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Ultrafilter convergence in stochastic analysis and mathematical finance

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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Abstract

Given a non-principal ultrafilter, we define and prove properties of ultralimits of measure spaces (including σ-algebras, filtrations and measures), random variables and discrete-time stochastic processes. Among other things, considering Brownian motion as the ultralimit of random walks, we define the stochastic integral as the ultralimit of sums involving the random walk and we show that solutions to stochastic differential equations can be written as the ultralimit of solutions to difference equations. We also show that the ultralimit of the Cox-Ross-Rubinstein model is the Black Scholes model.

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CHAPTER 1

Introduction

1.1 The purpose of this master's thesis

When it comes to convergence of stochastic variables and stochastic processes, a great deal of work has been done on weak convergence. A sequence of stochastic variables $\{X_n\}_{n\in\mathbb{N}}$, where each X_n is defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, is said to converge weakly (or converge in distribution) to a stochastic variable X defined on a probability space $(\Omega, \mathcal{F}, P)$ if $E_n[f(X_n)] \to E[f(X)]$ for all bounded continuous functions f. A disadvantage of weak convergence is that a given sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ may not converge weakly at all. Furthermore, even though we have weak convergence of random variables, we do not have a concept of weak convergence of measure spaces, only weak convergence of measures (and the measures may not weakly converge even if $\{X_n\}_{n\in\mathbb{N}}$ converges with respect to these measures).

In this thesis we will focus on what is called ultrafilter convergence. This allows us to define, given a sequence of probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$, a limiting probability space $(\Omega, \mathcal{F}, P)$, which is such that for a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$, with each X_n defined on $(\Omega_n, \mathcal{F}_n, P_n)$, given some weak assumptions, we have $E_n[X_n] \to E[X]$ with respect to the ultrafilter, where X is the ultralimit of $\{X_n\}_{n\in\mathbb{N}}$. This gives us the framework to define certain continuous-time stochastic processes living on the measure space $(\Omega, \{\mathcal{F}_t\}_{t\in I}, \mathcal{F}, P)$ as limits of discrete time processes living on measures spaces $(\Omega_n, \{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}, \mathcal{F}_n, P_n)$, where $(\Omega, \{\mathcal{F}_t\}_{t\in I}, \mathcal{F}, P)$ is the ultralimit of the spaces $(\Omega_n, \{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}, \mathcal{F}_n, P_n)$. We can then define the stochastic integral (with respect to Brownian motion) as a limit of sums of discrete-time stochastic processes and solutions to stochastic differential equations as limits of solutions to difference equations.

The notion of ultrafilter convergence of measure spaces, which relies on The Axiom of Choice, is not new and has been used in non-standard analysis (which is reliant on ultrafilter convergence), in particular in the treatment of Loeb-measures. Further work has been done within the field of non-standard stochastic analysis including (but not limited to) stochastic integrals and stochastic differential equations. Nonstandard analysis deals with what is called the extended real number line. In layman's terms this means that we define a number (or equivalently, a point) for each sequence of real numbers and let these "sequences" constitute the extended real number line, making room for infinitesimals ("infinitely small" numbers) that is used to derive analytic results.

What is new in this thesis is that we will use the notion of ultrafilter convergence within the standard universe only. That is, we will not make use of the extended real number line. We have tried to make this thesis as self-contained as possible. Only a prior knowledge of introductionary standard analysis and some basic knowledge of stochastic analysis is assumed.

1.2 An overview

The thesis is divided into six chapters, the first chapter being this introduction. In chapter two we first give the definitions of a non-principal ultrafilter and of ultrafilter convergence. We then present some basic properties of ultrafilter convergence before we define the ultralimit of measure spaces and show some properties of this measure space. We define the ultralimit of a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables and show that under some weak assumptions we have $E_n[X_n] \to E[X]$ and $E_n[X_n \mid \mathcal{G}_n] \to E[X \mid \mathcal{G}]$ with respect to the ultrafilter (where $\mathcal{G} \subseteq \mathcal{F}$ is the "ultralimit" of the $\mathcal{G}_n$'s, where $\mathcal{G}_n \subseteq \mathcal{F}_n$).

In chapter three we extend the notion of ultrafilter convergence of random variables to ultrafilter convergence of discrete-time stochastic processes. We call these discrete processes, for which there exists a continuous-time ultralimit, skeleton processes. In section one, we show that given our definition of ultrafilter convergence of skeleton processes, such an ultralimit must necessarily be continuous. We next define an ultralimit filtration and derive some basic properties of ultralimits of martingale (or sub-/supermartingale) skeleton processes before we show that we can define Brownian motion as the ultralimit of random walks. In section two we define a stochastic integral of an ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ (with respect to Brownian motion) as the ultralimit of discrete time stochastic integrals involving random walks given that $\{X_n\}_{n\in\mathbb{N}}$ satisfies some weak assumptions. Both the construction of Brownian motion and the construction of the stochastic integral is inspired by [And76].

Chapter four, which is focused on stochastic differential equations, is divided into two sections. In section one we use what we derived in chaper three to give conditions for which a stochastic differential equation has a strong solution in our given measure space. This section is inspired by [Kei84]. In section two we show that there exists weak solutions to some given stochastic differential equations.

In chapter five we give an overview of the terminology of mathematical finance and give a short description of mathematical modelling of a financial market in discrete time and in continuous time before we show that the Cox-Ross-Rubinstein models converges with respect to an ultrafilter to a Black Scholes model and prove the Black Scholes fair price of a European call option.

The last chapter, chapter six, is devoted to discussion. We summarize our work and express some afterthoughts.

1.3 My work

My work on this thesis has been as follows: Some times my advisor would give me problems to solve. These would be either specific or in the form of more loosely formulated questions. Other times, especially during the last semester, I would find problems on my own to solve. For some of the problems I was given by my advisor in the beginning, there already existed a solution in some book (although I never read these solutions myself), but mostly the problems had not been solved before. As already mentioned, some of the problems I was given to solve by my advisor were inspired by work in non-standard analysis, much of which I was not aware of before the end of writing this thesis. Also, for a few of the results that I chose to prove myself in my thesis there already exists a proof (although the proof is different since it's proved in a different setting).

In order to come up with problems and give proofs I had to study some (new to me) theory. At times my advisor would provide me with reading material, other times I would find reading materials on my own. Although most of this thesis is my own work, chapter two section one contains results and proofs that are well-known while a considerate portion of chapter two section two is based on [War12] (this bachelor thesis is a work in non-standard analysis and contained multiple errors). Although some of the results in chapter two section two is not completely my own work, I spent a considerable amount of time adapting the results to a standard universe, as well as correcting mistakes and filling out details. It is unfortunate that this thesis ([War12]) is no longer available online, but I have a copy should anyone be interested in seeing it.

For a full list of my work in this thesis, see the appendix.

CHAPTER 2

Preliminaries

2.1 Ultrafilter convergence of real numbers

Definition 2.1.1. An ultrafilter (on $\mathbb{N}$) is a family $\mathcal{U}$ of subsets of $\mathbb{N}$ such that the following holds true:

(i) If $F \in \mathcal{U}$ and $G \supseteq F$, then $G \in \mathcal{U}$.

(ii) If $F, G \in \mathcal{U}$, then $F \cap G \in \mathcal{U}$.

(iii) If $F \subseteq \mathbb{N}$, then precisely one of the subsets F, F^c is in $\mathcal{U}$.

We get a simple example of an ultrafilter by picking an $n \in \mathbb{N}$ and letting $\mathcal{U}_n = \{F \subseteq \mathbb{N} \mid n \in F\}$. This is called a *principal* ultrafilter. All other ultrafilters are called *non-principal*. It follows by Zorn's lemma that there exist non-principal ultrafilters on $\mathbb{N}$ (see [Alb+09]). We will in this thesis assume that $\mathcal{U}$ is a non-principal ultrafilter.

Proposition 2.1.2. *Assume that $\mathcal{U}$ is an ultrafilter.*

(i) $\mathbb{N} \in \mathcal{U}$, $\emptyset \notin \mathcal{U}$.

(ii) If $F \notin \mathcal{U}$ and $G \subseteq F$, then $G \notin \mathcal{U}$

(iii) If $F \notin \mathcal{U}, G \notin \mathcal{U}$, then $F \cup G \notin \mathcal{U}$

(iv) If F and G are disjoint, and $F \cup G \in \mathcal{U}$, then precisely one of the sets F, G is in $\mathcal{U}$.

(v) If the sets $F_1, F_2, ..., F_n$ are disjoint, and $F_1 \cup F_2 \cup ... \cup F_n \in \mathcal{U}$, then precisely one of the sets $F_1, F_2, ..., F_n$ is in $\mathcal{U}$

Proof. The proofs are direct consequences of Definition 2.1.1. The proof of (iii) follows from De Morgan's law, while the proof of (v) is just an induction argument that applies (iv). ∎

Lemma 2.1.3. *If $\mathcal{U}$ is a non-principal ultrafilter, and $I \subseteq \mathbb{N}$ is finite, then $I \notin \mathcal{U}$.*

Proof. For each $n \in I$, $\{n\} \notin \mathcal{U}$ and so $I = \cup_{n \in I}\{n\} \notin \mathcal{U}$ by Proposition 2.1.2 (iii). ∎

Definition 2.1.4. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{U}$-converges to $a \in \mathbb{R}$ if for all $\epsilon > 0$, the sets

$$F_\epsilon = \{n \in \mathbb{N} \mid |x_n - a| < \epsilon\}$$

are in $\mathcal{U}$. In that case we write $\lim_{\mathcal{U}} x_n = a$.

Proposition 2.1.5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ cannot $\mathcal{U}$-converge to more than one point $a \in \mathbb{R}$.*

Proof. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ $\mathcal{U}$-converges to $a_1, a_2 \in \mathbb{R}$ and that $a_1 \neq a_2$. Let $\epsilon = \frac{|a_1-a_2|}{2}$ and for $i = 1, 2$, let

$$F_\epsilon^{a_i} = \{n \in \mathbb{N} \mid |x_n - a_i| < \epsilon\}.$$

By Definition 2.1.1 (i), $F_\epsilon^{a_1} \cup F_\epsilon^{a_1} \in \mathcal{U}$, but by Proposition 2.1.2 (iv), since the sets $F_\epsilon^{a_1}, F_\epsilon^{a_2}$ are disjoint, only one of the sets $F_\epsilon^{a_1}, F_\epsilon^{a_2}$ is in $\mathcal{U}$, a contradiction by Definition 2.1.4. ∎

Definition 2.1.6. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called $\mathcal{U}$-bounded if there exists an $M \in \mathbb{R}$ such that

$$\{n \in \mathbb{N} \mid |x_n| \leq M\} \in \mathcal{U}.$$

Theorem 2.1.7. *Assume that $\{x_n\}_{n\in\mathbb{N}}$ is a $\mathcal{U}$-bounded sequence. For each $x \in \mathbb{R}$, we set*

$$G_x = \{n \in \mathbb{N} \mid x_n \leq x\}.$$

Then there is an $a \in \mathbb{R}$ such that $G_x \notin \mathcal{U}$ when $x < a$ and $G_x \in \mathcal{U}$ when $x > a$. Furthermore, $a = \lim_{\mathcal{U}} x_n$.

Proof. Let $I = \{x \in \mathbb{R} \mid G_x \in \mathcal{U}\}$. Since $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-bounded, there is an $M \in \mathbb{R}$ such that $\{n \in \mathbb{N} \mid |x_n| \leq M\} \in \mathcal{U}$, so $M \in I$. Since $M \in I$, I is non-empty. Since $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-bounded, $\inf I > -\infty$. Indeed, if $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-bounded by M, $\{n \in \mathbb{N} \mid |x_n| \leq M\} \cap G_x = \emptyset$ for all $x < -M$. Let $a = \inf I \in [-M, M]$ and notice that $G_x \in \mathcal{U}$ for $x > a$ (since there by definition of I is $m \in I$ such that $a \leq m < x$ so that $G_x \supset G_m$, and thus $x \in I$ for $x > a$) and $G_x \notin \mathcal{U}$ for $x < a$ (by definition of I). Now let $\epsilon > 0$. Then

$$\begin{aligned} F_\epsilon &= \{n \in \mathbb{N} \mid |x_n - a| < \epsilon\} \\ &= \{n \in \mathbb{N} \mid x_n < a + \epsilon\} \cap \{n \in \mathbb{N} \mid x_n > \ a - \epsilon\} \\ &\supseteq G_{a+\frac{\epsilon}{2}} \cap (G_{a-\epsilon})^c \in \mathcal{U}, \end{aligned}$$

hence $F_\epsilon \in \mathcal{U}$. So $\lim_{\mathcal{U}} x_n = a$. ∎

Definition 2.1.8. We set $\lim_{\mathcal{U}} x_n = \infty$ if there is no $x \in \mathbb{R}$ such that $G_x \in \mathcal{U}$, and we set $\lim_{\mathcal{U}} x_n = -\infty$ if $G_x \in \mathcal{U}$ for all $x \in \mathbb{R}$.

Corollary 2.1.9. *For any sequence $\{x_n\}_{n\in\mathbb{N}}$ there is an $a \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $\lim_{\mathcal{U}} x_n = a$.*

Proposition 2.1.10. *Suppose that $\lim_{\mathcal{U}} x_n = a$ and that*

$$\{n \in \mathbb{N} \mid x_n = y_n\} \in \mathcal{U}.$$

Then $\lim_{\mathcal{U}} y_n = a$.

Proof. Let $\epsilon > 0$. Then

$$\begin{aligned} F_\epsilon^y &= \{n \in \mathbb{N} \mid |y_n - a| < \epsilon\} \\ &\supseteq F_\epsilon^y \cap \{n \in \mathbb{N} \mid x_n = y_n\} \\ &= F_\epsilon^x \cap \{n \in \mathbb{N} \mid x_n = y_n\} \in \mathcal{U}. \end{aligned}$$

■

Proposition 2.1.11. *Suppose that $\{x_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} x_n = x \in \overline{\mathbb{R}}$. Then $\lim_{\mathcal{U}} x_n = x$.*

Proof. Suppose that $x \in \mathbb{R}$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, there is an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. By Lemma 2.1.3, any finite subset of $\mathbb{N}$ is not in $\mathcal{U}$, hence the complement of a finite subset of $\mathbb{N}$ is in $\mathcal{U}$. So $\{n \in \mathbb{N} \mid |x_n - x| < \epsilon\} \supseteq \{1, .., N-1\}^c \in \mathcal{U}$. Suppose that $x = \infty$. Then for each $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$. Thus $G_M = \{n \in \mathbb{N} \mid x_n \leq M\} \notin \mathcal{U}$ for each $M \in \mathbb{R}$, hence $\lim_{\mathcal{U}} x_n = x$. Similarly, if $x = -\infty$, $G_M \in \mathcal{U}$ for each $M \in \mathbb{R}$, hence $\lim_{\mathcal{U}} = x$. ■

The converse to Proposition 2.1.11 need not be true. Consider the sequence $1, -1, 1, -1, ...$ in $\mathbb{R}$. This sequence does not converge in the ordinary sense, but has a $\mathcal{U}$-limit, which is either 1 or -1 (depending on the ultrafilter $\mathcal{U}$).

Proposition 2.1.12. *Suppose that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences of real numbers such that $\lim_{\mathcal{U}} x_n = x \in$ and $\lim_{\mathcal{U}} y_n = y$. Suppose that either*

$$\begin{aligned} &x, y \in \mathbb{R} \\ &x \in \mathbb{R}, y \in \{-\infty, \infty\} \\ &x, y = \infty \\ &x, y = -\infty. \end{aligned}$$

Then

$$\lim_{\mathcal{U}} (x_n + y_n) = x + y.$$

Proof. Suppose $x, y \in \mathbb{R}$. Let $\epsilon > 0$. Then

$$\begin{aligned} F_\epsilon^{x+y} &= \{n \in \mathbb{N} \mid |x_n + y_n - (x+y)| < \epsilon\} \\ &\supseteq \{n \in \mathbb{N} \mid |x_n - x| < \frac{\epsilon}{2}\} \cap \{n \in \mathbb{N} \mid |y_n - y| < \frac{\epsilon}{2}\} \\ &= F_{\frac{\epsilon}{2}}^x \cap F_{\frac{\epsilon}{2}}^y \in \mathcal{U}. \end{aligned}$$

Suppose that $x \in \mathbb{R}$ and that $y = \infty$. Then

$$\begin{aligned} G_N^c &= \{n \in \mathbb{N} \mid x_n + y_n > N\} \\ &\supseteq \{n \in \mathbb{N} \mid y_n > N + 1 - x\} \cap \{n \in \mathbb{N} \mid |x_n - x| < 1\} \\ &= (G_{N+1-x}^y)^c \cap F_1^x \in \mathcal{U}, \end{aligned}$$

hence $\lim_{\mathcal{U}}(x_n + y_n) = x + y = \infty$. The proofs of the other cases are similar. ■

Proposition 2.1.13. *Suppose that* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *are sequences of real numbers such that* $\lim_{\mathcal{U}} x_n = x$ *and* $\lim_{\mathcal{U}} y_n = y$. *Suppose that either* $x, y \in \mathbb{R}$ *or* $x \in \mathbb{R}\setminus\{0\}$ *and* $y \in \{-\infty, \infty\}$ *or* $x, y = \infty$ *or* $x, y = -\infty$. *Then*

$$\lim_{\mathcal{U}} x_n y_n = xy.$$

Proof. Suppose that $x, y \in \mathbb{R}$. Let $\epsilon > 0$. Let

$$F^x_{\frac{1}{2(1+|y|)}} = \{n \in \mathbb{N} \mid |x_n - x| < \frac{\epsilon}{2(1+|x|)}\},$$

$$F^y_{\frac{1}{2(1+|x|)}} = \{n \in \mathbb{N} \mid |y_n - y| < \frac{\epsilon}{2(1+|y|)}\}$$

and

$$F^x_1 = \{n \in \mathbb{N} \mid |x_n - x| < 1\}.$$

Then, since

$$|x_n y_n - xy| \le |x_n y_n - x_n y| + |x_n y - xy| = |x_n||y_n - y| + |y||x_n - x|$$

we have that for $n \in F^x_{\frac{1}{2(1+|y|)}} \cap F^y_{\frac{1}{2(1+|x|)}} \cap F^x_1$,

$$|x_n y_n - xy| \le (1+|x|)|y_n - y| + (1+|y|)|x_n - x| < \epsilon.$$

Thus

$$\begin{aligned} F^{xy}_\epsilon &= \{n \in \mathbb{N} \mid |x_n y_n - xy| < \epsilon\} \\ &\supseteq F^x_{\frac{1}{2(1+|y|)}} \cap F^y_{\frac{1}{2(1+|x|)}} \cap F^x_1 \in \mathcal{U}. \end{aligned}$$

Suppose that $x \in \mathbb{R}, x > 0$ and $y = \infty$. Then

$$x_n y_n = (x_n - x) y_n + x y_n,$$

so that

$$\begin{aligned} G^c_N &= \{n \in \mathbb{N} \mid x_n y_n > N\} \\ &\supseteq \{n \in \mathbb{N} \mid |x_n - x| < \frac{x}{2}\} \cap \{n \in \mathbb{N} \mid \ y_n \ > \frac{2N}{x}\} \\ &= F^x_{\frac{x}{2}} \cap G_{\frac{2N}{x}} \in \mathcal{U}. \end{aligned}$$

The proofs of the other cases are similar. ■

Proposition 2.1.14. *Suppose that* $f : \mathbb{R} \to \overline{\mathbb{R}}$ *is continuous and that we have a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of real numbers such that* $\lim_{\mathcal{U}} x_n = x \in \mathbb{R}$. *Then* $\lim_{\mathcal{U}} f(x_n) = f(x)$.

Proof. Suppose $f(x) \in \mathbb{R}$. Let $\epsilon > 0$. Since f is continuous in x, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Then

$$\begin{aligned} F^{f(x)}_\epsilon &= \{n \in \mathbb{N} \mid |f(x) - f(x_n)| < \epsilon\} \\ &\supseteq \{n \in \mathbb{N} \mid |x_n - x| < \delta\} \\ &= F^x_\delta \in \mathcal{U}. \end{aligned}$$

Suppose $f(x) = \infty$ and let $N \in \mathbb{N}$. Then there is $\delta > 0$ such that $f(y) > N$ whenever $|x - y| < \delta$. Then

$$\begin{aligned} G_N^c &= \{n \in \mathbb{N} \mid f(x_n) > N\} \\ &\supseteq \{n \in \mathbb{N} \mid |x_n - x| < \delta\} \\ &= F_\delta^x \in \mathcal{U}. \end{aligned}$$

The proof of the case $f(x) = -\infty$ is similar.

■

Proposition 2.1.15. *Suppose that* $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is continuous and that we have a sequence* $\{x_n\}_{n \in \mathbb{N}}$ *of real numbers such that* $\lim_{\mathcal{U}} x_n = x \in \overline{\mathbb{R}}$. *Then* $\lim_{\mathcal{U}} f(x_n) = f(x)$.

Proof. Suppose $x \in \mathbb{R}$. As is Proposition 2.1.14 we get that $\lim_{\mathcal{U}} f(x_n) = f(x)$. So suppose $x = \infty$ and $f(x) \in \mathbb{R}$. Let $\epsilon > 0$. Since f is continuous, there is an $M \in \mathbb{R}$ such that $|f(x_n) - f(x)| < \epsilon$ whenever $x_n > M$. Since $\lim_{\mathcal{U}} x_n = \infty$, there is an $(G_M^x)^c \in \mathcal{U}$ such that $x_n > M$ whenever $n \in (G_M^x)^c$. Hence

$$\{n \in \mathbb{N} \mid |f(x_n) - f(x)| < \epsilon\} \supseteq \{n \in \mathbb{N} \mid x_n > M\} = (G_M^x)^c \in \mathcal{U}.$$

The other cases are similar. ■

2.2 Ultralimits of measure spaces

Suppose that we for each $n \in \mathbb{N}$ have a measure space $(\Omega_n, \mathcal{F}_n, \mu_n)$ and let $\prod_{n=1}^{\infty} \Omega_n$ be the space of all sequences $(\omega_1, \omega_2, ..., \omega_n, ...)$ where $\omega_n \in \Omega_n$ for all $n \in N$. We define an equivalence relation $\sim$ on $\prod_{n=1}^{\infty} \Omega_n$ by

$$\omega \sim \omega' \Leftrightarrow \{n \in \mathbb{N} \mid \omega_n = \omega_n'\} \in \mathcal{U}.$$

We let Ω be the set of all equivalence classes of $\sim$, and we let $[w_n]$ denote the equivalence class of $\{\omega_n\}_{n \in \mathbb{N}}$. Suppose that we for each $n \in \mathbb{N}$ have a function $X_n : \Omega_n \to \mathbb{R}$. Then we can define a function $[X_n] : \Omega \to \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$) by

$$[X_n]([\omega_n]) = \lim_{\mathcal{U}} X_n(\omega_n).$$

By Corollary 2.1.9, the product functions $[X_n]$ are well-defined.

We have a similar construction for sets: Suppose we have a set $A_n \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. Then we construct the *ultraproduct* $[A_n] \subset \Omega$ by

$$[\omega_n] \in [A_n] \Leftrightarrow \{n \in \mathbb{N} \mid \omega_n \in A_n\} \in \mathcal{U}.$$

We let $\mathcal{A}$ be the set of all ultraproducts, which are easily seen to be well-defined.

Proposition 2.2.1. *Let* $[A_n], [B_n] \in \mathcal{A}$.

(i) $[A_n]^c = \Omega \setminus [A_n] = [{A_n}^c]$

(ii) $[A_n] \cap [B_n] = [A_n \cap B_n]$

(iii) $[A_n] \cup [B_n] = [A_n \cup B_n]$

Proof. (i)

$$\begin{aligned}[\omega_n] \in [A_n]^c &\Leftrightarrow \{n \in \mathbb{N} \mid \omega_n \in A_n\} \notin \mathcal{U} \\ &\Leftrightarrow \{n \in \mathbb{N} \mid \omega_n \notin A_n\} \in \mathcal{U} \\ &\Leftrightarrow \{n \in \mathbb{N} \mid \omega_n \in {A_n}^c\} \in \mathcal{U} \\ &\Leftrightarrow [\omega_n] \in [{A_n}^c] \in \mathcal{A}\end{aligned}$$

(ii) We first prove that $[A_n \cap B_n] \subseteq [A_n] \cap [B_n]$: Suppose $[\omega_n] \in [A_n \cap B_n]$. Then, since

$$\begin{aligned}&\{n \in \mathbb{N} \mid \omega_n \in A_n \cap B_n\} \subseteq \{n \in \mathbb{N} \mid \omega_n \in A_n\} \text{ and} \\ &\{n \in \mathbb{N} \mid \omega_n \in A_n \cap B_n\} \subseteq \{n \in \mathbb{N} \mid \omega_n \in B_n\},\end{aligned}$$

$[\omega_n] \in [A_n] \cap [B_n]$.

We next prove that $[A_n] \cap [B_n] \subseteq [A_n \cap B_n]$: Suppose $\omega_n \in [A_n] \cap [B_n]$. Then

$$\{n \in \mathbb{N} \mid \omega_n \in A_n \cap B_n\} = \{n \in \mathbb{N} \mid \omega_n \in A_n\} \cap \{n \in \mathbb{N} \mid \omega_n \in B_n\} \in \mathcal{U},$$

hence $[\omega_n] \in [A_n \cap B_n]$.

(iii) This follows from De Morgan's law, using (i) and (ii).

■

Proposition 2.2.2. *$\mathcal{A}$ is an algebra, that is*

(i) $\emptyset \in \mathcal{A}$

(ii) If $A \in \mathcal{A}$, then $A^c = \Omega \setminus A \in \mathcal{A}$

(iii) If $C, D \in \mathcal{A}$, then $C \cup D \in \mathcal{A}$.

We define a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ by

$$\mu([A_n]) = \lim_{\mathcal{U}} \mu_n(A_n).$$

By Corollary 2.1.9, μ is well-defined.

Proposition 2.2.3. *μ is a finitely additive measure, that is*

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ *for all disjoint sets* $A, B \in \mathcal{A}$

Proof. (i) $\mu(\emptyset) = \lim_{\mathcal{U}} \mu_n(\emptyset) = 0$

(ii) Assume that $A \cap B = \emptyset$. We have that there for each $n \in \mathbb{N}$ are $A_n, B_n \in \mathcal{F}_n$ such that $A = [A_n]$ and $B = [B_n]$.

We argue that $G = \{n \in \mathbb{N} \mid A_n \cap B_n = \emptyset\} \in \mathcal{U}$. Suppose $G \notin \mathcal{U}$. Then $G^c = \{n \in \mathbb{N} \mid A_n \cap B_n \neq \emptyset\} \in \mathcal{U}$. We can then pick $[\omega_n] \in [A_n] \cap [B_n] = [A_n \cap B_n]$ by letting $\omega_n \in A_n \cap B_n$ for $n \in G^c$, a contradiction since $[A_n] \cap [B_n]$ is empty.

We can finally prove the assertion. Let $\epsilon > 0$. Then (remember that each μ_n is a measure)

$$\begin{aligned}
&\{n \in \mathbb{N} \mid |\mu_n(A_n + B_n) - (\mu(A) + \mu(B))| < \epsilon\} \\
&\supseteq \{n \in \mathbb{N} \mid |\mu_n(A_n + B_n) - (\mu(A) + \mu(B))| < \epsilon\} \cap G \\
&= \{n \in \mathbb{N} \mid |\mu_n(A_n) + \mu_n(B_n) - \mu(A) - \mu(B)| \ < \epsilon\} \cap G \\
&\supseteq \{n \in \mathbb{N} \mid |\mu_n(A_n) - \mu(A)| < \frac{\epsilon}{2}\} \cap \{n \in \mathbb{N} \mid |\mu_n(B_n) - \mu(B)| < \frac{\epsilon}{2}\} \cap G \in \mathcal{U}.
\end{aligned}$$

■

We want to extend μ to a measure. In order to do this we need to show that any countable union of ultraproducts that lies in $\mathcal{A}$ is actually a finite union of ultraproducts. To do this we need the following Theorem.

Theorem 2.2.4 (Countable Saturation Theorem)**.** *Suppose $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of ultraproducts such that*

$$\bigcap_{k=1}^{N} A_k \neq \emptyset$$

for all $N \in \mathbb{N}$. Then

$$\bigcap_{k\in\mathbb{N}} A_k \neq \emptyset.$$

Proof. We will construct an $[\omega_n] \in \Omega$ such that $[\omega_n] \in \bigcap_{k=1}^{N} A_k$ for each $N \in \mathbb{N}$ using a "diagonal" argument.

First we notice that, for $N \in \mathbb{N}$, $\bigcap_{k=1}^{N} A_k = \bigcap_{k=1}^{N}[A_{k,n}] = \left[\bigcap_{k=1}^{N} A_{k,n}\right]$, where $A_k = [A_{k,n}]$, and that

$$\left\{n \in \mathbb{N} \mid \bigcap_{k=1}^{N} A_{k,n} \neq \emptyset\right\} \in \mathcal{U}.$$

For each $n \in \mathbb{N}$, let $\ell_n = \sup\{\ell \in \{1, ..., n\} \mid \bigcap_{k=1}^{\ell} A_{k,n} \neq \emptyset\}$ and pick $\omega_n \in \bigcap_{k=1}^{\ell_n} A_{k,n}$. Then for $N \in \mathbb{N}$, by Lemma 2.1.3,

$$\{n \in \mathbb{N} \mid \omega_n \in \bigcap_{k=1}^{N} A_{k,n}\} \supseteq \{1, ..., N-1\}^c \cap \{n \in \mathbb{N} \mid \bigcap_{k=1}^{N} A_{k,n} \neq \emptyset\} \in \mathcal{U},$$

hence $[\omega_n] \in \bigcap_{k=1}^{N} A_k$. ■

Corollary 2.2.5. *Any countable union of ultraproducts in $\mathcal{A}$ is actually a finite union of ultraproducts.*

Proof. Suppose that $\bigcup_{k\in\mathbb{N}} A_k \in \mathcal{A}$. Let $B_k = \bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{j=1}^{k} A_j$. Then each B_k is an ultraproduct, $B_1 \supset B_2 \supset ...$ and $\bigcap_{k\in\mathbb{N}} B_k = \emptyset$. By Countable Saturation Theorem, there is an $N \in \mathbb{N}$ such that $B_N = \bigcap_{k=1}^{N} B_k = \emptyset$. But then $\bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k=1}^{N} A_k$. ■

Next we will need Caratheodory's Extension Theorem, which we will state here without proof (which can be found in [Lin18]). We will also use the notion of a premeasure.

Definition 2.2.6 (Premeasure)**.** A premeasure on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) if $A_1, A_2, ...$ is a countable collection of disjoint sets in $\mathcal{A}$ and if their union is contained in $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Theorem 2.2.7 (Caratheodory's Extension Theorem)**.** *Assume that $\mathcal{A}$ is an algebra and that μ is a premeasure on $\mathcal{A}$. Then the measure $\bar{\mu}$ generated by the outer measure construction is a complete measure extending μ. If μ is σ-finite, the extension is unique.*

Theorem 2.2.8. *μ can be extended to a (complete) measure $\bar{\mu}$.*

Proof. By Proposition 2.2.3, μ is a finitely additive measure. By Corollary 2.2.5, μ is a premeasure. By Caratheorody's Extension Theorem, μ can be extended to a (complete) measure $\bar{\mu}$ on $\mathcal{F}$, where $\mathcal{F}$ is the completion of the σ-algebra generated by $\mathcal{A}$. ■

(In a non-standard analysis setting this measure is called a Loeb measure.)

Definition 2.2.9. We call the measure space that we have constructed in this section for $(\Omega, \mathcal{F}, \bar{\mu})$.

We will from now on assume that each μ_n is a probability measure, making $\bar{\mu}$ a probability measure.

Lemma 2.2.10. *Let $\{A_k\}_{k\in\mathbb{N}}$ be an increasing family of sets, with each $A_k \in \mathcal{A}$, and let $A = \bigcup_{k\in\mathbb{N}} A_k$. Then there is a set $B \in \mathcal{A}$ such that*

(i) $A \subseteq B$

(ii) $\bar{\mu}(B) = \lim_{k\to\infty} \bar{\mu}(A_k)$

(iii) $\bar{\mu}(B \setminus A) = 0$

Proof. We will find a sequence $\{B_n\}_{n\in\mathbb{N}}$, with each $B_n \in \mathcal{F}_n$ such that $B = [B_n]$. We may assume that if $A_k = [A_{k,n}] \subset [A_{\ell,n}] = A_\ell$, then $A_{k,n} \subset A_{\ell,n}$ for each $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let

$$F_k = \{n \in \mathbb{N} \mid |\mu_n(A_{k,n}) - \mu(A_k)| < \frac{1}{k}\}.$$

If $n \notin F_1$, let $B_n = \Omega_n$. If $n \in \bigcap_{j=1}^k F_j$ for some $k \in \mathbb{N}$, let
$B_n = \bigcup_{k=1}^m A_{k,n} = A_{m,n}$, where $m = \sup\{k \leq n \mid n \in \bigcap_{j=1}^k F_j\}$.

Since (iii) follows from (i) and (ii), we only have to prove the first two assertions.

(i)
$$\{n \in \mathbb{N} \mid A_{k,n} \subseteq B_n\} \supseteq \bigcap_{j=1}^{k} F_j \cap \{1, ..., k-1\}^c \in \mathcal{U},$$
which implies that $A_k \subseteq B$ for each $k \in \mathbb{N}$. So $A \subseteq B$.

(ii) Pick $n_k \in \bigcap_{j=1}^k F_j \cap \{1, ..., k-1\}^c \cap \{n \in \mathbb{N} \mid |\mu_n(B_n) - \bar{\mu}(B)| < \frac{1}{k}\}$. Then $|\mu_{n_k}(B_{n_k}) - \bar{\mu}(B)| < \frac{1}{k}$, so we get that
$$\lim_{k\to\infty} \mu_{n_k}(B_{n_k}) = \lim_{\mathcal{U}} \mu_n(B_n) = \bar{\mu}(B).$$
Furthermore, $\mu_{n_k}(B_{n_k}) = \mu_{n_k}(A_{m_k,n_k})$, where $k \leq m_k \leq n_k$. We have that $\lim_{k\to\infty} \bar{\mu}(A_{m_k}) = \lim_{k\to\infty} \bar{\mu}(A_k)$. Since
$$|\mu_{n_k}(B_{n_k}) - \bar{\mu}(A_{m_k})| = |\mu_{n_k}(A_{m_k,n_k}) - \bar{\mu}(A_{m_k})| < \frac{1}{k},$$
we must have $\lim_{k\to\infty} \bar{\mu}(A_k) = \bar{\mu}(B)$.

■

Proposition 2.2.11. *Suppose that $F \in \mathcal{F}$ and that $\epsilon > 0$. Then there are ultraproducts A, B such that $A \subseteq F \subseteq B$ and*
$$\bar{\mu}(F \setminus A) < \epsilon \quad \text{and} \quad \bar{\mu}(B \setminus F) < \epsilon.$$

Proof. Let $F \in \mathcal{F}$ and cover F with a countable covering $\{A_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{A}$ such that
$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \bar{\mu}(A_k) < \bar{\mu}(F) + \epsilon.$$
By the outer measure construction, such a covering exists. By Lemma 2.2.10, there is $A \in \mathcal{A}$ such that $\bigcup_{k=1}^\infty A_k \subseteq A$ and $A \setminus \bigcup_{k=1}^\infty A_k$ has measure zero. Then
$$\begin{aligned} A \setminus F &= \left(\left(\bigcup_{k=1}^{\infty} A_k\right) \setminus F\right) \bigcup \left(\left(A \setminus \bigcup_{k=1}^{\infty} A_k\right) \setminus F\right) \\ &\subseteq \left(\left(\bigcup_{k=1}^{\infty} A_k\right) \setminus F\right) \bigcup \left(A \setminus \bigcup_{k=1}^{\infty} A_k\right), \end{aligned}$$
hence $\bar{\mu}(A \setminus F) \leq \bar{\mu}\left((\bigcup_{k=1}^\infty A_k) \setminus F\right) + \bar{\mu}(A \setminus \bigcup_{k=1}^\infty A_k) < \epsilon$.

Now cover F^c with a countable covering $\{C_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{A}$ such that
$$\bar{\mu}\left(\bigcup_{k=1}^{\infty} C_k\right) \leq \sum_{k=1}^{\infty} \bar{\mu}(C_k) < \bar{\mu}(F^c) + \epsilon.$$

By Lemma 2.2.10 there is $C \in \mathcal{A}$ such that $\bigcup_{k=1}^{\infty} C_k \subseteq C$ and $C \setminus \bigcup_{k=1}^{\infty} C_k$ has measure zero. Then $C^c \in \mathcal{F}$ and $C^c \subseteq (\bigcup_{k=1}^{\infty} C_k)^c \subseteq F$. Let $B = C^c$. Then $B \subseteq F$ and

$$\begin{aligned} F \setminus B &= F \setminus C^c \\ &= F \cap C \\ &= C \setminus F^c \\ &= \left(\left(\bigcup_{k=1}^{\infty} C_k\right) \setminus F^c\right) \bigcup \left(\left(C \setminus \left(\bigcup_{k=1}^{\infty} C_k\right)\right) \setminus F^c\right) \\ &\subseteq \left(\left(\bigcup_{k=1}^{\infty} C_k\right) \setminus F^c\right) \bigcup \left(C \setminus \left(\bigcup_{k=1}^{\infty} C_k\right)\right), \end{aligned}$$

hence $\bar{\mu}(F \setminus B) \leq \bar{\mu}((\bigcup_{k=1}^{\infty} C_k) \setminus F^c) + \bar{\mu}(C \setminus \bigcup_{k=1}^{\infty} C_k) < \epsilon$.

■

Corollary 2.2.12. *Suppose that $F \in \mathcal{F}$. Then there is an ultraproduct $A \in \mathcal{A}$ such that $\bar{\mu}(F \triangle A) = 0$.*

Proof. For each $k \in \mathbb{N}$, using Proposition 2.2.11, pick $A_k \in \mathcal{A}$ such that $A_k \subseteq A$ and $\bar{\mu}(F \setminus A_k) < \frac{1}{k}$. Then

$$\bar{\mu}(F \setminus \bigcup_{k=1}^{\infty} A_k) \leq \bar{\mu}(F \setminus A_n) < \frac{1}{n}$$

for each $n \in \mathbb{N}$, hence $\bar{\mu}(F \setminus \bigcup_{k=1}^{\infty} A_k) = 0$. By Lemma 2.2.10, there is an $A \in \mathcal{A}$ such that $A \supseteq \bigcup_{k=1}^{\infty} A_k$ and $\bar{\mu}(A \setminus \bigcup_{k=1}^{\infty} A_k) = 0$. Then

$$\begin{aligned} \bar{\mu}(F \triangle A) &= \bar{\mu}(F \setminus A) + \bar{\mu}(A \setminus F) \\ &\leq \bar{\mu}(F \setminus \bigcup_{k=1}^{\infty} A_k) + \bar{\mu}(A \setminus \bigcup_{k=1}^{\infty} A_k) = 0. \end{aligned}$$

■

We will next turn our attention towards integration theory.

Proposition 2.2.13. *Suppose that we for each $n \in \mathbb{N}$ have a probability space $(\Omega_n, \mathcal{F}_n, \mu_n)$. Suppose furthermore that we for each $n \in \mathbb{N}$ have a random variable $X_n : I_n \times \Omega_n \to \mathbb{R}$ that is measurable with respect to $\mathcal{F}_n$. Then $X = [X_n]$ is measurable with respect to $\mathcal{F}$.*

Proof. Since $X = X^+ - X^-$, it suffices to assume that $X \geq 0$. For each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, let $g_{m,n} : \Omega_n \to \mathbb{R}$ be defined by

$$g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{A_{m,n}^k}(\omega_n) + 2^m 1_{B_{m,n}}(\omega_n),$$

where $A_{m,n}^k = X_n^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$ and $B_{m,n} = X_n^{-1}([2^m, \infty])$. Then each $A_{m,n}^k \in \mathcal{F}_n$ and $B_{m,n} \in \mathcal{F}_n$. Let $g_m : \Omega \to \overline{\mathbb{R}}$ be defined by

$$g_m([\omega_n]) = \lim_{\mathcal{U}} g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{[A_{m,n}^k]}([\omega_n]) + 2^m 1_{[B_{m,n}]}([\omega_n]).$$

Then for all $\omega \in \Omega$, $\lim_{m\to\infty} g_m(\omega) = X(\omega)$. Since X is the pointwise limit of a sequence of $\mathcal{F}$-measurable variables, X is $\mathcal{F}$-measurable. ■

Theorem 2.2.14. *Suppose that $Y : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable. Then there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}_n$-measurable functions such that $X = [X_n]$ is equal to Y $\bar{\mu}$-almost everywhere.*

Proof. Let Y^+ and Y^- denote the positive and negative part of Y respectively. Since $Y = Y^+ - Y^-$, it suffices to prove the theorem for $Y \geq 0$. For each $m \in \mathbb{N}$, let $g_m : \Omega \to \mathbb{R}$ be defined by

$$g_m(\omega) = \sum_{k=0}^{2^{2m}-1} \frac{k}{2^m} 1_{E_m^k}(\omega) + 2^m 1_{\{\omega\in\Omega \ \mid \ Y(\omega)\geq 2^m\}}(\omega),$$

where $E_m^k = Y^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$. Then $\{g_m\}_{m\in\mathbb{N}}$ is a sequence of simple functions converging pointwise to Y. Using Proposition 2.2.12, we can for each E_m^k find an ultraproduct $[A_{m,n}^k] \in \mathcal{A}$ such that $\bar{\mu}(E_m^k \triangle [A_{m,n}^k]) = 0$. For each $m \in \mathbb{N}$, let $h_m : \Omega \to \mathbb{R}$ be defined by

$$h_m(\omega) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{[A_{m,n}^k]}(\omega) + 2^m 1_{\Omega \setminus \bigcup_{k=0}^{2^{2m}-1} [A_{m,n}^k]}(\omega).$$

Then, since a countable union of measure-zero sets has measure zero, $\{h_m\}_{m\in\mathbb{N}}$ is a sequence of measurable functions that converges pointwise to a function $Z : \Omega \to \overline{\mathbb{R}}$ that is equal to Y $\bar{\mu}$-almost everywhere.

Since we may disregard sets of measure zero, for fixed m, we may assume that $[A_{m,n}^k] \cap [A_{m,n}^\ell] = \emptyset$ for all $k, \ell \in \mathbb{N}$ with $k \neq \ell$. If $E_m^k = E_{m+1}^{2k} \cup E_{m+1}^{2k+1}$, we may assume that $[A_{m,n}^k] = [A_{m+1,n}^{2k}] \cup [A_{m+1,n}^{2k+1}]$. Moreover, if $[A_{m,n}^k] = [A_{m+1,n}^{2k}] \cup [A_{m+1,n}^{2k+1}]$ we may assume that $A_{m,n}^k = A_{m+1,n}^{2k} \cup A_{m+1,n}^{2k+1}$ for each $n \in \mathbb{N}$. We also note that if $[A_{\ell,n}^j] \subseteq [A_{m,n}^k]$, then we may assume that $A_{\ell,n}^j \subseteq A_{m,n}^k$ for each $n \in \mathbb{N}$.

Notice that for $[A_n] \in \Omega$, $1_{[A_n]}([\omega_n]) = \lim_{\mathcal{U}} 1_{A_n}(\omega_n)$ for all $[\omega_n] \in \Omega$, so that $1_{[A_n]} = [1_{A_n}]$. Hence $h_m = [h_{m,n}]$ is a product function. For each $n \in \mathbb{N}$, let $X_n : \Omega_n \to \mathbb{R}$ be defined by $X_n = h_{n,n}$. We will show that $X = [X_n]$ is equal to Z $\bar{\mu}$-almost everywhere.

Suppose $Z([\omega_n]) \in \mathbb{R}$ and let $\epsilon > 0$. Pick $N_1 \in \mathbb{N}$ large enough such that $g_m([\omega_n]) < 2^{N_1}$ for all $m \in \mathbb{N}$ such that $m \geq N_1$. Pick $N_2 \in \mathbb{N}$ such that $\frac{1}{2^{N_2}} < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and let

$$F = \left\{ n \in \mathbb{N} \mid |h_N, n(\omega_n) - h_N([\omega_n])| < \frac{1}{2^N} \right\} \in \mathcal{U}.$$

Then for all $m \in \mathbb{N}$ and all $n \in F$, $|h_{m,n}(\omega_n) - h_m([\omega_n])| < \frac{\epsilon}{2}$. Indeed, if $m \leq N$ and $n \in F$, then $|h_{N,n}(\omega_n) - h_N([\omega_n])| < \frac{1}{2^N}$, which means that $h_{N,n}(\omega_n) = h_N([\omega_n])$, which implies that $h_{m,n}(\omega_n) = h_m([\omega_n])$. If $m > N$ and $n \in F$, then, since $A^k_{N,n} = A^{2k}_{N+1,n} \cup A^{2k+1}_{N+1,n}$ and so on, $|h_{m,n}(\omega_n) - h_m([\omega_n])| < \frac{1}{2^N} < \frac{\epsilon}{2}$. So pick $M \in \mathbb{N}$ such that $|h_m([\omega_n]) - Z([\omega_n])| < \frac{\epsilon}{2}$ for all $m \geq M$. Then

$$\begin{aligned}
&\{n \in \mathbb{N} \mid |X_n(\omega_n) - Z([\omega_n])| < \epsilon\} \\
&\supseteq \left\{n \in \mathbb{N} \mid |X_n(\omega_n) - h_n([\omega_n])| < \frac{\epsilon}{2}\right\} \cap \left\{n \in \mathbb{N} \mid |h_n([\omega_n]) - Z([\omega_n])| < \frac{\epsilon}{2}\right\} \\
&\supseteq F \cap \{1, 2, ..., M-1\}^c \in \mathcal{U}.
\end{aligned}$$

Since $h_m = g_m$ for almost all $\omega \in \Omega$, by the definition of g_m, $Z(\omega) < \infty$ for almost all $\omega \in \Omega$. Hence we need not check that $\lim_{\mathcal{U}} X_n(\omega_n) = Z([\omega_n])$ when $Z([\omega_n]) = \infty$.

■

Theorem 2.2.15. *Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of μ_n-integrable functions such that $X = [X_n]$ is bounded. Then X is $\bar{\mu}$-integrable and*

$$\int_\Omega X \; d\bar{\mu} = \lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n.$$

Proof. Let X^+ and X^- denote the positive and negative parts of X, respectively. Since X is integrable if and only if both X^+ and X^- is integrable, is suffices to assume that $X \geq 0$. Suppose that X is bounded. Then there is $M \in \mathbb{N}$ and $G \in \mathcal{U}$ such that $|X_n| \leq M$ for all $n \in G$. Pick $m \in \mathbb{N}$ such that $2^m > M$ and define, for each $n \in \mathbb{N}$, $h_{m,n} : \Omega_n \to \mathbb{R}$ by

$$h_{m,n}(\omega_n) = \sum_{k=0}^{2^{2m}-1} \frac{k}{2^m} 1_{A^k_{m,n}}(\omega_n) + 2^m 1_{\{\omega_n \ | \ X_n(\omega_n) \geq 2^m\}}(\omega_n),$$

where $A^k_{m,n} = X_n{}^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$. Then $|h_{m,n}| \leq |X_n|$ for all $n \in \mathbb{N}$ and $|h_{m,n} - X_n| \leq \frac{1}{2^m}$ for all $n \in G$. We have that

$$\left| \int_{\Omega_n} h_{m,n} \; d\mu_n - \int_{\Omega_n} X_n \; d\mu_n \right| \leq \frac{1}{2^m}$$

for all $n \in G$, hence

$$\left| \lim_{\mathcal{U}} \int_{\Omega_n} h_{m,n} \; d\mu_n - \lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n \right| = \left| \lim_{\mathcal{U}} \int_{\Omega_n} (h_{m,n} - X_n) \; d\mu_n \right| \leq \frac{1}{2^m}.$$

Also,

$$\left| \int_\Omega [h_{m,n}] \; d\bar{\mu} - \int_\Omega X \; d\bar{\mu} \right| = \left| \int_\Omega [h_{m,n} - X_n] \; d\bar{\mu} \right| \leq \frac{1}{2^m}.$$

We shall show that

$$\int_\Omega [h_{m,n}] \; d\bar{\mu} = \lim_{\mathcal{U}} \int_{\Omega_n} h_{m,n} \; d\mu_n. \tag{2.1}$$

Since we can construct $[h_{m,n}]$ for m as large as we would like, this proves the theorem.

Notice that $[h_{m,n}]$ is a simple function. Indeed,

$$[h_{m,n}]([\omega_n]) = \sum_{k=0}^{2^{2m}-1} \frac{k}{2^m} 1_{[A^k_{m,n}]}([\omega_n]).$$

Since $[h_{m,n}]$ is a simple function, by linearity, it is enough to show (2.1) for characteristic functions. But this is just the definition of $\bar{\mu}$:

$$\int_\Omega 1_{[A_n]}\ d\bar{\mu} = \bar{\mu}([A_n]) = \lim_{\mathcal{U}} \mu(A_n) = \lim_{\mathcal{U}} \int_\Omega 1_{A_n}\ d\mu_n.$$

■

Lemma 2.2.16. *Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of μ_n-integrable functions and suppose $X = [X_n]$ is real-valued and $X_n \geq 0$ for each $n \in \mathbb{N}$. Then X is $\bar{\mu}$-integrable and*

$$\int_\Omega X\ d\bar{\mu} \leq \lim_{\mathcal{U}} \int_{\Omega_n} X_n\ d\mu_n. \tag{2.2}$$

Proof. For each $n, m \in \mathbb{N}$, construct the simple function $h_{m,n} : \Omega_n \to \overline{\mathbb{R}}$ as in Theorem 2.2.15. Then $h_{m,n}(\omega_n) \leq X_n(\omega_n)$ for each $n \in \mathbb{N}$ and $h_m : \Omega \to \overline{\mathbb{R}}$ defined by $h_m = [h_{m,n}]$ converges pointwise to X. We then have that for each $m \in \mathbb{N}$,

$$\begin{aligned} h_{m,n}(\omega_n) \leq X_n(\omega_n) \text{ for all } n \in \mathbb{N} &\Rightarrow \int_{\Omega_n} h_{m,n}\ d\mu_n \leq \int_{\Omega_n} X_n\ d\mu_n \text{ for all } n \in \mathbb{N} \\ &\Rightarrow \lim_{\mathcal{U}} \int_{\Omega_n} h_{m,n}\ d\mu_n \leq \lim_{\mathcal{U}} \int_{\Omega_n} X_n\ d\mu_n \\ &\Rightarrow \int_\Omega h_m\ d\bar{\mu} \leq \lim_{\mathcal{U}} \int_{\Omega_n} X_n\ d\mu_n, \end{aligned}$$

where the last inequality stems from the fact that h_m is bounded. By the Monotone Convergence Theorem,

$$\int_\Omega X\ d\bar{\mu} = \int_\Omega \lim_{m\to\infty} h_m\ d\bar{\mu} = \lim_{m\to\infty} \int_\Omega h_m\ d\bar{\mu} \leq \lim_{\mathcal{U}} \int_{\Omega_n} X_n\ d\mu_n.$$

■

We may not always have equality in (2.2) as the following example shows.

Example 2.2.17. For each $n \in \mathbb{N}$, let $\Omega_n = [0, 1]$ and define $X_n : \Omega_n \to \mathbb{R}$ by $X_n(\omega_n) = n1_{[1-\frac{1}{n},1]}(\omega_n)$. Let Ω_n be equipped with the Lebesgue σ-algebra and let μ_n denote the Lebesgue measure on $[0, 1]$. Then

$$\lim_{\mathcal{U}} \int_{\Omega_n} X_n\ d\mu_n = \lim_{\mathcal{U}} n\mu_n([1 - \frac{1}{n}, 1]) = 1.$$

Let $A = [\ [1 - \frac{1}{n}, 1]\ ]$. Then

$$\bar{\mu}(A) = \lim_{\mathcal{U}} \mu_n([1 - \frac{1}{n}, 1]) = 0.$$

Since

$$X([\omega_n]) = \lim_{\mathcal{U}} X_n(\omega_n) = \begin{cases} 0 & \text{if } [\omega_n] \in \Omega \setminus A \\ \infty & \text{otherwise,} \end{cases}$$

we have that $\int_\Omega X \; d\bar{\mu} = \int_{\Omega \setminus A} X \; d\bar{\mu} = 0$.

We want to find sufficient constraints on $\{X_n\}_{n\in\mathbb{N}}$ so that we can prove Theorem 2.2.15 for a broader class of functions X on Ω.

Definition 2.2.18 ($\mathcal{A}$-integrability)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of μ_n-integrable functions. We call this sequence $\mathcal{A}$-integrable if the following two conditions hold:

(i) $\lim_{\mathcal{U}} \int_{\Omega_n} |X_n| \; d\mu_n < \infty$

(ii) If $[A_n] \in \mathcal{A}$ and $\bar{\mu}([A_n]) = 0$, then $\lim_{\mathcal{U}} \int_{A_n} |X_n| \; d\mu_n = 0$

Theorem 2.2.19. *Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of μ_n-integrable functions. Consider the following two statements:*

(i) The sequence $\{X_n\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable

(ii) $X = [X_n]$ is $\bar{\mu}$-integrable and

$$\int_\Omega X \; d\bar{\mu} = \lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n$$

We always have (i) $\Rightarrow$ (ii). If $X_n \geq 0$ for each n, the two statements are equivalent.

Proof. (i) $\Rightarrow$ (ii):
Let X_n^+ and X_n^- denote the positive and negative parts of X_n, respectively. Then $\{X_n\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable if and only if both $\{X_n^+\}_{n\in\mathbb{N}}$ and $\{X_n^-\}_{n\in\mathbb{N}}$ are $\mathcal{A}$-integrable. It therefore suffices to assume that $X_n \geq 0$ for each $n \in \mathbb{N}$.

Suppose $\lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n < \infty$. Then, by Lemma 2.2.16, $\int_\Omega X \; d\bar{\mu} < \infty$. For each $k \in \mathbb{N}$ we have that $\lim_{\mathcal{U}} \int_{\Omega_n} X_n \wedge k \; d\mu_n = \int_\Omega X \wedge k \; d\bar{\mu} \leq \int_\Omega X \; d\bar{\mu}$.

Let $\epsilon > 0$ and notice that for each $k \in \mathbb{N}$,

$$\left\{ n \in \mathbb{N} \mid \int_{\Omega_n} X_n \wedge k \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon \right\} \in \mathcal{U}.$$

Construct the sequence $\{k_n\}_{n\in\mathbb{N}}$ of integers as follows:
If $\int_{\Omega_n} X_n \wedge k \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon$ for some $k \in \mathbb{N}$, let

$$k_n = \sup \left\{ k \in \{1, ..., n\} \mid \int_{\Omega_n} X_n \wedge k \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon \right\}.$$

Otherwise, let $k_n = 1$. Then $\{n \in \mathbb{N} \mid \int_{\Omega_n} X_n \wedge k_n \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon\} \in \mathcal{U}$, hence $\lim_{\mathcal{U}} \int_{\Omega_n} X_n \wedge k_n \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon$. Furthermore, for $N \in \mathbb{N}$,

$$\begin{aligned} &\{n \in \mathbb{N} \mid k_n \geq N\} \\ &= \left\{ n \in \mathbb{N} \mid \int_{\Omega_n} X_n \wedge N \; d\mu_n \leq \int_\Omega X \; d\bar{\mu} + \epsilon \right\} \bigcap \{1, 2, .., N-1\}^c \in \mathcal{U}, \end{aligned}$$

hence $\lim_{\mathcal{U}} k_n = \infty$.

For each $n \in \mathbb{N}$, let $A_n = \{\omega_n \in \Omega_n \mid X_n(\omega_n) > k_n\}$. By measurability, each $A_n \in \mathcal{F}_n$, so that $[A_n] \in \mathcal{A}$. We have that

$$\lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n \leq \lim_{\mathcal{U}} \int_{A_n} X_n \; d\mu_n + \lim_{\mathcal{U}} \int_{\Omega_n} X_n \wedge k_n \; d\mu_n$$

Since $X < \infty$ $\bar{\mu}$-almost everywhere (since $\int_\Omega X \; d\bar{\mu} < \infty$), $\bar{\mu}([A_n]) = 0$. By $\mathcal{A}$-integrability,

$$\lim_{\mathcal{U}} \int_{A_n} X_n \; d\mu_n = 0.$$

So we get that

$$\lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n \leq \int_\Omega X \; d\mu + \epsilon.$$

Since this holds for all $\epsilon > 0$, $\lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n \leq \int_\Omega X \; d\mu$, which combined with Lemma 2.2.16 gives the desired equality.

(ii) $\Rightarrow$ (i) if $X_n \geq 0$ for each n:
Suppose that (ii) holds. Then $\lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n = \int_\Omega X \; d\bar{\mu} < \infty$. Suppose $A = [A_n] \in \mathcal{A}$ and $\bar{\mu}([A_n]) = 0$. Then, by Lemma 2.2.16,

$$\int_\Omega X \; d\bar{\mu} = \int_{\Omega \setminus A} X \; d\bar{\mu} \leq \lim_{\mathcal{U}} \int_{\Omega_n \setminus A_n} X_n \; d\mu_n \leq \lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n = \int_\Omega X \; d\bar{\mu},$$

hence

$$\lim_{\mathcal{U}} \int_{\Omega_n \setminus A_n} X_n = \int_\Omega X \; d\bar{\mu}.$$

Also,

$$\begin{aligned}
\lim_{\mathcal{U}} \int_{\Omega_n \setminus A_n} X_n \; d\mu_n + \lim_{\mathcal{U}} \int_{A_n} X_n \; d\mu_n &= \lim_{\mathcal{U}} \int_{\Omega_n \setminus A_n} X_n \; d\mu_n + \int_{A_n} X_n \; d\mu_n \\
&= \lim_{\mathcal{U}} \int_{\Omega_n} X_n \; d\mu_n \\
&= \int_\Omega X \; d\bar{\mu}.
\end{aligned}$$

Combined we get that $\lim_{\mathcal{U}} \int_{A_n} X_n \; d\mu_n = 0$. ∎

Proposition 2.2.20. *Suppose there is a real number $p > 1$ such that $\lim_{\mathcal{U}} \int_{\Omega_n} |X_n|^p \; d\mu_n < \infty$. Then $\{X_n\}_{n \in \mathbb{N}}$ is $\mathcal{A}$-integrable.*

Proof. Let $q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality,

$$\begin{aligned}
\lim_{\mathcal{U}} \int_{\Omega_n} |X_n| \; d\mu_n &\leq \lim_{\mathcal{U}} \left(\int_{\Omega_n} 1^q \; d\mu_n \right)^{\frac{1}{q}} \left(\int_{\Omega_n} |X_n|^p \; d\mu_n \right)^{\frac{1}{p}} \\
&= \lim_{\mathcal{U}} \left(\int_{\Omega_n} |X_n|^p \; d\mu_n \right)^{\frac{1}{p}} < \infty.
\end{aligned}$$

Suppose $[A_n] \in \mathcal{A}$ and $\bar{\mu}([A_n]) = \lim_{\mathcal{U}} \mu_n(A_n) = 0$. Then, by Hölder's inequality,

$$\begin{aligned}
\lim_{\mathcal{U}} \int_{A_n} |X_n| \; d\mu_n &= \lim_{\mathcal{U}} \int_{\Omega_n} 1_{A_n} |X_n| \; d\mu_n \\
&\leq \lim_{\mathcal{U}} \left(\int_{\Omega_n} 1_{A_n}^q \; d\mu_n \right)^{\frac{1}{q}} \left(\int_{\Omega_n} |X_n|^p \; d\mu_n \right)^{\frac{1}{p}} \\
&= \lim_{\mathcal{U}} \mu_n(A_n)^{\frac{1}{q}} \left(\int_{\Omega_n} |X_n|^p \; d\mu_n \right)^{\frac{1}{p}} = 0.
\end{aligned}$$

■

Theorem 2.2.21. *Suppose $Y : I \times \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable and that*

$$\int_{\Omega} |X|^p \; d\bar{\mu} < \infty$$

for some $p \in [1, \infty)$. Then there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of $\mathcal{F}_n$-measurable functions such that $X = [X_n]$ is equal to Y $\bar{\mu}$-almost everywhere and such that

$$\lim_{\mathcal{U}} \int_{\Omega_n} |X_n|^p \; d\mu_n = \int_{\Omega} |Y|^p \; d\bar{\mu}_n.$$

Proof. By Theorem 2.2.14, there exists a sequence $\{Z_n\}_{n \in \mathbb{N}}$ of $\mathcal{F}_n$-measurable functions such that $Z = [Z_n]$ is equal to Y $\bar{\mu}$-almost everywhere. For each $k \in \mathbb{N}$, let $Z_k : \Omega \to \mathbb{R}$ be given by

$$Z_k(\omega) = \begin{cases} k & \text{if } Z(\omega) \geq k \\ -k & \text{if } Z(\omega) \leq -k \\ Z(\omega) & \text{else .} \end{cases}$$

For each $k \in \mathbb{N}$, for each $n \in \mathbb{N}$, let $Z_{k,n} : \Omega_n \to \mathbb{R}$ be given by

$$Z_{k,n}(\omega_n) = \begin{cases} k & \text{if } Z(\omega_n) \geq k \\ -k & \text{if } Z(\omega_n) \leq -k \\ Z_n(\omega_n) & \text{else .} \end{cases}$$

Then for each $k \in \mathbb{N}$ we have $Z_k = [Z_{k,n}]$. Furthermore, for each $\omega \in \Omega$, we have

$$Z(\omega) = \lim_{k \to \infty} Z_k(\omega).$$

By dominated convergence theorem (since Y and thus Z is in $L^p(\bar{\mu})$),

$$\lim_{k \to \infty} \int_{\Omega} |Z_k|^p \; d\bar{\mu} = \int_{\Omega} |Z|^p \; d\bar{\mu} = \int_{\Omega} |Y|^p \; d\bar{\mu}.$$

Moreover, by boundedness of $\{Z_{k,n}\}_{n \in \mathbb{N}}$, for each $k \in \mathbb{N}$ we have

$$\lim_{\mathcal{U}} \int_{\Omega_n} |Z_{k,n}|^p \; d\mu_n = \int_{\Omega} |Z_k|^p \; d\bar{\mu}.$$

We wish to construct a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that $X = [X_n] = Z$ and such that

$$\lim_{\mathcal{U}} \int_{\Omega_n} |X_n|^p \, d\mu_n = \int_{\Omega} |Z|^p \, d\bar{\mu}.$$

For each $n \in \mathbb{N}$, we choose X_n as follows: if

$$\left| \int_{\Omega_n} |Z_{1,n}|^p \, d\mu_n - \int_{\Omega} |Z_1|^p \, d\bar{\mu} \right| \geq 1,$$

let $X_n = Z_{1,n}$. Otherwise, let $X_n = Z_{k_n,n}$, where

$$k_n = \sup \left\{ k \in \{1, .., n\} \mid \left| \int_{\Omega_n} |Z_{j,n}|^p \, d\mu_n - \int_{\Omega} |Z_j|^p \, d\bar{\mu} \right| < \frac{1}{j} \text{ for all } j \leq k \right\}.$$

Then

$$\lim_{\mathcal{U}} \int_{\Omega_n} |X_n|^p \, d\mu_n = \int_{\Omega} |Z|^p \, d\bar{\mu}.$$

Indeed, let $\epsilon > 0$. Choose $K \in \mathbb{N}$ such that

$$\left| \int_{\Omega} |Z_k|^p \, d\bar{\mu} - \int_{\Omega} |Z|^p \, d\bar{\mu} \right| < \frac{\epsilon}{2}$$

for all $k \geq K$ and such that $\frac{1}{K} < \frac{\epsilon}{2}$. Let

$$F = \bigcap_{k=1}^{K} \left\{ n \in \mathbb{N} \mid \left| \int_{\Omega_n} |Z_{k,n}|^p \, d\mu_n - \int_{\Omega} |Z_k|^p \, d\bar{\mu} \right| < \frac{1}{k} \right\} \in \mathcal{U}.$$

Then for all $n \in F$,

$$\left| \int_{\Omega} |Z|^p \, d\bar{\mu} - \int_{\Omega_n} |X_n|^p \, d\mu_n \right| < \epsilon.$$

It remains to show that $X = [X_n] = Z$. Let $[\omega_n] \in \Omega$ and suppose $|Z([\omega_n])| < \infty$. Let $\epsilon > 0$. Choose $K \in \mathbb{N}$ such that $|Z([\omega_n])| < K - \epsilon$. Let

$$F_1 = \bigcap_{k=1}^{K} \left\{ n \in \mathbb{N} \mid \left| \int_{\Omega_n} |Z_{k,n}|^p \, d\mu_n - \int_{\Omega} |Z_k|^p \, d\bar{\mu} \right| < \frac{1}{k} \right\} \in \mathcal{U}$$

and

$$F_2 = \{ n \in \mathbb{N} \mid |Z_n(\omega_n) - Z([\omega_n])| < \epsilon \}.$$

Then for all $n \in F_1 \cap F_2$, $X_n = Z_{k_n,n}$ for some $k_n \geq K$ and we have

$$|X_n(\omega_n) - Z([\omega_n])| = |Z_{k_n}([\omega_n]) - Z_{k_n,n}(\omega_n)| = |Z_n(\omega_n) - Z([\omega_n])| < \epsilon.$$

Now suppose $Z([\omega_n]) = \infty$. Let $M \in \mathbb{N}$. Let

$$F = \bigcap_{k=1}^{M} \left\{ n \in \mathbb{N} \mid \left| \int_{\Omega_n} |Z_{k,n}|^p \, d\mu_n - \int_{\Omega} |Z_k|^p \, d\bar{\mu} \right| < \frac{1}{k} \right\} \in \mathcal{U}$$

and

$$G = \{ n \in \mathbb{N} \mid Z_n(\omega_n)| \geq M \} \in \mathcal{U}.$$

Then for all $n \in F \cap G$, $X_n = Z_{k_n,n}$ for some $k_n \geq M$ and we have

$$|X_n(\omega_n)| = |Z_{k_n,n}(\omega_n)| \geq M.$$

A similar argument shows that $Z([\omega_n]) = \lim_{\mathcal{U}} X_n(\omega_n)$ when $Z([\omega_n]) = -\infty$. ∎

Suppose that we have two measures $\bar{P}, \bar{Q}$ on $(\Omega, \mathcal{F})$. Then $\bar{P}$ is called *absolutely continuous* with respect to $\bar{Q}$, and we write $\bar{P} \ll \bar{Q}$, if $\bar{P}(E) = 0$ whenever $\bar{Q}(E) = 0$, $E \in \mathcal{F}$. If $\bar{P} \ll \bar{Q}$ and $\bar{Q} \ll \bar{P}$, we say that $\bar{P}$ and $\bar{Q}$ are *equivalent* measures.

Suppose that we for each $n \in \mathbb{N}$ have two measures P_n, Q_n on Ω_n such that $Q_n \ll P_n$. Then the ultralimit $\bar{Q}$ need not be absolutely continuous with respect to the ultralimit $\bar{P}$ as the following example shows.

Example 2.2.22. For each $n \in \mathbb{N}$, let $\Omega_n = [0,1]$ be equipped with the Lebesgue σ-algebra $\mathcal{F}_n$ and let μ_n denote the Lebesgue measure on Ω_n. Let $X_n : \Omega_n \to [0,\infty)$ be defined by $X_n(\omega_n) = \frac{n}{2n-1} 1_{[0,1-\frac{1}{2n}]}(\omega_n) + n 1_{[1-\frac{1}{2n},1]}(\omega_n)$. Define the measure λ_n on Ω_n by

$$\lambda_n(E) = \int_E X_n \, d\mu_n$$

for $E \in \mathcal{F}_n$. Then μ_n and λ_n are equivalent for each n. For each $n \in \mathbb{N}$, let $A_n = [1 - \frac{1}{2n}, 1] \in \mathcal{F}_n$. Then $\bar{\mu}([A_n]) = \lim_{\mathcal{U}} \mu_n(A_n) = 0$, but $\bar{\lambda}([A_n]) = \lim_{\mathcal{U}} \lambda_n(A_n) = \frac{1}{2}$.

In order to find sufficient constraints on $\{Q_n\}_{n\in\mathbb{N}}$ so that $\bar{Q}$ is absolutely continuous with respect to $\bar{P}$, we will need the notion of a Radon-Nikodym derivative.

Definition 2.2.23 (Radon-Nikodym derivative)**.** Let P_n, Q_n be probability measures on $(\Omega_n, \mathcal{F}_n)$ such that $Q_n \ll P_n$. The (unique up to P_n-null sets) nonnegative measurable function X_n such that

$$Q_n(E) = \int_E X_n \, dP_n$$

for all $E \in \mathcal{F}_n$ is called the Radon-Nikodym derivative of Q_n with respect to P_n and is denoted by $\frac{dQ_n}{dP_n}$.

As long as the measures P_n, Q_n are σ-finite, which is the case for probability measures, such a function exists (see [Ran02]).

Proposition 2.2.24. *Suppose that we for each $n \in \mathbb{N}$ have a probability space $(\Omega_n, \mathcal{F}_n)$ and that we for each n have two probability measures P_n, Q_n on Ω_n such that $Q_n \ll P_n$. Let $\frac{dQ_n}{dP_n}$ denote the Radon-Nikodym derivative of Q_n with respect to P_n. Then the ultralimit $\bar{Q}$ of $\{Q_n\}_{n\in\mathbb{N}}$ is absolutely continuous with respect to the ultralimit $\bar{P}$ of $\{P_n\}_{n\in\mathbb{N}}$ if and only if $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable.*

Proof. Suppose $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$ integrable. For all $[A_n]\in\mathcal{A}$,

$$Q([A_n]) = \lim_{\mathcal{U}} Q_n(A_n) = \lim_{\mathcal{U}} \int_{A_n} \frac{dQ_n}{dP_n} dP_n.$$

Since $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable,

$$Q([A_n]) = \int_{[A_n]} \frac{dQ}{dP}\, d\bar{P},$$

where $\frac{dQ}{dP} = [\frac{dQ_n}{dP_n}]$. By Caratheodory's extension theorem (by finiteness of Q), the extension of the measure Q on $\mathcal{A}$ to $\mathcal{F}$ is unique, hence

$$\bar{Q}(E) = \int_E \frac{dQ}{dP}\, d\bar{P}$$

for all $E\in\mathcal{F}$. Thus if $\bar{P}(E)=0$, then $\bar{Q}(E)=0$. So $\bar{Q}\ll\bar{P}$.

Conversely, suppose $\bar{Q}\ll\bar{P}$. Then $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$ integrable: Suppose $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is not $\mathcal{A}$-integrable. Then there exists $[A_n]\in\mathcal{A}$ such that $\bar{P}([A_n]) = 0$, but

$$\bar{Q}([A_n]) = \lim_{\mathcal{U}} \int_{A_n} \frac{dQ_n}{dP_n}\, dP_n \neq 0,$$

a contradiction. So $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable. ∎

It follows from Proposition 2.2.24 that is we for each $n\in\mathbb{N}$ have two equivalent measures P_n, Q_n on Ω_n, then the ultralimits $\bar{P},\bar{Q}$ are equivalent if and only if both $\{\frac{dP_n}{dQ_n}\}_{n\in\mathbb{N}}$ and $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ are $\mathcal{A}$-integrable.

We will end this section with a result about conditional expectation.

Proposition 2.2.25. *Suppose that we for each $n\in\mathbb{N}$ have a probability space $(\Omega_n,\mathcal{F}_n,P_n)$ and that we have used ultraproducts to construct a limit space $(\Omega,\mathcal{F},\bar{P})$. Suppose that we for each n have a sub-sigma-algebra $\mathcal{G}_n\subset\mathcal{F}_n$ and let $\mathcal{B}$ be the algebra consisting of all ultraproducts $[B_n]$ such that $B_n\subset\mathcal{G}_n$ for each $n\in\mathbb{N}$. Let $\mathcal{G}$ be the completion of the σ-algebra on Ω generated by $\mathcal{B}$. Suppose that we have an $\mathcal{A}$-integrable function $X=[X_n]$. Then*

$$E[X|\mathcal{G}]([\omega_n]) = \lim_{\mathcal{U}} E_n[X_n|\mathcal{G}_n](\omega_n)$$

for almost all $[\omega_n]\in\Omega$.

Proof. For each $n\in\mathbb{N}$, let $Y_n = E_n[X_n|\mathcal{G}_n]$ and let $Y:\Omega\to\overline{\mathbb{R}}$ be defined by $Y(\omega) = \lim_{\mathcal{U}} Y_n(\omega_n)$. We first show that Y is $\mathcal{G}$ measurable. Let $g_{m,n}:\Omega_n\to\mathbb{R}$ be defined by

$$g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{A^k_{m,n}}(\omega_n) + 2^m 1_{B_{m,n}}(\omega_n),$$

where $A^k_{m,n} = Y_n^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$ and $B_{m,n} = Y_n^{-1}([2^m, \infty])$. Let $g_m : \Omega \to \overline{\mathbb{R}}$ be defined by

$$g_m([\omega_n]) = \lim_{\mathcal{U}} g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{[A^k_{m,n}]}([\omega_n]) + 2^m 1_{[B_{m,n}]}([\omega_n]).$$

Then for all $\omega \in \Omega$, $\lim_{m\to\infty} g_m(\omega) = Y(\omega)$ Since Y is the pointwise limit of a sequence of $\mathcal{G}$-measurable functions, Y is $\mathcal{G}$-measurable.

We now show that $\int_G Y\ d\bar{P} = \int_G X\ d\bar{P}$ for all $G \in \mathcal{G}$. Let $G \in \mathcal{G}$. Then, by Corollary 2.2.12, there is $[B_n] \in \mathcal{B}$ such that $\bar{P}([B_n] \triangle G) = 0$. Let $\epsilon > 0$ and let

$$F^Y_{\frac{\epsilon}{2}} = \{n \in \mathbb{N} \mid \int_{[B_n]} Y\ d\bar{P} - \int_{B_n} Y_n\ dP_n| < \frac{\epsilon}{2}\}$$

and

$$F^X_{\frac{\epsilon}{2}} = \{n \in \mathbb{N} \mid \int_{[B_n]} X\ d\bar{P} - \int_{B_n} X_n\ dP_n| < \frac{\epsilon}{2}\},$$

which both are in $\mathcal{U}$ since $\{X_n\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable. Indeed, since $\{X_n\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable it follows by Jensen's inequality for conditional expectation (the absolute value function is convex - see [Doo12]) that

$$E[|E[X_n \mid \mathcal{G}_n]|] \leq E[E[|X_n| \mid \mathcal{G}_n]] = E[|X_n|]$$

and for any $B_n \in \mathcal{G}_n$ we have

$$\int_{B_n} |E_n[X_n \mid \mathcal{G}_n]|\ dP_n \leq \int_{B_n} E_n[|X_n| \mid \mathcal{G}_n]\ dP_n = \int_{B_n} |X_n|\ dP_n.$$

Pick $n \in F^X_{\frac{\epsilon}{2}} \cap F^Y_{\frac{\epsilon}{2}}$. Then we get that

$$\begin{aligned} &\left| \int_{[B_n]} Y\ d\bar{P} - \int_{[B_n]} X\ d\bar{P} \right| \\ &\leq \left| \int_{[B_n]} X\ d\bar{P} - \int_{B_n} X_n\ dP_n \right| + \left| \int_{[B_n]} Y\ d\bar{P} - \int_{B_n} Y_n\ dP_n \right| < \epsilon. \end{aligned}$$

So $\int_{[B_n]} Y\ d\bar{P} = \int_{[B_n]} X\ d\bar{P}$ and thus $\int_G Y\ d\bar{P} = \int_G X\ d\bar{P}$. ■

CHAPTER 3

Construction of the skeleton approximations

3.1 Skeleton processes

We assume that $I \subseteq [0, \infty)$ is a closed interval that can be either $[0, \infty)$ or a finite subinterval $[a, b]$. An *approximation sequence* $\mathcal{I}$ to I is a sequence $\{I_n\}_{n\in\mathbb{N}}$ of discrete subsets of I such that there for each $a \in I$ is a sequence $\{a_n\}_{n\in\mathbb{N}}$ where $a_n \in I_n$ and $a = \lim_{\mathcal{U}} a_n$.

We let $\mathcal{T}$ be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $a_n \in I_n$ for all $n \in \mathbb{N}$, and we let $T = \mathcal{T}/\mathcal{U}$. If $[a_n], [b_n] \in T$, we write $[a_n] \approx [b_n]$ if $\lim_{\mathcal{U}} a_n = \lim_{\mathcal{U}} b_n$.

Definition 3.1.1. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions $f_n : I_n \to \mathbb{R}$, we can define a function $f : T \to \overline{\mathbb{R}}$ by

$$f([a_n]) = \lim_{\mathcal{U}} f_n(a_n).$$

Definition 3.1.2. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions $f_n : I_n \to \mathbb{R}$ and let $f : T \to \overline{\mathbb{R}}$ be as in Definition 3.1.1. If $[a_n] \approx [b_n]$ implies that $f([a_n]) = f([b_n])$, we can define a function $\bar{f} : I \to \overline{\mathbb{R}}$ by

$$\bar{f}(s) = \lim_{\mathcal{U}} f_n(a_n)$$

for all sequences $\{a_n\}_{\mathbb{N}} \in \mathcal{T}$ such that $\lim_{\mathcal{U}} a_n = s$.

Proposition 3.1.3. *The function $\bar{f} : I \to \overline{\mathbb{R}}$ in Definition 3.1.2 is continuous.*

Proof. Suppose that $\{s_k\}_{k\in\mathbb{N}}$ is a sequence of numbers in $\mathbb{R}$ such that $\lim_{k\to\infty} s_k = s \in \mathbb{R}$. We show that $\limsup_{k\to\infty} \bar{f}(s_k) = \bar{f}(s)$. The proof that $\liminf_{k\to\infty} \bar{f}(s_k) = \bar{f}(s)$ is similar.

Let $r = \limsup_{k\to\infty} f(s_k)$ and suppose $r \in \mathbb{R}$. For each $k \in \mathbb{N}$, there is $s_k^* \in \{s_m \mid m \geq k\}$ such that

$$|\bar{f}(s_k^*) - \sup_{m\geq k} \bar{f}(s_m)| < \frac{1}{k}.$$

Furthermore, for each $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$, there is an $s^*_{k,n} \in I_n$ such that $s^*_k = \lim_{\mathcal{U}} s^*_{k,n}$ and thus $\bar{f}(s^*_k) = f([s^*_{k,n}])$. For each $n \in \mathbb{N}$, define a_n as follows: if

$$\left\{k \in \{1,..,n\} \mid |f_n(s^*_{k,n}) - \bar{f}(s^*_k)| < \frac{1}{k}\right\} \bigcap \left\{k \in \{1,..,n\} \mid |s^*_{k,n} - s^*_k| < \frac{1}{k}\right\}$$

is non-empty, let $a_n = s^*_{k,n}$, where

$$k = \sup \left\{ m \in \{1,..,n\} \mid |f_n(s^*_{j,n}) - \bar{f}(s^*_j)| < \frac{1}{j} \right.$$
$$\left. \text{and } |s^*_{j,n} - s^*_j| < \frac{1}{j} \text{ for all } j \in \{1,...,m\} \right\}.$$

Otherwise, let $a_n = s^*_{1,n}$. We will show that $\lim_{\mathcal{U}} f_n(a_n) = r$ and that $\lim_{\mathcal{U}} a_n = s$, so that $\limsup_{k\to\infty} \bar{f}(s_k) = f([a_n]) = \bar{f}(s)$.

Let $\epsilon > 0$. Pick $N_1 \in \mathbb{N}$ such that $|\sup_{m \geq k} \bar{f}(s_m) - r| < \frac{\epsilon}{3}$ for all $k \geq N_1$. Pick $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\epsilon}{3}$. Pick $N_3 \in \mathbb{N}$ such that $|s_k - s| < \frac{\epsilon}{2}$ for all $k \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$ and let

$$F = \{1,..,N-1\}^c \bigcap_{k=1}^{N} \left\{ n \in \mathbb{N} \mid |f_n(s^*_{k,n}) - \bar{f}(s^*_k)| < \frac{1}{k} \right\}$$
$$\bigcap \left\{ n \in \mathbb{N} \mid |s^*_{k,n} - s^*_k| < \frac{1}{k} \right\},$$

which is in $\mathcal{U}$. If $n \in F$, then $a_n = s^*_{k,n}$ for some $k \in \mathbb{N}$, $N \leq k \leq n$, so that

$$\begin{aligned} |f_n(a_n) - r| &= |f_n(s^*_{k,n}) - r| \\ &\leq |f_n(s^*_{k,n}) - \bar{f}(s^*_k)| + |\bar{f}(s^*_k) - \sup_{m \geq k} \bar{f}(s_m)| + |\sup_{m\geq k} \bar{f}(s_m) - r| \\ &< \epsilon. \end{aligned}$$

Hence $\{n \in \mathbb{N} \mid |f_n(a_n) - r| < \epsilon\} \supseteq F$. Furthermore, if $n \in F$, for some $k \geq N$ we have $a_n = s^*_{k,n}$ so that

$$|a_n - s| \leq |a_n - s^*_k| + |s^*_k - s| < \epsilon,$$

and thus $\{n \in \mathbb{N} \mid |a_n - s| < \epsilon\} \supseteq F$.

Now suppose that $r = \infty$. For each $k \in \mathbb{N}$ there is $s^*_k \in \{s_m \mid m \geq k\}$ such that $f(s^*_k) \geq k$. Define a_n as before and define F as before, only this time omitting N_1. We have that

$$\bar{f}(s) = f([a_n]) = \lim_{\mathcal{U}} f_n(a_n) = \lim_{\mathcal{U}} f_n(s^*_{k,n}),$$

for some $k \in \{1,..,n\}$. Since

$$f_n(s^*_{k,n}) \geq f(s^*_{k,n}) - \frac{1}{k} \geq k - \frac{1}{k}$$

on a set $F' \in \mathcal{U}$ and we have that $k \to \infty$ when $n \to \infty$, $\bar{f}(s) = r$. The proof of the case $r = -\infty$ is similar. ∎

Example 3.1.4. For each $n \in \mathbb{N}$, let $I_n = \{\frac{k}{n} \mid k = 0, 1, 2, 3, ...\}$, and let $f_n : I_n \to \mathbb{R}$ be defined by

$$f_n\left(\frac{k}{n}\right) = \left(1 + \frac{k}{n^2}\right)^n.$$

Then $\{I_n\}_{n\in\mathbb{N}}$ is an approximation sequence for $[0, \infty)$ and $\bar{f}$ is defined by $\bar{f}(x) = e^x$.

We will from now on assume that $I = [0, T]$ is a bounded interval and that I_n is finite for each $n \in \mathbb{N}$.

Definition 3.1.5. Suppose that we for each n have a stochastic process $X_n : I_n \times \Omega_n \to \mathbb{R}$. Then we define a stochastic process $X : T \times \Omega \to \overline{\mathbb{R}}$ by

$$X([a_n], [\omega_n]) = \lim_{\mathcal{U}} X_n(a_n, \omega_n).$$

Definition 3.1.6. Suppose that we for each n have a stochastic process $X_n : I_n \times \Omega_n \to \mathbb{R}$ and let $X : T \times \Omega \to \overline{\mathbb{R}}$ be defined as in Definition 3.1.5. Suppose that we for almost all $\omega \in \Omega$ have that $[a_n] \approx [b_n]$ implies that $X([a_n], \omega) = X([b_n], \omega)$. Then we can define a stochastic process $\bar{X} : I \times \Omega \to \overline{\mathbb{R}}$ such that for almost all $[\omega_n] \in \Omega$,

$$\bar{X}(s, [\omega_n]) = \lim_{\mathcal{U}} X_n(a_n, \omega_n)$$

for all sequences $\{a_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ with $\lim_{\mathcal{U}} a_n = s$ for all $s \in I$.

Notice that it follows by Proposition 3.1.3 that if we for each $n \in \mathbb{N}$ have a stochastic process $X_n : I_n \times \Omega_n \to \mathbb{R}$ and the ultralimit $\bar{X} : I \times \Omega \to \overline{\mathbb{R}}$ to $\{X_n\}_{n\in\mathbb{N}}$ exists for almost all $\omega \in \Omega$, then $\bar{X}$ is continuous for almost all $\omega \in \Omega$.

Next we want to define a filtration on Ω.

Definition 3.1.7. Suppose that we for each $n \in \mathbb{N}$ have an approximation sequence I_n to I and a filtered probability space $(\Omega_n, \{\mathcal{F}^n_{t_n}\}_{t_n \in I_n}, \mathcal{F}_n, P_n)$. Suppose we have constructed Ω as previously. For each $t \in I$, define a sequence $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$, where each $t_n = \inf\{r_n \in I_n \mid r_n \geq t\}$. Let $\mathcal{A}_t$ denote the algebra consisting of ultraproducts $[A_n]$, where each $A_n \in \mathcal{F}_{t_n}$. We define the filtration $\{\mathcal{F}_t\}_{t\in I}$ on Ω by letting $\mathcal{F}_t$ be the completion of the σ-algebra generated by $\mathcal{A}_t$. We let $\mathcal{F}$ be the completion of the σ-algebra generated by the algebra $\mathcal{A}$ consisting of ultraproducts $[A_n]$, where each $A_n \in \mathcal{F}_n$. Thus we get a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in I}, \mathcal{F}, \bar{P})$. We let $\{\mathcal{F}_{t+}\}_{t\in I}$ be the augmented right continuous filtration of $\{\mathcal{F}\}_{t\in I}$ (i.e. $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$).

Proposition 3.1.8. *Suppose that we for each $n \in \mathbb{N}$ have an approximation sequence I_n to I and a filtered probability space $(\Omega_n, \{\mathcal{F}^n_{t_n}\}_{t_n \in I_n}, \mathcal{F}_n, P_n)$. Suppose furthermore that we for each $n \in \mathbb{N}$ have a stochastic process $X_n : I_n \times \Omega_n \to \mathbb{R}$ that is adapted to the filtration $\{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}$. If the ultralimit $\bar{X} : I \times \Omega \to \overline{\mathbb{R}}$ to $\{X_n\}_{n\in\mathbb{N}}$ exists (and is continuous) for almost all $\omega \in \Omega$, then $\bar{X}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$.*

Proof. Since $\bar{X} = \bar{X}^+ - \bar{X}^-$, it suffices to assume that $\bar{X} \geq 0$. We have to show that for each $t \in I$, $\bar{X}_t$ is $\mathcal{F}_t$-measurable. For each $n \in \mathbb{N}$, let $t_n = \inf\{r_n \in I_n \mid r_n \geq t\}$. For each $n \in \mathbb{N}$ and $m \in \mathbb{N}$, let $g_{m,n} : \Omega_n \to \mathbb{R}$ be defined by

$$g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{A^k_{m,n}}(\omega_n) + 2^m 1_{B_{m,n}}(\omega_n),$$

where $A^k_{m,n} = (X_n(t_n))^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$ and $B_{m,n} = (X_n(t_n))^{-1}([2^m, \infty])$. Then each $A^k_{m,n} \in \mathcal{F}^n_{t_n}$ and $B_{m,n} \in \mathcal{F}^n_{t_n}$. Let $g_m : \Omega \to \overline{\mathbb{R}}$ be defined by

$$g_m([\omega_n]) = \lim_{\mathcal{U}} g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2m}-1} \frac{k}{2^m} 1_{[A^k_{m,n}]}([\omega_n]) + 2^m 1_{[B_{m,n}]}([\omega_n]).$$

Then for all $\omega \in \Omega$, $\lim_{m\to\infty} g_m(\omega) = \bar{X}(t, \omega)$. Since $\bar{X}_t$ is the pointwise limit of a sequence of $\mathcal{F}_t$-measurable variables, $\bar{X}_t$ is $\mathcal{F}_t$-measurable. ■

Definition 3.1.9 (Stopping times in continuous time)**.** In continuous time, we say that $\tau : \Omega \to I \cup \{\infty\}$ is a (strong) *stopping time* for $\{\mathcal{F}_t\}_{t\in I}$ if

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t \tag{3.1}$$

for all $t \in I$. If (3.1) holds with $<$ instead of $\leq$, τ is said to be a *weak* stopping time.

A (strong) stopping time is a weak stopping time, but the converse is not necessarily true. The following result can be found with proof in [Wei13].

Proposition 3.1.10. *$\tau : \Omega \to I \cup \{\infty\}$ is a weak stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if and only if τ is a strong stopping time with respect to the right continuous filtration $\{\mathcal{F}_{t+}\}_{t\in I}$.*

Definition 3.1.11 (Stopping time in discrete time)**.** In discrete time, we say that $\tau_n : \Omega_n \to I_n \cup \{\infty\}$ is a stopping time if

$$\{\omega_n \in \Omega_n \mid \tau_n(\omega_n) \leq t_n\} \in \mathcal{F}^n_{t_n}$$

for each $t_n \in I_n$.

Proposition 3.1.12. *Suppose that we for each $n \in \mathbb{N}$ have a probability space $(\Omega, \{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}, \mathcal{F}_n, P_n)$ and a stopping time $\tau_n : \Omega_n \to I_n \cap \{\infty\}$ with respect to the filtration $\{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}$. Then $\tau = [\tau_n]$ is a weak stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$.*

Proof. By Proposition 3.1.10 it suffices to prove that τ is a strong stopping time with respect to $\{\mathcal{F}_{t+}\}_{t\in I}$. Let $t \in I$. For each $n \in \mathbb{N}$ and each $m \in \mathbb{N}$, define $g_{m,n} : \Omega_n \to \mathbb{N}$ by

$$g_{m,n}(\omega_n) = \sum_{k=1}^{2^{2^m}-1} 1_{A^k_{m,n}}(\omega_n) + 1_{B_{m,n}}(\omega_n),$$

where $A^k_{m,n} = (\tau_n)^{-1}([\frac{k}{2^m}, \frac{k+1}{2^m}))$ and $B_{m,n} = (\tau_n)^{-1}([2^m, \infty])$. Let $g_m = [g_{m,n}]$. We shall now prove that for each $M \in \mathbb{N}$,

$$\{\tau \leq t\} = \bigcap_{m=M}^{\infty} g_m^{-1}([0,t]).$$

First we prove that $\{\tau \leq t\} \subseteq \bigcap_{m=M}^{\infty} g_m^{-1}([0,t])$. Suppose $\tau([\omega_n]) \leq t$. For each $n \in \mathbb{N}$, for each $m \in \mathbb{N}$, $g_{m,n} \leq \tau_n$. Thus $g_m([\omega_n]) \leq \tau([\omega_n]) \leq t$ for all $m \in \mathbb{N}$, in particular for all $m \geq M$.

The inclusion $\{\tau \leq t\} \supseteq \bigcap_{m=M}^{\infty} g_m^{-1}([0,t])$ follows from the defintion of the g_m's. Suppose $g_m(\omega) \leq t$ for all $m \geq M$. Then $\tau(\omega) = \lim_{m\to\infty} g_m(\omega) \leq t$.

Now choose $M \in \mathbb{N}$ such that $2^M > t$. For $m \geq M$, let $K_m = \sup\{k \in \mathbb{N} \mid \frac{k}{2^m} < t\}$. Then

$$g_m^{-1}([0,t]) = \bigcup_{k=0}^{K_m} [A^k_{m,n}] = [\bigcup_{k=0}^{K_m} A^k_{m,n}].$$

For each $n \in \mathbb{N}$, let $t_{m,n} = \inf\{r_n \in I_n \mid r_n \geq t + \frac{1}{2^m}\}$. For each $k \leq K_m$, we have that $A^k_{m,n} \in \mathcal{F}^n_{t_{m,n}}$ for each $n \in \mathbb{N}$ so that $[A^k_{m,n}] \in \mathcal{F}_{t+\frac{1}{2^m}}$. So for $M \in \mathbb{N}$ high enough we have $\{\tau \leq t\} \in \mathcal{F}_{t+\frac{1}{2^M}}$, Since this holds for each $M \in \mathbb{N}$ high enough, $\{\tau \leq t\} \in \mathcal{F}_{t+}$. ■

Proposition 3.1.13. *Suppose we have an interval I with an approximation sequence $\{I_n\}_{n\in\mathbb{N}}$. Suppose for each $n \in \mathbb{N}$ we have a filtered probability space $(\Omega_n, \{\mathcal{F}^n_{t_n}\}_{t_n \in I_n}, \mathcal{F}, P_n)$ and a process $X_n : I_n \times \Omega_n \to \mathbb{R}$ that is adapted to $\{\mathcal{F}^n_{t_n}\}_{t_n\in I_n}$. Suppose furthermore that the ultralimit $\bar{X} : I \times \Omega \to \overline{\mathbb{R}}$ to $\{X_n\}_{n\in\mathbb{N}}$ exists (and is continuous) for almost all $\omega \in \Omega$. Let $s, t \in I$ with $s < t$. For each $n \in \mathbb{N}$, let $t_n = \inf\{r_n \in I_n \mid r_n \geq t\}$ and $s_n = \inf\{r_n \in I_n \mid r_n \geq s\}$. If $\{X_n(t_n)\}_{n\in\mathbb{N}}$ is $\mathcal{A}_t$-integrable, then*

$$E[\bar{X}(t)|\mathcal{F}_s]([\omega_n]) = \lim_{\mathcal{U}} E_n[X_n(t_n)|\mathcal{F}^n_{s_n}](\omega_n)$$

for almost all $[\omega_n] \in \Omega$

Proof. This follows from Definition 3.1.7 and Proposition 2.2.25. ■

It follows by Proposition 3.1.13 that if we have a sequence of martingales (or supermartingales or submartingales, respectively) $X_n : I_n \times \Omega_n \to \mathbb{R}$ with respect to the filtrations $\{\mathcal{F}^n_{t_n}\}_{n\in\mathbb{N}}$, if the ultralimit $\bar{X} : I \times \Omega \to \mathbb{R}$ exists for almost all $[\omega_n] \in \Omega$ and $\{X_n(t_n)\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for any $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$, then $\bar{X}$ is a martingale (or supermartingale or submartingale, respectively) with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$. But if we do not have $\mathcal{A}$-integrability, $\bar{X}$ need not be a martingale. We do however have the following results. First we will give the following definition.

Definition 3.1.14 (Local martingale)**.** An $\mathcal{F}_t$-adapted stochastic process $\bar{X} : I \times \Omega \to \mathbb{R}$ is called a *local martingale* with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if there exists an increasing sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ (with respect to $\{\mathcal{F}_t\}_{t\in I}$) such that $\tau_k \to \infty$ for almost all $\omega \in \Omega$ as $k \to \infty$ and $\bar{X}(\cdot \wedge \tau_k)$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$ for each $k \in \mathbb{N}$.

Proposition 3.1.15. *Suppose that we for each $n \in \mathbb{N}$ have a martingale $X_n : I_n \times \Omega_n \to \mathbb{R}$ and suppose that the $\mathcal{U}$-limit $\bar{X} : I \times \Omega \to \mathbb{R}$ exists for all points $t \in I$ for almost all $\omega \in \Omega$. Suppose furthermore that $E[|\bar{X}(t)|] < \infty$ for all $t \in I$. For each $k \in \mathbb{N}$ and each $n \in \mathbb{N}$ define the stopping times $\tau_{k,n} : \Omega_n \to I_n \cup \{\infty\}$ by $\tau_{k,n} = \inf\{t_n \in I_n \mid |X_n| \geq k\}$. If*

$$\lim_{\mathcal{U}} E_n[\sup_{\substack{t_n \in I_n \\ 0 < t_n \leq \tau_{k,n}}} (X_n(t_n) - X_n(t_n - \Delta t_n))^2] < \infty$$

for each $k \in \mathbb{N}$, and if $\lim_{\mathcal{U}} E_n[X_n(0)] < \infty$, then $\bar{X}$ is a local martingale with respect to the filtration $\{\mathcal{F}_{t+}\}_{t \in I}$.

Proof. Let $\tau_k = [\tau_{k,n}]$. By Proposition 3.1.12, τ_k is a weak stopping time with respect to the filtration $\{\mathcal{F}_t\}_{t \in I}$ and we have that $\bar{X}(\cdot \wedge \tau_k)$ is the ultralimit to $\{X_n(\cdot \wedge \tau_{k,n})\}_{n \in \mathbb{N}}$ for almost all $[\omega_n] \in \Omega$. Then for each $k \in \mathbb{N}$, for any $\{t_n\}_{n \in \mathbb{N}} \in \mathcal{T}$,

$$\begin{aligned} &E_n[\sup_{t_n \in I_n} X_n(t_n \wedge \tau_{k,n})^2] \\ &\leq E_n[X_n(0)] + 2k^2 + 2E_n[\sup_{\substack{t_n \in I_n \\ 0 < t_n \leq \tau_{k,n}}} (X_n(t_n) - X_n(t_n - \Delta t_n))^2], \end{aligned}$$

hence $\{|X_n(t \wedge \tau_{k,n})|\}_{n \in \mathbb{N}}$ is $\mathcal{A}$-integrable by Proposition 2.2.20. By the same argument as in Proposition 3.1.13, $\bar{X}(\cdot \wedge \tau_k)$ is a martingale with respect to the filtration $\{\mathcal{F}_{t+}\}_{t \in I}$. Indeed, given $s, t \in I$ with $s < t$, for any $k \in \mathbb{N}$ high enough we have

$$\int_G \bar{X}(t \wedge \tau_k)\, d\bar{P} = \int_G \bar{X}((s + \frac{1}{k}) \wedge \tau_k)\, d\bar{P}$$

for all $G \in \mathcal{F}_{s+}$. Since $\bar{X}(s \wedge \tau_k)$ is dominated by $\sup_{t \in I} \bar{X}(t \wedge \tau_k)$ for all $s \in I$ and since

$$\sup_{t \in I} \bar{X}(t \wedge \tau_k) = \lim_{\mathcal{U}} \sup_{t_n \in I_n} X_n(t_n \wedge \tau_{k,n}),$$

by Lemma 2.2.16 we have

$$E[\sup_{t \in I} \bar{X}(t \wedge \tau_k)] \leq \lim_{\mathcal{U}} E_n[\sup_{t_n \in I_n} X_n(t_n \wedge \tau_{k,n})] < \infty.$$

So by dominated convergence theorem,

$$\int_G \bar{X}(t \wedge \tau_k)\, d\bar{P} = \int_G \bar{X}(s \wedge \tau_k)\, d\bar{P}.$$

Now we have that $\{\tau_k\}_{k \in \mathbb{N}}$ is an increasing sequence of stopping times (with respect to the filtration $\{\mathcal{F}_{t+}\}_{t \in I}$) such that $\tau_k \to \infty$ for almost all $\omega \in \Omega$ as $k \to \infty$. So $\bar{X}$ is a local martingale with respect to the filtration $\{\mathcal{F}_{t+}\}_{t \in I}$. ∎

Proposition 3.1.16. *Suppose that we for each $n \in \mathbb{N}$ have a martingale $X_n : I_n \times \Omega_n \to \mathbb{R}$ and suppose that the $\mathcal{U}$-limit $\bar{X} : I \times \Omega \to \mathbb{R}$ exists for all points $t \in I$ for almost all $\omega \in \Omega$. Suppose furthermore that $E[|\bar{X}(t)|] < \infty$*

for all $t \in I$. For each $k \in \mathbb{N}$ and each $n \in \mathbb{N}$ define the stopping times $\tau_{k,n} : \Omega_n \to I_n \cup \{\infty\}$ by $\tau_{k,n} = \inf\{t_n \in I_n \mid |X_n| \geq k\}$. If

$$\lim_{\mathcal{U}} E_n[\sup_{\substack{t_n \in I_n \\ 0 < t_n \leq \tau_{k,n}}} (X_n(t_n) - X_n(t_n - \Delta t_n))^2] < \infty$$

for each $k \in \mathbb{N}$, if $\lim_{\mathcal{U}} E_n[X_n(0)] < \infty$ and $\lim_{\mathcal{U}} E_n[|X_n(T)| \ln^+ |X_n(T)|] < \infty$, then $\bar{X}$ is a martingale with respect to the filtration $\{\mathcal{F}_{t+}\}_{t \in I}$.

Proof. By Proposition 3.1.15, X is a local martingale. Since $\sup_{t \in I} |\bar{X}(t)| = \lim_{\mathcal{U}} \sup_{t_n \in I_n} |X_n(t_n)|$ and since by Doobs martingale inequality (see [Doo53]) we have

$$E_n[\sup_{t_n \in I_n} |X_n(t_n)|] \leq \frac{e}{e-1}(1 + E_n[|X_n(T)| \ln^+ |X_n(T)|])$$

for each $n \in \mathbb{N}$, it follows by Lemma 2.2.16 that

$$E[\sup_{t \in I} |\bar{X}(t)|] \leq \lim_{\mathcal{U}} E_n[\sup_{t_n \in I_n} |X_n(t_n)|] < \infty.$$

Since we have $|\bar{X}(s \wedge \tau_k)| \leq \sup_{t \in I} |\bar{X}(t)|$ for any $s \in I$ for all τ_k, and since $\tau_k \to \infty$, by dominated convergence theorem for conditional expectation, since $\bar{X}$ is a local martingale, $\bar{X}$ is a martingale. Indeed for any $s, t \in I$ with $s < t$ we have

$$E_n[\bar{X}(t) \mid \mathcal{F}_s] = E_n[\lim_{k \to \infty} \bar{X}(t \wedge \tau_k) \mid \mathcal{F}_s] = \lim_{k \to \infty} E_n[\bar{X}(t \wedge \tau_k) \mid \mathcal{F}_s] = \bar{X}(s).$$

■

It follows by Fatou's lemma for conditional expectation (see [Doo12]) that if $\bar{X}$ is a local martingale (with respect to $\{\mathcal{F}_{t+}\}_{n \in \mathbb{N}}$) and $\bar{X} \geq 0$ for almost all $\omega \in \Omega$, then $\bar{X}$ is a supermartingale with respect to $\{\mathcal{F}_{t+}\}_{n \in \mathbb{N}}$. Similarly, if $\bar{X} \leq 0$ for almost all $\omega \in \Omega$, then $\bar{X}$ is a submartingale with respect to $\{\mathcal{F}_{t+}\}_{n \in \mathbb{N}}$. A related result is given below.

Proposition 3.1.17. *Suppose that we for each $n \in \mathbb{N}$ have a supermartingale $X_n : I_n \times \Omega_n \to \mathbb{R}$ and suppose that the $\mathcal{U}$-limit $\bar{X} : I \times \Omega \to \mathbb{R}$ exists for all points $t \in I$ for almost all $\omega \in \Omega$. Suppose furthermore that $E[|\bar{X}(t)|] < \infty$ for all $t \in I$. If there is a constant $K \in \mathbb{N}$ such that $\bar{X} \geq -K$, then $\bar{X}$ is a supermartingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in I}$. Similarly, suppose that we for each $n \in \mathbb{N}$ have a submartingale $X_n : I_n \times \Omega_n \to \mathbb{R}$ and suppose that the $\mathcal{U}$-limit $\bar{X} : I \times \Omega \to \mathbb{R}$ exists for all points $t \in I$ for almost all $\omega \in \Omega$. Suppose furthermore that $E[|\bar{X}(t)|] < \infty$ for all $t \in I$. If there is a constant $K \in \mathbb{N}$ such that $\bar{X} \leq K$, then $\bar{X}$ is a submartingale with respect to the filtration $\{\mathcal{F}_t\}_{t \in I}$.*

Proof. We first show the supermartingale argument. Suppose $n \in \mathbb{N}$, X_n is a supermartingale and suppose there is a constant $K \in \mathbb{N}$ such that $\bar{X} \geq -K$. For each $k \in \mathbb{N}$, by Jensen's inequality for conditional expectation (since the minimum function is concave - see [Doo12]), we have

$$E_n[X_n(t_n) \wedge k | \mathcal{F}^n_{s_n}] \leq X_n(s_n) \wedge k$$

for each $n \in \mathbb{N}$ and each $s_n, t_n \in I_n$ such that $s_n < t_n$. For each $k \in \mathbb{N}$ there is an $F \in \mathcal{U}$ such that $|X_n(t_n) \wedge k|$ is bounded by $\max\{K+1, k\}$ for all $n \in F$, hence $\{X_n(t_n) \wedge k\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for any $\{t_n\}_{n\in\mathbb{N}}$. So by Proposition 3.1.13, $\bar{X} \wedge k$ is a supermartingale for each $k \in \mathbb{N}$. By dominated convergence theorem for conditional expectation (see [Doo12]), since $\bar{X} \wedge k$ is dominated by $\bar{X}$ for any $k \in \mathbb{N}$, $\bar{X}$ is a supermartingale. Indeed, for any $s, t \in I$ with $s < t$ we have

$$\begin{aligned} E_n[\bar{X}(t) \mid \mathcal{F}_s] &= E_n[\lim_{k\to\infty} \bar{X}(t) \wedge k \mid \mathcal{F}_s] \\ &= \lim_{k\to\infty} E_n[\bar{X}(t) \wedge k \mid \mathcal{F}_s] \\ &\leq \lim_{k\to\infty} \bar{X}(s) \wedge k = \bar{X}(s). \end{aligned}$$

Now suppose that for each $n \in \mathbb{N}$, X_n is a submartingale and suppose there is a constant $K \in \mathbb{N}$ such that $\bar{X} \leq K$. Using the argument above for $-\bar{X}$ shows that $\bar{X}$ is a submartingale.

■

We will next construct skeleton processes (namely random walks) that converge to Brownian motion on a bounded interval with respect to an ultrafilter.

Theorem 3.1.18. *Let $I_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ and let Ω_n consist of all functions $\omega_n : I_n \to \{-1, 1\}$. Let P_n be the probability measure on Ω_n, which gives each ω_n the same weight. For each $n \in \mathbb{N}$ we define a random walk $X_n : I_n \times \Omega_n \to \mathbb{R}$ by setting $X_n(0, \omega) = 0$ and*

$$X_n(\frac{k_n}{n}, \omega_n) = \sum_{i=0}^{k_n - 1} \frac{\omega_n(\frac{i}{n})}{\sqrt{n}}$$

for $k_n > 0$. Then $\bar{X}$ exists and is a Brownian motion under $\bar{P}$ with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$.

Proof. We need to show that $\bar{X} : I \times \Omega \to \mathbb{R}$ as defined in Definition 3.1.6 exists so that $\bar{X}_t : \Omega \to \overline{\mathbb{R}}$ defined by $\bar{X}_t(\omega) = \bar{X}(t, \omega)$ is well-defined and that the following properties hold:

(i) $\bar{X}_0 = 0$.

(ii) The increment $\bar{X}_t - \bar{X}_s$ is normally distributed with mean 0 and variance $t - s$ for $t \geq s$.

(iii) The process $\{\bar{X}_t\}_{t\in[0,1]}$ has stationary increments, where each increment $\bar{X}_t - \bar{X}_s$ (where $t > s$) is independent of $\mathcal{F}_s$.

(iv) With probability 1, the function $t \to \bar{X}_t$ is continuous.

We will show all but the last part of (iii) in three steps, where we, given two sequences $\{\frac{k_{1,n}}{n}\}_{n\in\mathbb{N}}, \{\frac{k_{2,n}}{n}\}_{n\in\mathbb{N}} \in \mathcal{T}$ with $\lim_{\mathcal{U}} \frac{k_{1,n}}{n} \leq \lim_{\mathcal{U}} \frac{k_{2,n}}{n}$, determine the distribution of the variables $X([\frac{k_{1,n}}{n}])$ and $X([\frac{k_{2,n}}{n}]) - X([\frac{k_{1,n}}{n}])$.

Step 1: Let $t \in I$ and suppose $\frac{k_n}{n} \in I_n$ for each $n \in \mathbb{N}$ and $\lim_{\mathcal{U}} \frac{k_n}{n} = t$. Suppose that $\lim_{\mathcal{U}} k_n = \infty$. Pick a subsequence $\{\frac{k_{n_m}}{n_m}\}_{m\in\mathbb{N}}$ of $\{\frac{k_n}{n}\}_{n\in\mathbb{N}}$ such that $\lim_{m\to\infty} \frac{k_{n_m}}{n_m} = t$ and $\lim_{m\to\infty} k_{n_m} = \infty$. For each $i \leq n$, let $Y_{i,n} : \Omega_n \to \{-1,1\}$ be defined by $Y_{i,n}(\omega_n) = \omega_n(\frac{i}{n})$. Let $X_{\frac{k_n}{n}} : \Omega_n \to \mathbb{R}$ be defined by

$$X_{\frac{k_n}{n}}(\omega_n) = X_n(\frac{k_n}{n}, \omega_n) = \sum_{i=0}^{k_n-1} \frac{\omega_n(\frac{i}{n})}{\sqrt{n}} = \sum_{i=0}^{k_n-1} \frac{Y_{i,n}(\omega_n)}{\sqrt{n}}.$$

We have that the characteristic function

$$\begin{aligned}
\phi_{X_{\frac{k_{n_m}}{n_m}}}(u) &= E[e^{iuX_{\frac{k_{n_m}}{n_m}}}] \\
&= E[e^{iu(\sum_{j=0}^{k_{n_m}-1} Y_{j,n_m})}] \\
&= E[\prod_{j=0}^{k_{n_m}-1} e^{iu\frac{Y_{j,n_m}}{\sqrt{n_m}}}] \\
&= \prod_{j=0}^{k_{n_m}-1} E[e^{iu\frac{Y_{j,n_m}}{\sqrt{n_m}}}] && \text{by independence of the } Y_{j,n} \\
&= (\phi_{Y_{0,n_m}}(\frac{u}{\sqrt{n_m}}))^{k_{n_m}} && \text{since the } Y_{j,n} \text{ are identically distributed} \\
&= (1 - \frac{u^2}{2n_m} + o(n))^{k_{n_m}} && \text{by Taylor approximation.}
\end{aligned}$$

Let $0 < \epsilon < t$ and pick N large enough so that $|\frac{k_{n_m}}{n_m} - t| < \epsilon$ for all $m \geq N$. Then for $m \geq N$,

$$(1-(t+\epsilon)\frac{u^2}{2k_{n_m}}+o(n))^{k_{n_m}} \leq (1-\frac{k_{n_m}}{n_m}\frac{u^2}{2k_{n_m}}+o(n))^{k_{n_m}} \leq (1-(t-\epsilon)\frac{u^2}{2k_{n_m}}+o(n))^{k_{n_m}}.$$

Using a similar argument for the term $o(n)$, we have that

$$\lim_{m\to\infty}(1 - (t - \epsilon)\frac{u^2}{k_{n_m}} + o(n))^{k_{n_m}} = e^{-(t-\epsilon)\frac{u^2}{2}}.$$

Similarly,

$$\lim_{m\to\infty}(1 - (t + \epsilon)\frac{u^2}{k_{n_m}} + o(n))^{k_{n_m}} = e^{-(t+\epsilon)\frac{u^2}{2}}.$$

Hence we get that

$$\lim_{m\to\infty}(1 - \frac{u^2}{2n_m} + o(n))^{k_{n_m}} = e^{\frac{1}{2}tu^2}.$$

A random variable Z with the normal distribution has characteristic function $e^{iu\mu-\frac{1}{2}\sigma^2u^2}$, where $\mu = E[Z]$ and $\sigma^2 = Var(Z)$. Thus the sequence $\{X_{\frac{k_{n_m}}{n_m}}\}_{m\in\mathbb{N}}$ converges weakly to a random variable $Z \sim N(0,t)$.

Suppose that $\{k_n\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-bounded by M. Then $\lim_{\mathcal{U}} \frac{k_n}{n} = 0$ and

$\lim_{\mathcal{U}} |X_{\frac{k_n}{n}}(\omega_n)| \leq \lim_{\mathcal{U}} \frac{M}{\sqrt{n}} = 0.$

Step 2: Suppose we have a sequence $\{\frac{k_n}{n}\}_{n\in\mathbb{N}}$ as in Step 1 such that $\lim_{\mathcal{U}} k_n = \infty$ and $\lim_{\mathcal{U}} \frac{k_n}{n} = t$. We want to show that $X_{[\frac{k_n}{n}]} = X([\frac{k_n}{n}])$ is normally distributed with mean 0 and variance t. We show that the cumulative distribution function to $X_{[\frac{k_n}{n}]}$ is the cumulative distribution function for the normal distribution with mean 0 and variance t.

Let $a \in \mathbb{R}$ and $\epsilon > 0$. We have that

$$[X^{-1}_{\frac{k_n}{n}}([-\infty, a])] \subseteq X^{-1}_{[\frac{k_n}{n}]}([-\infty, a]) \subseteq [X^{-1}_{\frac{k_n}{n}}([-\infty, a+\epsilon])].$$

Pick a subsequence $\{\frac{k_{n_m}}{n_m}\}_{m\in\mathbb{N}}$ as in Step 1 such that $\lim_{m\to\infty} \frac{k_{n_m}}{n_m} = t$, $\lim_{m\to\infty} k_{n_m} = \infty$,

$$\lim_{m\to\infty} P_{n_m}(X_{\frac{k_{n_m}}{n_m}} \leq a) = \bar{P}([X_{\frac{k_n}{n}} \leq a])$$

and

$$\lim_{m\to\infty} P_{n_m}(X_{\frac{k_{n_m}}{n_m}} \leq a+\epsilon) = \bar{P}([X_{\frac{k_n}{n}} \leq a+\epsilon]).$$

We then have that

$$\begin{aligned} \bar{P}(X_{[\frac{k_n}{n}]} \leq a) - \bar{P}([X_{\frac{k_n}{n}} \leq a]) &\leq \bar{P}([X_{\frac{k_n}{n}} \leq a+\epsilon]) - \bar{P}([X_{\frac{k_n}{n}} \leq a]) \\ &= F_Z(a+\epsilon) - F_Z(a) \\ &< \epsilon \end{aligned}$$

where F_Z is the cumulative distribution function to $Z \sim N(0,t)$. Here we have used that $\{X_{n_m}\}_{m\in\mathbb{N}}$ converges weakly to Z if and only if the cumulative distribution to X_{n_m} converges pointwise to the cumulative distribution function to Z and that the cumulative distribution function to X_{n_m} in $a \in \mathbb{R}$ is given by $P_{n_m}(X_{\frac{k_{n_m}}{n_m}} \leq a)$.

Since this is true for all $\epsilon > 0$,

$$F_{X_{[\frac{k_n}{n}]}}(a) = \bar{P}(X_{[\frac{k_n}{n}]} \leq a) = \bar{P}([X_{\frac{k_n}{n}} \leq a]) = \lim_{\mathcal{U}} P_n(X_{\frac{k_n}{n}} \leq a) = \lim_{\mathcal{U}} F_{X_{\frac{k_n}{n}}}(a).$$

We can now pick a subsequence again as in Step 1, only this time ensuring that $\lim_{m\to\infty} F_{X_{\frac{k_{n_m}}{n_m}}}(a) = F_{X_{[\frac{k_n}{n}]}}(a)$. It then follows that $F_{X_{[\frac{k_n}{n}]}}(a) = F_Z(a)$. Since this holds for all $a \in \mathbb{R}$, $X_{[\frac{k_n}{n}]} \sim N(0,t)$. Notice that we have proved that for $a \in \mathbb{R}$, $\bar{P}(X_t \leq a) = \bar{P}([X_{\frac{k_n}{n}} \leq a])$. This will be useful later.

Step 3: Suppose that we have two sequences $\{\frac{k_{1,n}}{n}\}_{n\in\mathbb{N}}$ and $\{\frac{k_{2,n}}{n}\}_{n\in\mathbb{N}}$ such that $[\frac{k_{1,n}}{n}], [\frac{k_{2,n}}{n}] \in T$ and that $\lim_{\mathcal{U}} \frac{k_{1,n}}{n} = t \geq \lim_{\mathcal{U}} \frac{k_{2,n}}{n} = s$. Since

$$X([\frac{k_{1,n}}{n}], \omega) - X([\frac{k_{2,n}}{n}], \omega) = \lim_{\mathcal{U}}(X_n(\frac{k_{1,n}}{n}, \omega_n) - X_n(\frac{k_{2,n}}{n}, \omega_n))$$

for almost all $\omega \in \Omega$, notice that $X_n(\frac{k_{1,n}}{n}) - X_n(\frac{k_{2,n}}{n})$ has the same distribution as

$$\sum_{i=0}^{|k_{1,n}-k_{2,n}|-1} \frac{Y_{i,n}}{\sqrt{n}} = X_n(\frac{|k_{1,n}-k_{2,n}|-1}{n})$$

if we assume that $|k_{1,n} - k_{2,n}| > 0$. If we can show that $\bar{X}$ exists, then Step 1 and Step 2 shows that (ii) and the first part of (iii) holds. If we define $\bar{X}(0, \omega) = 0$ for all $\omega \in \Omega$, we have also proved (i).

We first need to show that $\bar{X}$ exists for almost all $\omega \in \Omega$. First we show that

$$\begin{aligned}&\{\omega \in \Omega \mid \lim_{\mathcal{U}} X_n(t_n) = \lim_{\mathcal{U}} X_n(s_n) \text{ for all}\\ &[s_n], [t_n] \in T \text{ such that } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t\}\end{aligned}$$

is measurable and has measure 1 for all $t \in I$. We have that

$$\begin{aligned}&\{\omega \in \Omega \mid \exists [t_n], [s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t\\ &\text{such that } \lim_{\mathcal{U}} X_n(t_n) \neq \lim_{\mathcal{U}} X_n(s_n)\}\\ &\subseteq \bigcup_{m \in \mathbb{N}} \{\omega \in \Omega \mid \exists [t_n], [s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t\\ &\text{such that } \lim_{\mathcal{U}} |X_n(t_n) - X_n(s_n)| \geq \frac{1}{m}\}\end{aligned}$$

For each $k \in \mathbb{N}$ we have

$$\begin{aligned}&\{\omega \in \Omega \mid \exists [t_n], [s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t\\ &\text{such that } \lim_{\mathcal{U}} |X_n(t_n) - X_n(s_n)| \geq \frac{1}{m}\}\\ &\subseteq \{\omega \in \Omega \mid \lim_{\mathcal{U}} \sup_{r_n \in [t-\frac{1}{k}, t+\frac{1}{k}] \cap I_n} |X_n(r_n) - X_n(u_n)| \geq \frac{1}{2m}\},\end{aligned}$$

where $u_n = \min\{s_n \in I_n \mid s_n \geq t - \frac{1}{k}\}$. Let $v_n = \max\{s_n \in I_n \mid s_n \leq t + \frac{1}{k}\}$. By Doob's martingale inequality (see [Doo53]),

$$\begin{aligned}&\bar{P}(\lim_{\mathcal{U}} \sup_{r_n \in [t-\frac{1}{k}, t+\frac{1}{k}] \cap I_n} |X_n(r_n) - X_n(u_n)| \geq \frac{1}{2m})\\ &\leq \lim_{\mathcal{U}} P_n(\sup_{r_n \in [t-\frac{1}{k}, t+\frac{1}{k}] \cap I_n} |X_n(r_n) - X_n(u_n)| \geq \frac{1}{4m})\\ &\leq 16m^2 \lim_{\mathcal{U}} E_n[(X_n(v_n) - X_n(u_n))^2]\\ &= 16m^2 E[(X([v_n]) - X([u_n]))^2]\\ &= 16m^2 \frac{2}{k}\end{aligned}$$

by Step 1,2 and 3. Hence

$$\begin{aligned}&\{\omega \in \Omega \mid \exists [t_n], [s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t\\ &\text{such that } \lim_{\mathcal{U}} |X_n(t_n) - X_n(s_n)| \geq \frac{1}{m}\}\end{aligned}$$

is contained in a set of measure 0 and is thus measurable with measure 0. Hence

$$\bar{P}(\exists [t_n], [s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t \text{ such that } \lim_{\mathcal{U}} X_n(t_n) \neq \lim_{\mathcal{U}} X_n(s_n)) = 0, \tag{3.2}$$

which is what we wanted to show. For $[r_n], [t_n] \in T$, we will write $[r_n] \leq [t_n]$ to mean that $\lim_{\mathcal{U}} r_n \leq \lim_{\mathcal{U}} t_n$. Similarly, we will write $k_1 \leq [t_n] \leq k_2$ to mean that $k_1 \leq \lim_{\mathcal{U}} t_n \leq k_2$. For $m, k \in \mathbb{N}$, let

$$\Omega_{m,k} = \left\{ \omega \in \Omega \mid \forall i \in \{0, 1, ..., k-1\} \left(\sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} \mid X([s_n], \omega) - X([r_n], \omega)| < \frac{1}{m} \right) \right\}.$$

In order to show that $\bar{X}$ exists and is continuous with probability 1, it suffices to show that the set

$$\Omega' = \bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \Omega_{m,k}$$

has measure 1. We have that

$$\begin{aligned}\Omega' = \{[\omega_n] \in \Omega \mid & \text{ for all } [t_n], [r_n] \in T, [t_n] \approx [r_n] \\ & \text{implies that } \lim_{\mathcal{U}} X_n(t_n, \omega_n) = \lim_{\mathcal{U}} X_n(r_n, \omega_n)\}.\end{aligned}$$

Indeed, if $[\omega_n] \in \Omega$ is such that for all $[t_n], [r_n] \in T$, $[t_n] \approx [r_n]$ implies that $\lim_{\mathcal{U}} X_n(t_n, \omega_n) = \lim_{\mathcal{U}} X_n(r_n, \omega_n)$, then by Proposition 3.1.3 $X([\omega_n])$ is uniformly continuous (since it's defined on a closed interval) and so $[\omega_n] \in \Omega'$. Conversely, if $[\omega_n] \in \Omega'$, then for each $m \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that

$$\sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X([s_n], [\omega_n]) - X([r_n], [\omega_n])| < \frac{1}{m}$$

for all $i \in \{0, 1, ..., k-1\}$. Thus $\lim_{\mathcal{U}} X_n(t_n, \omega_n) = \lim_{\mathcal{U}} X_n(r_n, \omega_n)$ for all $[r_n], [t_n] \in T$ such that $[r_n] \approx [t_n]$. We will show that

$$\Omega'^c = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Omega^c_{m,k}$$

has measure 0 (we will soon show that these sets are measurable). Suppose that $P(\Omega'^c) > 0$. Then there is $m_0 \in \mathbb{N}$ such that $\bar{P}(\bigcap_{k \in \mathbb{N}} \Omega^c_{m_0,k}) > 0$. We show that $\lim_{k \to \infty} P(\Omega^c_{m_0,k}) = 0$, which leads to a contradiction.

We have that

$$\begin{aligned}\Omega^c_{m_0,k} &= \left\{ \omega \in \Omega \mid \exists i \in \{0, 1, ..., k-1\} \text{ such that } \sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X(s, \omega) - X(\frac{i}{k}, \omega)| \geq \frac{1}{m_0} \right\} \\ &= \bigcup_{i=0}^{k-1} \left\{ \omega \in \Omega \mid \sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X(s, \omega) - X(\frac{i}{k}, \omega)| \geq \frac{1}{m_0} \right\}.\end{aligned}$$

For each $i \in \{0, ..., k\}$, let $[s_{n,i}] \in T$ be such that $\lim_{\mathcal{U}} s_{n,i} = \frac{i}{k}$. By (3.2),

$$\sup_{\substack{[r_n] \leq [t_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X([t_n],[\omega_n]) - X([r_n],[\omega_n])| = \lim_{\mathcal{U}} \sup_{s_{n,i} \leq t_n \leq s_{n,i+1}} |X_n(t_n,\omega_n) - X_n(s_{n,i},\omega_n)|$$

for almost all $[\omega_n] \in \Omega$ (this shows that the sets are measurable). We have

$$\begin{aligned}
\bar{P}(\Omega^c_{m_0,k}) &= \bar{P}\left(\bigcup_{i=0}^{k-1}\left\{\omega \in \Omega \mid \sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X([s_n],\omega) - X([r_n],\omega)| \geq \frac{1}{m_0}\right\}\right) \\
&\leq \sum_{i=0}^{k-1} \bar{P}\left(\left\{\omega \in \Omega \mid \sup_{\substack{[r_n] \leq [s_n] \leq \frac{i+1}{k} \\ \lim_{\mathcal{U}} r_n = \frac{i}{k}}} |X([s_n],\omega) - X([r_n],\omega)| \geq \frac{1}{m_0}\right\}\right) \\
&\leq \sum_{i=0}^{k-1} \lim_{\mathcal{U}} P_n\left(\left\{\omega_n \in \Omega_n \mid \sup_{s_{n,i} \leq s_n \leq s_{n,i+1}} |X_n(s_n,\omega_n) - X_n(s_{n,i},\omega_n)| \geq \frac{1}{2m_0}\right\}\right) \\
&\qquad \text{by (3.2)} \\
&\leq \sum_{i=0}^{k-1} \lim_{\mathcal{U}} 2P_n\left(\left\{\omega_n \in \Omega_n \mid |X_n(s_{n,i+1},\omega_n) - X_n(s_{n,i},\omega_n)| \geq \frac{1}{2m_0}\right\}\right) \\
&\qquad \text{by the reflection principle for random walks (see [MP10])} \\
&\leq \sum_{i=0}^{k-1} 2\bar{P}\left(\left\{\omega \in \Omega \mid |X([s_{n,i+1}],\omega) - X_n([s_{n,i}],\omega)| \geq \frac{1}{2m_0}\right\}\right) \\
&\leq 4m_0\sqrt{\frac{2}{k\pi}}e^{-\frac{k}{8m_0^2}} \to 0 \text{ as } k \to \infty
\end{aligned}$$

where the last inequality follows from a tail inequality for the normal distribution (see [Fel68]), which states that if $Z \sim N(0, \sigma^2)$, then

$$P(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}}\frac{\sigma}{t}e^{-\frac{t^2}{2\sigma^2}}.$$

So $\bar{X}$ exists for almost all $\omega \in \Omega$. By Proposition 3.1.3, X is t-continuous with probability one.

We are now ready to prove the last part of (iii). Let $t, s \in I$ with $t > s$. Suppose $a \in \mathbb{R}$ and $G \in \mathcal{F}_s$. We want to show that

$$\bar{P}(\{\bar{X}_t - \bar{X}_s \leq a\} \cap G) = \bar{P}(\{\bar{X}_t - \bar{X}_s \leq a\})\bar{P}(G).$$

For each $n \in \mathbb{N}$, let $t_n = \inf_{r_n \in I_n \mid r_n \geq t}$ and $s_n = \inf_{r_n \in I_n \mid r_n \geq s}$. By Corollary 2.2.12, there is $[A_n] \in \mathcal{A}_s$ such that $\bar{P}(G \triangle [A_n]) = 0$. By a previous observation we have that

$$[\{X_n(t_n) - X_n(s_n) \leq a\}] \subseteq \{\bar{X}_t - \bar{X}_s \leq a\}$$

and

$$\bar{P}(\{\bar{X}_t - \bar{X}_s \leq a\}) = \bar{P}[\{X_n(t_n) - X_n(s_n) \leq a\}].$$

Thus we have

$$\begin{aligned}
\bar{P}(\{\bar{X}_t - \bar{X}_s \le a\} \cap G) &= \int_\Omega 1_{\{\bar{X}_t - \bar{X}_s \le a\}} 1_G \; d\bar{P} \\
&= \int_\Omega 1_{[\{X_n(t_n) - X_n(s_n) \le a\}]} 1_{[A_n]} \; d\bar{P} \\
&= \lim_{\mathcal{U}} \int_{\Omega_n} 1_{\{X_n(t_n) - X_n(s_n) \le a\}} 1_{A_n} \; dP_n \\
&= \lim_{\mathcal{U}} P_n(\{X_n(t_n) - X_n(s_n) \le a\} \cap A_n) \\
&= \lim_{\mathcal{U}} P_n(\{X_n(t_n) - X_n(s_n) \le a\}) P_n(A_n) \\
&= \bar{P}([\{X_n(t_n) - X_n(s_n) \le a\}]) \bar{P}([A_n]) \\
&= \bar{P}(\{\bar{X}_t - \bar{X}_s \le a\}) \bar{P}(G),
\end{aligned}$$

which is what we wanted to show. So $\bar{X}_t - \bar{X}_s$ is independent from $\mathcal{F}_s$, i.e. the second part of (iii) holds.

■

Note that we can, by the same construction as above, construct Brownian motion on any interval $[0, T]$. Note also that we, by the same construction as above, doing some small adjustments, can define several different random walks on Ω_n, giving us different Brownian motions on Ω.

Proposition 3.1.19. *For each $n \in \mathbb{N}$, let B_n denote the process in Theorem 3.1.18 defined on the interval $I_n = \{\frac{kT}{n} \mid k \in \{0, ..., n\}\}$. Then, for $d \in \mathbb{N}$, for each $\{\frac{j_n T}{n}\}_{n \in \mathbb{N}} \in \mathcal{T}$, $\{B_n(\frac{j_n T}{n})^d\}_{n \in \mathbb{N}}$ and $\{e^{d\beta B_n(\frac{j_n}{n})}\}_{n \in \mathbb{N}}$ (where $\beta > 0$) are $\mathcal{A}$-integrable.*

Proof. Let $\{\frac{j_n T}{n}\}_{n \in \mathbb{N}} \in \mathcal{T}$ be a sequence of numbers. Showing that $\{e^{d\beta B_{\frac{j_n T}{n}}}\}_{n \in \mathbb{N}}$ is $\mathcal{A}$-integrable is equivalent to showing that $\{e^{\beta B_{\frac{j_n T}{n}}}\}_{n \in \mathbb{N}}$ is $\mathcal{A}$-integrable (since we assume that $\beta > 0$ is arbitrary). By Proposition 2.2.20, it suffices to show that there is a real number $p > 0$ such that

$$\lim_{\mathcal{U}} \int_{\Omega_n} e^{p\beta B_{\frac{j_n T}{n}}} \; dP_n < \infty.$$

Let $p = 2$. We will need the following useful inequality which we derive from Taylor series representation:

$$\frac{1}{2}(e^{-x} + e^x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \le \sum_{i=0}^{\infty} \frac{(\frac{x^2}{2})^i}{i!} = e^{\frac{x^2}{2}}. \tag{3.3}$$

We have that

$$\begin{aligned}
E_n[e^{2\beta B_{\frac{j_nT}{n}}}] &= E_n[e^{\sum_{i_n=0}^{j_nT} 2\beta\Delta B_{\frac{i_nT}{n}}}] \\
&= \prod_{i_n=0}^{j_n} E_n[e^{2\beta\Delta B_{\frac{i_nT}{n}}}] \qquad \text{by independence of the } \Delta B_{\frac{i_nT}{n}}\text{'s} \\
&= (\frac{1}{2}e^{2\beta\sqrt{\frac{T}{n}}} + \frac{1}{2}e^{-2\beta\sqrt{\frac{T}{n}}})^{j_n} \\
&\leq e^{\frac{2\beta^2 T j_n}{n}} \qquad \text{by (3.3).}
\end{aligned}$$

so that $\lim_{\mathcal{U}} E_n[e^{2\beta B_{\frac{j_nT}{n}}}] \leq e^{2\beta^2 T}$, which is what we wanted to show. To show that $\{B_n(\frac{j_nT}{n})^d\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable, notice that by symmetry of random walks,

$$E_n[|B_n(\frac{j_nT}{n})|^{d+1}] = 2E_n[1_{A_n}B_n(\frac{j_nT}{n})^{d+1}],$$

where $A_n = \{\omega_n \in \Omega \mid B_n(\frac{j_nT}{n}) \geq 0\}$. Thus

$$\begin{aligned}
E_n[|B_n(\frac{j_nT}{n})|^{d+1}] &= 2E_n[1_{A_n}B_n(\frac{j_nT}{n})^{d+1}] \\
&\leq 2E_n[1_{A_n}e^{(d+1)B_n(\frac{j_nT}{n})}] \\
&\leq 2E_n[e^{(d+1)B_n(\frac{j_nT}{n})}]
\end{aligned}$$

which proves that $\{B_n(\frac{j_nT}{n})^d\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable. ■

3.2 Stochastic integrals

Let $I_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ and let $B_n : I_n \times \Omega_n \to \mathbb{R}$ be the usual random walk

$$B_n(\frac{k}{n}, \omega_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{k-1}\omega_n(\frac{i}{n}).$$

We will from now on use a different notation for this expression, namely

$$B_n(t_n, \omega_n) = \frac{1}{\sqrt{n}}\sum_{s_n=0}^{t_n}\omega_n(s_n).$$

Notice that $s_n = t_n$ is not included in the last sum. We shall also use the notation

$$\Delta B_n(t_n, \omega_n) = B_n(t_n + \frac{1}{n}, \omega_n) - B(t_n, \omega_n).$$

This is the increment of B_n into the future.

If $X_n : I_n \times \Omega_n \to \mathbb{R}$ is another process, we define the *stochastic integral* $\int X_n \, dB_n$ to be the process given by

$$\left(\int X_n \, dB_n\right)(t_n, \omega_n) = \sum_{s_n=0}^{t_n} X_n(s_n, \omega_n)\Delta B_n(s_n, \omega_n).$$

We use the same summation convention as above: $s_n = t_n$ is *not* included in the sum.

Remark: To make notation easier, we will from now on omit the bar over the (well-defined) measures and processes that we construct on the measure space.

Proposition 3.2.1. *If $r_n, t_n \in I_n$, $r_n < t_n$, then*

$$B_n(t_n)^2 - B_n(r_n)^2 = 2\int_{r_n}^{t_n} B_n(s_n)\ dB_n(s_n) + (t_n - r_n)$$

Proof. We have that

$$\begin{aligned} B_n(t_n)^2 =& (B_n(t_n) - B_n(t_n - \Delta t_n) + B_n(t_n - \Delta t_n))^2 \\ =& (B_n(t_n) - B_n(t_n - \Delta t_n))^2 + B_n(t_n - \Delta t_n)^2 \\ &+ 2(B_n(t_n) - B_n(t_n - \Delta t_n))B_n(t_n - \Delta t_n) \\ =& \Delta t_n + B_n(t_n - \Delta t_n)^2 + 2B_n(t_n - \Delta t_n)\ \Delta B_n(t_n - \Delta t_n). \end{aligned}$$

Thus,

$$B_n(t_n)^2 - B_n(t_n - \Delta t_n)^2 = \Delta t_n + 2B_n(t_n - \Delta t_n)\ \Delta B_n(t_n - \Delta t_n).$$

Summing over we get

$$B_n(t_n)^2 - B_n(r_n)^2 = 2\int_{r_n}^{t_n} B_n(s_n)\ dB_n(s_n) + (t_n - r_n).$$

■

In general, the stochastic integral $\int X_n\ dB_n$ is not particularily well-mannered, so we will constrict ourself to the case where X_n is an *adapted* process. That X_n is adapted means that $X_n(t_n, \omega_n)$ only depends on the coin tosses that has happened before t_n, that is $\omega_n(0), \omega_n(\Delta t_n), \omega_n(2\Delta t_n), ..., \omega_n(t_n - \Delta t_n)$, where $\Delta t_n = \frac{1}{n}$, and not by the future coin tosses $\omega_n(t_n), \omega_n(t_n + \Delta t_n), ..., \omega_n(1)$. Mathematically this can be expressed by saying that if $\omega_n, \omega_n' \in \Omega_n$ with

$$\omega_n(0) = \omega_n'(0),\ \omega_n(\Delta t_n) = \omega_n'(\Delta t_n),\ \dots,\ \omega_n(t_n - \Delta t_n) = \omega_n'(t_n - \Delta t_n),$$

then $X_n(t_n, \omega_n) = X_n(t_n, \omega_n')$.

Proposition 3.2.2. *Suppose that X_n is adapted. If $r_n, t_n \in I_n$ and $r_n < t_n$ then*

$$E_n\left[\left(\int_{r_n}^{t_n} X_n\ dB_n\right)^2\right] = \int_{r_n}^{t_n} E_n[X_n(s_n)^2]\ ds_n,$$

where $\int_{r_n}^{t_n} E[X_n(s_n)^2]\ ds_n = \sum_{s_n=r_n}^{t_n} E[X_n(s_n)^2]\Delta t_n$.

Proof.

$$\begin{aligned}
E_n\left[\left(\int_{r_n}^{t_n} X_n\, dB_n\right)^2\right] &= E_n\left[\sum_{u_n=r_n}^{t_n}\sum_{v_n=r_n}^{t_n} X_n(u_n)X_n(v_n)\Delta B_n(u_n)\Delta B_n(v_n)\right] \\
&= \sum_{s_n=r_n}^{t_n} E_n[X_n(s_n)^2(\Delta B_n(s_n))^2] \\
&\text{since the increments of a random walk are independent} \\
&= \sum_{s_n=r_n}^{t_n} E_n[X_n(s)^2]\Delta s_n \\
&= \int_{r_n}^{t_n} E_n[X_n(s_n)^2]\, ds_n.
\end{aligned}$$

■

Proposition 3.2.3. *Suppose that ϕ is twice continuously differentiable. If $r_n, t_n \in I_n$ and $r_n < t_n$, then*

$$\begin{aligned}
&\phi(B_n(t_n,\omega_n)) - \phi(B_n(r_n,\omega_n)) \\
&= \int_{r_n}^{t_n} \phi(B_n(s_n,\omega_n))\, dB_n(s_n,\omega_n) + \frac{1}{2}\int_{r_n}^{t_n} \phi''(\theta_n(s_n,\omega_n))ds_n,
\end{aligned}$$

where $\theta_n(s_n,\omega_n)$ lies in the interval between $B_n(t_n,\omega_n)$ and $B_n(t_n+\Delta t_n,\omega_n)$.

Proof.

$$\begin{aligned}
&\phi(B_n(t_n+\Delta t_n,\omega_n)) - \phi(B_n(t_n,\omega_n)) \\
&= \phi'(B_n(t_n,\omega_n))(B_n(t_n+\Delta t_n,\omega_n) - B_n(t_n,\omega_n) + \frac{1}{2}\phi''(\xi)\Delta s_n \qquad (3.4)
\end{aligned}$$

by Taylor series representation, where ξ lies in the interval between $B_n(t_n,\omega_n)$ and $B_n(t_n+\Delta t_n,\omega_n)$. Hence

$$\begin{aligned}
&\phi(B_n(t_n,\omega_n)) - \phi(B_n(r_n,\omega_n)) \\
&= \sum_{s_n=r_n}^{t_n} \phi(B_n(s_n+\Delta s_n,\omega_n)) - \phi(B_n(s_n,\omega_n)) \\
&= \sum_{s_n=r_n}^{t_n} \phi'(B_n(s_n,\omega_n))(B_n(s_n+\Delta s_n,\omega_n) - B_n(s_n,\omega_n)) \\
&+ \frac{1}{2}\sum_{s_n=r_n}^{t_n} \phi''(\theta_n(s_n,\omega_n))\Delta s_n \qquad \text{by (3.4)} \\
&= \int_{r_n}^{t_n} \phi(B_n(s_n,\omega_n))\, dB_n(s_n,\omega_n) + \frac{1}{2}\int_{r_n}^{t_n} \phi''(\theta_n(s_n,\omega_n))\, ds_n,
\end{aligned}$$

where $\theta_n(s_n,\omega_n)$ lies in the interval between $B_n(t_n,\omega_n)$ and $B_n(t_n+\Delta t_n,\omega_n)$.

■

Theorem 3.2.4. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of adapted processes $X_n : I_n \times \Omega_n \to \mathbb{R}$ such that $\{X_n(s_n)^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for all $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{T}$. Suppose the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists for almost all $\omega \in \Omega$ for all but a finite number of $t \in I = [0,T]$. Then we may define the stochastic integral $\int X\ dB$ to be the ultralimit of $\{\int X_n\ dB_n\}_{n\in\mathbb{N}}$.*

Proof. For simplicity, we will let $I = [0,1]$ and show that

$$\lim_{\mathcal{U}} \int_0^1 X_n(s_n)\ dB_n(s_n) = \int_0^1 X(s)\ dB(s).$$

We first show the equality for simple functions. The simple functions on the form $\sum_{j=0}^{N} e_j 1_{[t_j,t_{j+1})}$, where the e_j's are random variables, are dense in $L^2(I \times \Omega, \mathcal{B}(I) \otimes \mathcal{F}, \mu \times P)$ (see [Øks03]). By linearity, it suffices to show the equality for functions of the form $e1_{[r,t)}$, where e is a random variable and $r,t \in I$ with $r < t$. We have that the stochastic integral

$$\int_0^1 e1_{[r,t)}\ dB = \int_r^t e\ dB = e(B_t - B_r).$$

Let $[r_n],[t_n] \in T$ such that $\lim_{\mathcal{U}} r_n = r$ and $\lim_{\mathcal{U}} t_n = t$. For each $n \in \mathbb{N}$, let e_n be an $\mathcal{F}_{r_n}$-adapted random variable such that $[e_n]$ is equal to e P-almost everywhere (by Theorem 2.2.21 such variables exist). Then, for $u \in I \setminus \{r,t\}$, for all $[u_n] \in T$ such that $\lim_{\mathcal{U}} u_n = u$ we have $\lim_{\mathcal{U}} e_n(\omega_n)1_{[r_n,t_n)}(u_n) = [e_n]([\omega_n])1_{[r,t)}(u)$ for all $[\omega_n] \in \Omega$. Then the ultralimit

$$\begin{aligned}
\int_0^1 e1_{[r,t)}\ dB &= \lim_{\mathcal{U}} e_n \int_0^1 1_{[r_n,t_n)}\ dB_n \\
&= \lim_{\mathcal{U}} e_n \sum_{s_n=r_n}^{t_n} \Delta B_{s_n} \\
&= \lim_{\mathcal{U}} e_n (B_{t_n} - B_{r_n}) \\
&= [e_n](B_t - B_r).
\end{aligned}$$

for almost all $[\omega_n] \in \Omega$, which is equal to $e(B_t - B_r)$ P-almost everywhere. Now suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of adapted processes $X_n : \Omega_n \times I_n \to \mathbb{R}$ such that $\{X_n(s_n)^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for each $[s_n] \in T$. To avoid confusion we will let $\mathcal{U} - \int_0^1 X\ dB$ denote the "ultralimit integral" and we will let $I - \int_0^1 X\ dB$ denote the Ito integral. Since the simple functions are dense in $L^2(I \times \Omega, \mathcal{B}(I) \otimes \mathcal{F}, \mu \times P)$, for $\epsilon > 0$, by Ito isometry (see [Øks03]), there is a simple function ϕ such that

$$E\left[\left(I - \int_0^1 X\ dB - \int_0^1 \phi\ dB\right)^2\right] = \int_0^1 E[(X-\phi)^2]\ ds < \epsilon.$$

Consider now the ultralimit $\mathcal{U} - \int_0^1 X\ dB$. For each $n \in \mathbb{N}$ there is a simple function $\phi_n : [0,1] \times \Omega_n \to \mathbb{R}$ such that ϕ is the ultralimit of $\{\phi_n\}_{n\in\mathbb{N}}$ for all but a finite number of $t \in [0,1]$ and such that the ultralimit $\int_0^1 \phi\ dB$ exists and is equal to the Ito integral. Indeed, by Theorem 2.2.21 such a sequence exists and we may find such a sequence such that $\{\phi_n(t_n)^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for all $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$. We want to show that

$$E\left[\left(\mathcal{U} - \int_0^1 X\ dB - \int_0^1 \phi\ dB\right)^2\right] = \lim_{\mathcal{U}} E_n\left[\left(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n\right)^2\right] \tag{3.5}$$

and that

$$\lim_{\mathcal{U}} E_n\left[\left(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n\right)^2\right] = \int_0^1 E[(X-\phi)^2]\ ds.$$

By Proposition 3.2.2,

$$E_n\left[\left(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n\right)^2\right] = \int_0^1 E_n[(X_n-\phi_n)^2]\ ds_n.$$

By $\mathcal{A}$-integrability of $\{X_n(s_n)^2\}_{n\in\mathbb{N}}$ and of $\{\phi_n(s_n)^2\}_{n\in\mathbb{N}}$ for each $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{T}$, $\left(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n\right)^2$ is $\mathcal{A}$-integrable and so (3.5) holds. Indeed, since

$$\lim_{\mathcal{U}} \sup_{s_n\in I_n} E_n[(X_n(s_n)-\phi_n(s_n))^2] < \infty,$$

$\lim_{\mathcal{U}} E_n[(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n)^2] < \infty$. Furthermore, if $[A_n] \in \mathcal{A}$ and $\lim_{\mathcal{U}} P_n(A_n) = 0$, then

$$\lim_{\mathcal{U}} \sup_{s_n\in I_n} E_n[1_{A_n}(X_n(s_n)-\phi_n(s_n))^2] = 0$$

and so $\lim_{\mathcal{U}} E_n[1_{A_n}(\int_0^1 X_n\ dB_n - \int_0^1 \phi_n\ dB_n)^2] = 0$.

Let $f_n : I_n \to \mathbb{R}$ be defined by

$$f_n(t_n) = E_n[(X_n(t_n)-\phi_n(t_n))^2]$$

and let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(t) = E[(X(t)-\phi(t))^2].$$

Let $\epsilon' > 0$. By Proposition 3.1.3, f is continuous in all but a finite number of points $t \in [0,1]$. Hence f is Riemann integrable. So there is an $N \in \mathbb{N}$ such that

$$\left|\int_0^1 f\ ds - \sum_{s_n=0}^{1} f(s_n)\Delta s_n\right| < \frac{\epsilon'}{2}$$

for all $n \geq N$. Suppose that $t_1, t_2, ..., t_{k-1}$ are the points of discontinuity of f and let $t_0 = 0$ and $t_k = 1$. Since f is continuous for all but a finite number of points, and since $f(t) < \infty$ for all $t \in [0,1]$, there is an $M_1 \in \mathbb{N}$ such that $f \leq M_1$. By $\mathcal{A}$-integrability, there is an $M_2 \in \mathbb{N}$ and a $G_{M_2} \in \mathcal{U}$ such that $f_n \leq M_2$ for all $n \in G_{M_2}$. Choose $m \in \mathbb{N}$ large enough so that

$\frac{2(k+1)(M_1+M_2)}{m} < \frac{\epsilon'}{4}$ and so that $\frac{2}{m} < |t_{i+1} - t_i|$ for each $i \in \{0, 1, ..., m-1\}$. Then for each interval $(t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m})$, for $i = 0, 1, ..., k-1$,

$$\left\{ n \in \mathbb{N} \mid |f_n(s_n) - f(s_n)| < \frac{\epsilon'}{4} \text{ for all } s_n \in I_n \cap (t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m}) \right\} \in \mathcal{U}.$$

Suppose not. Then there is an interval $(t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m})$ and an $F_i \in \mathcal{U}$ such that for each $n \in F_i$ there is $r_n \in I_n \cap (t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m})$ such that

$$|f_n(r_n) - f(r_n)| \geq \frac{\epsilon'}{4}.$$

We have that $\lim_{\mathcal{U}} r_n = r$ for some $r \in [t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m}]$ and $\lim_{\mathcal{U}} f_n(r_n) = f(r)$. Since Y is continuous in r, $\lim_{\mathcal{U}} f(r_n) = f(r)$. But now we arrive at a contradiction, since there is a $G \in \mathcal{U}$ such that

$$|f_n(r_n) - f(r_n)| \leq |f_n(r_n) - f(r)| + |f(r) - f(r_n)| < \frac{\epsilon'}{4}$$

for all $n \in G$. So we get that there is an $F \in \mathcal{U}$ such that

$$|E[(X(s_n) - \phi(s_n))^2] - E_n[(X_n(s_n) - \phi_n(s_n))^2]| < \frac{\epsilon'}{4}$$

for all $s_n \in \bigcup_{i=0}^{m} (t_i + \frac{1}{m}, t_{i+1} - \frac{1}{m}) \cap I_n$ for all $n \in F$. So for $n \in F \cap \{1, 2, ..., N-1\}^c \cap G_{M_2}$,

$$\left| \int_0^1 E[(X - \phi)^2] \; ds - \int_0^1 E_n[(X_n - \phi_n)^2] \; ds_n \right| < \epsilon'.$$

So

$$\lim_{\mathcal{U}} \int_0^1 E_n[(X_n - \phi_n)^2] \; ds_n = \int_0^1 E[(X - \phi)^2] \; ds,$$

hence

$$E\left[\left(\mathcal{U} - \int_0^1 X \; dB - \int_0^1 \phi \; dB \right)^2 \right] < \epsilon.$$

■

Note that the proof above generalizes: If X_n is as above, $\{t_n\}_{n\in\mathbb{N}}, \{s_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ and $\lim_{\mathcal{U}} t_n = t > \lim_{\mathcal{U}} s_n = s$, then $\lim_{\mathcal{U}} \int_{s_n}^{t_n} X_n \; dB_n = \int_s^t X \; dB$. Note also that the proof above also shows that, given a sequence of skeleton processes $X_n : I_n \times \Omega_n \to \mathbb{R}$ with ultralimit $X : I \times \Omega \to \mathbb{R}$ such that X is continuous in all but a finite number of points for each $\omega \in \Omega$, $\lim_{\mathcal{U}} \int_{s_n}^{t_n} X_n(r_n) \; dr_n = \int_s^t X(r) \; dr$, where $t = \lim_{\mathcal{U}} t_n$ and $s = \lim_{\mathcal{U}} s_n$.

Theorem 3.2.5. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of adapted processes $X_n : I_n \times \Omega_n \to \mathbb{R}$ such that $\{X_n(s_n)^4\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for all $\{s_n\}_{n\in\mathbb{N}} \in \mathcal{T}$. Suppose the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists for almost all $\omega \in \Omega$ for all but a finite number of $t \in [0,1]$. Then the ultralimit process $\int X \; dB$ to the discrete-time processes $\{\int X_n \; dB_n\}_{n\in\mathbb{N}}$ exists (and is continuous).*

Proof. First we show that

$$P(\lim_{\mathcal{U}} \int_0^{t_n} X_n(s_n)\ dB_n(s_n) = \lim_{\mathcal{U}} \int_0^{r_n} X_n(s_n)\ dB_n(s_n)$$
$$\text{for all } [t_n],[r_n] \in T \text{ such that } \lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} t_n = t) = 1$$

for all $t \in I$. We have that

$$\{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} X_n(s_n)\ dB_n(s_n) - \int_0^{r_n} X_n(s_n)\ dB_n(s_n))| \neq 0\}$$
$$\subseteq \bigcup_{m\in\mathbb{N}} \{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} X_n(s_n)\ dB_n(s_n) - \int_0^{r_n} X_n(s_n)\ dB_n(s_n))| \geq \frac{1}{m}\}.$$

For each $k \in \mathbb{N}$ we have

$$\{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} X_n(s_n)\ dB_n(s_n) - \int_0^{r_n} X_n(s_n)\ dB_n(s_n))| \geq \frac{1}{m}\}$$
$$\subseteq \{\omega \in \Omega \mid \lim_{\mathcal{U}} \sup_{r_n\in[t-\frac{1}{k},t+\frac{1}{k}]\cap I_n} |(\int_0^{r_n} X_n(s_n)\ dB_n(s_n)$$
$$- \int_0^{t-\frac{1}{k}} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{2m}\}.$$

By Doob's martingale inequality (see [Doo53]),

$$P(\lim_{\mathcal{U}} \sup_{r_n\in[t-\frac{1}{k},t+\frac{1}{k}]\cap I_n} |(\int_0^{r_n} X_n(s_n)\ dB_n(s_n)$$
$$- \int_0^{t-\frac{1}{k}} X_n(s_n)\ dB_n(s_n))| \geq \frac{1}{2m})$$
$$\leq \lim_{\mathcal{U}} P_n(\sup_{r_n\in[t-\frac{1}{k},t+\frac{1}{k}]\cap I_n} |(\int_0^{r_n} X_n(s_n)\ dB_n(s_n)$$
$$- \int_0^{t-\frac{1}{k}} X_n(s_n)\ dB_n(s_n))| \geq \frac{1}{4m})$$
$$\leq \lim_{\mathcal{U}} 16m^2 E_n[(\int_{t-\frac{1}{k}}^{t+\frac{1}{k}} X_n(s_n)\ dB_n(s_n))^2]$$
$$\leq \lim_{\mathcal{U}} 32m^2 \sup_{s_n\in I_n} E_n[X_n(s_n)^2]\frac{1}{k} = 32m^2 \sup_{s\in I} E_n[X(s)^2]\frac{1}{k}.$$

Since this is true for each $k \in \mathbb{N}$,

$$\{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} X_n(s_n)\ dB_n(s_n) - \int_0^{r_n} X_n(s_n)\ dB_n(s_n))| \geq \frac{1}{m}\}$$

is measurable and has measure 0, since it is the subset of a set of measure 0. Hence

$$\{\omega \in \Omega \mid \exists\, [t_n], [r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} X_n(s_n)\ dB_n(s_n) - \int_0^{r_n} X_n(s_n)\ dB_n(s_n))| \neq 0\}$$

is measurable and has measure 0. We want to show that

$$P(\lim_{\mathcal{U}} \int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) = \lim_{\mathcal{U}} \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n)$$
$$\text{whenever } [t_n] \approx [r_n] \text{ for all } [t_n], [r_n] \in T) = 1.$$

We will use the following notation: For $[t_n], [r_n] \in T$, we will write $[r_n] \leq [t_n]$ to mean that $\lim_{\mathcal{U}} r_n \leq \lim_{\mathcal{U}} t_n$. Similarly, we will write $k_1 \leq [t_n] \leq k_2$ to mean that $k_1 \leq \lim_{\mathcal{U}} t_n \leq k_2$. Let $Y : T \times \Omega \to \mathbb{R}$ be defined by

$$Y([t_n], [\omega_n]) = \lim_{\mathcal{U}} Y_n(t_n, \omega_n),$$

where

$$Y_n(t_n, \omega_n) = \left(\int_0^{t_n} X_n(s_n)\ dB_n(s_n)\right)(\omega_n).$$

We will proceed in a similar way to the proof of Theorem 3.1.18. For $m, k \in \mathbb{N}$, let

$$\Omega_{m,k} = \{[\omega_n] \in \Omega \mid \forall i \in \{0, 1, ..., k-1\}$$
$$\left.\left(\sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| < \frac{1}{m}\right)\right\}.$$

In order to show that $\bar{Y}$ exists as in Definition 3.1.6, it suffices to show that

$$\Omega' = \bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \Omega_{m,k}$$

has measure 1, since

$$\Omega' = \{[\omega_n] \in \Omega \mid \lim_{\mathcal{U}} Y_n(t_n, \omega_n) = \lim_{\mathcal{U}} Y_n(r_n, \omega_n)$$
$$\text{whenever } [t_n] \approx [r_n] \text{ for all } [t_n], [r_n] \in T\}.$$

We show that

$$\Omega'^c = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Omega_{m,k}^c$$

has measure 0 (we will soon show that these sets are indeed measurable). Suppose that $P(\Omega'^c) > 0$. Then there is $m_0 \in \mathbb{N}$ such that

$$P(\bigcap_{k \in \mathbb{N}} \Omega_{m_0,k}^c) > 0.$$

We show that $\lim_{k\to\infty} P(\Omega^c_{m_0,k}) = 0$, which leads to a contradiction. We have that

$$\Omega^c_{m_0,k} = \{[\omega_n] \in \Omega \mid \exists i \in \{0,1,...,k-1\} \text{ such that}$$

$$\left. \sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| \geq \frac{1}{m_0} \right\}$$

$$= \bigcup_{i=0}^{k-1} \left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| \geq \frac{1}{m_0} \right\}.$$

For each $i \in \{0,...,k\}$, let $\{s_{n,i}\}_{\in\mathbb{N}} \in \mathcal{T}$ be such that $\lim_{\mathcal{U}} s_{n,i} = \frac{iT}{k}$. Then, by the previous observation that

$$P(\{[\omega_n] \in \Omega \mid \exists [t_n],[s_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} s_n = t$$
$$\text{such that } \lim_{\mathcal{U}} Y_n(t_n,\omega_n) \neq \lim_{\mathcal{U}} Y_n(s_n,\omega_n)\}) = 0,$$

$$\sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| = \lim_{\mathcal{U}} \sup_{s_{n,i}\le t_n\le s_{n,i+1}} |Y_n(t_n,\omega_n) - Y_n(s_{n,i},\omega_n)|$$

for almost all $[\omega_n] \in \Omega$ (this shows that the sets are measurable). Then

$$P(\Omega^c_{m_0,k}) = P\left(\bigcup_{i=0}^{k-1} \left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| \geq \frac{1}{m_0} \right\} \right)$$

$$\leq \sum_{i=0}^{k-1} P\left(\left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| \geq \frac{1}{m_0} \right\} \right),$$

where we have used the old summation convention. We have that

$$P\left(\left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n]\le[t_n]\le\frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n],[\omega_n]) - Y([r_n],[\omega_n])| \geq \frac{1}{m_0} \right\} \right)$$

$$\leq \lim_{\mathcal{U}} P_n\left(\left\{ \omega_n \in \Omega_n \mid \sup_{s_{n,i}\le t_n\le s_{n,i+1}} |Y_n(t_n,\omega_n) - Y_n(s_{n,i},\omega_n)| \geq \frac{1}{2m_0} \right\} \right) \quad \text{by (4.3)}$$

$$\leq \lim_{\mathcal{U}} 16 m_0^4 E_n \left[\left(\sup_{s_{n,i}\le t_n\le s_{n,i+1}} |Y_n(t_n,\omega_n) - Y_n(s_{n,i},\omega_n)| \right)^4 \right]$$

by Markov's inequality (see [Bil08]).

Next we will use the Burkholder-Davis-Gundy inequality for martingales (see [BDG72]), which gives us that

$$E_n\left[\left(\sup_{s_{n,i}\le t_n\le s_{n,i+1}}|Y_n(t_n,\omega_n)-Y_n(s_{n,i},\omega_n)|\right)^4\right]$$
$$\le CE_n[[Y_n(t_n)-Y_n(s_{n,i})]^2_{t_n=s_{n,i+1}}],$$

for some $C\in\mathbb{R}$, where

$$[Y_n(t_n)-Y_n(s_{n,i})]_{t_n=s_{n,i+1}}=\sum_{s_n=s_{n,i}}^{s_{n,i+1}}X_n(s_n)^2\Delta s_n$$

is the quadratic variation of $Y_n(s_{n,i+1})-Y_n(s_{n,i})$. Thus

$$E_n\left[\left(\sup_{s_{n,i}\le t_n\le s_{n,i+1}}|Y_n(t_n,\omega_n)-Y_n(s_{n,i},\omega_n)|\right)^4\right]$$
$$\le C\sup_{t_n\in I_n}E_n[X_n(s_n)^4]|s_{n,i+1}-s_{n,i}|,$$

Hence

$$\begin{aligned}P(\Omega^c_{m_0,k})&\le\lim_{\mathcal{U}}16m_0kC\sup_{t_n\in I_n}E_n[X_n(s_n)^4]|s_{n,i+1}-s_{n,i}|^2\\&=16m_o^4C\sup_{s\in I}E[X(s)^4]T^2\frac{1}{k}\to 0\text{ as }k\to\infty,\end{aligned}$$

which is what we wanted to show. By Proposition 3.1.3, the ultralimit $\int X\ dB$ to $\{\int X_n\ dB_n\}_{n\in\mathbb{N}}$ exists and is continuous for almost all $\omega\in\Omega$. ■

Corollary 3.2.6. *Suppose that $\{B_n\}_{n\in\mathbb{N}}$ is a series or random walks. Let B denote the a Brownian motion we get by taking the ultralimit. Suppose that $r,t\in I$ and that $r<t$. Then*

$$B(t)-B(r)=2\int_r^t B(s)\ dB(s)+(t-r).$$

Proof. This follows from Proposition 3.2.1 and Theorem 3.2.4. ■

Corollary 3.2.7 (Simple Version of Ito's Formula)**.** *Suppose that ϕ is a twice continuously differentiable function. Suppose that $\{B_n\}_{n\in\mathbb{N}}$ is a series of random walks and let B denote the Brownian motion we get by taking the ultralimit. Suppose that $r,t\in I$ and that $r<t$. Furthermore suppose that $\{\phi'(B_n(t_n))^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for for each $\{t_n\}_{n\in\mathbb{N}}\in\mathcal{T}$. Then*

$$\phi(B(t))=\phi(B(r))+\int_r^t\phi'(B(s))\ dB(s)+\frac{1}{2}\int_r^t\phi''(B(s))\ ds.$$

Proof. This follows from Proposition 3.2.3 and Theorem 3.2.4. ■

CHAPTER 4

Stochastic differential equations

4.1 Strong solutions to stochastic differential equations

In thi section we will focus on strong solutions to stochastic differential equations, namely we will find criteria for when there exists a strong solutions to a stochastic differential equation. First we have the following result, which might look a little impractical at first, but will be useful later on.

Proposition 4.1.1. *Suppose that $a : I \times \mathbb{R} \to \mathbb{R}$ and $b : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions and that we have a stochastic differential equation*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad X(0) = x_0. \tag{4.1}$$

Suppose that for each $n \in \mathbb{N}$, X_n is the solution to the difference equation

$$\Delta X_n(t_n) = a(t_n, X_n(t_n)))\Delta t_n + b(t_n, X_n(t_n))\Delta B_n(t_n), \quad X_n(0) = x_0^n, \tag{4.2}$$

where $\lim_{\mathcal{U}} x_0^n = x_0$, and that the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists and is continuous for almost all $\omega \in \Omega$. If $\{b(t_n, X_n(t_n))^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for all $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$, then X is a solution to (4.1).

Proof. Suppose that X_n is the solution to (4.2) for each $n \in \mathbb{N}$, and that the ultralimit X of $\{X_n\}_{n\in\mathbb{N}}$ exists and is continuous for almost all $\omega \in \Omega$. Suppose that $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$ and $\lim_{\mathcal{U}} t_n = t \in I$. We have that

$$X_n(t_n) - X_n(0) = \int_0^{t_n} a(s_n, X_n(s_n))\ ds_n + \int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n)$$

By an argument similar to the one in Theorem 3.2.4,

$$\lim_{\mathcal{U}} \int_0^{t_n} a(s_n, X_n(s_n))\ ds_n = \int_0^{t} a(s, X(s))\ ds$$

and by Theorem 3.2.4, since $\{b(t_n, X_n(t_n))^2\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable for all $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$,

$$\lim_{\mathcal{U}} \int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) = \int_0^{t} b(s, X(s))\ dB(s).$$

Hence X is a solution to (4.1). ■

We want to find constraints on a and b such that the solution to (4.1) exists. Suppose that $a : I \times \mathbb{R} \to \mathbb{R}$ and $b : I \times \mathbb{R} \to \mathbb{R}$ are continuous, bounded functions and that we have a stochastic differential equation as in (4.1). Suppose that for each $n \in \mathbb{N}$, X_n is the solution to the difference equation (4.2). First we show that, given $t \in I$,

$$P(\lim_{\mathcal{U}} X_n(s_n) = \lim_{\mathcal{U}} X_n(t_n) \text{ for all } [t_n], [s_n] \in T \text{ such that}$$
$$\lim_{\mathcal{U}} s_n = \lim_{\mathcal{U}} t_n = t) = 1.$$

Since a and b are bounded, there is an $M \in \mathbb{R}$ such that $|a| \leq M$ and $|b| \leq M$. Since

$$|\int_{r_n}^{t_n} a(s_n, X_n(s_n))\ ds_n| \leq M|t_n - r_n|,$$

$$\lim_{\mathcal{U}} \int_0^{t_n} a(s_n, X_n(s_n))\ ds_n = \lim_{\mathcal{U}} \int_0^{r_n} a(s_n, X_n(s_n))\ ds_n$$

for all $[s_n], [t_n] \in T$ such that $\lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n$ for all $\omega \in \Omega$. Hence it suffices to check that

$$P(\lim_{\mathcal{U}} \int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) = \lim_{\mathcal{U}} \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n)$$
$$\text{for all } [t_n], [r_n] \in T \text{ such that } \lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} t_n = t) = 1$$

for all $t \in I$. We have that

$$\{\omega \in \Omega \mid \exists\ [t_n], [r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| \neq 0\}$$
$$\subseteq \bigcup_{m \in \mathbb{N}} \{\omega \in \Omega \mid \exists\ [t_n], [r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{m}\}.$$

For each $k \in \mathbb{N}$ we have

$$\{\omega \in \Omega \mid \exists\ [t_n], [r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that}$$
$$|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{m}\}$$
$$\subseteq \{\omega \in \Omega \mid \lim_{\mathcal{U}} \sup_{r_n \in [t-\frac{1}{k}, t+\frac{1}{k}] \cap I_n} |(\int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n)$$
$$- \int_0^{t-\frac{1}{k}} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{2m}\}.$$

By Doob's martingale inequality (see [Doo53]),

$$
\begin{aligned}
&P(\lim_{\mathcal{U}} \sup_{r_n\in[t-\frac{1}{k},t+\frac{1}{k}]\cap I_n} |(\int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n) \\
&- \int_0^{t-\frac{1}{k}} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{2m}) \\
&\leq \lim_{\mathcal{U}} P_n(\sup_{r_n\in[t-\frac{1}{k},t+\frac{1}{k}]\cap I_n} |(\int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n) \\
&- \int_0^{t-\frac{1}{k}} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{4m}) \\
&\leq \lim_{\mathcal{U}} 16m^2 E_n[(\int_{t-\frac{1}{k}}^{t+\frac{1}{k}} b(X_n(s_n), s_n)\ dB_n(s_n))^2] \leq 32m^2M^2\frac{1}{k}.
\end{aligned}
$$

Since this is true for each $k \in \mathbb{N}$,

$$
\begin{aligned}
&\{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that} \\
&|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| \geq \frac{1}{m}\}
\end{aligned}
$$

is measurable and has measure 0, since it is the subset of a set of measure 0. Hence

$$
\begin{aligned}
&\{\omega \in \Omega \mid \exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that} \\
&|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| \neq 0\}
\end{aligned}
$$

is measurable and has measure 0. Thus

$$
\begin{aligned}
&P(\exists\ [t_n],[r_n] \in T \text{ with } \lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n = t \text{ such that} \\
&|\lim_{\mathcal{U}}(\int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) - \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n))| = 0) = 1.
\end{aligned} \tag{4.3}
$$

We want to show that X exists and is continuous for almost all $\omega \in \Omega$. Since a is bounded,

$$
\lim_{\mathcal{U}} \int_0^{t_n} a(X_n(s_n))\ ds_n = \lim_{\mathcal{U}} \int_0^{r_n} a(X_n(s_n))\ ds_n
$$

whenever $\lim_{\mathcal{U}} t_n = \lim_{\mathcal{U}} r_n$. Hence it suffices to check that

$$
\begin{aligned}
&P(\lim_{\mathcal{U}} \int_0^{t_n} b(s_n, X_n(s_n))\ dB_n(s_n) = \lim_{\mathcal{U}} \int_0^{r_n} b(s_n, X_n(s_n))\ dB_n(s_n) \\
&\quad \text{whenever } [t_n] \approx [r_n] \text{ for all } [t_n],[r_n] \in T) = 1.
\end{aligned}
$$

We will use the following notation: For $[t_n], [r_n] \in T$, we will write $[r_n] \leq [t_n]$ to mean that $\lim_{\mathcal{U}} r_n \leq \lim_{\mathcal{U}} t_n$. Similarly, we will write $k_1 \leq [t_n] \leq k_2$ to mean that $k_1 \leq \lim_{\mathcal{U}} t_n \leq k_2$. Let $Y : T \times \Omega \to \mathbb{R}$ be defined by

$$Y([t_n], [\omega_n]) = \lim_{\mathcal{U}} Y_n(t_n, \omega_n),$$

where

$$Y_n(t_n, \omega_n) = \left(\int_0^{t_n} b(s_n, X_n(s_n)) \, dB_n(s_n) \right) (\omega_n).$$

We will proceed in a similar way to the proof of Theorem 3.1.18. For $m, k \in \mathbb{N}$, let

$$\Omega_{m,k} = \{[\omega_n] \in \Omega \mid \forall i \in \{0, 1, ..., k-1\} \left(\sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| < \frac{1}{m} \right) \Bigg\}.$$

In order to show that $\bar{Y}$ exists as in Definition 3.1.6, it suffices to show that

$$\Omega' = \bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \Omega_{m,k}$$

has measure 1, since

$$\begin{aligned} \Omega' = \{[\omega_n] \in \Omega \mid \ & \lim_{\mathcal{U}} Y_n(t_n, \omega_n) = \lim_{\mathcal{U}} Y_n(r_n, \omega_n) \\ & \text{whenever } [t_n] \approx [r_n] \text{ for all } [t_n], [r_n] \in T\}. \end{aligned}$$

We show that

$$\Omega'^c = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \Omega^c_{m,k}$$

has measure 0 (we will soon show that these sets are indeed measurable). Suppose that $P(\Omega'^c) > 0$. Then there is $m_0 \in \mathbb{N}$ such that

$$P(\bigcap_{k \in \mathbb{N}} \Omega^c_{m_0,k}) > 0.$$

We show that $\lim_{k \to \infty} P(\Omega^c_{m_0,k}) = 0$, which leads to a contradiction. We have that

$$\begin{aligned} \Omega^c_{m_0,k} &= \{[\omega_n] \in \Omega \mid \exists i \in \{0, 1, ..., k-1\} \text{ such that} \\ & \qquad \sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| \geq \frac{1}{m_0} \Bigg\} \\ &= \bigcup_{i=0}^{k-1} \left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| \geq \frac{1}{m_0} \right\}. \end{aligned}$$

For each $i \in \{0, ..., k\}$, let $\{s_{n,i}\}_{\in \mathbb{N}} \in \mathcal{T}$ be such that $\lim_{\mathcal{U}} s_{n,i} = \frac{iT}{k}$. Then, by (4.3),

$$\sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| = \lim_{\mathcal{U}} \sup_{s_{n,i} \leq t_n \leq s_{n,i+1}} |Y_n(t_n, \omega_n) - Y_n(s_{n,i}, \omega_n)|$$

for almost all $[\omega_n] \in \Omega$ (this shows that the sets are measurable). Then

$$\begin{aligned} P(\Omega^c_{m_0,k}) &= P\left(\bigcup_{i=0}^{k-1} \left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| \geq \frac{1}{m_0} \right\} \right) \\ &\leq \sum_{i=0}^{k-1} P\left(\left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| \geq \frac{1}{m_0} \right\} \right), \end{aligned}$$

where we have used the old summation convention. We have that

$$\begin{aligned} &P\left(\left\{ [\omega_n] \in \Omega \mid \sup_{\substack{[r_n] \leq [t_n] \leq \frac{(i+1)T}{k} \\ \lim_{\mathcal{U}} r_n = \frac{iT}{k}}} |Y([t_n], [\omega_n]) - Y([r_n], [\omega_n])| \geq \frac{1}{m_0} \right\} \right) \\ &\leq \lim_{\mathcal{U}} P_n \left(\left\{ \omega_n \in \Omega_n \mid \sup_{s_{n,i} \leq t_n \leq s_{n,i+1}} |Y_n(t_n, \omega_n) - Y_n(s_{n,i}, \omega_n)| \geq \frac{1}{2m_0} \right\} \right) \\ &\qquad \text{by (4.3)} \\ &\leq \lim_{\mathcal{U}} 16 m_0^4 E_n \left[\left(\sup_{s_{n,i} \leq t_n \leq s_{n,i+1}} |Y_n(t_n, \omega_n) - Y_n(s_{n,i}, \omega_n)| \right)^4 \right] \\ &\qquad \text{by Markov's inequality (see [Bil08]).} \end{aligned}$$

Next we will use the Burkholder-Davis-Gundy inequality for martingales (see [BDG72]), which gives us that

$$\begin{aligned} &E_n \left[\left(\sup_{s_{n,i} \leq t_n \leq s_{n,i+1}} |Y_n(t_n, \omega_n) - Y_n(s_{n,i}, \omega_n)| \right)^4 \right] \\ &\leq C E_n [[Y_n(t_n) - Y_n(s_{n,i})]^2_{t_n = s_{n,i+1}}], \end{aligned}$$

for some $C \in \mathbb{R}$, where

$$[Y_n(t_n) - Y_n(s_{n,i})]_{t_n = s_{n,i+1}} = \sum_{s_n = s_{n,i}}^{s_{n,i+1}} b(s_n, X_n(s_n))^2 \Delta s_n \leq M^2 |s_{n,i+1} - s_{n,i}|$$

is the quadratic variation of $Y_n(s_{n,i+1}) - Y_n(s_{n,i})$. Hence

$$P(\Omega^c_{m_0,k}) \leq \lim_{\mathcal{U}} 16 m_0 C k M^4 |s_{n,i+1} - s_{n,i}|^2 = 16 m_o^4 C M^4 T^2 \frac{1}{k} \to 0 \text{ as } k \to \infty,$$

which is what we wanted to show. By Proposition 3.1.3, the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists and is continuous for almost all $\omega \in \Omega$. Hence X is a solution to (4.1). We will formulate this as a lemma.

Lemma 4.1.2. *Suppose that $a : I \times \mathbb{R} \to \mathbb{R}$ and $b : I \times \mathbb{R} \to \mathbb{R}$ are bounded, continuous functions and that we have a stochastic differential equation*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad X(0) = x_0. \tag{4.4}$$

Suppose that for each $n \in \mathbb{N}$, X_n is the solution to the difference equation

$$\Delta X_n(t_n) = a(t_n, X_n(t_n)))\Delta t_n + b(t_n, X_n(t_n))\Delta B_n(t_n), \quad X_n(0) = x_0^n,$$

where $\lim_{\mathcal{U}} x_0^n = x_0$, Then the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists, is continuous and is a solution to (4.4) for almost all $\omega \in \Omega$.

Theorem 4.1.3. *Suppose that $a : I\times\mathbb{R} \to \mathbb{R}$ and $b : I\times\mathbb{R} \to \mathbb{R}$ are continuous functions with at most linear growth, i.e. there is a $C \in \mathbb{R}$ such that*

$$|a(t, x)| + |b(t, x)| \le C(1 + |x|).$$

Suppose we have a stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad X(0) = x_0. \tag{4.5}$$

and suppose that for each $n \in \mathbb{N}$, X_n is the solution to the difference equation

$$\Delta X_n(t_n) = a(t_n, X_n(t_n)))\Delta t_n + b(t_n, X_n(t_n))\Delta B_n(t_n), \quad X_n(0) = x_0^n$$

where $\lim_{\mathcal{U}} x_0^n = x_0$, Then the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ exists, is continuous and is a solution to (4.5) for almost all $\omega \in \Omega$.

Proof. Let

$$\begin{aligned}\Omega' =&\{\omega \in \Omega \mid X(\omega) \text{ exists and satisfies} \\ &X(t, \omega) = x_0 + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dB(s) \text{ for all } t \in I\}.\end{aligned}$$

For $m \in \mathbb{R}_+$, let $a^m : I \times \mathbb{R}$ and $b^m : I \times \mathbb{R}$ be defined by

$$a^m(s, x) = \begin{cases} a(s, x) & \text{if } |a(s, x)| \le m \\ m & \text{if } a(s, x) \ge m \\ -m & \text{if } a(s, x) \le -m \end{cases}$$

and

$$b^m(s, x) = \begin{cases} b(s, x) & \text{if } |b(s, x)| \le m \\ m & \text{if } b(s, x) \ge m \\ -m & \text{if } b(s, x) \le -m \end{cases}$$

For each $N \in \mathbb{N}$, let

$$\begin{aligned}\Omega_N =&\{\omega \in \Omega \mid X^N(\omega) \text{ exists and } X^N(t, \omega) \\ &= x_0 + \int_0^t a^{C(1+N)}(s, X^N(s, \omega))\, ds + \int_0^t b^{C(1+N)}(s, X^N(s, \omega))\, dB(s, \omega) \\ &\text{for all } t \in I\}.\end{aligned}$$

Then for each $N \in \mathbb{N}$,

$$\Omega'^c \subseteq \{\omega \in \Omega \mid \sup_{[t_n]\in T} |X([t_n],\omega)| \geq N\} \cup \Omega_N^c.$$

Indeed, if

$$\sup_{[t_n]\in T} |X([t_n],\omega)| < N,$$

then

$$|a|, |b| \leq C(1+N)$$

so that either $\omega \in \Omega_N^c$ or $X(\omega)$ exists and

$$\begin{aligned} X(\omega,t) &= x_0 + \int_0^t a^{C(1+N)}(s, X(s,\omega))\ ds + \int_0^t b^{C(1+N)}(s, X(s,\omega))\ dB(s,\omega) \\ &= x_0 + \int_0^t a(s, X(s,\omega))\ ds + \int_0^t b(s, X(s,\omega))\ dB(s,\omega) \end{aligned}$$

by Lemma 4.1.2. We have that

$$\begin{aligned} P(\sup_{[t_n]\in T} |X([t_n],\omega)| \geq N) &= P(\lim_{\mathcal{U}} \sup_{t_n \in I_n} |X_n(t_n)| \geq N) \\ &\leq \lim_{\mathcal{U}} P_n(\sup_{t_n\in I_n} |X_n(t_n)| \geq \frac{N}{2}). \end{aligned}$$

Furthermore, by Markov's inequality (see [Bil08]),

$$P_n(\sup_{t_n\in I_n} |X_n(t_n)| \geq \frac{N}{2}) \leq \frac{4}{N^2} E_n[\sup_{t_n\in I_n} X_n(t_n)^2].$$

We have that for each $u_n \in I_n$,

$$\begin{aligned} &E_n[\sup_{t_n\in I_n\cap[0,u_n]} X_n(t_n)^2] \\ &\leq 2(E_n[\sup_{t_n\in I_n\cap[0,u_n]} (\int_0^{t_n} a(s_n, X_n(s_n))\ ds_n)^2] + E_n[\sup_{t_n\in I_n\cap[0,u_n]} (\int_0^{t_n} b(s_n, X_n(s_n))\ dB(s_n))^2]), \end{aligned}$$

where

$$E_n[\sup_{t_n\in I_n\cap[0,u_n]} (\int_0^{t_n} a(s_n, X_n(s_n))\ ds_n)^2] \leq u_n \int_0^{u_n} E_n[a(s_n, X_n(s_n))^2] ds_n$$

by Hölder's inequality (see [Doo12]) and

$$E_n[\sup_{t_n\in I_n\cap[0,u_n]} (\int_0^{t_n} b(s_n, X_n(s_n))\ dB(s_n))^2] \leq 2\int_0^{u_n} E_n[b(s_n, X_n(s_n))^2]\ ds_n$$

by Doob's martingale inequality (see [Doo53]). Since

$$a(s_n, X_n(s_n))^2, b(s_n, X_n(s_n))^2 \leq 2C^2 + 2C^2 X_n(s_n)^2 \leq 2C^2 + 2C^2 \sup_{r_n\in I_n\cap[0,s_n]} X_n(r_n)^2$$

we have

$$E_n[\sup_{t_n\in I_n\cap[0,u_n]} X_n(t_n)^2] \leq C^2(4T(T+2) + C^2(4(T+2) \int_0^{u_n} E_n[\sup_{r_n\in I_n\cap[0,s_n]} X_n(r_n)^2]\ ds_n.$$

By Gronwall's inequality (see [Øks03]),

$$E_n[\sup_{t_n \in I_n} X_n(t_n)^2] \leq C^2 4T(T+2)e^{C^2 4T(T+2)}.$$

Hence

$$P(\sup_{[t_n] \in T} |X([t_n], \omega)| \geq N) \leq \frac{16C^2}{N^2} T(T+2)e^{C^2 4T(T+2)} \to 0 \text{ as } N \to \infty.$$

Since (by Lemma 4.1.2) $P(\Omega_N^c) = 0$ for all $N \in \mathbb{N}$, we get that Ω'^c is contained in a set of measure 0 and is thus measurable with measure zero. Hence $P(\Omega') = 1$, which is what we wanted to show. ■

Below is a widely known stochastic differential equation, whose (unique) solution is called geometric Brownian motion, which we will use later in this thesis.

Proposition 4.1.4. *Consider the stochastic differential equation*

$$dX(t) = \alpha dt + \beta dB(t), \quad X(0) = x_0, \tag{4.6}$$

where $\alpha, \beta \in \mathbb{R}$. Define a process $X_n : I_n \times \Omega_n \to \mathbb{R}$ by $X_n(0) = x_0 \in \mathbb{R}$ and

$$\Delta X_n(t_n, \omega_n) = X_n(t_n, \omega_n)(\alpha \Delta t_n + \beta \Delta B_n(t_n, \omega_n)) \text{ for all } t \in I_n.$$

Then

$$X_n(t_n, \omega_n) = x_0 \prod_{s_n=0}^{t_n - \Delta t_n} (1 + \alpha \Delta t_n + \beta \Delta B_n(s_n, \omega_n)).$$

Furthermore, when n is large, then

$$X_n(t_n, \omega_n) \approx x_0 e^{(\alpha - \frac{1}{2}\beta^2)t_n + \beta B_n(t_n, \omega_n)}$$

so that $X : I \times \Omega \to \mathbb{R}$ given by $X(t) = x_0 e^{(\alpha - \frac{1}{2}\beta^2)t_n + \beta B_n(t_n)}$ is a solution to (4.6).

Proof. We have that

$$\Delta X_n(t_n, \omega_n) = X_n(t_n, \omega_n)(\alpha \Delta t_n + \beta \Delta B_n(t_n, \omega_n)) \text{ for all } t_n \in I_n$$

and that

$$\Delta X_n(t_n, \omega_n) = X_n(t_n + \Delta t_n, \omega_n) - X_n(t_n, \omega_n),$$

hence

$$\begin{aligned} X_n(t_n + \Delta t_n) &= X_n(t_n, \omega_n)(1 + \alpha \Delta t_n + \beta \Delta B_n(t_n, \omega_n)) \\ &= X_n(t_n - \Delta t_n, \omega_n)(1 + \alpha \Delta t_n + \beta \Delta B_n(t_n, \omega_n)) \\ &\times (1 + \alpha \Delta t_n + \beta \Delta B_n(t_n - \Delta t_n, \omega_n)) \\ &= \ldots \\ &= x_0 \prod_{s_n=0}^{t_n} (1 + \alpha \Delta t_n + \beta \Delta B_n(s_n, \omega_n)). \end{aligned}$$

Let $Y_n : I_n \times \Omega_n \to \mathbb{R}$ be defined by

$$Y_n(t_n, \omega_n) = \ln(X_n(t_n, \omega_n)) = \ln(x_0) + \sum_{s_n=0}^{t_n} \ln(1 + \alpha\Delta t_n + \beta\Delta B_n(s_n, \omega_n)).$$

By Taylor series representation $\ln(1+x) \approx x - \frac{x^2}{2}$ when x is close to 0. So

$$\begin{aligned}\ln(1 + \alpha\Delta t_n + \beta\Delta B_n(s_n, \omega_n)) &\approx \alpha\Delta t_n + \beta\Delta B_n(s_n, \omega_n) \\ &\quad - \frac{1}{2}(\alpha\Delta t_n - \beta\Delta B_n(s_n, \omega_n))^2 \\ &= \alpha\Delta t_n + \beta\Delta B_n(s_n, \omega_n)) - \frac{1}{2}\alpha^2(\Delta t_n)^2 \\ &\quad - \frac{1}{2}\beta^2(\Delta B_n(s_n, \omega_n))^2 - \alpha\Delta t_n\beta\Delta B_n(s_n, \omega_n) \\ &\approx (\alpha - \frac{1}{2}\beta^2)\Delta t_n + \beta\Delta B_n(s_n, \omega_n)\end{aligned}$$

since the last two terms are of higher order and can be omitted. So we get that

$$\begin{aligned}Y_n(t_n, \omega_n) &= \ln(x_0) + \sum_{s_n=0}^{t_n} \ln(1 + \alpha\Delta t_n + \beta\Delta B_n(s_n, \omega_n)) \\ &\approx \ln(x_0) + \sum_{s_n=0}^{t_n} (\alpha - \frac{1}{2}\beta^2)\Delta t_n + \beta\Delta B_n(s_n, \omega_n) \\ &= \ln(x_0) + (\alpha - \frac{1}{2}\beta^2)t_n + \beta B_n(t_n, \omega_n).\end{aligned}$$

Therefore, when n is large,

$$X_n(t_n, \omega_n) \approx x_0 e^{(\alpha - \frac{1}{2}\beta^2)t_n + \beta B_n(t_n, \omega_n)}.$$

If $t = \lim_{\mathcal{U}} t_n$, then $\lim_{\mathcal{U}} \ln(X_n(t_n, \omega_n)) = \ln(x_0) + (\alpha - \frac{1}{2}\beta^2)t + \beta B(t, [\omega_n])$ and thus $\lim_{\mathcal{U}} X_n(t_n, \omega_n) = x_0 e^{(\alpha - \frac{1}{2}\beta^2)t + \beta B(t, [\omega_n])}$, where B is a Brownian motion. ■

4.2 Weak solutions to stochastic differential equations

In this section we will look at weak solutions to stochastic differential equations. We will use Girsanov's theorem, which we will state here without proof (which can be found in [Øks03]).

Theorem 4.2.1 (Girsanov's theorem for stochastic differential equations)**.** *Let $a, a' : I \times \mathbb{R} \to \mathbb{R}$ and $b : I \times \mathbb{R} \to \mathbb{R}$. Let X be an Ito process of the form*

$$dX_t = a'(t, X(t))dt + b(t, X(t))dB_t, \quad X(0) = x_0,$$

and suppose there exists functions $a : I \times \mathbb{R} \to \mathbb{R}$ and $\theta : I \times \Omega \to \mathbb{R}$ such that

$$b(t, X(t))\theta(t) = a'(t, X(t)) - a(t, X(t)).$$

Put

$$M(t) = e^{-\int_0^t \theta(s)\ dB_s - \frac{1}{2}\int_0^t \theta(s)^2\ ds}$$

and

$$dQ = M_T dP$$

on $\mathcal{F}$. Assume that M is a martingale (with respect to $\mathcal{F}$ and P). Then Q is a probability measure on $\mathcal{F}$, the process $\hat{B}$ defined by

$$\hat{B}(t) = \int_0^t \theta(s)\ ds + B(t)$$

is a Brownian motion with respect to Q and in terms of Q, the process X has the stochastic integral representation

$$dX_t = a(t, X(t))dt + b(t, X(t))d\hat{B}_t.$$

Suppose we have two differential equations

$$dX_t = a(X(t))dt + b(X(t))dB_t, \quad X(0) = x_0 \tag{4.7}$$

and

$$dX'_t = a'(X'(t))dt + b(X'(t))dB_t, \quad X'(0) = x_0 \tag{4.8}$$

and that we know the solution to (4.8) and that we wish to find out whether there exists a solution to (4.7). Girsanov's theorem says that if

$$\theta(s) = \frac{a'(X'(s)) - a(X'(s))}{b(X'(s))}$$

is well-defined and

$$M(t) = e^{\int_0^t \theta(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds}$$

is a martingale, then a weak solution to (4.7) exists.

We will consider stochastic differential equations of the form

$$dX_t = X(t)(\alpha + \gamma \ln X(t))dt + \beta X(t)dB_t, \quad X(0) = x_0 \tag{4.9}$$

for some constants $\alpha, \beta, \gamma \in \mathbb{R}$ with $\beta \neq 0$ and $x_0 > 0$. We will show that there exists a weak solution to (4.9) even though it does not satisfy the linear growth assumptions in Theorem 4.1.3.

First we solve the stochastic differential equation

$$dX_t = \alpha X(t)dt + \beta X(t)dB_t, \quad X(0) = x_0,$$

which has the solution

$$X(t) = x_0 e^{\beta B_t + (\alpha - \frac{1}{2}\beta^2)t}.$$

For each $n \in \mathbb{N}$ define $\theta_n : I_n \times \Omega_n \to \mathbb{R}$ by

$$\theta_n(t_n) = \frac{\gamma}{\beta}(\ln(x_0) + \beta B_n(t_n) + (\alpha - \frac{1}{2}\beta)t_n)$$

and $\theta'_n : I_n \times \Omega_n \to \mathbb{R}$ by

$$\theta'_n(t_n) = \begin{cases} \theta_n(t_n) & \text{if } |\theta_n(t_n)| \leq \frac{1}{2}\sqrt{n} \\ \frac{1}{2}\sqrt{n} & \text{if } \theta_n(t_n) \geq \frac{1}{2}\sqrt{n} \\ -\frac{1}{2}\sqrt{n} & \text{if } \theta_n(t_n) \leq -\frac{1}{2}\sqrt{n}. \end{cases}$$

We then have that there is a constant C such that

$$\theta'_n(t_n)^2 \leq \theta_n(t_n)^2 \leq C(1 + B_n(t_n)^2)$$

for all $t_n \in I_n$. Let $M_n : I_n \times \Omega_n \to \mathbb{N}$ be defined by

$$M_n(t_n) = \prod_{s_n=0}^{t_n - \Delta t_n} (1 + \theta'_n(s_n)\Delta B_n(s_n)).$$

and let $M : I \times \Omega \to \mathbb{R}$ denote the ultralimit. Then by Taylor approximation (using the same argument as in Proposition 4.1.4),

$$M(t) = e^{\int_0^t \theta(s)\ dB_s - \frac{1}{2}\int_0^t \theta(s)^2\ ds}$$

for almost all $\omega \in \Omega$, where θ is the ultralimit of $\{\theta_n\}_{n\in\mathbb{N}}$ which satisfies

$$\theta(s) = \frac{\alpha X(s) - X(s)(\alpha + \gamma \ln X(s))}{\beta X(s)} \tag{4.10}$$

for almost all $[\omega_n] \in \Omega$ (note that we can use the Taylor series argument since θ exists and is continuous for almost all $[\omega_n] \in \Omega$). Indeed, since the ultralimit θ of $\{\theta_n\}_{n\in\mathbb{N}}$ exists for almost all $[\omega_n] \in \Omega$, for almost all $[\omega_n] \in \Omega$, $\theta([\omega_n])$ is continuous for almost all $[\omega_n] \in \Omega$. If $[\omega_n] \in \Omega$ is such that θ is continuous, then $|\theta([\omega_n])|$ is bounded (since it's defined on a compact interval and since B takes values in $\mathbb{R}$) and so there is $G \in \mathcal{U}$ and $K \in \mathbb{N}$ such that $|\theta_n(\omega_n)| \leq K$ for all $n \in F$. Then for all $n \in F \cap \{1, .., 4K^2 - 1\}^c$, $\theta'_n(\omega_n) = \theta_n(\omega_n)$. So θ is the ultralimit of θ'_n. Since $\theta_n(t_n)^4$ is $\mathcal{A}$-integrable for any $\{t_n\}_{n\in\mathbb{N}}$, so is $\theta'_n(t_n)^4$ for any $\{t_n\}_{n\in\mathcal{T}}$. So we have that the ultralimit to $\{\int_0^{t_n} \theta'_n(s_n)\ dB_n(s_n)\}$ (which is $\int_0^t \theta(s)\ dB_s$, where $t = \lim_{\mathcal{U}} t_n$) exists and is continuous for almost all $[\omega_n] \in \Omega$. Again, since the ultralimit to $\{\theta'_n\}_{n\in\mathbb{N}}$ exists for almost all $[\omega_n] \in \Omega$, by the same argument as in Proposition 4.1.4 (since $\{\theta_n(\omega_n)'\}_{n\in\mathbb{N}}$ is $\mathcal{U}$-bounded for almost all $[\omega_n] \in \Omega$) we have that

$$\lim_{\mathcal{U}} \prod_{s_n=0}^{t_n - \Delta t_n} (1 + \theta'_n(s_n)\Delta B_n(s_n)) = e^{\int_0^t \theta(s)\ dB_s + \frac{1}{2}\int_0^t \theta(s)^2\ ds}$$

for any $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{\mathcal{U}} t_n = t$. Furthermore, since θ_n satisfies

$$\theta_n(s_n) = \frac{\alpha X_n(s_n) - X_n(s_n)(\alpha + \gamma \ln X_n(s_n))}{\beta X_n(s_n)},$$

where X_n is given is given in Proposition 4.1.4, we have that equation (4.10) holds.

By Proposition 3.1.15, since each M_n is a local martingale, we have that M is a local martingale. Indeed,

$$\begin{aligned}&E_n[\sup_{t_n \in I_n} \sup_{0<t_n\leq\tau_{k,n}} (M_n(t_n) - M_n(t_n - \Delta t_n))^2] \\ &\leq k^2 E_n[\theta'_n(t_n - \Delta t_n)^2 (\Delta B_n(t_n - \Delta t_n))^2] \leq \frac{1}{4}k^2\end{aligned}$$

and $E_n[M_n(0)] = 1$ for each $n \in \mathbb{N}$. We want to show that

$$\lim_{\mathcal{U}} E_n[M_n(T) \ln^+ M_n(T)] < \infty.$$

It then follows by Proposition 3.1.16 that M is a martingale. Let $\ln^- : (0, \infty) \to \mathbb{R}$ be defined by

$$\ln^-(x) = \ln(x) - \ln^+(x).$$

Then we have

$$E_n[M_n(T) \ln^+(M_n(T))] = E_n[M_n(T) \ln(M_n(T))] - E_n[M_n(T) \ln^-(M_n(T))].$$

Since $|x \ln(x)| \leq 1$ for $x < 1$, it suffices to check that

$$\lim_{\mathcal{U}} E_n[M_n(T) \ln(M_n(T))] < \infty.$$

Letting Q_n be the probability measure defined by $dQ_n = M_n(T)dP_n$, we consider the term

$$E_{Q_n}[\ln(1 + \theta'_n(r_n)\Delta B_n(r_n))]$$

for some $r_n \in I_n$, $r_n \leq T - \Delta t_n$. By Jensen's inequality, since ln is a concave function, we have

$$E_{Q_n}[\ln(1 + \theta'_n(r_n)\Delta B_n(r_n))] \leq \ln(1 + E_{Q_n}[\theta'_n(r_n)\Delta B_n(r_n)]).$$

We will next use the following theorem, which we will state here without proof (which can be found in [Øks03]):

Theorem 4.2.2 (Bayes' rule)**.** *Let $\mathcal{G}_n \subseteq \mathcal{F}_n$ be any sub-σ-algebra. For any $\mathcal{F}_n$-measurable random variable X_n, we have*

$$E_{Q_n}[X_n \mid \mathcal{G}_n] E_n\left[\frac{dQ_n}{dP_n} \mid \mathcal{G}_n\right] = E_n\left[X_n \frac{dQ_n}{dP_n} \mid \mathcal{G}_n\right].$$

Lemma 4.2.3.

$$E_{Q_n}[\theta'_n(r_n)\Delta B_n(r_n)] \mid \mathcal{F}^n_{r_n}] = \theta'_n(r_n)^2 \Delta t_n.$$

Proof. This follows by Theorem 4.2.2. Indeed, we have that

$$\begin{aligned}
E_n[M_n(T) \mid \mathcal{F}^n_{r_n}] &= E_n[\prod_{s_n=0}^{T-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) \mid \mathcal{F}^n_{r_n}] \\
&= \left(\prod_{s_n=0}^{r_n-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n))\right) \\
&\quad \times E_n\left[\prod_{s_n=r_n}^{T-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) \mid \mathcal{F}^n_{r_n}\right] \\
&= \prod_{s_n=0}^{r_n-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) = M_n(r_n).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E_n[\theta'_n(r_n)\Delta B_n(r_n)M_n(T)] &= E_n[\theta'_n(r_n)\Delta B_n(r_n)\prod_{s_n=0}^{T-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) \mid \mathcal{F}^n_{r_n}] \\
&= \left(\prod_{s_n=0}^{r_n-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n))\right) \\
&\quad \times E_n\left[\theta'_n(r_n)\Delta B_n(r_n)\prod_{s_n=r_n}^{T-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) \mid \mathcal{F}^n_{r_n}\right] \\
&= \left(\prod_{s_n=0}^{r_n-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n))\right) \\
&\quad \times E_n[(\theta'_n(r_n)\Delta B_n(r_n)+\theta'_n(r_n)^2\Delta t_n)] \\
&\quad \times E_n\left[\prod_{s_n=r_n+\Delta t_n}^{T-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n)) \mid \mathcal{F}^n_{r_n}\right] \\
&= \left(\prod_{s_n=0}^{r_n-\Delta t_n}(1+\theta'_n(s_n)\Delta B_n(s_n))\right)\theta'_n(r_n)^2\Delta t_n.
\end{aligned}$$

Hence

$$E_{Q_n}[\theta'_n(r_n)\Delta B_n(r_n)] = \frac{E_n[\theta'_n(r_n)\Delta B_n(r_n)M_n(T) \mid \mathcal{F}^n_{r_n}]}{E_n[M_n(T) \mid \mathcal{F}^n_{r_n}]} = \theta'_n(r_n)^2\Delta t_n.$$

■

By Lemma 4.2.3 we have

$$E_{Q_n}[\theta'_n(r_n)\Delta B_n(r_n)] = E_{Q_n}[\theta'_n(r_n)^2]\Delta t_n \le C(1+E_{Q_n}[B_n(r_n)^2])\Delta t_n$$

for some constant C by definition of θ_n. By Proposition 3.2.1 we have

$$B_n(r_n)^2 = \sum_{s_n=0}^{r_n} 2B_n(s_n)\Delta B_n(s_n) + r_n.$$

Again (using an argument similar to the one in Lemma 4.2.3), for high enough n, by conditional expectation, we have

$$E_{Q_n}[2B_n(s_n)\Delta B_n(s_n)] \leq C'(1 + E_{Q_n}[B_n(s_n)^2])\Delta t_n$$

for some constant C'. To summarize,

$$\begin{aligned} E_{Q_n}[\theta_n'(r_n)\Delta B_n(r_n)] &= E_{Q_n}[\theta_n'(r_n)^2]\Delta t_n \\ &\leq C(1 + E_{Q_n}[B_n(r_n)^2])\Delta t_n. \end{aligned}$$

Since we have

$$E_{Q_n}[B_n(r_n)^2] \leq \sum_{s_n=0}^{r_n} C'(1 + E_{Q_n}[B_n(s_n)^2])\Delta t_n + r_n),$$

by Gronwall's inequality,

$$E_{Q_n}[B_n(r_n)^2] \leq (C' + T)e^{C'T}.$$

(where T is the endpoint of the interval $[0, T]$) and so

$$E_{Q_n}[\theta_n'(r_n)\Delta B_n(r_n)] \leq C(1 + (C' + T)e^{C'T})\Delta t_n$$

So

$$E_{Q_n}[\ln(1 + \theta_n'(r_n)\Delta B_n(r_n))] \leq \ln(1 + C(1 + (C' + T)e^{C'T})\Delta t_n).$$

By Taylor approximation (using an argument similar to the one in Proposition 4.1.4),

$$\begin{aligned} \lim_{\mathcal{U}} \sum_{r_n=0}^{T} E_{Q_n}[\ln(1 + \theta_n'(r_n)\Delta B_n(r_n))] &\leq \lim_{\mathcal{U}} \sum_{r_n=0}^{T} \ln(1 + C(1 + (C' + T)e^{C'T})\Delta t_n) \\ &= C(1 + (C' + T)e^{C'T}T < \infty \end{aligned}$$

So M is a martingale. By Girsanov's theorem, a solution to (4.9) exists. We have thus proved the following result:

Proposition 4.2.4. *There exists a weak solution to the stochastic differential equation*

$$dX_t = X(t)(\alpha + \gamma \ln X(t))dt + \beta X(t)dB_t, \quad X(0) = x_0,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ *with* $\beta \neq 0$ *are constants and* $x_0 > 0$ *is constant.*

Proposition 4.2.4 also follows from Beněs condition, for which a proof can be found in [KL14].

Sometimes it can be useful to model a certain phenomenon in discrete time and take the limit to get a model for the behavior in continuous time. We will consider a particle moving along an axis and find a stochastic differential equation describing the position of the particle in continuous time. We start by discretizing the process. For each square number $n \in \mathbb{N}$ we let $\Delta x_n = \frac{1}{\sqrt{n}}$ and $S_n = \{0, \Delta x_n, ..., 1 - \Delta x_n, 1\}$ be a discretzation of the interval $[0, 1]$. As usual we let $I_n = \{0, \Delta t_n, 2\Delta t_n, ..., 1\}$, where $\Delta t_n = \frac{1}{n}$, be a discretization of the time interval. We assume that there is a function $\alpha : \mathbb{R} \to \mathbb{R}$ independent of n such that if the particle is in an inner point x_n of the interval at time $t_n \in I_n$, then the particle will be at the point $x_n + \Delta x_n$ at time $t_n + \Delta t_n$ with probability $\frac{1}{2}(1 + \alpha(x_n)\Delta x_n)$ and at the point $x_n - \Delta x_n$ at time $t_n + \Delta t_n$ with probability $\frac{1}{2}(1 - \alpha(x_n)\Delta x_n)$. We assume that if the particle is at the point 0 at time t_n, then the particle with be at the point Δx_n at time $t_n + \Delta t_n$. Similarly, we assume that if the particle is at the point 1 at time t_n, then the particle will be at the point $1 - \Delta x_n$ at time $t_n + \Delta t_n$. We will model the particle through a stochastic process X_n. At time $t = 0$ we assume that the process is at one of the points $x \in S_n$ with probability $\rho_n(x)$, that is $X_n(0) = Z_n$ is a random variable (taking values in S_n) distributed according to the probabilities given by ρ_n.

First we assume that the particle can move in the interval $\{-\sqrt{n}, -\sqrt{n} + \Delta x_n, ..., \sqrt{n} - \Delta x_n, \sqrt{n}, ..., \sqrt{n} + 1\}$ and that α is defined for all these point (this will make sense later on). For each path the particle may take there is a corresponding path the random walk B_n with starting value Z_n takes and vice versa. Both the particle x_n and the random walk B_n can either move up or down a distance of $\sqrt{\Delta t_n}$ during the time interval Δt_n. The only thing that separates them is the probability of going up and down. While the probability of going up for a random walk is the same as the probability of going down, namely $\frac{1}{2}$, the probability of the particle moving up is not necessarily the same as the probability of the particle moving down. Furthermore, the probability of the particle moving up or down is dependent on the current position of the particle. We want to find a measure Q_n such that the probability of B_n taking a certain path under Q_n is equal to the probability of x_n taking that path. We find Q_n by finding the Radon Nikodym derivative of Q_n with respect to P_n. We see that the Radon Nikodym derivative of Q_n with respect to P_n is

$$\frac{dQ_n}{dP_n} = \frac{\prod_{t_n=0}^{T-\Delta t_n} \frac{1}{2}(1 + \alpha(B_n(s_n))\Delta B_n(t_n))}{(\frac{1}{2})^n} = \prod_{t_n=0}^{T-\Delta t_n} (1 + \alpha(B_n(t_n))\Delta B_n(t_n)).$$

Then under Q_n the position of the particle is a solution to the difference equation

$$\Delta X_n(t_n) = \Delta B_n(t_n),$$

that is, $X_n(t_n) = B_n(t_n)$. If we can show that $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable, then by Proposition 3.1.13 $\frac{dQ}{dP} = [\frac{dQ_n}{dP_n}]$ is a martingale and so by Girsanov's theorem, if we let $Q = \frac{dQ}{dP}dP$, then $\hat{B} : I \times \Omega \to \mathbb{R}$ given by

$$\hat{B}(t) = B(t) - \int_0^t \alpha(X(s))ds$$

is a Brownian motion under Q (since we may define an ultrafilter on the square numbers) and so the ultralimit X to $\{X_n\}_{n\in\mathbb{N}}$ is a weak solution the stochastic differential equation

$$dX_t = \alpha(X_t)dt + d\hat{B}_t \quad X(0) = Z,$$

where $Z = [X_n(0)]$. Now, given α defined on the interval $[0, 1]$, we extend α to the entire real number line by letting α on an interval $[k, k+1]$ for $k \in \mathbb{Z} \setminus \{0\}$

be the negative of the reflection of α of the adjoining interval closest to the interval $[0, 1]$. Then on the entire real number line, X "will behave like the particle on the interval $[0, 1]$". We then say that the particle at time t is at the point $g(X(t))$, where g reflects X back to the original interval $[0, 1]$.

Now given a probability density f for the position of the particle on the interval $(0, 1)$, we would like to find α. We will assume that there exists a constant $K > 0$ such that $f \geq K$, and that f is differentiable on $(0, 1)$ with bounded derivative. Now let $\rho_n(x, t)$ denote the probability of the particle being in the point x, that is $x_n = x$, at time $t_n = t$. For $t_n \in I_n$ and $x_n \in S_n$ we have

$$\begin{aligned}\rho_n(x_n, t_n + \Delta t_n) &= \frac{1}{2}(1 + \alpha(x_n - \Delta x_n)\Delta x_n)\rho_n(x_n - \Delta x_n, t_n) \\ &\quad + \frac{1}{2}(1 - \alpha(x_n + \Delta x_n)\Delta x_n)\rho_n(x_n + \Delta x_n, t_n).\end{aligned}$$

If we assume that $\rho_n(\cdot, t_n) = \rho_n(\cdot, s_n)$ for $t_n, s_n \in I_n$, we say that ρ_n is an equilibrium state. In that case we write $\rho_n(x_n)$ instead of $\rho_n(x_n, t_n)$ and we have

$$\rho_n(x_n) = \frac{1}{2}(1 + \alpha(x_n - \Delta x_n)\Delta x_n) + \frac{1}{2}(1 - \alpha(x_n + \Delta x_n)\Delta x_n).$$

Rearranging this equation we have

$$\begin{aligned}\alpha(x_n + \Delta x_n)\rho_n(x_n + \Delta x_n) &= \alpha(x - \Delta x_n)\rho_n(x - \Delta x_n) \\ &\quad + \frac{\rho_n(x + \Delta x_n) - 2\rho_n(x_n) + \rho_n(x_n - \Delta x_n)}{\Delta x_n}.\end{aligned}$$

If we let $u(x_n) = \alpha(x_n)\rho_n(x_n)$ this can be written as

$$u(x_n + \Delta x_n) = u(x - \Delta x_n) + \frac{\rho_n(x - \Delta x_n) - 2\rho_n(x_n) + \rho_n(x_n - \Delta x_n)}{\Delta x_n}.$$

Summing over from 0 to $x_n = k_n \Delta x_n$ we have

$$\begin{aligned}&u((k_n + 1)\Delta x_n) + u(k_n \Delta x_n) - u(\Delta x_n) - u(0) \\ &= \frac{\rho_n((k_n + 1)\Delta x_n) - \rho_n(k_n \Delta x_n) - \rho_n(\Delta x_n) + \rho_n(0)}{\Delta x_n}.\end{aligned} \tag{4.11}$$

Now we know that

$$\begin{aligned}\rho_n(0) &= \frac{1}{2}(1 - \alpha(\Delta x_n)\Delta x_n)\rho_n(\Delta x_n) + \frac{1}{2}(1 + \alpha(-\Delta x_n)\Delta x_n)\rho_n(-\Delta x_n) \\ &= (1 - \alpha(\Delta x_n)\Delta x_n)\rho_n(\Delta x_n)\end{aligned}$$

(since we have reflected α). Rearranging this yields

$$u(\Delta x_n) = \frac{\rho_n(\Delta x_n) - \rho_n(0)}{\Delta x_n},$$

which substituted into (4.11) gives us

$$u((k_n + 1)\Delta x_n) + u(k_n \Delta x_n) - u(0) = \frac{\rho_n((k_n + 1)\Delta x_n) - \rho_n(k_n \Delta x_n)}{\Delta x_n}.$$

If we further assume that $\alpha(0) = \alpha(1) = 0$, we get

$$u((k_n+1)\Delta x_n) + u(k_n\Delta x_n) = \frac{\rho_n((k_n+1)\Delta x_n) - \rho_n(k_n\Delta x_n)}{\Delta x_n}.$$

Now we would like to have an approximation to ρ_n. A natural choice for ρ_n on the inner points of S_n is

$$\rho_n(x_n) = \int_{x_n - \frac{1}{2}\Delta x_n}^{x_n + \frac{1}{2}\Delta x_n} f(t)\ dt,$$

where f is the probability density of the particle's position. Now (4.11) can be written as

$$\begin{aligned} &u((k_n+1)\Delta x_n) + u(k_n\Delta x_n) \\ &= \frac{F((k_n+\frac{3}{2})\Delta x_n) - 2F((k_n+\frac{1}{2})\Delta x_n) + F((k_n - \frac{1}{2}\Delta x_n)\Delta x_n)}{\Delta x_n}, \end{aligned}$$

where F is the cumulative distribution function. We see that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in S_n such that $\lim_{\mathcal{U}} x_n = x \in (0,1)$, then if we divide (4.11) by Δx_n, we get that the $\mathcal{U}$-limit of this equation (using the approximation to the second derivative) is

$$2\alpha(x)f(x) = f'(x),$$

which gives us that

$$\alpha(x) = \frac{f'(x)}{2f(x)}$$

for all $x \in (0,1)$. Since f' is bounded on $(0,1)$ and since there exists a $K > 0$ such that $f \geq K$ by assumption, since $\alpha(0) = \alpha(1) = 0$, α is bounded. By an argument similar to the one in Proposition 3.1.19, $\{\frac{dQ_n}{dP_n}\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable. So we have found a stochastic differential equation (with a solution) to our problem, namely

$$dX_t = \alpha(X_t)dt + d\hat{B}_t, \quad X(0) = Z,$$

where Z has the probability density function f.

CHAPTER 5

Mathematical Finance

5.1 Terminology

An *asset* is a resource with economic value that is owned by an entity with the expectation that it will provide future benefit. A *financial asset* is a non-physical asset, an investment that derives value because of a contractual claim of what it represents. Examples of financial assets are *stock shares*, *bonds* and *bank accounts*. A *financial market* is an environment in which a range of financial assets are traded. A *security* is a tradable financial asset.

A *shareholder* in the *stock* of a company owns a part of the company including its assets and earnings. The value of a share will vary over time and cannot be determined in advance. We say that shares are *risky*, meaning their value may rise or fall in the future. The share price of a given stock is modelled mathematically by a stochastic process.

A *bond* is an investment in which the investor loans money to an entity. Bonds are issued by companies, governments or banks as a way to borrow money. The owner of a bond is entitled to receive from the issuer a fixed sum at the maturity date of the bond. In addition the bond will yield interest over its lifetime. Bonds (especially government bonds) tend to be less risky than stock, although they are not entirely risk free since a company or government may go bankrupt. We may make the assumption that a bond price is deterministic, i.e. risk-free.

A *derivative*, sometimes called a *derivative security*, is a financial asset whose value depends on one or more other more basic assets, called the *underlying securities*. It is a financial contract whose value at expiration date T is determined exactly by the price of the underlying financial assets at time T. Examples of derivatives are *call* and *put* options.

A *call* option is a contract giving the owner the right, but not the obligation, to buy a share of a given stock at a specified date at a specified *strike* price K. A *put* option is a contract giving the owner the right, but not the obligation, to sell a share of a given stock at a specified date at a specified *strike* price K. An *American option* give the owner the right to buy/sell at any time prior to or at expiry, while an *European option* give the owner the right to buy/sell at the expiry date.

A *portfolio* is a collection of financial assets such as stocks and bonds. We may call a portfolio a trading strategy. These terms will be interchanged throughout this chapter depending on the subject. A trading strategy is called *self-financing* if all trades are financed by selling or bying assets in the portfolio.

An *arbitrage* opportunity is a self-financing strategy for bying and selling shares and bonds at any times in the period $[0, T]$, with zero initial outlay, that provides a positive probability of profit, and no risk of loss. Since many wish to exploit arbitrage opportunities, if they exist in real markets, they must yield very small profit margins and be available for a very short time before the market prices adjust to eliminate them. We make the assumption that in a realistic market, no-one can guarentee a riskless profit, that is, there is no possibility of arbitrage. A model is said to be *viable* if there are no arbitrage opportunities.

5.2 Modelling a financial market

We will model a continuous-time financial market by an $(m+1)$-dimensional $\mathcal{F}_t$-adapted stochastic process $X : I \times \Omega \to \mathbb{R}^{m+1}$, $X(t) = (X_0(t), ..., X_m(t))$ for $t \in I$, where $X_0(t)$ denotes the value of the riskless asset at time $t \in I$ and $X_i(t)$, $i \in \{1, .., m\}$, denotes the price of the risky asset i at time $t \in I$. A price process $\{X_i(t)\}_{t \in I}$ is called *adapted* if $X_i(t)$ is $\mathcal{F}_t$-measurable for each $t \in I$. We will assume that the market is of the form

$$dX_0(t) = \rho(t, \omega) X_0(t) dt,$$

$X_0(0) = 1$, and

$$dX_i(t) = \mu_i(t, \omega) dt + \sigma_i(t, \omega) dB_i.$$

A portfolio in the continuous-time market $\{X(t)\}_{t \in I}$ is an $(m+1)$-dimensional $\mathcal{F}_t$-adapted process $\theta : I \times \Omega \to \mathbb{R}^{m+1}$, where $\theta(t) = (\theta_0(t), ..., \theta_m(t))$ for $t \in I$. A portfolio θ is called self-financing if

$$\int_0^T |\theta_0(s)\rho(s)X_0(s) + \sum_{i=1}^{m+1} \theta_i(s)\mu_i(s)| + (\sum_{i=1}^{m+1} \theta_i(s)\sigma_i(s))^2 \, ds < \infty$$

almost surely and the value process satisfies

$$V^\theta(t) = V^\theta(0) + \int_0^t \theta(s) \cdot dX(s)$$

for $t \in I$. A derivative with payoff D at time T is called *attainable* if there is a self-financing trading strategy θ such that $V^\theta(T, \omega) = D(\omega)$ for $\omega \in \Omega$. The trading strategy θ is then called a *replicating strategy* for D. A market model is called *complete* if every derivative is attainable.

The *discounted asset price* $\tilde{X}_i(t)$ at time t of an asset is the price of the asset, $X_i(t)$, at time t divided by the risk-less asset, $X_0(t)$, at time t, where $X_0(0) = 1$. A *risk-neutral* measure is a probability measure Q, equivalent to the real-world measure P, under which the discounted asset price is a martingale, i.e. $\{\frac{X_i(t)}{X_0(t)}\}_{t\in I}$ is a Q-martingale for all $i \in \{1,..,m\}$. A market is arbitrage free if and only if there exists a risk-neutral measure and a market is complete if and only if the risk-neutral measure is unique.

Analogously to the continuous-time case, we will model a discrete-time financial market by an $(m+1)$-dimensional $\mathcal{F}^n_{t_n}$-adapted stochastic process $X_n : I_n \times \Omega_n \to \mathbb{R}^{m+1}$, $X_n(t_n) = (X_{0,n}(t_n), ..., X_{m,n}(t_n))$ for $t_n \in I_n$, where $X_{0,n}(t_n)$ denotes the value of the riskless asset at time $t_n \in I_n$ and $X_{i,n}(t_n)$, $i \in \{1,..,m\}$, denotes the price of the risky asset i at time $t_n \in I_n$. A discrete process $\{\theta(t_n)\}_{t_n \in I_n}$ is called *predictable* if each $\theta(t_n)$ is $\mathcal{F}_{t_n - \Delta t_n}$-measurable. A portfolio in the discrete-time market $\{X_n(t_n)\}_{t_n\in I_n}$ is a predictable $(m+1)$-dimensional process $\theta_n : I_n \times \Omega_n \to \mathbb{R}^{m+1}$, where $\theta_n(t_n) = (\theta_{0,n}(t_n), ..., \theta_{m,n}(t_n))$ for $t_n \in I_n$. A portfolio is called self-financing if the value process satisfies

$$V_n^{\theta_n}(t_n) = V_n^{\theta_n}(0) + \sum_{i=0}^{m+1} \sum_{s_n=0}^{t_n} \theta_{i,n}(s_n)\Delta X_i(s_n).$$

The definitions of attainable derivatives, replicating strategies, complete markets discounted asset prices and risk neutral measures are the same for discrete-time as in continuous-time.

The goal of this chapter is to define continuous-time markets as limits of discrete-time markets.

5.3 Cox-Ross-Rubinstein Model

Let I be the interval $[0,T]$ and let $I_n = \{\frac{kT}{n} \mid k \in \mathbb{N}, k \leq n\}$. We consider a discrete-time multi-period financial market model called the Cox-Ross-Rubinstein model. Suppose we have a (riskless) bond or bank account with initial value 1 and that the value at time $t_n = \frac{k_n T}{n} \in I_n$ is $B_{0,n}(t_n) = (1+r_n)^{k_n}$, where the interest rate $r_n > -1$ is fixed and known at time 0. The model has a single (risky) stock with initial price $X_{1,n}(0) > 0$ and price process

$$X_{1,n}(t + \Delta t_n) = \begin{cases} u_n X_{1,n}(t) & \text{with probability } p \\ d_n X_{1,n}(t) & \text{with probability } 1-p, \end{cases}$$

where $t \in I_n$, $t < T$, and the parameters u_n and d_n are fixed and known at time 0, with $0 < d_n < u_n$. The underlying probability space is $(\Omega_n, \mathcal{F}_n, P_n)$, where

$$\begin{aligned} \Omega_n &= \{d_n, u_n\}^n \\ P_n(\{\omega_n\}) &= \tilde{P}_n(\{\omega^n_{t_1}\}) \times ... \times \tilde{P}_n(\{\omega^n_T\}) \text{ (where } \tilde{P}_n(\{u_n\}) = p_n \text{ , } \tilde{P}_n(\{d_n\}) = 1-p_n) \\ \mathcal{F}^n_0 &= \{\emptyset, \Omega_n\} \\ \mathcal{F}^n_{t_n} &= \sigma(X_{1,n}(t_1), ..., X_{1,n}(t_n)) \\ \mathcal{F}^n_T &= \mathcal{F}_n = \mathcal{P}(\Omega_n) \text{ (class of all subsets of } \Omega_n). \end{aligned}$$

The following results can be found with proofs in [CR12]:

Proposition 5.3.1. *The following statements are equivalent for the Cox-Ross-Rubinstein model with parameters $T, r_n, X_{1,n}(0), u_n$ and d_n.*

(i) The model is viable

(ii) $d_n < 1 + r_n < u_n$.

Proposition 5.3.2. *If a Cox-Ross-Rubinstein model is viable, then it admits a unique equivalent martingale measure* Q_n*. The one-step conditional risk-neutral probabilities are given by*

$$(q_n, 1 - q_n) = \left(\frac{1 + r_n - d_n}{u_n - d_n}, \frac{u - (1 + r_n)}{u_n - d_n} \right),$$

where q_n *is the* Q_n*-probability of going up and* $1 - q_n$ *is the probability of going down. For any* $\omega_n \in \Omega_n$*, if* $X_{1,n}(T, \omega_n) = X_{1,n}(0) u_n^s d_n^{n-s}$ *for some* $s \leq n$*, then* $Q_n(\omega_n) = q_n^s (1 - q_n)^{n-s}$*, which depends only on* $X_{1,n}(T, \omega_n)$ *and not the full price history of* ω_n*.*

Proposition 5.3.3. *If a Cox-Ross-Rubinstein model is viable, then it is complete. The unique fair price* D_{t_n} *at time* $t_n = \frac{k_n T}{n} \in I_n$ *of any derivative with payoff* D *at time* T *is*

$$D_{t_n} = V_n^{\Phi}(t_n) = (1 + r_n)^{k_n - n} E_{Q_n}(D | \mathcal{F}_{t_n}^n),$$

where $V_n^{\Phi}(t_n)$ *is a random variable describing the value of the the unique replicating strategy* Φ *for* D *at time* t_n *and* Q_n *is the unique equivalent martingale measure given in Proposition 5.3.2.*

Note that in a continuous time market, if the bond price function is X_0, then the value of a derivative at time t with payoff D at time T is $D_t = \frac{X_0(t)}{X_0(T)} E_Q[D | \mathcal{F}_t]$. We will use this later on.

A derivative D is called *path-independent* is there exists a *payoff function* $\hat{D}$ such that $D = \hat{D}(X_{1,n}(T))$. Suppose that D is a path-independent derivative. For any time s there are $\binom{n}{s}$ scenarios ω_n with the same final stock price $X_{1,n}(T, \omega_n) = X_{1,n}(0) u_n^s d_n^{n-s}$, and that each of the scenarios has risk-neutral probability $Q_n(\omega_n) = q_n^s (1 - q_n)^{n-s}$. Thus the Q_n-probability that the final stock price is equal to $X_{1,n}(0) u_n^s d_n^{n-s}$ can be written as

$$Q_n(\{\omega_n \in \Omega_n \mid X_{1,n}(T, \omega_n) = X_{1,n}(0) u_n^s d_n^{n-s}\}) = \binom{n}{s} q_n^s (1 - q_n)^{n-s}. \quad (5.1)$$

Hence we get that

$$D_0^n = (1 + r_n)^{-n} \sum_{s=0}^{n} \binom{n}{s} q_n^s (1 - q_n)^{n-s} \hat{D}(X_{1,n}(0) u^s d^{n-s}).$$

We will use the Cox-Ross-Rubinstein model to approximate the Black-Scholes model. The Black Scholes model is a continuous-time market model wih one bond and one stock over the interval $[0, T]$. The bond price at any time s is $X_0(s) = e^{rs}$ where r is the rate of *continuous compounding.* The stock price at any time $s > 0$ is

$$X_1(s) = X_1(0) e^{(\alpha - \frac{1}{2}\beta^2)s + \beta B_s},$$

where $\beta > 0$ is the *volatility* of the stock, the initial stock price $X_1(0) > 0$ is constant, and the process B_s is a Brownian motion. $X_1(s)$ is given by the differential equation

$$dX_1(s) = \alpha X_1(s)ds + \beta X_1(s)dB(s).$$

We will discretize this differential equation. Divide the interval $[0, T]$ into n intervals of length $\Delta t_n = \frac{T}{n}$. Let this be the time-steps of the n-th Cox-Ross-Rubinstein model. Let $X_{1,n}$ be given by

$$\Delta X_{1,n}(t) = \alpha X_{1,n}(t)\Delta t_n + \beta X_{1,n}(t)\Delta B_n(t).$$

with $X_{1,n}(0) = X_1(0)$. This gives

$$u_n = 1 + \alpha\Delta t_n + \beta\sqrt{\Delta t_n}$$

and

$$d_n = 1 + \alpha\Delta t_n - \beta\sqrt{\Delta t_n}.$$

Let the risk-free obligation be modelled by $X_{0,n}(\frac{kT}{n}) = (1 + \frac{rT}{n})^k$ for $\frac{kT}{n} \in I_n$, i.e. let the interest rate be $r_n = r\Delta t_n$.

Proposition 5.3.4. *Let N be the smallest integer such that $N > \frac{(r-\alpha)^2}{\beta^2}$.*

(i) The n-th Cox-Ross-Rubinstein model i viable if and only if $n \geq N$.

(ii) For $n \geq N$ the unique one-step conditional risk-neutral probabilities for the n-th Cox-Ross-Rubinstein model are $(q_n, 1 - q_n)$, where

$$q_n = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{1}{2} - \frac{\alpha - r}{2\beta}\sqrt{\Delta t_n}.$$

(iii) For $n \geq N$, the unique equivalent martingale measure Q_n in the n-th Cox-Ross-Rubinstein model satisfies

$$Q_n(\{\omega_n \in \Omega_n \mid X_{1,n}(T) = X_{1,n}(0)u_n^k d_n^{n-k}\}) = \binom{n}{k} q_n^k (1 - q_n)^{n-k}$$

for $k \leq n$.

Proof. By Proposition 5.3.1, the n-th Cox-Ross-Rubinstein model is viable if and only of

$$d_n < 1 + r_n < u_n,$$

which is equivalent to

$$1 + \alpha\Delta t_n - \beta\sqrt{\Delta t_n} < 1 + r\Delta t_n < 1 + \alpha\Delta t_n + \beta\sqrt{\Delta t_n},$$

which is equivalent to $-\beta\sqrt{\Delta t_n} < (r - \alpha)\Delta t_n < \beta\sqrt{\Delta t_n}$, which holds true if and only if $n > \frac{(r-\alpha)^2}{\beta^2}$. This gives (i). (ii) follows from Proposition 5.3.2 and (iii) follows from (5.1) ∎

We have that X_0 and X_1 are the ultralimits of $\{X_{0,n}\}_{n\in\mathbb{N}}$ and $\{X_{1,n}\}_{n\in\mathbb{N}}$, respectively, with respect to the measure P. Notice that B is not a Brownian motion under Q unless $\alpha = r$, but that $B(t)$ is normally distributed for each $t \in I$.

Suppose that D^n is a path-independent derivative with payoff function $\hat{D}$ in a viable Cox-Ross-Rubinstein model. By Proposition 5.3.3, its fair price at any time $t_n = \frac{k_n T}{n}$ is

$$D^n_{t_n} = (1 + r_n)^{k_n - n} E_{Q_n}[D^n|\mathcal{F}^n_{t_n}].$$

Suppose that $\{D^n\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable, with $\mathcal{U}$-limit $D = [D^n]$ and that $\{t_n\}_{n\in\mathbb{N}}$, with $t_n = \frac{k_n T}{n}$, is such that $\lim_{\mathcal{U}} t_n = t \in I$. Then

$$\lim_{\mathcal{U}} (1 + r_n)^{k_n - n} E_{Q_n}[D^n|\mathcal{F}^n_{t_n}] = e^{-r(T-t)} E_Q[D|\mathcal{F}_t]. \tag{5.2}$$

Since $\frac{X_{1,n}}{X_{0,n}}$ is a martingale under Q_n for each $n > \frac{(r-\alpha)^2}{\beta^2}$, we know that $\frac{X_1}{X_0}$ is a martingale under Q. If we can show that Q is equivalent to P, we know that Q is the unique (the Black Scholes model is complete) equivalent martingale measure and the fair price of D is given by (5.2).

Proposition 5.3.5. *Suppose that for each $n \in \mathbb{N}$, the measure P_n has probabilities $p_n = \frac{1}{2}$ and $1 - p_n = \frac{1}{2}$. Let P denote the ultralimit of $\{P_n\}_{n\in\mathbb{N}}$ and let Q denote the ultralimit of $\{Q_n\}_{n\in\mathbb{N}}$. Then P and Q are equivalent.*

Proof. Suppose that ω_n has $s_n \leq n$ ups and $n - s_n$ downs. Then the Radon-Nikodym derivatives of Q_n with respect to P_n in ω_n and of P_n with respect to Q_n in ω_n are, respectively,

$$\frac{dQ_n}{dP_n}(\omega_n) = \left(\frac{q_n}{p_n}\right)^{s_n} \left(\frac{1 - q_n}{1 - p_n}\right)^{n - s_n}$$

and

$$\frac{dP_n}{dQ_n}(\omega_n) = \left(\frac{p_n}{q_n}\right)^{s_n} \left(\frac{1 - p_n}{1 - q_n}\right)^{n - s_n}.$$

Using that $q_n = \frac{1}{2} - \frac{\alpha - r}{2\beta}\sqrt{t_n}$ and $1 - q_n = \frac{1}{2} + \frac{\alpha - r}{2\beta}\sqrt{t_n}$, this gives the expressions

$$\frac{dQ_n}{dP_n}(\omega_n) = \prod_{t\in I_n} (1 - \frac{\alpha - r}{\beta} \Delta B_n(\omega_n, t)) = e^{\sum_{t\in I_n} \ln(1 - \frac{\alpha - r}{\beta}\Delta B_n(\omega_n, t))}$$

and

$$\frac{dP_n}{dQ_n}(\omega_n) = \prod_{t\in I_n} \frac{1}{(1 - \frac{\alpha - r}{\beta}\Delta B_n(\omega_n, t))} = e^{-\sum_{t\in I_n} \ln(1 - \frac{\alpha - r}{\beta}\Delta B_n(\omega_n, t))}$$

for $\omega_n \in \Omega_n$. Using Taylor series expansion of the expressions $\ln(1 - \frac{\alpha - r}{\beta}\Delta B_n(\omega_n, t))$ as in Proposition 4.1.4, by the proof of Proposition 3.1.19, $\frac{dQ_n}{dP_n}$ and $\frac{dP_n}{dQ_n}$ are $\mathcal{A}$ integrable. It then follows by Proposition 2.2.24 that P and Q are equivalent. ∎

Theorem 5.3.6. *The unique fair price at time $t \in I$ of a European call option with strike K and expiry T in the Black Scholes model is*

$$D_t = N(d_1)X_1(t) - N(d_2)Ke^{-r(T-t)},$$

where N is the cumulative distribution function to the standard normal distribution and

$$d_1 = \frac{\ln \frac{X_1(t)}{K} + (r + \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln \frac{X_1(t)}{K} + (r - \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}}.$$

Proof. Suppose that $\{\frac{k_nT}{n}\}_{n\in\mathbb{N}} \in \mathcal{T}$ and that $\lim_{\mathcal{U}} \frac{k_nT}{n} = t$. Let

$$A_n(\omega_n) = \min\{i \in \mathbb{N}_0 \mid X_{1,n}(\frac{k_nT}{n}, \omega_n)u^i d^{n-k_n-i} > K\}.$$

Since the payoff function of the call option is $X_{1,n}(T) - K$, we have that

$$\begin{aligned}
&D^n_{\frac{k_nT}{n}}(\omega_n)\\
&= (1+r_n)^{k_n-n} E_{Q_n}[D^n|\mathcal{F}^n_{t_n}](\omega_n)\\
&= (1+r_n)^{k_n-n} \sum_{i=A_n(\omega_n)}^{n-k_n} \binom{n-k_n}{i} q^i(1-q)^{n-k_n-i}(X_{1,n}(\frac{k_nT}{n}, \omega_n))u^i d^{n-k_n-i} - K)\\
&= X_{1,n}(\omega_n, \frac{k_nT}{n}) \sum_{i=A_n(\omega_n)}^{n-k_n} \binom{n-k_n}{i} \hat{q}_n^i(1-\hat{q}_n)^{n-k_n-i}\\
&- K(1+r_n)^{k_n-n} \sum_{i=A_n(\omega_n)}^{n-k_n} \binom{n-k_n}{i} q_n^i(1-q_n)^{n-k_n-i},
\end{aligned}$$

where $\hat{q}_n = q_n\frac{u_n}{1+r_n}$ and $1-\hat{q}_n = (1-q_n)\frac{d_n}{1+r_n}$. Thus we have that

$$\begin{aligned}
D^n_{\frac{k_nT}{n}}(\omega_n) &= X_{1,n}(\omega_n, \frac{k_nT}{n})\hat{Q}_n(X_{1,n}(\omega_n, \frac{k_nT}{n})Y > K)\\
&- K(1+r_n)^{k_n-n}Q_n(X_{1,n}(\omega_n, \frac{k_nT}{n})Y > K),
\end{aligned}$$

where Y is binomially distributed with up u_n and down d_n, with parameters $n-k_n$ and q_n with respect to Q and parameters $n-k_n$ and $\hat{q}_n$ with respect to $\hat{Q_n}$. Hence

$$\begin{aligned}
D^n_{\frac{k_nT}{n}}(\omega_n) &= X_{1,n}(\omega_n, \frac{k_nT}{n})\hat{Q}_n(\frac{1}{Y} < \frac{X_{1,n}(\omega_n, \frac{k_nT}{n})}{K})\\
&- K(1+r_n)^{k_n-n}Q_n(\frac{1}{Y} < \frac{X_{1,n}(\omega_n, \frac{k_nT}{n})}{K})\\
&= X_{1,n}(\omega_n, \frac{k_nT}{n})\hat{Q}_n(-\ln(Y) < \ln(\frac{X_{1,n}(\omega_n, \frac{k_nT}{n})}{K}))\\
&- K(1+r_n)^{k_n-n}Q_n(-\ln(Y) < \ln(\frac{X_{1,n}(\omega_n, \frac{k_nT}{n})}{K})).
\end{aligned}$$

By Proposition 4.1.4, $\ln(Y) = (\alpha - \frac{1}{2}\beta^2)(T - \frac{k_nT}{n}) + \beta B_n(T - \frac{k_nT}{n}) + o(n)$. Thus we get

$$D^n_{\frac{k_nT}{n}}(\omega_n)$$
$$= X_{1,n}(\omega_n, \frac{k_nT}{n})\hat{Q}_n\left(-B_n(T - \frac{k_nT}{n}) + o(n) < \frac{\ln(\frac{X_{1,n}(\omega_n,\frac{k_nT}{n})}{K}) + (\alpha - \frac{1}{2}\beta^2)(T - \frac{k_nT}{n})}{\beta}\right)$$
$$- K(1+r_n)^{k_n-n}Q_n\left(-B_n(T - \frac{k_nT}{n}) + o(n) < \frac{\ln(\frac{X_{1,n}(\omega_n,\frac{k_nT}{n})}{K}) + (\alpha - \frac{1}{2}\beta^2)(T - \frac{k_nT}{n})}{\beta}\right).$$

By the argument given before Proposition 5.3.5, we have that

$$D_t([\omega_n]) = \lim_{\mathcal{U}} D^n_{\frac{k_nT}{n}}(\omega_n)$$
$$= X_1([\omega_n], t)\hat{Q}_n\left(-B(T-t) < \frac{\ln(\frac{X_1([\omega_n],t)}{K}) + (\alpha - \frac{1}{2}\beta^2)(T-t)}{\beta}\right)$$
$$- Ke^{-r(T-t)}Q_n\left(-B(T-t) < \frac{\ln(\frac{X_1([\omega_n],t)}{K}) + (\alpha - \frac{1}{2}\beta^2)(T-t)}{\beta}\right).$$

It follows by the proof of Theorem 3.1.18 that $B(T-t)$ is normally distributed with respect to Q and $\hat{Q}$. By Proposition 5.3.5, $\{\frac{dQ_n}{dP_n}^p\}_{n\in\mathbb{N}}$ and $\{\frac{d\hat{Q}_n}{dP_n}^p\}_{n\in\mathbb{N}}$ are $\mathcal{A}$-integrable for any $p > 1$. Thus $\{B_n(T - \frac{K_nT}{n})^d\}_{n\in\mathbb{N}}$ is $\mathcal{A}$-integrable with respect to Q and $\hat{Q}$ for any $d > 1$. Since $B(T-t)$ is normally distributed with respect to Q and $\hat{Q}$ with means

$$E_Q[B(T-t)] = \lim_{\mathcal{U}} E_{Q_n}[B_n(T - \frac{k_nT}{n})] = -\frac{\alpha - r}{\beta}(T-t)$$

and

$$E_{\hat{Q}}[B(T-T)] = \lim_{\mathcal{U}} E_{\hat{Q}_n}[B_n(T - \frac{k_nT}{n})] = \beta(T-t) - \frac{\alpha - r}{\beta}(T-t)$$

and variances

$$Var_Q(B(T-t)) = \lim_{\mathcal{U}} Var_{Q_n}(B_n(T - \frac{k_nT}{n})) = T - t$$

and

$$Var_{\hat{Q}}(B(T-t)) = \lim_{\mathcal{U}} Var_{\hat{Q}_n}(B_n(T - \frac{k_nT}{n})) = T - t,$$

(these values can also be found by going through the proof of Theorem 3.1.18) since a standard normal distributed variable is symmetric, we get that

$$D_t([\omega_n]) = N(d_1)X_1([\omega_n], t) - N(d_2)Ke^{-r(T-t)},$$

where N is the cumulative distribution function to the standard normal distribution and

$$d_1 = \frac{\ln\frac{X_1([\omega_n],t)}{K} + (r + \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln\frac{X_1([\omega_n],t)}{K} + (r - \frac{1}{2}\beta^2)(T-t)}{\beta\sqrt{T-t}}.$$

■

CHAPTER 6

Discussion

6.1 Summary

The purpose of this master thesis (as the title suggests) was to explore ultrafilter convergence in stochastic analysis and mathematical finance. In chapter one we introduced the topic and gave some motivations. In chapter two we focused on preliminaries (namely ultrafilter convergence of measure spaces and stochastic variables) in order to build the foundation of ultrafilter convergence of stochastic processes in chapter three. In chapter four we showed some applications to stochastic differential equations, while in chapter five we gave a proof of ultrafilter convergence of the Cox-Ross-Rubinstein models to the Black Scholes model in mathematical finance.

6.2 Afterthoughts and regrets

Working on this master thesis has been a fun and interesting experience. It has been challenging at times, but I have learned a lot during the year of working on this thesis. As I mentioned in the introduction, I sometimes had to come up with ideas on my own, but not all of them were fruitful. Transferring certain results from our probability space to other probability spaces proved to be challenging. Furthermore, on the probability space that we have created in this thesis, processes can "have a life of their own" and may not be the limit of discrete processes (in addition to the fact that a sequence of discrete processes may not have any ultralimit). This made some possible applications cumbersome. For example, chapter five originally consisted of one additional section that my advisor and I decided to omit since it did not add anything meaningful to the thesis. Even if I was able to come up with results in chapter five, they were restricted and only valid for our measure space. Furthermore, there were a few results in chapter three that I chose not to prove because I did not think they were interesting or useful.

As I finish this master thesis I do have a few regrets. I regret choosing to use some notation, but once it occured to me that I should have chosen differently, it was to late. For instance there is the unfortunate notion of "$\mathcal{A}$-integrability". I chose this notation as it was close to the notion used in [Cut04] (which uses "$\mathcal{S}$-integrability", where $\mathcal{S}$ stands for standard - a convention I was not aware of), but it soon (although not soon enough) occured to me that I should have chosen differently. A more appropriate notion would be one that incorporated the measure to which the sequence of variables is “$\mathcal{A}$-integrable“, such as ”P-integrability“, or better, one that incorporated both the measure and the algebra such as ”$(\mathcal{A}, P)$-integrability“.

Another regret is the fact that I eventually stopped using a ”bar“ over the stochastic processes in continuous time that were well-defined (defined on the regular real number line and not just the extended one). To be honest, I stopped using the ”bar“ because I forgot after a while to include it since I never included it in my own calculations done in my notes. Once I discovered that I had throughout chapter three section two omitted the notation, I decided to continue doing so. This solution works just fine since I believe the reader by then has grown accustomed to ultralimit processes, but I still feel that the thesis would be slightly more uniform had I chosen not to skip the extra notation.

Notation in general in this thesis has been difficult. I struggled for a long while to find a suitable notation with all those indices that I had to keep track of. I ended up changing my chosen notation a few times during the early stages of writing the thesis (when extra indices just kept popping up!), but I eventually settled on the one present today. I am not sure if it is the best choice, but I have not found any better alternative.

In the last stage of writing my thesis I discovered some weaknesses and gaps that I had to fill out. For example, one of the results that I have included is a product of some last minute work, namely Theorem 3.2.5. I chose to use the condition that $\{X_n(t_n)^4\}_{n\in\mathbb{N}}$ is ”$\mathcal{A}$-integrable” ($(\mathcal{A}, P)$-integrable!) for all $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$, but it is possible (and more natural) that the result should hold as well if $\{X_n(t_n)^2\}_{n\in\mathbb{N}}$ is ”$\mathcal{A}$-integrable” for all $\{t_n\}_{n\in\mathbb{N}} \in \mathcal{T}$. However, showing this would take time (time that I unfortunately did not have).

APPENDIX

A detailed list of my work

In the sections that follows I will list my work on this thesis (that I have done under guidance of my advisor). If I have come up with a result on my own, I will list it. If my advisor has given me a specific problem to solve or if I have found the result elsewhere, I will just list its proof.

My work in chapter 2

These are the results that I have proved for which I believe that a similar (or more or less identical) proof exists:

- The proof of Proposition 2.1.2
- The proof of Lemma 2.1.3
- The proof of Proposition 2.1.5
- The proof of Theorem 2.1.7
- The proof of Proposition 2.1.10
- The proof of Proposition 2.1.11
- Proposition 2.1.12 and its proof
- Proposition 2.1.13 and its proof
- Proposition 2.1.14 and its proof
- Proposition 2.1.15 and its proof
- The proof of Proposition 2.2.1
- The proof of Proposition 2.2.2
- The proof of Proposition 2.2.3
- The proof of Theorem 2.2.4
- The proof of Corollary 2.2.5
- The proof of Theorem 2.2.8

The following is my own work (for which I believe there does not exists a similar of identical proof):

- The proof of Proposition 2.2.10: I was inspired by a proof in [War12]
- The proof of Proposition 2.2.11
- The proof of Corollary 2.2.12
- Proposition 2.2.13 and its proof
- The proof of Theorem 2.2.14
- The proof of Theorem 2.2.15: I have corrected a proof in [War12], adapted it from a non-standard universe to a standard universe and filled in some details
- The proof of Lemma 2.2.16
- Example 2.2.17
- Definition 2.2.18: I was inspired by a definition in [Cut04]
- The proof of Theorem 2.2.19: I have corrected a proof in [War12], adapted it from a non-standard universe to a standard universe and filled in some details
- Proposition 2.2.20 and its proof
- Theorem 2.2.21 and its proof
- Example 2.2.22
- Proposition 2.2.24 and its proof
- Proposition 2.2.25 and its proof

My work in chapter 3

These are the results that I have proved for which I believe that a similar (or more or less identical) proof exists:

- The proof of Proposition 3.2.1
- The proof of Proposition 3.2.2
- The proof of Proposition 3.2.3

The following is my own work (for which I believe there does not exists a similar of identical proof):

- The proof of Proposition 3.1.3
- Definition 3.1.7
- Proposition 3.1.8 and its proof
- Proposition 3.1.12 and its proof

- Proposition 3.1.13 and its proof
- Proposition 3.1.15 and its proof
- Proposition 3.1.16 and its proof
- Proposition 3.1.17 and its proof
- The proof of Theorem 3.1.18
- The proof of Theorem 3.2.4
- Theorem 3.2.5 and its proof
- Corollary 3.2.6 and its proof
- Corollary 3.2.7 and its proof

My work in chapter 4

These are the results that I have proved for which I believe that a similar (or more or less identical) proof exists:

- The proof of Proposition 4.1.4

The following is my own work (for which I believe there does not exists a similar of identical proof):

- Proposition 4.1.1 and its proof
- The proof of Lemma 4.1.2
- The proof of Theorem 4.1.3
- Lemma 4.2.3 and its proof
- The proof of Proposition 4.2.4
- The remaining part (after Proposition 4.2.4) of chapter 4 section 2 (with some help from my advisor).

My work in chapter 5

Section 1 and 2: These have been written by me after reading texts referenced in the bibliography.

These are the results that I have proved for which I believe that a similar (or more or less identical) proof exists:

- The proof of Proposition 5.3.4

The following is my own work (for which I believe there does not exists a similar of identical proof):

- Proposition 5.3.5 and its proof
- The proof of Theorem 5.3.6

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