PROBABILISTIC BOUNDARIES OF FINITE EXTENSIONS OF QUANTUM GROUPS

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Abstract. Given a discrete quantum group H with a finite normal quantum subgroup G, we show that any positive, possibly unbounded, harmonic function on H with respect to an irreducible invariant random walk is G-invariant. This implies that, under suitable assumptions, the Poisson and Martin boundaries of H coincide with those of H/G. A similar result is also proved in the setting of exact sequences of C*-tensor categories. As an immediate application, we conclude that the boundaries of the duals of the group-theoretical easy quantum groups are classical.

Introduction

The study of probabilistic boundaries of quantum random walks was initiated in the 90s by Biane [1], who considered random walks on the duals of compact Lie groups, and by Izumi [4], who developed the Poisson boundary theory for discrete quantum groups. The Martin boundary theory for discrete quantum groups was later developed by Tuset and the second author [11]. Since then the boundaries have been computed in a number of cases, see e.g. [16, 18, 3]. The situation is particularly satisfactory for amenable quantum groups, where the Poisson boundaries have been identified for a large and important class of random walks [16, 13]. On the other hand, in the nonamenable case the duals of free unitary quantum groups remain the main example of a truly noncommutative computation [18].

In this note we consider probably the simplest example of discrete quantum groups that are neither commutative nor cocommutative, namely, the crossed products $\ell^\infty(H) = \ell^\infty(\Gamma) \rtimes S$, where Γ is a discrete group and S is a finite group acting on Γ by group automorphisms. They include the duals of the group-theoretical easy quantum groups recently studied in [15]. We show that under natural assumptions both boundaries of H coincide with the corresponding classical boundaries of Γ.

It should be noted that the Poisson boundaries of certain random walks on crossed products have been already studied in [5], see also [9], and shown to be isomorphic to crossed products of Poisson boundaries. There is no contradiction here, we could have obtained a similar result if we considered degenerate random walks on $\ell^\infty(\Gamma) \rtimes S$ that are trivial on the $C^*(S)$ part.

In fact, we formulate and prove our results in a more general setting than that of crossed products. We consider a discrete quantum group H with a finite normal quantum subgroup G, and show that under suitable assumptions the Poisson and Martin boundaries of H coincide with those of H/G. For Poisson boundaries of genuine groups this recovers a result of Kaimanovich [7] obtained as an application of his study of covering Markov operators. We also obtain similar results for exact sequences of C*-tensor categories in the framework of categorical random walks recently developed in [14].

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1. Invariance of harmonic functions under finite quantum groups

Let H be a discrete quantum group, with the von Neumann algebra $\ell^\infty(H)$ of bounded functions and comultiplication $\Delta_H\colon \ell^\infty(H)\to\ell^\infty(H)\bar\otimes\ell^\infty(H)$, see e.g. [19]. Recall that this implies that $\ell^\infty(H)$ is the ℓ^∞-direct sum $\ell^\infty\text{-}\bigoplus_{s\in I}B(H_s)$ of full matrix algebras, where I is the set of equivalence classes of irreducible representations of the dual compact quantum group $\hat H$. When H is a genuine group, we have $I=H$ and the comultiplication is given by $\Delta_H(f)(g,h)=f(gh)$ for $f\in\ell^\infty(H)$ and $g,h\in H$.

For a normal state $\phi\in\ell^\infty(H)_*$, consider the convolution operator

$$P_\phi\colon \ell^\infty(H)\to\ell^\infty(H),\quad P_\phi=(\phi\otimes\iota)\Delta_H.$$

An element $x\in\ell^\infty(H)$ is called *P_ϕ-harmonic* if $P_\phi(x)=x$. Note that since P_ϕ can be thought of as a matrix of completely positive maps $B(H_s)\to B(H_t)$, it also makes sense to talk about positive harmonic elements in the algebra $\prod_{s\in I}B(H_s)$ of all functions on H.

We denote by ϕ^n the convolution powers of ϕ, defined inductively by $\phi^{n+1}=(\phi^n\otimes\phi)\Delta_H$. Then $P^n_\phi=P_{\phi^n}$.

Assume now that G is a finite normal quantum subgroup of H. This means that $\ell^\infty(G)$ is finite dimensional and either of the following equivalent conditions is satisfied [17],[8]:

(i) we are given a surjective normal $*$-homomorphism $\pi\colon\ell^\infty(H)\to\ell^\infty(G)$ respecting the coproducts such that the fixed point algebra $\ell^\infty(G\backslash H)=\{x\mid\alpha_l(x)=1\otimes x\}$ under the left action $\alpha_l=(\pi\otimes\iota)\Delta_H$ of G on $\ell^\infty(H)$ coincides with the fixed point algebra $\ell^\infty(H/G)=\{x\mid\alpha_l(x)=x\otimes1\}$ under the right action $\alpha_r=(\iota\otimes\pi)\Delta_H$;

(ii) we are given an embedding $\mathbb{C}[\hat G]\to\mathbb{C}[\hat H]$ of the Hopf $*$-algebras of regular functions on the dual compact quantum groups such that $\mathbb{C}[\hat G]$ is invariant under the left and/or right adjoint action of the Hopf algebra $\mathbb{C}[\hat H]$ on itself.

Then the quotient discrete quantum group $\Gamma=H/G$ is defined by letting $\ell^\infty(\Gamma)=\ell^\infty(H/G)=\ell^\infty(G\backslash H)$ and the coproduct to be the restriction of Δ_H to $\ell^\infty(\Gamma)$.

Theorem 1.1. *Assume H is a discrete quantum group, G is a finite normal quantum subgroup of H, and ϕ is a generating normal state on $\ell^\infty(H)$, meaning that $\vee_{n\ge1}\operatorname{supp}\phi^n=1$. Then any positive, possibly unbounded, P_ϕ-harmonic function on H is G-invariant.*

For genuine discrete groups and bounded harmonic functions this was proved by Kaimanovich [7, Theorem 3.3.1 and Corollary 3] using measure-theoretic methods.

Remark 1.2. The left and right actions of G are both well-defined on the algebra of all functions on H. In order to see this, let us take (i) above as our main definition, so we assume that we are given a surjective normal $*$-homomorphism $\pi\colon\ell^\infty(H)\to\ell^\infty(G)$ respecting the coproducts. Recall that $\ell^\infty(H)=\ell^\infty\text{-}\bigoplus_{s\in I}B(H_s)$, where I is the set of equivalence classes of irreducible representations of $\hat H$. For every $s\in I$ fix a representative $U_s\in B(H_s)\otimes C(\hat H)$. Let p be the support of π, that is, we have $\ker\pi=(1-p)\ell^\infty(H)$. Then $p\ell^\infty(H)=\bigoplus_{s\in I_G}B(H_s)$ for a finite subset $I_G\subset I$, which we can identify with the set of equivalence classes of irreducible representations of $\hat G$. The tensor products of the representations U_s, $s\in I_G$, decompose according to the fusion rules of $\hat G$ (and, moreover, the subcategory of $\operatorname{Rep}\hat H$ generated by these representations can be identified with $\operatorname{Rep}\hat G$). It follows that by writing $r\sim s$ if there exists $t\in I_G$ such that U_s is a subrepresentation of $U_r\otimes U_t$, we get a well-defined equivalence relation on I with finite equivalence classes. If S is an equivalence class, then the action $\alpha_r=(\iota\otimes\pi)\Delta_H$ of G on $\ell^\infty(H)$ defines by restriction an action on $\bigoplus_{s\in S}B(H_s)$. From this we see that the action α_r of G on $\ell^\infty(H)$ extends in an obvious way to an action on the whole algebra $\prod_{s\in I}B(H_s)$ of functions on H. Similar considerations apply to the left action α_l. Note in passing that using the normality of G it can be checked that the corresponding equivalence relation on I is the same as for α_r, see also Section 3 for a stronger statement.

Proof of Theorem 1.1. Replacing ϕ by $\sum_{n\ge1}\phi^n/2^n$, which might only increase the set of harmonic elements, we may assume that ϕ is faithful.

Consider first bounded harmonic elements. As in Remark 1.2, consider the support p of $\pi\colon\ell^\infty(H)\to\ell^\infty(G)$. Define the states

$$\phi_1=\phi(p)^{-1}\phi(p\,\cdot)\ \text{ and }\ \phi_2=\phi(1-p)^{-1}\phi((1-p)\,\cdot).$$

Denote by ν the state on $\ell^\infty(G)$ such that $\phi_1(x)=\nu(\pi(x))$. Then, with $t=\phi(p)$, we have

$$P_\phi = tP_{\phi_1}+(1-t)P_{\phi_2} = t(\nu\otimes\iota)\alpha_l+(1-t)P_{\phi_2}.$$

Consider the conditional expectation

$$E=(h\otimes\iota)\alpha_l=(\iota\otimes h)\alpha_r\colon \ell^\infty(H)\to\ell^\infty(H/G),$$

where h is the Haar state on $C(G)$. It obviously commutes with P_{ϕ_1} and P_{ϕ_2}, so it suffices to show that there are no nonzero P_ϕ-harmonic elements in $\ker E$. Since P_{ϕ_2} is a contraction, for this, in turn, it suffices to show that the restriction of P_{ϕ_1} to $\ker E$ is a strict contraction.

Since $\ell^\infty(G)$ is finite dimensional and the state ν is faithful, there exists $\delta>0$ such that $\nu-\delta h\ge 0$. On $\ker E$ we have

$$P_{\phi_1}=(\nu\otimes\iota)\alpha_l=((\nu-\delta h)\otimes\iota)\alpha_l.$$

This implies that $\|P_{\phi_1}|_{\ker E}\|\le(\nu-\delta h)(1)=1-\delta$, which is what we need.

Assume now that a is an unbounded positive P_ϕ-harmonic element. By adding 1 we may assume that $a\ge 1$. Put $b=E(a)$. Then $b\ge 1$ is again a P_ϕ-harmonic element. Since it is G-invariant, it is also P_{ϕ_2}-harmonic. Since $h\ge(\dim C(G))^{-1}\varepsilon$, where ε is the counit on $C(G)$, we also have $b\ge(\dim C(G))^{-1}a$. It follows that the element $c=b^{-1/2}ab^{-1/2}$ is bounded.

Consider the Doob transform

$$P^b_\phi=b^{-1/2}P_\phi(b^{1/2}\cdot b^{1/2})b^{-1/2}$$

of P_ϕ defined by b. It is a well-defined ucp map on $\ell^\infty(H)$, and the element c is P^b_ϕ-harmonic. As the element b is G-invariant, the operator P^b_ϕ commutes with E and we have $P^b_{\phi_1}=P_{\phi_1}$, so that $P^b_\phi=tP_{\phi_1}+(1-t)P^b_{\phi_2}$. Now the same argument as in the first part of the proof applies and we conclude that $c\in\ell^\infty(H/G)$. Hence $a=b^{1/2}cb^{1/2}$ is G-invariant. □

Note that, as was already remarked by Kaimanovich [7], the normality condition in this theorem cannot be dropped. Indeed, otherwise the Poisson boundaries of free products of finite groups would be trivial, which is not true, since such free products are nonamenable except in a few trivial cases.

It is nevertheless tempting to think that Theorem 1.1 should be true in a greater generality and that under suitable irreducibility conditions any harmonic element with respect to a G-equivariant ucp map on a C*-algebra must be G-invariant. However, Theorem 4.3.3 in [7] shows that the question what such optimal conditions could be is quite delicate. We will strengthen Theorem 1.1 in a somewhat different direction by showing that the main part of the argument generalizes from finite to compact quantum groups.

Proposition 1.3. *Let $\alpha\colon A\to C(G)\otimes A$ be an action of a compact quantum group G with faithful Haar state on a unital C*-algebra A, and $P\colon A\to A$ be a ucp map satisfying the following properties:*

(i) *P commutes with the unique G-equivariant conditional expectation $E\colon A\to A^G$;*
(ii) *P can be written as a convex combination $tP_1+(1-t)P_2$, $0<t<1$, of two ucp maps such that $P_1=(\nu\otimes\iota)\alpha$ for some faithful state ν on $C(G)$.*

Then any P-harmonic element in A is G-invariant.

The proof is based on the following lemma.

Lemma 1.4. *Let G be a compact quantum group, ν be a faithful state on $C(G)$ and $U\in B(H_U)\otimes C(G)$ be a finite dimensional unitary representation without nonzero invariant vectors. Then $\|(\iota\otimes\nu)(U)\|<1$.*

Proof. Assume $\|(\iota\otimes\nu)(U)\|=1$. Then there exist unit vectors $\xi,\zeta\in H_U$ such that for the linear functional $\omega_{\xi,\zeta}=(\cdot\,\xi,\zeta)$ on $B(H_U)$ we have $(\omega_{\xi,\zeta}\otimes\nu)(U)=1$. In other words, $\nu(x)=1$ for the contraction $x=(\omega_{\xi,\zeta}\otimes\iota)(U)\in C(G)$. Since ν is a faithful state, it follows that $x=1$. Applying the counit ε on $\mathbb{C}[G]$ to the identity $(\omega_{\xi,\zeta}\otimes\iota)(U)=1$, we get $(\xi,\zeta)=1$, so $\xi=\zeta$, and then $U(\xi\otimes 1)=\xi\otimes 1$. Therefore ξ is an invariant vector, which is a contradiction. □

Proof of Proposition 1.3. As in the proof of Theorem 1.1, it suffices to show that if $x\in\ker E$ is P-harmonic, then $x=0$. Put $B=A^G$ and consider $\ker E$ as a right pre-Hilbert B-module with the inner product $\langle y,z\rangle=E(y^*z)$. Note that since the Haar state h on $C(G)$ is assumed to be faithful, the conditional expectation $E = (h \otimes \iota)\alpha\colon A \to B$ is faithful as well. We will show that $\psi(\langle x, x\rangle) = 0$ for any state ψ on B. Assume this is not the case for some ψ.

First of all note that by Schwarz's inequality for ucp maps we have

$$\langle P(y), P(y)\rangle = E(P(y)^*P(y)) \le EP(y^*y) = PE(y^*y) = P(\langle y, y\rangle) \tag{1.1}$$

for all $y \in \ker E$. In particular,

$$\psi(\langle x, x\rangle) \le \psi P(\langle x, x\rangle).$$

It follows that if we replace ψ by any weak* limit point of the states $n^{-1}\sum_{k=0}^{n-1}\psi P^k$ as $n \to \infty$, then this might only increase the value of ψ at $\langle x, x\rangle$. Therefore we may assume that ψ is P-invariant, or equivalently, P_2-invariant.

Consider now the Hilbert space H_ψ defined by the space $\ker E$ equipped with the pre-inner product $(y, z) = \psi(\langle z, y\rangle)$. Then H_ψ becomes a left unitary $C(G)$-comodule. In other words, there exists a unitary representation $U \in M(K(H_\psi) \otimes C(G))$ of G such that if $\Lambda_\psi\colon \ker E \to H_\psi$ denotes the canonical map, then

$$(\iota \otimes \Lambda_\psi)\alpha(y) = U^*_{21}(1 \otimes \Lambda_\psi(y))$$

for all $y \in \ker E$, and hence

$$\Lambda_\psi P_1(y) = (\iota \otimes \nu)(U^*)\Lambda_\psi(y).$$

The representation U has no nonzero invariant vectors, since there are no nonzero G-invariant vectors in $\ker E$. Decomposing U into a direct sum of finite dimensional representations, by Lemma 1.4 we conclude that

$$\|(\iota \otimes \nu)(U^*)\xi\| < \|\xi\|$$

for any nonzero vector $\xi \in H_\psi$. In particular, we have

$$\|\Lambda_\psi P_1(x)\| < \|\Lambda_\psi(x)\|. \tag{1.2}$$

On the other hand, by applying inequality (1.1) to $y = x$ and P_2 instead of P and using the P_2-invariance of ψ, we get

$$\|\Lambda_\psi P_2(x)\| \le \|\Lambda_\psi(x)\|.$$

But together with (1.2) this contradicts the equality $tP_1(x) + (1 - t)P_2(x) = x$. □

2. Probabilistic boundaries

Assume as in the previous section that H is a discrete quantum group and ϕ is a normal state on $\ell^\infty(H)$. Consider the space $H^\infty(H;\mu) \subset \ell^\infty(H)$ of bounded P_ϕ-harmonic elements. As was shown by Izumi [4, 6], it is a von Neumann algebra with the new product

$$x \cdot y = s^*\text{-}\lim_{n\to\infty} P^n_\phi(xy).$$

It is called (the algebra of bounded measurable functions on) the *Poisson boundary* of H. In this notation the first part of Theorem 1.1 states that if $G \subset H$ is a finite normal quantum subgroup, then for any generating normal state ϕ we have

$$H^\infty(H;\phi) = H^\infty(H/G;\bar\phi), \tag{2.1}$$

where $\bar\phi$ is the restriction of ϕ to $\ell^\infty(H/G) \subset \ell^\infty(H)$.

Recall next the definition of the Martin boundary [11]. For this we have to consider only normal states ϕ that are invariant under the left adjoint action of $\hat H$ on $\ell^\infty(H)$. In other words, if $\ell^\infty(H) = \ell^\infty\text{-}\bigoplus_{s\in I} B(H_s)$, then we consider the states of the form

$$\phi_\mu = \sum_{s\in I}\mu(s)\phi_s, \quad \phi_s = \frac{\mathrm{Tr}(\cdot\,\rho^{-1})}{\mathrm{Tr}(\rho^{-1})} \in B(H_s)^*,$$

where μ is a probability measure on I and ρ is the Woronowicz character f_1 for $\hat H$. These are precisely the states ϕ such that the operator P_ϕ leaves the center $\ell^\infty(I)$ of $\ell^\infty(H)$ globally invariant, so that it defines a classical random walk on I. To simplify the notation we will write P_μ for P_{ϕ_μ}.

Assume now that μ is generating, that is, ϕ_μ is generating. We also assume that the classical random walk on I is transient, or equivalently, the Green kernel

$$G_\mu\colon c_c(H)=\bigoplus_{s\in I}B(H_s)\to\ell^\infty(H),\quad G_\mu(x)=\sum_{n=0}^\infty P_\mu^n(x),$$

is well-defined. This is automatically the case if H is not of Kac type or, more generally, if the quantum dimension function is nonamenable. Denote by $I_0\in\ell^\infty(H)$ the unit in the matrix block corresponding to the counit, that is, I_0 is characterized by the property $xI_0=\varepsilon(x)I_0$ for $x\in\ell^\infty(H)$. Then the function $G_\mu(I_0)\in\ell^\infty(I)$ has no zeros, and the Martin kernel is defined as the completely positive map

$$K_\mu\colon c_c(H)\to\ell^\infty(H),\quad K_\mu(x)=G_\mu(I_0)^{-1}G_\mu(x).$$

The antipode defines an involution $s\mapsto\bar s$ on the set I. For a measure μ on I, we denote by $\check\mu$ the measure such that $\check\mu(s)=\mu(\bar s)$. If μ is transient, then $\check\mu$ is transient as well.

Consider the C*-subalgebra of $\ell^\infty(H)$ generated by $c_0(H)=c_0\text{-}\bigoplus_{s\in I}B(H_s)$ and $K_{\check\mu}(c_c(H))$. Its quotient by $c_0(H)$ is called the *Martin boundary* of H, and we denote it by $C(\partial H_{M,\mu})$.

Theorem 2.1. *Let H be a discrete quantum group, $\ell^\infty(H)=\ell^\infty\text{-}\bigoplus_{s\in I}B(H_s)$. Assume $G\subset H$ is a finite normal quantum subgroup. Consider the quotient quantum group H/G, $\ell^\infty(H/G)=\ell^\infty\text{-}\bigoplus_{t\in\bar I}B(H_t)$. Then for any transient generating finitely supported probability measure μ on I, the embedding $\ell^\infty(H/G)\hookrightarrow\ell^\infty(H)$ induces an isomorphism $C(\partial(H/G)_{M,\bar\mu})\cong C(\partial H_{M,\mu})$, where $\bar\mu$ is the measure on $\bar I$ characterized by $\phi_{\bar\mu}=\phi_\mu|_{\ell^\infty(H/G)}$.*

Proof. As in Section 1, consider the support p of the homomorphism $\pi\colon\ell^\infty(H)\to\ell^\infty(G)$ and the G-equivariant conditional expectation $E=(\iota\otimes h\pi)\Delta_H\colon\ell^\infty(H)\to\ell^\infty(H/G)$. We have $c_0(H)\cap\ell^\infty(H/G)=c_0(H/G)$, and E maps $c_c(H)$ onto $c_c(H/G)$. We claim that

$$K_{\check\mu}(x)-nK_{\check{\bar\mu}}(E(x))\in c_0(H)\ \text{ for any }\ x\in c_c(H),$$

where $n=\dim C(G)$. This obviously proves the theorem.

Note that $p\in\ell^\infty(H/G)$ is exactly the projection defining the counit, so

$$K_{\check{\bar\mu}}(x)=G_{\check\mu}(p)^{-1}G_{\check\mu}(x)=K_{\check\mu}(p)^{-1}K_{\check\mu}(x)\ \text{ for }\ x\in c_c(H/G)\subset c_c(H). \tag{2.2}$$

Note also that

$$E(I_0)=\frac{1}{n}p, \tag{2.3}$$

since $\pi(I_0)$ is the projection defining the counit on $C(G)$ and therefore we have $h(\pi(I_0))=1/n$, which can be seen by recalling that if we identify the C*-algebra $C(G)$ with a direct sum of full matrix algebras $B(H_i)$, then the Haar state is the sum of the traces $\frac{\dim H_i}{n}\operatorname{Tr}_{B(H_i)}$.

Next, let ψ be a right invariant Haar weight on $\ell^\infty(H)$. By [11, Theorem 3.3], for any state ω on $C(\partial H_{M,\mu})$ there exists a unique, possibly unbounded, positive P_μ-harmonic function x_ω on H such that

$$\psi(y\sigma^\psi_{-i/2}(x_\omega))=\omega(K_{\check\mu}(y))\ \text{ for all }\ y\in c_c(H),$$

where $\sigma^\psi_t=\operatorname{Ad}\rho^{-it}$ is the modular group of ψ. By Theorem 1.1 we have $E(x_\omega)=x_\omega$. Note also that E commutes with σ^ψ_t. It follows that if $y\in c_c(H)\cap\ker E$, then

$$\omega(K_{\check\mu}(y))=\psi(y\sigma^\psi_{-i/2}(x_\omega))=(\psi*h\pi)(y\sigma^\psi_{-i/2}(x_\omega))=\psi(E(y\sigma^\psi_{-i/2}(x_\omega)))=0.$$

Since this is true for any ω, we conclude that

$$K_{\check\mu}(y)\in c_0(H)\ \text{ for all }\ y\in c_c(H)\cap\ker E. \tag{2.4}$$

Now, (2.3) and (2.4) show that

$$n1-K_{\check\mu}(p)=K_{\check\mu}(nI_0-p)\in c_0(H).$$

Using (2.2) and again (2.4), for any $x\in c_c(H)$ we then get the following equalities modulo $c_0(H)$:

$$K_{\check\mu}(x)=K_{\check\mu}(E(x))=K_{\check\mu}(p)K_{\check{\bar\mu}}(E(x))=nK_{\check{\bar\mu}}(E(x)),$$

which proves our claim. ∎

Let us give a simple class of noncommutative examples where the above results can be applied.

Example 2.2. Let Γ be a discrete group and S be a finite group acting on Γ by group automorphisms $\gamma\mapsto s.\gamma$. We can then define a discrete quantum group H with the algebra of bounded measurable functions $\ell^\infty(H)=\ell^\infty(\Gamma)\rtimes S$ and the coproduct extending the usual coproducts on $\ell^\infty(\Gamma)$ and $C^*(S)$. This is a quantum group unless S is an abelian group acting trivially on Γ. The dual $G=\hat{S}$ is a normal quantum subgroup of H, with the structure homomorphism $\pi\colon \ell^\infty(H)\to\ell^\infty(G)=C^*(S)$ given by $f\lambda_s\mapsto f(e)\lambda_s$, and we have $H/G=\Gamma$. We thus see that, under suitable assumptions, the Poisson and Martin boundaries of H coincide with the corresponding classical boundaries of Γ.

Note that the $\hat{H}$-invariant normal states ϕ_μ are exactly the normal tracial states on $\ell^\infty(\Gamma)\rtimes S$. It is not difficult to show, see [10] for a more general statement, that such traces are given by S-invariant probability measures $\bar\mu$ on Γ and tracial states τ_γ on $C^*(S_\gamma)$, where $S_\gamma\subset S$ is the stabilizer of $\gamma\in\Gamma$, such that $\tau_{s.\gamma}(\lambda_s\cdot\lambda_s^*)=\tau_\gamma$. Namely, the trace ϕ_μ corresponding to a pair $(\bar\mu,(\tau_\gamma)_{\gamma\in\Gamma})$ is given by

$$\phi_\mu(f\lambda_s)=\sum_{\gamma\colon\ s\in S_\gamma}\bar\mu(\gamma)f(\gamma)\tau_\gamma(\lambda_s).$$

It can be checked that the trace ϕ_μ is faithful if and only if $\operatorname{supp}\bar\mu=\Gamma$ and every trace τ_γ is faithful. It follows then that, more generally, the trace ϕ_μ is generating if the set of elements $\gamma\in\operatorname{supp}\bar\mu$ such that τ_γ is faithful generates Γ as a semigroup. It is clear also that ϕ_μ is transient if and only if $\bar\mu$ is transient. This allows one to construct many examples where the assumptions of Theorem 2.1 are satisfied.

This class of discrete quantum groups H includes the duals of the group-theoretical easy quantum groups [15]. These duals are obtained by taking Γ to be a quotient of $(\mathbb{Z}/2\mathbb{Z})^{*n}$ and S to be the symmetric group S_n acting on Γ by permuting the generators.

3. Categorical analogue

In this section we will prove an analogue of Theorem 1.1 for C*-tensor categories. Our conventions are the same as in [12]. Briefly, we assume that the categories that we consider are small, closed under subobjects and finite direct sums, and the tensor units are simple unless explicitly stated otherwise. We also assume that the categories are strict. We denote the morphisms sets in a category $\mathcal{C}$ by $\mathcal{C}(U,V)$ and write $\mathcal{C}(U)$ for $\mathcal{C}(U,U)$.

Recall that for an object U in a C*-tensor category $\mathcal{C}$, the conjugate, or dual, object is an object $\bar U$ such that there exist morphisms $R\colon \mathbb{1}\to\bar U\otimes U$ and $\bar R\colon\mathbb{1}\to U\otimes\bar U$ solving the conjugate equations

$$(R^*\otimes\iota)(\iota\otimes\bar R)=\iota,\quad (\bar R^*\otimes\iota)(\iota\otimes R)=\iota.$$

The minimum of the numbers $\|R\|\,\|\bar R\|$ over all solutions is called the dimension of U. We denote the dimension by $d(U)$. A solution $(R,\bar R)$ is called standard if $\|R\|=\|\bar R\|=d(U)^{1/2}$. A category in which every object has a dual object is called rigid.

Fixing a standard solution $(R_U,\bar R_U)$ of the conjugate equations for every object U we can define maps

$$\mathcal{C}(U\otimes V,U\otimes W)\to\mathcal{C}(V,W),\quad T\mapsto(R_U^*\otimes\iota)(\iota\otimes T)(R_U\otimes\iota),$$

which are denoted by $\operatorname{Tr}_U\otimes\iota$ and called partial categorical traces. They are independent of the choice of standard solutions.

Recall next the notion of a harmonic natural transformation [14]. Fix objects U and V and consider the space $\operatorname{Nat}_b(\iota\otimes U,\iota\otimes V)$ of bounded natural transformation between the functors $\iota\otimes U$ and $\iota\otimes V$, that is, a uniformly bounded collection $\eta=(\eta_X)_X$ of natural in X morphisms $X\otimes U\to X\otimes V$. For every object W we then define an operator P_W on $\operatorname{Nat}_b(\iota\otimes U,\iota\otimes V)$ by

$$P_W(\eta)_X=d(W)^{-1}(\operatorname{Tr}_W\otimes\iota)(\eta_{W\otimes X}).$$

Consider the set $\operatorname{Irr}(\mathcal{C})$ of isomorphism classes of simple objects in $\mathcal{C}$, and choose for every $s\in\operatorname{Irr}(\mathcal{C})$ a representative U_s. For a probability measure μ on $\operatorname{Irr}(\mathcal{C})$ we put

$$P_\mu=\sum_s\mu(s)P_{U_s}.$$

A natural transformation $\eta \in \mathrm{Nat}_b(\iota\otimes U, \iota\otimes V)$ is called P_μ*-harmonic* if $P_\mu(\eta)=\eta$. If $U=V$ then it makes sense to also talk about unbounded positive P_μ-harmonic natural transformations.

Define convolution of measures on $\mathrm{Irr}(\mathcal{C})$ by

$$(\nu * \mu)(t) = \sum_{s,r} \nu(s)\mu(r) m^t_{sr} \frac{d(U_t)}{d(U_s)d(U_r)},$$

where m^t_{sr} is the multiplicity of U_t in $U_s\otimes U_r$. Then $P_\mu P_\nu = P_{\nu*\mu}$. We write μ^n for the nth convolution power of μ.

We next recall a few notions from [2], with obvious modifications needed in our C*-setting. Let $F\colon \mathcal{C}\to\mathcal{C}''$ be a unitary tensor functor between C*-tensor categories. The functor F is called *normal*, if for every object $U\in\mathcal{C}$ there exists a subobject U_0 such that $F(U_0)$ is the largest subobject of $F(U)$ which is trivial, that is, isomorphic to $\mathbb{1}^n$ for some n. We then denote by $\mathfrak{Ker}_F\subset\mathcal{C}$ the full subcategory consisting of objects U such that $F(U)$ is trivial. A sequence

$$\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

of unitary tensor functors is called *exact*, if

(a) F is dominant, that is, every object of $\mathcal{C}''$ is a subobject of $F(U)$ for some $U\in\mathcal{C}$;
(b) F is normal;
(c) i defines an equivalence between $\mathcal{C}'$ and $\mathfrak{Ker}_F$.

Given a C*-tensor category $\mathcal{C}$ and a full C*-tensor subcategory $\mathcal{C}'\subset\mathcal{C}$, let us say that $\mathcal{C}'$ is *normal* if there exists an exact sequence $\mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{C}''$, where i is the embedding functor. Let us also say that a natural transformation η between the functors $\iota\otimes U$ and $\iota\otimes V$ on $\mathcal{C}$ is $\mathcal{C}'$*-invariant*, if

$$\eta_{X\otimes Y} = \iota_X\otimes\eta_Y \ \text{ for all } \ X\in\mathcal{C}' \ \text{ and } \ Y\in\mathcal{C}.$$

We are now ready to formulate an analogue of Theorem 1.1.

Theorem 3.1. *Let $\mathcal{C}$ be a rigid C*-tensor category, $\mathcal{C}'\subset\mathcal{C}$ be a finite normal C*-tensor subcategory, and μ be a generating probability measure on $\mathrm{Irr}(\mathcal{C})$, meaning that $\cup_{n\ge1}\operatorname{supp}\mu^n = \mathrm{Irr}(\mathcal{C})$. Then any positive, possibly unbounded, P_μ-harmonic natural transformation $\eta\colon \iota\otimes U\to\iota\otimes U$ is $\mathcal{C}'$-invariant.*

Note that by [13, Theorem 4.1] this theorem generalizes Theorem 1.1 for $\hat{H}$-invariant ϕ, but not for arbitrary states, so formally these two results are independent. Not surprisingly, the proofs are nevertheless similar. But before we turn to the proof we need to formulate $\mathcal{C}'$-invariance in a more analytic way.

Define a probability measure h' on $\mathrm{Irr}(\mathcal{C}')$ by

$$h'(s) = \frac{d(U_s)^2}{d(\mathcal{C}')}, \ \text{ where } \ d(\mathcal{C}') = \sum_{t\in\mathrm{Irr}(\mathcal{C}')} d(U_t)^2.$$

It is known, and is easy to see using multiplicativity and additivity of the dimension function, that for any probability measure ν on $\mathrm{Irr}(\mathcal{C}')$ we have

$$\nu * h' = h' * \nu = h'. \tag{3.1}$$

Since we can identify $\mathrm{Irr}(\mathcal{C}')$ with a subset of $\mathrm{Irr}(\mathcal{C})$, we can also view h' as a measure on $\mathrm{Irr}(\mathcal{C})$.

Lemma 3.2. *For a full finite rigid C*-tensor subcategory $\mathcal{C}'$ of a rigid C*-tensor category $\mathcal{C}$, a natural transformation $\eta\colon\iota\otimes U\to\iota\otimes U$ is $\mathcal{C}'$-invariant if and only if $P_{h'}(\eta)=\eta$.*

Proof. If η is $\mathcal{C}'$-invariant, then obviously $P_\nu(\eta)=\eta$ for any probability measure ν on $\mathrm{Irr}(\mathcal{C}')$, in particular, for h'. Conversely, assume $P_{h'}(\eta)=\eta$. It suffices to show that $\eta_X=\iota_X\otimes\eta_{\mathbb{1}}$ for all $X\in\mathcal{C}'$, since by applying this statement to the natural transformation $(\eta_{Z\otimes Y})_{Z\in\mathcal{C}}\colon \iota\otimes Y\otimes U\to\iota\otimes Y\otimes U$ (which was denoted by $\iota_Y\otimes\eta$ in [14]) we then get $\eta_{X\otimes Y}=\iota_X\otimes\eta_Y$ for all $X\in\mathcal{C}'$, as required. For this, in turn, consider $\iota\otimes U$ as a functor $\mathcal{C}'\to\mathcal{C}$. Then $(\eta_X)_{X\in\mathcal{C}'}$ is an endomorphism of this functor, while $P_{h'}$ can be considered as an operator on the space of such endomorphisms. By [14, Proposition 2.4], when $\mathcal{C}=\mathcal{C}'$, the subspace of P_ν-invariant endomorphisms consists of the elements $(\iota_X\otimes T)_{X\in\mathcal{C}'}$, with $T\in\mathcal{C}(U)$, for any generating probability measure ν on $\mathrm{Irr}(\mathcal{C}')$. The same proof works in general, so $\eta_X=\iota_X\otimes\eta_{\mathbb{1}}$ for all $X\in\mathcal{C}'$. ☐

Assume now that we are in the setting of Theorem 3.1 and consider the corresponding exact sequence $\mathcal{C}' \to \mathcal{C} \xrightarrow{F} \mathcal{C}''$. Since $F(X)$ is trivial for every $X \in \mathcal{C}'$, $F|_{\mathcal{C}'}$ can be considered as a unitary fiber functor. This already implies that the dimension function on $\mathcal{C}'$ is integral and that $\mathcal{C}'$ can be identified with $\operatorname{Rep} G$ for a finite quantum group G. Consider the object

$$A = \bigoplus_{s \in \operatorname{Irr}(\mathcal{C}')} U_s^{d(U_s)} \in \mathcal{C}'.$$

It is shown in [2, Section 5.2] that A admits the structure of a commutative central algebra in $\mathcal{C}$. In particular, $A \otimes Y \cong Y \otimes A$ for all $Y \in \mathcal{C}$, which implies that

$$\nu * h' = h' * \nu \tag{3.2}$$

for any probability measure ν on $\operatorname{Irr}(\mathcal{C})$.

Proof of Theorem 3.1. The proof goes along the same lines as that of Theorem 1.1. We will only consider the case of bounded natural transformations, the general case is dealt with similarly to the second part of that proof.

We may assume that $\operatorname{supp} \mu = \operatorname{Irr}(\mathcal{C})$. We can then write μ as a convex combination $t\mu_1 + (1-t)\mu_2$ of two measures, with $0 < t < 1$, μ_1 supported on $\operatorname{Irr}(\mathcal{C}')$ and μ_2 on $\operatorname{Irr}(\mathcal{C}) \setminus \operatorname{Irr}(\mathcal{C}')$. By (3.1) and Lemma 3.2, the operator $E = P_{h'}$ defines a projection onto the space of $\mathcal{C}'$-invariant natural transformations. By (3.2) this projection commutes with P_{μ_1} and P_{μ_2}. Therefore it suffices to show that the restriction of P_{μ_1} to $\ker E$ is a strict contraction. Since $\operatorname{supp} \mu_1 = \operatorname{Irr}(\mathcal{C}')$, there exists $\delta > 0$ such that $\mu_1 - \delta h'$ is a positive measure. Since $P_{\mu_1} = P_{\mu_1 - \delta h'}$ on $\ker E$, it follows then that the norm of the restriction of P_{μ_1} to $\ker E$ is bounded by $(\mu_1 - \delta h')(\operatorname{Irr}(\mathcal{C})) = 1 - \delta$. ☐

Remark 3.3. Theorem 1.1 can be formulated by saying that, under its assumptions, any positive harmonic function on H arises from that on H/G. In a similar way Theorem 3.1 implies that any positive harmonic natural transformation $\iota \otimes U \to \iota \otimes U$ of functors on $\mathcal{C}$ arises from a natural transformation $\iota \otimes F(U) \to \iota \otimes F(U)$ of functors on $\mathcal{C}''$. In other words, we claim that if an endomorphism $\eta = (\eta_X)_{X \in \mathcal{C}}$ of $\iota \otimes U$ is $\mathcal{C}'$-invariant, then the collection of morphisms

$$F(X) \otimes F(U) \xrightarrow{F_2} F(X \otimes U) \xrightarrow{F(\eta_X)} F(X \otimes U) \xrightarrow{F_2^{-1}} F(X) \otimes F(U)$$

defines, necessarily uniquely, an endomorphism of $\iota \otimes F(U)$. Indeed, by [2, Corollary 5.8], the category $\mathcal{C}''$ can be identified with the category $A\text{-}\operatorname{mod}_{\mathcal{C}}$ of left A-modules in $\mathcal{C}$ and then the functor F is given by $F(X) = A \otimes X$. Note that any left A-module can also be considered as an A-bimodule by the commutativity of the central algebra A, and this turns $A\text{-}\operatorname{mod}_{\mathcal{C}}$ into a tensor category with the tensor product $\otimes_A$. It follows that our claim is equivalent to the statement that for any $\mathcal{C}'$-invariant η the morphisms $\iota_A \otimes \eta_X \colon A \otimes X \otimes U \to A \otimes X \otimes U$ are natural with respect to the A-module morphisms $A \otimes X \to A \otimes Y$. But this is clear, as $\iota_A \otimes \eta_X = \eta_{A \otimes X}$.

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