Optimal spline spaces for L^2 n-width problems with boundary conditions

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Abstract

In this paper we show that, with respect to the L^2 norm, three classes of functions in $H^r(0,1)$, defined by certain boundary conditions, admit optimal spline spaces of all degrees $\geq r-1$, and all these spline spaces have uniform knots.

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1 Introduction

Recently there has been renewed interest in using splines of maximal smoothness, i.e. of smoothness C^{d-1} for splines of degree d, as finite elements for solving PDEs. This is one of the main ideas behind isogeometric analysis [1, 4, 2, 10]. This raises the issue of how good these splines are at approximating functions of a certain smoothness class, especially with respect to approximation in the L^2 norm. This was answered to some extent by Melkman and Micchelli [7] who studied the L^2 approximation of functions u in the Sobolev space

$$H^r = H^r(0,1) = \{u \in L^2(0,1) : u^{(\alpha)} \in L^2(0,1), \quad \alpha = 1,2,\ldots,r\},$$

and measured the error relative to the L^2 norm of $u^{(r)}$. They showed that from this point of view there are two spaces of splines that are *optimal*, one of degree $r-1$, the other of degree $2r-1$. Later it was shown in [3] that these two spaces are just the first two of a whole sequence of optimal spline spaces of degrees $lr-1$, $l = 1,2,3,\ldots$. In the case $r = 1$ there is therefore an optimal spline space of every degree, but whether this is true for $r \geq 2$ is an open question.

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In this paper we study the related problem of approximating functions in H^r subject to certain boundary conditions. Specifically, we look at

$$\begin{aligned}
H_0^r &= \{u \in H^r : \quad u^{(k)}(0) = u^{(k)}(1) = 0, \quad 0 \le k < r, \quad k \text{ even}\}, \\
H_1^r &= \{u \in H^r : \quad u^{(k)}(0) = u^{(k)}(1) = 0, \quad 0 \le k < r, \quad k \text{ odd}\}, \\
H_2^r &= \{u \in H^r : \quad u^{(k)}(0) = u^{(l)}(1) = 0, \quad 0 \le k, l < r, \quad k \text{ even}, \quad l \text{ odd}\}.
\end{aligned}$$

Our main result is to show that for all $r \ge 1$, the spaces H_i^r, $i = 0, 1, 2$, admit optimal spline spaces of *all* degrees $\ge r - 1$. This is very similar to the numerical results reported by Evans et al. [2] regarding the degrees of the spline spaces, however their paper considered other boundary conditions (periodic conditions or no conditions).

The derivations in [7] and [3] were based on the use of an integral operator K that represents integration r times. Roughly speaking, and ignoring what happens at the boundary of the interval, if X_n is an optimal space of splines of some degree d, then the space $K(X_n)$, i.e., the space generated by integrating the splines in X_n, r times, is also an optimal space, consisting of splines of degree $d + r$.

In contrast, in this paper we work only with an integral operator K that represents a *single* integration. We generate optimal spline spaces for H_i^r, $i = 0, 1, 2$, by applying K, i.e., *one* integration, both to the initial Sobolev space H_i^1 and its optimal spline space, X_n, of degree 0. This approach works for H_i^r, $i = 0, 1, 2$, because, unlike H^r itself, when we apply (the right) K to the functions in H_i^r we get back a similar space, with r increased by one.

The optimal spline spaces we obtain have the same type of boundary conditions (odd or even derivatives are zero at the ends of the interval) as the spaces H_i^r themselves. The splines also have uniform knots, thus making them convenient to use in practice. In particular, some of the spline spaces corresponding to H_1^r are precisely the 'reduced spline spaces' studied recently by Takacs and Takacs [10, Section 5] (see also the end of Section 3 in this paper). They proved approximation estimates and inverse inequalities for these spaces, with a view to constructing fast iterative methods for solving PDEs in the framework of isogeometric analysis.

2 Kolmogorov n-widths

We start by formulating the concept of optimality in terms of Kolmogorov n-widths [9]. Denote the norm and inner product on $L^2 = L^2(0, 1)$ by

$$\|f\|^2 = (f, f), \qquad (f, g) = \int_0^1 f(t) g(t)\, dt,$$

for real-valued functions f and g. For a subset A of L^2, and an n-dimensional subspace X_n of L^2, let

$$E(A, X_n) = \sup_{u \in A} \inf_{v \in X_n} \|u - v\|$$

be the distance to A from X_n relative to the L^2 norm. Then the Kolmogorov L^2 n-width of A is defined by

$$d_n(A) = \inf_{X_n} E(A, X_n).$$

A subspace X_n is called an optimal space for A provided that

$$d_n(A) = E(A, X_n).$$

Now, consider the function classes

$$A_i^r = \{u \in H_i^r : \|u^{(r)}\| \leq 1\}, \quad i = 0, 1, 2. \tag{1}$$

By looking at $u/\|u^{(r)}\|$, for functions $u \in H_i^r$, we have for any n-dimensional subspace X_n of L^2,

$$\|u - P_n u\| \leq E(A_i^r, X_n)\|u^{(r)}\|,$$

where P_n denotes the L^2 projection onto X_n. Moreover, if X_n is an optimal subspace for A_i^r, then

$$\|u - P_n u\| \leq d_n(A_i^r)\|u^{(r)}\|,$$

and $d_n(A_i^r)$ is the least possible constant over all n-dimensional subspaces X_n.

3 Main results

We first describe the n-widths for A_i^r in (1) and the optimal subspaces based on eigenfunctions. We will show

Theorem 1 *For any integer $r \geq 1$, the n-widths of A_i^r, $i = 0, 1, 2$, are*

$$d_n(A_0^r) = \frac{1}{(n+1)^r \pi^r}, \qquad d_n(A_1^r) = \frac{1}{(n\pi)^r}, \qquad d_n(A_2^r) = \frac{1}{(n+\frac{1}{2})^r \pi^r}. \tag{2}$$

Furthermore, the spaces

$$[\sin \pi x, \sin 2\pi x, \ldots, \sin n\pi x], \tag{3}$$

$$[1, \cos \pi x, \cos 2\pi x, \ldots, \cos(n-1)\pi x], \tag{4}$$

$$[\sin(1/2)\pi x, \sin(3/2)\pi x, \ldots, \sin(n-1/2)\pi x] \tag{5}$$

are optimal n-dimensional spaces for, respectively, A_0^r, A_1^r and A_2^r.

Here, $[\cdots]$ denotes the span of a set of functions. The result for A_1^1 was shown by Kolmogorov [6]. With r an even number the result for A_0^r was shown in [3]. The remaining cases will be shown in Sections 7 and 8.

Now, let us describe the optimal spline spaces for these sets. Suppose $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$ is a knot vector such that

$$0 < \tau_1 < \cdots < \tau_m < 1,$$

and let $I_0 = [0, \tau_1)$, $I_j = [\tau_j, \tau_{j+1})$, $j = 1, \ldots, m-1$, and $I_m = [\tau_m, 1]$. For any $d \geq 0$, let Π_d be the space of polynomials of degree at most d. Then we define the spline space $S_{d,\boldsymbol{\tau}}$ by

$$S_{d,\boldsymbol{\tau}} = \{s \in C^{d-1}[0,1] : s|_{I_j} \in \Pi_d,\ j = 0, 1, \ldots, m\},$$

which has dimension $m+d+1$. We now define the three n-dimensional spline spaces $S_{d,i}$, for $i=0,1,2$, by

$$\begin{aligned} S_{d,0} &= \{s \in S_{d,\boldsymbol{\tau}_0} : s^{(k)}(0) = s^{(k)}(1) = 0, \quad 0 \le k \le d, \quad k \text{ even}\}, \\ S_{d,1} &= \{s \in S_{d,\boldsymbol{\tau}_1} : s^{(k)}(0) = s^{(k)}(1) = 0, \quad 0 \le k \le d, \quad k \text{ odd}\}, \\ S_{d,2} &= \{s \in S_{d,\boldsymbol{\tau}_2} : s^{(k)}(0) = s^{(l)}(1) = 0, \quad 0 \le k,l \le d, \quad k \text{ even}, \quad l \text{ odd}\}, \end{aligned} \tag{6}$$

where the knot vectors $\boldsymbol{\tau}_i$ for $i=0,1,2$, are given as

$$\begin{aligned} \boldsymbol{\tau}_0 &= \begin{cases} (\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}), & d \text{ odd}, \\ (\frac{1/2}{n+1}, \frac{3/2}{n+1}, \ldots, \frac{n+1/2}{n+1}), & d \text{ even}, \end{cases} \\ \boldsymbol{\tau}_1 &= \begin{cases} (\frac{1/2}{n}, \frac{3/2}{n}, \ldots, \frac{n-1/2}{n}), & d \text{ odd}, \\ (\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}), & d \text{ even}, \end{cases} \\ \boldsymbol{\tau}_2 &= \begin{cases} (\frac{1}{2n+1}, \frac{3}{2n+1}, \ldots, \frac{2n-1}{2n+1}), & d \text{ odd}, \\ (\frac{2}{2n+1}, \frac{4}{2n+1}, \ldots, \frac{2n}{2n+1}), & d \text{ even}. \end{cases} \end{aligned} \tag{7}$$

All these knot vectors have equidistant knots, but if we extend them to include the endpoints of $[0,1]$, the first and last knot intervals of these extended knot vectors sometimes have half the length of the interior ones. Examples of these knot vectors are shown in Figures 2, 3 and 4. Our main result is then the following.

Theorem 2 *Suppose $r \ge 1$. Then for any $i = 0,1,2$, the spline spaces $S_{d,i}$ are optimal n-dimensional spaces for the set A_i^r for any $d \ge r-1$.*

The case A_1^1 was shown in [3, Theorem 2]. On the other hand, the case A_0^r is a generalization of [3, Theorem 1] since that theorem only treated even r and spline spaces of degrees $lr-1$ for $l=1,2,\ldots$, thus leaving gaps between the degrees. When the degree d is even, the spaces $S_{d,1}$, whose common extended knot vector is equidistant, are the 'reduced spline spaces' of Takacs and Takacs [10, Section 5]. They have also derived approximation results regarding these spaces, using Fourier analysis. We can see from Theorem 2 and (2) that, for even d, the constant $\sqrt{2}$ in [10, Corollary 5.1] can be replaced by the optimal constant $1/\pi$.

4 Sets defined by kernels

We need some properties of kernels, and so this section is similar to [3, Section 3]. The starting point of the analysis is to represent the lowest order function classes A_i^1, $i=0,1,2$, in the form

$$A = K(B) = \{Kf : \|f\| \le 1\}, \tag{8}$$

where B is the unit ball in L^2, and K is the integral operator

$$Kf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

As in [7] we use the notation $K(x, y)$ for the kernel of K. We only consider kernels $K(x, y)$ that are continuous or piecewise continuous for $x, y \in [0, 1]$. Observe that for A in (8) and any n-dimensional subspace X_n of L^2,

$$E(A, X_n) = \sup_{\|f\|\leq 1} \|(I - P_n)Kf\| = \|(I - P_n)K\|_2, \tag{9}$$

where P_n is the orthogonal projection onto X_n, and $\|\cdot\|_2$ denotes the operator norm induced by the L^2 norm for functions.

We will denote by K^* the adjoint, or dual, of the operator K, defined by

$$(f, K^*g) = (Kf, g).$$

The kernel of K^* is $K^*(x, y) = K(y, x)$. Similar to matrix multiplication, the kernel of the composition of two integral operators K and L is

$$(KL)(x, y) = (K(x, \cdot), L(\cdot, y)).$$

The operator K^*K, being self-adjoint and positive semi-definite, has eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \geq 0, \tag{10}$$

and corresponding orthogonal eigenfunctions

$$K^*K\phi_n = \lambda_n\phi_n, \qquad n = 1, 2, \ldots. \tag{11}$$

If we further define $\psi_n = K\phi_n$, then

$$KK^*\psi_n = \lambda_n\psi_n, \qquad n = 1, 2, \ldots, \tag{12}$$

and the ψ_n are also orthogonal. The square roots of the λ_n are known as the s-numbers of K (or K^*). With these definitions we obtain [9, p. 6 or p. 65]:

Theorem 3 $d_n(A) = \lambda_{n+1}^{1/2}$, *and the space* $[\psi_1, \ldots, \psi_n]$ *is optimal for* A.

5 Totally Positive Kernels

Melkman and Micchelli [7] proved that if K is nondegenerate totally positive (NTP) [9, p. 108], then there are in fact two other optimal subspaces for A. Specifically, if K is NTP it follows from a theorem of Kellogg [9, p. 109] that the eigenvalues of K^*K and KK^* in (11) and (12) are positive and simple, $\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots > 0$, and the eigenfunctions ϕ_{n+1} and ψ_{n+1} have exactly n simple zeros in $(0, 1)$,

$$\phi_{n+1}(\xi_j) = \psi_{n+1}(\eta_j) = 0, \quad j = 1, 2, \ldots, n,$$

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1, \qquad 0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1.$$

Melkman and Micchelli [7, Theorem 2.3] then proved that the spaces

$$\begin{aligned} X_n^0 &= [K(\cdot, \xi_1), \ldots, K(\cdot, \xi_n)], \\ X_n^1 &= [(KK^*)(\cdot, \eta_1), \ldots, (KK^*)(\cdot, \eta_n)] \end{aligned} \tag{13}$$

are optimal for A. Using a duality technique that we will discuss in the next section it was later shown in [3, Theorem 5] that, given these two optimal spaces, there is an optimal space X_n^d, for all $d = 0, 1, 2, \ldots$, where

$$X_n^d = \begin{cases} [(KK^*)^i K(\cdot, \xi_1), \ldots, (KK^*)^i K(\cdot, \xi_n)], & d = 2i, \\ [(KK^*)^{i+1}(\cdot, \eta_1), \ldots, (KK^*)^{i+1}(\cdot, \eta_n)], & d = 2i + 1. \end{cases} \tag{14}$$

Melkman and Micchelli also constructed two optimal subspaces for the set A even when K is not NTP, but for K satisfying some related properties. We will deal with such a situation in Section 8.

6 Further optimality results

In this section we describe how optimal subspaces for the set A in (8) can be used to find optimal subspaces for sets of the form $K^*(A)$, $KK^*(A)$, and so on. The results here will hold for any integral operator K.

To ease notation we define two function classes A^r and A_*^r, for $r \geq 1$, by

$$A^r = \begin{cases} (KK^*)^i K(B), & r = 2i + 1, \\ (KK^*)^i(B), & r = 2i, \end{cases} \qquad A_*^r = \begin{cases} (K^*K)^i K^*(B), & r = 2i + 1, \\ (K^*K)^i(B), & r = 2i. \end{cases} \tag{15}$$

Observe that both A^r and A_*^r are defined by alternately applying the operators K and K^*, r times, to the unit ball B, with K always being the left-most operator for A^r, and K^* always being the left-most operator for A_*^r. Since $A^1 = A$, we will write A_* when referring to A_*^1. As we shall see momentarily the duality between the operators K and K^* will play an important role for the sets A^r and A_*^r, and especially their respective optimal subspaces. In some sense their optimal subspaces could be considered 'dual' to each other.

Since eigenvalues of powers of KK^* (and K^*K) are just powers of the λ_n in (10), with the same corresponding eigenfunction, it follows that the n-widths of the sets A^r and A_*^r are given by

$$d_n(A_*^r) = d_n(A^r) = d_n(A)^r, \tag{16}$$

and the space $[\psi_1, \ldots, \psi_n]$ in Theorem 3 is optimal for A^r, and the space $[\phi_1, \ldots, \phi_n]$ is optimal for A_*^r. As a tool for finding further optimal subspaces for A^r and A_*^r, with $r \geq 2$, we start with the following lemma.

Lemma 1 *For any integral operator K, let $L = KK^*$. If X_n and Y_n are any subspaces of L^2, then*

$$E(A^r, L^i(X_n)) \leq E(A, X_n) E(A^{r-1}, L^i(X_n)), \qquad r = 2i + 1,$$
$$E(A^r, L^{i-1}K(Y_n)) \leq E(A_*, Y_n) E(A^{r-1}, L^{i-1}K(Y_n)), \qquad r = 2i,$$

for $r \geq 2$.

Proof. First assume $r = 2i + 1$, for $i \geq 1$. From the definition of A^r we have $A^r = L^i K(B)$ and $A^{r-1} = L^i(B)$.

Let P_n be the L^2 projection onto X_n, and let Q_n be L^2 projection onto $L^i(X_n)$. Then

$$L^i P_n K f \in L^i(X_n),$$

for all $f \in L^2$, and so

$$(I - Q_n)L^i P_n K = 0.$$

Thus, by using equation (9), we find that

$$\begin{aligned} E(A^r, L^i(X_n)) &= \|(I - Q_n)L^i K\|_2 = \|(I - Q_n)L^i K - (I - Q_n)L^i P_n K\|_2, \\ &= \|(I - Q_n)L^i(I - P_n)K\|_2 \leq \|(I - Q_n)L^i\|_2 \|(I - P_n)K\|_2, \\ &= E(A^{r-1}, L^i(X_n))E(A, X_n). \end{aligned}$$

Next, assume $r = 2i$, for $i \geq 1$. Then $A^r = L^i(B)$ and $A^{r-1} = L^{i-1}K(B)$. In this case, let P_n be the L^2 projection onto Y_n, and let Q_n be L^2 projection onto $L^{i-1}K(Y_n)$. Then, as before,

$$(I - Q_n)L^{i-1}K P_n K^* = 0,$$

and the result follows by an almost identical argument as in the previous case. □

Now suppose that X_n^0 is an optimal n-dimensional subspace for A, and Y_n^0 is an optimal n-dimensional subspace for A_*. With these two subspaces one can generate a whole sequence of subspaces X_n^d and Y_n^d, by

$$X_n^d = K(Y_n^{d-1}), \qquad Y_n^d = K^*(X_n^{d-1}), \tag{17}$$

for all $d = 1, 2, 3, \ldots$, and it follows from [3, Lemma 1] that all the X_n^d are optimal for the n-width of $A^1 = A$, and all the Y_n^d are optimal for the n-width of $A_*^1 = A_*$. Note that for $d > 0$, the spaces X_n^d and Y_n^d could in general have dimension less than n, but they are still optimal for the n-width problem. In fact, if X_n^d or Y_n^d have dimension m, $0 \leq m < n$, then $d_m(A)$ must equal $d_n(A)$ by definition of the n-width.

Next, we consider A^r and A_*^r for $r \geq 2$.

Lemma 2 *Suppose the subspace X_n^0 is optimal for A and Y_n^0 is optimal for A_*. Then, for $r \geq 2$,*

$$E(A^r, X_n^d) \leq d_n(A)E(A^{r-1}, X_n^d), \tag{18}$$

$$E(A_*^r, Y_n^d) \leq d_n(A)E(A_*^{r-1}, Y_n^d), \tag{19}$$

for all $d \geq r - 1$.

Proof. We start by proving inequality (18). Let $L = KK^*$. First, assume $r = 2i + 1$, for $i \geq 1$. It then follows from (17) that $X_n^d = L^i(X_n^{d-r+1})$ for $d \geq r - 1$, and so the result follows from Lemma 1, with $X_n = X_n^{d-r+1}$, since X_n^{d-r+1} is optimal for A.

Next, assume $r = 2i$, for $i \geq 1$. It then follows from (17) that $X_n^d = L^i K(Y_n^{d-r+1})$ for $d \geq r - 1$, and so the result follows from Lemma 1, with $Y_n = Y_n^{d-r+1}$, since Y_n^{d-r+1} is optimal for A_* and $d_n(A_*) = d_n(A)$.

Inequality (19) then follows from the same argument if we interchange the roles of K and K^*. □

Using Lemma 2, we now obtain optimality results for A^r and A_*^r, for all $r \geq 1$.

Theorem 4 *Suppose the subspace X_n^0 is optimal for A and Y_n^0 is optimal for A_*. Then, for $r \geq 1$,*

- *the subspaces X_n^d in (17) are optimal for the n-width of A^r, and*
- *the subspaces Y_n^d in (17) are optimal for the n-width of A_*^r,*

for all $d \geq r - 1$.

Proof. The case $r = 1$ follows from [3, Lemma 1]. For $r \geq 2$ the result for the X_n^d follows from inequality (18) in Lemma 2, equation (16) and induction on r, since $d_n(A)^r = d_n(A) d_n(A)^{r-1}$. Similarly, now using inequality (19) in Lemma 2, we get the result for the Y_n^d as well. □

X_n^0	Y_n^1	X_n^2	Y_n^3	$\cdots$
A^1	A_*^1	A^1	A_*^1	$\cdots$
	A_*^2	A^2	A_*^2	$\cdots$
		A^3	A_*^3	$\cdots$
			A_*^4	$\cdots$

Y_n^0	X_n^1	Y_n^2	X_n^3	$\cdots$
A_*^1	A^1	A_*^1	A^1	$\cdots$
	A^2	A_*^2	A^2	$\cdots$
		A_*^3	A^3	$\cdots$
			A^4	$\cdots$

Figure 1: Optimality results.

We have summarized the statement of Theorem 4 in Figure 1. Under the assumption of Theorem 4 on X_n^0 and Y_n^0, all the spaces (above the line) in the two tables are optimal for all the function classes below them. Optimality of X_n^0 for A^1 implies optimality of Y_n^1 for A_*^1 by [3, Lemma 1], and so on along the first row (below the line) in the left table. Then, by Lemma 2, optimality of X_n^0 for A^1, and Y_n^1 for A_*^1, imply optimality of Y_n^1 for A_*^2, and so on along the second row. Optimality of X_n^0 for A^1, and X_n^2 for A^2, imply optimality of X_n^2 for A^3, and so on along the third row. Similarly for the right table.

Let us now turn back to the case where K is NTP. The subspace X_n^0 in (13) is optimal for A, and since K being NTP is equivalent to K^* being NTP, we also have that the subspace

$$Y_n^0 = [K^*(\cdot, \eta_1), \ldots, K^*(\cdot, \eta_n)] \tag{20}$$

is optimal for A_*, and so we can apply Theorem 4. The subspaces X_n^d in (17) are in this case the same as those in equation (14). Since the eigenvalues (10) (and thus also the n-widths) are strictly decreasing whenever K is NTP, the subspaces X_n^d and Y_n^d are in this case also n-dimensional for all $d \geq 0$.

7 Mixed boundary conditions

In this section we study the n-width problem for the function class A_2^r in (1). Consider the operator K given by

$$Kf(x) = \int_0^x f(y)dy = \int_0^1 K(x, y) f(y) dy, \tag{21}$$

whose kernel is

$$K(x,y) = \begin{cases} 0 & x < y, \\ 1 & x \geq y. \end{cases} \tag{22}$$

Using the equality $K^*(x,y) = K(y,x)$, we find that

$$K^* f(x) = \int_x^1 f(y)dy. \tag{23}$$

Thus K represents integration from the left, while K^* represents integration from the right.

From (21) we see that the set A_2^1 in equation (1) can be expressed as

$$A_2^1 = \{u \in H^1 : \|u'\| \leq 1,\ u(0) = 0\} = \{\int_0^x f(y)dy : \|f\| \leq 1\} = K(B).$$

To see that the remaining A_2^r can be expressed in terms of K and K^*, it is convenient to recognize the kernel of the composition KK^* as the Green's function for a boundary value problem, whose eigenfunctions we will need later anyway [in equation (27)].

Lemma 3 *If $u(x) = KK^* f(x)$ then u is the unique solution to the boundary value problem*

$$-u''(x) = f(x), \quad u(0) = u'(1) = 0. \tag{24}$$

Proof. We see from (21) and (23) that for any h,

$$\begin{aligned} (Kh)'(x) &= h(x), \\ (K^*h)'(x) &= -h(x). \end{aligned} \tag{25}$$

So, if $u(x) = KK^* f(x)$ then $-u''(x) = f(x)$. For the left boundary condition, from (21), we find that

$$u(0) = (KK^* f)(0) = 0.$$

For the right boundary condition, from (25) and (23),

$$u'(1) = (KK^* f)'(1) = (K^* f)(1) = 0.$$

To see that u is unique, suppose $f = 0$ in (24). Then u must be a linear function, but to satisfy the boundary conditions we must have $u = 0$. □

By applying the above lemma to functions f in B and $K(B)$ respectively and repeating the procedure i times, we find that

$$A_2^{2i} = (KK^*)^i(B), \qquad A_2^{2i+1} = (KK^*)^i K(B),$$

where A_2^{2i} and A_2^{2i+1} are as in equation (1). Observe that the left-most operator for the function class A_2^r is always K, and so A_2^r is an instance of A^r in (15).

7.1 Proof of Theorem 1 for A_2^r

In analogy to Lemma 3 we have, for the other composition K^*K,

Lemma 4 *If $u(x) = K^*Kf(x)$ then u is the unique solution to the boundary value problem*

$$-u''(x) = f(x), \quad u'(0) = u(1) = 0.$$

From Lemma 4, we see that the eigenvalues and eigenfunctions of K^*K are

$$\lambda_n = \frac{1}{(n-1/2)^2\pi^2}, \qquad \phi_n(x) = \cos(n-1/2)\pi x, \qquad n = 1, 2, \ldots. \tag{26}$$

From Lemma 3, the operator KK^* has the same eigenvalues, but the eigenfunctions are

$$\psi_n(x) = \sin(n-1/2)\pi x, \qquad n = 1, 2, \ldots. \tag{27}$$

So, by Theorem 3, the n-width of A_2^1 is as given in equation (2) and an optimal subspace is as given in (5). The analogous results for A_2^r, $r > 1$, follow from equation (16).

7.2 Proof of Theorem 2 for A_2^r

We have already seen that the function class A_2^r is the function class A^r in (15) when K has a kernel as given in (22). Since it is well known that this choice of K is NTP [5, p. 16], we can apply Theorem 4 to the spaces X_n^0 in (13) and Y_n^0 in (20). All that remains to show is that the optimal subspaces X_n^d generated as in equation (17) are the spline spaces we claim.

The zeros ξ_j of $\phi_{n+1}(x)$ in (26) are the knots in the even degree case for the knot vector $\boldsymbol{\tau}_2$ in equation (7), and the zeros η_j of $\psi_{n+1}(x)$ in (27) are the knots in the odd degree case. Thus, X_n^0 in (13), with the kernel of K as in equation (22), is equal to

$$X_n^0 = [K(\cdot, \xi_1), \ldots, K(\cdot, \xi_n)] = S_{0,2},$$

where $S_{0,2}$ is the piecewise constant spline space given in equation (6). To find X_n^1 we perform a simple calculation to see that

$$KK^*(x,y) = (K(x,\cdot), K(y,\cdot)) = \begin{cases} x, & x < y, \\ y, & x > y, \end{cases}$$

and so, $X_n^1 = K(Y_n^0) = [(KK^*)(\cdot, \eta_1), \ldots, (KK^*)(\cdot, \eta_n)] = S_{1,2}$, the piecewise linear spline space given in equation (6). The remaining X_n^d, for $d \geq 2$, can be found by using the fact that $X_n^{d+2} = KK^*(X_n^d)$ and applying Lemma 3, since the derivative of a spline is a spline on the same knot vector of one degree lower.

Remark: We note that interchanging the roles of K and K^* shows that the subspaces Y_n^d are optimal for the sets defined by interchanging the boundary conditions in A_2^r, i.e., odd derivatives set to zero at the left-hand side, and even derivatives set to zero at the right-hand side. One finds that the subspaces Y_n^d are equal to their corresponding 'dual' subspace X_n^d, just with interchanged boundary conditions and interchanged knots (i.e., replacing the even degree case for $\boldsymbol{\tau}_2$ in (7) with the odd degree case, and vice versa).

8 Symmetric boundary conditions

In this section we study the n-width problems for the remaining function classes A_0^r and A_1^r in (1). Let K_1 be the operator given by

$$K_1 = (I - Q)K, \tag{28}$$

where Q is the orthogonal projection onto the constant functions, Π_0, and K is again the operator (21). From [7] we know that the set A_1^1 given in equation (1) can be written as the orthogonal sum

$$A_1^1 = \Pi_0 \oplus K_1(B).$$

It follows from [9, Chap. IV, Sec. 3.2] that the set A_0^1 in equation (1) can be written as

$$\begin{aligned} A_0^1 &= \{u \in H^1 : \|u'\| \le 1, \quad u(0) = u(1) = 0\}, \\ &= \{K^* f : \|f\| \le 1, \quad f \perp 1\} = K^*(I - Q)(B) = K_1^*(B). \end{aligned}$$

The kernel of $K_1 K_1^*$ is the Green's function to the boundary value problem

$$-u''(x) = f(x), \quad u'(0) = u'(1) = 0, \quad u, f \perp 1 \tag{29}$$

(see e.g. [3, Lemma 4]). Using equation (29) i times and then adding back the constants we find that,

$$A_1^{2i} = \Pi_0 \oplus (K_1 K_1^*)^i(B), \qquad A_1^{2i+1} = \Pi_0 \oplus (K_1 K_1^*)^i K_1(B), \tag{30}$$

where A_1^{2i} and A_1^{2i+1} are as in equation (1). The kernel of $K_1^* K_1$ is the Green's function to the boundary value problem

$$-u''(x) = f(x), \quad u(0) = u(1) = 0 \tag{31}$$

(see e.g. [3, Lemma 3]). Then, using equation (31) i times we find that,

$$A_0^{2i} = (K_1^* K_1)^i(B), \qquad A_0^{2i+1} = (K_1^* K_1)^i K_1^*(B),$$

where A_0^{2i} and A_0^{2i+1} are as in equation (1). Observe that the left-most operator for the function class A_0^r is always K_1^*, and so A_0^r is an instance of A_*^r in (15). The function class A_1^r, on the other hand, is not quite an instance of A^r in (15), but it is of the form $\Pi \oplus A^r$.

8.1 Proof of Theorem 1 for A_0^r and A_1^r

From equation (31) we see that the eigenvalues and eigenfunctions of $K_1^*K_1$ are

$$\lambda_n = \frac{1}{(n\pi)^2}, \qquad \phi_n(x) = \sin(n\pi x), \qquad n = 1, 2, \dots. \tag{32}$$

The operator $K_1K_1^*$ has the same eigenvalues, but the eigenfunctions are

$$\psi_n(x) = \cos(n\pi x), \qquad n = 1, 2, \dots. \tag{33}$$

So, by Theorem 3, the n-widths of both $A_0^1 = K_1^*(B)$ and $K_1(B)$ are equal to $d_n(A_0^1)$ in equation (2). An optimal n-dimensional subspace for A_0^1 is as given in (3), and an optimal n-dimensional subspace for $K_1(B)$ is $[\cos(\pi x), \cos(2\pi x), \dots, \cos(n\pi x)]$. Since this subspace is orthogonal to Π_0 it follows that an optimal $(n+1)$-dimensional subspace for $A_1^1 = \Pi_0 \oplus K_1(B)$ is

$$[1, \cos(\pi x), \cos(2\pi x), \dots, \cos(n\pi x)],$$

thus showing that (4) is an optimal n-dimensional space, and that the n-width of A_1^1 is as given in (2). Pay special attention to this index-shift caused by Π_0: the n-width of $K_1(B)$ is equal to $\lambda_{n+1}^{1/2}$, but the n-width of A_1^1 is equal to $\lambda_n^{1/2}$ in (32). As before, the analogous results for A_0^r and A_1^r, $r > 1$, follow from equation (16).

8.2 Proof of Theorem 2 for A_0^r and A_1^r

To prove Theorem 2 for A_0^r and A_1^r we will use Theorem 4 with K_1 playing the role of the generic operator K. We must therefore identify the first optimal space X_n^0 for $K_1(B)$ and the first optimal space Y_n^0 for $A_0^1 = K_1^*(B)$. Unlike K in equation (21), K_1 is not NTP (specifically, it is not totally positive) and this creates an extra challenge compared with subsection 7.2. Fortunately, as shown in [7, 8] the operator $K_1^*K_1$ is in fact NTP, and we can make use of this and other results in [7, Section 5]. Specifically, we have from [7, Theorem 5.1] that

$$X_n^0 = [K_1(\cdot, \xi_1), \dots, K_1(\cdot, \xi_n)]$$

is an optimal subspace for the n-width of $K_1(B)$, where the ξ_j, for $j = 1, 2, \dots, n$, are the n zeros of $\phi_{n+1}(x)$ in (32). Observe, that these ξ_j's are the knots in the odd degree case for the knot vector $\boldsymbol{\tau}_0$ in (7).

Now, we consider Y_n^0. First, let η_j, for $j = 1, 2, \dots, n+1$, be the $n+1$ zeros of $\psi_{n+1}(x)$ in (33), which are the knots in the even degree case of $\boldsymbol{\tau}_0$ in (7). Additionally, let J be the interpolation operator from $C[0, 1]$ to Π_0 determined by interpolating at η_1, and define the operator

$$\overline{K}_1 = (I - J)K,$$

where K still is the operator (21). If we let Y_n^0 be the n-dimensional space

$$Y_n^0 = [\overline{K}_1^*(\cdot, \eta_2), \dots, \overline{K}_1^*(\cdot, \eta_{n+1})], \tag{34}$$

then the proof of [7, Theorem 5.1] (or [9, Theorem 5.11 p. 121]) contains the following important result.

Lemma 5 *If P_n is the orthogonal projection onto the space Y_n^0, and λ_{n+1} is as in* (32), *then*

$$\sup_{\|f\|\leq 1} \|\overline{K}_1(I-P_n)f\| \leq \lambda_{n+1}^{1/2}. \tag{35}$$

Proof. The operator K_1 is a special case of the operator K_1 in [9] (as explained on page 124). It therefore satisfies the assumptions of [9, Theorem 5.11 p. 121], and inequality (35) is then proved on page 122. Note the shift in index, the above $n+1$ corresponds to n in [9, Theorem 5.11]. □

Using this inequality we can show the following.

Theorem 5 *The space Y_n^0 is optimal for $A_0^1 = K_1^*(B)$.*

Proof. Let P_n be the orthogonal projection onto Y_n^0 in (34). To prove that Y_n^0 is an optimal subspace for A_0^1, we need to show that

$$E(A_0^1, Y_n^0) \leq d_n(A_0^1),$$

or equivalently,

$$\|(I-P_n)K_1^*\|_2 \leq \lambda_{n+1}^{1/2},$$

with λ_{n+1} as given in (32). First observe that

$$\|(I-P_n)K_1^*\|_2 = \|K_1(I-P_n)\|_2 = \|(I-Q)K(I-P_n)\|_2.$$

Next, since both J and Q are projections onto the constants, Π_0, but only Q is the orthogonal projection, we must have

$$\|(I-Q)K(I-P_n)\|_2 \leq \|(I-J)K(I-P_n)\|_2 = \|\overline{K}_1(I-P_n)\|_2.$$

Hence, the result follows from Lemma 5. □

Remark: Melkman and Micchelli [7, Theorem 5.1] used the inequality in Lemma 5 to directly conclude that the $(n+1)$-dimensional space

$$\Pi_0 + [(\overline{K}_1\,\overline{K}_1^*)(\cdot,\eta_2),\ldots,(\overline{K}_1\,\overline{K}_1^*)(\cdot,\eta_{n+1})],$$

is optimal for the set $A_1^1 = \Pi_0 \oplus K_1(B)$. On the other hand, from [3, Lemma 1] and the above Theorem 5, it follows that $K_1(Y_n^0)$ is an optimal space for $K_1(B)$, and so

$$\Pi_0 \oplus K_1(Y_n^0) = \Pi_0 \oplus [(K_1\overline{K}_1^*)(\cdot,\eta_2),\ldots,(K_1\overline{K}_1^*)(\cdot,\eta_{n+1})],$$

is optimal for A_1^1. This is consistent with their result, since the difference

$$(K_1\overline{K}_1^*)(\cdot,\eta_j) - (\overline{K}_1\,\overline{K}_1^*)(\cdot,\eta_j),$$

is a constant for any $j = 2,\ldots,n+1$.

We now have the first optimal space X_n^0 for $K_1(B)$ and the first optimal space Y_n^0 for $A_0^1 = K_1^*(B)$, and so we can apply Theorem 4. To do this let us express A_1^r in equation (30) as

$$A_1^r = \Pi_0 \oplus \tilde{A}_1^r.$$

Now, if X_n^d and Y_n^d are generated as in (17) with K_1 playing the role of the generic K, then it follows from Theorem 4 that, for all $r \geq 1$,

- the n-dimensional spaces X_n^d are optimal for the n-width of $\tilde{A}_1^r$, and
- the n-dimensional spaces Y_n^d are optimal for the n-width of A_0^r,

for all $d \geq r-1$. Note further that these spaces are all n-dimensional since the n-widths in (2) are strictly decreasing. Moreover, since both $X_n^d \perp \Pi_0$ and $\tilde{A}_1^r \perp \Pi_0$, we find that the $(n+1)$-dimensional spaces $\Pi_0 \oplus X_n^d$ are optimal for $A_1^r = \Pi_0 \oplus \tilde{A}_1^r$ for $d \geq r-1$.

The remaining task is to recognize the spaces $\Pi_0 \oplus X_n^d$ and Y_n^d as spline spaces. As already stated, the optimal spaces $\Pi_0 \oplus X_n^d$ were identified in [3] and we have the equality

$$S_{d,1} = \Pi_0 \oplus X_{n-1}^d,$$

where $S_{d,1}$ is the n-dimensional space defined in (6). However, only the spline spaces Y_n^d when d is odd were found in [3]. In that case we have

$$S_{d,0} = Y_n^d, \tag{36}$$

with $S_{d,0}$ also as in (6). Now, using the definition of $\overline{K}_1$ we find that the kernel $\overline{K}_1^*(x,y) = \overline{K}_1(y,x)$ is equal to

$$\overline{K}_1^*(x,y) = \begin{cases} 0, & x < \eta_1, \\ 1, & \eta_1 < x < y, \\ 0, & x > y, \end{cases}$$

for $y > \eta_1$. The space Y_n^0 in equation (34) is then the space of piecewise constant splines with knots η_j, $j = 1, \ldots, n+1$, that vanish on the intervals $[0, \eta_1)$ and $(\eta_{n+1}, 1]$. Since $Y_n^2 = K_1^* K_1(Y_n^0)$, and so on, we know from (31) that equation (36) also holds in the case of d even. This proves Theorem 2 for A_0^r and A_1^r.

9 Basis functions

In this section we describe how to create a local basis for the spline spaces $S_{d,i}$, $i = 0, 1, 2$. First consider $i = 1$. An explanation of how to construct a local basis for $S_{d,1}$ (with d even) is presented in [10]. The basic idea consists of three parts. Start with our uniform knot vector $\boldsymbol{\tau}_1$ in (7) and extend it to a uniform knot vector on the whole real line. Second, construct all the B-splines on this infinite knot vector that have non-zero support on $(0, 1)$. Third, identify the B-splines that cross the boundary and add them together in pairs, chosen in such a manner that the symmetry of uniform B-splines ensures the boundary conditions (all odd derivatives set to zero) are satisfied. Figure 2 shows the basis functions for $S_{d,1}$ of degree 0 to 3 with knot-distance 0.2 ($n = 5$).

Next, we consider $i = 0$. Constructing a basis for $S_{d,0}$ can be done by essentially the same procedure as for $S_{d,1}$. Instead of adding pairs of B-splines together we take differences. The symmetry of uniform B-splines will again ensure that the boundary conditions (all even derivatives set to zero) are satisfied. Figure 3 shows the basis functions for $S_{d,0}$ of degree 0 to 3 with knot-distance 0.2 ($n = 4$).

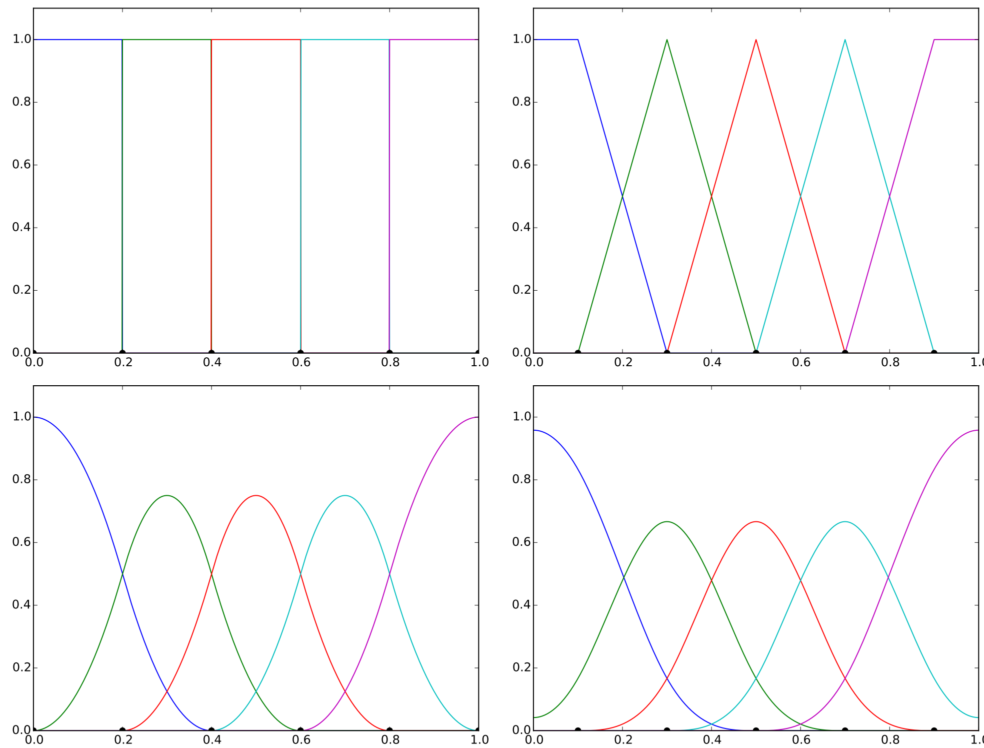

Figure 2: Basis functions for $S_{d,1}$.

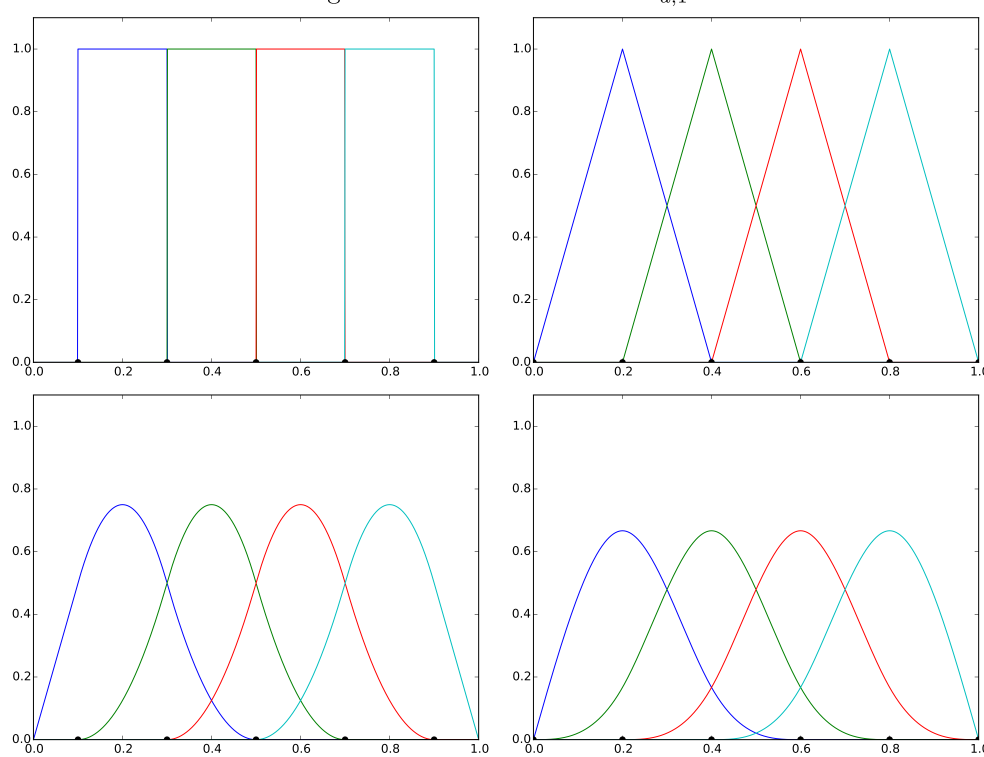

Figure 3: Basis functions for $S_{d,0}$.

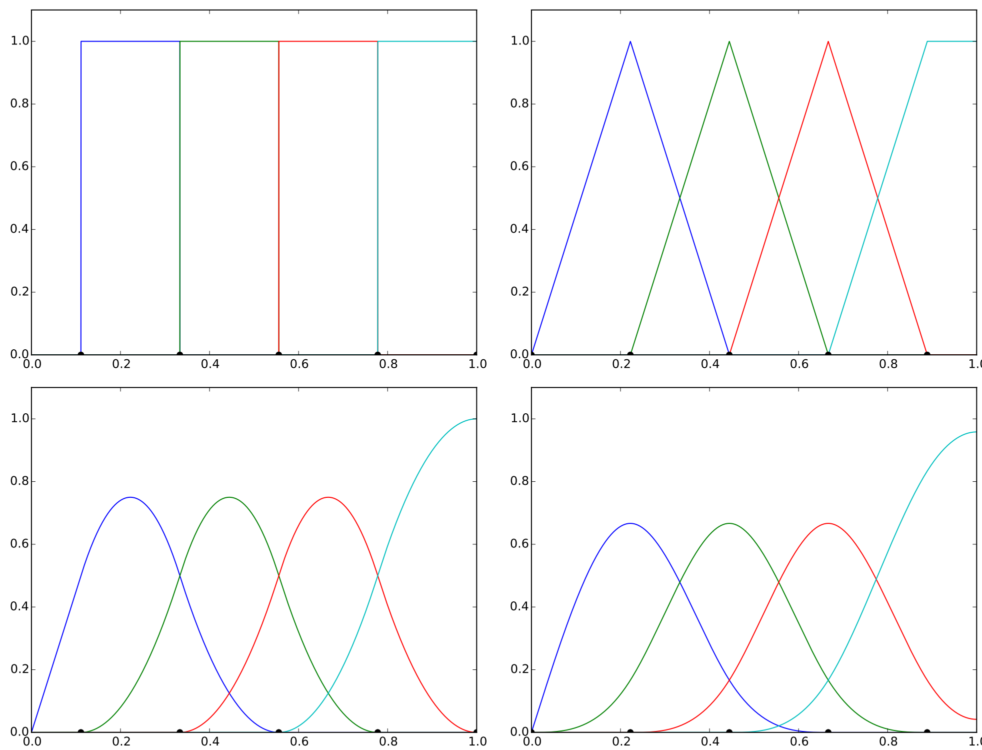

Figure 4: Basis functions for $S_{d,2}$.

Regarding $i = 2$, adding pairs of B-splines together on the right-hand side and subtracting on the left-hand side will give a basis for $S_{d,2}$. Figure 4 shows the basis functions for $S_{d,2}$ of degree 0 to 3 with knot-distance 2/9 ($n = 4$).

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