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Geometry of Higher Dimensional Representations of Noncommutative Algebras

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis we begin by looking at polynomials of m $n \times n$ matrices invariant under simultaneous conjugation. Thereafter we connect the invariant polynomials to simple representations of noncommutative plane curves, foremost in dimension 1 and 2. We also give an algorithm to find the trace of an arbitrary polynomial in two 2×2 matrices. To get a more complete understanding of the simple representations, we consider extensions. In Chapter 5 we give a picture of the 2-dimensional simple representations in the case when the algebra is given by $k\langle x, y\rangle/I$, where I is generated by $f(x, y) = x^2 + y^2 - 1 + \delta[x, y]$ with $\delta \in k$.

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CHAPTER 1

Introduction

Let $X_1, \dots, X_m$ be m $n \times n$ matrices over a field k of characteristic 0. For an invertible $n \times n$ matrix P, it then makes sense to talk about simultaneous conjugation of the X_i's by P, i.e., P operates on the m matrices X_i by

$$(X_1, \dots, X_m) \mapsto (PX_1P^{-1}, \dots, PX_mP^{-1}).$$

Let $k[x^i_{\alpha,\beta}]$ be the ring generated over k by the coefficients of the matrices $X_1, \dots, X_m$.

In 1969 M. Artin [Art69] conjectured that any polynomial in $k[x^i_{\alpha,\beta}]$ invariant under simultaneous conjugation by all invertible matrices, is a polynomial in the elements $\operatorname{tr}(X_{i_1}X_{i_2}\cdots X_{i_r})$. It seems that this was already known in Russia, see [Kir67]. However, it was proved independently by Procesi [Pro76] and Razmyslov [Raz74] in the 1970s.

Both these authors introduce trace identities and show their close relationship to the Cayley–Hamilton theorem, a theorem that informally says that any square matrix over a commutative ring satisfies its own characteristic polynomial, see, e.g., pages 23-24 of [Cay58].

Procesi found generators of the invariant ring and relations between them, and also gave an upper bound for the degree of elements needed to generate the invariant ring. As a consequence of Procesi's work, we know that for two 2×2 matrices X and Y, the invariant ring is generated over k by the traces of X, Y, XY, X^2 and Y^2.

The study of the invariant ring, finding it generators for arbitrary many matrices of arbitrary order, is an ongoing process. There are still attempts to sharpen the upper bound given by Procesi, see, e.g., [For02].

However, in this thesis the case of two 2×2 matrices is the one of greatest interest, due to the connection to 2-dimensional simple representations of the free algebra in two variables over k, $k\langle x, y\rangle$.

A representation of an k-algebra A is loosely said a homomorphism $\phi\colon A \to M_n(k)$, where $M_n(k)$ is the ring of $n \times n$ matrices over k. A representation is said to be simple if it is surjective, and two representations are isomorphic if their corresponding matrices are conjugates of each other.

We will study the isomorphism classes of simple n-dimensional representations, denoted $\mathrm{Simp}_n(A)$, under the action of $\mathrm{PGl}_n(k)$, the projective linear group. In Chapter 3 we see that when $n = 2$, the simple representations form a subset of the spectrum of the invariant ring for two 2×2 matrices. Hence $\mathrm{Simp}_2(A)$ is a subset of $\mathbb{A}^5$.

This gives us the framework to work with specific algebras on the form $k\langle x, y\rangle/I$, where I is an ideal, and looking at the simple representations of that algebra. This is done in Chapter 3.

A more complete picture of the simple representations can be given through extensions. The theory of extension modules will give a clearer insight into the boundary of the set of 2-dimensional simple representations. In Chapter 4 we look at the dimension of extensions of 1-dimensional representations.

In this thesis we try to connect the invariant ring to the 2-dimensional simple representations and to extension theory. In Chapter 5 we seek to give a geometric model of the 2-dimensional simple representations when $A = k\langle x, y\rangle/I$ where I is the ideal generated by $f(x, y) = x^2 + y^2 - 1 + \delta[x, y]$. Here $[x, y] = xy - yx$ and $\delta \in k$. It gives a connection between Chapter 3 and Chapter 4.

1.1 Outline

The rest of the thesis is organised as follows:

Chapter 2 introduces the trace ring and presents Procesi's proof of Artin's conjecture, with some elaborations, examples and modifications.

Chapter 3 introduces representations and gives the setup to find $\mathrm{Simp}_n(A)$, before looking at some examples finding $\mathrm{Simp}_2(A)$ of some algebras a. In Section 3.2 we try to find a formula for what happens to the trace of a monomial in two 2×2 matrices X and Y when multiplied with either X or Y. We end the chapter with an algorithm to find the trace of an arbitrary polynomial in X and Y.

Chapter 4 starts with a brief look at the Ext-functor. Thereafter we find a truncated resolution so that we can calculate $\dim_k \mathrm{Ext}^1_A(k(p_1), k(p_2))$ for two 1-dimensional representations $k(p_1)$ and $k(p_2)$. We end the chapter by looking at some examples.

Chapter 5 tries to give a connection between the preceding chapters. We look at the algebra $A = k\langle x, y\rangle/I$ where I is the ideal generated by $f(x, y) = x^2 + y^2 - 1 + \delta[x, y]$. The goal is to give a geometric model of the 2-dimensional simple representations of this algebra. This results in a close connection between Chapter 3 and Chapter 4. Finally we give a picture of the simple representations of dimension 2.

Computations are done with [Matlab]. Figures are made with Ti*k*Z, [Matlab] and [POV].

CHAPTER 2

The Trace Ring

In this chapter, we lay the foundations for what comes next. We define and find the trace ring, a specific kind of invariant ring. This ring will be used frequently in the coming chapters.

This chapter follows the first five sections of [Pro76]. The main theorems and results can be found there, and we try to give an overview with what is of interest to us. We will to a large extent follow Procesi's approach, with some changes and elaborations. Throughout this chapter, and later in the thesis, we will sometimes abbreviate End_k to End and $\otimes_k$ to $\otimes$ when the underlying field k is already specified.

2.1 Generators

Let k be a field of characteristic 0. We set $G = \mathrm{Gl}_n(k)$ to be the general linear group, and denote the ring of $n \times n$ matrices over k by $M_n(k)$ or M_n if the field is already specified. The group G acts on matrices by conjugation and on i-tuples of $n \times n$ matrices by simultaneous conjugation. That is, for $g \in G$ and $A = (A_1, \dots, A_i)$, where $A_j \in M_n$ for $1 \leq j \leq i$, we have

$$g * (A_1, A_2, \dots, A_i) = (gA_1g^{-1}, gA_2g^{-1}, \dots, gA_ig^{-1}).$$

What will be of interest to us is the ring of polynomial functions on M_n^i (i-tuples of $n \times n$ matrices) invariant under the action of G.

Definition 2.1. The ring of polynomial functions on M_n^i that are invariant under the action of $G = \mathrm{Gl}_n(k)$ by simultaneous conjugation is denoted by $T_{i,n}$ and is called the trace ring.

As a set, we have $T_{i,n} = \{f \colon M_n^i \to k \mid f \text{ is a polynomial and } f(A) = f(g * A)\ \forall A \in M_n^i,\ g \in G\}$. The reason we call it the trace ring will become apparent eventually.

Example 2.2. Let tr denote the usual trace function of a square matrix. Since $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(A^{-1}AB) = \mathrm{tr}(B)$ for all $A \in G,\ B \in M_n$, we have that the trace function is invariant under the action of G.

To describe the trace ring, we will first find out what kind of polynomial functions are invariant under the action of G.

Let $V \simeq k^n$ be an n-dimensional vector space and denote its dual, the linear maps from V to k, by V^*. We adopt the notation $\underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} = V^{\otimes i}$.

After choosing a basis for the vector space V, we have the standard isomorphism between the endomorphism ring of V and the n-dimensional matrix ring, $\mathrm{End}(V) \simeq M_n(k)$. We also have an isomorphism

$$\gamma\colon \mathrm{End}(V)^{\otimes i} \to \mathrm{End}(V^{\otimes i})$$

given by

$$\gamma(A_1 \otimes \cdots \otimes A_i)(v_1 \otimes \cdots \otimes v_i) = A_1 v_1 \otimes \cdots \otimes A_i v_i.$$

Thus we can identify $M_n^{\otimes i}$ with $\mathrm{End}(V^{\otimes i})$. The group G can be embedded into $\mathrm{End}(V^{\otimes i})$ using the action

$$A * (v_1 \otimes \cdots \otimes v_i) = Av_1 \otimes \cdots \otimes Av_i,$$

which turns the matrix $A \in G$ into an element of $\mathrm{End}(V^{\otimes i})$.

Our first objective is to classify all multilinear invariants from $V^{*\otimes i} \otimes V^{\otimes i}$ to k. The idea is the following:

First, we find all G-linear transformations of $V^{\otimes i}$, i.e., the centralizer of $\mathrm{End}(V^{\otimes i})$ in G, $C_{\mathrm{End}(V^{\otimes i})}(G)$. Then we find an isomorphism of G-spaces

$$\pi\colon (V^{*\otimes i} \otimes V^{\otimes i})^* \simeq \mathrm{End}(V^{\otimes i}),$$

before using this to identify the G-linear transformations of $V^{\otimes i}$ with linear maps $(V^{*\otimes i} \otimes V^{\otimes i})^* \to k$ invariant under G.

For $\sigma \in S_i$, the symmetric group on i letters, we define the endomorphism

$$\lambda_\sigma(v_1 \otimes \cdots \otimes v_i) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(i)}.$$

Then we have

$$\begin{aligned} g * (\lambda_\sigma(v_1 \otimes \cdots \otimes v_i)) &= g * \big(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(i)}\big) \\ &= gv_{\sigma^{-1}(1)} \otimes \cdots \otimes gv_{\sigma^{-1}(i)} \\ &= \lambda_\sigma(gv_1 \otimes \cdots \otimes gv_i) \\ &= \lambda_\sigma(g * (v_1 \otimes \cdots \otimes v_i)), \end{aligned}$$

and so $\lambda_\sigma g = g\lambda_\sigma$. Hence $\lambda_\sigma \in C_{\mathrm{End}(V^{\otimes i})}(G)$, which is the algebra of G-linear transformations of $V^{\otimes i}$. In fact, the endomorphisms on the form λ_σ spans $C_{\mathrm{End}(V^{\otimes i})}(G)$ as a vector space, see, e.g., [Wey39].

Now we want to establish an isomorphism

$$\pi\colon (V^{*\otimes i} \otimes V^{\otimes i})^* \simeq \mathrm{End}(V^{\otimes i}).$$

This will be done in several steps. Firstly, for the vector space V we have that $V \simeq V^*$. We also have that $V^{\otimes i *} \simeq V^{*\otimes i}$ since everything is finite dimensional, and an isomorphism $\mathrm{End}(V) \simeq V^* \otimes V$ given by

$$\langle \phi, u\rangle v = (\phi \otimes v)(u),$$

where $\langle \phi, u \rangle = \phi(u)$.

Since $\mathrm{End}(V^{\otimes i})$ is a vector space, we thus have

$$\mathrm{End}(V^{\otimes i}) \simeq \mathrm{End}(V^{\otimes i})^*$$

and

$$\mathrm{End}(V^{\otimes i}) \simeq V^{\otimes i *} \otimes V^{\otimes i}.$$

Combining these, we have

$$\mathrm{End}(V^{\otimes i}) \simeq (V^{\otimes i *} \otimes V^{\otimes i})^* \simeq (V^{*\otimes i} \otimes V^{\otimes i})^*,$$

as wanted.

Let $\pi \colon (V^{*\otimes i} \otimes V^{\otimes i})^* \simeq \mathrm{End}(V^{\otimes i})$ be the canonical identification. Then π is obtained from the nondegenerate pairing

$$\mathrm{End}(V^{\otimes i}) \times V^{*\otimes i} \otimes V^{\otimes i} \to k$$

given by

$$\langle \lambda, \phi_1 \otimes \cdots \otimes \phi_i \otimes v_1 \otimes \cdots \otimes v_i \rangle = \langle \phi_1 \otimes \cdots \otimes \phi_i, \lambda(v_1 \otimes \cdots \otimes v_i) \rangle.$$

Here $\langle \phi_1 \otimes \cdots \otimes \phi_i, \lambda(v_1 \otimes \cdots \otimes v_i) \rangle$ means the evaluation of $\phi_1 \otimes \cdots \otimes \phi_i \in V^{*\otimes i} \simeq (V^{\otimes i})^*$ on the vector $\lambda(v_1 \otimes \cdots \otimes v_i) \in V^{\otimes i}$.

Now we can use π to identify the space of G-linear endomorphisms, which we already know what is, with the space of linear maps $(V^{*\otimes i} \otimes V^{\otimes i}) \to k$, i.e., the G-invariant vectors of $(V^{*\otimes i} \otimes V^{\otimes i})^*$.

We are set up to prove a part of the so-called first fundamental theorem.

Theorem 2.3. *Any multilinear invariant $\gamma \colon V^{*\otimes i} \otimes V^{\otimes i} \to k$ is a linear combination of the invariants*

$$\prod_{j=1}^{i} \langle \phi_{\sigma(j)}, X_j \rangle = \mu_\sigma \left(\phi_1 \otimes \cdots \otimes \phi_i \otimes X_1 \otimes \cdots \otimes X_i \right).$$

Proof. In the setting above, we need to find out what the λ_σ's correspond to under π. We get

$$\begin{aligned} \langle \lambda_\sigma, \phi_1 \otimes \cdots \otimes \phi_i \otimes X_1 \otimes \cdots \otimes X_i \rangle &= \langle \phi_1 \otimes \cdots \otimes \phi_i \otimes X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(i)} \rangle \\ &= \prod_{j=1}^{i} \langle \phi_j, X_{\sigma^{-1}(j)} \rangle \\ &= \prod_{j=1}^{i} \langle \phi_{\sigma(j)}, X_j \rangle. \end{aligned}$$

Defining

$$\mu_\sigma \left(\phi_1 \otimes \cdots \otimes \phi_i \otimes X_1 \otimes \cdots \otimes X_i \right) = \prod_{j=1}^{i} \langle \phi_{\sigma(j)}, X_j \rangle$$

gives the theorem. ∎

The next step in our quest for polynomial invariants under simultaneous conjugation, is to translate this result into a result about matrices instead of the tensor product $V^{*\otimes i} \otimes V^{\otimes i}$.

The isomorphism $\mathrm{End}(V) \simeq V^* \otimes V$ combined with the isomorphism $M_n \simeq \mathrm{End}(V)$, gives

$$M_n^{\otimes i} \simeq (V^* \otimes V)^{\otimes i} \simeq V^{*\otimes i} \otimes V^{\otimes i}.$$

This will be used to find the invariants we are looking for. However, first we need a couple of identities which will be used to prove Theorem 2.5.

Lemma 2.4. *Let $\phi, \psi \in V^*$ and $u, v \in V$. Then we have:*

$$(i)\ \phi \otimes v \cdot \psi \otimes u = \phi \otimes \langle \psi, v \rangle u \tag{2.1}$$

$$(ii)\ \mathrm{tr}(\phi \otimes v) = \langle \phi, v \rangle. \tag{2.2}$$

Proof. We let e_i be a basis for V and e_i^* be a basis for V^*. Then we can write $\phi = \sum_{i=1}^n a_i e_i^*$, $\psi = \sum_{i=1}^n a_i' e_i^*$, $v = \sum_{i=1}^n b_i e_i$ and $u = \sum_{i=1}^n b_i' e_i$.

(i): This is tedious and does not provide any real insight, so we take an example when $n = 2$.
We calculate the two sides of the equation using the isomorphism $M_n \simeq V^* \otimes V$. If we let the vectors of V^* be the columns, and the vectors of V be the rows, the left hand side becomes

$$\phi \otimes v \cdot \psi \otimes u = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix} \begin{bmatrix} a_1' b_1' & a_1' b_2' \\ a_2' b_1' & a_2' b_2' \end{bmatrix}.$$

The right hand side becomes

$$\begin{aligned} \phi \otimes \langle \psi, v \rangle u &= \phi \otimes (a_1' b_i + a_2' b_2) u \\ &= \phi \otimes ((a_1' b_1 + a_2' b_2) b_1' e_1 + (a_2' b_2) b_2' e_2) \\ &= \begin{bmatrix} a_1 b_1' (a_1' b_1 + a_2' b_2) & a_1 b_2' (a_1' b_1 + a_2' b_2) \\ a_2 b_1' (a_1' b_1 + a_2' b_2) & a_2 b_2' (a_1' b_1 + a_2' b_2) \end{bmatrix}. \end{aligned}$$

Thus the left hand side and the right hand side coincide.

(ii): We have

$$\begin{aligned} \mathrm{tr}(\phi \otimes v) &= \sum_{i=1}^n (\phi \otimes v)_{ii} \\ &= \sum_{i=1}^n \phi(e_i) v_i \\ &= \phi\left(\sum_{i=1}^n e_i \cdot v_i\right) \\ &= \phi(v) = \langle \phi, v \rangle. \end{aligned}$$

■

Letting $\sigma = (i_1 i_2 \cdots i_k)(j_1 j_2 \cdots j_l) \cdots (t_1 t_2 \cdots t_e) \in S_i$ be decomposed into disjoint cycles, we can now "translate" Theorem 2.3 into the language of matrices.

Theorem 2.5. *For $A_1, \ldots, A_i \in M_n$, we have*

$$\mu_\sigma(A_1 \otimes \cdots \otimes A_i) = \operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_k}) \operatorname{tr}(A_{j_1} \cdots A_{j_l}) \cdots \operatorname{tr}(A_{t_1} \cdots A_{t_e}).$$

Proof. Since we have multilinear maps on both sides of the equality, we can reduce the tensor products from sums to single terms, i.e., $A_j = \phi_j \otimes X_j$. Hence we have

$$\begin{aligned}\mu_\sigma(A_1 \otimes \cdots A_i) =& \mu_\sigma(\phi_1 \otimes \cdots \otimes \phi_i \otimes X_1 \cdots \otimes X_i) \\ =& \prod_{j=1}^{i} \langle \phi_{\sigma(j)}, X_j \rangle \\ =& \langle \phi_{i_2}, X_{i_1} \rangle \langle \phi_{i_3}, X_{i_2} \rangle \cdots \langle \phi_{i_k}, X_{i_{k-1}} \rangle \langle \phi_{i_1}, X_{i_k} \rangle \cdot \\ & \langle \phi_{j_2}, X_{j_1} \rangle \cdots \langle \phi_{j_1}, X_{j_l} \rangle \cdots \langle \phi_{t_1}, X_{t_e} \rangle.\end{aligned}$$

Let $M = \langle \phi_{i_2}, X_{i_1} \rangle \langle \phi_{i_3}, X_{i_2} \rangle \cdots \langle \phi_{i_k}, X_{i_{k-1}} \rangle \langle \phi_{i_1}, X_{i_k} \rangle$. By repeated use of (2.1), we get

$$\begin{aligned}A_{i_1} A_{i_2} \cdots A_{i_k} &= (\phi_{i_1} \otimes X_{i_1})(\phi_{i_2} \otimes X_{i_2}) \cdots (\phi_{i_k} \otimes X_{i_k}) \\ &\overset{(2.1)}{=} \langle \phi_{i_2}, X_{i_1} \rangle \langle \phi_{i_3}, X_{i_2} \rangle \cdots \langle \phi_{i_k}, X_{i_{k-1}} \rangle \phi_{i_1} \otimes X_{i_k}.\end{aligned}$$

Taking the trace on both sides, (2.2) gives

$$\begin{aligned}\operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_k}) &= \operatorname{tr}(\langle \phi_{i_2}, X_{i_1} \rangle \langle \phi_{i_3}, X_{i_2} \rangle \cdots \langle \phi_{i_k}, X_{i_{k-1}} \rangle \phi_{i_1} \otimes X_{i_k}) \\ &\overset{(2.2)}{=} \langle \phi_{i_2}, X_{i_1} \rangle \langle \phi_{i_3}, X_{i_2} \rangle \cdots \langle \phi_{i_k}, X_{i_{k-1}} \rangle \langle \phi_{i_1}, X_{i_k} \rangle = M,\end{aligned}$$

and the theorem follows. ■

Theorem 2.3 and Theorem 2.5 leads to a handy theorem about the polynomial invariants of a collection of matrices.

Theorem 2.6. *Any polynomial invariant of i $n \times n$ matrices $A_1, \ldots, A_i$ is a polynomial in the invariants $\operatorname{tr}(A_{i_1} A_{i_2} \cdots A_{i_k})$, where A_{i_j} is one of the matrices $A_1, \ldots, A_i$ and $A_{i_1} A_{i_2} \cdots A_{i_k}$ runs through all possible monomials.*

This theorem narrows the search for invariants considerably. The result gives us a good idea of what the polynomials look like. For instance, $f(X) = X^2$ can not be an invariant unless we can write X^2 as a polynomial in traces of monomials of matrices. Thus, we have ruled out a lot of functions. A problem, however, is that there are still plenty left.

2.2 Finiteness and Relations

Having found the polynomials, we turn our attention to finding relations between the generators and to a finiteness result.

We start with the last, which is a special case of more general algebraic results, see [Pro76].

Theorem 2.7. *The ring $T_{i,n}$ is generated over k by the elements* $\operatorname{tr}\left(A_{i_1}\cdots A_{i_j}\right)$, *where $j \leq 2^n - 1$.*

This of course is a powerful tool when looking to describe all invariants, because it narrows down the possibilities for functions considerably. In some cases we can now find the trace ring quite easily. For example, we can calculate $T_{1,2}$ and $T_{2,2}$.

Example 2.8. Theorem 2.7, together with the Cayley–Hamilton theorem, immediately gives that $T_{1,2} = k[t_X, t_{X^2}]$ for a matrix X. Together with the cyclic invariance of the trace, we get that $T_{2,2} = k[t_X, t_Y, t_{X^2}, t_{XY}, t_{Y^2}]$ for the two matrices X and Y. Since we have the relation

$$d_X = \frac{1}{2}\left(t_X^2 - t_{X^2}\right),$$

we also have that $T_{2,2} \simeq k[t_X, t_Y, t_{XY}, d_X, d_Y]$. In the sequel, we will work with the version of $T_{2,2}$ which best fits the purpose.

It is well known that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A+B) = \operatorname{tr}(A)+\operatorname{tr}(B)$. It would of course be of great interest to find all relations between a monomial M and $\operatorname{tr}(M)$, where M is a monomial in the matrix variables $X_1, \ldots, X_i, \ldots$.

For this purpose, we consider the formal polynomial ring T generated by the symbols $\operatorname{Tr}(X_{i_1}X_{i_2}\cdots X_{i_k})$. We say that $\operatorname{Tr}(M) = \operatorname{Tr}(N)$ if and only if N can be obtained by a cyclic permutation of M.

Definition 2.9. An element $f \in T$ is called a commutative trace polynomial in the variables X_i, and we write it $f(X_1, X_2, \ldots, X_i, \ldots)$.

We will consider a function $\pi\colon T \to T_{\infty,n}$, where $T_{\infty,n}$ is the invariants of infinitely many matrices. Since we have no relations among the elements in T, the kernel of π will be the relations between the invariants. We fix an integer n and consider the space M_n^∞ of sequences of $n \times n$ matrices $(A_1, \ldots, A_i, \ldots)$ where almost all A_j is zero.

We will define π on the generators and then extend it to all of T. Let $f \in T$ be on the form $f(X_1, \ldots, X_i, \ldots) = \operatorname{Tr}(X_{i_1}X_{i_2}\cdots X_{i_k})$, hence a generator for T. To it, we associate an invariant $\bar{f} \in T_{\infty,n}$ given by $\bar{f}(X_1, \ldots, X_i, \ldots) = \operatorname{tr}(X_{i_1}X_{i_2}\cdots X_{i_k})$. By extending this process to all of T, we have the desired map $\pi\colon T \to T_{\infty,n}$.

Definition 2.10. Elements in the kernel of π are called trace identities of $n \times n$ matrices.

These are the elements we are interested in, since they give the possible relations between the generators of the trace ring.

Let $\sigma = (i_1 i_2 \cdots i_k)(j_1 j_2 \cdots j_l)\cdots(t_1 t_2 \cdots t_e)$ be an element of S_m written as a product of disjoint cycles. We can then define an element $\Phi_\sigma \in T$ by

$$\Phi_\sigma(X_1, \ldots, X_m) = \operatorname{Tr}\left(X_{i_1}\cdots X_{i_k}\right)\operatorname{Tr}\left(X_{j_1}\cdots X_{j_l}\right)\cdots\operatorname{Tr}\left(X_{t_1}\cdots X_{t_e}\right).$$

For $A_1, \ldots, A_m \in M_n$, we have

$$\pi(\Phi_\sigma) = \bar{\Phi}_\sigma(A_1, \ldots, A_m) = \mu_\sigma(A_1 \otimes \cdots \otimes A_m).$$

Definition 2.11. $F(X_1, \dots, X_{n+1}) = \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\Phi_\sigma$ is called the fundamental trace identity.

Here $\operatorname{sgn}(\sigma) = 1$ if σ is even and $\operatorname{sgn}(\sigma) = -1$ if σ is odd.

As indicated by the name, $F(X_1, \dots, X_{n+1})$ plays an important role in describing the trace identities.

Before stating the main theorem, we need some definitions.

A Young diagram is a finite collection of boxes arranged in left-adjusted rows such that the rows are in nonincreasing order from the top.

A Young tableau is obtained from a Young diagram by filling the boxes with symbols, in our case positive integers. The tableau is on standard form if the integers in each row and column increases.

We will se examples of tableaux a bit later.

Definition 2.12. Let S_m be the symmetric group on m letters and let λ be a Young tableau corresponding to a numbered partition of m. Let

$$P_\lambda = \{g \in S_m \mid g \text{ preserves each row of } \lambda\}$$

and

$$Q_\lambda = \{g \in S_m \mid g \text{ preserves each column of } \lambda\}.$$

Put

$$a_\lambda = \sum_{g \in P_\lambda} e_g$$

and

$$b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) e_g,$$

where e_g is the unit vector corresponding to g. Then the Young symmetrizer corresponding to the Young tableau λ is defined to be

$$c_\lambda = a_\lambda b_\lambda = \sum_{g \in P_\lambda, h \in Q_\lambda} \operatorname{sgn}(h) e_{gh}.$$

Using Young tableaux, Procesi shows two important results.

Theorem 2.13. *An element $\sum_{\sigma \in S_m} \alpha_\sigma \Phi_\sigma$ is a trace identity for $n \times n$ matrices if and only if the element $\sum \alpha_\sigma \sigma$ belongs to the ideal of the group algebra of S_m spanned by the Young symmetrizers relative to the diagrams with at least $n+1$ rows.*

In particular we have the fundamental trace identity, corresponding to the Young diagram with one column and $n+1$ rows.

Corollary 2.14. *A multilinear trace identity of degree $n+1$ in $n+1$ variables is a scalar multiple of $F(X_1, \dots, X_{n+1})$.*

In fact, the fundamental trace identity generates the ideal $\ker \pi$.

Theorem 2.15. *The ideal* $\ker\pi$ *is generated by the elements* $F(M_1, \dots, M_{n+1})$*, where the* M_i*'s run over all possible monomials.*

This is all well and good, but also a bit mysterious. To make it a bit more tangible, we will look at some examples.

To start with, let $n = 1$. Thus we are looking at 1×1 matrices, or just scalars. In this case, the fundamental trace identity becomes

$$F(X_1, X_2) = \Phi_{Id} - \Phi_{(12)} = t_{X_1} t_{X_2} - t_{X_1 X_2} = X_1 X_2 - X_1 X_2 = 0,$$

which of course is true. It corresponds to the Young tableau $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$.

Let now $n = 2$. The fundamental trace identity then becomes

$$\begin{aligned} F(X_1, X_2, X_3) =&\Phi_{id} - \Phi_{(12)} - \Phi_{(13)} - \Phi_{(23)} + \Phi_{(123)} + \Phi_{(132)} \\ =&t_{X_1} t_{X_2} t_{X_3} - t_{X_1 X_2} t_{X_3} - t_{X_1 X_3} t_{X_2} \\ &- t_{X_1} t_{X_2 X_3} + t_{X_1 X_2 X_3} + t_{X_1 X_3 X_2}, \end{aligned}$$

where id denotes the identity permutation. It corresponds to the Young tableau $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$.

We can also look at an example illustrating the converse way of part one of Theorem 2.13. Let $n = 2$ and $m = 4$. Since we are supposed to have at least $n + 1 = 3$ rows, there are two possible Young tableaux:

$$i) \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad ii) \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}$$

Let us now consider tableau $i)$ above. We get that $P_\lambda = \{id, (14)\}$, while $Q_\lambda = \{id, (12), (13), (23), (123), (132)\}$. Thus we get

$$a_\lambda = id + (14)$$

and

$$b_\lambda = id - (12) - (13) - (23) + (123) + (132).$$

Hence the Young symmetrizer is

$$\begin{aligned} c_\lambda = &\, id - (12) - (13) - (23) - (123) - (132) \\ &+ (14) - (142) - (143) - (14)(23) + (1423) + (1432). \end{aligned}$$

Translated to the world of matrices and trace identities, it corresponds to

$$\begin{aligned} &\Phi_{id} - \Phi_{(12)} - \Phi_{(13)} - \Phi_{(23)} - \Phi_{(123)} - \Phi_{(132)} \\ &+ \Phi_{(14)} - \Phi_{(142)} - \Phi_{(143)} - \Phi_{(14)(23)} - \Phi_{(1423)} - \Phi_{(1432)}. \end{aligned}$$

Using symbolic calculation, for instance with [Matlab], this can be shown to be 0. Hence, looking at the Young symmetrizer relative to a Young tableau with at least $n+1$ rows and thereafter translating to the world of matrices, we get a trace identity.

Repeating the process with tableau *ii*), we get that P_λ is just the identity, while Q_λ is the whole of S_4. Thus the Young symmetrizer becomes

$$c_\lambda = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) e_\sigma.$$

Translating to matrices and the functions Φ, and then using for instance [Matlab], we get that this is in fact 0 as well.

Theorem 2.15 gives us what the kernel of π is generated by, that is, what generates all the relations among the invariants. We look at a couple of examples.

First assume $n=2$. The fundamental trace identity is

$$\begin{aligned} F(X_1, X_2, X_3) = {} & t_{X_1} t_{X_2} t_{X_3} - t_{X_1 X_2} t_{X_3} - t_{X_1 X_3} t_{X_2} \\ & - t_{X_2 X_3} t_{X_1} + t_{X_1 X_2 X_3} + t_{X_1 X_3 X_2}. \end{aligned}$$

Let now $X_1 = X_2 = X_3 = X$. Then we get

$$F(X, X, X) = t_X^3 - 3 t_{X^2} t_X + 2 t_{X^3} = 0.$$

This is in fact exactly the same as we get from Cayley–Hamilton after multiplying with X and taking traces. Using two different matrices, X and Y, we get

$$\begin{aligned} F(X, X, Y) &= F(X, Y, X) = F(Y, X, X) \\ &= t_X^2 t_Y + 2 t_{X^2 Y} - 2 t_{XY} t_X - t_{X^2} t_Y = 0. \end{aligned}$$

By swapping X and Y in the expressions for $F(X, X, X)$ and $F(X, X, Y)$, we get all the fundamental trace identities for two 2×2 matrices.

Let now $n = 3$. Then we are looking at S_4 which has 24 elements, making it tedious to work with. But it is not particularly difficult to calculate $F(M_1, M_2, M_3, M_4)$ when $M_1 = M_2 = M_3 = M_4 = X$. We have that S_4 consists of the elements in Table 2.1.

Permutations	**Type**
id	product of 1-cycles
(12), (13), (14), (23), (24), (34)	2-cycles
(12)(34), (13)(24), (14)(23)	product of 2-cycles
(123), (124), (132), (134), (142), (143), (234), (243)	3-cycles
(1234), (1243), (1324), (1342), (1423), (1432)	4-cycles

Table 2.1: Symmetric group on four letters.

The 4-cycles and the 2-cycles have sgn $= -1$. We also have

$$\Phi_\sigma = \begin{cases} t_{X^2}t_X^2 & \text{if } \sigma \text{ is a 2-cycle} \\ t_{X^2}^2 & \text{if } \sigma \text{ is a product of 2-cycles} \\ t_{X^3}t_X & \text{if } \sigma \text{ is a 3-cycle} \\ t_{X^4} & \text{if } \sigma \text{ is a 4-cycle} \\ t_X^4 & \text{if } \sigma = id. \end{cases}$$

Hence we get

$$F(X,X,X,X) = \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma)\Phi_\sigma = t_X^4 - 6t_{X^2}t_X^2 + 3t_{X^2}^2 + 8t_{X^3}t_X - 6t_{X^4} = 0.$$

Again, we see the resemblance to the Cayley–Hamilton theorem. Using the Faddeev–LeVerrier algorithm explained in [FF63], we have that the characteristic polynomial of a 3×3 matrix X applied to the matrix itself, is

$$P_3(X) = X^3 - t_X X^2 - \frac{1}{2}\left(t_{X^2} - t_X^2\right)X + \frac{1}{6}\left(-2t_{X^3} + 3t_X t_{X^2} - t_X^3\right)I.$$

This turns out, as in the $n = 2$ case, to lead to the trace identity above. Multiplying with X and taking traces, gives us

$$t_{X^4} - \frac{4}{3}t_X t_{X^3} - \frac{1}{2}t_{X^2}^2 + t_X^2 t_{X^2} - \frac{1}{6}t_X^4 = 0.$$

Multiplying with -6, we obtain the same formula as $F(X,X,X,X)$.

Let us now assume that we have two different 3×3 matrices X and Y. Then we have

$$\begin{aligned} F(X,X,X,Y) = F(X,X,Y,X) = F(X,Y,X,X) &= F(Y,X,X,X) \\ &= t_X^3 t_Y - 3t_{X^2}t_X t_Y - 3t_X^2 t_{XY} + 3t_{X^2}t_{XY} \\ &+ 2t_{X^3}t_Y + 6t_{X^2Y}t_X - 6t_{X^3Y}. \end{aligned}$$

and

$$\begin{aligned} F(X,X,Y,Y) = F(X,Y,Y,X) = F(X,Y,X,Y) &= F(Y,Y,X,X) \\ &= t_X^2 t_Y^2 - 4t_{XY}t_X t_Y - t_{X^2}t_Y^2 - t_X^2 t_{Y^2} + 2t_{XY}^2 + t_{X^2}t_{Y^2} \\ &+ 4t_{X^2Y}t_Y + 4t_{Y^2X}t_X - 2t_{(XY)^2} - 4t_{X^2Y^2}. \end{aligned}$$

We could of course proceed in this manner, but it would require a lot of calculation without too much gain.

Results in this chapter give us much information about the invariants. We know that it is a polynomial in traces, and that it is generated by traces of $2^n - 1$ matrices or less. We also have generators for the relations between the polynomials, but since it includes the symmetric group, it quickly gets big when n grows. It is therefore not trivial to find the minimal set of generators for the invariants. Even for $n = 3$ this proves to be a difficult task.

CHAPTER 3

2-Dimensional Simple Representations

From now on we will work with representations of algebras. In this chapter we are mainly focusing on 2-dimensional simple representations of noncommutative plane curves and connect them to the trace ring from Chapter 2. The main influence is [JLS04]. In Section 3.2 we give some new results and proofs regarding the trace of monomials, while we in Section 3.3 give an algorithm to find the trace of a polynomial in two variables X and Y, where X and Y are 2×2 matrices. However, we start with a short introduction to representations.

Let k be an algebraically closed field of characteristic 0, and let A be a finitely generated associative k-algebra. We can give a k-vector space M a left A-module structure by a k-algebra homomorphism

$$\phi\colon A \to \mathrm{End}_k(M).$$

By choosing a basis for M, we have that $\mathrm{End}_k(M) \simeq M_n(k)$.

Definition 3.1. A k-algebra homomorphism

$$\phi\colon A \to \mathrm{End}_k(M) \simeq M_n(k)$$

is called an n-dimensional representation of A.

Hence a module structure on a k-vector space is essentially the same as a representation.

Two representations gives isomorphic A-module structures if they differ by an inner automorphism of $\mathrm{End}_k(M)$. Translated from $\mathrm{End}_k(M)$ to $M_n(k)$, we have that two representations are isomorphic if the corresponding matrices are conjugates of each other.

What will be of most interest for us, are the simple representations.

Definition 3.2. A representation is called simple if its corresponding homomorphism is surjective.

The isomorphism classes under the action of $\mathrm{PGl}_n(k)$, the projective linear group, of simple n-dimensional representations of A is denoted by $\mathrm{Simp}_n(A)$.

Example 3.3. Let $S = k\langle x_1, \dots, x_m\rangle$ be the free algebra on m generators. Then we have that $\mathrm{Simp}_1(S) \simeq \mathbb{A}^m$ since every x_i is sent to an element of k.

3.1 Setup and Examples

Our main focus will be 1 and 2-dimensional representations. However, we start by looking at $\mathrm{Simp}_n(A)$, before calculating some examples of 2-dimensional simple representations.

Set $S = k\langle x_1, \dots, x_m\rangle$. A representation

$$\phi\colon S \to M_n(k)$$

of S is then given by m $n \times n$ matrices $\phi(x_1), \dots, \phi(x_m)$. Often it is not of interest just looking at S, but at a quotient of S. Let $A = k\langle x_1, \dots, x_m\rangle/I$, where $I = (f_1, \dots, f_r)$. Of course, if we were to translate S into the world of matrices through ϕ, we would also need to translate the relations from I into the world of matrices. Then, instead of relations between the generators of S, we will get relations between the entries of matrices. To get the idea right, we look at a short example.

Example 3.4. Assume $A = k\langle x, y\rangle/(x^2 - y^3)$, and that $\phi\colon A \to M_2(k)$ is a representation. Let $X = \phi(x)$ and $Y = \phi(y)$. Then we would like to translate the condition $x^2 = y^3$ to $X^2 = Y^3$, which would constitute relations among the entries of the two matrices instead of relation between x and y.

Let now again $A = S/I$ as above. To the representation ϕ, we associate the point $\left\{a^i_{p,q}\right\} \in \mathbb{A}^{mn^2}$, $i = 1, \dots, m$, $p, q = 1, \dots, n$, given by $\phi(x_i) = (a^i_{p,q})$, i.e., a matrix. Putting

$$\Gamma(A) = \Gamma = k[x^i_{p,q}]/(\tilde{f}_1, \dots, \tilde{f}_r),$$

where $\tilde{f}_j = f_j\left((x^1_{p,q}), \dots, (x^m_{p,q})\right)$, we have that $\mathrm{Spec}(\Gamma)$ is equal to the set of k-algebra homomorphism from A to $M_n(k)$, which we denote by $\mathcal{X}^n_A$. That is, $\mathrm{Spec}(\Gamma) = \mathcal{X}^n_A$. Notice that $\mathrm{PGl}_n(k)$ acts on both $\mathcal{X}^n_A$ and Γ.

Let

$$\mathrm{Repr}_n(A) = \mathcal{X}^n_A / \mathrm{PGl}_n(k).$$

The underlying set of $\mathrm{Repr}_n(A)$ is the set of isomorphism classes of n-dimensional representations of A. $\mathrm{Simp}_n(A)$ is then the subset of $\mathrm{Repr}_n(A)$ consisting of isomorphism classes of simple representations.

Let us now concentrate on $S = k\langle x_1, \dots, x_m\rangle$. A representation $\phi\colon S \to M_n(k)$ is then given by m $n \times n$ matrices $\phi(x_1), \dots, \phi(x_m)$, and we can identify $\mathcal{X}^n_A$ with the affine space $\mathbb{A}^{mn^2}$ as before. Now $\Gamma \simeq k[x^i_{p,q}]$, and $\mathrm{PGl}_n(k)$ acts on Γ by conjugation. Procesi [Pro76] proved the conjecture of Artin [Art69, p. 558] that the trace ring $T_{m,n}$ is precisely the subring $\Gamma^{\mathrm{PGl}_n(k)}$ of $k[x^i_{p,q}]$, as established in Chapter 2. See a more detailed version in [JLS04].

Hence, for a k-algebra $A = S/I$, we have that $\mathrm{Spec}(\Gamma(A))$ is a subset of $\mathrm{Spec}(T_{m,n})$ and that $\mathrm{Simp}_n(A)$ is a subset of $\mathrm{Spec}(\Gamma(A))$. Thus $\mathrm{Simp}_n(A)$ will be a subset of $\mathrm{Spec}(T_{m,n})$.

This gives us a tool to calculate with representations, since we have a lot of information about matrices, especially when n is small. Another important tool, mainly when $n = 2$, is the Formanek center of n-central polynomials. The Formanek center is a subset $F_n(S) \subset S$ such that a representation $\phi\colon S \to M_n(k)$ is simple if and only if $\phi(F_n(S)) \neq 0$. Usually the Formanek center is very difficult to compute, but in the case $n = 2$, it is not too hard.

Let now $S = k\langle x, y\rangle$ and let M be a left S-module given by the representation

$$\phi\colon S \to M_2(k).$$

Then M is simple if and only if ϕ is surjective. That is, M is simple if and only if $X = \phi(x)$ and $Y = \phi(y)$ generate $M_2(k)$ as a k-algebra. The following result gives a way to find out when this is the case.

Proposition 3.5. *X and Y generates $M_2(k)$ as a k-algebra if and only if* $\det([X, Y]) \neq 0$.

Proof. The first statement is equivalent to the fact that the set $\{I, X, Y, XY\}$ is linearly independent. Letting

$$XY = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{3.1}$$

and letting A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ x_{11} & x_{12} & x_{21} & x_{22} \\ y_{11} & y_{12} & y_{21} & y_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \end{bmatrix},$$

the statement is equivalent to $\det(A) \neq 0$. But a calculation shows that $\det(A) = \det([X, Y])$, proving the proposition. ∎

For future calculations, we need a handy lemma about 2×2 matrices. This will be used a lot when doing concrete examples.

Lemma 3.6. *For any 2×2 matrices X and Y, we have*

$$\begin{aligned} i)\ & X^2 = t_X X - d_X I \\ ii)\ & X^3 = (t_X^2 - d_X)X - t_X d_X I \\ iii)\ & d_{X+Y} - d_X - d_Y = t_X t_Y - t_{XY} \\ iv)\ & YX = -XY + t_X Y + t_Y X + \Delta t_{XY} I, \end{aligned}$$

where $\Delta t_{XY} = t_{XY} - t_X t_Y$.

Denoting $\det(M)$ by d_M and $\operatorname{tr}(M)$ by t_M for a matrix M, calculations yield

$$\begin{aligned} d_{[X,Y]} &= -\frac{1}{4}\left((2t_{XY} - t_X t_Y)^2 - (t_X^2 - 4d_X)(t_Y^2 - 4d_Y)\right) \\ &= -\left(t_{XY}(t_{XY} - t_X t_Y) + t_{X^2} d_Y + t_{Y^2} d_X\right). \end{aligned}$$

This is the Formanek center.

M being simple is equivalent to the Formanek center being nonzero. When we have two 2×2 matrices, X and Y, we know that the trace ring is

$$T_{2,2} = k[t_X, t_Y, t_{XY}, d_X, d_Y].$$

Hence $\mathrm{Simp}_2(S)$ will be an open subset of the affine 5-space $\mathrm{Spec}(T_{2,2}) \simeq \mathbb{A}^5$.

When giving a description of the 2-dimensional simple representations of some noncommutative plane curves, it is important to remember Lemma 3.6.

Let $A = k\langle x, y\rangle/(f)$, and put $\Gamma = k[x^i_{p,q}]/(\tilde{f})$ as before. For a 2-dimensional representation $\phi\colon k\langle x, y\rangle \to M_2(\Gamma)$, we set $X = \phi(x)$ and $Y = \phi(y)$. Then we have

$$\phi(f) = c_1 XY + c_2 X + c_3 Y + c_4 I, \;\; c_i \in \Gamma,$$

since we can express YX by I, X, Y and XY by Lemma 3.6. Assume that ϕ corresponds to a closed point $\gamma \in \mathrm{Spec}(\Gamma)$. For ϕ to be a simple representation, we need $c_i(\gamma) = 0$ for $i = 1, \ldots, 4$ and $d_{[X,Y]}(\gamma) \neq 0$. We are now ready to see some examples.

Example 3.7. Let $\delta \in k$ be arbitrary, and let $f_\delta = x^2 + y^2 - 1 + \delta[x, y]$. We are going to consider $A = k\langle x, y\rangle/(f_\delta)$, making use of the identities established in Lemma 3.6 several times. We get

$$\begin{aligned} \phi(x^2 + y^2 - 1 + \delta[x, y]) &= X^2 + Y^2 - I + \delta XY - \delta YX \\ &= t_X X - d_X I + t_Y Y - d_Y I - I + \delta XY \\ &\quad\; - \delta(-XY + t_X Y + t_Y X + \Delta t_{XY} I) \\ &= I(-\delta\Delta t_{XY} - d_X - d_Y - 1) + X(t_X - \delta t_Y) \\ &\quad\; + Y(t_Y - \delta t_X) + XY(2\delta) \end{aligned}$$

For ϕ to be simple, we need that the coefficients of I, X, Y and XY are zero. Hence $\mathrm{Simp}_2(A_\delta) = \emptyset$ if $\delta \neq 0$. Putting $\delta = 0$, we are left with

$$t_X = t_Y = d_X + d_Y + 1 = 0,$$

so $\mathrm{Spec}\left(\Gamma\left(A_{\delta=0}\right)\right) = V(t_X, t_Y, d_X + d_Y + 1) \subset \mathbb{A}^5$. Hence $\mathrm{Simp}_2(A_{\delta=0})$ is the open subscheme of $\mathrm{Spec}\left(\Gamma\left(A_{\delta=0}\right)\right)$ given by the nonvanishing of the Formanek center. Since both X and Y are traceless, Cayley–Hamilton gives us $X^2 = -d_X I$ and $Y^2 = -d_Y I$. Thus the Formanek center becomes

$$\begin{aligned} -t_{XY}^2 - t_{X^2} d_Y - t_{Y^2} d_X &= -t_{XY}^2 + 2d_X d_Y + 2d_Y d_X \\ &= -t_{XY}^2 + 4d_X d_Y \\ &= -t_{XY}^2 - (2d_X + 1)^2 + 1. \end{aligned}$$

Hence we have

$$\mathrm{Simp}_2(A_{\delta=0}) = V(-t_{XY}^2 - (2d_X + 1)^2 + 1)^C \subset \mathrm{Spec}(\Gamma(A_{\delta=0})).$$

In this example we see that the trace ring is generated by t_{XY} and d_X.

We will continue working with this example in the next chapters.

Example 3.8. Let now $f_{\delta x} = x^2 + y^2 - 1 + \delta x^2 y - \delta xyx = x^2 + y^2 - 1 + \delta x[x, y]$, and consider the same setting as above. Then we have

$$\begin{aligned}\phi(f_{\delta x}) &= X^2 + Y^2 - I + \delta X^2 Y - \delta XYX \\ &= t_X X - d_X I + t_Y Y - d_Y I - I + \delta(t_X X - d_X I)Y \\ &\quad - \delta X(-XY + t_X Y + t_Y X + \Delta t_{XY} I) \\ &= I(-d_X - d_Y - 1 + \delta t_Y d_X) + X(t_X - \delta \Delta t_{XY} - \delta t_X t_Y) \\ &\quad + Y(t_Y - 2\delta d_X) + XY(\delta t_X).\end{aligned}$$

Assume that $\delta \neq 0$. Then $t_X = 0$, and we have the equations

$$d_X + d_Y + 1 - \delta t_Y d_X = \delta \Delta t_{XY} = t_Y - 2\delta d_X = 0.$$

Thus $t_{XY} = 0$, and the trace ring is generated by d_X, while the Formanek center becomes

$$-(t_{X^2} d_Y + t_{Y^2} d_X) = 4 d_X \left(\delta^2 d_X^2 - d_X - 1\right).$$

Hence we have that

$$\operatorname{Spec}(\Gamma(A_\delta)) = V\left(t_X, t_{XY}, d_X + d_Y + 1 - \delta t_Y d_X, t_Y - 2\delta d_X\right) \subset \mathbb{A}^5,$$

and

$$\operatorname{Simp}_2(A_\delta) = V\left(4 d_X \left(\delta^2 d_X^2 - d_X - 1\right)\right)^C \subset \operatorname{Spec}(\Gamma(A_\delta)).$$

If we were to assume that also $t_Y = 0$, then for ϕ to correspond to a simple module, we would need

$$d_X + d_Y + 1 = \delta \Delta t_{XY} = 2\delta d_X = 0.$$

Thus $d_X = 0$ and $d_Y = -1$, so the trace ring would just be k.

In this scenario however, the Formanek center becomes $-t_{X^2}$. But we have

$$X^2 = t_X X - d_X I = 0,$$

so $t_{X^2} = 0$. Hence $\delta \neq 0$ while $t_X = t_Y = 0$ means that ϕ can not correspond to a simple module.

The $\delta = 0$ part was covered in the previous example.

Example 3.9. Let $f_{\delta y} = x^2 + y^2 - 1 + \delta yxy - \delta y^2 x = x^2 + y^2 - 1 + \delta y[x, y]$, and consider the same setting as in the above examples. We get

$$\begin{aligned}\phi(f_{\delta y}) &= X^2 + Y^2 - I + \delta YXY - \delta Y^2 X \\ &= t_X X - d_X I + t_Y Y - d_Y I - I \\ &\quad + \delta(-XY + t_X Y + t_Y X + \Delta t_{XY} I)Y - \delta(t_Y Y - d_Y I)X \\ &= I\left(-d_X - d_Y - 1 - \delta t_X d_Y - \delta t_Y \Delta t_{XY}\right) + X\left(t_X + 2\delta d_Y - \delta t_Y^2\right) \\ &\quad + Y\left(t_Y + \delta \Delta t_{XY}\right) + XY\left(\delta t_Y\right).\end{aligned}$$

Since $\delta = 0$ is already covered, we assume that $\delta \neq 0$. Then $t_Y = 0$ from the coefficient of XY, which again means that $t_{XY} = 0$ from the condition on Y. Then we are left with the equations

$$t_X + 2\delta d_Y = -d_X - d_Y - 1 - \delta t_X d_Y = 0.$$

The trace ring is thus generated by d_Y.

Since $t_Y = 0$, we have that $Y^2 = -d_Y I$, so the Formanek center becomes

$$\begin{aligned} -t_{X^2} d_Y - t_{Y^2} d_X &= d_Y(4d_X - t_X^2) \\ &= 4d_Y \left(\delta^2 d_Y^2 - d_Y - 1\right). \end{aligned}$$

Hence $\text{Spec}(A_{\delta \neq 0}) = V(t_Y, t_{XY}, t_X + 2\delta d_Y, -d_X - d_Y - 1 - \delta t_X d_Y)$ and $\text{Simp}_2(A_{\delta \neq 0})$ is the open subscheme given by the nonvanishing of

$$d_Y(4d_X - t_X^2) = 4d_Y \left(\delta^2 d_Y^2 - d_Y - 1\right).$$

Remark 3.10. If we were to assume that also $t_X = 0$, the Formanek center would be identically equal to 0, and there would not be any simple representations.

These three examples illustrate how different the noncommutative world is in comparison to the commutative world. In these examples we would for instance not need to deal with separate cases for $\delta = 0$ and $\delta \neq 0$, since the expression $\delta[x, y]$ would be 0.

3.2 Finding Traces

Given an n-dimensional simple representation of an algebra on the form $k\langle x_1, \ldots, x_m \rangle / I$, we would like to find a way to find the trace ring without an awful amount of calculations, but this turns out to be quite difficult. We start with $m = 1$ and $n = 2$, giving first a recursive formula for the trace of one 2×2 matrix, before finding a formula over the reals.

Assume that X is a 2×2 matrix, and let $u = \text{tr}(X)$ and $v = \text{tr}(X^2)$. We know that these generate the trace ring when considering only one matrix. We are interested in a formula for the trace of X^n.

Theorem 3.11. *For $n \geq 2$, we have*

$$t_{X^n} = u t_{X^{n-1}} + \frac{1}{2} t_{X^{n-2}} \left(v - u^2\right)$$

Proof. For $n = 2$ and $n = 3$ this follows from Cayley–Hamilton, see Lemma 3.6. Suppose now that $n > 3$. Then we have

$$\begin{aligned} X^n = X^{n-3} X^3 &= X^{n-3} \left(t_X X^2 - \frac{1}{2} \left(t_X^2 - t_{X^2}\right) X \right) \\ &= X^{n-3} \left(u X^2 - \frac{1}{2} \left(u^2 - v\right) X \right) \\ &= u X^{n-1} + \frac{1}{2} \left(v - u^2\right) X^{n-2}. \end{aligned}$$

Taking traces on both sides, we obtain

$$t_{X^n} = ut_{X^{n-1}} + \frac{1}{2}t_{X^{n-2}}\left(v - u^2\right)$$

as wanted. ■

From now on, we let X be a real 2×2 matrix. The above yields the following characteristic equation of the difference equation:

$$r^2 - ur - \frac{1}{2}\left(v - u^2\right) = 0.$$

The solutions of the characteristic equation are given by

$$r = \frac{u \pm \sqrt{2v - u^2}}{2}.$$

The solution of the difference equation depends on the sign of $2v - u^2$. When $2v - u^2 > 0$, we write the solution on the form

$$t_{X^n} = A\left(\frac{u + \sqrt{2v - u^2}}{2}\right)^n + B\left(\frac{u - \sqrt{2v - u^2}}{2}\right)^n.$$

Through the help of [MATLAB], we obtain that

$$A = B = 1.$$

Thus, when $2v - u^2 > 0$, we have the solution

$$t_{X^n} = \left(\frac{u + \sqrt{2v - u^2}}{2}\right)^n + \left(\frac{u - \sqrt{2v - u^2}}{2}\right)^n.$$

If we assume that $2v = u^2$, we have the general formula

$$t_{X^n} = (A + Bn)\left(\frac{u}{2}\right)^n.$$

Using [MATLAB] to find A and B, we get

$$t_{X^n} = \left(\frac{4(u^2 - v)}{u^2} + \frac{2(2v - u^2)}{u^2}n\right)\left(\frac{u}{2}\right)^n = 2\left(\frac{u}{2}\right)^n,$$

where we have used that $2v = u^2$.

Putting $2v - u^2 = 0$ in the expression we got when assuming $2v - u^2 > 0$, we see that these two coincide. Hence we do not need to split in two separate cases.

Assume now that $2v - u^2 < 0$. Then the general formula for solving the difference equation is

$$t_{X^n} = Ar^n \cos(\phi n) + Br^n \sin(\phi n),$$

where $r = \sqrt{\frac{1}{2}(u - v^2)}$ and $\phi = \arccos\left(\frac{u}{2r}\right)$. [MATLAB] then gives us that $A = 2$ and $B = 0$. Hence we end up with the formula

$$t_{X^n} = 2r^n \cos(\phi n).$$

We summarize it all:

Theorem 3.12. *Let $X \in M_2(\mathbb{R})$, and let $u = \operatorname{tr}(X)$, $v = \operatorname{tr}(X^2)$. Then, for $n \geq 1$, we have*

$$t_{X^n} = \begin{cases} \left(\frac{u+\sqrt{2v-u^2}}{2}\right)^n + \left(\frac{u-\sqrt{2v-u^2}}{2}\right)^n & \text{if } 2v - u^2 \geq 0 \\ 2r^n \cos(\phi n) & \text{if } 2v - u^2 < 0, \end{cases}$$

where $r = \sqrt{\frac{1}{2}(u - v^2)}$ and $\phi = \arccos\left(\frac{u}{2r}\right)$.

While this case is not too difficult, it becomes considerably more complicated when we have two 2×2 matrices.

Let us first go back to the more general setting, to see what we are interested in, and why.

Let $f \in k\langle x, y\rangle$ be the generator of the ideal I, and let $A = k\langle x, y\rangle / I$. Consider a representation $\phi\colon A \to M_2(\Gamma)$ with $X = \phi(X)$, $Y = \phi(Y)$ as previously, and let $F = \phi(f)$. We are interested in the trace ring corresponding to A, so we are looking for which relations the ideal I induces in the matrix ring. Since $t_{FX} = t_{XF}$, $t_{YF} = t_{FY}$ and Cayley–Hamilton reduces X^2 and Y^2, we are left with being interested in $t_F, t_{XF}, t_{YF}, t_{XYF}$ and t_{YXF}, which all should be zero.

If we for a monomial M could find a relation between the trace of M and M multiplied with X or Y, we could easily deduce the traces above from t_F. We have, unfortunately, not found a way to completely determine it. Instead, we will show some results, before including an algorithm to calculate the trace of a polynomial.

We start by finding recursive formulas for matrices on the form X^nY, XY^n and X^nY^m, where we assume that n and m is greater than 1. We are going to need Lemma 3.6, namely

$$XY = -YX + t_XY + t_YX + \Delta t_{XY}I.$$

Proposition 3.13. *For two 2×2 matrices X and Y, and $n \geq 2$, we have*

$$t_{X^nY} = \frac{1}{2}\left(t_Xt_{X^{n-1}Y} + t_{X^n}t_Y + \Delta t_{XY}t_{X^{n-1}}\right).$$

Proof. We have

$$\begin{aligned} X^nY &= X^{n-1}XY \\ &= X^{n-1}\left(-YX + t_XY + t_YX + \Delta t_{XY}I\right) \\ &= -X^{n-1}YX + t_XX^{n-1}Y + t_YX^n + \Delta t_{XY}X^{n-1}. \end{aligned}$$

Taking traces on both sides, and using that $t_{X^{n-1}YX} = t_{X^nY}$, gives the result. ∎

For XY^n a similar calculation, after rewriting

$$XY^n = (XY)Y^{n-1} = \left(-YX + t_XY + t_YX + \Delta t_{XY}I\right)Y^{n-1},$$

yields the following.

Proposition 3.14. *For two* 2×2 *matrices* X *and* Y*, and* $n \geq 2$*, we have*

$$t_{XY^n} = \frac{1}{2}\left(t_X t_{Y^n} + t_Y t_{XY^{n-1}} + \Delta t_{XY} t_{Y^{n-1}}\right).$$

We can now find a recursive formula for $t_{X^mY^n}$.

Proposition 3.15. *For two* 2×2 *matrices* X *and* Y*, and* $m, n \geq 2$*, we have*

$$\begin{aligned} t_{X^mY^n} &= t_X t_{X^{m-1}Y^n} + \frac{1}{2}\left(t_{X^2} - t_X^2\right) t_{X^{m-2}Y^n} \\ &= t_Y t_{X^mY^{n-1}} + \frac{1}{2}\left(t_{Y^2} - t_Y^2\right) t_{X^mY^{n-2}}. \end{aligned}$$

Proof. For $m \geq 2$, we have

$$\begin{aligned} X^mY^n &= X^{m-2}(X^2)Y^n \\ &= X^{m-2}\left(t_X X + \frac{1}{2}\left(t_{X^2} - t_X^2\right) I\right) Y^n \\ &= t_X X^{m-1}Y^n + \frac{1}{2}\left(t_{X^2} - t_X^2\right) X^{m-2}Y^n. \end{aligned}$$

Taking traces on both sides gives the wanted relation. Similarly, for $n \geq 2$

$$\begin{aligned} X^mY^n &= X^m(Y^2)Y^{n-2} \\ &= X^m\left(t_Y Y + \frac{1}{2}\left(t_{Y^2} - t_Y^2\right) I\right) Y^{n-2} \\ &= t_Y X^mY^{n-1} + \frac{1}{2}\left(t_{Y^2} - t_Y^2\right) X^mY^{n-2}. \end{aligned}$$

Taking trace on both sides gives the equality. ■

The cases $m = 0$, $m = 1$, $n = 0$, and $n = 1$ are solved previously, giving us a system of recursive formulas to find $t_{X^mY^n}$.

Let M now be a monomial in X and Y. In the following propositions, we adopt the notation $M\frac{1}{X}$ to mean that we take away the far right X in the expression of M, and similarly $\frac{1}{X}M$ means that we take away the X to the left in the expression of M. The notation is only meant to be used when removing one of the end variables, either that to the far left or the one to the far right. We include an example to make it clearer.

Example 3.16. Let $M = XYXY$. Then we have

$$\frac{1}{X}M = YXY$$

and

$$M\frac{1}{Y} = XYX,$$

while we would not write $\frac{1}{Y}M$ or $M\frac{1}{X}$.

In the following propositions we assume $n_1, m_1, n_i, m_i > 0$.

Proposition 3.17. *Let $M = X^{n_1}Y^{m_1} \cdots X^{n_i}Y^{m_i}$. Then we have*

(*i*)

$$\begin{aligned} t_{MX} &= -t_{M\frac{1}{Y}XY} + t_X t_M + t_Y t_{M\frac{1}{Y}X} + \Delta t_{XY} t_{M\frac{1}{Y}} \\ &= t_X t_M - \frac{1}{2}\left(t_X^2 - t_{X^2}\right) t_{\frac{1}{X}M}. \end{aligned}$$

(*ii*)

$$\begin{aligned} t_{MY} &= t_Y t_M - \frac{1}{2}\left(t_Y^2 - t_{Y^2}\right) t_{M\frac{1}{Y}} \\ &= -t_{XY\frac{1}{X}M} + t_X t_{Y\frac{1}{X}M} + t_Y t_M + \Delta t_{XY} t_{\frac{1}{X}M}. \end{aligned}$$

Proof.

(*i*) We multiply with X from left, i.e., we calculate XM.

$$\begin{aligned} XM &= XX^{n_1}Y^{m_1} \cdots X^{n_i}Y^{m_i} \\ &= X^2X^{n_1-1}Y^{m_1} \cdots X^{n_i}Y^{m_i} \\ &= \left(t_X X - \frac{1}{2}\left(t_X^2 - t_{X^2}\right)\right) X^{n_1-1}Y^{m_1} \cdots X^{n_i}Y^{m_i} \\ &= t_X M - \frac{1}{2}\left(t_X^2 - t_{X^2}\right) \frac{1}{X} M. \end{aligned}$$

Taking traces on both sides and remembering that $t_{XM} = t_{MX}$, gives the desired result. For the other expression we calculate MX.

$$\begin{aligned} MX &= X^{n_1}Y^{m_1} \cdots X^{n_i}Y^{m_i-1}(YX) \\ &= X^{n_1}Y^{m_1} \cdots X^{n_i}Y^{m_i-1}(-XY + t_Y X + t_X Y + \Delta t_{XY} I) \\ &= -M\frac{1}{Y}XY + t_X M + t_Y M\frac{1}{Y}X + \Delta t_{XY} M\frac{1}{Y}. \end{aligned}$$

Taking traces then gives the result.

(*ii*) This is proved in similar fashion, calculating YM and MY and using the expressions for Y^2 and YX from Lemma 3.6. ∎

Proposition 3.18. *Let $M = X^{n_1}Y^{m_1} \cdots Y^{m_{i-1}}X^{n_i}$. Then we have*

(*i*)

$$t_{MX} = t_X t_M - \frac{1}{2}\left(t_X^2 - t_{X^2}\right) t_{\frac{1}{X}M}.$$

(*ii*)

$$\begin{aligned} t_{MY} &= -t_{M\frac{1}{X}YX} + t_X t_{M\frac{1}{X}Y} + t_Y t_M + \Delta t_{XY} t_{M\frac{1}{X}} \\ &= -t_{XY\frac{1}{X}M} + t_X t_{Y\frac{1}{X}M} + t_Y t_M + \Delta t_{XY} t_{\frac{1}{X}M}. \end{aligned}$$

Proof.

(i) This is proved like i) in Proposition 3.17, i.e., by multiplying with X from either one of the sides.

(ii) We have

$$\begin{aligned} MY &= X^{n_1}Y^{m_1}\cdots Y^{m_{i-1}}X^{n_i}Y \\ &= M\frac{1}{X}(XY) \\ &= M\frac{1}{X}(-YX + t_X Y + t_Y X + \Delta t_{XY} I) \\ &= -M\frac{1}{X}YX + t_X M\frac{1}{X}Y + t_Y M + \Delta t_{XY} M\frac{1}{X}. \end{aligned}$$

Taking traces on both sides gives the result. A similar calculation done with YM yields the other equality. ■

From Proposition 3.18 we get the equality

$$t_{M\frac{1}{X}YX} + t_X t_{M\frac{1}{X}Y} = -t_{XY\frac{1}{X}M} + t_X t_{Y\frac{1}{X}M}$$

for a monomial M starting and ending with X.

It turns out that it is more complicated to handle a monomial which starts and ends with the same variable than a monomial starting and ending with different variables when investigating what happens when multiplying with a variable. In the last case we get a neat formula, while for the first case it is a bit more messy.

For instance, if we know the trace of the two monomials $X^{n_1}Y^{m_1}\cdots X$ and $X^{n_1}Y^{m_1}\cdots X^2$, there is no problem finding the trace of $X^{n_1}Y^{m_1}\cdots X^{n_i}$, where $n_i > 2$. The problem starts when multiplying by Y.

3.3 Algorithm

There is, however, a not very complicated algorithm to break down the polynomial to pieces for which the traces are known, assuming the generators of the trace ring are given.

We will first give a description of the algorithm, before giving a step-by-step guide of it. We end the chapter with a flow diagram of the algorithm.

The input to the algorithm is a polynomial, call it P. We will need three variables or positions, which will be updated throughout the algorithm. We denote this triple by $(-,-,-)$, and we will call each $-$ for a storage place. So the input is $(-,P,-)$ for a polynomial P. The first place will be used for the expression we are working with, while the last it the output place, where we eventually will have a polynomial we can take the trace of.

The first step of the algorithm is to take out the first monomial of the polynomial. Then we check if it is on the form x, y, xy, yx or just a constant. If so, we already know its trace, and we store it in the third storage place. In the second storage place, we subtract the monomial from the polynomial we started with and go to step 1. If the monomial is not on a form for which we know know the trace, we try to break it down.

The first step then, is to make sure that the first and last variable of the monomial is different through a cyclic permutation. Since the trace is invariant under cyclic permutation, this is safe. When that is done, we use Cayley–Hamilton on the powers of x's and y's.

If Cayley–Hamilton changes the monomial, we save the new expression of the monomial together with the original polynomial subtracted the monomial, and go back to start with that as the input polynomial.

If Cayley–Hamilton does not change the monomial, we repeat it looking for powers of xy. Again, we either proceed or store the information and start from the top.

When none of the steps does anything to any of the monomials of the new polynomial, we take the trace of the polynomial in the last storage place. This can be done, since we have broken down the polynomial to pieces for which we know the trace.

Written step by step, in short, we have the following algorithm:

Algorithm 3.19. The following algorithm calculates the trace of an input polynomial P, using three variables of storage, denoted $(-,-,-)$. So the input is $(-,P,-)$, the output is the last dash, while the first is an intermediate variable.

1: Take the first monomial, M, out of the polynomial, and save $(M, P \setminus M, -)$.

2a: Check the length of M. If it is less than three, and M is not on the form x^2 or y^2, go to 2b. Else proceed to 3a.

2b: Save $(-, P \setminus M, * + M)$ and go to step 1.

3a: Check if the first variable is equal to the last variable of M. If so, go to 3b. If not, proceed to 4a.

3b: Make a cyclic permutation until the first and last variable are equal.

4a: Go through the polynomial and check if there is any power of 2 of x and/or y. If so, go to step 4b. Else proceed to 5a.

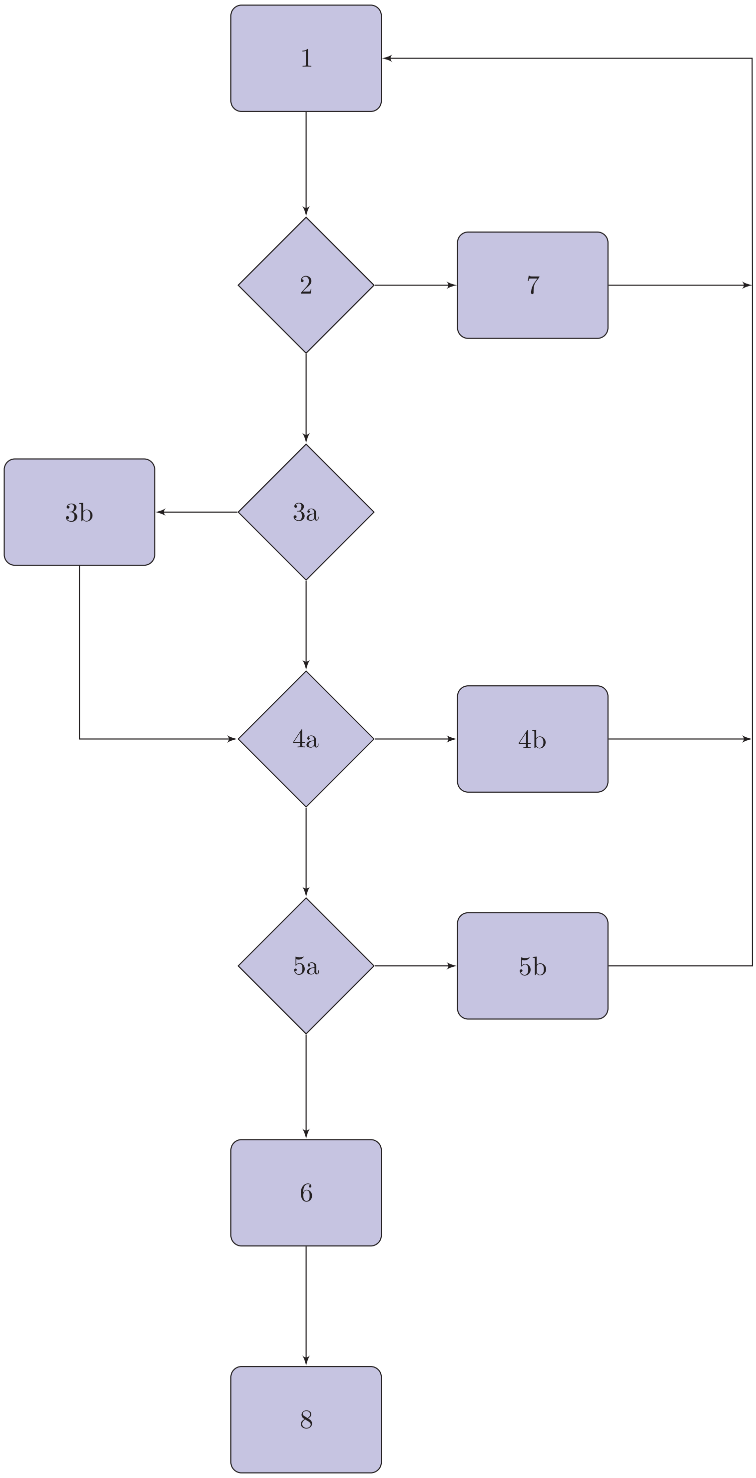
4b: Use Cayley–Hamilton on the powers found in 4a. Store as $(-, C\text{-}H(M) + P \setminus M, *)$ and go to step 1.

5a: Repeat 4a, but look for powers of xy instead. Go to 5b if there is any, and 6 else.

5b: Do 4b to the powers of xy.

6: Take the trace of the last storage place.

We have the following flow diagram to illustrate the algorithm.



CHAPTER 4

Dimension of Extensions

The (commutative version of the) Jacobian matrix is well known and frequently used. Trying to generalize it in the noncommutative world is not straightforward. Since multiplication is no longer commutative, we have to concentrate on either left or right multiplication. Here we choose to work with left modules, ideals, etc. The goal is to use the new Jacobian matrix to find a projective resolution, so we can calculate

$$\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)),$$

where $A = k\langle x_1, \ldots, x_m \rangle / I$, I is a two-sided ideal, and the p's are points in $\mathbb{A}^m$. By $k(p)$ we mean an A-module with underlying set k and module operation defined by $x_i \to a_i$.

We will look at 1-dimensional representations, derivations and decompositions of functions to find the resolution as in [JLS04] However, we start with the few things we need to know about resolutions.

4.1 Constructing a Truncated Resolution

Definition 4.1. A projective resolution of an A-module M is an exact sequence

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

where each P_i is projective.

Lemma 4.2 (See 2.2.1 in [Wei94])**.** *An A-module is free if and only if it is a direct summand of a free A-module.*

This means that an exact sequence on the form

$$A^r \xrightarrow{f} A^m \xrightarrow{g} A \xrightarrow{h} k(p) \to 0,$$

where $k(p)$ is an A-module, is a so-called truncated projective resolution. This is exactly the kind of sequence we are going to construct, finding functions f, g and h that make the sequence exact.

Consider now a sequence as in Definition 4.1. Applying the functor $\operatorname{Hom}_A(-, N)$ to the sequence, we obtain a chain complex

$$\operatorname{Hom}_A(P_0, N) \xrightarrow{d^0} \operatorname{Hom}_A(P_1, N) \xrightarrow{d^1} \operatorname{Hom}_A(P_2, N) \xrightarrow{d^2} \ldots. \qquad (4.1)$$

We are now ready to define Ext.

Definition 4.3. The Ext functor of a projective resolutions is the cohomology of (4.1), i.e., we have

$$\operatorname{Ext}^i_A(M,N)_P = \ker(d^i)/\operatorname{im}(d^{i-1}),\ i \geq 1.$$

We are now going to find functions such that

$$A^r \xrightarrow{f} A^m \xrightarrow{g} A \xrightarrow{h} k(p) \to 0,$$

becomes exact.

Let $S = k\langle x_1, \dots, x_m \rangle$ be the free algebra on m noncommuting variables. Remember that we have $\operatorname{Simp}_1(S) \simeq \mathbb{A}^m$. Hence when we say that a point corresponds to a representation, we use this isomorphism as the correspondence.

Let f be a polynomial of S and let $\phi_p \colon S \to k(p)$ be a 1-dimensional representation corresponding to a point $p \in \operatorname{Simp}_1(S)$ such that $\phi_p(f) = f(p) = 0$. Then the representation is given by $\phi_p(x_i) = a_i$ for $a_i \in k$, $i = 1, \dots, m$, i.e., $p = (a_1, \dots, a_m) \in \mathbb{A}^m$.

Definition 4.4. Let $f \in S$ and let ϕ_p be the 1-dimensional representation corresponding to a point $p = (a_1, \dots, a_m) \in \mathbb{A}^m$. A left decomposition of f with respect to $k(p)$ is an equality

$$f = \sum_{k=1}^{m} f_{k,p}(x_k - a_k),$$

where $f_{k,p} \in S$.

Theorem 4.5. *For any $f \in S$ and any $\phi_p \colon S \to k(p)$ such that $\phi_p(f) = f(p) = 0$, there exists a unique left decomposition of f with respect to $k(p)$.*

Proof. For any monomial m in f of positive degree, we can write

$$m = m'x_i = m'(x_i - a_i) + a_i m',$$

where m' is of less degree than m and x_i is the rightmost variable in m. By, if necessary, repeating the procedure with the $a_i m'$ term, we can then inductively write

$$f(x) = f(p) + \sum_{k=1}^{m} f_{k,p}(x_k - a_k) = \sum_{k=1}^{m} f_{k,p}(x_k - a_k),$$

since $f(p) = 0$.

The set $\{x_k - a_k\}_{k=1,\dots,m}$ is a free generating set for $k\langle x_1, \dots, x_m \rangle$, so the decomposition is unique. ∎

The uniqueness of the left decomposition is important, and we can in fact find expressions for the $f_{k,p}$'s. They are the noncommutative left partial derivatives of f with respect to a 1-dimensional representation $k(p)$, i.e., the linear form $D_i(-;p)$ defined on S such that $D_i(a;p) = 0$ for all $a \in k$, $D_i(x_j;p) = \delta_{ij}$ and

$$D_i(fg;p) = fD_i(g;p) + D_i(f;p)g(p)$$

for a product fg.

Example 4.6. Let $f(x,y) = x^3 + xy^2 - x$. Then we have

$$D_x(f;(a,b)) = x^2 + ax + a^2 + b^2 - 1$$

and

$$D_y(f;(a,b)) = xb + xy$$

This of course differs from the commutative partial derivatives of analysis, which would have resulted in

$$D_x(f;(a,b)) = 3a^2 + b^2 - 1$$

and

$$D_y(f;(a,b)) = 2ab.$$

With this noncommutative derivation, we see that the end result is in A, i.e., we can still have variables left after evaluating in a point.

Definition 4.7. The element $D_k(f;p)$ is called the noncommutative left k-th partial derivative of f with respect to the 1-dimensional representation $k(p)$.

Theorem 4.8. *Let $f \in S$ and let ϕ_p be a 1-dimensional representation corresponding to a point $p = (a_1, \ldots, a_m) \in \mathbb{A}^m$. Then the left composition of f with respect to $k(p)$ is given by*

$$f = \sum_{k=1}^{m} D_k(f;p)(x_k - a_k).$$

Proof. The crucial property is that $D_i(fg;p) = fD_i(g;p) + D_i(f;p)g(p)$. For a product $f_{k,p}(x_k - a_k)$, we thus have

$$D_{x_i}(f_{k,p}(x_k - a_k);p) = D_{x_i}(f_{k,p};p)(a_k - a_k) + f_{k,p}D_{x_i}(x_k - a_k).$$

Using Theorem 4.5, we can write $f = \sum_{k=1}^{m} f_{k,p}(x_k - a_k)$. Applying D_{x_i} on both sides, we obtain

$$\begin{aligned} D_{x_i}(f;p) &= D_{x_i}\left(\sum_{k=1}^{m} f_{k,p}(x_k - a_k);p\right) \\ &= \sum_{k=1}^{m} \left(D_{x_i}(f_{k,p};p)(a_k - a_k)\ \ f_{k,p}D_{x_i}(x_k - a_k;p)\right) \\ &= \sum_{k=1}^{m} f_{k,p}D_{x_i}(x_k - a_k;p) \\ &= f_{i,p}. \end{aligned}$$

Hence $f_{i,p} = D_{x_i}(f;p)$ as wanted. ∎

To construct the Jacobi matrix, we consider a two-sided ideal I of S. Let $f_1, \dots, f_r$ be a set of generators for I and put $A = S/I$. Also, let ϕ_p be a representation of S corresponding to a point $p \in \mathrm{Simp}_1(S/I)$.

By $J_i(I; f_1, \dots, f_r; p)$ we denote the left ideal of A generated by the images of the i-th partial derivatives of the f_j's.

It would of course be troublesome if the generators of I should effect the Jacobi matrix. Fortunately, this is not the case, as the following lemma shows.

Proposition 4.9. *The ideal $J_i(I; f_1, \dots, f_r; p)$ is independent of the choice of generators for the ideal I.*

Proof. Let f be an arbitrary element of I. Since I is a two-sided ideal, f can be written on the form

$$f = \sum_{j,k} \alpha_{j,k} f_j \beta_{j,k},$$

where $\alpha_{j,k}, \beta_{j,k} \in S$.

Now we have

$$\begin{aligned}
D_i(f;p) = D_i &\left(\sum_{j,k} \alpha_{j,k} f_j \beta_{j,k}; p \right) \\
&= \sum_{j,k} \big(D_i(\alpha_{j,k};p)\phi_p(f_j\beta_{j,k}) + \alpha_{j,k} D_i(f_j;p)\phi_p(\beta_{j,k}) + \alpha_{j,k} f_j D_i(\beta_{j,k};p) \big) \\
&= \sum_{j,k} \big(\alpha_{j,k} D_i(f_j;p)\phi_p(\beta_{j,k}) + \alpha_{j,k} f_j D_i(\beta_{j,k};p) \big),
\end{aligned}$$

using that $\phi_p\left(f_j \beta_{j,k}\right) = 0$. Thus $D_i(f;p) \in J_i(I; f_1, \dots, f_r; p) + I$. ■

Since $J_i(I; f_1, \dots, f_r; p)$ is independent of the choice of generators, we will denote it by $J_i(I;p)$.

Definition 4.10. The ideal $J_i(I;p)$ is called the i-th noncommutative Jacobi ideal of I with respect to $k(p)$. The matrix $J(I;p) = (J_i(f_j;p))_{i,j}$ is called the noncommutative Jacobi matrix of the presentation $I = (f_1, \dots, f_r)$.

We are now ready to prove the truncated resolution of A.

Theorem 4.11. *The sequence*

$$A^r \xrightarrow{J(I;p)} A^m \xrightarrow{(\bar{x}-\bar{a})} A \xrightarrow{\phi_p} k(p) \to 0$$

of left A-modules is exact, where the two first maps are multiplication with the matrix $J(I;p)$ and the column vector $\bar{x} - \bar{a}$ from right, and $\bar{a} = (a_1, \dots, a_m)$.

Proof. First we prove exactness at A^m.

$\mathbf{im(J(I;p))} \subseteq \mathbf{ker(\bar{x}-\bar{a})}$:
It suffices to show that the composition $(\bar{x}-\bar{a}) \circ J(I;p)$ is the zero map. Let $(g_1,\dots,g_r) \in A^r$ be arbitrary. Then we have

$$(g_1,\dots,g_r)\begin{bmatrix} D_1(f_1;p) & \cdots & D_m(f_1;p) \\ \vdots & & \vdots \\ D_1(f_r;p) & \cdots & D_m(f_r;p)\end{bmatrix}\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_m-a_m\end{bmatrix}$$

$$= \left[\textstyle\sum_{i=1}^r g_i D_1(f_i;p),\dots,\sum_{i=1}^r g_i D_m(f_i;p)\right]\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_m-a_m\end{bmatrix}$$

$$= g_1\sum_{j=1}^m D_j(f_1;p)(x_j-a_j)+\cdots+g_r\sum_{j=1}^m D_j(f_r;p)(x_j-a_j)$$

$$= g_1f_1+g_2f_2+\cdots+g_rf_r \in I,$$

where the last equality comes from the uniqueness of left decomposition of the f_j's.

$\mathbf{ker(\bar{x}-\bar{a})} \subseteq \mathbf{im(J(I;p))}$:
An element g of $\operatorname{im}(\bar{x}-\bar{a})$ is on the form $g=\sum_{i=1}^m r_i(x_i-a_i)$. By the uniqueness of the left decomposition, we have $r_i = D_i(g;p)$. Let $g \in \ker(\bar{x}-\bar{a})$. Then g must be in the ideal I. Thus we can write $g=\sum_{j,k}\alpha_{j,k}f_j\beta_{j,k}$ where the α's and β's are in S. We get

$$D_i(g;p) = D_i\left(\sum_{j,k}\alpha_{j,k}f_j\beta_{j,k}\right)$$

$$= \sum_{j,k}\left(D_i(\alpha_{j,k};p)\phi_p(f_j\beta_{j,k})+\alpha_{j,k}D_i(f_j;p)\phi_p(\beta_{j,k})+\alpha_{j,k}f_jD_i(\beta_{j,k};p)\right).$$

Modulo I, we then have that $r_i=\sum_j s_jD_i(f_j;p)$, where $s_j=\sum_k\alpha_{j,k}\phi_p(\beta_{j,k})$. Hence

$$g=\sum_i\left(\sum_j s_jD_i(f_j;p)\right)(x_i-a_i),$$

so $g \in \operatorname{im}(J(I;p))$. Thus $\ker(\bar{x}-\bar{a}) = \operatorname{im}(J(I;p))$.

We now show exactness at A.

$\mathbf{im(\bar{x}-\bar{a})} \subseteq \mathbf{ker(\phi_p)}$:

Let $g = \sum_i r_i(x_i - a_i) \in \text{im}(\bar{x} - \bar{a})$. Then we have $\phi_p(g) = g(p) = 0$, as wanted.

$\mathbf{ker}(\boldsymbol{\phi_p}) \subseteq \mathbf{im}(\bar{\mathbf{x}} - \bar{\mathbf{a}})$: This follows from Theorem 4.5. Hence the sequence is exact at A.

Exactness at $k(p)$ follows from the fact that ϕ_p is surjective since it is simple. Thus the sequence is exact. ■

For a second representation $\phi_2 \colon S \to k(p_2)$, we write $J(I; p_1)(p_2)$ for the evaluation of the Jacobi matrix with respect to $k(p_1)$ in p_2.

To get the desired theorem about the dimension of Ext^1_A, we will need a version of Schur's lemma.

Lemma 4.12 (Schur's lemma)**.** *Let V_1, V_2 be two representations of an algebra A over k. Let $\phi \colon V_1 \to V_2$ be a nonzero homomorphism of representations. Then we have*

(i) If V_1 is irreducible, ϕ is injective.

(ii) If V_2 is irreducible, ϕ is surjective.

Proof.

(i) Let $K = \ker \phi$. It is a subrepresentation of V_1. Since ϕ is nonzero, $K \neq V_1$. Hence $K = 0$, so ϕ is injective.

(ii) im ϕ is a subrepresentation of V_2 and can not be zero, since ϕ is nonzero. Hence im $\phi = V_2$. ■

When both V_1 and V_2 are irreducible, we thus have that the only nonzero homomorphisms between them must be isomorphisms.

Hence if $k(p_1)$ and $k(p_2)$ are two 1-dimensional representations corresponding to $p_1, p_2 \in \text{Simp}_1(A)$, we either have $p_1 = p_2$ or that there is no nonzero homomorphism between $k(p_1)$ and $k(p_2)$.

The resolution in Theorem 4.11 now gives us the following result.

Theorem 4.13. *Let $A = k\langle x_1, \ldots, x_m\rangle / I$ be a k-algebra, and let ϕ_1, ϕ_2 be 1-dimensional representations corresponding to $p_1, p_2 \in Simp_1(A)$. Then we have*

$$\dim_k \text{Ext}^1_A(k(p_1), k(p_2)) = \begin{cases} m - 1 - rk\ J(I; p_1)(p_2) & \text{if } p_1 \neq p_2 \\ m - rk\ J(I; p_1)(p_2) & \text{if } p_1 = p_2. \end{cases}$$

Proof. Using $\text{Hom}_A(-, k(p_2))$ on the exact sequence of Theorem 4.11, we obtain the sequence

$$\begin{aligned} 0 \to \text{Hom}_A(k(p_1), k(p_2)) &\to \text{Hom}_A(A, k(p_2)) \\ &\xrightarrow{f} \text{Hom}_A(A^m, k(p_2)) \xrightarrow{g} \text{Hom}_A(A^r, k(p_2)). \end{aligned}$$

Now $\operatorname{Ext}^1_A(k(p_1), k(p_2))$ is defined to be $\ker g/\operatorname{im} f$. Since both $k(p_1)$ and $k(p_2)$ are irreducible representations, Schur's lemma gives us that the only homomorphisms between them are isomorphisms and the 0 map. Hence

$$\dim_k \operatorname{Hom}_A(k(p_1), k(p_2)) = \begin{cases} 0 & \text{if } p_1 \neq p_2 \\ 1 & \text{if } p_1 = p_2. \end{cases}$$

The result now follows since $\dim_k(\operatorname{im} f)$ is either 0 or 1 depending on p_1 and p_2, while $\dim_k(\ker g) = m - \operatorname{rk} J(I; p_1)(p_2)$. ■

4.2 Examples

For some curves we can now find requirements for $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2))$ to be nonzero, which are what we are interested in. We will always assume that the points p_i is in $\operatorname{Simp}_1(A)$, such that if $I = (f)$, then $f(p_i) = 0$.

Example 4.14 (Standard circle)**.** Let I be the ideal of $k\langle x, y\rangle$ generated by $f(x, y) = x^2 + y^2 - 1$ and put $p_1 = (a, b)$. Then the partial derivatives in p becomes

$$D_1(f; p_1) = D_1(x^2; p_1) = x + a$$

and

$$D_2(f; p_1) = D_2(y^2; p_1) = y + b.$$

We now get the Jacobian matrix $J(I; p_1) = \begin{bmatrix} x + a & y + b \end{bmatrix}$. Putting $p_2 = (c, d)$, the Jacobian matrix becomes

$$J(I; p_1)(p_2) = \begin{bmatrix} c + a & d + b \end{bmatrix}.$$

We see that setting $c = -a$ and $d = -b$ gives $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) = 1$. Also, setting $p_2 = p_1 \neq (0, 0)$, we get that $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) = 1$. For $p_1 = p_2 = (0, 0)$ we have that $J(I; p_1)(p_2)$ is the zero matrix, so $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) = 2$. Any other choice of p_2 will however give dimension equal to 0. To conclude, we have

$$\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) = \begin{cases} 2 & \text{if } p_1 = p_2 = (0, 0) \\ 1 & \text{if } p_1 = p_2 \neq (0, 0) \text{ or } p_2 = -p_1 \\ 0 & \text{else.} \end{cases}$$

Example 4.15 (Noncommutative version of circle)**.** Let $f(x, y) = x^2 + y^2 - 1 + \delta[x, y] = x^2 + y^2 - 1 + \delta xy - \delta yx$ and let I be the ideal generated by f. Since $\delta = 0$ is covered in Example 4.14, we assume $\delta \neq 0$. Setting $p_1 = (a, b)$ and $p_2 = (c, d)$, we obtain, as previously,

$$J = J(I; p_1)(p_2) = \begin{bmatrix} a + c + \delta(b - d) & b + d + \delta(c - a) \end{bmatrix}.$$

Let first $p_1 = p_2$. Then $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0$ since the Jacobi matrix has rank at most 1.

Assume now $p_1 \neq p_2$. For $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0$, we then need that the rank of J is 0, i.e., that J is the zero matrix. This is equivalent to

$$\delta = \frac{a+c}{d-b} = \frac{b+d}{a-c},$$

given that the fraction is defined. Notice that we can not have both $a = c$ and $b = d$. Hence at least one of the fractions is always defined.

Since $f(p_1) = f(p_2) = 0$, p_1 and p_2 are points on the unit circle, so we can write them with polar coordinates, say $p_1 = (a, b) = (\cos\omega_1, \sin\omega_1)$ and $p_2 = (\cos\omega_2, \sin\omega_2)$. Then we have

$$\delta = \frac{\cos\omega_1 + \cos\omega_2}{\sin\omega_2 - \sin\omega_1} = \frac{\sin\omega_1 + \sin\omega_2}{\cos\omega_1 - \cos\omega_2} = -\cot\left(\frac{\omega_1 - \omega_2}{2}\right).$$

The assumption that $\delta = 0$, gives us

$$\delta = -\frac{1}{\tan\left(\frac{\omega_1-\omega_2}{2}\right)} \Rightarrow \omega_1 - \omega_2 = 2\tan^{-1}\left(\frac{-1}{\delta}\right).$$

Thus δ, the scalar in front of the commutator, gives rise to the angle between two points corresponding to modules having nonvanishing Ext^1_A.

By taking δ to be specific values, we can calculate the angle we need between the points p_1 and p_2 to make the rank of the Jacobian equal to 0.

For instance, setting $\delta = 1$, gives

$$\omega_1 - \omega_2 = 2\tan^{-1}(-1) = -\frac{\pi}{2}.$$

Hence we need the points p_1 and p_2 to have an angle of $\frac{\pi}{2}$ between them for the rank of the Jacobian to be 0. Similarly $\delta = -1$ gives

$$\omega_1 - \omega_2 = \frac{\pi}{2}.$$

We will come back to this example in Chapter 5.

Example 4.16 (Elliptic curve)**.** Let $f(x, y) = x^3 + ax + b - y^2 + \delta[x, y]$, $a, b \in k$, and let I be the ideal generated by f, that is, an elliptic curve. By setting $p_1 = (r, s)$, we obtain

$$J(I; p_1) = \begin{bmatrix} x^2 + rx + r^2 + a + \delta b - \delta y & \delta x - \delta r - y - s \end{bmatrix}.$$

By Theorem 4.13 we have that $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0$ if $p_1 = p_2$, since the matrix has rank 0 or 1.

Assume $p_1 \neq p_2$. We want the Jacobian to be the zero matrix. We have a quadratic relation in x, which gives us

$$x = \frac{\delta^2 - r \pm \sqrt{\delta^4 - 6\delta^2 r - 3r^2 - 4(a + \delta b + \delta s)}}{2}$$

and

$$y = \delta x - \delta r - s = \delta(x - r) - s.$$

Setting $\mathcal{D} = \delta^4 - 6\delta^2 r - 3r^2 - 4(a + \delta b + \delta s)$ to be the discriminant, we get the solutions

$$x_1 = \frac{1}{2}\left(\delta^2 - u + \sqrt{\mathcal{D}}\right), \quad y_1 = \delta\left(x_1 - r\right) - s$$

and

$$x_2 = \frac{1}{2}\left(\delta^2 - u - \sqrt{\mathcal{D}}\right), \quad y_2 = \delta\left(x_2 - r\right) - s.$$

Calling the solutions q_1 and q_2, we have

$$q_i = (x_i, y_i) = (x_i, \delta(x_i - r) - s) = (r, -s) + (x_i - r)(1, \delta),\ i = 1, 2.$$

According to the group law on an elliptic curve, we have $-p_1 = (r, -s)$. Hence the points $-p_1, q_1$ and q_2 are collinear, i.e., lies on the same line. Thus, for $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0$, we need that $-p_1$ and p_2 lie on the same line, or that they are the equal.

CHAPTER 5

A Geometric Model of the Circle

Let $f(x,y) = x^2 + y^2 - 1 + \delta[x,y] = x^2 + y^2 - 1 + \delta xy - \delta yx$, let I be the ideal generated by f and put $A = k\langle x,y\rangle/I$, as we have done previously.

The purpose of this chapter is to study the geometry of the trace ring, the 2-dimensional simple representations and connect it to the extension modules. The goal is to give a closer relation between the previous chapters.

5.1 A Bit More About Ext

We start by looking closer at extensions. The reason for this is that we need a better understanding of how to represent a module which is an extension.

Definition 5.1. Let M and N be A-modules. An extension ξ of M by N is an exact sequence

$$0 \to N \to E \to M \to 0.$$

We will call the A-module E an extension module. Two such extensions, ξ and ξ', are said to be equivalent if there exists a commutative diagram

$$\begin{array}{lccccccccc}
\xi: & 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow \simeq & & \| & & \\
\xi': & 0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

It can be shown, see, e.g., theorem 3.4.3 in [Wei94], that there is a 1-1 correspondence

$$\{\text{equivalence classes of extensions of } M \text{ by } N\} \overset{1-1}{\longleftrightarrow} \operatorname{Ext}^1_A(M,N).$$

We could also define Ext through derivations. A derivation of A is k-linear map $D\colon A \to \operatorname{Hom}_k(M,N)$ such that $D(ab)(m) = D(a)b(m) + aD(b)(m)$ for all a, $b \in A$, $m \in M$.

If there exists an element $m \in M$ such that $D(a) = am - ma$ for all $a \in A$, we say that D is an inner derivation. Letting $\operatorname{Der}_k(A, \operatorname{Hom}_k(M,N))$ denote the set of derivations and $\mathrm{Inder}_k(A, \mathrm{Hom}_k(M, N))$ denote the set of inner derivations, we have, see, e.g., Lemma 9.2.1 in [Wei94], that

$$\mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) \simeq \mathrm{Der}_k(A, \mathrm{Hom}_k(M, N)) \big/ \mathrm{Inder}_k(A, \mathrm{Hom}_k(M, N)).$$

Thus an extension module is a derivation. The reason we are interested in this, is because it will help us understand what a representation of an extension module looks like.

Let us now be a little more concrete. Let A be a k-algebra, and let $k(p_1), k(p_2)$ be as before, say $k(p_i)$ takes an element $r \in A$ to r_i, $i = 1, 2$. Assume we have an extension

$$0 \to k(p_1) \xrightarrow{f} V \xrightarrow{g} k(p_2) \to 0,$$

and suppose we have a representation $\phi\colon A \to \mathrm{End}(V) \simeq M_2(k)$. V can be thought of as a direct sum $k(p_1) \oplus k(p_2)$ with some elements identified by an equivalence relation, since f is injective and g is surjective. Let now $f(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $g\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = b$.

The goal is to show that ϕ represents r by an upper triangular matrix, with a derivation in the top right corner.

When we work in $k(p_1)$, we have that r is represented by r_1, in V r is represented by $\phi(r)$, and in $k(p_2)$ it is represented by r_2. We let $r*$ denote the action of r, i.e., $r*$ is the element r is represented by in the module we work within. For instance, in V we have

$$r * \begin{bmatrix} a \\ b \end{bmatrix} = \phi(r) \begin{bmatrix} a \\ b \end{bmatrix}.$$

Let now $\phi(r) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Then we have

$$f(r_1) = \begin{bmatrix} r_1 \\ 0 \end{bmatrix},$$

and on the other hand

$$f(r_1) = r * f(1) = \phi(r) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}.$$

Hence $x_{11} = r_1$ and $x_{21} = 0$, so

$$\phi(r) = \begin{bmatrix} r_1 & s \\ 0 & t \end{bmatrix}$$

for some $s, t \in k$. We continue by determining t.

We have that

$$g\left(\phi(r) \begin{bmatrix} a \\ b \end{bmatrix}\right) = tb$$

and

$$g\left(\phi(r)\begin{bmatrix}a\\b\end{bmatrix}\right) = r * g\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = r_2 b.$$

Thus $r_2 b = tb$, so $t = r_2$, which leaves us with

$$\phi(r) = \begin{bmatrix}r_1 & s\\0 & r_2\end{bmatrix}.$$

Let now r' be a different element of A, and let $k(p_i)$ take r' to r'_i. Then, by the same argumentation as previously,

$$\phi(r') = \begin{bmatrix}r'_1 & s'\\0 & r'_2\end{bmatrix}.$$

Thus

$$\phi(rr') = \phi(r)\phi(r') = \begin{bmatrix}r_1 & s\\0 & r_2\end{bmatrix}\begin{bmatrix}r'_1 & s'\\0 & r'_2\end{bmatrix} = \begin{bmatrix}r_1 r'_1 & r_1 s' + r'_2 s\\0 & r_2 r'_2\end{bmatrix}.$$

Setting $s = \gamma(r)$, we see that $\gamma(rr') = r_1\gamma(r') + \gamma(r)r'_2$. This means that γ is a derivation.

If we have an extension module of $k(p_1)$ by $k(p_2)$, it can therefore be represented by an upper triangular matrix. This will come in handy later on.

5.2 Modules Which Can Be Deformed to Simple Modules

We want to explore the boundary of the 2-dimensional simple representations by two different approaches. One where we look at the trace ring using Chapter 3, and one where we look at modules which can be deformed to simple modules. We start with the last, and it is here Ext^1_A comes into play.

Definition 5.2. Let $\mathcal{V} = \{V_1, \ldots, V_r\}$ be simple A-modules. The directed extension graph $Q\mathcal{V}$ of $\mathcal{V}$ has the modules V_i as its vertices and there is an arrow from V_i to V_j if and only if $\mathrm{Ext}^1_A(V_i, V_j) \neq 0$.

As usual, a cycle is a path starting and ending in the same vertex, while we call a cycle complete if it containts all vertices of $Q\mathcal{V}$. We call a module E simplifiable if it can be deformed into a simple module.

We also need to know what the support of a module is. Suppose E is a nonsimple, indecomposable A-module with a finite composition series. Let

$$E = E_r \to E_{r-1} \to \cdots \to E_1 \to E_0 = 0$$

be a composition series of E, and put $V_i = \ker\{E_i \to E_{i-1}\}$. Then the set $\{V_1, \ldots, V_r\} = \mathrm{Supp}(E)$ is called the support of E.

Theory presented in section 6 of [JLS04] gives us the following result:

Theorem 5.3 ([JLS04])**.** *Let A be an associative k-algebra and E an indecomposable A-module, such that $\mathrm{End}_A(E) \simeq k$. Let $\mathcal{V} = \mathrm{Supp}(E) = \{V_1, \ldots, V_r\}$ be the associated simple modules, and let $Q\mathcal{V}$ be the extension graph. If E is simplifiable, then there exists a complete cycle in $Q\mathcal{V}$.*

So, what does this mean for us? First we note that this is not a theorem that says that there exist simple modules. It says that if E is simplifiable, there must exist nonzero extensions of all its associated simple modules. This means that the boundary of the simple representations is the modules which can be deformed into a simple module.

5.3 The Geometry

Now we will become a bit more concrete, looking at our specific situation with $f = x^2 + y^2 - 1 + \delta[x, y]$, $I = (f)$ and $A = k\langle x, y\rangle/I$. Let $p_1, p_2 \in \mathbb{A}^2$, and consider two 1-dimensional representations ϕ_1, ϕ_2 corresponding to $k(p_1)$ and $k(p_2)$.

We are interested in p_1 and p_2 such that $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0$ and $\dim_k \operatorname{Ext}^1_A(k(p_2), k(p_1)) \neq 0$.

By Example 4.14 and Example 4.15 we know the condition for the dimension of Ext^1_A to be nonzero: For $\delta = 0$, we have that we either have $p_1 = p_2$ or $p_2 = -p_1$. For $\delta \neq 0$, we have

$$c = \frac{(\delta^2 - 1)a}{\delta^2 + 1} - \frac{2\delta b}{\delta^2 + 1} \quad \text{and} \quad d = \frac{(\delta^2 - 1)b}{\delta^2 + 1} + \frac{2\delta a}{\delta^2 + 1},$$

where $p_1 = (a, b)$ and $p_2 = (c, d)$. Finding an expression for δ, as we did in Example 4.15, shows that δ corresponds to an angle.

Both p_1 and p_2 lie on the circle, since $f(p_i) = 0$ for $i = 1, 2$. Letting p_1 lie on one circle, say C^1, and letting p_2 lie on another circle, say C^2, we can by parametrizing C^1 also parametrize C^2 through δ, such that $\dim_k \operatorname{Ext}^1_A(k(p), k(p')) \neq 0$ for $p \in C^1$, $p' \in C^2$. Since we have that $S^1 \times S^1$ is the torus, we now have curves going around the torus for each δ.

Since we have $\dim_k \operatorname{Ext}^1_A(k(p), k(p')) \neq 0$, we have extension modules. These extension modules, which correspond to a point on the curve on the torus, are the ones we are interested in, as they constitute the boundary of the simple, 2-dimensional representations. Since there does not exist any such representations for $\delta \neq 0$ per Example 3.7, we now concentrate on $\delta = 0$.

Then the curve on the torus will be one of the two nontrivial Villarceau circles, while the other Villarceau circle will be obtained by orienting one of the circles the opposite way, see Figure 5.1b.

Now we have that $\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0 \iff p_2 = p_1$ or $p_2 = -p_1$. By symmetry, we thus have

$$\dim_k \operatorname{Ext}^1_A(k(p_1), k(p_2)) \neq 0 \iff \dim_k \operatorname{Ext}^1_A(k(p_2), k(p_1)) \neq 0.$$

Theorem 5.3 tells us that the simplifiable modules of these extension modules, are the boundary of the open set $\operatorname{Simp}_2(A_{\delta=0})$.

Now we move our focus to the trace ring. We work within the spectrum of the 5-dimensional trace ring $T_{2,2} = k[t_X, t_Y, t_{XY}, d_X, d_Y]$. Inside $\operatorname{Spec}(T_{2,2}) = \mathbb{A}^5$, we know that we find the 2-dimensional simple representations. They are given by the nonvanishing of the Formanek center, so they are an open subscheme of $T_{2,2}$, as seen in Section 3.1.

The boundary of this open subset of $\mathbb{A}^5$ and the boundary given by the simplifiable modules, should in some way correspond to each other. That is, the boundary of $\text{Simp}_2(A_{\delta=0})$ should somehow be glued onto the curve on the torus over $\delta = 0$.

Let us take a more thorough look at the trace geometry, keeping Example 3.7 and Example 4.14 in mind.

Previous calculations, see Example 3.7, have shown that

$$\text{Spec}(\Gamma(A_{\delta=0})) \simeq V(t_X, t_Y, d_X + d_Y + 1) \subseteq \text{Spec}(T_{2,2}).$$

Hence

$$\text{Spec}(\Gamma(A_{\delta=0})) \simeq \text{Spec}\left(k[t_{XY}, d_X, d_Y]/(d_X + d_Y + 1)\right).$$

The equation $d_X + d_Y + 1 = 0$ gives us a plane in t_{XY}, d_X, d_Y-space, and the general theory gives us that the set of simple, 2-dimensional representations is an open subset of this plane. We have calculated that these representations are given by

$$\text{Simp}_2(A_{\delta=0}) \simeq V\left(t_{XY}^2 + (2d_X + 1)^2 - 1\right)^C \subseteq \text{Spec}(A_{\delta=0}).$$

Looking at the equation $t_{XY}^2 + (2d_X + 1)^2 - 1 = 0$, we have an ellipse in the t_{XY}, d_X-plane. Letting the last variable, d_Y, be free, we have an elliptic cylinder in 3-space. This cylinder will intersect the plane given by $d_X + d_Y + 1 = 0$ in an ellipse. The plane minus the intersection with the cylinder, will then be $\text{Simp}_2(A_{\delta=0})$, while the intersection will be the boundary. Hence the intersection consists of extension modules which can be deformed into simple modules.

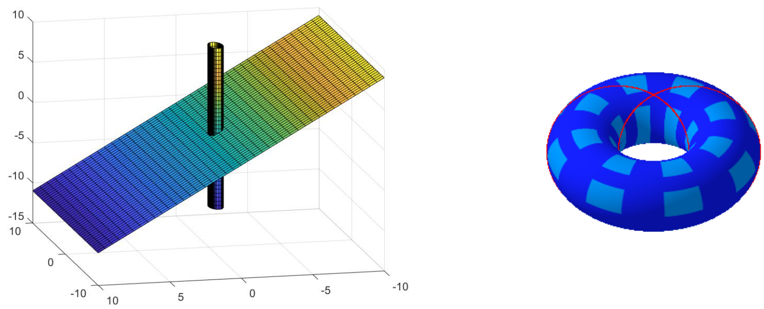
Thus we have points on circles, say (x_1, x_2) and $(-x_1, -x_2)$ giving the combined point $(x_1, x_2, -x_1, -x_2)$ on the torus. The point $(x_1, x_2, -x_1, -x_2)$ corresponds to nonzero extension modules. Hence it corresponds to two 2×2 matrices,

$$X = \begin{bmatrix} x_1 & x \\ 0 & -x_1 \end{bmatrix}, \; Y = \begin{bmatrix} x_2 & y \\ 0 & -x_2 \end{bmatrix},$$

as we saw earlier in the chapter.

Then $d_X = -x_1^2$ and $t_{XY} = 2x_1x_2$. Since these are the variables defining the boundary of $\text{Simp}_2(A_{\delta=0})$, we have the following correspondence between the curve on the torus and the boundary of the simple, 2-dimensional representations:

The point $(x_1, x_2, -x_1, -x_2)$ on the torus corresponds to the representation $d_X = -x_1^2$ and $t_{XY} = 2x_1x_2$. This gives a 2 to 1 correspondence, since the point $(-x_1, -x_2, x_1, x_2)$ gives the same d_X and t_{XY}. That we have a 2 to 1 correspondence should come as no surprise as the trace function sees the eigenvectors of a matrix, but can not separate them.



(a) The plane and the elliptic cylinder. (b) The torus with the two Villarceau circles.

Figure 5.1: One of the red circles on the torus is going to be glued twice around the ellipse formed by the intersection between the plane and the elliptic cylinder.

We end this discussion with a real picture of the plane and the elliptic cylinder and a picture of the torus with one curve corresponding to points giving nonzero Ext^1_A (the other circle is the other Villarceau circle), see Figure 5.1 Here the elliptic intersection of the plane and the elliptic cylinder is going to be glued twice around one of the red circles of the torus.

When $\delta \neq 0$, we have that $\mathrm{Simp}_2(A_\delta) = \emptyset$. Then Theorem 5.3 tells us that we can not have any simplifiable extension modules of dimension 2. Thus even though there exist points p_1 and p_2 such that $\dim_k \mathrm{Ext}^1_A(k(p_1), k(p_2)) \neq 0$ and $\dim_k \mathrm{Ext}^1_A(k(p_2), k(p_1)) \neq 0$, the extension module will not be simplifiable. There is however the possibility that there exist simplifiable modules of higher dimension. We will not investigate this here, but the framework is in the previous chapters.

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