denoted by $\subseteq$, is defined in the following way:

$$x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y).$$

So, x is a subset of y if and only if every member of x is a member of y. This way, whenever we write $x \subseteq y$, we can always replace it with the more cumbersome $\forall z (z \in x \rightarrow z \in y)$ to get rid of the defined symbol.

1.1.2   The Axioms and Axiom Schemas of ZFC

ZFC can be presented in different ways.[1] Here we mainly follow Enderton (1977) and Kunen (2013), and we also give both a formal version and an informal gloss, sometimes with additional comments, of each axiom or axiom scheme. When stating the axioms and axiom schemas we will also help ourselves to defined notions whenever convenient:

Extensionality axiom. $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$
Informally: Any two sets are identical if they share all members.

Empty set axiom.[2] $\exists x \forall y (y \notin x)$
Informally: There is an empty set.
Comment: Furthermore, there is a unique such set: since any two sets x and y having no members trivially share all members, then, by Extensionality, $x = y$. The empty set is denoted by $\varnothing$.

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[1]For some examples, see Enderton (1977), Kunen (1980), Jech (2003) and Kunen (2013).

[2]This axiom can be eschewed assuming there is at least one thing in the domain. Under that assumption one can prove the existence of an empty set using the axioms of Separation. Uniqueness of the empty set follows from Extensionality.

Pairing axiom. $\forall x \forall y \exists y \forall w (w \in y \leftrightarrow w = x \lor w = y)$

Informally: For any two sets x and y there is a set containing just x and y as members.

Union axiom. $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \land w \in x))$

Informally: For any set x, there is a set that has as its only elements any member of a member of x.

Powerset axiom. $\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)$

Informally: For any set x there is a set of all subsets of x, called the powerset of x. We write $\mathscr{P}(x)$ to denote the powerset of x.

Separation axiom scheme.[3] For each formula ϕ in $\mathscr{L}_\in$ without y free, the universal closure (since ϕ might contain parameters) of the following is an axiom:

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \phi(z))$$

Informally: For any set x and condition ϕ, one can "separate" out the ϕ's in x into another set y.

Infinity axiom. $\exists x (\varnothing \in x \land \forall y (y \in x \rightarrow y \cup \{y\} \in x))$

Informally: There is a set with $\varnothing$ as a member and such that if x is a member, then the union of x and its singleton $\{x\}$ is also a member. Thus, there is an infinite set.

Foundation axiom. $\forall x (x \neq \varnothing \rightarrow \exists y (y \in x \land y \cap x = \varnothing))$

Informally: Every nonempty set contains a member that is disjoint from it.

Comment: The axiom rules out the existence of certain types of sets, such as self-membered sets ($x \in x$) and infinite descending membership sequences ($... \in x_2 \in x_1 \in x_0$).

Replacement axiom scheme. For each formula $\phi(p, q)$ in $\mathscr{L}_\in$ without y free, the universal closure (since $\phi(p, q)$ might contain parameters) of the following is an axiom:

$$\forall x [(\forall u \in x) \forall v \forall w (\phi(u, v) \land \phi(u, w) \rightarrow v = w) \rightarrow \exists y \forall z (z \in y \leftrightarrow \exists t (t \in x \land \phi(t, z)))]$$

Informally: The antecedent of the main conditional states that $\phi(p, q)$ is a functional condition. So, each instance of the axiom scheme says that if ϕ defines a function, then the image of any set under ϕ is also a set. That is, we can "replace" any member t of a set with the value of t under the functional relation defined by ϕ and still get a set.

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[3]This being an axiom scheme, there is a separation axiom for each ϕ in $\mathscr{L}_\in$, which yields infinitely many axioms. The same holds for Replacement.

Choice axiom.[4]

$$\forall F(\varnothing \notin F \wedge \forall v \in F \forall w \in F(v \neq w \to v \cap w = \varnothing) \to \exists c \forall x \in F \exists y(c \cap x = \{y\}))$$

Informally: For any set F of pairwise-disjoint nonempty sets, there is a set containing exactly one member from each member of F.

Comment: Here is a picture meant to roughly illustrate the axiom: Imagine that you have a bag F full of nonempty bags $x_0, x_1, x_2...$, then you can always take another bag c for which you choose as members exactly one thing from each bag $x_0, x_1, x_2...$ in the bag F.

These axioms and axiom schemas make up ZFC. The theory is extremely powerful, as noted in the introduction of this thesis, and acts as a framework theory for (almost all of) conventional mathematics. Set theory also deals with its own interesting subject matter as a theory of the infinite that goes far beyond the needs of other branches of mathematics. From this point and onwards I will assume familiarity with ZFC and the basic results and notions of set theory, such as the development of the theory of ordinals.

1.1.3   The Cumulative Hierarchy

What kind of structure or domain are these axioms meant to describe? The traditional answer is that ZFC set theory concerns a structure in which more and more sets are generated bottom-up in a well-ordered sequence of stages. At each stage every collection or plurality of objects formed at an earlier stage is formed into a set, thus generating new objects on which the operation of set-formation can act to form additional sets at subsequent stages.

In pure set theory one starts at the bottom stage where one collects together every object available prior to that stage into a set, and since there are no objects available prior to the bottom stage in pure set theory we form $\varnothing$. From there on a hierarchy of sets is built up along the ordinals by iterating the powerset operation at successor stages and the union operation at limit stages.

Formally, we define the cumulative hierarchy of sets by transfinite recursion on the class of ordinals in the following way:

$$V_0 = \varnothing$$

$$V_{\alpha+1} = \mathscr{P}(V_\alpha)$$

$$V_\lambda = \bigcup_{\beta < \lambda} V_\beta \quad \text{if } \lambda \text{ is a limit ordinal.}$$

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[4]This is but one formulation of the Choice axiom to which there are many equivalent statements, some of which are strikingly different from the one stated here. For more on the axiom and its equivalencies, see Enderton (1977:181–4) and Moore (1982).

Furthermore, it follows from the Foundation axiom that for every set x there is some α such that $x \subseteq V_\alpha$ and $x \in V_{\alpha+1}$ (see Enderton 1977:205–6). So, according to ZFC, every set appears somewhere in the cumulative hierarchy. The cumulative hierarchy is also known as the universe of all sets and is often denoted by V.

Every stage V_α is itself a set, but the universe V is not a set as it is a theorem of ZFC that there is no set to which every set belongs. The existence of such a set would lead to an inconsistency by the following argument: If we had a set of all sets, $V = \{x \mid x = x\}$, then by Separation there would be a set of all sets that do not contain themselves, $R = \{x \in V \mid x \notin x\}$. R is just the Russell set familiar from Russell's famous paradox. Is R a member of itself? Well, $R \in R \leftrightarrow R \in V \wedge R \notin R$. Since V is meant to be the set of all sets and R would have to be in V if it were a set at all, this reduces to $R \in R \leftrightarrow R \notin R$, which is obviously inconsistent. So, there is no set of all sets.

The standard approach is to think of V as a proper class. Informally, a proper class is a collection of objects that are "too many" to form a set. V itself is often characterized in the following way (see for example Jech 2003:64):

$$V = \bigcup V_\alpha \quad \text{for } \alpha \in ON.^5$$

Intuitively, V is just the union of all stages. Of course, this cannot be the same as "union" in the sense of ZFC. If it were, then V would be a set and since, as we noted above, every set is a member of some stage V_α, V would be a set to which every set belongs. So, in the language of ZFC our characterization of V is ill-formed and fails to pick out any object. This can be remedied, however: Either we move to a theory which incorporates proper classes, such as V and ON, as part of the domain together with a language that defines union and membership so that they can be used in the way done above, or we think of proper classes as formulas expressing conditions that a given set may meet. In the former case V and ON would be objects (or pluralities or concepts; the options are many) we could pick out in our formal theory and the characterization above would count as a formal definition.[6] In the latter case we could think of V as a universal condition, such as $x = x$, and ON as the condition that x be an ordinal. In that case, $x \in V$ and

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[5]ON is the class of all ordinals. Since there is no set of all ordinals, by the Burali-Forti paradox, ON is also a proper class.

[6]Some more detail: One approach is to assume a domain where everything is a class and then define a set predicate $S(x)$ as $\exists C(x \in C)$, that is, x is a set if and only if there is a class C of which x is a member. A class A is a proper class if and only if $\neg \exists C(A \in C)$. The schemas of ZFC can now be given as sentences by quantifying over all classes. In addition one has a Class comprehension principle, which is the universal closure of

$$\exists X \forall y (y \in X \leftrightarrow \phi(y)).$$

If we restrict the bound variables in ϕ so that they may only range over sets, we get the proper class theory known as NBG. If we allow the bound variables in ϕ to range over all classes, we get MK. The first is a conservative extension of ZFC, while the second is not. In any case, we get the class V of all sets, defined by $\forall x(S(x) \to x \in V)$.

$x \in ON$ would be read as abbreviations of the open formulas 'x is self-identical' and 'x is an ordinal' (for more on this see Kunen 1980:24).

By looking at the way we define V and its stages, we can identify two "dimensions" to the cumulative hierarchy. First, since the hierarchy is built up along the ordinals, the "height" of the hierarchy is determined by the extent of the class of ordinals. Second, since successor stages are obtained by forming the set of all subsets of the previous stage, the "width" of the hierarchy is determined by what subsets are assigned to a set by the powerset operation. So, at least partly, how clear our picture of V is depends on the clarity of our understanding of the class of ordinals and the powerset operation.

1.2   Independence

One foundational role that some take ZFC to play is as a final court of appeal for questions about the provability or unprovability of mathematical statements, that is, what we in the introduction called providing a **Shared Standard** (see Maddy 2017:296–8). Given a mathematical statement ψ we look at its set theoretic surrogate ϕ and ask if ZFC can prove ϕ or, by proving $\neg\phi$, disprove ϕ. For most mathematical statements of interest the answer will be one of the two: provable from ZFC or disprovable from ZFC. Unfortunately, for many open mathematical problems and conjectures, both within set theory and from other branches of mathematics, the answer is neither (assuming ZFC is consistent). For all its glory, ZFC is incomplete in significant ways.

A sentence ϕ that is neither provable nor disprovable from a first order axiomatic theory T is said to be *independent* of T. The standard way of showing that a given ϕ is independent of a theory T is by constructing at least two models M, N such that $M \models T + \phi$ and $N \models T + \neg\phi$. By the soundness of first order logic, if T proves ϕ, then if $M \models T$, then $M \models \phi$. So, by contraposition, given a model M such that $M \models T + \phi$, then T does not prove $\neg\phi$, and given a model N such that $N \models T + \neg\phi$, then T does not prove ϕ.

It follows from the completeness theorem for first order logic that a theory like ZFC has a model if and only if it is consistent, that is, if no contradiction is provable from it. On the standard approach to models, a model of ZFC would have to be a pair $\langle M, E \rangle$ where M is a nonempty set and E a binary relation on M such that all the axioms of ZFC are true when we let the variables that appear in the axioms range over M and interpret $\in$ as E. Now, the second incompleteness theorem states that a theory like ZFC, if it is consistent, cannot prove the formal statement $CON^{\text{ZFC}} \in \mathscr{L}_{\in}$ expressing the consistency of ZFC. So, if ZFC is consistent, ZFC cannot prove the existence of a model of ZFC. Therefore, if one wants to show that a certain statement ϕ is independent of ZFC using ZFC as the background theory, one cannot straightforwardly construct two models M, N such that ZFC proves that $M \models \text{ZFC} + \phi$ and $N \models \text{ZFC} + \neg\phi$, as this would

be a proof of the consistency of ZFC from ZFC. Instead, one must assume that ZFC is consistent, and therefore has a model, and on this assumption construct the desired models. Or, by contraposing, show that ZFC is inconsistent if ZFC + the statement of interest is inconsistent. These proofs are called relative consistency proofs. If both ZFC + ϕ and ZFC + $\neg\phi$ are consistent relative to the assumed consistency of ZFC, then ϕ is independent of ZFC.

We mentioned earlier that a whole range of mathematical statements have been shown to be independent of ZFC. We now turn to looking at one of these in more detail.

1.2.1 The Continuum Hypothesis

The most famous statement independent of ZFC is probably the continuum hypothesis (CH), a suggested answer to a rather naturally occurring question concerning the cardinality of the set of real numbers. The statement was first conjectured to be true by Georg Cantor, one of the founders of set theory, in the latter half of the nineteenth century, and later shown to be independent of ZFC through the work of Kurt Gödel and Paul Cohen. In this section we state CH, and in the two sections after that, we briefly review the methods that Gödel and Cohen used to establish the independence of CH from ZFC.

The cardinal number κ of a set x denotes the size of that set. Two sets x and y are said to have the same size if they are *equinumerous*, written $x \approx y$, which means that there is a one-to-one correspondence between x and y (we have a one-to-one correspondence between two sets if there is a one-to-one function from x onto y). So we can say that $|x| = |y|$ iff $x \approx y$, where $|x|$ denotes the cardinality of x.[7] For any finite-sized set, the cardinal number of that set will be one of the natural numbers. But the set of all natural numbers, $\omega = \{0, 1, 2, 3, ...\}$, is an infinite set. What, then, should we say about the cardinal number of ω? One of the strengths of set theory, as it originated in the work of Cantor, is that it allows us to speak of the sizes of infinite sets.

The cardinal number of ω is denoted by $\aleph_0$, which is the least infinite cardinal. In fact, many infinite-sized sets have cardinality $\aleph_0$, like the set of all odd numbers, the set of all integers, the set of all rational numbers, and many more. But there are also infinite sets with cardinalities strictly greater than $\aleph_0$. One such set is the set of real numbers $\mathbb{R}$. It can be shown that $\mathbb{R} \approx \mathscr{P}(\omega)$, where $\mathscr{P}(\omega)$ is the powerset of the set of natural numbers. By Cantor's theorem (6B(b) in Enderton 1977), which states that for any given set x, the powerset of x is strictly greater than x, that is, $\forall x(|x| < |\mathscr{P}(x)|)$, we know that $\aleph_0 < |\mathbb{R}|$. Since it also holds that $|\mathbb{R}| = 2^{\aleph_0}$, we can write $\aleph_0 < 2^{\aleph_0}$.

To sum up what we have so far:

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[7]Ultimately, one defines the cardinal of a single set x to be the least ordinal α equinumerous to x; one can then prove the definition above. A cardinal number can be thought of as a special kind of ordinal, a so-called initial ordinal, which is an ordinal with the property of not being equinumerous to any smaller ordinal (for more see Enderton 1977:Chapter 7).

1. $|\omega| = \aleph_0$

2. $|\mathbb{R}| = 2^{\aleph_0}$

3. $\aleph_0 < 2^{\aleph_0}$

We are now ready to state CH. Here is a rather simple question to ask when faced with the third statement on the list: is there any cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$? In other words, is there any size in-between the size of the natural numbers and the size of the real numbers? Cantor hypothesized that the answer is negative; there is no such κ. If we define $\aleph_1$ to be the least infinite cardinal such that $\aleph_0 < \aleph_1$, CH can be stated as

(CH) $$2^{\aleph_0} = \aleph_1.$$

The negation of CH says that $2^{\aleph_0}$ does not equal $\aleph_1$. Since $2^{\aleph_0}$ is strictly greater than $\aleph_0$, then, according to ¬CH, there must be at least one cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$. The *generalized* CH (GCH) states that for any infinite cardinal κ there is no cardinal number λ such that $\kappa < \lambda < 2^{\kappa}$. This can also be stated as: for any α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. This statement entails CH.

1.2.2 Gödel's *L*

The first step towards showing that CH is independent of ZFC was taken by Gödel in the late 1930s. By assuming that ZF (ZFC without the Choice axiom) is consistent, he built a model of ZFC in which GCH (and therefore CH) holds, which establishes that

$$\text{Con(ZF)} \rightarrow \text{Con(ZFC + GCH)}.$$

So, if ZF is consistent, then Choice cannot be disproved from ZF and CH cannot be disproved from either ZF or ZFC.

The proper class model Gödel constructed is known as the constructible hierarchy, denoted by L. In a similar manner to V, L is defined by transfinite recursion on the class of ordinals starting with $\varnothing$. The important difference is that, at successor stages, instead of taking the full powerset of V_α to get $V_{\alpha+1}$, one only takes the subsets of L_α definable from L_α using elements of L_α as parameters. In general, a set x is definable over A iff there is a $\phi \in \mathscr{L}_\in$ and parameters $a_1, ..., a_n \in A$ such that $x = \{y \in A \mid A \models \phi(y, a_1, ..., a_n)\}$. The set of subsets definable over a set A we call the definable powerset of A and denote with $\mathscr{D}(A)$ (for more details on how to define L see Kunen 1980:165–6 and Jech 2003:175).

The definitions of the stages of L can thus be stated as:

$$L_0 = \varnothing$$
$$L_{\alpha+1} = \mathscr{D}(L_\alpha)$$
$$L_\lambda = \bigcup_{\beta < \lambda} L_\beta \quad \text{if } \lambda \text{ is a limit ordinal.}$$

Furthermore, we let L be the proper class

$$L = \bigcup L_\alpha \quad \text{for } \alpha \in ON.$$

Assume ZF has a model $M = \langle D_M, E_1 \rangle$. Then we can produce a model $N = \langle D_N, E_2 \rangle$ such that the domain D_N is a subset of D_M such that the members satisfy the definition of being constructible and the interpretation E_2 of $\in$ in N is $E_1 \cap D_N \times D_N$ (that is, we keep the interpretation of $\in$ fixed except for restricting the relation to the domain of N). Furthermore, N contains all the ordinals of M. We can prove that N is a model of ZF. It will also be a model of the statement $V = L$ which says that every set is constructible. From ZF $+$ $V = L$ one can prove the Choice axiom and GCH, so Choice and GCH also hold in N. To summarize we have

$$\text{Con(ZF)} \to \text{Con(ZF} + V = L),$$
$$\text{ZF} + V = L \vdash \text{Choice} + \text{GCH}$$

So,

$$\text{Con(ZF)} \to \text{Con(ZFC} + \text{GCH)}.$$

It follows from this that if ZFC is consistent, then ZFC $+$ CH is consistent, so there can be no proof of ¬CH from ZFC.

If we compare L with V with regards to the "dimensions" mentioned earlier, we see that the height of L is equal to the height of V, as the generation of stages in L is carried out along the ordinals in the same manner as with the stages of V. The case of width is more complicated. Starting with L_0 up to and including L_ω, the stages of L and V are equally wide since $\mathscr{D}(x) = \mathscr{P}(x)$ for finite x, so $L_\alpha = V_\alpha$ for $\alpha \leq \omega$. However, at successor stages $L_{\alpha+1}$ of stages L_α with $\omega \leq \alpha$, we have that $L_{\alpha+1} \subsetneq V_{\alpha+1}$, so each successor stage after ω is thinner in L compared to the stages in V. This might suggest that the V_α's will outgrow the L_α's and that L cannot be equal to V, but this is not implied. The trick is to realize that although a given set $x \in V_{\alpha+1}$ may not be a definable subset of L_α, and thus not a member of $L_{\alpha+1}$, x may be a definable subset of some later stage L_β, and in that case $x \in L_{\beta+1}$. This way, any set that is not formed in L at the stage where it is formed in V could possibly be retrieved at a later stage in L, allowing that L could be equal to V. In fact, as we will see, the statement $V = L$ is not only relatively consistent

with ZFC but also independent of ZFC.[8]

One last thing to note about L is that it is an example of a so-called inner model in set theory. More generally, we say that given an extension of ZF, T_1 (which could be ZF itself), and a theory T_2 (possibly the same as T_1) also stated in $\mathscr{L}_\in$, N is a model of T_2 and inner (in M) if M is a model of T_1 and N is such that the domain of N is a transitive class of the domain of M, the interpretation of $\in$ in N is equal to the interpretation of $\in$ in M restricted to N and N contains all the ordinals of M.

1.2.3 Forcing

The second and last step towards showing that CH is independent of ZFC was taken by Cohen in the early 1960s. Developing and using the technique known as forcing, he showed how to construct, given a model of ZFC, a model of ZFC + ¬CH, thus

$$\text{Con}(ZFC) \to \text{Con}(ZFC + \neg CH).$$

So, if ZFC is consistent, then CH cannot be proved from ZFC. Cohen also showed that the negation of the Choice axiom is relatively consistent with ZF, so the Choice axiom is independent of ZF.

With L, and inner models more generally, one assumes that there is a model of, say, ZFC and then obtains a substructure of that model and investigates what theory might hold there. With forcing the idea is to start with some model of ZFC and then obtain an expansion of that model which might model additional claims in the language of set theory. We superficially and very briefly describe one approach to forcing below, involving the use of countable transitive models of (finite sub-theories of) ZFC. For more detailed expositions of both this approach and others one can consult Kunen (1980, 2013), Jech (2003) and Bell (2005). The original method and results are given by Cohen (1966) and an attempt at a more accessible presentation can be found in Chow (2009).

The method of forcing allows us to construct from a given countable transitive model M of ZFC, a generic extension $M[G]$ which is also a model of ZFC and, depending on how we carry out the construction, further claims in the language of set theory. A model is transitive if every member y of a member x of M is also a member of M ($y \in x \land x \in M \to y \in M$); the assumption of transitivity does not matter much for our purposes but we note that it simplifies many parts of the proof of M and $M[G]$ being models of ZFC as many set theoretic notions are absolute for transitive classes. Although ZFC proves the existence of uncountable sets, it follows from the downward direction of

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[8]Having said that, many set theorists and philosophers of mathematics would probably argue that $V \neq L$. One reason is that $V = L$ is inconsistent with the existence of relatively weak large large cardinals, such as a measurable cardinal. For more discussion of $V = L$ and a case against the statement, see Maddy (1993) and (1997:216–32).

the Löwenheim-Skolem theorem and the Mostowski collapse lemma that if ZFC has a well-founded model, then ZFC has a countable transitive model. The main reason one works with countable M is that if one does not assume that M is countable, then it is not guaranteed that there will exist the right kind of G to add to M, which is a so-called filter generic over M, but assuming that M is countable it is easy to prove the existence of such a filter.

The standard approach to set forcing, then, is to assume the existence of a countable transitive model M of ZFC called a ground model. We then find a partially ordered set with a maximal element $\mathbb{P} = \langle P, \leq_{\mathbb{p}}, \mathbb{1}_{\mathbb{p}} \rangle \in M$ such that P is the domain, and $\leq_{\mathbb{p}}$ is the ordering on P with a maximal element $\mathbb{1}_{\mathbb{p}}$. We then use this partially ordered set to define a new set G such that G is a filter on $\mathbb{P}$ intersecting every subset of M that is dense in $\mathbb{P}$; in that case we say that G is $\mathbb{P}$-*generic* over M.[9] It follows from this that $G \notin M$. Eschewing a lot of technical detail, we adjoin G to M in a controlled manner to get the forcing extension $M[G]$. By varying $\mathbb{P}$ one can, with great freedom, model different claims in the language of set theory. If all this is done with care, the resulting $M[G]$ will be a model of ZFC with the same ordinals as M and, importantly for our purposes, M is a proper subset of $M[G]$, so there are sets in $M[G]$ that are not in M.

Intuitively, we are adding subsets to the stages of M in such a manner that we preserve ZFC yet alter the truth value of many set theoretic statements. In this sense, the forcing extension $M[G]$ will be "wider" than the ground model M we started with. Cohen used the technique to show that assuming that there is a countable transitive model M of ZFC $+ V = L$, there is a forcing extension $M[G]$ such that it is a model of ZFC $+ V \neq L +$ ¬CH. In particular, Cohen gave an $M[G]$ such that $2^{\aleph_0} = \aleph_2$ in $M[G]$.[10] We can think of this forcing as adding $\aleph_2$ new subsets of ω. This result means that

$$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con(ZFC} + V \neq L)$$

and

$$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con(ZFC} + \neg\mathrm{CH}).$$

Since Gödel showed that $\mathrm{Con(ZF)} \rightarrow \mathrm{Con(ZFC} + V = L)$ and Cohen showed that $\mathrm{Con(ZFC} + V = L) \rightarrow \mathrm{Con(ZFC} + V \neq L + \neg\mathrm{CH})$, we get $\mathrm{Con(ZF)} \rightarrow \mathrm{Con(ZFC} + V \neq L + \neg\mathrm{CH})$, and from that the two statements above follow. The underlying reasoning on the countable transitive model approach is that, by features of the models and the

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[9] Let $\mathbb{P} = \langle P, \leq_{\mathbb{p}} \rangle$ be a partially ordered set. A subset x of M is *dense* in $\mathbb{P}$ if for all $p \in P$, there is a $q \in x$ such that $q \leq_{\mathbb{p}} p$. $G \subset P$ is a *filter* in P iff (a) $\forall p, q \in G \exists r \in G(r \leq_{\mathbb{p}} p \wedge r \leq_{\mathbb{p}} q)$, and (b) $\forall p \in G \forall q \in P(p \leq_{\mathbb{p}} q \rightarrow q \in G)$.

[10] More generally, we can force $2^{\aleph_0}$ to equal any cardinal with uncountable cofinality, including $\aleph_7, \aleph_{\omega_1}$ and even $\aleph_{2^{\aleph_0}}$. The restriction to cardinals of uncountable cofinality is due to König's Theorem which states that for κ greater than or equal to 2 and any infinite cardinal λ, the cofinality of κ^λ is strictly greater than λ. For more details, see Kunen (2013:75).

finiteness of proofs, if ZFC + ¬CH proves a contradiction, that is, is inconsistent, then ZFC proves a contradiction on its own (see Kunen 2013:281–2). So, conversely, if ZFC is consistent, then ZFC + ¬CH is as well. Therefore, there is no proof of CH or $V = L$ from ZFC, and combining this with the results obtained by Gödel's L, both statements are independent of ZFC.

Since Gödel's and Cohen's introduction of the methods, the techniques of defining inner models and forcing extensions have been used to show that many more or less natural and interesting mathematical statements from several different branches of mathematics are independent of ZFC. Examples include Whitehead's problem in group theory, the Borel conjecture in measure theory and Kaplansky's conjecture on Banach algebras. This is done by considering a diverse variety of models of set theory, as Joel David Hamkins points out:

> A large part of set theory over the past half-century has been about constructing as many different models of set theory as possible, often to exhibit precise features or to have specific relationships with other models. Would you like to live in a universe where CH holds, but $\Diamond$ fails? Or where $2^{\aleph_n} = \aleph_{n+2}$ for every natural number n? Would you like to have rigid Suslin trees? Would you like every Aronszajn tree to be special? Do you want a weakly compact cardinal κ for which $\Diamond_\kappa(\text{REG})$ fails? Set theorists build models to order. (Hamkins 2012:418)

One strikingly simple and reasonable-sounding statement independent of ZFC (discussed by Hamkins (2015:142–3)) is the statement

$$|x| < |y| \rightarrow |\mathscr{P}(x)| < |\mathscr{P}(y)|.$$

That is, if the cardinality of y is greater than that of x, then the cardinality of the powerset or the number of subsets of y is greater than that of the powerset or number of subsets of x. The statement is implied by GCH but can fail in certain forcing extensions. For example, there are models where the statement $2^\omega = 2^{\omega_1}$, known as Luzin's hypothesis, holds. In such a model, the powerset of the natural numbers stands in a one-one-correspondence with the powerset of an uncountable set, ω_1. These and further examples show that ZFC is incomplete in a significant way both regarding questions within set theory and from other branches of mathematics.

1.3   Gödel's Program

In 1900 the mathematician David Hilbert presented twenty-three then unsolved problems in mathematics for which he thought that mathematicians ought to find an answer and through which advancements in the science of mathematics may be expected. On the top

of that list stood CH. As we now know CH is independent of the most widely accepted foundational theory for mathematics, ZFC, so the question of whether CH holds or not cannot be decided in that system. ZFC doesn't have the answer. What to do in light of this fact?

Many mathematicians think that the question of CH is "solved" by the independence result: it has no answer; there is nothing more to say. This could be spelled out by saying that a question about whether a mathematical statement holds or not has a definite and unique answer if it is decidable from ZFC, that is, ZFC proves it or its negation. Otherwise, if the statement is undecidable from, that is, independent of, ZFC, then it has no unique answer and if it has no unique answer, it has no answer at all.[11] Of course, the more interesting mathematical statements we prove independent of ZFC, the less palatable this position potentially becomes. For example, if some of the most famous and important unsolved problems in mathematics, such as Goldbach's conjecture or Riemann's hypothesis, were to be proven independent of ZFC, it is less likely that the working mathematician would go along with the response just sketched.

A different reaction came from Kurt Gödel, one of the central figures in the development of the independence results and one of the most influential mathematicians of the twentieth century. Gödel, who was strongly realist about the subject-matter of mathematics, thought that the independence results only show us that our axioms are incomplete in their description of mathematical reality. In 1947, before Cohen had established that CH is independent of ZFC, Gödel anticipated the result yet saw it as no solution. Worth quoting at length, he wrote:

> [E]ven if one should succeed in proving its [CH's] undemonstrability [...] this would [...] by no means settle the question definitively. Only someone who [...] denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture [CH] must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained. (Gödel 1947:519–20)

Gödel suggests a program of research with the goal of reducing the incompleteness in set theory by justifying new and stronger axioms that are to be added to ZFC. The test case

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[11]A slightly different yet related reaction which we will discuss more in this thesis, is to say that there is a lot more to be said about CH and how it might hold or fail, it is just that there is more than one justified answer to the question of CH; the statement is both true and false relative to different universes of set theory.

for the success of the program is CH. This program of research is often called Gödel's program and lives on today in the work of several contemporary set theorists, such as W. Hugh Woodin. It has led to the formulation of many different axiom candidates proposed in addition to the axioms of ZFC, many of which are incompatible with each other.

The goal of Gödel's program is to reduce the amount of incompleteness in set theory for mathematically interesting $\phi \in \mathscr{L}_\in$ such that ϕ is independent of ZFC, by extending ZFC. Of course, there are trivial ways of doing this. For example, we could take the theory ZFC $+ \phi$ or the theory ZFC $+ \neg\phi$, both of which easily prove or disprove ϕ. So the point of the program is to extend ZFC in a non-trivial and justified manner. This introduces a component into the program clearly amenable to philosophical discussion: What is it for an axiom to be justified? What evidence can we give in favor of an axiom? We will return to these questions in Chapter 3.

Following Koellner (2006), we might divide Gödel's program, implicit in the way Gödel himself presented it, into two parts; the one more general than the other. First, the narrow sense of the program is a program of reducing incompleteness by so-called large cardinal axioms. There is no precise definition of what counts as a large cardinal axiom, but, to give a rough characterization, many large cardinal axioms Λ are statements asserting the existence of a cardinal κ such that κ cannot be obtained by the operations for generating larger and larger cardinals in ZFC; in other words their existence cannot be proved from ZFC (for a comprehensive and detailed treatment of large cardinals, see Kanamori (2003)). Several large cardinal axioms have been formulated and many of them decide statements independent of ZFC. For example, by the assumption that there is an inaccessible cardinal κ, one can prove the consistency of ZFC. Koellner (2006) argues that the program for large cardinals has been very successful "below CH" but that it breaks down at that point as no known large cardinal axiom decides CH.

This leads us to the wider sense of the program: reducing incompleteness by any new axioms for set theory in general. The goal of this second and more general sense of Gödel's program is nicely summarized by Steel (2014) as:

> Decide mathematically interesting questions independent of ZFC in well-justified extensions of ZFC.

If we strengthen this to well-justified and true or correct theories, a question of monism vs. pluralism arises: Say that $\phi \in \mathscr{L}_\in$ is *satisfactorily* decidable iff there is at least one well-justified and correct extension of ZFC, T, such that T either proves ϕ or $\neg\phi$. Say that ϕ is *uniquely* decidable iff ϕ is satisfactorily decidable and for all well-justified and correct extensions of ZFC, T_1, if T_1 proves ϕ, then there is no well-justified and correct extension of ZFC, T_2, such that T_2 proves $\neg\phi$ and if T_1 proves $\neg\phi$, then there is no well-justified and correct extension of ZFC, T_3, such that T_3 proves ϕ. Do the satisfactorily and uniquely decidable $\phi \in \mathscr{L}_\in$ coincide? Someone who answers "yes", we may call a

set theoretic monist, someone who answers "no" and thinks that there are satisfactorily decidable ϕ that are not uniquely decidable, we may call a set-theoretic pluralist. Monism and pluralism are thus understood as postions regarding correct theories of set theory.

It seems clear that Gödel proposed his program in a monist spirit and that many of the set theorists involved in the program are committed to something like monism. That is, if the question of whether a statement such as CH holds or fails is to be decided in well-justified and correct extensions of ZFC, so that we can reduce the incompleteness in set theory, it ought to be done so uniquely. In the next chapter we will investigate a view that might be taken to give up the uniqueness requirement and holds that many of the statements independent of ZFC, such as CH, have in fact already been established to be satisfactorily and non-uniquely decidable, thereby reducing the incompleteness in set theory in a different manner. The view does so by positing a multiverse of set theoretic universes.

Chapter 2

Multiversism in Set Theory

> [I] do not agree with the pure Platonic view that the interesting problems in set theory can be decided, that we just have to discover the additional axiom. My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality"[...].
>
> S. Shelah (2003), "Logical Dreams"

ZFC is significantly or, as Andrzej Mostowski once put it, "hopelessly" incomplete (reported in Lakatos 1967:93). Furthermore, if ZFC is consistent, there are many consistent ways of extending the theory; some of which are mutually incompatible. This is what the model theory of set theory has shown us, as we saw in the last chapter. For example, it is consistent with ZFC that the cardinality of the real numbers takes on a wide range of values in different models of ZFC. But among all the possible ways of extending ZFC, should we expect one to be privileged in some way? Is there one and only one way that correctly unfurls our concept of set embodied in the cumulative hierarchy? Is there only one universe of sets to describe?

In this chapter we turn to the main philosophical question of this thesis, namely whether or not there is one and only one definite universe in set theory or if there are many distinct set theoretic universes. We start by briefly describing a one universe view and its role in Gödel's program; motivating the search for new axioms. After this, we look at an alternative view. It arises in part out of the pervasiveness and seeming unresolvability of the independence phenomenon in set theory. It posits that there are many universes of sets in which different and sometimes incompatible extensions of ZFC hold. As such, we have satisfactory answers to many of the statements undecided by ZFC, although not unique answers. In the end, we consider some of the main objections to the multiverse view. The aim of the chapter is to establish the multiverse view as a philosophically interesting and coherent understanding of set theory without defending the view any further.

2.1 Universism

The point of Gödel's program is to formulate and justify new axioms for set theory that reduce the incompleteness of ZFC. The *monist* thinks that among all the possible extensions of ZFC, there is only one correct theory of sets. The universe view or *universism* can be seen as a view of sets and the determinacy of set talk that underpins this expectation about theory.

As we saw earlier, for Gödel the program of new axioms was motivated by the thought that set theory is about a well-determined reality in which independent statements such as CH either hold or not. Understood this way, the significant incompleteness of ZFC only shows us that the axioms of ZFC are incomplete in their description of set theoretic reality.

We could spell this out a bit more using the notion of the cumulative hierarchy V. In the language of ZFC we can define the stages of V and from ZFC prove their existence. But the model theory of set theory has shown us that many features of V are left underspecified by ZFC. For a wide class of models M of ZFC, we can find models of ZFC such that they are taller than M or wider than M.[1] One way of understanding Gödel's view is as holding that there is a maximal and unique universe V of all sets. ZFC tries to describe V but the theory fails to give a complete description. Still, since the universe exists and there is only one, CH and other independent statements must be either true or false in that universe. The uniquely correct set theory, then, is the theory containing all true statements about V.

This way, the one universe view naturally motivates a search for a new axiom that will give us a more complete description of V. One of the foremost contributors to the search for new axioms, W. Hugh Woodin, commenting on the prospects for finding such an axiom, puts it thus:

> A far stronger view [...] which I also currently hold [...], is that there *must* be such an axiom and in understanding it we will understand why it is essentially unique and therefore true. Further this new axiom will in a transparent fashion both settle the classical questions of combinatorial set theory where to date independence has been the rule and explain the large cardinal hierarchy.

And

> In other words, we would have come to a conception of the transfinite universe which is as clear and unambiguous as our conception of the fragment V_ω, the universe of the finite integers. (Woodin 2011:17)

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[1] I did not discuss height extensions of models of set theory in the last chapter. For our purposes it is sufficient to note that similarly to how forcing allows us to expand models of ZFC in width, there are techniques for expanding models of ZFC in height.

Although the quotes as I present them contain no arguments for the one universe view, the point is that at least some set theorists subscribe to the one universe view and see it as an important motivation for the search for new axioms.

The one universe view invites an analogy between mathematics and natural science. In physics, for example, most scientists believe there is an independently existing and unique physical domain that we are studying; yet no theory we have come up with so far is seen as anything near complete in its description of that domain. So the business of physics as a science is to improve our theories by whatever legitimate methods of justification we have at our hands. If we have come up with a sensible question regarding the physical domain and our theory does not tell us the answer, then business as usual is to try to find out what the answer is. Similarly, a proponent of the one universe view in set theory holds that there independently exists a unique set theoretic domain, and although our theory does not completely capture it, we can try to formulate and justify more complete theories. This analogy was invoked and developed by Gödel (1944). The analogy isn't perfect: for example, in set theory, the universist would regard ZFC as incomplete but all the same *correct* in its description of V, while in natural science, we would probably take our best scientific theories to be neither complete nor correct, but more or less accurate in their description of their domains of study. We will return to this analogy and matters of justification in Chapter 3.

But what if the uniqueness of the set theoretic domain fails? We now turn to an alternative response to the independence phenomenon in set theory, namely the view that there are many distinct set theoretic universes.

2.2   Multiversism

The central idea of multiverse views in set theory is that there are many, somehow equally legitimate, universes of sets. Motivated by the range of different model constructions for set theory, these universes are thought to be different in various interesting ways. For example, in some universes CH holds and in others the statement fails. In asking questions about sets we must consequently ask our questions and seek answers relative to what kind of universe of sets we want to consider.

Although explicit defense and more detailed development of multiverse views in set theory are quite recent, precursor ideas of some kind of pluralism in set theory can be found. After having established the relative consistency of ZFC + GCH with ZF by the construction of L, Gödel gave a lecture at Göttingen where he concluded by suggesting that "it is very plausible that with $[V = L]$ one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems," (Gödel 1939:155) presumably constructible and non-constructible set theory. Gödel later recanted this view in his (1947/1964). After Cohen's proof of the independence of CH from ZFC and the

establishment of a wide array of other such results, other set theorists voiced similar opinions more firmly. For example, in 1965 Mostowski, commenting on the independence results in a talk, argued "that there are several essentially different notions of set which are equally admissible as the intuitive basis for set theory" (Mostowski 1965:82). Furthermore, reminiscent of thoughts underlying contemporary multiverse views, he claimed that

> [m]odels constructed by Gödel and Cohen are important [...] because they show us various possibilities which are open to us when we want to make more precise the intuitions underlying the notion of a set. [...] Probably we shall have in the future essentially different intuitive notions of sets just as we have different notions of space, and will base our discussions of sets on axioms which correspond to the kind of sets which we want to study. (Mostowski 1965:94)

In the discussion, the mathematician László Kalmár concurred, adding "I guess that in the future we shall say as naturally 'let us take a set theory S' as we take now a group G or a field F" (in Lakatos 1967:105).[2] Similarly, Cohen and Hersh (1967) distinguish, in analogy with Euclidean and non-Euclidean geometry, between Cantorian and non-Cantorian set theory.

In arguing against the existence of multiple universes of sets, Martin (2001:14) notes that it is hard to criticize "the view that the independence proofs by forcing show that there are many universes of sets" because "the view, though often expressed in conversation, is rarely expounded in print." Today, however, there are several set theorists and philosophers of mathematics articulating and defending some form of multiversism, attesting to the recency of these views. There is the Hyperuniverse program of Sy-David Friedman (see Arrigoni and Friedman 2013), the development of a language and theory of the so-called generic multiverse by John Steel (2014), and the more radical multiverse view defended by Joel David Hamkins (2012, 2015) who defends (in print) exactly the kind of view Martin describes. Others, such as Saharon Shelah (2003), quoted at the outset of this chapter, have also expressed similar ideas. There are also more general and philosophical precursor pluralist views in the work of Balaguer (1995, 1998) and Field (1998) which we return to briefly below. We cannot hope to review and discuss all these views here (for a survey of different views, see Antos et al. (2015)), and therefore choose to look in more detail at the most ardent, radical and explicitly philosophical defense of a multiverse view, which is Hamkins' multiversism.

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[2]Interestingly, these early pluralists understood the pluralism as a problem for the foundational aspirations of set theory, in contrast to contemporary pluralists such as Hamkins. We return to this issue below.

2.2.1 Hamkins' Multiversism

Hamkins asserts the existence of many universes of sets, and he characterizes his multiverse view in the following way:

> [T]here are diverse distinct concepts of set, each instantiated in a corresponding set theoretic universe, which exhibit diverse set theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. (Hamkins 2012:416–17)

He adds:

> Part of my goal [is] to tease apart two often-blurred aspects of set-theoretic Platonism, namely, to separate the claim that the set theoretic universe has a real mathematical existence from the claim that it is unique. The multiverse perspective is meant to affirm the realist position, while denying the uniqueness of our set-theoretic background concept. (Hamkins 2015:137)

Furthermore, Hamkins thinks that many of these universes have already been studied, somewhat indirectly, via models of set theory, such as L and $M[G]$.

> Our most powerful set-theoretic tools, such as forcing, ultrapowers, and canonical inner models are most naturally understood as methods of constructing alternative set-theoretic possibilities. (Hamkins 2012:418)

So, there are many set theoretic universes, and the model theoretic methods in set theory give us a handle on characterizing and studying them.

One might respond that inner models do not pose that much of a problem for the one universe view: If there was a maximal and unique universe of sets V, and we let that serve as our model, we could still obtain many inner models as substructures of V, that is, by restricting our view, so to speak, of V to only certain parts of it. What Hamkins take as much more problematic for universism is to explain the use of forcing in general and to avoid the sense that there are universes outside of V.

Forcing allows us to expand models M of ZFC, to get forcing extensions $M[G]$. As we saw in Chapter 1, if M is a model of ZFC and $\mathbb{P} \in M$ is a suitable forcing notion, then, if G is an M-generic filter over $\mathbb{P}$, then $G \notin M$. Consequently, if one thinks that there is a maximal and unique universe V, there can be no V-generic filters G over non-trivial forcing notions in V, as such a G would have to be outside of V. Thus there can be no forcing extension of the universe.

Hamkins regards this as an overly restrictive view of the forcing method. Although there is no such G in V, Hamkins takes the different formal approaches to forcing, especially approaches to forcing which involves the use of class models of ZFC, as ways of simulating $V[G]$ inside V, suggesting that there must be universes outside of V. The

universist might take these formal techniques for extending models of ZFC as tricks that give rise to a mistaken sense that there is something outside of V. Hamkins, on the other hand, take the formal techniques to suggest that there *is* something outside of V:

> The multiverse view [...] takes this use of forcing at face value, as evidence that there actually are V-generic filters and the corresponding universes $V[G]$ to which they give rise, existing outside the universe. This is a claim that we cannot prove within set theory, but [it] makes sense of our experience [...] by positing as a philosophical claim the actual existence of the generic objects which forcing comes so close to grasping, without actually grasping. (Hamkins 2012:425)

Adding to this, one might say that the development of forcing techniques has given us an insight into a general mathematical phenomenon and a robust sense of what it means for there to be a forcing extension of V, even though we cannot state that there exists one directly from within our foundational theory. This is analogous to how we might come to be justified in believing that our foundational theory is consistent, even though that theory could not prove its own consistency. Thus, Hamkins' multiversism is a philosophical position trying to make sense of the model theory of set theory. It understands the independence phenomenon not as the discovery that our theory is incomplete in its description of a unique universe of all sets, but rather as the discovery of many kinds of universes of sets satisfying different theories.

Hamkins compares the situation in set theory with the situation in geometry after the middle of the 19th century. The analogy revolves around the Parallel postulate:

> **Euclidean Parallel Postulate:** For every line l and for every point P that does not lie on l there exists a unique line m through P that is parallel to l.[3]

For about two millennia geometers hoped to prove the postulate, which is equivalent to Euclid's fifth postulate, from Euclid's other four postulates, known as absolute geometry. In the nineteenth century the statement was discovered to be independent of absolute geometry. This subsequently led to a bifurcation in the study of geometry into Euclidean geometry, where the postulate holds, and non-Euclidean geometry, where the postulate fails; either because there is no line parallel to l through P (Elliptic geometry) or because there is more than one line parallel to l through P (Hyperbolic geometry). According to the mathematician Marvin J. Greenberg:

> This discovery shattered the traditional conception of geometry as the true description of physical space. Mainly through the influence of David Hilbert's

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[3]Where, in the 2-dimensional case, two lines l and m are defined to be parallel if they do not intersect, that is, if no point lies on both of them.

Grundlagen der Geometrie, a new conception emerged in which the existence of many equally consistent geometries was acknowledged[...]. (Greenberg 1994:xi)

The point is that in geometry, mathematicians accepted the different kinds of spaces characterized by the models used to show that the parallel postulate is independent of absolute geometry as equally mathematically legitimate or real. Hamkins thinks that the same is happening in set theory and that "the multiverse view now makes the same step in set theory that geometers ultimately made long ago, namely to accept the alternative worlds as fully real" (Hamkins 2012:426).

Although Hamkins presents a highly interesting and fascinating view, drawing on a wealth of knowledge about technical developments in set theory, the view is somewhat philosophically underdeveloped. For example, Hamkins often varies between talking about universes, models, and concepts of sets, without being very explicit about how they relate to each other: sometimes they are simply identified with each other but at other times they are thought of as distinct entities. In the next section we try to spell out the philosophical components of the view a bit further.

2.2.2   Multiversism as Plentiful Platonism

The universist platonist view in set theory asserts the independent existence of a determinate universe of sets. Statements in the language of set theory are either true or false of that universe, and thus uniquely true or false, and our constant symbols and interpreted variables refer to sets in the universe. Furthermore, there are unique concepts of set and membership, which are just the concepts that pick out all the sets in the universe and how those are structured by the membership-relation.

I take Hamkins to defend the extension of this standard platonism to include many distinct universes of sets and equally many distinct concepts of set. That is, there are many universes of sets all independently existing of us. Within each there exist sets of a certain kind, namely the sets of that universe. So, for each universe of sets there is a concept of set. Statements in the language of set theory are either true or false of a given universe, relative to the corresponding concept of set, and under that concept the constant symbols and interpreted variables refer to sets in the universe. Therefore, statements in the language of set theory need not be uniquely true or false, as a given statement can be false relative to one concept of set and true relative to another. All the same, on Hamkins' multiversism (understood this way), the basic account of the ontology of sets and its relation with truth and reference in set talk mirrors the universist account of the same phenomena, supplemented with many universes and several set concepts.

On this understanding, Hamkins is defending a more detailed and technically motivated version of what Balaguer (1995, 1998) has defended under the name of "full-blooded

platonism" and Field (1998) explored as "plenitudinous platonism." Maddy (1997:196) calls it "plentiful platonism" and characterizes the view as the claim that there exists a universe of sets corresponding to every consistent extension of ZFC. Balaguer's and Field's views are much more general, less technically developed and more philosophically motivated than Hamkins' view. Balaguer (1995) proposes his view as an account of how we can acquire knowledge of mathematical objects. Roughly speaking, his view is that all mathematical objects that could exist (that is, are logically possible) actually do exist. Then, according to Balaguer, as long as we can consistently conceive of a mathematical object or structure, we will have an accurate representation of some mathematical object or structure, thus knowledge, since every such object or structure that is logically possible does in fact exist. Field (1998) explores the view as a way for a realist to characterize and make sense of indeterminacy in set theory. He characterizes the view as "whenever you have a consistent theory of pure mathematics[...], then there are mathematical objects that satisfy that theory under a *perfectly standard satisfaction relation*" (Field 1998:292). If we take every consistent extension of ZFC as our range of theories, then there will be, for example, CH and ¬CH universes. This explains why CH is not uniquely true or false.

What makes Hamkins' view interesting is how it ties these philosophical claims to the technical developments in set theory such as forcing. For example, Hamkins' view predicts that ordinary set talk is ambiguous between, or involves a cluster of, different set concepts, picking out different kinds of sets. A natural way of understanding Hamkins' view of the model theory of set theory is as providing tools for sharpening the ambiguous concept of set or narrowing the cluster of set concepts in various ways. Although the methods do not give us completely sharpened concepts, they still illuminate in informative ways what kind of universes there are and what statements that hold there, such as with Gödel's L and Cohen's $M[G]$. As part of this, the model-theoretic tools provide us with our primary way of epistemically accessing the multiverse. If we can, from a model M of, say, ZFC, obtain a model N of ZFC + ϕ by the use of for example inner models or forcing extensions, then we are justified in believing that there exists a ZFC + ϕ universe. This way the different kinds of model constructions we are familiar with, although not necessarily the universes themselves, allow us to glimpse into the multiverse and characterize what kind of universes there are.

Understood as a form of plentiful platonism, Hamkins' multiversism thus gives us an account of the ontology of sets, together with ways of understanding set talk and the epistemology of set theory, that interact with technical developments in the model theory of set theory in interesting ways.

2.2.3 Strengths of Multiversism

We have seen that multiversism is an interesting alternative to the traditional set theoretic platonism of Gödel and other universists. Below we summarize what one may, depending on other commitments, take to be nice features of the view:

A simple and intuitive account of forcing

First, we have seen that the multiverse view backs a "naive" and straightforward understanding of forcing (and extensions more generally) in set theory. For example, Kunen (2013) starts his explanation of the idea behind forcing in the proof of the relative consistency of ¬CH with ZFC like this:

> *Very naively,* we step outside our universe and create some ideal universe $N \supsetneq V$ that has sufficiently many subsets of ω so that CH fails. (Kunen 2013:244)

Of course, this informal idea needs to be worked out somehow formally, either with the use of countable transitive models or class models. But at least it supports Hamkins' contention that, informally, the idea behind the forcing method is to allow us to study "ideal" universes of sets which might be different in interesting ways, and that the multiverse view in some way is taking the use of forcing at face value.

To this, the universist is liable to respond that this is too naive; talk about "going outside V" is just a manner of speaking, which does not really make sense on the one universe view. Furthermore, writing things like $N \supsetneq V$ or $V[G]$ is an abuse of notation used for mere expediency and should always be understood as strictly speaking referring to an extension $M[G]$ of a *model M* of set theory, both of which exist in V. As Hamkins himself notes, the universist "can insist on an absolute background universe V, regarding all forcing extensions and other models as curious complex simulations within it" (Hamkins 2012:426).

From the multiverse perspective, however, viewing the forcing extensions of a universe as *merely* simulated inside it leads to unnatural and complex contortions in the informal interpretation and understanding of the results involving them. The multiverse view makes sense of these forcing extensions and our results about them in a simpler manner by asserting that they in fact exist as independent universes of sets. So, when it seems like we are starting with a universe of sets where certain statements hold, and then, via forcing, look at a different universe where other statements hold, that is in fact exactly what we are doing according to the multiversist; the forcing extensions are not illusions. As such, multiversism can be seen as a kind of naive or straightforward realism about forcing extensions. Although the universist can account for forcing extensions as well, the multiversist account is arguably simpler and more intuitive.

Accounting for indeterminacy

Second, to some it will be attractive that the view explains indeterminacy in set theory while retaining realism about its subject matter. Some mathematicians think that CH and other statements independent of ZFC are somehow vague or indeterminate (see for example Feferman (2011)). In a loose sense, we can say that a formula ϕ in the language of set theory has indeterminate truth-value if it is neither true nor false or can be correctly interpreted as true and correctly interpreted as false.

Multiversism can be seen as giving an account of indeterminacy: Since for mathematical $\phi \in \mathscr{L}_\in$ independent of ZFC, ZFC $+ \phi$ and ZFC $+ \neg\phi$ will both be consistent extensions of ZFC (if ZFC is consistent), there will be, on quite a radical understanding of the multiverse view, universes where ϕ holds and universes where it fails. So, the formulas in $\mathscr{L}_\in$ independent of ZFC lack a unique truth-value because they are true relative to some universes of sets and false relative to other universes. So in the loose sense characterized above, such ϕ are indeterminate. Furthermore, multiversism does this on a realist understanding of the subject matter of set theory.

Unlike some versions of indeterminacy views, however, Hamkins' multiversism claims that we can sharpen or pick out, using techniques from the model theory of sets, more determinate concepts of sets, on which the truth-value of many relevant ϕ is settled. This leads us nicely to our next point.

Reducing incompleteness

Third, the view reduces incompleteness in set theory in the following way: again taking quite a radical understanding of the multiverse, for any ϕ independent of ZFC, there will be ZFC $+ \phi$ universes and ZFC $+ \neg\phi$ universes. As such, the set of statements T true in any of those universes is a correct theory of sets that decides ϕ. Furthermore, by obtaining models of ZFC $+ \phi$ or ZFC $+ \neg\phi$, that is, establishing the independence of ϕ, we are justified in believing in the corresponding universes and consequently the correctness of its theory. Understood this way, the model theory of set theory has shown us that for many ϕ, such as CH, there are well-justified and correct extensions of ZFC, such that these extensions decide ϕ. So, on the multiverse view, we have that a large variety of ϕ independent of ZFC are what we earlier called satisfactorily decidable. But these will not be uniquely decidable, as for the ϕ we have shown independent of ZFC, we must have obtained models of both ϕ and its negation, thereby having correct and well-justified yet incompatible extensions of ZFC. Thus, multiversism leads to a form of pluralism about set theory.

This way, the view settles many questions universists regard as open. As part of its strengths it does so in a substantive way that does not violate the sense that these statements lack a unique truth-value. Take CH for example. On the multiverse view,

according to Hamkins (2015), the question of CH is settled by our detailed knowledge of how it behaves in different models of set theory. This way the multiverse view takes seriously CH as an interesting mathematical statement and the expectation that the question of whether it holds or not should be answered. But, in contrast with the universe view, the multiverse view holds that this question can be answered in more than one well-justified and correct way; CH is true relative to some universes of sets and false relative to others, similarly to how the parallel postulate holds in certain kinds of geometrical spaces but fails in others. Some of these universes may be more interesting or fascinating or nicer than others, but they are all equally real.

2.3 Objections to Multiversism

At this stage, I hope the reader have been given some sense of why multiversism might be an attractive view in the philosophy of set theory. Let us now turn to assess some objections that might be raised against the view.

2.3.1 Bloated Ontology

The multiverse view as we understand it posits a vast plethora of universes, each of which contain their own infinity of sets. Is this just too much? The principle known as Occam's razor tells us not to multiply entities beyond necessity, and it is a call for ontological parsimony in our theorizing. One objection to the multiverse view simply says that the view has violated this principle.

To this, the multiversist can reply that Occam's razor seems to have no purchase when it comes to abstract entities. Burgess (1998) argues by looking at several smaller case studies that reasoning conforming to Occam's razor applied to abstract objects plays no role in scientists' assessment of different theories. He suggests that, in science quite generally, one can postulate abstract entities as far as scientifically convenient. This seems to be even more so the case in mathematics in particular, where there is great freedom in asserting the existence of different kinds of entities and structures as long as the resulting mathematics is interesting. As Koellner (2009a) puts it: in mathematics, ontology is cheap.

This argument against Occam's razor in mathematics can be seen as a naturalistic argument by taking scientific practice as its starting point: since mathematicians themselves do not seem to care about ontological parsimony in mathematics, neither should philosophers.

2.3.2   Conceptual Issues

Central to Hamkins' account are the various models of set theory we can construct from a given model. Controversially, Hamkins claims that to each of these models there exists a universe of sets satisfying the theory that holds in the model. Thus, there are several correct theories of sets.

We saw earlier that Mostowski predicted that there would be some kind of pluralism in set theory arising from essentially different intuitive notions of sets. Kalmár agreed and predicted a great freedom in choosing different set theories as a points of departure for further study; a freedom supported to a great extent by Hamkins' view. But in response to Kalmár, Mostowski qualifies his pluralist remarks:

> I can only add that various set theories which will perhaps appear in the future must be based on firm intuitive basis. Otherwise it is hard to see what would be their role. Thus it is not at all clear that one will have such a degree of freedom in the choice of a set theory as one has at present in, say, group theory. (Mostowski 1965:105–6)

Here Mostowski is restricting the extent of acceptable set theories to those that have a "firm intuitive basis" or in other words theories that have some thought or conception behind them.

The supplementation of the model theory with different intuitive conceptions of space seems to have been the case in geometry. The development of non-Euclidean models was accompanied by an increased sense of there being different intuitive mathematical conceptions of space that could be used to motivate alternative theories of space to the traditional Euclidean theory. As Hamkins puts it: "In time, however, geometers gained experience in the alternative geometries, developing intuitions about what it is like to live in them, and gradually they accepted the alternatives as geometrically meaningful" (Hamkins 2012:425–6). The point is that supplying an intuitive conception of space to understand models and theories of, say, hyperbolic geometry, gives us reasons to regard such spaces as equally mathematically real as the standard Euclidean space.

To drive home the point that having a model might not be sufficient for accepting a corresponding mathematical reality, we could compare this with number theory. Although mathematicians speak of a standard model $\mathfrak{N} = \langle \mathbb{N}, +, \times, <, E \rangle$ for first-order arithmetic, one can easily construct non-standard models. We could, for example using the compactness theorem for first-order logic or the Löwenheim-Skolem theorems, construct a model $\mathfrak{A}$ which although elementarily equivalent to $\mathfrak{N}$, that is, they model all the same ϕ in the language of number theory, is not isomorphic to $\mathfrak{N}$, for example because the domain of $\mathfrak{A}$ contains more elements than the domain of $\mathfrak{N}$. More generally, there are models of PA, the standard first-order axiomatic theory of arithmetic, which are not isomorphic to the standard model.

The point to make with the non-standard models is that although most mathematicians would accept their existence as model-theoretic entities, it is not given that they would accept that there exists a corresponding universe of non-standard numbers to each non-standard model. The main reason for this seems to be that these models seem to have no intuitive conception of numbers behind them.

If Mostowski is right that one must not only have model-constructions but also an intuitive basis to accept a set theory as correct, a challenge to Hamkins' radical pluralism is to find an intuitive basis or conception behind the different models and consequently incompatible theories he accepts. Trying to overcome the challenge might severely limit the multiverse view. For example, some models seems to have a thought behind them which allow us to form an intuitive picture of them, such as Gödel's L where we build the hierarchy in the "thinnest" possible way. Or the way we obtain models where CH fail by "widening" the hierarchy. But there are also some non-intuitive models. For example, assuming ZFC is consistent, we can construct countable, ill-founded models of ZFC. To give intuitive characterizations of the universe picked out by such models might be much more difficult.

2.3.3 Quasi-Categoricity

Another common response against claims about several universes of sets is to argue that set theory is about a particular structure by appealing to the quasi-categoricity result tracing back to Zermelo (1930). Such a line of argument can be found in Kreisel (1965), Martin (2001) and Isaacson (2011).

A theory is called *categorical* if any two structures which satisfy it are isomorphic. In other words, the theory picks out a unique structure up to isomorphism. Due to the Löwenheim-Skolem theorems, there are no first-order theories T with infinite models that are categorical. So, for example PA or ZFC are not categorical. In the case of arithmetic, most mathematicians have the sense that there is a unique natural number-structure. The standard move to defend this view is to formulate a second-order theory of arithmetic, PA2, which one can show is categorical. Although the exact philosophical implications of a categoricity result like this can be discussed and disputed (see Meadows 2013), many take this as vindicating the view that arithmetic is about a specific structure (at least up to isomorphism).

Zermelo (1930) adopts a second-order formulation of ZFC, ZFC2, where the axioms schemas of ZFC are replaced with single axioms in second-order language. So, for example, the Separation schema:

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi(z))$$

gets replaced with

$$\forall F \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge Fz).$$

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Similarly with Replacement. The idea behind Zermelo's quasi-categoricity result is to build up an isomorphism between two models M and N of ZFC2 in a stepwise manner. Supposing that M and N are isomorphic up to some stage α, we can extend the isomorphism to stages $\alpha + 1$. Roughly, we assume that M and N meet what Martin (2001) calls "the concept of set" which include a maximal understanding of width, taking *all* subsets of sets when forming powersets in the structures, and a maximal understanding of height, claiming that the structures contain *all* ordinals. One can then use these assumptions to argue that for any two M and N with isomorphic initial segments, one can extend the isomorphism at successor and limit stages, and this will be done equally far. We end up with an argument to the effect that M and N must be isomorphic (if they have any isomorphic initial segments for a given rank α).

To the multiversist, the problem with this argument to establish that there is a unique set theoretic structure (up to isomorphism) is that it begs the question. To give the argument one must assume, in giving the semantics for the second-order language, absolute notions of powerset and infinity. Why assume that there are absolute notions of powerset and infinity? Koellner (2013) gives the following response on behalf of the multiversist:

> But this doesn't get any traction with the advocate of the multiverse since it presupposes absolute conceptions of powerset and infinity and it presupposes that there is a single, univocal conception of set. The advocate of the multiverse will argue that the above argument is circular. "True if one presupposes that there is a univocal conception of set, one which has absolute notions of powerset and infinity, then one can run the categoricity argument. But that just presupposes in the meta-language what one set out to establish. One gets out what one puts in." (Koellner 2013:11)

As we have seen, someone like Hamkins thinks that there are universes of sets of differing height and width, and consequently, the notions of all subsets of a given infinite set and all ordinals have no absolute characterization.

One might interpret the second-order variables in the language of ZFC2 in different ways; using for example concepts, classes, or pluralities of sets. And depending on prior beliefs about the determinacy of those notions one might hope to make the categoricity argument work. For someone like Hamkins, however, the response will generally be the same: these notions are not any clearer than the notion of sets with regards to for example questions about the width or height of infinite structures. To take a specific instance, Hamkins would have to claim that the "powerplurality" (that is, the plurality of all subpluralities) of ω is no more determinate than $\mathscr{P}(\omega)$. So, in particular, CH cannot be shown to be determinate by assuming an absolute notion of powerplurality for infinite pluralities.

2.3.4 New Intractable Questions

One of the nice features of the multiverse view is that it reduces incompleteness in the way described above and allows us to answer other questions as well through the study of various models of ZFC. This makes many questions about statements in the language of ZFC highly tractable. But, of course, new questions arise about the multiverse itself. If these are as intractable as the old questions were on the universe conception, then one could doubt how much progress that has been made.

For example, we might ask what universes there are in the multiverse. Hamkins' vision is quite expansive:

> The background idea of the multiverse [...] is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion to only ZFC models, as we can include models of weaker theories, ZF, ZF$^-$, KP, and so on, perhaps even down to second-order number theory, as this is set theoretic in a sense. [...] We want to consider that the multiverse is as big as we can imagine. (Hamkins 2012:436–7)

Others hold more restricted views. So, here the different multiverse views disagree. For example, Steel (2014) does not think that the L of a given universe constitutes a universe itself, while Hamkins accepts the following principle:

> **Realizability Principle.** For any universe V, if W is a model of set theory and definable or interpreted in V, then W is a universe. (Hamkins 2012:437)[4]

So, in particular, for any universe V, the L of that universe is itself a universe. The point here is not so much to investigate why Steel and Hamkins think so differently

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[4]Hamkins accept more generally the following principles in an attempt to characterize the multiverse:

Realizability Principle. For any universe V, if W is a model of set theory and definable or interpreted in V, then W is a universe.

Forcing Extension Principle. For any universe V and any forcing notion $\mathbb{P}$ in V, there is a forcing extension $V[G]$, where $G \subseteq \mathbb{P}$ is V-generic.

Reflection Axiom. For every universe V, there is a much taller universe W with an ordinal θ for which $V \precsim W_\theta \prec W$.

Countability Principle. Every universe V is countable from the perspective of another, better universe W.

Well-foundedness Mirage. Every universe V is ill-founded from the perspective of another, better universe.

Reverse Embedding Axiom. For every universe V and every embedding $j : V \to M$ in V, there is a universe W and embedding h

$$W \xrightarrow{h} V \xrightarrow{j} M$$

such that j is the iterate of h.

Absorption into L. Every universe V is a countable transitive model in another universe W satisfying $V = L$.

These are not meant to be formalized in a first-order theory, but proposed as informal principles of the multiverse. We return to the question of a formal theory of the multiverse briefly below.

about the multiverse, but rather to showcase that new and tough questions about the multiverse itself arise; the potentially different answers to which might be hard to justify and adjudicate between.

Related to this is the worry that the pluralism will push up to the next level as well. Hamkins often speaks as if there is a determinate multiverse. There is "*the* multiverse" and we have "glimpsed" into it through our model-theoretic methods but our knowledge of it is incomplete. Now, take for example the question: is the L of a given universe itself a universe? A one definite multiverse person thinks that either "yes" or "no," and we can try to find out which answer is correct. But maybe there are many multiverses? Maybe in some of them L of any given V is a universe, and in others not? As we can see, a question of monism vs. pluralism about our account of the multiverse itself arises. This can lead to familiar self-undermination arguments. Say there is no convincing argument for a one definite multiverse view. Adopting a many multiverses view, in Hamkinsian spirit, we would like to go as big as possible, having as many different multiverses as possible. Well, the narrowest multiverse among all the multiverses we accept will contain only one universe V. So, the *universe* view will come out as a legitimate view to hold! Thus, Hamkins would probably do best in trying to develop and defend a one definite multiverse view. But I will note that the question of one vs. many multiverses in set theory seems as intractable a question as they come.

Further questions that arise in the context of multiversism are: What statements $\phi \in \mathscr{L}_\in$ are true in all universes? How do you justify that a statement ϕ is true across the universe? These questions will be discussed more in the next chapter.

2.3.5   Fragmented Foundations

The early pluralists seemed to have held that pluralism is incompatible with set theory as a foundation. Mostowski claims: "Of course if there are a multitude of set-theories then none of them can claim the central place in mathematics." (1965:94). Intuitively, foundations require unity, and the multiverse view is in many ways giving up on that. Hamkins, on the other hand, thinks that:

> [T]he multiverse view does not undermine the claim that set theory serves an ontological foundation for mathematics, since one expects to find all the familiar classical mathematical objects and structures inside any one of the universes of the multiverse[...]. (Hamkins 2012:417)

Of course, in line with the last section, the question of whether or not this expectation is reasonable is a substantive question about the multiverse. But granting that expectation, Hamkins can be taken as claiming that at least as a **Generous Arena** the multiverse can still serve as a foundation, since any one of the universes can act as such an arena as far as classical mathematics goes.

Maddy (2017), in her review of the multiverse view in relation to the different foundational roles she considers as important, largely agrees with this judgement. She agrees that most of the roles, such as **Generous Arena**, **Shared Standard**, **Elucidation** and **Risk Assessment** will carry over. Take **Risk Assessment**, for example: Hamkins claims that one is, for different purposes, free to focus on certain parts of the multiverse, say, parts with universes satisfying very strong theories such as ZFC + large cardinal axioms (2012:436). So, for the purpose of risk assessment the multiverse provides the theories we need. Maddy's main issue with the multiverse view is the need for and potential problems with what a theory of the multiverse itself will look like, in relation to something to play the role of a **Meta-mathematical Corral**. Maddy is worried whether or not such a multiverse theory can be formulated without relying on a prior, background set theory. For example, Hamkins accepts the following principle:

> **Forcing Extension Principle.** For any universe V and any forcing notion $\mathbb{P}$ in V, there is a forcing extension $V[G]$, where $G \subseteq \mathbb{P}$ is V-generic.

Can such a principle be made sense of without substantial amounts of set theory? Hamkins' response is that this is exactly why we shouldn't expect a first-order theory of the multiverse itself on the multiverse view, as such a theory will presuppose a specific universe and set theory, and the whole point of the multiverse view is that there are universes outside a given universe (Hamkins 2012:436). Thus, Hamkins may have to give up on the hope of having a unique **Meta-mathematical Corral**. One option at this point is to take what one can get in terms of foundational roles and claim that this is the sense in which set theory is a foundation.

So, maybe, set theory could still play a foundational role even under a multiverse view. All the same, a different response, to avoid the question altogether, is to say that set theory need not be a foundation at all. For example, one could think that the issue of foundations in mathematics is outdated, that mathematics isn't really in need of a foundation, or maybe it is that some other approach works better. On such a view the primary role of set theory is as a mathematical theory of infinity. On such an understanding multiversism may be more plausible. That there might be different correct accounts of the mathematically infinite, such as the different pictures of transfinite cardinal exponentiation that arise from whether GCH holds or the different ways it could fail, given the highly abstract nature of the subject matter, seems not so implausible.

2.4   Chapter Conclusion

We have assessed multiversism in set theory, primarily Hamkins' view, while often comparing it with the one universe view. We have seen that multiversism has certain strengths: it gives a simpler and more straightforward informal account of forcing, it gives an account

of indeterminacy in set theory without giving up realism, and reduces incompleteness in set theory (albeit in a pluralistic manner). Weaknesses of the view include potential problems in providing intuitive conceptions of the different universes in the multiverse (at least on the radical view of Hamkins), raising new intractable questions about the multiverse itself and potentially having to play the role of a reduced foundational theory.

There are fundamental conceptual disagreements between multiversism and universism which are hard to challenge dialectically once you take a side. The case of the quasi-categoricity results nicely illustrates this, where, as Koellner puts it, one gets out what one puts in. So, the debate will most probably be settled (if at all) by a detailed and nuanced assessment of several different features of the views and how these features tie to other philosophical, meta-mathematical and mathematical considerations that goes far beyond what has been done in this chapter. Therefore, I make no endorsement here of either view. Still, I hope to have established multiversism as an interesting conception of set theory worthy of further study. In the next chapter, I will assume multiversism and its core claim which I take to be:

(**Multiversism**) There are many distinct universes of sets.

We then explore potential consequences of that view for set theoretical methodology and practice.

Chapter 3

Multiversism and Mathematical Evidence

We have seen how multiversism effects a pluralistic reduction in incompleteness. But on the traditional understanding of Gödel's program, the point is to formulate and justify stronger axioms for set theory so that we can *uniquely* decide interesting statements in the language of set theory that are independent of ZFC. How can one justify new axioms or basic principles for set theory? This question has pushed set theorists and philosophers of mathematics to give accounts of what different kinds of evidence one could give in support of new axioms and then go on to legitimize those kinds of evidence. Broadly speaking, mathematical evidence is divided into two kinds: intrinsic and extrinsic. In this chapter we will explore the relationship between these kinds of evidence and multiversism in set theory. In particular, we will investigate in what way multiversism can be seen as undermining certain uses of intrinsic and extrinsic evidence.

I start by introducing in more detail the two kinds of evidence which set theorists and philosophers of mathematics have claimed can be used to argue for the adoption of new axioms. I briefly touch upon how multiversism makes the appeal to intrinsic evidence more difficult. Then I look in more detail at the consequences for extrinsic evidence. I introduce a principle of my own, the criterion of match, to the effect that what counts as proper methodology within a field of study depends in part on the nature of that which is studied. I argue that conditional on the criterion of match and multiversism the use of abductive reasoning, an important form of extrinsic evidence, to establish the truth of new basic set theoretic principles is not legitimate. The conclusion is that, quite generally, to justify basic principles of sets under multiversism would be very challenging. After that I touch upon the issue of whether or not the debate between universists and multiversists has any real impact on set theoretic practice.

3.1 Evidence in Set Theory

The point of Gödel's program is to extend ZFC with an additional axiom (or axioms), so as to uniquely decide mathematically interesting statements in the language of set theory independent of ZFC. Given a universist explication of the expected monism, the addition of further axioms is supposed to yield a more complete description of set theoretical reality. Thus, we can only add axioms we are well-justified to hold as true. This gives rise to an interesting methodological question: how to justify axioms?

It is common to distinguish between two kinds of evidence that can be used to justify set theoretical axioms: intrinsic and extrinsic evidence. Something along the lines of this distinction was introduced and discussed by Gödel (1947/164), and a fuller account of these kinds of justification in set theory has been subsequently developed, for example in the work of Maddy (1988a, 1988b, 1990, 1997, 2011) and Koellner (2006, 2009b). Let us look at a rough characterization of the two kinds of evidence.

Intrinsic evidence: We have intrinsic evidence in set theory when our intuitions about sets or our analysis of the concept of set support a given axiom candidate. This might be dubbed the traditional view of the justification of axioms. Intuitive and conceptual evidence is unified by a kind of directness, that is, there is some immediate grasp of the grounds for justification of the given claim. So, in contemplating sets or the concept of set, certain things seem to hold in a direct and immediate manner and this gives intrinsic evidence. For example, it seems inherent to the concept of set that if x and y share all members, then x is identical to y. Accordingly, Extensionality is supported by intrinsic evidence.

Extrinsic evidence: There can also be indirect evidence for mathematical statements, it is claimed, not grounded in our intuitions or the concept of set. We have extrinsic evidence when a given axiom candidate is theoretically fruitful, for example by systematizing our theory, by simplifying certain proofs, by better explaining the intrinsic evidence (the intuitive and conceptual data), or by having otherwise verifiable consequences in set theory or other branches of mathematics. As a historical example, certain arguments in favor of the Choice axiom rely on such evidence. As Maddy (1988a:488) points out, Zermelo, although he thought that the axiom was intuitively evident, bolstered his defense of it by listing "fundamental" and "elementary" theorems provable from the axiom in order to argue that it is necessary for a successful development of set theory to accept the axiom.

The distinction between intrinsic and extrinsic evidence comes with certain problematic features. First, the line between the two kinds of evidence is somewhat vague and sensitive to different understandings of the concept of set. This has led to different judgements about whether particular axioms are primarily intrinsically evident or extrinsically so. For example, Boolos (1971) thinks that the Replacement scheme and the Choice ax-

iom cannot be inferred from the concept of set, while Gödel (1947/1964) seems to think that at least Replacement is motivated by the concept of set. Second, the relative importance of the two kinds of evidence is debatable. Although Gödel seems to accept both on an equal footing, others have argued that either intrinsic evidence or extrinsic evidence is generally more important (an example of an extrinisicalist view of mathematics is found in the last chapter of Maddy (2011), while intrinsicalist ideas can be found in Tiles (1989:208) and Tait (2001:96)). A related question is the *legitimacy* of the different kinds of evidence and methods in set theory. In particular, we can ask whether or not certain applications of the two kinds of evidence counts as legitimate or illegitimate, and, if one way or the other, why that is the case. This is a call to better understand the conditions that must obtain for variants of these kinds of evidence to count as legitimate.

We now turn to how multiversism undermines the legitimacy of these kinds of justification. Since philosophers sympathetic to universism, like Koellner, have already argued for the weakness of intrinsic evidence in going beyond ZFC, we only briefly touch upon how multiversism makes matters worse.

3.2 Problems with Evidence in Set Theory

Say that a statement $\phi \in \mathscr{L}_\in$ is absolutely true, if it is true in all universes of sets. Then according to the universist, ϕ is absolutely true if it is true in V. According to the multiversist, on the other hand, ϕ is absolutely true if it is true across the multiverse, that is, true in each universe. Let us call an axiom that is absolutely true a basic principle of sets.

Note that multiversism does not preclude that there are basic principles of sets or that we could discover new basic principles of sets. If we could justify that something was true across the multiverse, we would be in such a situation. In fact, most multiversist positions accept a certain amount of axioms as absolutely true and their consequences would thus be uniquely decided. One example is a multiverse where every universe satisfies either theories with Choice independent of them or theories with Choice. Such a multiverse would uniquely decide theorems that follow from Choice. Throughout the subsequent discussion, to simplify things, we will assume that both universists and multiversists accept ZFC as a list of basic principles of sets.

Bracketing for now the question of justifying ZFC as basic principles of sets, I will now argue that multiversism undermines the use of both kinds of evidence to establish *new* basic principles of sets. After that we return to the question whether this extends to ZFC itself or not.

3.2.1 Intrinsic Evidence

Intrinsic evidence is intuitive or conceptual evidence. This might be the traditional form of justifying axioms and basic principles in mathematics. In set theory, people have argued that there seems to be no forthcoming intuitive or conceptual evidence that will support principles ϕ that when added to ZFC will effect a significant reduction in incompleteness. To these theorists, it seems ZFC exhausts our conception of sets, and any stronger theory must be justified on the grounds of extrinsic evidence. Koellner (2006, 2009b), in particular, has argued that the use of intrinsic evidence is limited in the search for new axioms by tying intrinsic evidence to so-called reflection principles[1] which he argues are too weak to decide the statements Gödel's program seeks to decide, such as $V = L$ and CH.

Of course, multiversism undermines the use of intrinsic evidence to uniquely decide such statements even further – and radically so. To the multiversist there is a whole range of legitimate concepts of sets instantiated in different universes. As we contemplate a set concept and come to the judgement that ϕ is intuitively true, there is no guarantee that there is no other concept of set such that $\neg\phi$ is true under that concept. To take a specific instance, the multiverse view predicts that there will be legitimate conceptions of set, for example what Gödel called "sets as extensions of definable properties", such that $V = L$ is intrinsically evident, from which GCH follows, but also legitimate conceptions, for example what Gödel called "sets as arbitrary multitudes", such that $V \neq L$ has intrinsic evidence in its favor, under which GCH might fail. So, the multiversist can use intrinsic evidence to argue that statements are satisfactorily decidable. But, in general, the use of intrinsic evidence to uniquely decide basic statements in $\mathscr{L}_{\in}$ will become suspicious as it is uncertain that what seems to follow from some concept of sets, will do so from any concept of sets.

3.2.2 Extrinsic Evidence

Extrinsic evidence is really a term covering a great range of different kinds of indirect evidence in favor of theories, such as induction, abduction, theoretical virtues like simplicity, parsimony, fertility, elegance, and so on. An important form of extrinsic evidence is what might be called abductive reasoning. To establish a principle by abduction, also called inference to the best explanation, is to show that the principle in question best explains

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[1]Roughly speaking, such principles assert that anything true in V fails to fully characterize the universe as it is already true in some initial segment of V. Schematically, the principles are stated in the following form, where ϕ is some condition, x an arbitrary parameter and ϕ^{α} and x^{α} the relativization of quantifiers and parameters to V_{α}:

$$V \models \phi(x) \rightarrow \exists\alpha V_{\alpha} \models \phi^{\alpha}(x^{\alpha}).$$

(for example by having as a consequence) something already granted, and preferably, at least in mathematics, to show that the principle is necessary to derive that which is granted. Now, abductive reasoning is usually seen as a staple of sound method in the natural sciences and its general legitimacy in that case unassailable. But what counts as legitimate abductive reasoning in the case of mathematics might not be straightforwardly importable from natural science. I now turn to examining why interesting uses of abductive reasoning in set theory pertaining to Gödel's program might fail.

The criterion of match

The question of why a certain method counts as legitimate or not within a scientific field can be difficult to answer. To start studying the conditions under which abductive reasoning might fail I will assume a principle I call the criterion of match. The set theorist's question is "what are the basic principles that hold of sets?", and our metatheoretical question is "what is the proper method for answering the set theorist's question?".

I think that the right answer to the metatheoretical question hinges in part on the nature of the domain which we theorize about. I therefore suggest the following principle:

> **(Criterion of Match)** What counts as proper method is determined, in part, by the nature of that to which it will be applied.

The point of this principle is to establish a link between the ontology of that which is studied and proper methodology within a given scientific field. In slogan form, "the method must fit!".

The principle generalizes, I think, a quite common sentiment in science, namely that the ways things are should influence the way we study it. Although it can be hard to distinguish in a principled manner between substantive claims and methodological claims, I do think that methodological claims and maxims often depend on or assume, either explicitly or implicitly, substantive claims. Thus, one way to undermine a methodological claim is to undermine substantive claims on which it rests.

We should be naturalists enough to allow the justification for the substantive assumptions used in a scientific inquiry to come primarily from the science in question itself. For example, the entities of modern-day physics are assumed to be able to enter into statistical or causal mechanical relations with our experimental instruments. This grounds the legitimacy of the experimental method in physics. The best evidence, however, for this assumption about the entities of physics comes from physics itself, utilizing the said method.

But before having such reassurance, a potential problem is that there could be a *mismatch* between method and ontology. Turning to set theory and Gödel's program, we can ask what prior assumptions are needed to justify the legitimacy of abductive reasoning and whether or not they can be undermined.

Gödel, inspired by Russell (see Gödel 1944), famously argued for the legitimacy of extrinsic evidence in mathematics and logic by an analogy with physics and natural science more generally, which we touched upon at the start of chapter 2. The role of more basic principles in physics is, plausibly, to explain (and make predictions about) the data set our observations constitute. In mathematics, Gödel contends, some axioms, especially those with a very small degree of intrinsic evidence, get justified in a similar manner; by having as consequence and explaining the more directly given mathematical facts. We can see the argument as consisting of two premises: 1) Abductive reasoning to fundamental principles is part of proper methodology in physics, and 2) what makes this proper methodology in physics is also the case in mathematics. The conclusion is that the same kind of justificatory process is part of proper methodology in mathematics.

The way I have stated it, this is a weak argument in favor of the use abductive reasoning in set theory. As we have discussed in detail, much more can be said about how the ontology of mathematics is similar or different in important respects from the ontology of physics, and this might affect the plausibility of the argument. One salient example to us is whether or not there is a unique domain of sets that stands in suitable parallel to the unique physical domain assumed in physical theorizing. Multiversism, of course, claims that this is not the case. In the next section we investigate how multiversism might block certain important uses of abductive reasoning. In particular, how, under the multiverse view, the ontology of sets is such that the use of abductive reasoning to the absolute truth of new set theoretic axioms is illegitimate.

The multiverse and abductive reasoning

The ontological claim of multiversism is that set theoretical reality is fractured into different, distinct universes. The key aspect of this feature of the view is that certain set theoretical claims (both axioms and theorems) might not be uniformly true across the multiverse. One way to understand this indeterminacy is as being due to a pluralism of set concepts that corresponds to each universe. Thus, set theoretical claims might differ in truth value under different set concepts. What consequences could this have for proper methodology?

A suggestive discussion can be found in Hamkins (2015). We mentioned earlier that he thinks that set-theoreticians are already more or less explicitly embracing the view that there are CH and ¬CH universes. Furthermore, he claims that if a proposed set theoretic principle was shown to entail either CH or ¬CH, this would prevent set-theorists from accepting the proposed principle as a basic set-theoretic principle *no matter how much extrinsic or philosophical evidence it has in its favor*. The reason is that they have extensive experience with universes where the opposite claim holds. How can it be a consequence of a basic principle of set theory that CH holds, when we have seen universes

where ¬CH holds, and vice versa?

I think we can generalize Hamkins' thinking in an argument based on the multiverse ontology of sets to challenge certain potential uses of abductive reasoning in set theory, especially inferring to the absolute truth of some new axiom candidate that extends ZFC. To sum up prior to giving the argument, we have a certain substantive claim about the ontology of sets:

(Multiversism) There are many distinct universes of sets.

And a claim about the relationship between proper methodology and substantive claims:

(Criterion of Match) What counts as proper method is determined, in part, by the nature of that to which it will be applied.

I now show how together these two claims can be used to argue against certain uses of abductive reasoning in set theory.

Simply put, the argument is that the truth of the ontological claim that there is a set theoretic multiverse, which is a philosophical claim, blocks abductive reasoning to statements in the language of set theory, particularly new axiom candidates. Inferring to the absolute *truth*, that is, true in all universes, of a first-order set theoretic axiom candidate because it has certain fruitful or verifiable consequences, will not in general be legitimate if there are many distinct set theoretic universes. This is not a complete rejection of abductive reasoning in relation to sets. The best way to argue for multiversism itself seems to be by abductive-like reasoning, for example by pointing out that the multiverse claim best explains developments in mathematical practice and the intuitions of mathematicians. Still, if the argument goes through, the use of abductive reasoning (and maybe extrinsic evidence more generally) in support of a large class of interesting set theoretic claims, the axiom candidates, will be defeated by a certain ontological assumption about the subject matter.

Take two competing extensions of ZFC: ZFC + ϕ and ZFC + ψ (where ϕ and ψ stand for arbitrary axiom candidates). Accepting the one extension over the other is no trivial matter under Gödel's program; the added axiom is supposed to be absolutely true. Say ZFC + ϕ has a certain amount of seeming explanatory force, maybe it decides some of the independent statements of ZFC in which we have a prior interest. Say ZFC + ψ has similar support, but let us say that it has less of it. All else being equal, should this give you reason to think that ZFC + ϕ is absolutely true as opposed to ZFC + ψ? Given the truth of the multiverse claim, the answer is "no", because there most likely will be set theoretic universes where ZFC + ψ holds as opposed to ZFC + ϕ. The factors that support the abductive step might give you reasons for being particularly interested in "ZFC + ϕ"-universes, but, in line with the criterion of match, they do not legitimize the further step of claiming ϕ to be a basic principle of sets.

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Of course, under a radical form of multiversism, this will generally be the case for any new axiom candidate beyond whatever the multiversist take to be basic principles of sets, say, ZFC, as a requirement is that such a candidate axiom be independent of ZFC (there is no point in adding a statement that is decided by ZFC, because that will either lead to a contradiction or add something already provable from the axioms). Since for such a statement to be independent there must be a model where it holds and another model where its negation holds, the radical multiversist will assert the existence of the corresponding universes.

Therefore I take it that the truth of multiversism blocks approaches to discovering new set theoretic axioms that are absolutely true. But, returning to an earlier topic, how does the multiverse claim itself get justified? The best way to reconstruct Hamkins' own arguments for the existence of a multiverse is partly as a case of abductive reasoning. By looking at assumptions implicit in mathematical practice and technical results, Hamkins tries to persuade us that the existence of the multiverse *best* explains the current state of set theory, in particular the proliferation of different set theoretic models and the failure of attempts to settle independent claims. As such, the inference runs from the practice and results of the given science, to an underlying ontological assumption that explains that practice and those results. These are probably not the strongest arguments in favor of the view, however. The best way to strengthen these arguments would be to supplement this admittedly weak evidence in favor of the multiverse view with evidence based on a positive and intuitive conception of mathematical reality that supports the view directly.

The upshot, anyway, is that assuming the substantive and philosophical claim that there are many distinct universes of sets blocks appeals to a certain form of extrinsic evidence at the first-order level, that is, inferences to the absolute truth of proposed axiom candidates based on their fruitfulness, set-theoretic consequences, and so on. This kind of methodology would be ruled out by the criterion of match, as being out of step with the nature of set theoretic reality.

The multiverse and Occam's razor

Let us return briefly to the question of Occam's razor in mathematics. Since we are now assuming multiversism to be true, this enables a very different argument against that principle in mathematics compared to the naturalistic argument given before: Parsimony does not just fail as a theoretical virtue in mathematics because working mathematicians in fact seem to disregard Occam's razor. We can argue, under the combined forces of **Multiversism** and **Criterion of Match** that neither mathematicians nor philosophers of mathematics *ought* to adhere to Occam's razor due to the picture of the ontology of abstract reality offered by multiversism. On this understanding, we have in effect discovered *why* the principle fails in this case; not only *that* it fails.

3.2.3   Summary

Combined, these arguments show that it is extremely hard to justify new basic principles of sets on the multiverse conception of set theory. This is because both the legitimacy of intrinsic and extrinsic evidence is undermined. These forms of justification might still have roles to play, however, even to the multiversist.

Conceptual or intuitive evidence that a universe candidate has certain features that cohere with a given concept of set might help justify the assertion that such a universe exists. As such, intrinsic evidence flowing from different concepts of sets can help meet Mostowski's challenge raised in chapter 2. Extrinsic evidence can be useful in articulating why we would like to focus on only parts of the multiverse for different purposes. So, for example, we might want to focus on parts of the multiverse where ZFC + large cardinal axioms hold, as this gives us a way of measuring consistency strength. To the multiversist this nice feature of such strong theories does not suggest that they are absolutely true, but the features can still be used to motivate a purpose-dependent restricted view of the multiverse.

Still, to argue for new basic principles of sets is out of the question. So, the prospects of uniquely deciding further ϕ in the language of set theory seem bleak. This is of course to be expected from multiversism, but we have seen in more detail why this is so.

3.2.4   Objections

We now consider two objections to the arguments undermining the use of intrinsic and extrinsic evidence to establish new basic principles of sets on the assumption of multiversism.

Against the criterion of match

The first objection addresses the argument against extrinsic evidence via the criterion of match. One could simply try to deny the criterion of match in the case of set theory. This is an interesting strategy. For example, I take it that Maddy (2011) develops a view in this spirit when claiming that both Thin realism, Arealism and Objectivism are all appropriate starting points for the defense of extrinsic justifications in set theory (Maddy 2011:134).

More generally, strongly pragmatist approaches to the philosophy of science have it that proper methodology within a given scientific field is not hostage to the kind of philosophical and ontological considerations the criterion of match demands. In the case of set theory, one would hold that many parts of proper set theoretic methodology are not grounded in the metaphysics of sets. The hard part is to spell out this idea without oneself taking a stance on the metaphysics of sets. Still, the idea opens up an interesting

venue of research into how methodological principles and maxims, both in set theory and other parts of science, might not be grounded in the ontology of what we study at all, but in other features relevant to the scientific activity in question.

But, at least for me, there is a lingering sense that for there to be a well-defined notion of proper methodology within a scientific enterprise aiming at truth, there must be a relationship between what makes that method legitimate and the nature of that to which it is applied.

Intractable questions, revisited

Second, to the universist the arguments to the effect that it will be almost impossible to justify basic principles of sets will suggest that multiversism is an inherently unstable position. We mentioned earlier that multiversism might give rise to new intractable questions. The upshot of the arguments so far is that it is extremely difficult to justify new basic principles of sets either by use of intrinsic evidence or extrinsic evidence. So questions about whether a given ϕ is true in all universes seem to have become intractable.

This might be self-effacing to the multiversist who accepts a list of basic principles of sets, say, ZFC. This is because these issues re-opens old questions about justifying ZFC with a vengeance as earlier answers might have presupposed a unique concept of set and a unique universe. For example, if Choice is primarily established by extrinsic evidence, how are we sure that there aren't ¬Choice universes in the multiverse? If the multiversist, faced with this question, wanted to argue that Choice indeed is a basic principle of sets, it seems this have become intractable as well because of the considerations above. Of course, for some axioms this might not be so problematic; maybe there are interesting Choiceless hierarchies of sets or hierarchies where Replacement fails. But if the multiversist is unable to push back on any axiom, she might end up accepting for each axiom ϕ of ZFC, that there are ϕ and ¬ϕ-universes. This seems highly implausible; can a ¬Extensionality-hierarchy be called a universe of sets at all? The problem is that the multiversist, once we think about it, might be unable to justify anything as a basic principle of sets.

The response must be to find a non-arbitrary cut-off point for how radical the multiverse view can be. For example, maybe it is a minimal requirement that each universe satisfy enough set theory to reconstruct a delimited part of classical mathematics we have independent reason to think cannot fail. Or maybe we can use the finite/infinite or countable/uncountable distinctions to argue that facts about the finite or countable must be absolutely true while facts about the infinite or uncountable might be radically indeterminate. The point is that there are ways for the multiversist to try to push back and find some firm ground for characterizing the multiverse in an interesting way.

All the same, I take it that, even if multiversism fails, we have laid bare an important and intricate interplay between ontology and method, particularly abductive reasoning,

in set theory. A serious study of such interactions is important to any discipline with proper scientific ambitions. This needs to be done internally to the discipline in question. There is no prior guarantee that abstract set theoretic reality behaves in the same way as the concrete reality of natural science. So, simpler arguments by analogy like that of Gödel (1944) will not cut it.

We now turn to the question of whether the debate between the universist and the multiversist has any potentially real impact on set theoretical practice at all.

3.3 The Potential Fruitfulness of Multiversism vs. Universism

We have explored a fundamental disagreement about the subject matter of set theory – universism vs. multiversism. Furthermore, we have seen how the latter view has consequences for what counts as legitimate forms of justification in the adoption of new basic principles of sets. All the same, one might argue that this has very little real impact on set theoretic practice; that both views can interpret further developments under their own fundamental view of the field.

Suppose set theorists started to unanimously work in and accept a stronger theory than ZFC. They might give what seems like intrinsic or extrinsic evidence in favor of the additional axiom or axioms. Linnebo (2017:181–2) suggests that both the universist and the multiversist can interpret this development in a way consistent with their position. Roughly, to the universist, set theorists have in fact been successful in justifying new basic principles of sets true in V and thereby gotten the more complete description of the universe they seek. To the multiversist, however, what has happened is that set theorists at large have decided to focus on only certain parts of the multiverse under some sharpened concept of sets where the stronger theory holds, for whatever reasons or purposes they invoke in the adoption of the stronger theory. But to the multiversist this doesn't make all the other universes go away as independently existing entities. It is just that for some reason or other set theorists aren't interested in those structures.[2] Linnebo (who frames the discussion in terms of 'monism' vs. 'pluralism') concludes:

> So long as both interpretations are available, mathematical practice can proceed unaffected by the question of whether monism or pluralism is right. These reflections suggest that the question of pluralism matters less to mathematical practice than one might initially have thought. (Linnebo 2017:182)

Although I agree that, most likely, these primarily philosophical questions and resulting views will, and maybe should, have little effect on the practice of set theorists, I do think

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[2]Alternatively, the multiversist may say, more analogous with the universist, that they are accepting a stronger list of basic principles of sets which results in a narrower multiverse.

that the views come with different expectations and guidelines for how further practice should develop.

As heuristic devices in the further development of set theory, we can expect universism and multiversism to play quite different roles. Universism has arguably played a great role in inspiring set theorists to formulate new and stronger axioms of sets by emphasizing what they understand as our lack of knowledge of the universe. But at the same time universism has also constrained the development by expecting a unified theory of V. For example, there have been proposed large cardinal axioms that are inconsistent with the Choice axiom. The universist who thinks that Choice is a basic principle of sets is then liable to discard any such theory. As set theorists have been unable to resolve or unanimously accept stronger theories of sets that decide things like CH, multiversism offers a different picture which might inspire set theorists to view all of the different theories they have and might come up with as equally legitimate. Thus one can move on to explore these theories with greater freedom, without worrying so much about unifying everything in V. So, a universe where Choice fails because of a certain large cardinal axiom holding there might be interesting to explore and one can do this without giving up on universes where Choice hold. Slightly related to this, universism expects a stabilized set of consequences of stronger theories, that is, universism cannot, ultimately, countenance divergent sets of consequences or incompatible theories and will thus work to avoid such things. Multiversists, on the other hand, have no impetus to do so, and are happy to allow divergent sets of consequences and incompatible theories as the end result of set theoretic developments.

Arguments for or against views in the philosophy of mathematics often go from practice to philosophy. That is, the philosophical views are assessed by how well they cohere with and explain the practice as we find it. The fact that universism and multiversism might have some impact on set theoretic practice suggests a different kind of assessment of the views in addition to the standard one, namely asking which one is more fruitful for further practice. Thus, we might try to assess the views on how well they might influence or change practice in fruitful or better ways. I will not attempt the task here but leave it open as a question for further study.

3.4   Chapter Conclusion

I have argued that the idea that we can justify the adoption of different extended first-order axiomatizations of ZFC as absolutely true over others by means of intrinsic or extrinsic evidence, although it might seem reasonable at first glance, can be problematic due to certain assumptions about the ontology of sets required to sustain that methodology. We saw that there is a view of set theoretical reality, multiversism, that invalidates the straightforward use of such a methodology. But we also saw that setting such a high

standard for justification might be problematic to the multiversist who wants to argue that at least some axioms are true across the multiverse.

In the end, if one denies the multiverse conception but accepts the criterion of match, one is left with the task, assuming intrinsic evidence is fine, of defending the method of extrinsic evidence, in particular abductive reasoning in set theory. I hope to have established clearly that such a defense must seriously engage with questions about the nature of the abstract part of reality which is the subject matter of set theory.

Furthermore, we have seen to what degree universism and multiversism can be expected to influence practice in different ways. Although I agree that they will probably do so to a small degree, they still might have the potential to do so to such a degree that we might try to measure the success of these philosophical views not only by how well they explain practice but also in what way they might fruitfully influence practice.

Conclusion

Where does this leave us? We started this thesis with general questions about the unity or disunity of mathematical reality. I proposed to sharpen these questions by looking at the most famous attempt at giving a foundational theory for mathematics, namely set theory. A natural view is that the domain of set theory gives us the unified domain one might seek: the universe of all sets V.

We have seen that the independence phenomenon in set theory challenges the degree to which we understand V. The universist thinks these results show us that our knowledge of V is limited but argues that there are ways of trying to increase that knowledge. But there is an alternative interpretation of these results. The multiversist claims that this shows us that there are many different universes of sets, and they are all equally real. So, within our foundational theory one can defend views to the effect that there is a unified mathematical reality but also that this is not the case. We then explored and assessed strengths and weaknesses of multiversism in more detail, and concluded that the view is interesting and philosophically coherent but did not endorse it any further.

After that we turned to arguments challenging the use of different kinds of evidence in mathematics to establish new basic principles of sets, which is a goal in Gödel's program, on the assumption of multiversism. Multiversism seems to entail that it would be nigh impossible to justify that something is a basic principle of sets by appealing to the traditional forms of evidence. An objection is that this consequence might undermine multiversism itself by making the standards of justifications so high that no axiom, even those of ZFC, can be argued to be a basic principle of sets, which is highly implausible. After this, it was also suggested that the debate between universism and multiversism might not only be assessed by the views' success in explaining set theoretcal practice as we find it but also to what degree they might influence further practice in fruitful ways.

This leaves a lot of the terrain unexplored. I therefore end this thesis by formulating some topics which would be interesting to explore in more detail in further research.

Topics for Further Research

At the beginning of the thesis we saw some very general reasons for thinking that mathematical reality cannot be a delimited and unified whole due the richness of mathematical

structures and the open-endedness of mathematical operations. My first suggestion for a line of further inquiry is whether these thoughts could be developed into a more intuitive and natural motivation for accepting multiversism in set theory in addition to the abductive-like reasons Hamkins gives.

In this thesis I have been careful not to conflate pluralism with multiversism and monism with universism, because one can probably develop monist and pluralist positions in other ways. I have focused on realist versions of monism and pluralism, namely universism and multiversism. A further line of inquiry is to bring anti-realist versions of monism and pluralism into the fray. They might illuminate the discussion further in interesting ways. Although I have defended the importance of studying realist pluralism, which multiversism offers, both for its own sake and how it might challenge and shed light on unclarities with realist monism, I guess that many would find a less robust form of realist, maybe even anti-realist, pluralism more attractive.

Even though Hamkins tries to argue that we shouldn't expect a first-order theory of the multiverse, it would still be nice to have a formal theory of the multiverse in some form or another. Exploring candidates for such a theory would probably shed further light on both strengths and weaknesses of multiversism.

Another issue we have raised, which have received little to no attention elsewhere, is whether or not adopting multiversism puts us in an equally bad or even worse epistemic position than universism by engendering new intractable questions. For example, is there only one multiverse or are there many? What is true across (a) the multiverse and how can we even justify that something is true across the multiverse? If the multiversist could give well-motivated answers to these questions, this would probably lead us a step forward. On the other hand, these might be the questions that end up sinking multiversism.

When arguing against the legitimacy of extrinsic evidence on the assumption of multiversism, I invoked a **Criterion of Match**. I also suggested that some might object to this principle. An interesting line of further research is to investigate this criterion and its role in science more generally. Are there parts of science and methodological principles or maxims that do not seem to conform to the criterion?

Finally, I suggest studying the degree of fruitfulness of universism and multiversism as distinct heuristic devices for further set theoretical practice, although, it would probably be difficult to assess whether or not further developments in set theory connect with these philosophical views in any substantial way. Yet, one would think that if practitioners thought they had stumbled upon the discovery of a vast multiverse of different universes of sets with all kinds of strange features, this would have at least some repercussions for what they do next.

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