Nonlinear Compactness Effects of Scalar Conservation Law

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Contents

Chapter 1

Introduction

Nonlinear compactness effects of the scalar conservation law is the topic of this thesis. This thesis is closely related to nonlinear regularizing effects of scalar conservation law recently introduced by Golse [23, 24]. We consider one-dimensional scalar conservation law

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}_+ \tag{1.1}$$

with initial condition

$$u(x, 0) = u_0(x), \tag{1.2}$$

where f is the nonlinear flux function, and u represents the conserved quantity (see [4, 12, 19, 29]) .

For nonlinear fluxes, the solution of (1.1) and (1.2) may become discontinuous in finite time [12, 19, 27, 29]). Even if the initial data is smooth in finite time, characteristic may intersect [11, 19]) i.e. shock may be formed. Such solutions are called shock waves, and shock waves are always defined as weak solutions. So it is necessary to consider weak solutions. But there may be more than one weak solutions [4]. Weak solutions are not necessarily unique [27]. So we need to pick out the correct (psychically relevant) solution among several weak solutions. This mechanism or an additional condition is called entropy condition [12, 19, 27, 29].

Viscous regularization [29, 51, 53] is one of the most common entropy condition, where the given conservation law (1.1) is replaced by

$$u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0, \tag{1.3}$$

and the physically reasonable weak solution of (1.1) must be obtained as the limit of unique classical solutions u^ϵ of (1.3) [4].

Equation (1.3) is also known as a parabolic equation[12]. Here ϵu_{xx} models the effects of diffusion neglected in (1.1). This also means to say that any physically relevant solution must be obtained as a limit (when $\epsilon \to 0$) of solution u^ϵ to (1.3). The second order operator ϵu^ϵ_{xx} is added because the resulting equation provides unique and smooth solutions without shocks. If u^ϵ is a smooth solution of equation (1.3), then this gives raise to a sequence $\{u^\epsilon\}_{\epsilon>0}$. After obtaining sequence of approximate solution, we show convergence by using the Rellich-Kandrachov theorem [19] and also by the Kolmogorov compactness theorem[29].

In Chapter 2, we discuss scalar conservation laws, Burger's equation, weak solutions, and entropy solutions. In Chapter 3, we state and prove uniqueness and existence of entropy solutions for the given conservation law [5, 15, 33, 36, 44, 50, 52]. We use the Kolmogorov compactness theorem [29] to show the convergence of the approximate solutions.

Reformulation of scalar conservation law is also possible which, generalizes the notion of entropy solutions. This is called the kinetic formulation of scalar conservation laws [40]. In Chapter 4, we discuss viscous kinetic formulation and also state and prove existence and uniqueness results with the help of BV estimates. See also [31, 40, 45].

The main goal of this thesis is to prove convergence of sequence of approximate solutions by exploring a new compactness method, introduced resently by Golse. This is the topic of Chapter 5.

As primary sources, I have used [12, 19, 29, 32, 40]. For detailed understanding of the concepts and examples in this thesis we refer to [4, 8, 25, 36, 53].

Chapter 2

Scalar conservation law

A system of partial differential equation, where at least one equation is not linear is called a non-linear system.

One example of partial a nonlinear differential equation is

$$u_t + uu_x = 0.$$

This equation is known as inviscid Burger's equation, an example of a conservation law [19, 27].

In general, conservation law can be written as

$$u_t + f(u)_x = 0, \tag{2.1}$$

for a given nonlinear function $f : \mathbb{R} \to \mathbb{R}$.

If we integrate the above equation over a given interval $[a, b]$, we get

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)\,dx &= \int_a^b u_t(x,t)\,dx\\ &= -\int_a^b f(u(x,t))_x dx\\ &= f(u(a,t) - f(u(b,t))\\ &= \text{inflow at } a - \text{outflow at } b,\end{aligned}$$

i.e., the rate of change in the amount of u in $[a, b]$ is provided by the difference of the fluxes $f(u(a,t))$ and $f(u(b,t))$. This shows that the amount of u is neither created nor destroyed . This means that the total amount of u inside $[a, b]$ can be changed only because of the flow of u across boundary points [29].

Apart from Burgers equation, conservation laws arise in a wide variety of models.

Traffic Flow Problemma[12, 19, 29]:

Suppose we have a one way street, where there is no entry and exist of cars, and with traffic in one direction only.

Let us assume,

$$\begin{aligned}u(x,t) &= \text{density of the cars at point } x \text{ and with time } t,\\ f &= \text{number of the car passing at } x \text{ at time } t \text{ and}\\ N &= \text{the total number of cars.}\end{aligned}$$

Then N between point a and b at time t can be expressed by

$$N = \int_a^b u(x,t)\,\mathrm{d}x$$

Assume that $f = f(u)$ is a function of car density u. The rate of change between the points a and b with time t can be written as,

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)dx &= \int_a^b u_t(x,t)dx \\ &= f(u(a,t)) - f(u(b,t)) \\ &= -\int_a^b f(u)_x dx,\end{aligned}$$

i.e.,

$$\int_a^b u_t + f(u)_x dx = 0,$$

holds for any a,b. Which says that the density of the car $u(x,t)$, satisfies the PDE

$$u_t + [f(u)]_x = 0$$

Here we have assumed that the density of the car is continuously differentiable function.

2.1 Weak Solutions

This section discusses the notion of weak solutions [19]. We know that if the solution u is not differential function (discontinuous), we must introduce the notion of weak solution.

One peculiar consequence of nonlinearity is that if the solution initially is smooth, the solution may develop discontinuity at later times. Characteristic may intersects in finite time (see example (2.2) below). The solution then becomes discontinuous. Discontinuities are known as shock waves. When such models exist then new form of solution (weak solution) is defined.

Example 2.1 (Burger's Equation). *Consider the inviscid Burger's equation[19]*

$$u_t + (\frac{u^2}{2})_x = 0. \tag{2.2}$$

Or equivalently

$$u_t + uu_x = 0 \quad in \quad \mathbb{R} \times (0, \infty)$$
$$u(x,0) = u_0(x) \quad in \quad \mathbb{R} \times t = 0$$

with initial data

$$u_0(x) = \begin{cases} 1, & \textit{if } x \leq 0, \\ 1 - x, & \textit{if } 0 \leq x \leq 1, \\ 0, & \textit{if } x \geq 1. \end{cases}$$

The characteristic is a basic technique for solving PDEs. So we use this method to solve (2.2).

By the chain rule,

$$\frac{du}{ds} = \frac{du}{dt}\frac{dt}{ds} + \frac{du}{dx}\frac{dx}{ds}.$$

Compering it with equation (2.2)*, we get*

$$\frac{dx}{ds} = u, \frac{du}{ds} = 0, \textit{ and } \frac{dt}{ds} = 1.$$

Here

$$\frac{dx}{ds} = u \implies \frac{dx}{ds} = u \implies x = us + x_0.$$

Also

$$\frac{du}{ds} = 0 \implies \frac{du}{ds} = \underbrace{u_t\frac{dt}{ds} + \frac{du}{dx}\frac{dx}{ds}} = 0.$$

Consequently u is constant along the line $x(s)$, i.e.,

$$u(x(s), t(s)) = u_0(x_0) = constant.$$

With the initial conditions $x(0) = x_0$, $t(0) = 0$, we obtain

$$x(s) = x_0 + u_0(x_0)s.$$

Therefore

$$x(s) = \begin{cases} x_0 + s, & \text{if } x_0 \leq 1; \\ x_0 + s(1 - x_0), & \text{if } x_0 \in [0, 1]; \\ x_0, & x_0 \geq 1, \end{cases}$$

and

$$x_0 = \begin{cases} x - s, & \text{if } x_0 \leq 1; \\ \dfrac{1 - x}{1 - s}, & \text{if } x_0 \in [0, 1]; \\ x, & x_0 \geq 1. \end{cases}$$

We know,

$$u(x, s) = u_0(x_0).$$

Therefore

$$u(x_0) = \begin{cases} 1, & \text{if } x - s \leq 0; \\ 1 - \dfrac{1 - x}{1 - s}, & \text{if } 0 \leq \frac{1-x}{1-s} \leq 1; \\ 0, & x \geq 1, \end{cases}$$

$$= \begin{cases} 1, & \text{if } x \leq s; \\ \dfrac{1 - x}{1 - s}, & \text{if } s \leq x \leq 1; \\ 0, & x \geq 1. \end{cases}$$

Put $t = s$. Then for $t \leq 1$, our solution is given by,

$$u(x, t) = \begin{cases} 1, & \text{if } x \leq t; \\ \dfrac{1 - x}{1 - t}, & \text{if } t \leq x \leq 1; \\ 0, & x \geq 1. \end{cases} \tag{2.3}$$

This method breaks down, since the characteristics curves cross at $(x, t) = (1, 1)$.

Hence smooth or classical solution may not always exists [19]. So there arises a question how to define u for $t \geq 1$?

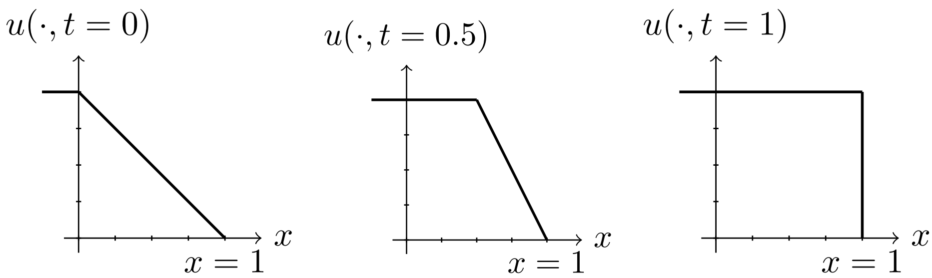
Here characteristic lines cross at $(x,t) = (1,1)$, so the shock curve is given by,

$$(x-1) = \frac{1}{2}.(t-1) \implies x = s(t) = \frac{t+1}{2}.$$

Therefore for $t \geq 1$, we obtain

$$u(x,t) = \begin{cases} 1, & \text{if } x < s(t) = \frac{t+1}{2}; \\ 0, & x > s(t) = \frac{t+1}{2}. \end{cases} \tag{2.4}$$

Hence solution for $t \leq 1$ is given by (2.3) *while for $t \geq 1$ the solution is given by* (2.4) *[19, 34].*



(a) at $t = 0$ (b) at $t = 0.5$ (c) at $t = 1$

Figure 2.1: Solution of burgers equation in different time

Remark 2.2. *The physically relevant solution of Burger's equation* (2.2) *should be obtained as the $\epsilon \to 0$ limit of the unique classical solution u^ϵ of the parabolic equation (PCL),*

$$u_t + uu_x = \epsilon u_{xx}$$

Although the solution of the conservation laws are in general discontinuous, the solution of (PCL) are always smooth and thus a unique classical solution [12, 19].

Motivation for weak formulation:

Suppose u be classical solution of given initial value problem (2.1). Let us denote by $C_c^\infty(R \times [0,\infty))$ the function $\phi \in C^\infty(R^2)$ that vanishes outsides of a compact subset in $R \times [0,\infty)$ [19].

Multiplying (1.1), (1.2) by smooth test function ϕ and using the integration by parts, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty}\int_0^{\infty} (u_t + f(u)_x)\phi \, dtdx \\ &= \int_a^b \int_0^T u_t\phi dtdx + \int_a^b \int_0^T f(u)_x \, \phi \, dtdx \\ &= -\int_a^b \int_0^T (u\phi_t + f(u)\phi_x)dx - \int_a^b u_0\phi(x,0)dx \end{aligned}$$

Consequently,

$$\int_{-\infty}^{\infty}\int_0^{\infty} (u\phi_t + f(u)\phi_x)dtdx + \int_{-\infty}^{\infty} (u_0(x)\phi(x,0)dx = 0.$$

This is called the weak form of the given conservation law (2.1) [19].

Definition 2.3 (Weak solution)**.** *$u \in L^\infty$ is called a weak solution of the conservation law* (2.1)*, with given initial data $u_0 \in L^\infty(R)$ if*

$$\int_{-\infty}^{\infty}\int_0^{\infty} (u\phi_t + f(u)\phi_x)dtdx + \int_{-\infty}^{\infty} (u_0(x)\phi(x,0)dx = 0$$

holds for all $\phi \in C_c^\infty(R \times [0,\infty))$, ($\phi$ is positive test function).

Remark 2.4.

i. *Classical solution of* (2.1) *satisfies the weak form [19, 48].*

ii. *The weak solutions needn't be smooth and not even be continuous [27].*

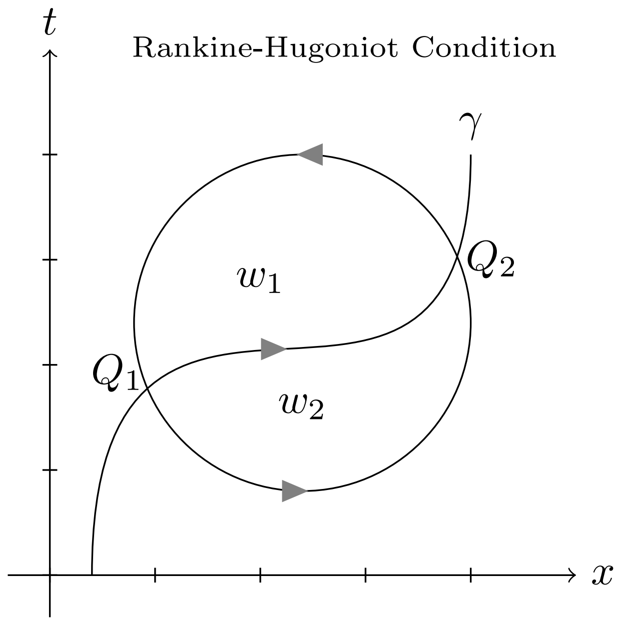
At the point of jump, one condition must be satisfied, and that condition is called the Rankine-Hugoniot jump condition [19, 27]. It is derived in the following subsection.

2.1.1 Rankine-Hugnoit jump condition

Theorem 2.5. *If u is a weak solution, then*

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Where s represents the shock speed [19].



Consider open region W of $\mathbb{R} \times (0, \infty)$, where u is smooth on either side of a smooth curve $\gamma : x = x(t)$. Let W_1 and W_2 be the left part and the right part of the curve with $W = W_1 \cup W_2$. We assume that u is weak solution of (1.1), (1.2), and that u and its first derivative are uniformly continuous in W_1 and W_2. So we must have,

$$\begin{aligned} u_t + f(u)_x &= 0 \quad \text{in} \quad W_2 \\ u_t + f(u)_x &= 0 \quad \text{in} \quad W_1. \end{aligned}$$

Let $\phi \in C_c^\infty(W)$. From the weak formulation, we obtain

$$\begin{aligned} 0 &= \int\int_W (u\phi_t + f(u)\phi_x)\, dt\, dx \\ &= \int\int_{W_1} (u\phi_t + f(u)\phi_x) dt dx + \int\int_{W_2} (u\phi_t + f(u)\phi_x)\, dt\, dx. \end{aligned}$$

Since u is C^1 in W_1 and W_2, so the divergence theorem [37] provides,

$$\begin{aligned} \int\int_{W_j} (u\phi_t + f(u)\phi_x) dt dx &= \int\int_{W_j} (u\phi)_t + (f(u)\phi)_x dt dx \\ &= \int_{\partial W_j} \phi(-u dx + f(u) dt), \end{aligned}$$

where $j = l, 2$. Along γ, the line integrals $\int_{\partial W_j}(\dots)$ are not zero:

$$\int_{\partial W_1} \phi(-u dx + f(u) dt) = \int_{Q_1}^{Q_2} \phi(-u_l dx + f(u_l) dt)$$

$$\begin{aligned} \int_{\partial W_2} \phi(-u dx + f(u) dt) &= \int_{Q_2}^{Q_1} \phi(-u_r dx + f(u_r) dt) \\ &= -\int_{Q_1}^{Q_2} \phi(-u_r dx + f(u_r) dt). \end{aligned}$$

Combining, we obtain

$$\int_\gamma \phi(-(u_l - u_r) dx + (f(u_l) - f(u_r)) dt) = 0.$$

This equality holds true for all test functions [32, lecture set 1].

Consequently,

$$s = \frac{dx}{dt} = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{2.5}$$

Here $f(u_l) - f(u_r)$ and $u_l - u_r$ represents the jump of $f(u)$ and u accross the curve respectively.

Definition 2.6. *Equation* (2.5) *is called the Rankine-Hugonoit jump condition [19, 29].*

Solutions that satisfy the condition (2.5) are weak solutions [21]. But the Rankine Hugoniot condition doesn't guarantee uniqueness [16]. So we need to study the entropy condition to choose the correct solution.

2.1.2 Entropy condition and entropy solution

Nonuniqueness
We may have more than one weak solutions [13, 38].

Example 2.7. *Again let us take Burgers equation* (2.2) *with initial condition* $u_0(x) = w(x)$. *Where*

$$w(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0. \end{cases}$$

Using the method of characteristic as in example (2.2)*, we get* $x(s) = g(x_0)s + x_0$. *Here,*

$$\text{If } x_0 < 0, \text{ then } g(x_0) = 0. \text{ So } x = x_0$$
$$\text{If } x_0 > 0, \text{ then } g(x_0) = 1. \text{ So } x = s + x_0.$$

This shows that there is no crossing of the characteristics but it is not able to *provide information within* $\{0 < x < t\}$*. Let,*

$$u_a(x,t) = \begin{cases} 0, & \text{if } x < \frac{t}{2}; \\ 1, & \text{if } x > \frac{t}{2}. \end{cases}$$

Here $u_a(x,t)$ *is classical solution on either side of the curve of discontinuity. The Rankine-Hugoniot jump condition is satisfied at the discontinuity, i.e,*

$$speed = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{2}.$$

So $u_a(x,t)$ *is one possible weak solution.*

We also have

$$u_b(x,t) = \begin{cases} 1, & \text{if } x > t; \\ \frac{x}{t}, & \text{if } 0 < x < t; \\ 0, & x < 0. \end{cases}$$

This is another possible weak solution. Such type of solution is called a Rarefraction wave [19].

Hence we noticed that weak solutions are not necessarily unique. But we want to pick out the correct and physically relevant solution among many weak solutions. Some mechanism or technique is required for this process and this is called the entropy condition [19].

2.1.3 Entropy function and entropy flux

Let us consider a general convex function $\eta : \mathbb{R} \to \mathbb{R}$, with $\eta = \eta(u)$ and $\eta''(.) \geq 0$. Here η is called the entropy function, and the corresponding entropy flux function, $q : R \to R$ is defined as

$$q' = \eta' f'$$
$$\text{i.e., } q(u) = \int_0^u \eta'(z) f'(z) dz.$$

Definition 2.8 (Entropy pair). *A Pair of functions (η, q), as defined above is called entropy pair.*

Now definition of entropy solution can be stated as follows [17, 19, 51]:

Definition 2.9 (Entropy solution). *Let $u \in L^\infty$ is weak solution of* (2.1), *then u is called an entropy solution of this conservation law if for all convex entropies pair (η, q), we have*

$$\int_{-\infty}^{\infty} \int_0^{\infty} (\eta(u)\phi_t + q(u)_x \phi_x) dt dx + \int_{-\infty}^{\infty} (\eta(u_0(x))\phi(x,0) dx \geq 0 \quad (2.6)$$

holds for all $\phi \in C_c^\infty(R \times [0,\infty))$. Where ϕ is positive test function . The inequality (2.6) *also can be represented as,*

$$\eta(u)_t + q(u)_x \leq 0 \quad in\, D'(\mathbb{R} \times [0,\infty)). \quad (2.7)$$

Remark 2.10.

1. *In the entropy solution energy is non increasing in time $(t > 0)$ [20].*
2. *Inequality* (2.7) *is known as the entropy inequality.*

2.1.4 Kruzkov entropy solution

For any fixed $c \in R$, the function η defined by

$$\eta(u, c) = |u - c|$$

is called the Kruzkov entropy function [28]. The corresponding entropy flux is denoted by $q(u;c)$ and is defined by

$$q(u;c) = sgn(u-c)(f(u) - f(c)),$$

where

$$sgn(u-c) = \begin{cases} -1, & \text{if } u < c; \\ 0, & \text{if } u = c; \\ 1, & u > c. \end{cases}$$

Moreover, if f is monotone, then

$$q(u;c) = |f(u) - f(c)|.$$

The definition of the Kruzkov entropy solution [33] can be stated as follows:

Definition 2.11 (Kruzkov entropy solution)**.** *$u \in L^\infty$ is known as kruzkov entropy solution if,*

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}(|u-c|\,\phi_t + sgn(u-c)(f(u) - f(c)\phi_x)dtdx \\ + \int_{-\infty}^{\infty}|u_0 - c|\,\phi(x,0)dx \geq 0$$

for all $c \in R$ and for all $\phi \in C_c^\infty(R \times [0,\infty))$. Here ϕ is a positive test function.

If u is Kruzkov entropy solution, then u is automatically a weak solution [22, 29]. And if entropy condition (2.6) is satisfied, then the Kruzkov entropy condition automatically holds and vice-versa. Moreover, according to the Kruzkov theorem [33] if u and v are two kruzkov entropy solutions, then for $t > 0$:

$$\|u(.,t) - v(.,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1(\mathbb{R})}. \tag{2.8}$$

Remark 2.12. *Inequality* (2.8) *called the* L_1*-contraction principle, which provides continuous dependence on the initial condition in* L^1 *[27].*

This principle implies the uniqueness of the entropy solution. This is discussed in the next chapter [6, 12, 19, 29, 33].

Chapter 3

Existence and Uniqueness of Entropy Solution

In this chapter, we will show that there exists a unique entropy solution of conservation law

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(x,0) = u_0(x) \end{aligned} \tag{3.1}$$

3.1 Uniqueness

Motivation:
If u and v are two sloutions of (3.1), then we must have

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \tag{3.2}$$

and

$$v_t + f(v)_x = 0, \quad v(x,0) = v_0(x). \tag{3.3}$$

Case I: Assume u and v are classical solutions of (3.1).

Subtracting (3.2) and (3.3), we obtain

$$(u-v)_t + (f(u)-f(v))_x = 0. \tag{3.4}$$

Multiplying both sides of (3.4) by $\text{sgn}(u-v)$ implies

$$(u-v)_t \times \text{sgn}(u-v) + (f(u)-f(v))_x \times \text{sgn}(u-v) = 0.$$

Now using the chain rule yields

$$\begin{aligned} |u-v|_t + (\text{sgn}(u-v)(f(u)-f(v))_x &= \text{sgn}'(u-v)(f(u)-f(v)) \\ &= 0. \end{aligned}$$

Again integrating with respect to x implies

$$\frac{d}{dt}\int_R |u(x,t)-v(x,t)| \, \mathrm{d}x = 0.$$

Hence if $u(.,0) = v(.,0)$, then $u(.t) = v(.,t)$ for positive time t.

Case II: If entropy solutions exists, then they must be unique [41]. But chain rule cannot be applied to the entropy solution, so we use Kruzkov-doubling of variables method [33] to show the uniqueness of the entropy solution.

Theorem 3.1 (Kruzkov Uniqueness Theorem)**.** *Suppose u and v are two kruzkov entropy solutions (KES) of* (3.1)*, where $u, v \in L^\infty$, and also $u, v \in C(\mathbb{R}_+; L^1)$. Then for positive time t,*

$$\|u(.,t) - v(.,t)\|_{L^1(R)} \leq \|u_0 - v_0\|_{L^1(R)}. \tag{3.5}$$

Proof. Since u is Kruzkov entropy solution

$$\iint |u-c|\,\phi_t + \operatorname{sgn}(u-c)(f(u)-f(c))\phi_x\,dtdx + \int |u_0 - c\,|\,\phi|_{t=0}\,dx \geq 0. \tag{3.6}$$

Similarly,since v is a kruzkoz entropy solution,

$$\iint |v-c|\,\phi_t + \operatorname{sgn}(v-c)(f(v)-f(c))\phi_x\,dtdx + \int |v_0 - c|\,\phi\,|_{t=0}\,dx \geq 0. \tag{3.7}$$

Let $q(u;c) = \operatorname{sgn}(u-c)(f(u)-f(c))$, and let $c = v(y,s)$ be constant in (3.6). Here $v(y,s)$ is considered as entropy solution of

$$v_s + f(v)_y = 0,$$
$$v(y,0) = v_0(y).$$

Taking the test function $\phi = \phi(x,t,y,s)$, and integrating (3.6) over y, s we obtain

$$\iiiint |u(x,t) - v(y,s)|\,\phi_t + q(u(x,t),v(y,s))\phi_x dtdxdsdy + \iiint |u_0(x) - v(y,s)|\,\phi\,|_{t=0}\,dxdyds \geq 0 \tag{3.8}$$

Similarly with $v = v(y,s)$ as entropy solution and using $c = u(x,t)$ as constant in (3.7), we obtain

$$\iiiint |v(y,s) - u(x,t)|\, \phi_s + q(v(y,s), u(x,t))\phi_y dtdxdsdy + \iiint |v_0(y) - u(x,t)|\, \phi\, |_{s=0}\, dtdxdy \geq 0. \tag{3.9}$$

We know that

$$|e - f| = |f - e|, \quad q(e,f) = q(f,e).$$

Together with this rule, adding the inequalities (3.8) and (3.9) yields

$$\iiiint |\, u(x,t) - v(y,s)\, |\, (\phi_t + \phi_s) + q(u(x,t), v(y,s))(\phi_x + \phi_y) dtdxdsdy + \underbrace{\iiint |u_0(x) - v(y,s)|\, \phi\, |_{t=0}\, dxdyds} + \underbrace{\iiint |v_0(y) - u(x,t)|\, \phi\, |_{s=0}\, dtdxdy} \geq 0. \tag{3.10}$$

Choose a special test function for (3.10),

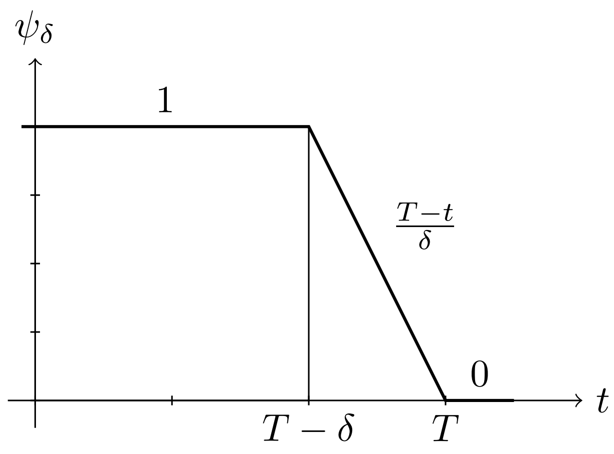


Figure 3.1: Figure representing ψ_δ

$$\phi = \varphi_\delta(t)\omega_\epsilon(t-s)\omega_\epsilon(x-y).$$

Where

$$\varphi_\delta(t) = \min(1, \max(0, \frac{T-t}{\delta}))$$

for fixed $T > 0$, and $\delta > 0$ is a small number and ω is a mollifier in $C_c^\infty(R)$ satisfying

$$\omega_\epsilon(x) = \frac{1}{\epsilon}\omega(\frac{x}{\epsilon}),$$

$0 \leq \omega(x) \leq 1$, $supp(\omega) \subset [-1,1]$, and $\int \omega dx = 1$.

Partially differentiating implies,

$$\phi_x + \phi_y = 0 \text{ and}$$
$$\phi_t + \phi_s = \varphi'_\delta(t)\omega_\epsilon(t-s)\omega_\epsilon(x-y).$$

Using this in inequality (3.10), we have

$$\begin{aligned}\iiiint |u-v|\,\varphi'_\delta(t)\omega_\epsilon(t-s)\omega_\epsilon(x-y)dtdxdsdy \\ + \iiint |u_0-v|\,\varphi_\delta(0)\omega_\epsilon(0-s)\omega_\epsilon(x-y)dxdyds \\ + \iiint |v_0-u|\,\varphi_\delta(t)\omega_\epsilon(t-0)\omega_\epsilon(x-y)dtdxdy \geq 0.\end{aligned}$$

Now sending $\epsilon \to 0$, we arrive at

$$\iint |u-v|\,\varphi'_\delta(t)dtdx + \int_R |u_0(x)-v_0(x)|\,dx \geq 0,$$

That is,

$$-\frac{1}{\delta}\int_{T-\delta}^{T} \| u(.,t)-v(.,t) \|_{L^1(R)} + \| u_0 - v_0 \|_{L^1(R)} \geq 0,$$

using the expression for $\varphi'_\delta(t)$)

Sending $\delta \to 0$ gives

$$\| u(.,t) - v(.,t) \|_{L^1(R)} \leq \| u_0 - v_0 \|_{L^1(R)}$$

Which concludes the proof [12, 29] (see set 4 of [32]). □

Note:

Large part of this chapter is based on [12, 19, 29, 32]. See also [6, 20].

3.2 Compact Embedding Of The Sobolev Space

We know that the solution of a partial differential equation often belongs to the Sobolev space. In this section, we explain about the Sobolev space[19].

Definition 3.2 (Sobolev space). *Let Ω be the open subset of $\mathbb{R}^n$, and r be non-negative integer and $1 \leq p \leq \infty$. Then the Sobolev space is denoted by $W^{r,p}(\Omega)$, and is defined symbolically by*

$$W^{r,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), 0 \leq \alpha \leq r\}.$$

The norm on Sobolev space can be denoted and defined as follows:

$$\|v\|_{W^{r,p}(\Omega)} = (\sum_{|\alpha| \leq r} \int_\Omega |D^\alpha v|^p \, dx)^{\frac{1}{p}}, 1 \leq p < \infty,$$

while for $p = \infty$,

$$\|v\|_{W^{r,\infty}(\Omega)} = \sum_{|\alpha| \leq r} \text{ess sup}_\Omega |D^\alpha v| \,.$$

Definition 3.3 (Compact Embedding). *Suppose A_1 and A_2 be two Banach spaces, we can say that A_1 is compactly embedded into A_2 if and only if*

1. $\|x\|_{A_2} \leq c \|x\|_{A_1}$ *, for all* $x \in A_1$*, and for some constant* c*.*

2. *Each bounded sequence in* A_1 *is pre-compact in* A_2*:* $A_1 \subset\subset A_2$*.*

For more result see ([19, 49]).

Theorem 3.4. *Let* Ω *be bounded open domain in* $\mathbb{R}^n$ *with* C^1 *boundary. Assume* $1 \leq p < n$*. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for all $1 \leq q < p^*$*, where* $p^* = \dfrac{np}{n-p}$*. This theorem is also known as the* ***Rellich -Kondorchov compactness theorem [19]***

Remark 3.5. *In the thesis, we use Sobolev space with* $p = 1$*. Hence for* $p = 1$ *the Rellich -Kondorchov compactness theorem* (3.4) *implies that* $W^{1,1}$ *is compactly embedded in* $L^1(\Omega)$*.*

3.2.1 Compactness

There are different types of methods to show compactness [7]. Some special types of compactness methods are mentioned below:

Suppose $\{u^\epsilon(x,t)\}_{\epsilon>0}$ is a sequence of function.

L^1 **compactness:**

a) u^ϵ satisfies

$$\|u^\epsilon\|_{Loc^1} \leq k,$$

where $k > 0$ doesn't depend on ϵ.

b) $\{u^\epsilon\}_{\epsilon>0}$ is equi-continious in $L^1_{loc}(\mathbb{R}^{d+1}_+)$ (for any compact subset K of $\mathbb{R}^{d+1}_+$,

$$\int\int_K |u^\epsilon(x+\Delta x, y+\Delta y)| \, dxdt \to 0(uniformly)$$

as $\Delta x, \Delta y \to 0$), then there exists a subsequence u^{ϵ_j} such that

$$u^{\epsilon_j} \to u$$

in $L^1_{loc}(\mathbb{R}^{d+1}_+)$ as $j \to \infty$.

BV compactness:

If u^ϵ satisfies

$$\|u^\epsilon\|_{L^\infty} \le k \|u_0\|_{L^\infty} = M$$

and

$$TV(u^\epsilon) \le kTV(u_0),$$

where k is a constant that doesn't depend on ϵ, then there exists a subsequence u^{ϵ_j} such that

$$u^{\epsilon_j} \to u$$

almost everywhere, as $j \to \infty$ (Helly's theorem [29]).

Kolmogorov compactness:

Theorem 3.6. *Let us suppose that Ω is open subset of $\mathbb{R}^n$, and $K \subset L^p(\Omega)$, where $1 \le p < \infty$. Then K is relatively compact if and only if,*

a) K is bounded in $L^p(\Omega)$ i.e. $sup_{v \in K} \|v\|_p < \infty$.

b) For some modulus of continuity ω, $\|v(., +\epsilon) - v\| \le \omega(|\epsilon|)$. i.e, independently of $v \in K$ (we suppose v is 0 outside Ω).

c)

$$\lim_{\theta \to \infty} \int_{\{x \in \Omega / |x| \ge \theta\}} |v(x)|^p \, dx = 0$$

uniformly for $v \in K$ [29].

Definition 3.7 (Weak Convergence). *Suppose Ω is an open, smooth and bounded subset of $\mathbb{R}^N$ with $N \geq 2$. Let*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Where p' is the conjugate exponent and let $1 \leq p \leq \infty$.

A sequence $\{u^n\}_{n\geq 1}$ of $L^p(\Omega)$ is said to converge weakly to u in $L^p(\Omega)$, if

$$\int_\Omega u_n v dx \to \int_\Omega u v dx$$

for all $v \in L^{p'}(\Omega)$, which is denoted as follows

$$u_n \rightharpoonup u$$

in $L^p(\Omega)$ [17, 18].

Definition 3.8 (Weak-$*$Convergence). *A sequence $\{u^n\}_{n\geq 1} \subset L^p(\Omega)$ is said to converge weak-$*$ to u in $L^p(\Omega)$, if*

$$\int_\Omega u_n v dx \to \int_\Omega u v dx$$

for all $v \in L^1(\Omega)$. This is denoted as

$$u_n \overset{*}{\rightharpoonup} u$$

in $L^\infty(\Omega)$ [17, 18, 19].

Remark 3.9 (Final Remarks). *We shall use the Rellich Compactness Theorem* (3.4) *and the Kolmogorov Compactness Theorem* (3.6) *to show the convergence of approximate solution of* (3.1). *Large part of this section is based on [17, 18, 19]*

3.3 Existence

Let $\{u^\epsilon\}_{\epsilon>0}$ be a sequence of the classical solutions of

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \quad \epsilon > 0 \tag{3.11}$$

with

$$u^\epsilon(x,0) = u_0^\epsilon(x). \tag{3.12}$$

Remark 3.10. *If we can show that* $\| u^\epsilon \|_{L^\infty} \leq M$ *(maximum principle) and* $u^\epsilon \to u$ *almost everywhere when* $\epsilon \to 0$*, then the limit* u *is an entropy solution (lecture set 5 of [32]).*

Theorem 3.11. *Suppose* $u_0 \in L^\infty$, $f \in C^2(\mathbb{R})$ *with* $f'' > 0$*, then there exists a weak solution* u *satisfying*

1. $\| u^\epsilon \|_{L^\infty} \leq \| u_0 \| = M < \infty$
2. $u^\epsilon \to u$ *almost everywhere as* $\epsilon \to 0$.
3. *The limit* u *satisfies the entropy condition* $\eta(u^\epsilon)_t + q(u^\epsilon)_x \leq 0$ *in the sense of distribution.*

Remark 3.12. *The maximum Principle*

$$\|u^\epsilon\|_{L^\infty} \leq \|u_0\| = M < \infty, \tag{3.13}$$

where m *is independent of* ϵ*, doesn't provide pre-compactness of* $\{u^\epsilon\}_\epsilon > 0$ *in any* L^p *space for* $1 \leq p \leq \infty$*. So we need to control on the derivative* $\frac{\partial u^\epsilon}{\partial x}$ *in some* L^p *space . Therefore we expect* $u^\epsilon \to u$*, where* u *solves the given conservation law* (3.11), (3.12).

Proof. We will prove that

$$\int_{\mathbb{R}} |u_x^\epsilon| \, dx \leq c \tag{3.14}$$

and

$$\int_R |u_t^\epsilon| \, dx \leq c. \tag{3.15}$$

Step 1:

The goal is to establish (3.14): Partially differentiating both sides of (3.11) with respect to x yields

$$\frac{\partial}{\partial x} \mid u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon,$$

i.e.,

$$(u_x^\epsilon)_t + (f'(u_\epsilon) u_x^\epsilon)_x = \epsilon (u_x^\epsilon)_{xx}. \tag{3.16}$$

Let us set $A^\epsilon = u_x^\epsilon$. Then above equation (3.16) becomes

$$A_t^\epsilon + (f'(u^\epsilon) A^\epsilon)_x = \epsilon A_{xx}^\epsilon.$$

Multiplying both sides by $\operatorname{sgn}(A^\epsilon)$ implies

$$Sgn(A^\epsilon) \mid A_t^\epsilon + (f'(u^\epsilon) A^\epsilon)_x = \epsilon A_{xx}^\epsilon,$$

$$|A^\epsilon|_t + (f'(u^\epsilon\,|A^\epsilon|)_x - f' \operatorname{sgn}'(A^\epsilon) A_x^\epsilon = \epsilon\,|A^\epsilon|_{xx} - \epsilon \operatorname{sgn}'(A^\epsilon)(A_x^\epsilon)^2 \tag{3.17}$$

We know

$$f' \operatorname{sgn}'(A^\epsilon) A_x^\epsilon = 0$$

and also

$$\epsilon \operatorname{sgn}'(A^\epsilon)(A_x^\epsilon)^2 \geq 0.$$

Rewriting (3.17) gives

$$|A^\epsilon|_t + (f'(u^\epsilon\,|A^\epsilon|)_x \leq \epsilon\,|A^\epsilon|_{xx}$$

Integrating with respect to x, assuming $|u|, |u_x| \to 0$ as $|x| \to \infty$ implies,

$$\frac{d}{dt}\int_{\mathbb{R}} |A^{\epsilon}(x,t)|\, dx \leq 0.$$

So,

$$\int_{\mathbb{R}} |A^{\epsilon}(x,t)|\, dx = \int_{R} |u_x^{\epsilon}|\, dx \leq \int_{R} |(u_0)_x|\, dx \leq C.$$

Hence,

$$\int_{R} |u_x^{\epsilon}|\, dx \leq C.$$

Which is the required bound (3.14).

Step 2:

We want to establish (3.15). Partially differentiating both sides of (3.11), (3.12) with respect to t implies

$$\frac{\partial}{\partial t} \mid u_t^{\epsilon} + f(u^{\epsilon})_x = \epsilon u_{xx}^{\epsilon},$$

i.e.,

$$(u_t^{\epsilon})_t + (f'(u_{\epsilon})u_t^{\epsilon})_x = \epsilon (u_t^{\epsilon})_{xx}. \tag{3.18}$$

Let us denote $B^{\epsilon} = u_t^{\epsilon}$ for simplicity. Then above equation (3.18) becomes

$$(B^{\epsilon})_t + (f'(u^{\epsilon})B\epsilon)_x = \epsilon B_{xx}^{\epsilon}.$$

Same as before multiplying both sides by $\text{sgn}(B^{\epsilon})$, we obtain

$$Sgn(B^{\epsilon}) \mid B_t^{\epsilon} + (f'(u^{\epsilon})B^{\epsilon})_x = \epsilon B_{xx}^{\epsilon}.$$

Using chain rule similarly as above, we get

$$|B^{\epsilon}|_t + (f'(u^{\epsilon}\, |B^{\epsilon}|)_x \leq \epsilon\, |B^{\epsilon}|_{xx}.$$

Now integrating provides,

$$\frac{d}{dt}\int_{\mathbb{R}} |B^{\epsilon}(x,t)|\,dx \leq 0.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}} |B^{\epsilon}(x,t)|\,dx = \int_{\mathbb{R}} |u_t^{\epsilon}|\,dx &\leq \int_{\mathbb{R}} |(u_t^{\epsilon}(x,0))|\,dx \\ &= \int_{\mathbb{R}} |f(u_0)_x - \epsilon(u_0)_{xx}| \\ &\leq C,\end{aligned}$$

So we obtain

$$\int_{\mathbb{R}} |u_t^{\epsilon}|\,dx \leq C.$$

Hence the sequence $\{u^{\epsilon}\}_{\epsilon>0}$ is bounded in in $L^{\infty} \cap W^{1,1}$. It is clear from the Rellich Kundrachov Theorem (3.4) that $W^{1,1}$ is compactly embedded in $L^1(\Omega)$. So any bounded sequence in $W^{1,1}(\Omega)$ contains a subsequence which converges in L^1.

This means that there exists a subsequence $\{u^{\epsilon_j}\}_{j=1}^{\infty} \subset \{u^{\epsilon}\}_{\epsilon>0}$ and limit point u in $L^{\infty} \cap L^1$ such that $u^{\epsilon_j} \to u$ in L^1 as $j \to \infty$.

It is time to show that the limit function u is an entropy solution.

Multiplying both side of (3.11) by $\eta'(u^{\epsilon)}$, we obtain

$$\eta'(u^{\epsilon)}(u_t^{\epsilon}) + \eta'(u^{\epsilon)}f(u^{\epsilon})_x = \eta'(u^{\epsilon})\epsilon(u_{xx}^{\epsilon}).$$

We know that

$$\eta'(u^{\epsilon})u_t^{\epsilon} = \eta(u^{\epsilon})_t$$

Also

$$\begin{aligned}\eta'(u^{\epsilon})\epsilon(u_{xx}^{\epsilon}) &= \epsilon(\eta'(u^{\epsilon})u_x^{\epsilon})_x - \epsilon\eta''(u^{\epsilon})(u_x\epsilon)^2 \\ &= \epsilon\eta(u^{\epsilon})_{xx} - \epsilon\eta''(u^{\epsilon})(u_x^{\epsilon})^2,\end{aligned}$$

and
$$\eta'(u^\epsilon)f'(u^\epsilon)u^\epsilon_x = q(u^\epsilon)_x.$$

Rewriting implies

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon\eta(u^\epsilon)_{xx} - \epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2.$$

We know η is convex. So dissipation $\epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2$ must be positive i.e.,

$$\epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2 \geq 0.$$

Therefore
$$\eta(u^\epsilon)_t + q(u^\epsilon)_x \leq \epsilon\eta(u^\epsilon)_{xx} \tag{3.19}$$

The weak formulation of (3.19) is

$$\int_{-\infty}^{\infty}\int_0^{\infty}(\eta(u^\epsilon)\phi_t)dtdx + \int_{-\infty}^{\infty}\int_0^{\infty}(q(u^\epsilon)\phi_x)dtdx + \int_{-\infty}^{\infty}\eta(u_0(x))\phi(x,0)dx$$
$$\geq -\epsilon\int_{-\infty}^{\infty}\int_0^{\infty}\eta(u^\epsilon)\phi_{xx}dtdx,$$

for all $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$, here ϕ is positive test function.

Using that $u^\epsilon \to u$ a.e., $\|u^\epsilon\|_\infty \leq M$ and using the dominance convergence theorem, we get

$$\int_{-\infty}^{\infty}\int_0^{\infty}(\eta(u)\phi_t)dtdx + \int_{-\infty}^{\infty}\int_0^{\infty}(q(u)\phi_x)dtdx \geq 0,$$

so
$$\int_{-\infty}^{\infty}\int_0^{\infty}\eta(u)\phi_t + q(u)\phi_x dtdx \geq 0,$$

Hence u satisfies the entropy inequality [32, lecture set 5] □

Remark 3.13. *Alternatively, we can also show the convergence using the Kolmogorov Compactness Theorem [29]. This approach is presented in the next subsection.*

3.3.1 Compactness and Convergence

In this subsection, we discuss and verify some theorems and lemmas before explaning the convergence.

Remark 3.14. *If $u \in L^1(\mathbb{R})$ and*

$$TV(u) = \overset{lim\,sup}{\epsilon \to 0} \int_{\mathbb{R}} \frac{u(x+\epsilon) - u(x)}{\epsilon} dx < \infty,$$

then we say $u = u(x) \in BV$ [29].

Theorem 3.15. *Let us assume that f is lipschitz cotinious function and $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then the entropy solution $u = u(x,t)$ of* (3.1)*, satisfies the following:*

a) $\|u(x,t)\| \leq \|u_0\|_\infty$ *(maximum principle)*

b) $\|u(.,t) - v(.,t)\|_1 \leq \|u_0 - v_0\|_1$ *(L^1 contraction principle).*

c) If $u_0 \in BV(\mathbb{R})$, then $u(.,t) \in BV(R)$ and $TV(u(x,t)) \leq TV(u_0)$ (total variation diminishing).

d) u_0 monotone implies $u(.,t)$monotone (monotonocity preservation).

e) Suppose $v_0 \in BV \cap L^1$ and $u_0 \leq v_0$. Let $v = v(x,t)$ be another entropy solution with $v_0 = v(x,0)$. Then $u(.,t) \leq v(.,t)$.

f) Lipschitz continuity in time ($\|u(.,t) - u(.,r)\|_1 \leq ||f||_{lip}.T.v.(u_0)|\,t-r|$ for all $r,t \in [0,\infty)$ [29].

Proof a:

Remark 3.16. *First we obtain the maximum princile ($\|u^\epsilon(x,t)\| \leq \|u_0\|_\infty$) for* (3.11), (3.12)*, since using limit as $\epsilon \to 0$, provides $\|u\|_{L^1} \leq \|u_0\|_{L^1}$*

Consider a auxiliary function

$$G(x,t) = u(x,t) - \frac{\eta}{2}(2t + (\eta x)^2)).$$

Then $G = G(x,t)$ attains its maximum at some (x_0, t_0) on $\mathbb{R} \times [0,T]$ (when $|x| \to \infty$, $G \to -\infty$). Then

$$G(x_0,t_0) = u(x_0,t_0) - \frac{\eta}{2}(2t_0 + (\eta x_0)^2)) \geq u_0(0),$$

i.e.,

$$2u(x_0,t_0) - 2\eta t_0 - \eta^3 x_0^2 \geq 2u_0(0),$$

so

$$\eta^3 x_0^2 \leq 2(u(x_0,t_0) - \eta t_0 - u_0(0)) \leq O(1), \tag{3.20}$$

is independent of η (according to the construction u is bounded on $\mathbb{R} \times [0,T]$).
Let us suppose that $0 < t_0 \leq T$. At the maximum, we get $u_x(x_0,t_0) = \eta^3 x_0$, $u_{xx}(x_0,t_0) \leq \eta^3$, and $u_t(x_0,t_0) \geq \eta$.
Inequality (3.20) implies

$$f'(u(x_0,t_0))u_x(x_0,t_0) - \epsilon u_{xx}(x_0,t_0) + u_t(x_0,t_0) \geq \eta - \epsilon\eta^3 - O(1)\eta^{\frac{3}{2}} > 0$$

for sufficiently small η. Which contradicts our assumption that the maximum was obtained for $t > 0$. So

$$u(x,t) - \frac{\eta}{2}(2t + (\eta x)^2)) \leq \sup_x(u_o(x) - \frac{\eta^3 x^2}{2}) \leq \sup_x(u_o(x)).$$

This implies $u \leq \sup u_0$ and when we take η negative, we get $u \geq \inf u_o$. Hence we must have $\|u\| \leq \|u_0\|_\infty$. $\square$

Proof b: If we take $v_0 \equiv 0$, then $v \equiv 0$ is the unique entropy solution.

According to the L^1 stability (2.8), we get

$$\|u\|_{L^1} \leq \|u_0\|_{L^1}$$

□

Proof c. Suppose that $u(x,t)$ be the entropy solution of (3.1). Let $v(x,t)$ be the entropy solution of (3.1) with initial data $v_0(x) = u_0(x+\epsilon)$. Here ϵ is the small number. According to uniqueness of entropy solution, we must have

$$u(x+\epsilon, t) = v(x,t).$$

From L^1 stability, we have

$$\int_{\mathbb{R}} |u(x+\epsilon,t) - u(x,t)|\, dx \leq \int_{\mathbb{R}} |u_0(x+\epsilon) - u_0(x)|\, dx \leq \epsilon TV(u_0),$$

i.e,

$$\int_{\mathbb{R}} \frac{|u(x+\epsilon,t) - u(x,t)|}{\epsilon} dx \leq TV(u_0).$$

Thus

$$TV(u(.,t) \leq TV(u_0).$$

Finally, let us prove the Lipschitz continuity in time.

Suppose we have $0 < r < t < T$, and let β_ϵ be an approximation to the characteristic function of the interval $[r,t]$, so that

$$\lim_{\epsilon \to 0} \beta_\epsilon = \chi_{[r,t]},$$

almost everywhere. Let us define

$$\psi_\epsilon(x,t) = \beta_\epsilon(t)\phi(x),$$

where ϕ is a smooth function with compact support. Now inserting it into the weak formulation, we obtain,

$$\int_0^T \int_{\mathbb{R}} u\psi_{\epsilon,t} + f(u)\psi_{\epsilon,x} dxdt + \int_{\mathbb{R}} \psi_{\epsilon(x,0)} u_0(x)dx = 0.$$

Sending $\epsilon \to 0$,

$$\int_{\mathbb{R}} \phi(x)(u(x,t) - u(x,r)dx + \int_r^t \int_{\mathbb{R}} \phi_x f(u)dxdr = 0$$

i.e.,

$$\int \phi(x)(u(x,t) - u(x,r)dx = -\int_r^t \int \phi_x f(u)dxdr$$

Now

$$\|u(.,t) - u(.r)\| = |\phi| \overset{sup}{\leq} 1 \int \phi(x)(u(x,t) - u(x,r)dx.$$

Using value

$$\|u(.,t) - u(.r)\| = |\phi| \overset{sup}{\leq} 1 \int_r^t \int -\phi(x)_x f(u)dxdr \leq \int_r^t Mf(u)dr \leq TV(u_0) \|f\|_{Lip} (t-r)$$

Hence

$$\|u(.,t) - u(.r)\|_1 \leq M \|f\|_{Lip} |t-r|,$$

[29, 39]. □

Lemma 3.17. *Suppose u^ϵ be the solution of* (3.11), (3.12), *where $u_0 \in L^1(\mathbb{R}^n)$, which takes values in $[a,b] = M$. Then*

$$\int_{\mathbb{R}^n} |u_0(x+y) - u_0(x)| \, dx \leq \omega(|y|), \quad y \in \mathbb{R}^n, \tag{3.21}$$

for some modulus of continuity ω with $\omega(s) \downarrow 0$ when $s \downarrow 0$.
we can find a constant (say k), which depends on M such that for any $t > 0$,

$$\int_{\mathbb{R}^n} |u^\epsilon(x+y,t) - u^\epsilon(x,t)| \, dx \leq \omega(|y|) \tag{3.22}$$

$$\int_{\mathbb{R}^n} u^\epsilon(x,t+h) - u^\epsilon(x,t) \mid dx \leq k(h^{\frac{2}{3}} + \epsilon h^{\frac{1}{3}}) \|u_0\|_{L^1} + 2\omega(h^{\frac{1}{3}}), h > 0[12]. \tag{3.23}$$

Proof. Let us fix $t > 0$. The function $\bar{u}^\epsilon(x,t) = u^\epsilon(x+y,t)$ is solution of with initial condition $\bar{u_0}(x) = u_0(x+y)$ for any $y \in \mathbb{R}^n$.

We know according to the L^1 contraction principle,

$$\int_{\mathbb{R}^n} |u^\epsilon(x+y,t) - u^\epsilon(x,t)dx| \leq \int_{\mathbb{R}^n} |u_0(x+y,t) - u_0(x)|$$

Thus (3.22) follows easily.

Let us prove (3.23).

We want to normalize f by subtracting f(0). So we can suppose without loss of generality that $f(0) = 0$. Let fix $h > 0$, and ϕ be a smooth bounded function, which is defined on $\mathbb{R}^n$. Multiplying (3.11) by ϕ and integrating obtained equation on $\mathbb{R}^n \times (t, t+h)$, we get

$$\int_{\mathbb{R}^n} \phi[u^\epsilon(x+y,t) - u^\epsilon(x,t)]dx = \int_t^{t+h} \int_{\mathbb{R}^n} \phi_x f(u^\epsilon(x,s)) + \epsilon \phi_{xx} u^\epsilon(x,s)dxds. \tag{3.24}$$

Put $w(x) = u^\epsilon(x,t+h) - u^\epsilon(x,t)$. If we insert $\phi(x) = \operatorname{sgn}(w(x))$ in (3.24) we may obtain (3.23). But we know that sgn is a discontinuous function. So we first mollify it as follows

$$\phi(x) = \int_{\mathbb{R}^n} h^{\frac{-n}{3}} \psi(\frac{x-z}{h^{\frac{1}{3}}}) \operatorname{sgn}(w(z))dz. \tag{3.25}$$

Here ψ is a smooth and nonnegative function on $\mathbb{R}$, with support contained in $[-n^{1/2}, n^{-1/2}]$ and having total mass 1. Here $|\phi_x| \leq c_1 h^{-1/3}$, $|\phi_{xx}| \leq c_2 h^{-2/3}$. Now according to the L^1 contraction principle ,

$$\|u(.,s)\|_{L^1(\mathbb{R}^n)} \leq \|u_0(.)\|_{L^1(\mathbb{R}^n)}$$

Therefore (3.24) implies

$$\int_{\mathbb{R}^n} \phi(x)w(x)dx \leq (ch^{2/3} + c\epsilon h^{1/3}) \|u_0\|_{L^1(\mathbb{R}^n)}. \tag{3.26}$$

On the other hand

$$\begin{aligned} |w(x)| - w(x)\,\mathrm{sgn}(w(z)) &= |w(x)| - |w(z)| + [w(z) - w(x)]\,\mathrm{sgn}(w(z)) \\ &\leq 2\,|w(x) - w(z)| \end{aligned}$$

So from (3.25), we get

$$\begin{aligned} |w(x)| - \phi(x)w(x) &= \int_{\mathbb{R}^n} h^{-n/3}\psi(\frac{x-z}{h^{1/3}})[|w(x)| - w(x)\,\mathrm{sgn}(w(z))]dz \\ &\leq 2\int_{|\xi|<1} \psi(\xi)\Big|w(x) - w(x - h^{1/3}\xi)\Big|d\xi \end{aligned} \tag{3.27}$$

Now combining (3.25), (3.27), (3.26), (3.22), we obtain (3.23). □

Remark 3.18.

1. *According to (3.23), regarding $\{u^\epsilon\}_{\epsilon>0}$ as a family in $C^0[\mathbb{R}_+; L^1(\mathbb{R}^n)$, is uniformly equi-continious.*
2. *According to equation (3.22), for any fix $t > 0$, $\{u^\epsilon(., t)\}_{\epsilon>0}$ is contained in a compact set of $L_{loc}(\mathbb{R}^n)$ [12].*

Convergence of the approximate aolution

From the Theorem 3.15 and the Lemma 3.17, we can apply the Kolmogorov compactness theorem 3.6. The Kolmogorov theorem for any sequence there exists a subsequence still denoted by $\{\epsilon_j\}$ with $\epsilon_j \to 0$ when $j \to \infty$ such that

$$u^{\epsilon_j} \to u \text{ in } L^1_{loc}(\mathbb{R}^n)$$

(See [2] and [3] for more).

Chapter 4

Kinetic Formulation Of Scalar Conservation Law

A reformulation of (3.1) is also possible, which generalises the notion of the entropy solution [54]. This is called the kinetic formulation of (3.1). The main feature of the kinetic formulation is its linearity[27]. From a mathematical point of view, dealing with linear differential equations are much easier. So in this chapter, we concentrate on the kinetic formulation of (3.1), which converts the nonlinear conservation law into a linear form. The kinetic formulation of conservation law was introduced by Lions, Perthame, and Tadmor [40]. By following the approach of Perthame [46], we define the Kinetic formulation, χ-function, the uniqueness and existence theorem, and convergence results of (3.1).

Remark 4.1. *Using the kinetic formulation the existence of a unique entropy solution of conservation law can be proved pure in the L^1 setting.*

4.1 χ-function

The χ-function plays very important role In the kinetic formulation. This is also known as indicator function.

Definition 4.2 (χ-function)**.** *The χ-function is denoted by $\chi(\xi, u)$ and is defined by*

$$\chi(\xi,u) = \begin{cases} +1, & \text{if } \xi \in (0,u), u > 0\,; \\ -1, & \text{if } \xi \in (u,0), u < 0\,; \\ 0, & \text{if otherwise.} \end{cases}$$

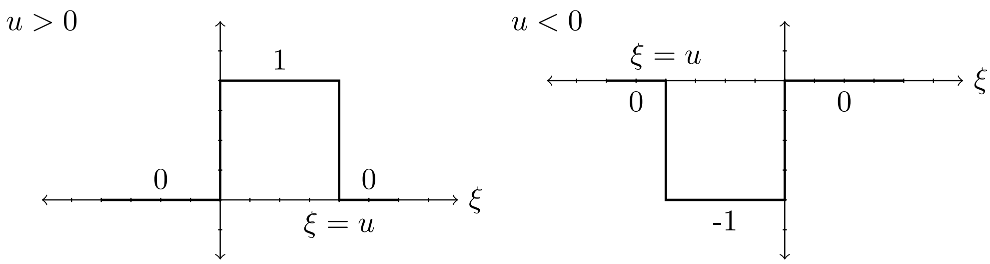


Figure 4.1: Figure of χ-function

It has some important properties [46, 54] stated in the next lemma.

Lemma 4.3. *Suppose that $S : \mathbb{R} \to \mathbb{R}$ locally Lipschitz, then*

1.
$$\int_{\mathbb{R}} S'(\xi)\chi(\xi,u)d\xi = S(u) - S(0).$$

2.
$$\int_{\mathbb{R}} |\chi(\xi,u) - \chi(\xi,v|\, d\xi = |u - v|\,.$$

Proof. Let us prove (1) first. If $u < 0$, then

$$\int_R S'(\xi)\chi(\xi,u)d\xi = -\int_u^0 S'(\xi)d\xi = S(u) - S(0).$$

If $u > 0$, then

$$\int_R S'(\xi)\chi(\xi,u)d\xi = \int_0^u S'(\xi)d\xi = S(u) - S(0).$$

Now we prove (2).

Let $u > v$. then

$$|\chi(\xi,u) - \chi(\xi,v)| = \begin{cases} 1, & \text{if } u < \xi < v\,; \\ 0, & \text{if otherwise.} \end{cases}$$

So

$$\int_R |\chi(\xi,u) - \chi(\xi,v)| = v - u.$$

Again if $u < v$, then

$$\int_R |\chi(\xi,u) - \chi(\xi,v)|\, d\xi = \int_u^v -1 d\xi = -(v-u).$$

□

Remark 4.4.

We also have $\int_R \chi(\xi,u)d\xi = u$*, and in the distributional sense*

1.
$$\frac{\partial}{\partial u}\chi(\xi,u) = \delta(\xi - u) \quad if \quad \xi \neq 0$$

2.
$$\frac{\partial}{\partial \xi}\chi(\xi,u) = \delta(\xi) - \delta(\xi - u).$$

Nonlinear function and their weak limits [46]

To know further properties related with χ-function, let us suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence of functions bounded in $L^{\infty}(\mathbb{R}^d)$. Also assume that $\{u_n\}_{n=1}^{\infty}$ is locally weakly compact in $L^1(\mathbb{R}^d)$. This means that there exists a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and a limit u which belongs to $L^1_{loc}(\mathbb{R})$ such that

$$u_{n_i} \to u$$

in $L^1_{loc}(\mathbb{R})$. This can also be written as

$$\int_A u_{n_i} v dx \to \int_A uv dx,$$

for all $v \in L^\infty(A)$ and for all balls $A \subset \mathbb{R}^d$. Without loss of generality let us assume that $u_n \rightharpoonup u$ in $L^1_{loc}(\mathbb{R})$ (not just a subsequence). We know that $|\chi(\xi, u_n)| \leq 1$. So we can assume that

$$\chi(\xi, u_n) \overset{*}{\rightharpoonup} f(x, \xi)$$

in $L^\infty(\mathbb{R}^{d+1})$ for some limit $f = f(x, \xi) \in L^\infty(\mathbb{R}^{d+1})$.

Our main target is to show that f is the object of "minimal complexity".

Theorem 4.5. *Suppose we have*

$$f \in L^1(\mathbb{R}), \textit{ and } |f(\xi)| = \operatorname{sgn}(\xi) f(\xi) \leq 1. \tag{4.1}$$

Set

$$u = \int_{\mathbb{R}} f d\xi. \tag{4.2}$$

Then

1. *there exists unique $m \in C_0(\mathbb{R})$ satisfying*

$$\chi(\xi, u) - f(\xi) = \frac{\partial}{\partial \xi} m(\xi),$$

   where $m \geq 0$ and

$$\|m\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}.$$

2. *Let $S : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous convex function and $u \in \mathbb{R}$. Then the value of the optimization*

$$\inf\left\{\int_{\mathbb{R}} S'(\xi)f(\xi)d\xi : f \in L^1(\mathbb{R}), |f(\xi)| \leq 1, u = \int_{\mathbb{R}} f(\xi)d\xi\right\} \qquad (4.3)$$

is $S(u) - S(0)$, which is obtained at $\chi(\xi, u) = f$.

*This theorem is known as **Gibb's principle** [46]*

Remark 4.6. *The χ-function can be viewed as minimizer for a Gibb's functional.*

In what follows, we collect some other important properties (lemmas) [46]:

Lemma 4.7. *Suppose f satisfies* (4.1), (4.2) *such that*

$$\chi(\xi, u) - f(\xi) = -\frac{\partial}{\partial\xi}m(\xi),$$

where $m \in C_0(\mathbb{R})$, $m \geq 0$, and $u \in \mathbb{R}$. Then $m = o$, and $f = \chi(\xi, u)$.

Lemma 4.8. *For all balls A, the weak limit $f = f(x, \xi)$ of $\chi(\xi, u_n)$ satisfies*

1. $|f(\xi)| = sgn\xi f(\xi) \leq 1$,
2. $f(x, \xi) \in L^1(A \times \mathbb{R}_\xi)$,
3. *and there exists a nonnegative measure $\nu_x(\xi)$ in (x, ξ) and $x \mapsto \int_{\mathbb{R}} v(\xi)d\nu_x(\xi)$ is measurable for all $v \in C_0(\mathbb{R})$, such that*

$$\frac{\partial}{\partial\xi}f(x, \xi) = \delta(\xi) - \nu_x(\xi).$$

Moreover,

$$1 = \int_{\mathbb{R}} \nu_x(\xi)d\xi = \int_{\mathbb{R}} d\nu_x(\xi) = \nu(\mathbb{R}).$$

Proof: see [46].

Lemma 4.9. *Suppose that in $L^1(\mathbb{R}^d)$, u_n is relatively locally weakly compact. Then the weak limits $u(x)$ and $f(x, \xi)$ satisfy*

1. $\int_{\mathbb{R}} f(x, \xi)d\xi = u(x)$

2. $\int_{\mathbb{R}} |f(x,\xi)|\, d\xi = \overset{w-lim}{n \to \infty} |u_n| \in L^1_{loc}(\mathbb{R})$

3. $\int_{\mathbb{R}} S'(\xi) f(x,\xi) d\xi = \overset{w-lim}{n \to \infty} S(u_n(x)).$

4. $u_n \to u$ *strongly* $\Longleftrightarrow f = \chi(\xi, u).$

Here $S : \mathbb{R} \to \mathbb{R}$ *is globally Lipschitz Continuous and satisfy* $S(0) = 0$ *[46].*

Proof 1: We want to obtain

$$\int_{\mathbb{R}} f(x,\xi) d\xi = u(x).$$

Without loss of generality, let us decompose $u_n(x)$ as

$$u_n(x) = \int_{-r}^{r} \chi(\xi, u_n(x)) d\xi + \int_{r}^{\infty} \chi(\xi, u_n(x) d\xi + \int_{-\infty}^{r} \chi(\xi, u_n(x)) d\xi. \quad (4.4)$$

Here r is a positive real number. Take

$$v_n^r(x) = \int_{r}^{\infty} \chi(\xi, u_n(x) d\xi + \int_{-\infty}^{r} \chi(\xi, u_n(x)) d\xi.$$

Then (4.4) becomes

$$u_n(x) = \int_{-r}^{r} \chi(\xi, u_n(x)) d\xi + v_n^r(x). \quad (4.5)$$

Now according to the definition of $\chi(\xi, u_n(x))$,

$$|v_n^r(x)| = \begin{cases} |u_n(x)| - r, & \text{if } |u_n(x)| - r > 0\,; \\ 0, & \text{if otherwise.} \end{cases}$$

Let us write the term on the right hand side (RHS) as $(|u_n(x)| - r)^+$, so

$$|v_n^r(x)| = (|u_n(x)| - r)^+.$$

According to the assumption, $\{u_n(x)\}_{n=1}^{\infty}$ is locally weakly compact and

$$\int_{K\cap\{x:|u_n(x)|>r\}} (|u_n(x)|-r)^+ dx \leq \int_{K\cap\{x:|u_n(x)|>r\}} 2\,|u_n(x)|\,dx$$
$$\to 0 \quad \text{as} \quad r\to\infty.$$

So for any $K\subset\mathbb{R}$,

$$\int_K |v_n^r(x)|\,dx = \int_{K\cap\{x:|u_n(x)|>r\}} (|\,u_n(x)\,|-r)dx$$
$$\to 0 \quad \text{as} \quad r\to\infty.$$

Thus

$$u_n(x) = \int_{-r}^{r} \chi(\xi, u_n(x))d\xi + v_n^r(x).$$

$$u(x) = \int_{-r}^{r} \chi(\xi, u_n(x))d\xi + v^r(x)$$

Note that in (4.5) the terms containing χ and u_n passes weak limits. So $V_n^r(x)$ also has a weak limit. We denoted it by v^r in $L^1(\mathbb{R}^d)$. And this satisfies

$$\int_K |v^r|\,dx \leq O\Big(\frac{1}{R}\Big) \to 0 \quad \text{as} \quad r\to\infty.$$

So when $r\to\infty$, we get

$$u(x) = \int_{\mathbb{R}} f(x,\xi)d\xi,$$

for some f in $L_{loc,x}(L^1_\xi)$. □

Proof 2: We can prove 2 in exactly the same way as 1, but starting from

$$|u_n(x)| = \int_{-r}^{r} \operatorname{sgn}(\xi)\chi(\xi, u_n(x))d\xi + |v_n^r(x)|\,,$$

since $\operatorname{sgn}(\xi)f = |f|$. □

Proof 3: We prove 3 following the same process as in 1, but starting from

$$S(u_n(x)) = \int_{-r}^{r} S'(\xi)\chi(\xi, u_n(x))d\xi + \int_{r}^{\infty} S'(\xi)\chi(\xi, u_n(x)d\xi \\ + \int_{-\infty}^{r} S'(\xi)\chi(\xi, u_n(x))d\xi \tag{4.6}$$

Let

$$v_n^r(x) = \int_{r}^{\infty} S'(\xi)\chi(\xi, u_n(x)d\xi + \int_{-\infty}^{r} S'(\xi)\chi(\xi, u_n(x))d\xi.$$

So (4.6) becomes

$$S(u_n(x)) = \int_{-r}^{r} S'\xi\chi(\xi, u_n(x))d\xi + v_n^r(x). \tag{4.7}$$

We know

$$\int_K |v_n^r(x)|\, dx \to 0$$

as $r \to \infty$ and

$$|v_n^r| \leq \left\|S'\right\|_{L^\infty} (|u_n| - r)^+.$$

Using weak limits as in 1,we get

$$n \overset{w-lim}{\to} \infty S(u_n(x)) = \int_{\mathbb{R}} S'(\xi)f(x,\xi)d\xi.$$

□

Proof 4: According to 3, we know that

$$n \overset{w-lim}{\to} \infty S(u_n(x)) = \int_{\mathbb{R}} S'(\xi)f(x,\xi)d\xi.$$

$$n \overset{w-lim}{\to} \infty S(u_n(x)) = \int_{\mathbb{R}} S'(\xi)f(x,\xi)d\xi \geq \int_{\mathbb{R}} S'(\xi)\chi(\xi, u)d\xi \\ = S(u) - S(0) = S(u).$$

With equality $\iff f = \chi(\xi, u)$, i.e., f is a χ function. Using remark(4.10) below

$$\int_K (u_n - u)^2 dx \leq \int_K (S(u_n) - S(u))dx + \int_K S'(u)(u_n - u)dx$$

converges to 0 when $n \to \infty$. □

Remark 4.10.

$$n \overset{w-lim}{\rightarrow} \infty S(u_n(x)) = S(u)$$

$\implies$ *$u_n \to u$ in $L^1(K)$ for the bounded domain K, provided $S \in C^2$ strictly convex. According to the Taylor series [47]*

$$S(u_n) - S(u) = S'(u)(u_n - u) + \frac{S''(w)(u_n - u)^2}{2},$$

since S is strictly convex

$$S'(u)(u_n - u) + \frac{S''(w)(u_n - u)^2}{2} \geq S'(u)(u_n - u) + c(u_n - u)^2$$

for some $w(x,t) = w$ lying between $u_n(x)$and $u(x,t)$. We have also $u_n \rightharpoonup u$ in $L^1(K)$, $S' \in L^\infty(K)$ [32, lecture set 6]).

4.2 Viscous Kinetic Formulation

The kinetic formulation can be derived by the use of different methods. In this thesis, we use vanishing viscosity methods.

Derivation:

Let us consider scalar conservation law

$$u_t + B(u)_x = 0 \tag{4.8}$$

with

$$u(x, 0) = u_0(x).$$

Let the flux function $B : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $a(u) = B'(u)$ and $a \in L^\infty_{loc}(\mathbb{R})$. Let u^ϵ solve

$$u^\epsilon_t + B(u^\epsilon)_x = \epsilon u^\epsilon_{xx}.$$

Assume S' is compactly supported, $S \in C^2$, $S'' \geq 0$. Aslo denote $q = S'B'$. We know that $a = B'$ and from the viscous conservation law it follows that

$$S(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon S(u^\epsilon)_{xx} - \epsilon S''(u^\epsilon)(u^\epsilon_x)^2. \tag{4.9}$$

With the help of lemma (4.3)

$$\int_{\mathbb{R}} S'(\xi)\chi(\xi, u^\epsilon)d\xi = S(u^\epsilon) - S(0)$$

and

$$\int_{\mathbb{R}} S'(\xi)a(\xi)\chi(\xi, u^\epsilon)d\xi = \int_{\mathbb{R}} S'(\xi)B'(\xi)\chi(\xi, u^\epsilon)d\xi = q(u^\epsilon) - q(0).$$

Thus ,

$$\begin{aligned} &\int_{\mathbb{R}} S'(\xi)\Big(\chi(\xi, u^\epsilon)_t + a(\xi)\chi(\xi, u^\epsilon)_x - \epsilon\chi(\xi, u^\epsilon)_{xx}\Big)d\xi \\ &= -\int_{\mathbb{R}} \delta(\xi - u^\epsilon)S''(\xi)\epsilon(u^\epsilon_x)^2 d\xi. \end{aligned} \tag{4.10}$$

Suppose the the right hand side as

$$Z = -\int_{\mathbb{R}} \delta(\xi - u^\epsilon)S''(\xi)\epsilon(u^\epsilon_x)^2 d\xi = -\int_{\mathbb{R}} S''(\xi)m^\epsilon(x, t, \xi)d\xi,$$

where

$$m^\epsilon = \delta(\xi - u^\epsilon)\epsilon(u^\epsilon_x)^2.$$

Note:
Here m^ϵ is a positive measure having finite mass such that $m^\epsilon \in M(\mathbb{R}_x \times \mathbb{R}_t \times \mathbb{R}_\xi)$.

Let us prove this. Let $S(u) = \frac{u^2}{2}$. Assume $|u|$, $|u_x| \to 0$ when $|x| \to \infty$

therefore $S'' = 1$. Here we assume that $u_0 = u(x,0) \in L^2(\mathbb{R})$.

Integrating (4.9),

we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_{\mathbb{R}} m^{\epsilon}(x,t,\xi)dxdtd\xi = \int_{\mathbb{R}_+}\int_{\mathbb{R}} \epsilon(u_x^{\epsilon})^2 dxdt$$
$$\leq \int_{\mathbb{R}} \frac{(u_(x,0))^2}{2} dx < \infty.$$

Assuming that S' is compactly supported. Integration by parts yields

$$Z = \int_{\mathbb{R}} S'(\xi) \frac{\partial}{\partial \xi} m^{\epsilon}(x,t,\xi)d\xi.$$

Now using this in (4.10),

$$\int_{\mathbb{R}} S'(\xi)\Big(\chi(\xi,u^{\epsilon})_t + a(\xi)\chi(\xi,u^{\epsilon})_x - \epsilon\chi(\xi,u^{\epsilon})_{xx} - \frac{\partial}{\partial \xi} m^{\epsilon}(x,t,\xi)\Big)d\xi = 0.$$

This is true for all test functions $S'(\xi)$. So

$$\chi(\xi,u^{\epsilon})_t + a(\xi)\chi(\xi,u^{\epsilon})_x - \epsilon\chi(\xi,u^{\epsilon})_{xx} - \frac{\partial}{\partial \xi} m^{\epsilon}(x,t,\xi) = 0.$$

We write more compactly

$$\chi_t^{\epsilon} + a(\xi)\chi_x^{\epsilon} = \epsilon\chi_{xx}^{\epsilon} + \frac{\partial}{\partial \xi} m^{\epsilon}.$$

We have used $\chi^{\epsilon} = \chi(\xi,u^{\epsilon})_t$. We know that $\{m^{\epsilon}\}_{\epsilon>0}$ is bounded in $M(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$. So we can suppose that there exists a bounded measure $m(x,t,\xi) \geq 0$ such that $m^{\epsilon}(x,t,\xi) \overset{*}{\rightharpoonup} m(x,t,\xi)$ in $M(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ i.e., for all test functions $\phi \in C_0(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$, we have

$$\iiint \phi m^{\epsilon} dxdtd\xi \overset{\epsilon\downarrow 0}{\to} \iiint \phi m dxdtd\xi,$$

where $\phi = \phi(x,t,\xi)$ and $m = m(x,t,\xi)$. Suppose that $u^\epsilon \overset{\epsilon\downarrow 0}{\to} u$ in L^1_{loc}. Then $\chi^\epsilon \overset{\epsilon\downarrow 0}{\to} \chi(\xi, u(x,t)$ in L^1_{loc}.

Hence, $\chi = \chi(\xi, u)$ satisfies

$$\chi_t + a(\xi)\chi_x = \frac{\partial}{\partial \xi} m \quad (weakly),$$

[32, lecture set 7].

Theorem 4.11. *Suppose $u \in (\mathbb{R}_+ : L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times \mathbb{R}_+)$. Then u is called the entropy solution of* (4.8) *if and only if there exists a non negative bounded measure $m = m(x,t,\xi)$, $m \in C_0(\mathbb{R}_\xi; M(\mathbb{R} \times \mathbb{R}_+)weak^*)$ such that*

$$\chi(\xi, u(x,t))_t + a(\xi)\chi(\xi, u(x,t)_x = \frac{\partial m}{\partial \xi}, \tag{4.11}$$

with

$$\chi(\xi, u(x,0) = \chi_0 = \chi(\xi, u_0(x)), \tag{4.12}$$

holds weakly[27, 40, 46].

Lemma 4.12. *a) (bound on the total mass)*

$$\int_{\mathbb{R}_\xi \times \mathbb{R} \times \mathbb{R}_+} dm(t,x,\xi) \leq \int_{\mathbb{R}} \frac{{u_0}^2}{2} dx.$$

We know that $u_0 \in L^1 \cap L^\infty$. So $\int_{\mathbb{R}} \frac{{u_0}^2}{2} dx$ must be finite. Hence

$$m(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_\xi) < \infty.$$

b) For

$$\xi < -\left\| (u_0)_- \right\|_{L^\infty(\mathbb{R})}$$

or for

$$\xi > \left\| (u_0)_+ \right\|_{L^\infty(\mathbb{R})},$$

we have $m(t,x,\xi) = 0$.

c) For all $T \geq 0$,

$$\int_{\mathbb{R}\times[0,T]} dm(t,x,\xi) \in C_0(\mathbb{R}),$$

and

$$\int_{\mathbb{R}\times\mathbb{R}_+} dm(t,x) \leq \mu(\xi)$$

for some $\mu \in L_0^\infty(\mathbb{R})$, where

$$\mu(\xi) = 1_{\xi\leq 0} \,||\, (u_0-\xi)_- \,|\, \Big|_{L^1(\mathbb{R})} + 1_{\xi\geq 0}\Big| \,|(u_0-\xi)_+|\, |_{L^1(\mathbb{R})} < \infty.$$

This is because

$$1_{\xi\leq 0} \left\|(u_0-\xi)_-\right\|_{L^1(\mathbb{R})} + 1_{\xi\geq 0} \left\|(u_0-\xi)_+\right\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}.$$

d) Let us consider a bounded domain $\Omega \subset \mathbb{R}_+ \times \mathbb{R}$. Let us assume that $u \in W^{1,1}(\Omega)$. Then on $\Omega \times \mathbb{R}_\xi$, we have $m \equiv 0$

Proof: see page 60 of [46].

4.2.1 Regularization of generalized kinetic solution

Uniqueness and Existence

In this subsection, some backgrounds which are very essential for the proof of the existence and uniqueness for the solution of the kinetic formulation are discussed and derived. Some fundamental steps are needed to prove the uniqueness [46]:

Set

$$m_\epsilon(x,t,\xi) = m * \phi_\epsilon(x,t)$$

$$f_\epsilon(x,t,\xi) = f * \phi_\epsilon(x,t),$$

where

$$\phi_\epsilon(x,t) = \frac{1}{\epsilon_1}\phi_1(\frac{t}{\epsilon_1})\frac{1}{\epsilon_2}\phi_2(\frac{x}{\epsilon_2}),$$

ϕ_1 and ϕ_2 are non-negative and used as normalized regularizing kernel, $\phi_1, \phi_2 \in C^\infty$. Here we have also chosen

$$\int \phi_1 = \int \phi_2 = 1.$$

Also $supp(\phi_1) \subset [-1,0]$, $sup(\phi_2) \in [-1,1]$ and $\epsilon = (\epsilon_1, \epsilon_2)$. By the linearity of the kinetic formulation, regularisation [46] can be performed as follows:
Consider the problem

$$\frac{\partial f_\epsilon}{\partial t} + a(\xi)\frac{\partial f_\epsilon}{\partial \xi} = \frac{\partial m_\epsilon}{\partial \xi} \tag{4.13}$$

with

$$f(x,o,\xi) = \chi(\xi, u_0(x))[46].$$

Lemma 4.13. *a) In* (x,t,ξ)*,* m_ϵ *is Lipschitz-Continuous and equation* (4.13)*, holds for every* ξ *in the classical sense.*

b) Furthermore for $\phi \geq 0$*,* $\phi \in C_c^\infty(\mathbb{R} \times [0,\infty))$*,*

$$0 \leq \int_0^\infty \int_{\mathbb{R}} m_\epsilon \phi(x,t)dxdt \leq \mu(\xi)\, \|\phi\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}$$

and

$$\xi \mapsto \int_0^\infty \int_{\mathbb{R}} m_\epsilon \phi(x,t)dxdt$$

is Lipschitz continuous.

Moreover, m_ϵ *is bounded in* $C_0(\mathbb{R} \cap W^{1,\infty}(\mathbb{R})$ *in the sense that*

$$||\,\xi \mapsto \int_0^\infty \int_{\mathbb{R}} m_\epsilon(x,t,\xi)\phi(x,t)dxdt\,||_{L^\infty(\mathbb{R})} \leq C$$

and

$$||\,\frac{\partial}{\partial \xi}(\xi \mapsto \int_0^\infty \int_{\mathbb{R}} m_\epsilon(x,t,\xi)\phi(x,t)dxdt)\,||_{L^\infty(\mathbb{R})} \leq C.$$

c) $\mu(\xi) \in L_0^\infty(\mathbb{R}_\xi)$ *such that*

$$\mu(\xi) = \begin{cases} \|(u_0 - \xi)^-\|_{L^1(\mathbb{R})} \\ \|(u_0 - \xi)^+\|_{L^1(\mathbb{R})} \text{ [46]}. \end{cases}$$

.

Lemma 4.14. *Same as the assumption and notation in Lemma 4.13, suppose* $\frac{\epsilon_1}{\epsilon_2} \to 0$ *as* $\epsilon_1, \epsilon_2 \to 0$ *then*

$$f_\epsilon(x, 0, \xi) \to \chi(\xi, u_0(x))$$

in $L^p(\mathbb{R} \times \mathbb{R}_\xi)$ *for* $1 \leq p < \infty$.

Proof: See [46].
Note:

This lemma 4.14 also says that at $t = 0$ the regularised solution is strongly convergent.

Remark 4.15. *Let*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} m dx dt \leq \mu(\xi) \in L_0^\infty(\mathbb{R}).$$

This bound expresses the mass conservation for positive time. Integrating the equation in x,t and ξ *gives*

$$\int_{\mathbb{R}\times(-\mathbb{R},\mathbb{R})} |f_\epsilon(x, T, \xi)| - \chi(\xi, u_0(x)) d\xi dx$$
$$= - \int_0^T \int_{\mathbb{R}} (m(x, t, R) - m(x, t, -R)) dx dt \to 0$$

as $R \to \infty$ *[46].*

Here $x \in \mathbb{R}$, $\xi \in (-\mathbb{R}, \mathbb{R})$,$t \in (0, T)$.

4.2.2 Generalized Kinetic solution

Suppose we have

$$f_t + a(\xi) f_x = \frac{\partial m}{\partial \xi} \tag{4.14}$$

with

$$f(x, t = 0, \xi) = \chi(\xi, u_0(x)). \tag{4.15}$$

Here $u_0 \in L^1(\mathbb{R})$ and m is a bounded positive measure.

Definition 4.16 (Weak solution)**.** *For all positive test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_\xi)$, f is called a weak solution of above equation (4.14), (4.15) if*

$$\iiint f(\phi_t + a(\xi)\phi_x)\, dx\, dt\, d\xi + \iint \chi(\xi, u_0(x)\phi(x, t = 0, \xi) dx d\xi$$
$$= \iiint m \frac{\partial \phi}{\partial \xi}\, dx\, dt\, d\xi,$$

Where

$$f = f(x, t, \xi)$$
$$\phi = \phi(x, t, \xi)$$
$$m = m(x, t, \xi)$$
$$\text{and } f \in L^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_\xi).$$

Definition 4.17 (Generalized Kinetic Solution)**.**
A function $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R} \times \mathbb{R}_\xi)$ is called a generalised kinetic solution (GKS) of (4.8)*, if f is a weak solution for some positive measure $m \geq 0$ such that the following conditions holds for some function $\mu(\xi)$ and for some non-negative measure $\nu(x, t, \xi)$:*

a)

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} m dx dt \leq \mu(\xi) \in L_0^\infty(\mathbb{R}); \tag{4.16}$$

b)

$$\frac{\partial f}{\partial \xi} = \delta(\xi) - \nu(x,t,\xi);$$

c)

$$|f| = \text{sgn}(\xi) f(x,t,\xi) \leq 1.$$

If $u = u(x,t)$ is entropy solution of(4.8), then $f = f(x,t,\xi) = \chi(\xi; u(x,t))$ is a Kinetic solution [46].

Remark 4.18. *For positive R, we know*

$$f(x,t,R_+) - f(x,t,R_-) = 1 - \int_{R_-}^{R_+} d\nu(x,t,\xi.$$

But for almost all (x,t) and $f = f(x,t,\xi) \in L^1(\mathbb{R}_\xi)$*, the term*

$$f(x,t,R_+) - f(x,t,R_-)$$

vanishes

$$\lim_{R\to\infty} (f(x,t,R_+) - f(x,t,R_-) = 0.$$

So

$$\int_{\mathbb{R}_\xi} d\nu(x,t,\xi) = \int_{\mathbb{R}_\xi} \nu(x,t,\xi) d\xi = 1.$$

4.3 Uniqueness and existence

Theorem 4.19. *Suppose* $f = f(x,t,\xi)$ *be a generalized kinetic solution of* (4.8)*, where we assume* $u_0 \in L^1(\mathbb{R})$*. Then*

a) For some function u(x,t), f is a χ*-function i.e.,* $f = f(x,t,\xi) = \chi(\xi; u(x,t))$*.*

b) Furthermore as $t \downarrow 0$*, we get*

i

$$f(x,t,\xi) \to \chi(\xi, u(x,0)) \; in \; L^1(\mathbb{R} \times \mathbb{R}_\xi),$$

ii

$$\int_0^t \int_{\mathbb{R}} m(x,r,\xi)dxdr \to 0\, for\, all\, \xi$$

iii

$$m(x,t,\xi = u(x,t)) = 0$$

c) *Finally, suppose $u(x,t)$ and $v(x,t)$ are two kinetic solutions of* (4.8)*, with initial data $u_0(x)$ and $v_0(x)$, respectively. Then for almost all $t > 0$,*

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)|\, dx \le \int_{\mathbb{R}} |u_0(x) - v_0(x)|\, dx$$

This is also known as the contraction principle theorem.

Proof: See theorem 4.3.1 of [46] for detail.

Proof. To verify $f = \chi(\xi, u)$ function for some function u, it is enough to show

$$\int_{\mathbb{R}} \int_{\mathbb{R}_\xi} (|f| - f^2) d\xi dx = 0,$$

since this equation conclude that f takes values +1,-1, and 0. Recall that

$$\frac{\partial f}{\partial \xi} = \delta(\xi) - \nu_{x,t}(\xi),$$

where $\nu_{x,t}(\xi)$ is positive probability measure. We conclude $u(x,t)$

$$f(x,t,\xi) = \chi(\xi; u(x,t) for u(x,t).$$

We shall derive the following two equations:

$$\frac{d}{dt} \int_{\mathbb{R}} \int_{\mathbb{R}_\xi} |f_\epsilon|\, d\xi dx = -\int_{\mathbb{R}} 2m_\epsilon(x,t,0)dx \tag{4.17}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}_\xi} {f_\epsilon}^2 d\xi dx = -\int_{\mathbb{R}} 2m_\epsilon(x,t,0)dx + \iint m_\epsilon \nu_{x,t} * \phi_\epsilon d\xi dx \qquad (4.18)$$

First step:

Because of linearity, f_ϵ solves the PDE

$$\frac{\partial}{\partial t}f_\epsilon + a(\xi)\frac{\partial}{\partial x}f_\epsilon = \frac{\partial}{\partial \xi}m_\epsilon. \qquad (4.19)$$

Here f_ϵ and m_ϵ are the regularized functions. Multiplying both sides of (4.19) by $\text{sgn}(\xi)$ mplies

$$\frac{\partial}{\partial t}|f_\epsilon| + a(\xi)\frac{\partial}{\partial x}|f_\epsilon| = \text{sgn}(\xi)\frac{\partial}{\partial \xi}m_\epsilon,$$

Since $\text{sgn}(\xi)f_\epsilon = |f_\epsilon|$. Integrating in x, t, ξ, we get

$$\int_{\mathbb{R}\times(-\mathbb{R},\mathbb{R})} |f_\epsilon(x,T,\xi)|\, d\xi dx - \int_{\mathbb{R}\times(-\mathbb{R},\mathbb{R})} f_\epsilon(x,0,\xi)\mid d\xi dx$$
$$= \int_0^T \int_{\mathbb{R}\times(-\mathbb{R},\mathbb{R})} \text{sgn}(\xi)\frac{\partial}{\partial \xi}m_\epsilon d\xi dx dt$$

Since according to lemma 4.13, m_ϵ is Lipschitz continious and bounded. So we can use integration by parts here. Using remark 4.15,

we get

$$\int_0^T \int_{\mathbb{R}\times(-\mathbb{R},\mathbb{R})} \text{sgn}(\xi)\frac{\partial}{\partial \xi}m_\epsilon d\xi dx dt$$
$$= \int_0^T \int_{\mathbb{R}} \text{sgn}(\xi)\{m_\epsilon(x,t,R) - m_\epsilon(x,t,-R)\}dxdt$$
$$- \int_0^T \int_{\mathbb{R}} 2m_\epsilon(x,t,0)dxdt.$$

But

$$\int_0^T \int_{\mathbb{R}} \text{sgn}(\xi)\{m_\epsilon(x,t,R) - m_\epsilon(x,t,-R)\}dxdt \to 0$$

as $\mathbb{R} \to \infty$.

Therefore we get

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}_\xi} |f_\epsilon|\, d\xi dx = -\int_{\mathbb{R}} 2m_\epsilon(x,t,0)dx$$

in the weak sense. Which is our required equation (4.17)

Second Step:

In this step we want to obtain equation (4.18). So let us multiply both sides of (4.19) by f_ϵ,

$$f_\epsilon(\frac{\partial}{\partial t}f_\epsilon + a(\xi)\frac{\partial}{\partial x}f_\epsilon) = f_\epsilon\frac{\partial}{\partial \xi}m_\epsilon,$$

We know f_ϵ is smooth in x, t, so chain rule can be applied here. Thus

$$\frac{d}{dt}\iint {f_\epsilon}^2 d(\xi)dx + \iint a(\xi)\frac{\partial}{\partial x}{f_\epsilon}^2 d(\xi)dx = \iint 2f_\epsilon\frac{\partial}{\partial \xi}m_\epsilon d\xi. \tag{4.20}$$

We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}_\xi} a(\xi)\frac{\partial}{\partial x}(f_\epsilon)^2 d(\xi)dx = 0,$$

$$\frac{\partial f}{\partial \xi} = \delta(\xi) - \nu_{x,t}(\xi),$$

$$\text{and } \iint 2f_\epsilon\frac{\partial}{\partial \xi}m_\epsilon d\xi = -\iint 2m_\epsilon\frac{\partial}{\partial \xi}f_\epsilon d\xi.$$

So with these relation (4.20) becomes

$$\frac{d}{dt}\iint (f_\epsilon)^2 d(\xi)dx = -\int 2m_\epsilon(x,t,0)dx + 2\iint m_\epsilon(\nu_{x,t} * \phi_\epsilon)d(\xi)dx,$$

which is our required equation (4.18).

Final Step :

Subtracting (4.17) and (4.18), we get

$$\frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}_\xi}|f_\epsilon(x,t,\xi)|\,d\xi dx - \frac{d}{dt}\int_{\mathbb{R}}\int_{\mathbb{R}_\xi}f_\epsilon(x,t,\xi)^2 d\xi dx$$
$$= -2\int_{\mathbb{R}}\int_{\mathbb{R}_\xi}m_\epsilon(x,t,\xi)\nu_{x,t}*\phi_\epsilon d\xi dx$$
$$\text{i.e., } \frac{d}{dt}\iint|f_\epsilon(x,t,\xi)| - f_\epsilon(x,t,\xi)^2 d\xi dx$$
$$+2\iint m_\epsilon(x,t,\xi)\nu_{x,t}*\phi_\epsilon d\xi dx = 0.$$

So for all most all $t > 0$,

$$\iint|f_\epsilon(x,t,\xi)| - f_\epsilon(x,t,\xi)^2 d\xi dx + 2\int_0^t\iint m_\epsilon(x,t,\xi)\nu_{x,t}*\phi_\epsilon d\xi dx$$
$$= \iint|f_\epsilon(x,0,\xi)|) - f_\epsilon(x,0,\xi)^2 d\xi dx. \tag{4.21}$$

According to lemma 4.14, we know that $f_\epsilon(x,0,\xi) \to \chi(\xi, u_0(x)$ as $\epsilon \to 0$, so the the term in right hand of (4.21)becomes

$$\iint|f_\epsilon(x,0,\xi)| - f_\epsilon(x,0,\xi)^2 d\xi dx$$
$$\to \iint|\chi(\xi,u_0(x)| - \chi(\xi,u_0(x)^2 d\xi dx.$$

We know $|\chi(\xi,u_0(x)| - \chi(\xi,u_0(x)^2 = 0$. Therefore

$$\iint|f_\epsilon(x,0,\xi)| - f_\epsilon(x,0,\xi)^2 d\xi dx = 0.$$

Now when $\epsilon \to 0$

$$\lim_{\epsilon\to 0} 2\int_0^t\iint m_\epsilon(x,t,\xi)\nu_{x,t}*\phi_\epsilon d\xi dx = 0$$

and

$$\lim_{\epsilon\to 0}\int(|f_\epsilon(x,t,\xi)|) - f_\epsilon(x,t,\xi)^2 d\xi dx = 0.$$

Hence

$$\int_{\mathbb{R}} \int_{\mathbb{R}_\xi} (|f| - f^2) d\xi dx = 0.$$

This means

$$f = \begin{cases} 1 \\ -1 \\ 0 \end{cases}$$

So we must have $f(x,t:\xi) = \chi(\xi, u(x,t))$ for some function $u(x,t)$ [46], lecture set 9 of [32]. □

Existence proof:

Let $\{u_\epsilon\}_{\epsilon>o}$ be a sequence of classical solutions of the parabolic conservation law

$$(u^\epsilon)_t + B(u^\epsilon)_x = \epsilon(u^\epsilon_{xx})$$

with initial data

$$u^\epsilon(x,0) = u_0(x \in u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

.

Note:

Here we consider the case without BV estimates.

We assume on L^∞ as priori estimate: $\|u^\epsilon\|_{L^\infty} \leq C$, for all $\epsilon \geq 0$.

Accordingly, we can assume,

a)

$$\int_o^\infty \int_{-\infty}^\infty u^\epsilon \phi \, dx \, dt \overset{\epsilon \downarrow 0}{\to} \int_o^\infty \int_{-\infty}^\infty u\phi dx dt.$$

We can also write this as $u^\epsilon \overset{*}{\rightharpoonup} u$ in L^∞ as $\epsilon \to 0$.

Again we know also that$\|B(u^\epsilon)\|_{L^\infty} \leq C_B$ (here C_B is a constant, which is indepndent of ϵ). So we can assume

b) $B(u^\epsilon) \overset{*}{\rightharpoonup} \bar{B}$ in $L^\infty(\mathbb{R} \times \mathbb{R}_+)$ for some limit $\bar{f}$

c) for each $\epsilon > 0$,

$$\int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_t + B(u^\epsilon)\phi_x dxdt = 0.$$

Using a) and b) in c), sending $\epsilon \to 0$ gives

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + \bar{B}(t,x) dxdt = 0$$

and this provides weak form of

$$u_t + \bar{B}_x = 0.$$

Here we don't know whether $\bar{B} = B(u)$ or not? So this is not our pde. We need strong convergence to conclude this.

Recall the kinetic formulation,

$$\chi(\xi, u^\epsilon(x,t))_t + a(\xi)\chi(\xi, u^\epsilon(x,t)_x = \epsilon\chi(\xi, u^\epsilon)_{xx} + \frac{\partial m^\epsilon}{\partial \xi} \tag{4.22}$$

with initial condition

$$\chi(\xi, u(x,0) = \chi_0 = \chi(\xi, u_0(x)), \tag{4.23}$$

which holds weakly[27]. Here $a(\xi) = B'(\xi)$, $m^\epsilon = \delta(\xi - u^\epsilon).\epsilon((u^\epsilon)_x)^2$ as before.

Since,

$$|\chi(\xi, u^\epsilon)| \leq 1$$

for all x,t, ξ. So we can assume $\chi(\xi, u^\epsilon) \xrightharpoonup{*} f(x,t,\xi)$ as $\epsilon \to 0$, for some limit f.

Furthermore

$$
\begin{aligned}
|f(x,t,\xi)| &= \text{sgn}(\xi) f(x,t,\xi) \leq 1 \\
\frac{\partial}{\partial \xi} f(x,\xi) &= \delta(\xi) - \nu_{x,t(\xi)},
\end{aligned}
$$

For some positive probability measure $\nu_{x,t(\xi)}$.

In addition,

$$
\|m^\epsilon\|_{L^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_\xi)} \leq \int_{\mathbb{R}} \frac{(u_0)^2}{2} < \infty,
$$

and

$$
m^\epsilon \xrightharpoonup{*} m
$$

in the sense of measure as $\epsilon \to 0$. Hence sending $\epsilon \downarrow 0$ in (4.22), we get

$$
\begin{aligned}
f_t + a(\xi) f_x &= \frac{\partial m}{\partial \xi} \\
f\,|_{t=0} &= \chi_0 = \chi(\xi, u_0).
\end{aligned}
$$

From Theorem 4.19, we know that f is a χ-function: $f = \chi(\xi, u(x,t))$ for some function $u(x,t)$.

Also from above Lemma 4.9, a sequence which converges weakly to a χ-function converges strongly. Hence

$$
u^\epsilon \to u \quad in \quad L^1_{loc}
$$

when $\epsilon \downarrow 0$, and u solves (4.8) [32, lecture set 9].

Final remarks:

Large part of this section is based on [32, 46]. See [27, 32, 40, 54] for detail understanding.

Chapter 5

Nonlinear compactness effects

The main part of the thesis is presented in this chapter. It is based on the paper introduced by Golse [1, 24].

Let us consider the one dimensional Scalar conservation law

$$u_t + f(u)_x = 0,\ \mathrm{x} \in \mathbb{R}\,, \mathrm{t} > 0, \tag{5.1}$$

with initial condition

$$u(x, 0) = u_0(x). \tag{5.2}$$

Let us assume that $f \in C^2(\mathbb{R})$, $a > 0$, and $f'' \geq a$. Without loss of generality we also assume that $f(0) = f'(0) = 0$.

Remark 5.1.

a If f is linear, then the solution of (5.1) is $u = u_0(x - kt)$. Consequently, the solution u inherits the regularity of the initial data.

b The situation is different in the non-linear case. It was already mentioned in previous chapters that non-linearity makes the solution loose , say, C^1 regularity. So non-linear regularizing (compactness) effects have been studied with *great interest by many authors.*

Let us discuss some of historical backgrounds on this issue [1, 24, 30]:

5.1 Historical developments

1. The Lax-Oleinik one-sided estimate [35, 36, 43],

$$u_x \leq \frac{1}{bt}$$

   implies that for $t > 0$ and $x \in \mathbb{R}$, $u \in BV_{loc}((-\infty, \infty) \times \mathbb{R}_+^*)$ in the sense of distribution. This result is available for one-dimensional conservation law with $f'' > b > 0$.

2. Using the kinetic formulation of the scalar conservation law with the velocity averaging regularity method, E. Tadmor, B. Perthame and P. L Lions [40] proved that for $r < 5/3$ and $1 \leq p < 3/2$, u belongs to $W_{loc}^{r,p}((-\infty, \infty) \times \mathbb{R}_+^*))$. Later in 2002, P.E Jabin and B. Perthame [31] slightly changed the previous result to $r < 1/3$ and $1 \leq p < 5/2$. Their theory is not able to capture the BV regularity provided by the Lax Oleinik one-sided estimate.

3. Just using the fact that the entropy production is a bounded radon measure, C. DeLellis and M. Westdickenberg [14] proved that we cannot find better regularity than we cannot find better regularity than $B_\infty^{1/s,s}$ for $s \geq 3$ or $B_s^{1/s,3}$ for $1 \leq s < 3$.

4. Recently Golse has proved nonlinear regularizing effects of scalar conservation law (5.1), which also gives the regularity due to in the DeLellis and Westdickenberg. This result doesn't use the positivity of the entropy production.

5.2 Golse's results

Theorem 5.2 (Golse [24]). *Let u be the unique entropy solution of the conservation law (5.1) with $f'' \geq a > 0$ and $f(0) = 0$. Then for $u_0(x) \in L^\infty(\mathbb{R})$, which satisfies $u_0(x) = 0$ for $|x| \geq \mathbb{R}$, the entropy solution belongs to $B^{1/4,4}_{\infty,loc}((-\infty,\infty) \times \mathbb{R}^*_+)$. More precisely,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi(x,t)^2 \left| u(x,t) - u(x+y,t+s) \right|^4 dxdt = O(|y| + |s|),$$

for every $\chi \in C^1_c((-\infty,\infty) \times \mathbb{R}^*_+)$ [1, 24].

Remark 5.3. *Here $B^{1/4,4}_{\infty,loc}((-\infty,\infty) \times \mathbb{R}^*_+)$ represents the Besove space estimate [49].*

Proof. We know that the entropy solution u satisfies the entropy inequality

$$\eta(u)_t + q(u)_x \leq 0,$$

for convex entropy pairs (η, q). In particular,

$$\begin{aligned} u_t + f(u)_x &= 0 \\ \frac{1}{2}(u^2)_t + q(u)_x &= -\mu, \end{aligned}$$

for some nonnegative bounded radon measure [51] μ Satisfying

$$\mu(\mathbb{R}_+ \times \mathbb{R}) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \mu \leq \int_{\mathbb{R}} \frac{|u_0|^2}{2} < \infty.$$

Step 1: We want to use div-curl argument [42, 51] (see also [1, 10, 20]).

Let us denote $\tau_{(y,s)}\phi(x,t) = \phi(x-y,t-s)$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, here J represents the rotation of $\pi/2$.

Moreover, set

$$A = \begin{pmatrix} u \\ f(u) \end{pmatrix} \text{ and } B = (\tau_{(y,s)} - I) \begin{pmatrix} \frac{(u)^2}{2} \\ q(u) \end{pmatrix}.$$

We have $B, A \in L^{\infty}_{x,t}$,

$$div_{x,t} A = u(_t + f(u)_x = 0,$$

and

$$div_{x,t} B = \mu - \tau_{(y,s)} \mu.$$

In particular there exists $\pi \in Lip(\mathbb{R} \times \mathbb{R}^*_+)$ such that $A = J \bigtriangledown_{x,t} \pi$

Now using integration by parts, we get

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi^2 B \cdot J(\tau_{(y,s)} A - A) dt dx &= - \int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi^2 B \cdot \bigtriangledown_{x,t} (\tau_{(y,s)} \pi - \pi) dt dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \bigtriangledown_{x,t} \chi^2 \cdot B(\tau_{(y,s)} \pi - \pi) dt dx \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi^2 (\tau_{(y,s)} \pi - \pi)(\mu - \tau_{(y,s)} \mu) dt dx. \end{aligned}$$

Here $\tau_{(y,s)} A = [u(x-y, t-s), f(u(x-y, t-s))]$ and so on. Since π is Lipchitz continuous, we must have

$$\left\| \tau_{(y,s)} \pi - \pi) \right\| \leq Lip(\pi)(|s| + |y|)$$

Hence we obtain the upper bound

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi^2 B.J(\tau_{(y,s)} A - A) dt dx \\ &\quad \leq (\left\| \bigtriangledown_{x,t} \chi^2 \right\|_{L^1} \|B\|_{L^\infty} + 2 \left\| \chi^2 \right\|_{L^\infty} \iint |\mu|) \\ &\quad .Lip(\pi)(|y| + |s|). \end{aligned}$$

The above computation yields an estimate of the form

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+}\chi^2\big((\tau_{(y,s)}u - u)(\tau_{(y,s)}q(u) - q(u))$$
$$- \frac{1}{2}(\tau_{(y,s)}u^2 - u^2)(\tau_{(y,s)}f(u) - f(u))\big)dtdx$$
$$\leq C(|y| + |s|). \tag{5.3}$$

This is the required upper bound.

Step 2: In this step, we want to obtain a lower bound for the integrand in the left hand side of (5.3).

Remark 5.4. *There in remark 30 of [51], can find the inequality*

$$(g - h)(\psi(g) - \psi(h)) \geq (\phi(g) - \phi(h))(f(g) - f(h)), \tag{5.4}$$

for all g,h $\in \mathbb{R}$. Where ϕ is strictly convex entropy. Also f, and ϕ both are convex c^2 function on $\mathbb{R}$ satisfying $\psi' = f'\phi'$. (also see [20]),

Lemma 5.5. *Suppose $f \in C^2(\mathbb{R})$ satisfies $f''(z) \geq a > 0$ for all $z \in \mathbb{R}$. Then for every $w, z \in \mathbb{R}$, we have*

$$(w - z)(q(w) - q(z)) - \frac{1}{2}(w^2 - z^2)(f(w) - f(z))$$
$$\geq \frac{a}{12}|w - z|^4. \tag{5.5}$$

Proof. Let us rewrite (5.4) by putting $q = \psi$, $\phi(u) = \eta(u) = \frac{u^2}{2}$, $\phi = \eta$ $w = g$, and $z = h$. We have already known that $q'(w) = wf'(w)$, for all $w \in \mathbb{R}$.
Without loss of generality let us assume $z < w$. The left hand side of this (5.5) can be expressed in integrand form as

$$\begin{aligned}
&(w-z)(q(w)-q(z)) - \frac{1}{2}(w^2-z^2)(f(w)-f(z)) \\
&\quad = \int_z^w d\xi \int_z^w sf'(s)ds - \int_z^w \xi d\xi \int_z^w f'(s)ds \\
&\quad = \int_z^w \int_z^w (s-\xi)f'(s)d\xi ds \\
&\quad = \frac{1}{2}\int_z^w \int_z^w (s-\xi)(f'(s)-f'(\xi))d\xi ds \qquad (5.6) \\
&\quad \geq \frac{a}{2}\int_z^w \int_z^w (s-\xi)^2 d\xi ds = \frac{a}{2}|w-z|^4 \qquad (5.7)
\end{aligned}$$

where we have used the assumption $f'' \geq a > 0$ and mean value theorem (lemma 5.2 of [26], [25]). □

Replaceing w and z by $u(x-y,t-s)$ and $u(x,t)$ in (5.7) implies

$$\begin{aligned}
&(u(x-y,t-s)-u(x,t))(q(u(x-y,t-s))-q(u(x,t))) \\
&-\frac{1}{2}(u(x-y,t-s)^2-u(x,t)^2)(f(u(x-y,t-s))-f(u(x,t))) \\
&\geq \frac{a}{12}|u^\epsilon(x-y,t-s)-u^\epsilon(x,t)|^4. \qquad (5.8)
\end{aligned}$$

Step 3: Combining the results obtained in step 1 and step 2, we get

$$\frac{a}{12}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \chi^2 \left|\tau_{(y,s)}u - u\right|^4 dtdx \leq C(|y|+|s|).$$

□

5.3 Motivation

We have already proved that the physically reasonable weak solution of the conservation law (5.1) must be obtained as the limit of the unique classical solutions u^ϵ of the parabolic equation

$$u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0 \tag{5.9}$$

with

$$u(x,0) = u_0(x) \tag{5.10}$$

Motivated by the theorem (5.6) (introduced by F. Golse [1, 24]) want to obtain our goal of this paper i.e.,

Theorem 5.6. *Let u^ϵ be the unique classical solution (5.9) and* (5.10) *with $f'' \geq a > 0$ and $f(0) = 0$. Let us assume that $u_0(x) \in L^\infty(\mathbb{R})$, $u_0(x) = 0$ for $|x| \geq \mathbb{R}$. Then u^ϵ belongs to $B^{1/4,4}_{\infty,loc}((-\infty,\infty) \times \mathbb{R}^*_+)$. More precisely,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \chi(x,t)^2 \left| u^\epsilon(x,t) - u^\epsilon(x+y,t+s) \right|^4 dxdt = O(|y| + |s|),$$

for every $\chi \in C^1_c((-\infty,\infty) \times \mathbb{R}^*_+)$.

Proof. Let us multiply both sides of (5.9) by $\eta'(u^\epsilon)$, we get

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon\eta(u^\epsilon)_{xx} - \epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2. \tag{5.11}$$

Since η is convex, then the dissipation[9] term $\epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2$ is positive (see [7]).

Set

$$\mu := \epsilon\eta(u^\epsilon)_{xx} - \epsilon\eta''(u^\epsilon)(u^\epsilon_x)^2.$$

Then the equation (5.11), becomes

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = -\mu. \tag{5.12}$$

We take $\eta(u) = \frac{u^2}{2}$, and integrating over $\mathbb{R} \times \mathbb{R}_+$, there by obtaining

$$\epsilon \int_{\mathbb{R}} \int_{\mathbb{R}_+} (u_x^\epsilon)^2 dt dx = \frac{1}{2} \int_{\mathbb{R}} u_0(x)^2 \leq C. \tag{5.13}$$

Note that

$$\begin{aligned} \left| \epsilon \iint \eta(u^\epsilon)_{xx} \phi dt dx \right| &\leq C_1 \iint |u_x^\epsilon| \, |\phi_x| \, dt dx \\ &\leq C\sqrt{\epsilon} \left\| \sqrt{\epsilon} u_x^\epsilon \right\|_{L^2} \|\phi\|_{H_0^1} \qquad \text{using5.13} \\ &\leq C\sqrt{\epsilon} \, \|\phi\|_{H_0^1}, \end{aligned}$$

for any $\phi = \phi(x,t) \in H_0^1(\Omega)$, and $\Omega \subset\subset \mathbb{R}_+^2$. Here C is independent of ϵ. Therefore

$$\|\epsilon \eta(u^\epsilon)_{xx}\|_{H_{loc}^{-1}(\mathbb{R}_+^2)} \leq C\sqrt{\epsilon}.$$

It also follows that

$$\left\| \epsilon \eta''(u^\epsilon)(u_x^\epsilon)^2 \right\|_{L_{loc}^1(\mathbb{R}_+^2)} \leq C_1 \left\| \sqrt{\epsilon} u_x^\epsilon \right\|_{L_{loc}^2(\mathbb{R}_+^2)} \leq C,$$

which implies

$$\epsilon \eta''(u^\epsilon)(u_x^\epsilon)^2$$

is compact in $W_{loc}^{-1,p}(\mathbb{R}_+^2)$ for $p \in (1,2)$, $\epsilon > 0$ (cf. compact embedding theorem).

Hence for $1 < p < 2$, we have that $\eta(u^\epsilon)_t + q(u^\epsilon)_x$ is compact in $W_{loc}^{-1,p}(\mathbb{R}_+^2)$. On the other hand $\eta(u^\epsilon)_t + q(u^\epsilon)_x$ is bounded in $W_{loc}^{-1,\infty}(\mathbb{R}_+^2)$. Therefore $\eta(u^\epsilon)_t + q(u^\epsilon)_x$ is compact in $H_{loc}^{-1}(\mathbb{R}_+^2)$ [7].

Sketch of proof: Let us denote $\tau_{(y,s)}\phi(x,t) = \phi(x-y, t-s)$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Step1: Let $A = \begin{pmatrix} u^\epsilon \\ f(u^\epsilon) \end{pmatrix}$ and $B = (\tau_{(y,s)} - I) \begin{pmatrix} \frac{(u^\epsilon)^2}{2} \\ q(u^\epsilon) \end{pmatrix}$.

We have $B, A \in L^\infty_{x,t}$, and also [20]

$$\mathrm{div}_{x,t} A = u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx},$$
$$\mathrm{div}_{x,t} B = \mu - \tau_{(y,s)}\mu.$$

Using the Div-Curl argument, (see remark5.7 below, [15]) we obtain the bound

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} \chi^2 B.J(\tau_{(y,s)}A - A)dxdt \leq C(|s| + |t|)$$

Which yields an estimate of the of the form

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}_+} \chi^2\big((\tau_{(y,s)}u^\epsilon - u^\epsilon)(\tau_{(y,s)}q(u^\epsilon) - q(u^\epsilon))\\ &- \frac{1}{2}(\tau_{(y,s)}(u^\epsilon)^2 - (u^\epsilon)^2)(\tau_{(y,s)}f(u^\epsilon) - f(u^\epsilon))\big)dtdx\\ &\leq C(|y| + |s|), \end{aligned} \tag{5.14}$$

Step 2: From Lemma 5.5,

$$(w - z)(q(w) - q(z)) - \frac{1}{2}(w^2 - z^2)(f(w) - f(z)) \geq \frac{a}{12}|w - z|^4.$$

Let us replace w and z by $u^\epsilon(x - y, t - s)$ and $u^\epsilon(x, t)$, respectively. Then

$$\begin{aligned}&(u^\epsilon(x - y, t - s) - u^\epsilon(x, t))(q(u^\epsilon(x - y, t - s)) - q(u^\epsilon(x, t)))\\ &- \frac{1}{2}(u^\epsilon(x - y, t - s)^2 - u^\epsilon(x, t)^2)(f(u^\epsilon(x - y, t - s)) - f(u^\epsilon(x, t)))\\ &\geq \frac{a}{12}|u^\epsilon(x - y, t - s) - u^\epsilon(x, t)|^4. \end{aligned} \tag{5.15}$$

Step 3: Using (5.15)in (5.14), we obtain

$$\frac{a}{12}\int_{\mathbb{R}}\int_{\mathbb{R}_+}\chi^2\left|\tau_{(y,s)}u^{\epsilon}-u^{\epsilon}\right|^4 dxdt \leq C\left(|y|+|s|\right).$$

This is the required Besov space estimate for the solutions of the parabolic equation (5.9).

□

Remark 5.7. *Let Ω be open subset of $\mathbb{R}_+^* \times \mathbb{R}$ and $\chi \in C_c^\infty(\Omega)$. Suppose $A = A(t,x)$ and $B = B(t,x) \in \mathbb{R}^2$ be two vector fields satisfying $A, B \in L^\infty(\Omega, \mathbb{R}^2)$. Then Div-curl bilinear inequality implies*

$$\begin{aligned}\left|\iint_{\Omega}\chi^2 A\cdot JBdtdx\right| \leq\; & \|\chi A\|_{L^p(\Omega)}\|\chi div_{t,x}B\|_{W^{-1,p'}(\Omega)} \\ & + \|\chi B\|_{L^p(\Omega)}\|\chi div_{t,x}A\|_{W^{-1,p'}(\Omega)} \\ & + \|\chi A\|_{L^2(\Omega)}\|\nabla_{t,x}\chi B\|_{H^{-1}(\Omega)} \\ & + \|\chi B\|_{L^2(\Omega)}\|\nabla_{t,x}\chi A\|_{H^{-1}(\Omega)} \qquad (5.16)\end{aligned}$$

[24]

5.4 Conclusion

The nonlinear compactness effects of the scalar conservation law is the topic of this thesis. The main target of this thesis is to prove convergence of sequence of approximate solutions by exploring a new compactness method introduced recently by Golse in [24]. We started by considering one-dimensional scalar conservation laws, $u_t + f(u)_x = 0$, where the flux function f is nonlinear. Giving different examples, we tried to make clear that such equations may admit many weak solutions. Which raised the need to introduce a technique to choose the physically correct solution.

We discussed in detail how entropy inequalities may support that purpose. If such entropy solutions exist, then they must be unique. We used the Kruzkov Uniqueness Theorem to show the uniqueness of the entropy solution.

To prove existence of entropy solutions, we must produce a sequence of approximate solutions. To obtain the sequence of approximate solutions, we used the vanishing viscosity method. The interesting step in the existence prove is to show compactness since the physically correct solution of the conservation law must be obtained as the unique limit of the viscous regularization (when $\epsilon \to 0$). We have shown how the sequence of approximate solution converges by using the Rellich-Kandrachov theorem and also by using the Kolmogorov compactness theorem. We also studied the kinetic formulation of the scalar conservation laws, which generalizes the notion of entropy solutions. We introduced the χ-function to turn the nonlinear conservation law into a linear one. We proved the uniqueness and existence theorem, relying on weak compactness techniques.

Due to the presence of nonlinearity, there are some regularizing effects on the solution. Non-linear regularizing (compactness) effects have been studied with great interest by many author. We discussed the recent work by F.Golse [24]. We rediscovered how he has proved his claim. We proved a new theorem (5.6) for parabolic conservation laws by using Golse's result for the viscosity method. Hence reviewing this new theorem from the point of view of the Kolmogorov compactness theorem provides a new convergence method (nonlinear) for approximate solutions.

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