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Simulation of Greeks of financial claims in Markets with Memory

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This master's thesis is submitted under the master's programme *Computational Science and Engineering*, with programme option *Computational Science*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 30 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In thesis we discuss a new "derivative-free" formula for the computation of the price sensitivity,"Delta" with respect to the past given in [2]. This can be achieved by an appropriate relationship between the Malliavin derivative and a functional directional derivative. Further, we develop a novel numerical implementation method with respect to the representation for the "Delta". As an example we compute the "Delta" for specific claims in the case of a labor income model with memory, by using Monte Carlo techniques.

Key words and phrases: Greeks, Malliavin Calculus, sensitivity analysis, stochastic differential delay equation, stochastic functional differential equation, Skorohod integral.

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Notation and Symbols

We shall make use of the following notation in this thesis, and some of the notation will be introduced as we go along.

Spaces

$\mathbb{N}$	the set of all natural number, $\{1, 2, 3, \dots\}$.
$\mathbb{R}^d$	the set of all d-dimentional column vectors with real entries.
$\mathbb{R}^{d \times m}$	all $d \times m$ matrices with real entries.
$L^2[a, b]$	Hilbert space.
$C^2([0, \infty), \mathbb{R})$	twice continuous differentiable on $[0, \infty) \times \mathbb{R}$ with continuous extensions of the partial derivatives to $[0, \infty) \times \mathbb{R}$.
$L^2(\Omega)$	Hilbert space of square integrable real-valued random variable on Ω with inner product $< X, Y >= E(XY)$.

Other notation

$< x, y >$	$< x, y >= \sum_{i=1}^{d} x_i y_i$, the inner product on $\mathbb{R}^d$.
$\lvert \cdot \rvert$	the Eclidean norm in $\mathbb{R}^d$.
$\lVert \cdot \rVert$	L^2-norm.
$\mathbb{1}_A$	the indicator function of the event A. If A occurs then $\mathbb{1}_A = 1$, otherwise $\mathbb{1}_A = 0$.
$\mathcal{B}(\mathbb{R}^d)$	the Borel σ-algebra on $\mathbb{R}^d$.
A^c	complement of event A.
$\mathcal{N}$	the family of all null sets.
$X \stackrel{d}{=} Y$	the stochastic variables X and Y are equal in distribution.
$X \sim \Theta$	the stochastic variable X is Θ-distributed.
$P << Q$	the probability measure P is absolutely continuos with respect to the probability measure Q.
$P \sim Q$	P and Q are equivalent probability measures.
$E_Q[\cdot]$	the expectation under the probability measure Q.
$\forall$	for all.
$\square$	marks the end of a proof.

Abbreviations

a.s.	almost surely, with probability 1.
i.i.d.	independent and identically distributed.
SDE	stochastic differential equation.
SDDE	stochastic delay differential equation.
SFDE	stochastic functional differential equation.
SLLN	strong law of large numbers.

Contents

Chapter 1

Introduction

In recent years there has been an increased interest among scholars and practitioners in the financial mathematics and economics literature to better understand the impact of memory presence in stock prices, commodities and other assets or goods. To understand the effects, stochastic models that take market memory into account have been developed.

Models considering memory presence have been e.g. used to explain the phenomenon of random cyclical fluctuations in markets, see [11]. On the other hand fluctuations may also be due to violation of market efficiency theory where inside who have access to financial information prior to the beginning of the trading period. See [17], where the author uses stochastic delay equations for the modeling of the latter effects. See also [1] and the references therein.

In this thesis we will study the price sensitivities of financial claims also called "Greeks", in markets with memory. These are quantities representing the market sensitivities of financial derivatives to the variation of the model parameters. The main case here will be the analysis of the so-called "Delta", which measures the asset price sensitivity with respect to input data η from the past and which typically takes the form

$$\Delta(\eta) := \frac{\partial}{\partial \eta} p(\eta),$$

where

$$p(\eta) = E_{Q^{\eta}}[\frac{\Phi({}^{\eta}S_T)}{{}^{\eta}N(T)}]$$

is the price of the claim (or option) $\Phi({}^{\eta}S_T)$ with respect to the underlying asset process ${}^{\eta}S_t, 0 \leq t \leq T$ at maturity T. Here Φ is the pay-off function, ${}^{\eta}N(t), 0 \leq t \leq T$ the numéraire (based e.g. on the discounting process), and Q^{η} a certain probability measure (e.g. risk neutral measure). We assume that ${}^{\eta}S_t, 0 \leq t \leq T$ is a commodity or stock price process on a market with "memory" η, described by a stochastic delay equation, [13].

The main objective of the master thesis is the computation of the "Delta" of option prices with respect to a specific market model with memory by using a new "derivative-free" representation of price sensitivities (Bismut-Elworthy-Li formula), based on Malliavin calculus [2].

Objectives of the thesis
The objectives of this thesis are the following:

- Discussion of a new Bismut-Elworthy-Li formula for the computation of "Deltas" of option price with respect to a model for labor income, [3].
- Development of a new numerical implementation method with respect to the representation formula for the "Delta".
- Implementation of the numerical method in the case of specific claims based on a stochastic labor income model with memory.

Outline of the thesis The thesis is structured such that it should be self-contained for the reader. Therefore, have we given all the necessary tools to be able to read and understand the contents of this thesis, as we go along.

Chapter 2 serves as an introduction to basic probability theory and other important concepts to be used later on. Chapter 3 is dedicated to the Malliavin calculus and applications, and chapter 4 is devoted to stochastic differential delay equations and their applications to finance. Chapter 5 is aimed to discuss the sensitivity of claims with respect to the initial paths of solutions to stochastic delay equations. In chapter 6 we introduce a new numerical method for the implementation of the Bismut-Elworthy-Li formula based on Malliavin calculus. Moreover we simulate specific sensitivities with respect to that formula in the case of a labor income model. Finally in 7 we give a summary of our results and discuss ideas for future research work. Chapters 4 and 5 are the motivation for chapter 6 and address the question here, why we are interested in computing "Deltas". The statistical background and some proofs can be found in Appendix A. The statistical software R will be used through the thesis, and the computer code will be found in Appendix B.

Chapter 2

Probability theory

In this chapter we will give an introduction to selected parts of probability theory and stochastic analysis. This will be used throughout the thesis, and will be beneficial for the reader to be familiar with.

2.1 Brownian Motion

A Brownian motion moves so rapidly and irregularly that almost all of its sample paths are nowhere differentiable. A process like this is very important as it provides an easy way of modeling the "noisy" part of a model, and will be used in problems encountered in this thesis. The purpose of this section is to briefly treat the mathematical definition and construction of Brownian motion. Stock price is an example where we try to model a phenomenon that we can not be certain of how it evolves over time.

Definition 2.1.1. (Brownian Motion). A stochastic process $B = \{B_t\}_{0 \leq t \leq T}$ on the probability space $(\Omega, \mathcal{F}, P)$ is called Brownian Motion if the following properties hold:

(i) $B_0 = 0$, P-a.e. (A.2.7).

(ii) B has independent increments:
$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \dots, B_{t_n} - B_{t_{n-1}}$ independent if
$0 \leq t_0 < t_1 < \cdots < t_n \leq T$.

(iii) B has Gaussian increments:
$B_t - B_s \stackrel{d}{=} B_{t-s}, \qquad 0 \leq s < t$ and
$B_{t-s} \sim \mathcal{N}(0, t-s)$.

Note here that the Brownian motion is defined without its dimension, which means it is one-dimensional. The paths of Brownian motion are also easy to simulate, and we arrive at the following Algorithm. Figure 2.1 is an example of such a path of Brownian motion. This will be used later in this thesis, when we try to solve the problem numerically.

Algorithm 2.1 Path of Brownian motion

Data: time horizon T; partition n
$\Delta t \leftarrow T/n$ ▷ Subinterval width
generate $\xi_i \sim \mathcal{N}(0,1),\ i = 0,\dots,n-1$ ▷ $\xi \sim$ a Gaussian stochastic variable
$B_0 \leftarrow 0$
for $i = 0,\dots,n-1$ **do**
    $B_{t_{i+1}} \leftarrow B_{t_i} + \xi_i\sqrt{\Delta t}$
return $\{B_{t_i}\}_{i=0}^{n}$

Here we assume $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ with

$$\begin{aligned} |\Pi| &= \sup_{0\le i\le n-1} |t_{i+1} - t_i| \\ &= \Delta t \\ &= T/n \end{aligned}$$

such that

$$t_i := i\Delta t, \qquad i = 0,\dots,n.$$

By letting ξ be a standard Gaussian stochastic variable, we have from the Gaussian increments of the Brownian motion:

$$B_{t_{i+1}} - B_{t_i} \stackrel{d}{=} \xi\sqrt{\Delta t}, \qquad i = 0,\dots,n-1,$$

such that

$$B_{t_{i+1}} = B_{t_i} + (B_{t_{i+1}} - B_{t_i}) \stackrel{d}{=} B_{t_i} + \xi\sqrt{\Delta t}, \qquad i = 0,\dots,n-1.$$

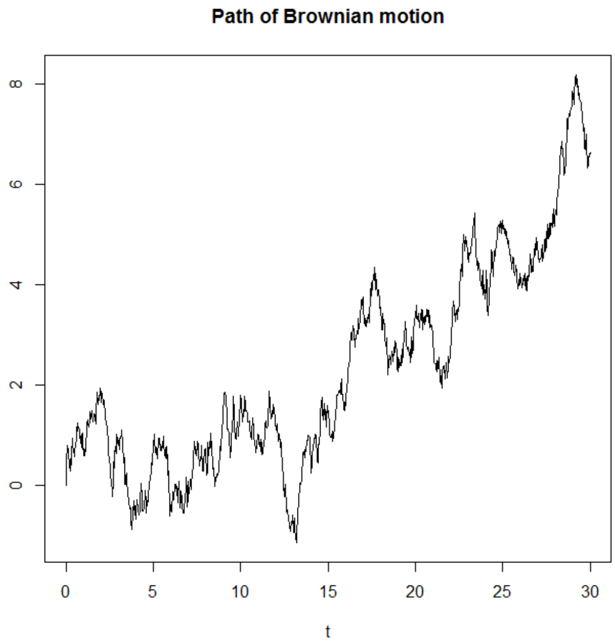

Figure 2.1: *Brownian motion is plotted with equidistant partitioning* Π *of* $[0, 30]$ *with* $|\Pi| = \Delta t = 3/100$.

2.2 Itô' Integral and Itô Formula

In this chapter we will define the Itô integral and discuss the Itô formula, which constitute the foundation of stochastic analysis. The Itô formula is a sort of chain rule in connection with Itô calculus, and can only be interpreted in the integral form

$$\int_0^t f(t, \omega) dB_t(\omega). \tag{2.1}$$

Let us introduce some basic definitions first.

Definition 2.2.1. (Filtration $\mathcal{F}_t$). Let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be a family of σ-algebras on $(\Omega, \mathcal{F}, P)$ such that

$$\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \qquad (\subset \mathcal{F})$$

for all $0 \leq t_1 \leq t_2 \leq T$. Then $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is called filtration on $(\Omega, \mathcal{F}, P)$.

Definition 2.2.2. (Adaptedness). A process $(X_t)_{t \in T}$ on $(\Omega, \mathcal{F}, P)$ for an interval T is called adapted to the filtration $\{\mathcal{F}_t\}_{t \in T}$, if X_t is $\{\mathcal{F}_t\}$-measurable for every $t \in T$.

Definition 2.2.3. Let $\mathcal{L}^2([a,b],\Omega)$ denote the class of functions $f(t,\omega)$, satisfying the following:

(i) f(t) is adapted to the filtration $\{\mathcal{F}_t\}$ and,

(ii) $\int_a^b |f(t)|^2 dt < \infty$ a.s.

Definition 2.2.4. (Martingale). A stochastic process $M_t = \{M_t\}_{t\geq 0}$ is called a martingale with respect to the filtration $\{\mathcal{F}_t\}$ if,

(i) M_t is $\{\mathcal{F}_t\}$-measurable $\quad \forall\ t$,

(ii) $E[|M_t|] < \infty \quad \forall\ t$, and

(iii) $E[M_t|\mathcal{F}_s] = M_s \quad \forall\ s \leq t$.

Definition 2.2.5. (Local martingale). An $\{\mathcal{F}_t\}$-adapted stochastic process $(X_t)_{a\leq t\leq b}$, is called a local martingale with respect to $\{\mathcal{F}_t\}$ if there exists a sequence of stopping times $\{\rho_n\}_{n=1}^{\infty}$ such that

1. ρ_n increases monotonically to b a.s. as $n \to \infty$,
2. for each n, $X_{t\wedge\rho_n}$ is a martingale with respect to $\{\mathcal{F}_t : a \leq t \leq b\}$.

By choosing $\rho_n = b$ we have that a martingale is a local martingale, but a local martingale may not be a martingale. For this we need the following theorem from [12].

Definition 2.2.6. Let $L^2_{ad}([a,b]\times\Omega)$ be a class of processes

$$f(t,\omega) : [0,\infty)\times\Omega \to \mathbb{R}$$

on a probability space $(\Omega,\mathcal{F},P)$, such that

(1) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.

(2) $f(t,\omega)$ is $\{\mathcal{F}_t\}$-adapted, where $\{\mathcal{F}_t\}$ is generated by the Brownian motion and the P-null sets.

(3) $E[\int_a^b f(t,\omega)^2 dt] < \infty$.

Theorem 2.2.7. (Martingale property). Let $f \in L^2_{ad}([a,b]\times\Omega)$. Then the stochastic process

$$X_t = \int_a^t f(s,\omega)dB_s(\omega), \qquad a \leq t \leq b, \tag{2.2}$$

is a martingale with respect to the filtration $\{\mathcal{F}_t : a \leq t \leq b\}$.

Proof. Given in Appendix A.3.

∎

Definition 2.2.8. (Semimartingale). Let $S = (S_t)_{0 \le t \le T}$ be a $\{\mathcal{F}_t\}$-adapted process. Then S is a semimartingale if:

$$S_t = S_0 + M_t + V_t, \qquad 0 \le t \le T,$$

where $M = (M_t)_{0 \le t \le T}$ is a $\{\mathcal{F}_t\}$-adapted local martingale, and $V = (V_t)_{0 \le t \le T}$ is a $\{\mathcal{F}_t\}$-adapted process with finite variation over $[0, T]$.

Definition 2.2.9. (Stochastic integral of elementary process). Let $(Y_t)_{0 \le t \le T}$ be a process of the form

$$Y_s = \sum_{i=1}^{n-1} \tau_i \mathbb{1}^{(s)}_{[t_i, t_{i+1}]}, \qquad 0 \le t \le T,$$

where $0 = t_0 < t_1 < \cdots < t_n = T$.

Assuming that Y_T is a random variable on $(\Omega, \mathcal{F}_T, P)$ and τ_i is a random variable on $(\Omega, \mathcal{F}_{t_i}, P), i = 1, \ldots, n$ such that

$$\max_{i=1}^{n} |\tau_i| \le C < \infty$$

for a constant C. Then $Y_t, 0 \le t \le T$ is called elementary process.

With these definition in hand, let us look at the Itô-integral of Y_t.

Definition 2.2.10. (Itô integral). A measurable stochastic process Y_s on $(\Omega, \mathcal{F}_T, P)$ is called Itô integrable on $[0, T]$, if:

1. Y_s is adapted with respect to a filtration $\{\mathcal{F}_t\}$, which is generated by the Brownian motion and the P-null sets, and
2. $\int_0^T E[Y_s^2] ds < \infty$.

For the above processes Y_s it is known that there exist elementary processes $Y_t^{(n)}$, $n \ge 1$ such that

$$E\left[\int_0^T (Y_s^{(n)} - Y_s)^2 ds\right] \to 0.$$

The latter implies the existence of a random variable X, such that

$$\text{Var}\left[\int_0^T Y_s^{(n)} dB_s - X\right] \to 0.$$

The random variable X is called Itô-integral, or stochastic integral of Y_t with respect to B_t and we write

$$\int_0^T Y_s dB_s = X.$$

Then we have the following property of Itô-integrals

i) $\int_0^T (\alpha Y_s + \beta Z_s) dB_s = \alpha \int_0^T Y_s dB_s + \beta \int_0^T Z_s dB_s$ (linearity).

ii) $E[\int_0^T Y_s dB_s] = 0$.

iii) Itô isometry:

$$\mathrm{Var}\left[\int_0^T Y_s dB_s\right] = E\left[\int_0^T Y_s^2 ds\right]. \tag{2.3}$$

iv) Define $M_t = \int_0^t Y_s dB_s$. Then the process M_t is martingale with respect to $\{\mathcal{F}_t\}$, that is

$$E[M_t|\mathcal{F}_t] = M_s, \quad t \geq s.$$

v) There exists a continuous version of $M_t = \int_0^t Y_s dB_s$.
We may assume that $(t \mapsto M_t)$ is continuous P-a.e.

2.2.11 The Itô Formula

The Itô formula may serve as a tool to evaluate stochastic integrals.

Theorem 2.2.12. (Itô Formula for Brownian Motion). Assume that the Brownian motion B_t starts at x, and let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Then,

$$f(B_t) = f(x) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_t) ds. \tag{2.4}$$

Definition 2.2.13. Let B_t be one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A process X_t is an Itô process if there exist an Itô integrable stochastic process Y_t and an adapted process Z_t, such that

$$X_t = x + \int_0^t Z_s ds + \int_0^t Y_s dB_s, \qquad 0 \leq t \leq T, \tag{2.5}$$

where we assume that

$$E[\int_0^t |Z_s| ds] < \infty, \quad t \geq 0,$$

and this leads to the adaptedness of X_t. For $Z_t = 0$ the semimartingale X_t reduces to an Itô integral, which is a martingale.

We have the following shorthand notation for equation (2.5), given as

$$dX_t = Zdt + YdB_t. \tag{2.6}$$

Theorem 2.2.14. (General Itô Formula). Let X_t be an Itô process of the form (2.5), and assume that $g(t,x)$ is a function which is once continuously differentiable in t and twice continuously differentiable in x. Then

$$\begin{aligned} g(t, X_t) = g(0,x) + \int_0^t Y_s \frac{\partial g(s,X_s)}{\partial x} dB_s \\ = \int_0^t \frac{\partial g(s,X_s)}{\partial t} + Z_s \frac{\partial g(s,X_s)}{\partial x} + \frac{1}{2} Y_s^2 \frac{\partial^2 g(s,X_s)}{\partial x^2} ds. \end{aligned} \tag{2.7}$$

Proof. Given in Appendix A.3 □

Theorem 2.2.15. (Itegration by parts). Let

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dB_s,$$

and

$$Y_t = Y_0 + \int_0^t \tilde{K}_s + \int_0^t \tilde{H}_s dB_s$$

be Itô processes, then

$$X_t \cdot Y_t = X_0 \cdot Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t H_s \cdot \tilde{H}_s ds$$

where the last part is the quadratic variation $< X, Y >_t$ of X and Y.

To prove the Wiener Itô expansion later in chapter 3, we need the following Itô Representation Theorem.

Theorem 2.2.16. (The Itô representation theorem). Let $F \in L^2(\mathcal{F}_T, P)$, then there exists a unique stochastic process $f(t,\omega) \in L^2_{ad}([0,T] \times \Omega)$ such that

$$F(\omega) = E[F] + \int_0^T f(t,\omega) dB(t). \tag{2.8}$$

Theorem 2.2.17. (Martingale representation theorem). There exists a unique stochastic process $g(s,\omega)$ such that $g \in \mathcal{L}^2([a,b],\Omega)$ for all $t \geq 0$ and

$$M_t(\omega) = E[M_0] + \int_0^t g(s,\omega) dB(s), \qquad \text{a.s. for all } t \geq 0.$$

The following theorem is a tool to explicitly construct risk neutral measures Q i financial applications.

Theorem 2.2.18. (Girsanov's theorem). Assuming $(X_t)_{0\leq t\leq T}$ to be a real valued $\{\mathcal{F}_t\}$-adapted process on the probability space $(\Omega, \mathcal{F}, P)$, and letting Y_t be an Itô process of the form

$$Y_t = B_t + \int_0^t X_s ds, \qquad 0 \leq t \leq T.$$

Define the process

$$Z_t = e^{-\int_0^t X_s dB_s - \frac{1}{2}\int_0^t |X_s|^2 ds}, \qquad 0 \leq t \leq T. \tag{2.9}$$

Assuming that X_t satisfies the Novikov condition, that is

$$E[e^{\frac{1}{2}\int_0^T |X_s|^2 ds}] < \infty. \tag{2.10}$$

Then the Girsanov's transformation Q of the measure is P defined by the probability measure

$$Q(A) := E[\mathbb{1}_A \cdot Z_T]. \tag{2.11}$$

Then Y_t is a Brownian motion under Q, so Y_t has independent and normal stationary increments with respect to Q.

2.3 Monte Carlo method

Stochastic modeling, takes one or more random variables to predict the future outcome. Computerized mathematical simulation techniques such as the Monte Carlo method offers a unique insight into processes, which are not directly observable in physical experiments.

The Monte Carlo technique relies on repeated random sampling to obtain numerical results, and evaluate portfolios. We use this technique to approximate the solution to our problem later on.

Say we have a transformation $\delta(\cdot)$ of a stochastic variable ξ, where ξ has some probability distribution Θ. Then by sampling repeatedly from that distribution Θ, we can approximate the solution. The following theorem is the foundation of Monte Carlo techniques.

Theorem 2.3.1. (Kolmogorov's Strong Law of Large Numbers). Assume that

$$E[|X_1|] < \infty,$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. stochastic variables with values in $\mathbb{R}$, then

$$\frac{1}{l}\sum_{n=1}^{l} X_n \underset{l\to\infty}{\longrightarrow} E[X_1], \qquad a.s.$$

To solve stochastic problems numerically, the given Theorem 2.3.1 or Strong Law of Large Numbers (SLLN) are key results. Then we may approximate the mean of $\delta(\xi)$ by:

$$E[\delta(\xi)] \approx \bar{\delta} := \frac{1}{l} \sum_{i=1}^{l} \delta(\xi_i), \tag{2.12}$$

where $l \in \mathbb{N}$ is large and $\{\xi_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. stochastic variables with distribution Θ. Let us give this in the following Algorithm 2.2.

Algorithm 2.2 Monte Carlo simulation

1: **Data:** Function $\delta(\cdot)$; distribution Θ; fixed $l \in \mathbb{N}$
2: generate $\xi_i \sim \Theta, \qquad i = 0, \dots, l$
3: $\bar{\delta} \leftarrow \frac{1}{l} \sum_{i=1}^{l} \delta(\xi_i)$
4: **return** $\bar{Z}$

Chapter 3

Malliavin Calculus

Malliavin calculus is known as the stochastic calculus of variations. More precisely, computations of sensitivity parameters of option price also known as "Greeks". The Malliavin calculus was introduced by Paul Malliavin in the 1970´s. His aim was to give a probabilistic proof of Hörmander´s theorem. [15] When Paul Malliavin introduced the infinite-dimensional calculus in 1978, his motivation was to deal with Brownian motion and the application to regularity results for solutions of SDEs.

This chapter is mainly based on [6] and [4]. We aim to state central theorems and definitions, which will be in hand to discuss our objectives of this thesis. The first section 3.1, will describe the construction of the Wiener-Itô chaos expansion. In section 3.2 the Skorohod integral will be defined and Section 3.3 will be fundamental for the development of the Malliavin calculus. In Section 3.4 we will give an important result from the efforts of the previous sections. Further in section 3.5 the Clark-Ocone Formula will be stated, and finally in section 3.6 result for the "Greeks" will be presented.

3.1 Wiener-Itô Chaos Expansion

Letting $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space and letting $W = W_t = W(\omega, t), \omega \in \Omega$ be a one-dimensional Brownian motion (Wiener process) with respect to P as in Definition 2.1.1. Further, the integral of a deterministic function $f \in L^2[0,T]$ over a fixed, finite interval $[0,T]$ with respect to Brownian motion,

$$I(f) = \int_0^T f(t)dW(t),$$

as a Wiener integral. Then we have that this Wiener integral is measurable with respect to the Brownian σ-algebra.

Definition 3.1.1. (Symmetric function). A real function $g : T^n \to \mathbb{R}$ is called symmetric if

$$g(t_{\sigma_1}, \dots, t_{\sigma_n}) = g(t_1, \dots, t_n), \tag{3.1}$$

for all permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of $\{1, 2, \dots, n\}$.

For a function f, $\tilde{f}$ denotes the symmetrization of f given by

$$\tilde{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \dots, t_{\sigma_n}). \tag{3.2}$$

$\tilde{f} = f$, if and only if f is symmetric. Further, $\tilde{L}^2([0,T]^n) \subset L^2([0,T]^n)$ denotes the space of symmetric square integrable Borel functions on $[0,T]^n$.

Example 3.1.2. The symmetrization $\tilde{f}$ of the function

$$f(t_1, t_2) = t_1^2 + t_2 sin(t_1), \qquad (t_1, t_2) \in [0,T]^2,$$

is

$$\begin{aligned}\tilde{f}(t_1, t_2) &= \frac{1}{2!} \sum_{\sigma} t_1^2 + t_2 sin(t_1) \\ &= \frac{1}{2}[t_1^2 + t_2^2 + t_2 sin(t_1) + t_1 sin(t_2)], \in [0,T]^2\end{aligned}$$

for $n = 2$ and $\sigma \in S_2 = \{(1,2)(2,1)\}$.

Definition 3.1.3. If $g \in \tilde{L}^2([0,T]^n)$ we define

$$I_n(g) = \int_{[0,T]^n} g(t_1, \dots, t_n) dW(t_1) \dots dW(t_n) = n! J_n(g),$$

$J_n(g)$ is defined to be the n-fold iterated Itô integral:

$$J_n(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \dots, t_n) dW(t_1) dW(t_2) \dots dW(t_{n-1}) dW(t_n),$$

and because of the construction of Itô integrals, $J_n(f)$ belongs to $L^2(P)$ which is the space of square integrable random variables. Then we have the following proposition.

Proposition 3.1.4. Let $f \in L^2([0,T]^n),\ n \geq 1$. Then

(1) $I_n(f) = I_n(\tilde{f})$, where $\tilde{f}$ is the symmetrization of f.

(2) $E[I_n(f)] = 0$.

(3) $E[I_n(f)^2] = n! \|\tilde{f}\|^2_{L^2([0,T]^n)}$.

With this we are finally able to state the following on the Wiener-Itô chaos expansion.

Theorem 3.1.5. (The Wiener-Itô chaos expansion). Let ξ be a $\{\mathcal{F}_T\}$ -measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0,T]^n)$ such that

$$\xi = \sum_{n=0}^{\infty} I_n(f_n) \tag{3.3}$$

where the convergence is in $L^2(P)$. Moreover, we have the isometry

$$\|\xi\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^n)}. \tag{3.4}$$

Sketch of proof. Use the Itô representation theorem 2.2.16 to write:

$$\xi = E[\xi] + \int_0^T \varphi_1(s_1) dW(s_1), \tag{3.5}$$

where $\varphi(s_1),\ 0 \le s_1 \le T$, is $\{\mathcal{F}_t\}$-adapted such that

$$E\Big[\int_0^T \varphi^2(s_1) ds_1\Big] \le E[\xi^2]. \tag{3.6}$$

Apply the Itô isometry again to $\{\mathcal{F}_t\}$-adapted processes $\varphi_1(s_1), \varphi_2(s_2, s_1), \dots, \varphi_{n+1}(s_{n+1}, s_n, \dots, s_1)$ for $0 \le s_{n+1} \le s_n \le \cdots \le s_1 \le T$.

Define

$$\begin{aligned} g_0 &= E[\xi], \\ g_1(s_1) &= E[\varphi_1(s_1)], \\ g_2(s_2, s_1) &= E[\varphi_2(s_2, s_1)], \\ &\vdots, \\ g_{n+1}(s_{n+1}, \dots, s_1) &= E[\varphi_1(s_{n+1}, \dots, s_1)]. \end{aligned}$$

Then after n steps

$$\xi = \sum_{k=0}^{n} J_k(g_k) + \int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)},$$

where the expression

$$\int_{Sn+1} \varphi_{n+1} dW^{\otimes(n+1)} := \int_0^T \int_0^{t_{n+1}} \cdots \int_0^{t_2} \varphi_{n+1}(t_1, \dots, t_{n+1})(t_1) \dots dW(t_{n+1})$$

is the $(n+1)$-fold iterated integral of φ_{n+1}.

The second part of the sum above converges to zero, if we extend g_n to $[0,T]^n$ by putting

$$g_n(t_1, \dots, t_n) = 0, \qquad (t_1, \dots, t_n) \in [0,T]^n \backslash S_n.$$

Now by defining $f_n := \tilde{g}_n$ to be the symmetrization of g_n, we have

$$\begin{aligned} I_n(f_n) &= n! J_n(f_n) \\ &= n! J(\tilde{g}_n) \\ &= J_n(g_n). \end{aligned}$$

□

3.2 Skorohod integral

In this section the aim is to go further and look at an extension of the Itô integral to integrands that not necessarily are adapted to the filtration $\{\mathcal{F}_t\}$, namely the *Skorohod integral*. The Skorohod integral is a stochastic integral developed by A. Skorohod in 1975 [14].

Letting $u(t) = u(t, \omega)$ for $t \in [0, T]$ and $\omega \in \Omega$ be a measurable stochastic process such that,

(i) $u(t)$ is a $\{\mathcal{F}_T\}$-measurable random variable and,

(ii) $E[u^2(t)] \leq \infty$ for all $t \in [0, T]$.

With this we can apply the Wiener–Itô chaos expansion (3.3) to the random variable $u(t) = u(t, \omega)$. Then for each $t \in [0, T]$ there are symmetric functions

$$f_{n,t} = f_{n,t}(t_1, \dots, t_n), (t_1, \dots, t_n) \in [0, T]^n$$

in $\tilde{L}^2([0, T]^n), n \in \mathbb{N}$ such that $u(t)$ has the chaos expansion

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

Considering f_n as a function of $n+1$ variables, with the functions $f_{n,t}, n \in \mathbb{N}$ depend on the parameter $t \in [0, T]$. Hence, we can write

$$f_n(t_1, \dots, t_n, t_{n+1}) = f_n(t_1, \dots, t_n, t) := f_{n,t}(t_1, \dots, t_n).$$

The symmetrization $\tilde{f}_n$ of f_n is given by

$$\begin{aligned} \tilde{f}_n(t_1, \dots, t_{n+1}) = \frac{1}{n+1} \big[& f_n(t_1, \dots, t_{n+1}) + f_n(t_2, \dots, t_{n+1}, t_1) \\ & + \dots + f_n(t_1, \dots, t_{n-1}, t_{n+1}, t_n) \big]. \end{aligned} \tag{3.7}$$

With this we can define the Skorohod integral from [6]:

Definition 3.2.1. (Skorohod integral). Let $u(t), t \in [0, T]$, be a measurable stochastic process such that for all $t \in [0, T]$ the random variable $u(t)$ is $\{\mathcal{F}_T\}$ -measurable satisfying the conditions above and $E[\int_0^T u^2(t) dt] < \infty$. Let its Wiener–Itô chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t)).$$

Then the Skorohod integral of u is defined by

$$\delta(u) := \int_0^T u(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \tag{3.8}$$

when convergent in $L^2(P)$. Here $\tilde{f}_n, n \in \mathbb{N}$ are the symmetric functions (3.7) derived from $f_n(\cdot, t), n \in \mathbb{N}$.

We say that u is Skorohod integrable, and we write $u \in Dom(\delta)$ if the series in (3.8) converges in $L^2(P)$.

Remark 3.2.2. A stochastic process u belongs to $Dom(\delta)$ iff.:

$$E[\delta(u)^2] = \sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty. \tag{3.9}$$

Now let us state the following theorem:

Theorem 3.2.3. Let $u = u(t),\ t \in [0, T]$, be an Itô integrable process. Then u is Skorohod integrable and its Skorohod integral coincides with the Itô integral such that,

$$\int_0^T u(t)\delta W(t) = \int_0^T u(t)d(t). \tag{3.10}$$

3.3 The Malliavin Derivative

The Malliavin derivative can be constructed in several ways. In this section the construction is based on the chaos expansion given above. The following definition is the Malliavin derivative.

Definition 3.3.1. (The Malliavin derivative). Let $F \in L^2(P)$ be $\{\mathcal{F}_T\}$ - measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2([0, T]^n),\ n \in \mathbb{N}$ are symmetric functions. Then we say that $F \in \mathbb{D}_{1,2}$ if

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn!\ \|f_n\|^2_{L^2([0,T]^n)} < \infty. \tag{3.11}$$

If $F \in \mathbb{D}_{1,2}$, define the Malliavin derivative $D_t F$ of F at time t as the expansion

$$D_t F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0, T]. \tag{3.12}$$

Before going any further we need to state some fundamental results for Malliavin derivatives borrowed from [6].

Theorem 3.3.2. (Closability of the Malliavin derivative). Suppose $F \in L^2(P)$ and $F_k \in \mathbb{D}_{1,2}, k \in \mathbb{N}$, such that

- $F_k \to F, k \to \infty$, in $L^2(P)$
- $\{D_t F_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$, where λ is the Lebesgue measure.

Then $F \in \mathbb{D}_{1,2}$ and $D_t F_k \to D_t F, k \to \infty$, in $L^2(P \times \lambda)$.

Theorem 3.3.3. (Product rule for the Malliavin derivative). Suppose $F_1, F_2 \in \mathbb{D}_{1,2}^0$. Here $\mathbb{D}_{1,2}^0$ is the set of all $F \in L^2(P)$, whose chaos expansion has only finitely many terms. Then $F_1, F_2 \in \mathbb{D}_{1,2}$ and the product $F_1 F_2 \in \mathbb{D}_{1,2}$ with

$$D_t(F_1 F_2) = F_1 D_t F_2 + F_2 D_t F_1. \tag{3.13}$$

Let us now consider the case when $f_n = f^{\otimes n}$ for some $f \in L^2([0, T])$, that is

$$f_n(t_1, \ldots, t_n) = f(t_1) \cdots f(t_n).$$

Here $\otimes$ denotes the tensor power, and gives us the following definition.

Definition 3.3.4. (Tensor product). The tensor product $f \otimes g$ of two functions f and g is defined as

$$(f \otimes g)(x_1, x_2) = f(x_1) g(x_2),$$

and the symmetrized tensor product $f \hat{\otimes} g$ is the symmetrization of $f \otimes g$. Then we have

$$I_n(f_n) = \|f\|^n h_n\left(\frac{\theta}{\|f\|}\right) \tag{3.14}$$

where $\|f\| = \|f\|_{L^2([0,T])}, \theta = \int_0^T f(t) dW(t)$ and the Hermite polynomials h_n of n order is defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}), \qquad x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

A basic property of the Hermite polynomials is that

$$h'_n(x) = nh_{n-1}(x). \tag{3.15}$$

We have from (3.12) that

$$\begin{aligned} D_t I_n(f_n) &= nI_{n-1}(f_n(\cdot, t)) \\ &= nI_{n-1}(f^{\otimes(n-1)})f(t) \\ &= nf^{n-1}h_{n-1}\left(\frac{\theta}{\|f\|}\right)f(t). \end{aligned} \tag{3.16}$$

Then we have that

$$D_t h_n\left(\frac{\theta}{\|f\|}\right) = h'_n(x)\left(\frac{\theta}{\|f\|}\right)\left(\frac{f(t)}{\|f\|}\right), \tag{3.17}$$

and by choosing $n = 1$, we get

$$D_t \int_0^T f(s)dW(s) = f(t). \tag{3.18}$$

Similarly by (3.15) and induction for $n = 2, 3 \ldots$, we have

$$D_t\Big(\int_0^T f(s)dW(s)\Big)^n = n\Big(\int_0^T f(s)dW(s)\Big)^{n-1} f(t). \tag{3.19}$$

3.4 Chain rule

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function in $\mathcal{C}^1$ with bounded partial derivatives. For fixed $p \geq 1$ and $F = (F^1, \ldots, F^d)$ a random vector such that $F^i \in \mathbb{D}_{1,2}$ for any $i = 1, \ldots, d$. Then $g(F) \in \mathbb{D}_{1,2}$, and

$$D(g(F)) = \sum_{i=1}^{d} \partial_i g(F) DF^i.$$

This can be extended in the case where g is a Lipschitz function [14].

Proposition 3.4.1. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function, that is for some constant $K > 0$,

$$|g(x) - g(y)| \leq K \parallel x - y \parallel$$

for all x,y $\in \mathbb{R}^d$. Suppose that $F = (F^1, \ldots, F^d)$ a random vector such that $F^i \in \mathbb{D}_{1,2}$ for any $i = 1, \ldots, d$. Then $g(F) \in \mathbb{D}_{1,2}$, and there exists a random vector $G = (G_1, \ldots, G_d)$ bounded by K such that

$$D(g(F)) = \sum_{i=1}^{d} G_i DF^i$$

and if $g \in \mathcal{C}^1(\mathbb{R}^d)$, then

$$D(g(F)) = \sum_{i=1}^{d} \partial_i g(F) DF^i.$$

With this let us state the following rule, for one dimension:

Theorem 3.4.2. (Chain rule). Let $F \in \mathbb{D}_{1,2}$ and $g \in C^1(\mathbb{R})$ with bounded derivative. Then $g(F) \in \mathbb{D}_{1,2}$ and

$$D_t g(F) = g'(F) D_t F. \tag{3.20}$$

where $g'(x)$ is the derivative of $g(x)$.

With all these results, we are now able to state the relationships between the Malliavin derivative and the Skorohod integral. The following theorem shows that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.4.3. (Duality formula). Let $F \in \mathbb{D}_{1,2}$ be $\{\mathcal{F}_T\}$-measurable and let u still be a Skorohod integrable stochastic process. Then

$$E\Big[F \int_0^T u(t)\delta W(t)\Big] = E\Big[\int_0^T u(t) D_t F dt\Big].$$

Proof. Given in Appendix A.3 □

Theorem 3.4.4. (Integration by parts). Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then

$$F \int_0^T u(t)\delta W(t) = \int_0^T F u(t)\delta W(t) + \int_0^T u(t) D_t F dt. \tag{3.21}$$

Proof. Given in Appendix A.3 □

3.4.5 A Fundamental Theorem

Finally we have the fundamental theorem, which gives us a useful connection between differentiation and Skorohod integration.

Theorem 3.4.6. (The fundamental theorem). Let $u(s)$ for $s \in [0,T]$, be a stochastic process such that

$$E\Big[\int_0^T u^2(s) ds\Big] < \infty \tag{3.22}$$

and assume that for all $s, t \in [0,T]$, $u(s) \in \mathbb{D}_{1,2}$ and $D_t u \in Dom(\delta)$ is Skorohod integrable. Moreover, assume

$$E\Big[\int_0^T (\delta(D_t u))^2 dt\Big] < \infty. \tag{3.23}$$

Then $\int_0^T u(s)\delta W(s)$ is well-defined and belongs to the space $\mathbb{D}_{1,2}$, and

$$D_t\Big(\int_0^T u(s)\delta W(s)\Big) = \int_0^T D_t u(s)\delta W(s) + u(t). \tag{3.24}$$

Proof. First of all we need to prove (3.24) with the help of the symmetrization function (3.1.1), and the prove that $\delta(u)$ is well-defined and belongs to $\mathbb{D}_{1,2}$ then finally prove (3.24)

The detailed proof of Theorem 3.4.6 is given in A.3 □

3.5 The Clark-Ocone Formula

In this section we will give some generalization of the Clark-Ocone formula. In our case, this is a central result in the application of the sensitivity analysis. The Clark–Ocone formula is also used in the application to hedging in mathematical finance.

The following result shows that any random variable $F \in \mathbb{D}_{1,2}$ can be written as the sum of its expectation and a stochastic integral of conditional expectations (Definition A.2.3) of its Malliavin derivative.

Theorem 3.5.1. (The Clark–Ocone formula). Let $F \in \mathbb{D}_{1,2}$ be $\{\mathcal{F}_T\}$ -measurable. Then

$$F = E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t). \tag{3.25}$$

Proof. For those interested, it is given as proof of Theorem 3.11 in [6]

□

3.5.2 The Clark–Ocone Formula under Change of Measure

This section will consider the Clark–Ocone formula under change of measure. Assuming F to be a $\{\mathcal{F}_T\}$-measurable random variable, then the Clark–Ocone formula expresses F as a stochastic integral with respect to a process of the form

$$\widetilde{W}(t) = \int_0^t u(s)ds + W(t) \qquad 0 \le t \le T \tag{3.26}$$

where $u(s),\ s \in [0,T]$, is a given $\{\mathcal{F}_t\}$-adapted stochastic process satisfying the Novikov condition (2.10). Then by Girsanov's theorem 2.2.18, the process

$\widetilde{W}(t) = \widetilde{W}(\omega, t)$, for $\omega \in \Omega,\ t \in [0, T]$, is a Wiener process (with respect to the filtration $\{\mathcal{F}_t\}$) under the new probability measure Q defined on $(\Omega, \mathcal{F}_T)$ by

$$Q(d\omega) = Z(T, \omega)P(d\omega), \tag{3.27}$$

where $Z(t)$ is defined as in (2.9).

From [6] we have the following Theorem

Theorem 3.5.3. (The Clark-Ocone formula under change of measure). Let $F \in \mathbb{D}_{1,2}$ be $\{\mathcal{F}_T\}$ -measurable. Suppose that

$$E_Q[|F|] < \infty \tag{3.28}$$

$$E_Q\Big[\int_0^T |D_t F|^2 dt\Big] < \infty. \tag{3.29}$$

Also assume that $u(s) \in \mathbb{D}_{1,2}$ for all s, $Z(T)F \in \mathbb{D}_{1,2}$ and

$$E_Q\Big[|F| \int_0^T \Big(\int_0^T D_t u(s) dW(s) + \int_0^T u(s) D_t u(s) ds\Big)^2 dt\Big] < \infty. \tag{3.30}$$

Then

$$F = E_Q[F] + \int_0^T E_Q\Big[(D_t F - F \int_t^T D_t u(s) d\widetilde{W}(s))|\mathcal{F}_t\Big] d\widetilde{W}(t). \tag{3.31}$$

Note here that we let E_Q denote the expectation with respect to the new probability measure Q, while $E_P = E$ denotes the expectation with respect to P.

3.6 Application to Sensitivity Analysis and Computation of the "Greeks"

The Greeks are defined as the collection of statistical values that measure the risk involved in an options contract in relation to certain underlying variables. In other words it is the derivative of the option price with respect to any of its parameters of the model (see [7]).

Considering the price of an option V_0 of strike K and maturity T depends on five parameters, such as (x, r, σ, T, K), where x is the premium, r is the interest rates, and σ the volatility. The Greeks are then the partial derivatives of V_0 with respect to these parameters. Hence, the most popular Greeks are:

- "Delta" measures the sensitivity to changes in the initial price x of the underlying asset:
  $\Delta = \frac{\delta V}{\delta x}$.
- "Gamma" measures the rate of change in the "Delta":
  $\Gamma = \frac{\delta^2 V}{\delta x^2}$.

- "Rho" measures the sensitivity to the applicable interest rate r:
  $\rho = \frac{\delta V}{\delta r}$.
- "Theta" measures the sensitivity to the amount of time to expiration date:
  $\Theta = \frac{\delta V}{\delta T}$.
- "Vega" measures the sensitivity to volatility σ:
  $\nu = \frac{\delta V}{\delta \sigma}$.

The name *Greeks* was given because these quantities often are denoted by Greek letters.

Given that V is computed as an expectation, the Greeks are basically derivatives of expectations. In [6] it is shown that the Greeks computation based on Malliavin calculus is in many situations better than, that based on the so called density method.

3.6.1 Delta

Let us have a closer look at the "Delta". In our case we would like to study the Greek Delta which is connected with the so-called Δ-hedging, and considering one-dimensional processes. Let us look at a market model consisting of the following assets:

$$\text{risk free asset} \begin{cases} dS_0(t) = \rho(t)S_0(t)dt \\ S_0(0) = 1 \end{cases} \quad \text{risky asset} \begin{cases} dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dW(t)] \\ S_1(0) = x > 0 \end{cases}$$

where we assume that $\rho(t) = \rho$ is constant and the coefficients μ and σ are Markovian, such that $\mu(t) = \mu(S_1(t))$ and $\sigma(t) = \sigma(S_1(t)) \neq 0, 0 \leq t \leq T$. By replicating an $\{\mathcal{F}_T\}$-measurable Markovian payoff, such as

$$F = \varphi(S_1(T)),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded, then we can try to find a self-financing portfolio $\theta(t) = (\theta_0(t), \theta_1(t))_{0 \leq t \leq T}$ and a function $(f(t,x))_{0 \leq t \leq T}, x > 0$. Such that the value process $V^\theta(t)$ given by

$$V^\theta(t) = \theta_0(t)S_0(t) + \theta_1(t)S_1(t), \qquad 0 \leq t \leq T$$

is of the form

$$V^\theta(t) = f(t, S_1(t)), \qquad t \in [0,T].$$

Note here that $\theta(t)$ is called self-financing if

$$dV^\theta(t) = \theta_0(t)dS_0(t) + \theta_1(t)dS_1(t).$$

By using Itô formula in Theorem 2.2.14, we get

$$dV(t) = \frac{\partial f}{\partial t}(t, S_1(t))dt + \frac{\partial f}{\partial x}(t, S_1(t))dS_1(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t)dt, \tag{3.32}$$

and since θ is self-financing we can write

$$dV^{\theta}(t) = \theta_0(t)S_0(t)\rho dt + \theta_1(t)dS_1(t). \tag{3.33}$$

Now by comparing the two equations above (3.32) and (3.33) we get

$$\begin{aligned}\theta_0(t)S_0(t)\rho + \theta_1(t)S_1(t)\mu(S_1(t)) &= \frac{\partial f}{\partial t}(t, S_1(t)) + \frac{\partial f}{\partial x}(t, S_1(t))S_1(t)\mu(S_1(t)) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t),\end{aligned} \tag{3.34}$$

and

$$\theta_1(t)\sigma(S_1(t))S_1(t) = \frac{\partial f}{\partial x}(t, S_1(t))\sigma(S_1(t))S_1(t). \tag{3.35}$$

Here we have that (3.35) holds if and only if

$$\theta_1(t) = \frac{\partial f}{\partial x}(t, S_1(t)) \qquad \text{(the "}\Delta\text{-hedge")}, \tag{3.36}$$

by substituting this to (3.34) we get

$$[f(t, S_1(t)) - S_1\frac{\partial f}{\partial x}(t, S_1(t))]\rho = \frac{\partial f}{\partial t}(t, S_1(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_1(t))\sigma^2(S_1(t))S_1^2(t), \tag{3.37}$$

where $f(t, S_1(t))$ must satisfy the Black–Scholes equation, that is

$$\begin{cases}\frac{\partial f}{\partial x}(t,x) = -\rho f(t,x) + \rho x\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(t,x) = 0, \quad t < T \\ f(T,x) = \varphi(x).\end{cases} \tag{3.38}$$

By using the Feynman–Kac formula (see [10]), we get that the solution of this equation is

$$\begin{aligned}f(t, S_1(t)) &= E^x[e^{-\rho(T-t)}\varphi(X(T-t))]|_{x=S_1(t)} \\ &= e^{-\rho(T-t)}E^x[\varphi(X(T-t))|_{x=S_1(t)}.\end{aligned}$$

Here $X(t) = (X^x(t))_{0\le t\le T}$, is the solution of the stochastic differential equation:

$$dX(t) = X(t)[\rho dt + \sigma(X(t))dW(t)]; \quad X(0) = x > 0.$$

Therefore, to compute the "Δ-hedge" $\theta_1(t),\ t \in [0,T]$, we need to compute

$$\begin{aligned}\frac{\partial f}{\partial x}(t,x) &= e^{-\rho(T-t)}\frac{\partial}{\partial x}E^x[\varphi(X(T-t))] \\ &= e^{-\rho(T-t)}\frac{\partial}{\partial x}E[\varphi(X^x(T-t))].\end{aligned} \tag{3.39}$$

For numerical computations, φ may be discontinuous as e.g. in the case of binary options or may not be smooth. However by using Malliavin calculus to transform the expression (3.39), gives us a form that is more suitable for numerical computations. Let us present this approach.

First we consider a general Itô diffusion $X^x(t),\ t \geq 0$ given by

$$dX^x(t) = b(X^x(t))dt + \sigma(X^x(t))dW(t), \quad X^x(0) = x \in \mathbb{R},$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are given functions in $C^1(\mathbb{R})$ and $\sigma(x) \neq 0$ for all $x \in \mathbb{R}$. Then we have the first variation process:

$$Y(t) := \frac{\partial}{\partial x} X^x(t), \qquad t \geq 0,$$

which satisfies

$$dY(t) = b'(X^x(t))Y(t)dt + \sigma'(X^x(t))Y(t)dW(t), \qquad Y(0) = 1,$$

that is,

$$Y(t) = exp\{\int_0^t [b'(X^x(u)) - \frac{1}{2}\sigma'(X^x(u))^2]du + \int_0^t \sigma'(X^x(u))dW(u)\}. \quad (3.40)$$

For a fixed $T > 0$ we define:

$$g(x) = E^x[\varphi(X(T))] = E[\varphi(X^x(T))].$$

Then we obtain the following theorem.

Theorem 3.6.2. (Malliavin weight). Let $a(t), t \in [0, T]$, be a continuous deterministic function such that

$$\int_0^T a(t)dt = 1.$$

Then

$$g'(x) = E^x[\varphi(X(T)) \int_0^T \pi^{\Delta}]. \quad (3.41)$$

The random variable in (3.41) is defined as

$$\pi^{\Delta} = \int_0^T a(t)\sigma^{-1}(X(t))Y(t)dW(t),$$

and is a so-called Malliavin weight.

This Malliavin weight is central in chapter 5, where we discuss a formula considering the presence of memory.

Chapter 4

Stochastic differential delay equations and applications to finance

This chapter is aimed to have a closer look at stochastic delay equation in connection with finance. The latter will be useful in view of the next chapter.

Time delay and random effects in economics and finance is not unknown. Several authors have tried to explain this, and everyone has their own explanation such as:

- random cyclical factors
- unstable economic system
- time delayed influence

Time delayed influence causes periodic fluctuations, and such delays should obviously affect the price dynamics (see [11]).

Let us consider the simplest stochastic differential delay equation (SDDE) under the Banach space $C([-r,0],\mathbb{R})$, [13]:

$$\left.\begin{aligned} dx(t) &= x(t-r)dW(t) \qquad 0 < t \leq r \\ x_0 &= \eta \in C([-r,0],\mathbb{R}). \end{aligned}\right\} \tag{4.1}$$

W(t) is still a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.

Example 4.0.1. Considering the SDDE (4.1) for the ordinary case where $r = 0$, and applying the Itô calculus we get the following solution:

$$x(t) = e^{W(t)-\frac{1}{2}t}, \quad t \in \mathbb{R}.$$

For $\{{}^{\eta}x_t : t > 0\}$ and through the initial path $\eta \in C$, the trajectory field of (4.1) is generated by the unique solution ${}^{\eta}x_t \in L^2(\Omega, C)$. It is solved by 'integrating' over steps of lengths r:

$$
{}^{\eta}x(t) = \begin{cases} \eta(0) + \int_0^t \eta(u-r)dW(u) & 0 \le t \le r \\ \eta(t) \qquad t \in [-r, 0]. \end{cases}
$$

4.1 A delayed Black and Scholes formula

In what follows we aim at discussing the applications of stochastic delay equation to mathematical finance.

From [1] we have an explicit formula for pricing European call options, where the underlying stock price satisfies a nonlinear SDDEs. An European call option can only be exercised at the maturity date. Further, the market here is complete and the model maintains the no-arbitrage property[1].

Having the fair price of a call option, it is interesting to consider the effect of the past. Here we assume that the stock price satisfies a stochastic functional differential equation (SFDE), which are substantially stochastic differential equations with coefficients depending on the past history of the dynamic itself. Several articles on this subject are mentioned in [2], and will be partially handled in chapter 5.

Now let us look at a stock, where the price at time t is modeled by a stochastic process $S(t)$ satisfying the following SDDE. This process is defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$, such that

$$
\left.\begin{aligned} dS(t) &= \mu S(t-a)S(t)dt + g(S(t-b))S(t)dW(t), \qquad t \in [0,T] \\ S(t) &= \varphi(t), \qquad t \in [-L, 0], \end{aligned}\right\} \tag{4.2}
$$

where the process W is a one-dimensional standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ and a, b, μ and T are positive constants with $L := \max\{a, b\}$. The function $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. The space $C([-L, 0], \mathbb{R})$ of all continuous functions $\eta : [-L, 0] \to \mathbb{R}$ is a Banach space.

The initial process $\varphi : \Omega \to C([-L, 0], \mathbb{R})$ is $\mathcal{F}_0$-measurable with respect to the Borel σ-algebra of $C([-L, 0], \mathbb{R})$.

From Theorem 1 in [1] we have that the equation above (4.2) admits a pathwise unique solution S, where $S(t) > 0$ almost surely for all $t \ge 0$, if $\varphi(0) > 0$ almost surely.

Having a self-financing strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ consisting of holding $\pi_S(t)$ units of the stock and $\pi_B(t)$ units of the bond at time t, we end up with the fair price $V(t)$ of an option on the stock evolving as described by the SDDE (4.2) to be:

$$
V(t) = e^{-r(T-t)} E_Q[X|\mathcal{F}_t^S], \qquad t \in [0, T]
$$

[1] *No-arbitrage:* There is no opportunity to get risk-free profit.

at each $t \in [0, T]$ a.s, such that the market satisfies the no-arbitrage property, and the contingent claim X is attainable, such that the market $\{B(t), S(t) : t \in [0, T]\}$ is complete. Note here that Q is the (local) martingale measure from Girsanov's transformation 2.2.18, depending on both the delayed drift and the volatility coefficient of the stock price.

4.2 Stochastic labor income

Another study of delayed dynamics is discussed in [3]. Here the authors consider a standard complete market model of securities with prices evolving as geometric Brownian motions (GBM), but the dynamics of the contingent claims is described by an (SFDE).

A practical example here is the stochastic labor income, and the valuation of human capital. The market value of human capital can be derived by risk-neutral valuation, and the labor income is spanned by tradable assets. In chapter 6 we will suggest to introduce delay terms in income dynamics. This is based on empirical evidence on wage rigidity, (e.g.,[3]). Then the income dynamics will adjust slowly to financial market shocks by introducing delayed drift and volatility coefficients in a GBM model.

Let us assume that the labor income follows the SFDE with delay of a GBM model given below:

$$\left.\begin{aligned} dX^0(t) &= [X^0\mu^0 + \int_{-r}^{0} X^0(t+s)\phi(ds)]dt \\ &+ \left[X^0(t)(\sigma^0)^T + \begin{pmatrix} \int_{-r}^{0} X^0(t+s)\varphi^1(ds) \\ \vdots \\ \int_{-r}^{0} X^0(t+s)\varphi^n(ds) \end{pmatrix}^T \right] dZ(t) \\ X^0(0) &= x^0 \\ X^0(s) &= x^1(s) \text{ for } s \in [-r, 0), \end{aligned}\right\} \tag{4.3}$$

where Z is an n-dimensional Brownian motion, $\mu^0 \in \mathbb{R}_{>0}$ and $\sigma^0 \in \mathbb{R}^n$. Here we denote $\mathbb{R}^n_{>0}$ for the set $(x^i) \in \mathbb{R}^n : \ x^i > 0, \ i = 1, \dots, n$. Moreover, ϕ, φ^i are signed measures of bounded variation on $[-r, 0]$, with $i = 1, \dots, n$, and $x^0 \in \mathbb{R}_{>0}$ and $x^1 \in L([-r, 0]; \mathbb{R}_{>0})$. This type of equation which is different from equation (4.2) also admits an unique (strong) solution.

Chapter 5

SFDE and sensitivity to their initial path

5.1 Introduction

Market inefficiency and the fact that traders use past prices as a guide to decision making, induces memory effects that may be held responsible for market bubbles and crashes. Several stochastic models deal with delay and memory in different areas, ranging from biology to finance. For instance, we looked at the delayed response in the price of financial assets in the previous chapter.

Let us now look at a general stochastic dynamic model involving delay or memory effects. More precisely we will consider stochastic functional differential equations (SFDE). When choosing such a model we also have to consider the model risk, in view of prediction and forecast. One way to manage this is by studying the sensitivity to the initial condition, also know as the Delta in the terminology of mathematical finance. In such a situation we may redefine the Delta to be defined as a functional directional derivative. In view of numerical computations we need to look at a representation formula without requiring that the evaluation or payoff function is differentiable. This is achieved by an appropriate relationship between the Malliavin derivative and functional directional derivatives.

In our case we need to redefine the Delta from a single initial point as in the standard stochastic differential equations in section 3.6, to be the initial condition as initial path. By this we are dealing with sensitivity to the initial path, which is very new and for the first time handled in [2].

Let us here consider the SFDE:

$$\left.\begin{aligned} dx(t) &= f(t,x(t),x_t)dt + g(t,x(t),x_t)dW(t), \quad t\in[0,T] \\ (x(0),x_0) &= \eta, \end{aligned}\right\} \tag{5.1}$$

where $x(t)$ is the evaluation at time t of the solution process, $x_t = (u \mapsto x(t+u)$ is the segment of past and η is the initial path. Next section will handle this equation in detail. Further, let us look at the evaluation $p(\eta)$ at $t=0$ of some value $\Phi(^{\eta}x(T),^{\eta}x_T)$ at $t=T$ of a functional Φ of the model:

$$p(\eta) = E[\Phi(^\eta x(T),^\eta x_T)]. \tag{5.2}$$

Here we assume the dependence on the initial path η by an anticipated superindex. This type of evaluation are typical in the pricing of financial derivatives. Then, financial contracts with payoff Ψ written on an underlying asset with price dynamics S is given by a SFDE such as (5.1). A fair price is then given from the classical non arbitrage pricing rule:

$$\begin{aligned}
P_{risk-neutral}(\eta) &= E_{^\eta Q}\Big[\frac{\Psi(^\eta S(T),^\eta S_T)}{N(T)}\Big] \\
&= E\Big[^\eta Z(T)\frac{\Psi(^\eta S(T),^\eta S_T)}{N(T)}\Big],
\end{aligned}$$

where $^\eta Z(T) = \frac{d^\eta Q}{P}$ is the Radon-Nikodym derivative (Theorem A.2.10) of the risk-neutral probability measure $^\eta Q$ depending on η and $N(T)$ is used as a discount factor.

Moreover the benchmark approach to pricing, a non-arbitrage fair price depending on the initial path η is given in this form:

$$P_{benchmark}(\eta) = E\Big[\frac{\Psi(^\eta S(T),^\eta S_T)}{^\eta G(T)}\Big].$$

$^\eta G(T)$ is the value of an appropriate benchmark process. This denominator is used in discounting and guaranteeing that P is an appropriate pricing measure.

Both pricing approaches given above can be represented as (5.2). Then the sensitivity to the initial condition can be measured by:

$$\frac{\partial}{\partial\eta}p(\eta) = \frac{\partial}{\partial\eta}E[\Phi(^\eta x(T),^\eta x_T)],$$

with the payoff functional Φ and the "Delta" redefined as a functional directional derivative.

Further in this chapter Malliavin calculus is used to derive a formula considering the presence of memory, and the derivative is itself represented as an expectation of the product of the functional Φ and a Malliavin weight (Theorem 3.6.2).

In order to obtain such a formula, we have to study the relationship between functional Fréchet derivatives and Malliavin derivatives. The technique here is based on *the randomization of the initial path condition*, which is again based on the use of an independent Brownian noise.

5.2 Stochastic functional differential equation

Let us first discuss the background of SFDE's, by looking at the general case.

The model: Letting the probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be complete, where the filtration satisfies the usual assumptions (Definition 2.2.1) and is such that $\mathcal{F} = \{\mathcal{F}_T\}$. We also consider $W = W(t,\omega);\ \omega \in \Omega,\ t \in [0,T]$ to be an m-dimensional standard $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion (Definition 2.2.6), with $T \in [0,\infty)$.

We are mainly interested in stochastic processes $x : [-r,T] \times \Omega \to \mathbb{R}^d, r \geq 0$, with finite second order moments and almost surely continuous sample paths. Then, we can consider x as a random variable $x : \Omega \to \mathscr{C}([-r,T],\mathbb{R}^d)$ in $L^2(\Omega, \mathscr{C}([-r,T],\mathbb{R}^d))$. In fact, we can look at x as

$$x : \Omega \to \mathscr{C}([-r,T],\mathbb{R}^d) \hookrightarrow L^2([-r,T],\mathbb{R}^d) \hookrightarrow \mathbb{R}^d \times L^2([-r,T],\mathbb{R}^d),$$

where $\hookrightarrow$ is a notation for continuous embeddings.

Further we will denote $u \in [0,T]$ as $M_2([-r,u],\mathbb{R}^d) := \mathbb{R}^d \times L^2([-r,T],\mathbb{R}^d)$ such that the Delfour-Mitter space is endowed with the norm given below:

$$\|(v,\theta)\|_{M_2} = (|v|^2 + \|\theta\|_2^2)^{\frac{1}{2}}, \quad (v,\theta) \in M_2([-r,u],\mathbb{R}^d). \tag{5.3}$$

Here $\|\cdot\|_2$ is the L^2-norm and $|\cdot|$ the Euclidean norm in $\mathbb{R}^d$. For simplicity, let $M_2 := M_2([-r,0],\mathbb{R}^d)$. This space endowed with the L^2-norm (5.3) has a Hilbert space structure which allows for a Fourier representation of its elements. On the other hand, the point 0 plays an important role, as we will see later on. Therefore it is needed to distinguish between two processes in $L^2([-r,0],\mathbb{R}^d)$, that have different images at the point 0.

Generally the spaces $M_2([-r,u],\mathbb{R}^d)$ are also natural to use since they agree with the corresponding spaces of continuous functions $\mathscr{C}([-r,u],\mathbb{R}^d)$ completed with respect to the L^2-norm given above by (5.3). In fact we can take the natural injection $i(\varphi(\cdot)) = (\varphi(u), \varphi(\cdot)1_{[-r,u)})$ for a $\varphi \in \mathscr{C}([-r,u],\mathbb{R}^d)$ and closing it.

The above mentioned results, lead to the random process $x : \Omega \times [-r,u] \to \mathbb{R}^d$ as a random variable.

$$x : \Omega \to M_2([-r,u],\mathbb{R}^d)$$

in $L^2(\Omega, M_2([-r,u],\mathbb{R}^d))$,
such as

$$\|X\|_{L^2(\Omega,M_2([-r,u],\mathbb{R}^d))} = \Big(\int_\Omega \|X(\omega)\|^2_{M_2([-r,u],\mathbb{R}^d)} P(d\omega)\Big)^{\frac{1}{2}} < \infty.$$

As mentioned earlier, to deal with memory and delay we may use the segment of x. Then we have a process x, some delay gap $r > 0$, and a specified time $t \in [0,T]$. Further the segment of x in the past time interval $[t-r,t]$ is denoted by $x_t(\omega,\cdot) : [-r,0] \to \mathbb{R}^d$ and can be defined as

$$x_t(\omega,s) = x(\omega,t+s), \qquad s \in [-r,0].$$

Here the segment x_0 for time $t = 0$ is the initial path, and contains all the information about the process from before $t = 0$. In general $x_t(\omega, \cdot)$ is the segment of the ω-trajectory of the process x, and covers all the information of the past down to time $t - r$.

$x_t(\omega)$ can be seen as an element of $L^2([-r, 0], \mathbb{R}^d)$ for $t \in [0, T]$, by assuming $\omega \in \Omega$ and $x(\cdot, \omega) \in L^2([-r, T], \mathbb{R}^d)$. Note the couple $(x(t), x_t)$ is a $\{\mathcal{F}_t\}$-measurable random variable with values in M_2, given $w \in \Omega$.

Considering an $\mathcal{F}_0$-measurable random variable $\eta \in L^2(\Omega, M_2)$. Again for simplicity we denote $\mathbb{M}_2 := L^2(\Omega, M_2)$. As above (5.1), a stochastic functional differential equation (SFDE), can be written as

$$\left.\begin{aligned} dx(t) &= f(t, x(t), x_t)dt + g(t, x(t), x_t)dW(t), \quad t \in [0, T] \\ (x(0), x_0) &= \eta \in \mathbb{M}_2, \end{aligned}\right\} \tag{5.4}$$

where the functionals $f : [0, T] \times M_2 \to \mathbb{R}^d$ and $g : [0, T] \times M_2 \to L(\mathbb{R}^m, \mathbb{R}^d)$.

Here with appropriate hypotheses on the functionals f and g, we may get existence and uniqueness of the strong solution (in the sense of L^2) of the SFDE (5.4) given above. The solution is a process $x \in L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$ admitting an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted modification, such that $x \in L^2_A(\Omega, M_2([-r, T], \mathbb{R}^d))$ for $L^2_A(\Omega, M_2([-r, T], \mathbb{R}^d))$ being the subspace of $L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$.

Two processes $x^1, x^2 \in L^2(\Omega, M_2([-r, T], \mathbb{R}^d))$ are unique in the L^2-sense, if

$$\|x_1 - x_2\|_{L^2(\Omega, M_2([-r,T],\mathbb{R}^d))} = 0.$$

Let us give the suitable hypotheses of existence and uniqueness in our case.

Hypotheses: Existence and Uniqueness (EU):

(EU1) (Local Lipschitzianity). The drift and the diffusion functionals f and g are Lipschitz on bounded sets in the second variable uniformly with respect to the first, i.e., for each integer $n \geq 0$, there is a Lipschitz constant L_n independent of $t \in [0, T]$ such that,

$$|f(t, \varphi_1) - f(t, \varphi_2)|_{\mathbb{R}^d} + \|g(t, \varphi_1) - g(t, \varphi_2)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leq L_n \|\varphi_1 - \varphi_2\|_{M_2}$$

for all $t \in [0, T]$ and functions $\varphi_1, \varphi_2 \in M_2$ such that

$$\|\varphi_1\|_{M_2}, \|\varphi_2\|_{M_2} \leq n.$$

(EU2) (Linear growths). There exists a constant $C > 0$ such that,

$$|f(t, \Psi)|_{\mathbb{R}^d} + \|g(t, \Psi)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leq C(1 + \|\Psi\|_{M_2})$$

for all $t \in [0, T]$ and $\Psi \in M_2$.

Existence and uniqueness of solutions of SFDE's have also been studied by Mohammed in [13].

Theorem 5.2.1. (Existence and Uniqueness).
Given Hypotheses **(EU)** on the coefficients f and g and the initial condition $\eta \in M_2$, the SFDE (5.4) has a solution ${}^{\eta}x \in L^2_A(\Omega, M_2([-r,T], \mathbb{R}^d))$ which is unique in the sense of L^2. The solution is a process ${}^{\eta}x : \Omega \times [-r,T] \to \mathbb{R}^d$ such that

i) ${}^{\eta}x(t) = \eta(t), \qquad t \in [-r, 0]$.

ii) ${}^{\eta}x(\omega) \in M_2([-r,T], \mathbb{R}^d) \qquad \omega$-a.s.

iii) For every $t \in [0,T]$, ${}^{\eta}x(t) : \Omega \to \mathbb{R}^d$ is $\{\mathcal{F}_t\}$-measurable.

From this it makes sense to write the solution as

$$
{}^{\eta}x(t) = \begin{cases} \eta(0) + \int_0^t f(u, {}^{\eta}x(u), {}^{\eta}x_u,)du + \int_0^t g(u, {}^{\eta}x(u), {}^{\eta}x_u,)dW(u), & t \in [0,T] \\ \eta(t), \ t \in [-r, 0]. \end{cases}
$$

The integrals here are well defined and the process below, belongs to $\mathbb{M}_2$ and is adapted.

$$
(\omega, t) \mapsto ({}^{\eta}x(t,\omega), {}^{\eta}x(\omega)).
$$

Here ${}^{\eta}x$ represents the solution starting at time 0 with initial condition $\eta \in \mathbb{M}_2$. Since x is continuous with respect to time and adapted, its composition with the deterministic coefficients f and g is adapted as well.

If we for a change consider the same dynamics but starting at a later time, $s \in (0,T]$ with initial condition $\eta \in \mathbb{M}_2$:

$$
\left.\begin{aligned} dx(t) &= f(t, x(t), x_t)dt + g(t, x(t), x_t)dW(t), \quad t \in [s,T] \\ (x(t)) &= \eta(t-s), \qquad t \in [s-r, s], \end{aligned}\right\} \tag{5.5}
$$

and again considering under the hypotheses (**EU**) the SFDE (5.5) has the following solution

$$
{}^{\eta}x^s(t) = \begin{cases} \eta(0) + \int_s^t f(u, \ {}^{\eta}x^s(u), \ {}^{\eta}x^s_u)du + \int_s^t g(u, \ {}^{\eta}x^s(u), \ {}^{\eta}x^s_u)dW(u), & t \in [s,T] \\ \eta(t-s), t \in [s-r, s]. \end{cases}
$$

The right-hand side superindex in ${}^{\eta}x^s$ denotes the starting time. In relation to the solution to (5.5) starting at any time s, let us introduce the following notation for later use

$$
X^s_t(\eta, \omega) := X(s,t,\eta,\omega) := ({}^{\eta}x^s(t,\omega), {}^{\eta}x^s_t(\omega)), \qquad \omega \in \Omega, s \le t. \tag{5.6}
$$

The evaluation operator is defined as:

$$
\rho_0 : M_2 \to \mathbb{R}^d, \qquad \rho_0\varphi := v \text{ for any } \varphi = (v, \theta) \in M_2.
$$

From this, we observe that the random variable ${}^{\eta}x^s(t)$ is an evaluation at 0 of the process $X^s_t(\eta),\ t \in [s,T]$.

Our main aim in this chapter is to study the influence of the initial path η on the functionals of the solution associated with equation (5.4). For this we need to guarantee that there exists a differentiable stochastic flow for (5.4).

Suppose we have two Banach spaces E and F, letting $U \subseteq E$ be an open set and $k \in \mathbb{N}$. Then we write $L^k(E,F)$ for the space of continuous k-multilinear operators $A : E^k \to F$ endowed with the uniform norm

$$\|A\|_{L^k(E,F)} := \text{sub}\{\|A(\nu_1,\dots,\nu_k)\|_F,\ \|\nu_i\|_E \le 1,\ i = 1,\dots,k\}.$$

From this, an operator $f : U \to F$ is said to be of class $\mathscr{C}^{k,\delta}$ if it is C^k and $D^k f : U \to L^k(E,F)$ is δ - Hölder continuous on bounded sets in U, see Definition A.1.4. Moreover, $f : U \to F$ is said to be of class $\mathscr{C}^{k,\delta}_b$ if it is $C^k, D^k f : U \to L^k(E,F)$ is δ -Hölder continuous on U, and all its derivatives $D^j f, 1 \le j \le k$ are globally bounded on U. The derivative D is taken in the Fréchet sense.

First, we consider the special case of SFDEs when g is actually a function

$$g(t,(\varphi(0),\varphi(\cdot))) = g(t,\varphi(0)), \qquad \varphi = (\varphi(0),\varphi(\cdot)) \in \mathbb{M}_2$$

such that g is $[0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$.

Let us give the following definition of a stochastic flow for the sake of completeness.

Definition 5.2.2. (Stochastic flow).
Denote $S([0,T]) := \{s,t \in [0,T] : 0 \le s < t < T\}$, and let E be a Banach space. A stochastic $\mathscr{C}^{k,\delta}$ -semiflow on E is a measurable mapping

$$X : S([0,T]) \times E \times \Omega \to E$$

satisfying the following properties:

(i) For each $\omega \in \Omega$, the map $X(\cdot,\cdot,\cdot,\omega) : S([0,T]) \times E \to E$ is continuous.

(ii) For fixed $(s,t,\omega) \in S([0,T]) \times \Omega$ the map $X(s,t,\cdot,\omega) : E \to E$ is $\mathscr{C}^{k,\delta}$.

(iii) For $0 \le s \le u \le t,\ \omega \in \Omega$ and $x \in E$, the property

$$X(s,t,\eta,\omega) = X(u,t,X(s,u,\eta,\omega),\omega)$$

holds.

(iv) For all $(t,\eta,\omega) \in [0,T] \times E \times \Omega$, one has $X(t,t,\eta,\omega) = \eta$.

Further in the sequel, we consider the space $E = M_2$.

Hypotheses (FlowS):

(FlowS1) The function $f : [0,T] \times M_2 \to \mathbb{R}^d$ is jointly continuous; the map $M_2 \ni \varphi \mapsto f(t,\varphi)$ is Lipschitz on bounded sets in M_2 and $\mathscr{C}^{1,\delta}$ uniformly in t (i.e. the δ -Hölder constant is uniformly bounded in $t \in [0,T]$) for some $\delta \in (0,1]$.

(FlowS2) The function $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is jointly continuous; the map $\mathbb{R}^d \ni v \mapsto g(t,v)$ is $\mathscr{C}_b^{2,\delta}$ uniformly in t.

(FlowS3) One of the following conditions is satisfied:

(a) There exist $C > 0$ and $\gamma \in [0,1)$ such that

$$|f(t,\varphi)| \le C(1 + \|\varphi\|_{M_2}^{\gamma})$$

$\forall\, t \in [0,T]$ and all $\varphi \in M_2$.

(b) For all $t \in [0,T]$ and $\varphi \in M_2$, one has $f(t,\varphi,\omega) = f(t,\varphi(0),\omega)$. Moreover, it exists $r_0 \in (0,r)$ such that

$$f(t,\varphi,\omega) = f(t,\widetilde{\varphi},\omega)$$

$\forall\, t \in [0,T]$ and all $\widetilde{\varphi}$ such that $\varphi(\cdot)1_{[-r,-r_0]}(\cdot) = \widetilde{\varphi}(\cdot)1_{[-r,-r_0]}(\cdot)$.

(c) For all $\omega \in \Omega$

$$\sup_{t\in[0,T]} \|(D\Psi(t,\nu,\omega))^{-1}\|_{M_2} < \infty,$$

where $\Psi(t,\nu)$ is defined by the stochastic differential equation

$$\begin{cases} d\Psi(t,\nu) = g(t,\Psi(t,\nu))dW(t) \\ \Psi(0,\nu) = \nu. \end{cases}$$

And, there exists a constant C such that

$$|f(t,\varphi)| \le C(1 + \|\varphi\|_{M_2})$$

$\forall\, t \in [0,T]$ and $\varphi \in M_2$.

Then we have the theorem below.

Theorem 5.2.3. Under Hypotheses **(EU)** and **(FlowS)**, $X_t^s(\eta,\omega)$ defined as in (5.6) is a $\mathscr{C}^{1,\epsilon}$-semiflow for every $\epsilon \in (0,\delta)$.

Hypotheses (Flow):

(Flow1) f satisfies **(FlowS1)** and there exists a constant C such that

$$|f(t,\varphi)| \le C(1 + \|\varphi\|_{M_2})$$

$\forall\, t \in [0,T]$ and $\varphi \in M_2$.

(Flow2) $g(t,\varphi)$ is of the following form

$$g(t,\varphi) = \bar{g}(t,\nu,\tilde{g}(\theta)), \qquad t \in [0,T], \qquad \varphi = (\nu,\theta) \in M_2$$

where $\bar{g}$ satisfies the following conditions:

(a) The function $\bar{g}[0,T] \times \mathbb{R}^{d+k} \to \mathbb{R}^{d\times m}$ is jointly continuous; the map $\mathbb{R}^{d+k} \ni y \mapsto \bar{g}(t,y)$ is $\mathscr{C}_b^{2,\delta}$ uniformly in t.

(b) For each $\nu \in \mathbb{R}^{d+k}$, let $\{\Psi(t,\nu)\}_{t\in[0,T]}$ solve the stochastic differential equation

$$\Psi(t,\nu) = \nu + \begin{pmatrix} \int_0^t \bar{g}(s,\Psi(s,\nu))dW(s) \\ 0 \end{pmatrix},$$

with null-vector in $\mathbb{R}^k$. Then $\Psi(t,\nu)$ is Fréchet differentiable with respect to ν and the Jacobi-matrix $D\Psi(t,\nu)$ is invertible and fulfills, for all $\omega \in \Omega$,

$$\sup_{\substack{t\in[0,T] \\ \nu\in\mathbb{R}^{d+k}}} \|D\Psi^{-1}(t,\nu,\omega)\| < \infty,$$

where $\|\cdot\|$ denotes any matrix norm, and $\tilde{g} : L^2([-r,0],\mathbb{R}^d) \to \mathbb{R}^k$ satisfies the following conditions:

(c) It exists a jointly continuous function $h : [0,T] \times M_2 \to \mathbb{R}^k$ such that for each $\widetilde{\varphi} \in L^2([-r,T],\mathbb{R}^d)$

$$\tilde{g}(\widetilde{\varphi}_t) = \tilde{g}(\widetilde{\varphi}_0) + \int_0^t h(s,(\widetilde{\varphi}(s),\widetilde{\varphi}_s))ds,$$

where $\widetilde{\varphi}_t \in L^2([-r,0],\mathbb{R}^d)$ is the segment at t of a representative of $\widetilde{\varphi}$.

(d) $M_2 \ni \varphi \mapsto h(t,\varphi)$ is Lipschitz on bounded sets in M_2, uniformly with respect to $t \in [0,T]$ and $\mathscr{C}^{1,\delta}$ uniformly in t.

Corollary 5.2.4. Under Hypotheses **(Flow)**, the solution $X_t^s(\eta) = X(s,t,\eta,\omega)$, $\omega \in \Omega$, $t \geq s$ to (5.5) is a $\mathscr{C}^{1,\epsilon}$ -semiflow for every $\epsilon \in (0,\delta)$. In particular, $\varphi \mapsto X(s,t,\varphi,\omega)$ is C^1 in the Fréchet sense.

5.3 Sensitivity analysis to the initial path condition

In this section we consider a stochastic process x satisfying the dynamics (5.4), and the coefficients f and g satisfy the conditions **(EU)** and **(Flow)**. Then we are able to achieve our goal, namely the computation of the sensitivity of evaluations of the form:

$$p(\eta) = E[\Phi(X_T^0(\eta))] = E[\Phi(^\eta x(T), ^\eta x_T)], \qquad \eta \in \mathbb{M}_2 \tag{5.7}$$

with the initial path $^\eta x$ in the model. Here, $\Phi : M_2 \to \mathbb{R}$ is such that $\Phi(X_T^0(^\eta)) \in L^2(\Omega,\mathbb{R})$. Moreover, the sensitivity will be interpreted as the directional derivative, that is

$$\partial_h p(\eta) := \frac{d}{d\epsilon} p(\eta + \epsilon h)|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{p(\eta+\epsilon h) - p(\eta)}{\epsilon}, \qquad h \in M_2. \tag{5.8}$$

As we introduced, our final goal is to give a representation of $\partial_h p(\eta)$ in which the function Φ is not directly differentiated. This can be achieved by representing the sensitivity parameter Delta by means of weights. See, e.g. the Malliavin weight introduced in [8] where the classical case without memory is discussed. In our case there is a need to impose some stronger regularity conditions on f and g, through the following hypotheses (H):

Hypotheses (H:)

(H1) (Global Lipschitzianity) $\varphi \mapsto f(t,\varphi), \varphi \mapsto g(t,\varphi)$ globally Lipschitz uniformly in t with Lipschitz constants L_f and L_g, i.e.

$$|f(t,\varphi_1) - f(t,\varphi_2)|_{\mathbb{R}^d} \leq L_f \|\varphi_1 - \varphi_2\|_{M_2} \tag{5.9}$$

$$|g(t,\varphi_1) - g(t,\varphi_2)|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq L_g \|\varphi_1 - \varphi_2\|_{M_2} \tag{5.10}$$

$\forall\ t \in [0,T]$ and $\varphi_1, \varphi_2 \in M_2$.

(H2) (Lipschitzianity of the Fréchet derivatives) $\varphi \mapsto Df(t,\varphi), \varphi \mapsto Dg(t,\varphi)$ are globally Lipschitz uniformly in t with Lipschitz constants L_{Df} and L_{Dg}, i.e.

$$\|Df(t,\varphi_1) - Df(t,\varphi_2)\| \leq L_{Df} \|\varphi_1 - \varphi_2\|_{M_2} \tag{5.11}$$

$$\|Dg(t,\varphi_1) - Dg(t,\varphi_2)\| \leq L_{Dg} \|\varphi_1 - \varphi_2\|_{M_2} \tag{5.12}$$

$\forall\ t \in [0,T]$, $\varphi_1, \varphi_2 \in M_2$ and the stochastic $\mathscr{C}^{1,1}$-semiflow is again denoted by X.

Since we want to study the directional derivative $\partial_h p(\eta)$ in (5.8). One possible approach here is to randomizing the initial condition η, and find a relationship between the Fréchet derivative $DX_T^0(\eta)$ and the Malliavin derivative of the X_T^0. Where $DX_T^0(\eta)$ is applied to a direction $h \in \mathbb{M}_2$ and X_T^0 with the randomized starting condition.

5.3.1 Randomization of the initial condition and the Malliavin derivative

We have the m-dimensional Wiener process W that drives the SFDE in (5.4), defined on the probability space $(\Omega^W, \mathcal{F}^W, P^W)$. Then, let us define an isonormal Gaussian process $\mathbb{B}$ on $L^2([-r,0],\mathbb{R})$ and probability space $(\Omega^{\mathbb{B}}, \mathcal{F}^{\mathbb{B}}, P^{\mathbb{B}})$. Moreover, W and $\mathbb{B}$ are independent such that $(\Omega, \mathcal{F}, P) = (\Omega^W \times \Omega^{\mathbb{B}}, \mathcal{F}^W \otimes \mathcal{F}^{\mathbb{B}}, P^W \otimes P^{\mathbb{B}})$, and we define $\Omega = \Omega^W \times \Omega^{\mathbb{B}}$.

From now on we shall work with, the Malliavin and Skorohod calculus with respect to the isonormal Gaussian process $\mathbb{B}$. In fact, for $\mathbb{B}$ we define the Malliavin derivative operator $\mathscr{D}$ and the Skorohod integral operator δ. For immediate use, let us give the following Lemma. This gives us the link between the Malliavin derivative of a segment and vice versa.

Lemma 5.3.2. If $X_t^0(\eta) = ({}^\eta x(t), {}^\eta x_t) \in \mathbb{M}_2$ is Malliavin differentiable for all $t \geq 0$, then, for all $s \geq 0$,

$$\mathscr{D}_s {}^\eta x_t = \{\mathscr{D}_s {}^\eta x(t+u), u \in [-r, 0]\}$$

and

$$\mathscr{D}_s X_t^0(\eta) = (\mathscr{D}_s {}^\eta x(t), \mathscr{D}_s {}^\eta x(t+\cdot)) \in \mathbb{M}_2.$$

To study the relationship between the Malliavin derivatives and Fréchet derivatives, we need to consider the chain rule for the Malliavin derivative in $\mathbb{M}_2$.

If DX_T^0 is bounded, i.e for all $\omega = (\omega^W, \omega^{\mathbb{B}}) \in \Omega$ and

$$\sup_{\eta \in \mathbb{M}_2} \|DX_T^0(\eta(\omega), \omega^W)\| < \infty$$

, then the chain rule given above (3.20) gives us that

$$\mathscr{D}_s X_T^0(\eta(\omega^W, \omega^{\mathbb{B}}), \omega^W) = DX_T^0(\eta(\omega^W, \omega^{\mathbb{B}}), \omega^W)[\mathscr{D}_s \eta(\omega^W, \omega^{\mathbb{B}})],$$

here the Malliavin derivative only acts on $\omega^{\mathbb{B}}$.

What about when DX_T^0 is unbounded? This can be handled by applying $\mathscr{D}_s$ directly to the dynamics given by the equation (5.4), and then we get the following theorem.

Theorem 5.3.3. Have that Hypotheses **(EU)**, **(Flow)** and **(H)** are fulfilled. Let $X_\cdot^0(\eta) \in L^2(\Omega; \mathbb{M}_2([-r, T], \mathbb{R}^d))$ be the solution of (5.4), then

$$\mathscr{D}_s X_T^0(\eta) = DX_T^0(\eta)[\mathscr{D}_s \eta] \qquad (w, s)\text{- almost everywhere.} \tag{5.13}$$

To consider the randomization of the initial condition, we need to define ξ. In this case ξ is an $\mathbb{R}$-valued functional of $\mathbb{B}$, non-zero P-a.s. More precisely, ξ is a random variable independent of W, and we choose it to be Malliavin differentiable with respect to $\mathbb{B}$ with $\mathscr{D}_s \xi \neq 0$ for almost all (ω, s). With this we let η to be the *true* (i.e. not randomized) initial condition, and h the direction. Then $\eta, h \in \mathbb{M}_2$ are random variables on Ω^W, such that $\eta(w) = \eta(\omega^W), h(\omega) = h(\omega^W)$. Further we denote $\eta, h \in \mathbb{M}_2(\omega^W)$, where $\mathbb{M}_2(\omega^W)$ is the space of the random variables in $\mathbb{M}_2$ only depending on $\omega^W \in \Omega^W$). For simplicity we have $\widetilde{\eta} := \eta - h$, and later h will be differentiated.

Corollary 5.3.4. Let Hypotheses **(EU)**, **(Flow)** and **(H)** be fulfilled. Let $X_\cdot^0(\widetilde{\eta} + \lambda\xi\mu) \in L^2(\Omega; M_2([-r, T], \mathbb{R}^d))$ be the solution of (5.4) with initial condition $\widetilde{\eta} + \lambda\xi\mu \in \mathbb{M}_2$, where $\lambda \in \mathbb{R}$. Then we have

$$\mathscr{D}_s X_T^0(\widetilde{\eta}(\omega^W) + \lambda\xi(\omega^{\mathbb{B}})h(\omega^W)) = DX_T^0(\widetilde{\eta}(\omega^W) + \lambda\xi(\omega^{\mathbb{B}})h(\omega^W))[\lambda\mathscr{D}_s\xi(\omega^{\mathbb{B}})h(\omega^W)], \tag{5.14}$$

(ω,s)-almost everywhere, and we have the following short hand notation:

$$\mathscr{D}_s X_T^0(\tilde{\eta} + \lambda\xi\mu) = DX_T^0(\tilde{\eta} + \lambda\xi\mu)[\lambda\mathscr{D}_s\xi h]. \tag{5.15}$$

Let us now give a derivative free representation of the expectation of the Fréchet derivative of $\Phi \circ X_T^0$ at η in direction h in terms of a Skorohod integral, δ. Later, we will use this to get a representation for the derivative of $p(\eta)$ in direction h.

Theorem 5.3.5. Let Hypotheses **(EU)**, **(Flow)** and **(H)** be satisfied and let Φ be Fréchet differentiable. Furthermore, let $a \in L^2([-r, 0], \mathbb{R})$ be such that $\int_{-r}^0 a(s)ds = 1$. If $a(\cdot)\frac{\xi}{\mathscr{D}.\xi}$ is Skorohod integrable and if the Skorohod integral below with the evaluation at $\lambda = \frac{1}{\xi} \in \mathbb{R}$ are well defined, then following relation holds

$$E[D(\Phi \circ X_T^0)(\eta)[h]] = -E\Big[\Big\{\delta\Big(\Phi(X_T^0(\tilde{\eta} + \lambda\xi\mu))a(\cdot)\frac{\xi}{\mathscr{D}.\xi}\Big)\Big\}|_{\lambda=\frac{1}{\xi}}\Big]. \tag{5.16}$$

Proof. Given in A.3. □

Further, let us look at the representation formula for Delta under a suitable choice of the randomization. Since $\mathscr{D}_s\xi = \xi$ for all $s \in [-r, 0]$, an interesting choice of randomization will be $\xi = \exp\{\mathbb{B}(1_{[-r,0]})\}$ where

$$\begin{aligned}\|\delta(u(\cdot, \lambda_1)) - \delta(u(\cdot, \lambda_2))\|^2_{L^2(\Omega)}& \\ \le \|a\|^2_{L^2([-r,0])}(\|\Phi(X_T^0(\tilde{\eta} + \lambda_1\xi\mu)) - \Phi(X_T^0(\tilde{\eta} + \lambda_2\xi\mu))\|^2_{L^2(\Omega)}& \qquad (5.17)\\ + \|\mathscr{D}\{\Phi(X_T^0(\tilde{\eta} + \lambda_1\xi\mu)) - \Phi(X_T^0(\tilde{\eta} + \lambda_2\xi\mu))\}\|^2_{L^2(\Omega\times[-r,0])}).&\end{aligned}$$

then let the following hypotheses be fulfilled for equation (5.17)

Hypotheses (A): Assuming the Fréchet differentiable Φ and its derivative $D\Phi$ to be globally Lipschitz with Lipschitz constant L_Φ, C^1 and $L_{D\Phi}$ respectively.

Lemma 5.3.6. Under Hypotheses **(EU), (Flow), (H)** and **(A)**, we have the sensitivity to the initial path, Delta in direction $h \in M_2$ to be

$$\partial_h p(\eta) = E[D(\Phi \circ X_T^0)(\eta)[h]]. \tag{5.18}$$

Then with this result, we are finally able to give a derivative free representation formula for the directional derivatives of $p(\eta)$.

Theorem 5.3.7 (Representation formula)**.** Let Hypotheses **(EU), (Flow), (H)** and **(A)** be fulfilled. Let $a \in L^2([-r, 0], \mathbb{R})$ be such that $\int_{-r}^0 a(s)ds = 1$ and let $\xi = \exp\{\mathbb{B}(1_{[-r,0]})\}$. Then the directional derivatives of p have representation:

$$\partial_h p(\eta) = -E\Big[\Big\{\delta\Big(\Phi(X_T^0(\tilde{\eta} + \lambda\xi\mu))a(\cdot)\Big)\Big\}\Big|_{\lambda=\frac{1}{\xi}}\Big]. \tag{5.19}$$

Proof. Given in [2], Appendix. □

Chapter 6

Application

By combining our central theories and findings in the previous chapters, we aim at developing in this chapter a new numerical method for the approximation of the Delta in Theorem 5.3.7.

6.1 Simulation of the representation formula for the Delta

In chapter 4, we discussed the SDDE in the connection with financial application. More precisely we considered the labor income following SFDE's with delay. Here in this section we want to simulate the representation formula in connection with the model for labor income, capturing slow adjustment of labor income to market shooks.

Then we have the following linear stochastic delay equation:

$$X^0(t) = \eta(0) + \int_0^t \Big(\int_{-r}^0 X^0(u+s)\phi(u)du + X^0(s) \Big) ds + W(t). \tag{6.1}$$

Here $(W(t))_{0\leq t\leq T}$ is a one-dimensional Brownian motion and $\phi \in L^2([-r,0])$ In order to solve this stochastic delay equation (6.1), we may resort to techniques from spectral theory, see e.g [5].

In the view of simulation, we assume that ϕ in (6.1) is equal to zero, then we have the following labor income model

$$X^0(t) = \eta(0) + \int_0^t x^0(s)ds + W(t). \tag{6.2}$$

By using Itô's formula in Theorem 2.2.14, we get the following solution of (6.2)

$$x^0(t) = \begin{cases} e^t\eta(0) + \int_0^t e^{-(s-t)}dW(s), & 0 \leq t \leq T \\ \eta(t), \qquad -r \leq t \leq 0. \end{cases}$$

Further we denote $\mathbb{L}^{1,2}$ to be the Hilbert space of processes $u \in L^2([-r,0]\times\Omega)$, such that $u(t)$ is Malliavin differentiable for almost all t, and there exists a measurable version of the two-parameter process $D_s u_t$, such that

$$E\Big[\int_{-r}^{0}\int_{-r}^{0}(D_s u_t)^2 ds dt\Big] < \infty. \tag{6.3}$$

Moreover, in order to approximate the representation for the Delta in Theorem 5.3.7 we need the following auxiliary result from [14]:

Theorem 6.1.1. Let $u \in \mathbb{L}^{1,2}$ and $\pi = \{-r = t_0 < t_1 < \cdots < t_n = 0\}$ denoted by an arbitrary partition of the interval $[-r,0]$, with mesh $|\pi| := \sup_{0\le i\le n-1}|t_{i+1}-t_i|$, and we set

$$Z_\pi = \sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\Big(\int_{t_i}^{t_{i+1}} E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^C}]ds\Big)(B_{t_{i+1}}-B_{t_i}), \tag{6.4}$$

where $\mathcal{F}_{[t_i,t_{i+1}]^C}$ is the σ-algebra generated by $B_t - B_s$, here the interval $(s,t]$ is disjoint with $[t_i,t_{i+1}]$. Then

$$Z_\pi \underset{|\pi|\to 0}{\longrightarrow} \delta(u) \qquad \text{in } L^2(\Omega).$$

We can now apply Theorem 6.1.1 to Theorem 5.3.7 as follows if:

$$u_t := \Phi(X_T^0(\widetilde{\eta}+\lambda\xi h))a(t) \in \mathbb{L}^{1,2},$$

then

$$\delta(u) \approx Z_\pi \qquad \text{for small } |\pi|$$

where we choose $\widetilde{\eta} = \eta - h$.

In our situation, we may e.g. choose the payoff functions:

$$\begin{aligned}
&\Phi : M_2 \to \mathbb{R} \text{ as}\\
&\Phi(g) = \int_{-r}^{0} g(u)du \text{ or,} &(6.5)\\
&\Phi(g) = \exp\Big(-\Big(\int_{-r}^{0} g(u)du\Big)^2\Big). &(6.6)
\end{aligned}$$

Further let us consider a simple case by assuming that,
$\widetilde{\eta} \equiv 1, h \equiv 1$ and $a(t) \equiv \frac{1}{r}$ on $[-r,0]$.
So if $T > r$ and Φ is given as (6.5) then

$$\begin{aligned}
u_t &= \Phi(X_T^0(\widetilde{\eta}+\lambda\xi h))a(t)\\
&= \int_{-r}^{0}\Big(\lambda\xi e^{T+u} + \int_0^{T+u} e^{-(s-(T+u))}dW(s)\Big)du\frac{1}{r}, &(6.7)
\end{aligned}$$

where $\xi = e^{-B_{-r}}$.

Hence,

$$E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^C}] = \int_{-r}^{0} \Big(\lambda e^{T+u} E[\xi|\mathcal{F}_{[t_i,t_{i+1}]^C}] + E\Big[\int_0^{T+u} e^{-(s-(T+u))} dW(s)|\mathcal{F}_{[t_i,t_{i+1}]^C}\Big]\Big) du \frac{1}{r}.$$

Since the conditional expectation above is only defined on the probability space with respect to B, we can treat the stochastic integral with respect to W as a constant and we get that

$$E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^C}] = \int_{-r}^{0} \left\{ \Big(\lambda e^{T+u} E[\xi|\mathcal{F}_{[t_i,t_{i+1}]^C}]\Big) \frac{1}{r} + \int_0^{T+u} e^{-(s-(T+u))} dW(s) \frac{1}{r} \right\} du.$$

On the other hand, we have that

$$\xi = e^{B_{t_{i+1}} - B_{t_i}} e^{B_{t_i} - B_{-r}} e^{-B_{t_{i+1}}}.$$

So

$$E[\xi|\mathcal{F}_{[t_i,t_{i+1}]^C}] = E[e^{B_{t_{i+1}} - B_{t_i}} |\mathcal{F}_{[t_i,t_{i+1}]^C}] e^{B_{t_i} - B_{-r}} e^{-B_{t_{i+1}}}.$$

Hence because of independence, we get that

$$E[\xi|\mathcal{F}_{[t_i,t_{i+1}]^C}] = e^{-\frac{1}{2}(t_{i+1}-t_i)^2} e^{B_{t_i} - B_{-r}} e^{-B_{t_{i+1}}}.$$

Thus

$$E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^C}] = \lambda e^{-\frac{1}{2}(t_{i+1}-t_i)^2} e^{B_{t_i} - B_{-r}} e^{-B_{t_{i+1}}} e^T (1 - e^{-r}) \frac{1}{r} + \int_{-r}^{0} \Big(\int_0^{T+u} e^{-(s-(T+u))} dW(s) \frac{1}{r} \Big) du.$$

Then Theorem 6.1.1 yields that

$$\begin{aligned} \delta(u) \approx \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \Big(\int_{t_i}^{t_{i+1}} & \lambda e^{-\frac{1}{2}(t_{i+1}-t_i)^2} e^{B_{t_i} - B_{-r}} e^{-B_{t_{i+1}}} e^T (1 - e^{-r}) \frac{1}{r} \\ & + \int_{-r}^{0} \Big(\int_0^{T+u} e^{-(s-(T+u))} dW(s) \frac{1}{r} \Big) du \, ds \Big) (B_{t_{i+1}} - B_{t_i}) \\ & =: Z_\pi. \end{aligned} \tag{6.8}$$

Altogether, we obtain from the representation formula, Theorem 5.3.7 that

$$\partial_h P(\eta) \approx -E[Z_\pi(\lambda)|_{\lambda = e^{B_{-r}}}], \tag{6.9}$$

provided that $|\pi|$ is small enough.

Our final step is to use the Monte Carlo method, to approximate the sensitivity to the initial path (6.9)

$$\partial_h P(\eta) \approx -\frac{1}{N}\sum_{i=1}^{N} Y_i,$$

where $Y_i, i = 1, \ldots, N$ are i.i.d. copies of the random variable

$$Y := Z_\pi(\lambda)\big|_{\lambda=e^{B-r}}.$$

6.1.2 Simulation procedure

Before simulating the sensitivity, we need to simulate paths of Brownian motion B. In the following we explain this procedure in more detail.

Step 1. We simulate e.g. 20 paths of the Brownian motion as in algorithm 2.1.

Step 2. Approximate $\delta(u)$ as in (6.8) for e.g. $r = 30$ (years) and $\lambda = e^{B_{-30}}$, with the simulated paths in Step 1.

Step 3. We approximate $\partial_h P(\eta)$, by Monte Carlo techniques.

By following this procedure for the given parameters, Step 1. is presented in Figure 2.1, but now with 20 paths and then the sensitivity to the initial path is:

$$\partial_h P(\eta) \approx 4.54 \cdot 10^{13}.$$

In this type of simulation it may be interesting to vary the parameters. Here we try to vary r along with the number of paths. The table 6.1, below gives us that the sensitivity increases as r is increased, and by increasing the paths the sensitivity decreases and stabilize to a certain degree, for the representative r.

Delay/Paths	2 paths	20 paths	100 paths	100 000 paths
$r = 1$	0.013	0.017	0.002	0.001
$r = 2$	0.097	0.098	0.004	0.003
$r = 5$	3.359	4.271	0.177	0.249
$r = 10$	1727.722	1682.09	93.346	83.878
$r = 20$	285334116	294843028	10342662	10968885
$r = 30$	$4.70 \cdot 10^{13}$	$4.54 \cdot 10^{13}$	$2.10 \cdot 10^{12}$	$2.36 \cdot 10^{12}$
$r = 50$	$1.46 \cdot 10^{24}$	$1.27 \cdot 10^{24}$	$6.61 \cdot 10^{22}$	$6.01 \cdot 10^{22}$
$r = 80$	$7.54 \cdot 10^{39}$	$5.11 \cdot 10^{39}$	$2.62 \cdot 10^{38}$	$2.38 \cdot 10^{38}$

Table 6.1: *Approximation of* $\partial_h P(\eta)$, *for* $r = 1, 2, 5, 10, 20, 30, 50, 80$ *and number of paths* $= 2, 20, 100, 100\ 000$.

We mention here that one may also use other choices of the function $a(t)$ in connection with our simulations.

Principally with some more expenditure in connection with Theorem 6.1.1, we can also do the same approximation of the representation formula for the Delta in the case of the payoff Φ in (6.6). Indeed, in the case of (6.6) we get more precisely the following:

Using Fubini's theorem for stochastic integrals, we observe that

$$\begin{aligned}
&\int_{-r}^{0} \int_{0}^{T+u} e^{-(s-(T+u))} dW(s) \frac{1}{r} du \\
&= \int_{0}^{T} \int_{-r}^{0} \mathbb{1}_{[0,T+u]}^{(s)} e^{-(s-(T+u))} du dW(s) \\
&= H_T.
\end{aligned}$$

On the other hand, we have that

$$-\Big(\int_{-r}^{0}\lambda\xi e^{T+u}du+H_T\Big)^2=-\lambda^2\xi^2\Big(\int_{-r}^{0}e^{T+u}du\Big)^2-2\lambda\xi\int_{-r}^{0}e^{T+u}du\cdot H_T-(H_T)^2.$$

Hence

$$U_t=\exp\Big(-\lambda^2\xi^2\Big(\int_{-r}^{0}e^{T+u}du\Big)^2-2\lambda\xi\int_{-r}^{0}e^{T+u}du\cdot H_T-H_T^2\Big)\frac{1}{r}.$$

Then, using the independence of increments of B and the fact that we can treat H_T as a constant, we find that

$$E[u_s|\mathcal{F}_{[t_i,t_{i+1}]^C}]=\Big\{F_i(x,y)\Big|_{x=B_{t_i}-B_{-r},\ y=-B_{t_{i+1}}}\Big\}e^{-H_T^2}\frac{1}{r},$$

with

$$F_i(x,y):=E\Big[\exp\Big\{-e^{2(x+y)}e^{2(B_{t_{i+1}}-B_{t_i})}\lambda^2\Big(\int_{-r}^{0}e^{T+u}du\Big)^2-2e^{x+y}e^{B_{t_{i+1}}-B_{t_i}}\lambda\int_{-r}^{0}e^{T+u}du\cdot H_T\Big\}\Big].$$

So it follows from Theorem 6.1.1 that

$$\begin{aligned}\delta(u)&\approx\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\Big\{F_i(x,y)\Big|_{x=B_{t_i}-B_{-r},\ y=-B_{t_{i+1}}}\Big\}e^{-H_T^2}\ \frac{1}{r}ds\ (B_{t_{i+1}}-B_{t_i})\\&=Z_\pi(\lambda).\end{aligned}$$

As before we obtain from Theorem 5.3.7 that

$$\partial_h P(\eta)\approx -E[Z_\pi(\lambda)|_{\lambda=e^{B_{-r}}}]$$

for $|\pi|$ small enough.

Finally by applying the Monte Carlo method to the latter expectation (on the product probability space with respect to B and W) we can approximate $\partial_h P(\eta)$.

Remark 6.1.3. Clearly a disadvantage of our new method here for the approximation of the Delta representation in Theorem 5.3.7 is the lack of general error estimates with respect to the Skorohod integral δ in the literature.

Chapter 7

Conclusion and Discussion

7.1 Conclusion

One of the aims of the thesis was to develop a new method for the computation of the Delta of option prices with respect to a specific market model with memory by using a new "derivative-free" representation of price sensitivities (Bismut-Elworthy-Li formula), based on the Malliavin calculus [2]. In order to do so, we first discussed a new Bismut-Elworthy-Li formula. Further in Chapter 6 we introduced a novel numerical implementation approach with respect to the representation formula for Delta, and simulated the sensitivity in the case of specific claims based on a stochastic labor income model with memory. Here the income dynamics adjusts slowly to financial market shocks [3]. As we see from our specific simulation in table 6.1 the sensitivity seems to a certain degree stabilize, when we increase the number of paths.

7.2 Challenges

One of the challenges of my thesis I have been faced with, was the difficulty of simulations of solutions of stochastic functional differential equations. The literature treated in connection with such type of equations is rather scarce. Further, the numerical expenditure with respect to those equations is very high, in general. Therefore we confined ourselves in this thesis to simple models to explain the main principles of our simulation technique. Another challenge we were coping with in this thesis was the lack of error estimates with respect to our simulation approach, which are very difficult to obtain in general.

7.3 Further work

Clearly, there are several possibilities to extend this thesis. First of all we may consider different parameters, another function $a(t)$ or other payoff functions Φ.

The second extension could be in the direction of deriving general error estimates for our implementation method based on the Skorohod integral δ in Theorem 5.3.7.

The third extension would be to look at the case of models with discontinuous noise. This means that there are jumps in the behavior of what we are attempting to model. To deal with such a behavior, one could e.g. resort to driving noises given by a certain class of Lèvy processes.

Appendix A

Preliminaries - Probability Theory

A.1 Measure theory

A measure *measures* the size of sets. Not all sets can be measured in this sense, but on the other hand the class of sets which are measurable should be sufficiently rich. In particular we want to keep measurability of sets if we perform simple operations like taking the complement or taking (countable) unions and intersections. This leads to the following definition.

Definition A.1.1. (σ-algebra). Let Ω be a non-empty set and let $\mathcal{F}$ be a collection of subsets of Ω. Then $\mathcal{F}$ is called a σ-algebra over Ω, if the following holds:

a) $\Omega \in \mathcal{F}$.

b) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$. [1]

c) For every sequence $A_1, A_2, \dots$ of elements of $\mathcal{F}$ the union $\cup_{n=1}^{\infty} A_n$ is also a member (i.e. an element) of $\mathcal{F}$.

The pair $(\Omega, \mathcal{F})$, where Ω is a non-empty set and $\mathcal{F}$ is a σ-algebra on Ω, is called a measurable space.

Definition A.1.2. (Measure). Let Ω be a set and let $\mathcal{F}$ be a σ-algebra on Ω. A map $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if the following properties are satisfied:

(i) $\mu(\emptyset) = 0$

(ii) μ is σ-additive, i.e. for every sequence of pairwise disjoint sets $\mathcal{F}_n \in \mathcal{F}$ one has

$$\mu\Big(\bigcup_{n=0}^{\infty} \mathcal{F}_n\Big) = \sum_{n=0}^{\infty} \mu(\mathcal{F}_n).$$

[1] A^c denotes the complement of A in Ω.

Null-sets are very useful because for most purposes one can ignore everything that happens on them.

Definition A.1.3. (Null set). Let $(\Omega, \mathcal{F}, \mu\ )$ be a measure space. We call a set $A \subset \Omega$ null-set or, more precisely a μ-null-set if there is some $N \in \mathcal{F}$ with $\mu(N) = 0$ and $A \subset N$.

Definition A.1.4. (Hölder continuity). Let $f : (S^1, \rho^1) \to (S^2, \rho^2)$ be a function between two complete metric spaces S^1, S^2 with respective matrices ρ^1, ρ^2, is called Hölder continuous with exponent α iff.

$$\sup_{s \neq t} \left\{ \frac{(f(s), f(t))}{\rho^2(s,t)^\alpha} : s, t \in S^1, \rho^1(s,t) < \infty \right\} < \infty.$$

It is called locally Hölder continuity if and only if it is Hölder continuity on every bounded set.

A.2 Probability theory

Throughout this thesis we considered a probability space $(\Omega, \mathcal{F}, P)$ which is complete. Here:

- Ω is the sample space; the set of all outcomes of some random experiment.
- $\mathcal{F}$ is a σ-algebra on Ω, containing all events that might occur.
- P is a probability measure on $(\Omega, \mathcal{F})$, if P is a measure such that,
  - (i) $P : \mathcal{F} \to [0, 1]$ and
  - (ii) $P[\Omega] = 1$.

$(\Omega, \mathcal{F}, P)$ is a complete probability space if any subset of a P-null set is an event in $\mathcal{F}$.

Remark A.2.1. ($\mathcal{B}$-Borel set). Let $\mathcal{B}$ denote the Borel σ-algebra on Ω generated by $\mathcal{U}$, the collection of all open subsets of Ω. Then $B \in \mathcal{B}$ are called Borel sets.

Definition A.2.2. (Random variable). Let $(\Omega, \mathcal{F}, P)$ denote a given complete probability space. A random variable X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. Every random variable induces a probability measure μ_X on $\mathbb{R}^n$, defined by

$$\mu_X(B) = P(X^{-1}(B)),$$

here μ_X is called the distribution of X.

Definition A.2.3. (Conditional expectation with respect to $\mathcal{A}$).
Let X be a random variable such that $E[|X|] < \infty$ and $\mathcal{A} \sup \mathcal{F}$ a σ-algebra which in a financial context may represent the entirety of market information up to time $t < T$ maturity. Then the expected value of X given $\mathcal{A}$ is the unique random variable Y such that

$$E[\mathbb{1}_A \cdot X] = E[\mathbb{1}_A \cdot Y] \qquad \forall A \in \mathcal{A}.$$

Proposition A.2.4. (Properties of $E[X|\mathcal{A}]$).

(i) $E[\alpha X + \beta Y|\mathcal{A}] = \alpha E[X|\mathcal{A}] + \beta E[Y|\mathcal{A}]$ linearity.

(ii) $E[E[X|\mathcal{A}]] = E[X]$.

(iii) $E[X|\mathcal{A}] = X$, if X is a random variable on $(\Omega, \mathcal{A}, P)$.

(iv) $E[X|\mathcal{A}] = E[X]$, if X is independent of $\mathcal{A}$, i.e.
$P(\{X \leq t\} \cap A) = P(X \leq t) \cdot P(A) \forall t \in \mathbb{R}, A \in \mathcal{A}$.

(v) $E[X|\mathcal{A}] = E[E[X|\mathcal{B}]|\mathcal{A}]$, if $\mathcal{A} \subseteq \mathcal{B}$ is σ-algebra.

Definition A.2.5. (Stochastic process). A stochastic process is a parameterized collection of random variables

$$\{X_t\}_{t \in T},$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.

The parameter space T in the definition above can be a closed or half-open interval on the real line, i.e. $[a, b]$ or $[0, \infty)$, or even subsets of $\mathbb{R}^n$ for $n \geq 1$. For every fixed $t \in T$ we have a random variable

$$\omega \mapsto X_t(\omega), \qquad \omega \in \Omega.$$

Fixing $\omega \in \Omega$, however, gives the path of X_t

$$t \mapsto X_t(\omega), \qquad t \in T.$$

Hence, the parameter t is usually interpreted as time, and $X_t(\omega)$ as the position of a particle ω at a given time t. The author in [12] defines X_t on the product space $T \times \Omega$ and uses the notation $X(t, \omega)$. With this notation the process can be viewed as a function of two variables

$$X(t, \omega) : T \times \Omega \to \mathbb{R}^n.$$

This is often a convenient interpretation, since it is crucial in stochastic analysis $X(t, \omega)$ being jointly measurable in (t, ω) (Øksendal, p. 8,[16])

Definition A.2.6. (P-almost surely). Considering the family $\mathcal{N}$ as a collection of all "possible" events. For all events outside the family $\mathcal{N}$ of all P-null sets, then property holds P-almost surely (P-a.s.).

Definition A.2.7. (P-almost everywhere). We say a property Π holds P-almost everywhere (P-a.e.) if there exist a null set $N \in \mathcal{F}$ such that Π holds for all $\omega \in N^c = \Omega - N$.

The following Theorems are borrowed from [18].

Definition A.2.8. (Gaussian Process). Let a stochastic process $\{X_t\}_{t \in T}$ is called Gaussian, iff for any choice of

(i) $t_1, \dots, t_n \in T$, and

(ii) $c_1, \dots, c_n \in \mathbb{R}$.

Then for any $n \geq 0$, the random variable $\sum_{k=1}^{n} c_k X_{t_k}$ is Gaussian.

Definition A.2.9. (Isonormal Gaussian Process). Let H be a real separable Hilbert space, then a Hilbert space isometry $\eta : H \to \mathcal{H} \subset L^2(\Omega, \mathcal{F}, P)$, with

$$E[\eta(h)\eta(g)] = < h, g >, \forall g, h \in H,$$

is called an isonormal Gaussian Process, where $< h, g >$ denotes the inner product of it.

Let us now look at some useful tools used in application of probability theory. When we examine a problem it may be useful to change the probability measure. The following theorem is borrowed from Jacod, Protter [9] (Theorem 28.3).

Theorem A.2.10. (Radon-Nikodym Theorem). Let $\mathbb{Q}$ be a finite measure on $(\Omega, \mathcal{F})$, such that that $\mathbb{Q} \ll P$, i.e. Q is absolutely continuous with respect to P. Then there exists a unique integrable positive stochastic variable Λ such that

$$\mathbb{Q}[A] = E[1_A \Lambda], \qquad A \in \mathcal{F}.$$

We will often write

$$\Lambda = \frac{d\mathbb{Q}}{dP},$$

and refer to this as the Radon-Nikodym derivative.

As next theorem we state the Bayes's theorem for conditional expectation.

Theorem A.2.11. (The Bayes rule). Let $(\Omega, \mathcal{F}, P)$ be the probability space, and X is an integrable stochastic variable on it. Assuming $\mathcal{H}$ to be a sub-σ-algebra of $\mathcal{F}$, and $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} >> P$ and

$$\Lambda = \frac{dP}{d\mathbb{Q}}.$$

Then

$$E[X|\mathcal{H}] \cdot E^{\mathbb{Q}}[\Lambda|\mathcal{H}] = E^{\mathbb{Q}}[X\Lambda|\mathcal{H}], \qquad a.s. \tag{A.1}$$

A.3 Proofs

Proofs in Chapter 2

Proof of Theorem 2.2.7. First of all we need to consider the case where f is an step stochastic process, for any $a \leq s \leq t \leq b$, such that

$$E[X_t|\mathcal{F}_s] = X_s, \qquad \text{almost surely.}$$

Since $X_t = X_s + \int_s^t f(u)dB(u)$, we need to show that

$$E\Big[\int_s^t f(u)dB(u)|\mathcal{F}_s)\Big] = 0, \qquad \text{almost surely.} \tag{A.2}$$

Then we consider f given as

$$f(u,\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega)\mathbb{1}_{(t_{i-1},t_i]}(u), \qquad s = t_0 < t_1 < \cdots < t_n = t$$

where ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and belongs to $L^2(\Omega)$. This gives us that

$$\int_s^t f(u)dB(u) = \sum_{i=1}^{n} \xi_{i-1}(B(t_i) - B(t_{i-1})), \qquad \text{for any } i = 1, 2, \ldots, n$$

such that

$$\begin{aligned}
&E[\xi_{i-1}(B(t_i) - B(t_{i-1}))|\mathcal{F}_s] \\
&\qquad = E[E[\xi_{i-1}(B(t_i) - B(t_{i-1}))|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s] \\
&\qquad = E[\xi_{i-1}E[(B(t_i) - B(t_{i-1}))|\mathcal{F}_{t_{i-1}}]|\mathcal{F}_s] \\
&\qquad = 0
\end{aligned}$$

since $E[(B(t_i) - B(t_{i-1}))|\mathcal{F}_{t_{i-1}}] = 0$, it follows that equation (A.2) holds. Now by letting $f \in L^2_{ad}([a,b] \times \Omega)$ and take a sequence $\{f_n\}$ of step stochastic process, such that

$$\lim_{n\to\infty} \int_a^b E[|f(u) - f_n(u)|^2]du = 0.$$

Then for each n we define a stochastic process

$$X_t^{(n)} = \int_a^t f_n(u)dB(u).$$

Here the first case, $X_t^{(n)}$ is a martingale, and for $s < t$ we write

$$X_t - X_s = (X_t - X_t^{(n)}) + (X_t^{(n)} - X_s^{(n)}) + (X_s^{(n)} - X_s),$$

and then by taking the conditional, we get

$$E[X_t - X_s|\mathcal{F}_s] = E[X_t - X_t^{(n)}|\mathcal{F}_s] + E[X_s^{(n)} - X_s|\mathcal{F}_s]. \tag{A.3}$$

By applying Theorem 4.3.5 in [12], we get

$$\begin{aligned} E[\ |E[X_t - X_t^{(n)}|\mathcal{F}_s]\ |^2] &\leq \int_a^t E[|f(u) - f_n(u)|^2]du \\ &\leq \int_a^b E[|f(u) - f_n(u)|^2]du \\ &\to 0, \text{ as } n \to \infty, \end{aligned}$$

since

$$E[\ |E[X_t - X_t^{(n)}|\mathcal{F}_s]\ |^2] \leq E[|X_t - X_t^{(n)}|^2].$$

Thus by taking a subsequence we see that $E[X_t|\mathcal{F}_s] = X_s$, almost surely since

$$\begin{aligned} E[X_t - X_t^{(n)}|\mathcal{F}_s], &\text{converges a.s to } 0 \\ E[X_s - X_s^{(n)}|\mathcal{F}_s] &\to 0 \text{ a.s.} \\ \text{and by (A.3) } E[X_t - X_s|\mathcal{F}_s] &= 0 \text{ a.s.} \end{aligned}$$

Thus X_t is a martingale.

□

Sketch of the proof of Theorem 2.2.14. We have the Itô process:

$$dX_t = Zdt + YdB_t.$$

Let $g(t,x) \in C^2([0,\infty),\mathbb{R})$. Then

$$U_t = g(t, X_t)$$

is again an Itô process, and

$$dU_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dx + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2 \tag{A.4}$$

by substituting the first Itô process into equation (A.4), where $(dX_t)^2 = (dX_t)\cdot(dX_t)$ is computed according to the following rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \qquad dB_t \cdot dB_t = dt, \tag{A.5}$$

we get:

$$\begin{aligned} U_t = g(t,X_t) = g(0,X_0) + \int_0^t \Big(\frac{\partial g}{\partial s}(s,X_s) + Z_s\frac{\partial g}{\partial x}(s,X_s) + \frac{1}{2}Y_s^2\frac{\partial^2 g}{\partial x^2}(t,X_s)(dX_s)\Big)ds \\ + \int_0^t Y_s \cdot \frac{\partial g}{\partial x}(s,X_s)dB_s. \end{aligned} \tag{A.6}$$

This is still an Itô process. Further we assume that $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are bounded and we obtain the general case by approximating by C^2 functions g_n such that $g_n, \frac{\partial g_n}{\partial t}, \frac{\partial g_n}{\partial x}$ and $\frac{\partial^2 g_n}{\partial x^2}$ are bounded for each n and converge uniformly on compact subsets of $[0,\infty) \times \mathbb{R}$ to $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$, respectively.

Moreover $Z_s = Z(s,\omega)$ and $Y_s = Y(s,\omega)$ are elementary functions. By using Taylor's theorem we get:

$$
\begin{aligned}
g(t, X_t) &= g(0, X_0) + \sum_j \Delta g(t_j, X_j) \\
&= g(0, X_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta X_j + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 \\
&+ \sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 + \sum_j R_j \qquad \text{(A.7)}
\end{aligned}
$$

$\Delta t_j = t_{j+1} - t_j, \Delta X_j = X_{t_{j+1}} - X_{t_j}, \Delta g(t_j, X_{t_j}) = g(t_{j+1}, X_{t_{j+1}}) - g(t_j, X_j)$ and $R_j = o(|\Delta t_j|^2 + |\Delta X_j|^2)$ for all j.

If $\Delta t_j \to 0$ then

$$
\sum_j \frac{\partial g}{\partial t}\Delta t_j = \sum_j \frac{\partial g}{\partial t}(t_j, X_j)\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds \qquad \text{(A.8)}
$$

$$
\sum_j \frac{\partial g}{\partial x}\Delta X_j = \sum_j \frac{\partial g}{\partial x}(t_j, X_j)\Delta X_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s \qquad \text{(A.9)}
$$

since u and v are elementary, we get:

$$
\begin{aligned}
\sum_j \frac{\partial^2 g}{\partial X^2}(\Delta X_j)^2 = \sum_j \frac{\partial^2 g}{\partial x^2} Z_j^2 (\Delta t_j)^2 + 2\sum_j \frac{\partial^2 g}{\partial x^2} Z_j Y_j (\Delta t_j)(\Delta B_j)^2 \\
+ 2\sum_j \frac{\partial^2 g}{\partial x^2} Y_j^2 (\Delta B_j)^2. \qquad \text{(A.10)}
\end{aligned}
$$

We see here that the two first terms tends to 0, when $\Delta t_j \to 0$ and the last term tends to:

$$
\int_0^t \frac{\partial^2 g}{\partial x^2} Y^2 ds \qquad \text{in } L^2(P), \text{ as } \Delta t_j \to 0.
$$

Last but not least, the proves above also supports that $\sum R_j \to 0$ as $\Delta t_j \to 0$. This completes the proof of the Itô formula.

□

Proofs in Chapter 3

Proof of The Duality Theorem 3.4.3. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and, for all t, $u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))$ be the chaos expansions of F and $u(t)$, respectively. Then

$$\begin{aligned}
E\Big[F\int_0^T u(t)\delta W(t)\Big] &= E\Big[\sum_{n=0}^{\infty} I_n(f_n)\int_0^T \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\delta W(t)\Big] \\
&= E\Big[\sum_{n=0}^{\infty} I_n(f_n)\sum_{k=0}^{\infty} I_{k+1}(\tilde{g}_k)\Big] \\
&= E\Big[\sum_{k=0}^{\infty} I_{k+1}(f_{k+1})I_{k+1}(\tilde{g}_k)\Big] \\
&= \sum_{k=0}^{\infty}(k+1)!\int_{[0,T]^{k+1}} f_{k+1}(x)\tilde{g}_k(x)dx \\
&= \sum_{k=0}^{\infty}(k+1)!(f_{k+1},\tilde{g}_k)_{L^2([0,T]^{k+1})} \qquad \text{(A.11)}
\end{aligned}$$

$\tilde{g}_k$ is the symmetrization of g_k as a function of $n+1$ variables. On the other hand we have:

$$\begin{aligned}
E\Big[\int_0^T u(t)D_tF dt\Big] &= E\Big[\int_0^T \Big(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\Big)\Big(\sum_{n=0}^{\infty} I_{n-1}(f_n(\cdot,t))\Big)dt\Big] \\
&= \int_0^T \sum_{k=0}^{\infty} E[(k+1)I_k(g_k(\cdot,t))I_k(f_{k+1}(\cdot,t))]dt \\
&= \int_0^T \sum_{k=0}^{\infty}(k+1)k!(f_{k+1}(\cdot,t),g_k(\cdot,t))_{L^2([0,T]^k)}dt \\
&= \sum_{k=0}^{\infty}(k+1)!(f_{k+1},g_k)_{L^2([0,T]^{k+1})}. \qquad \text{(A.12)}
\end{aligned}$$

And

$$\begin{aligned}
(f_{k+1},\tilde{g}_k) &= \int_0^T (f_{k+1}(\cdot,t),\tilde{g}_k(\cdot,t))_{L^2([0,T]^k)}dt \\
&= \frac{1}{k+1}\sum_{j=0}^{k+1}\int_0^T (f_{k+1}(\cdot,t_j),g_k(\cdot,t_j))_{L^2([0,T]^k)}dt_j \\
&= \int_0^T (f_{k+1}(\cdot,t),g_k(\cdot,t))_{L^2([0,T]^k)}dt \\
&= (f_{k+1},g_k)_{L^2([0,T]^{k+1})}. \qquad \text{(A.13)}
\end{aligned}$$

Then by combining (A.13) with (A.11) and (A.12), completes the proof of the Duality formula. □

Proof of Integration by parts, Theorem 3.4.4. Here we need Theorem 3.3.3 and 3.4.3 in hand. first we assume that $F \in \mathbb{D}^0_{1,2}$ from Theorem 3.3.3, then we choose $G \in \mathbb{D}^0_{1,2}$ and get

$$\begin{aligned} E\Big[G\int_0^T Fu(t)\delta W(t)\Big] &= E\Big[\int_0^T Fu(t)D_tGdt\Big] \\ &= E\Big[GF\int_0^T u(t)\delta W(t)\Big] - E\Big[G\int_0^T u(t)D_tFdt\Big]. \end{aligned}$$

Since the set of all $G \in \mathbb{D}^0_{1,2}$ is dense in $L^2(P)$, it follows that

$$F\int_0^T u(t)\delta W(t) = \int_0^T Fu(t)\delta W(t) + \int_0^T u(t)D_tFdt \qquad \text{P-a.s.}$$

Then we have that the result follows for general $F \in \mathbb{D}_{1,2}$ by approximating F by $F^{(n)} \in \mathbb{D}^0_{1,2}$ such that

$$F^{(n)} \to F \text{ in } L^2(P) \text{ and } D_tF^{(n)} \to D_tF \text{ in } L^2(P \times \lambda), \text{ for } n \to \infty.$$

□

Proof of Fundamental Theorem 3.4.6. Letting

$$u(s) = I_n(f_n(\cdot, s)),$$

and for symmetric $f_n(t_1, \dots, t_n, s)$ with respect to $t_1, \dots, t_n$, we get.

$$\int_0^T u(s)\delta W(s) = I_{n+1}[\tilde{f}_n]$$

where $\tilde{f}_n$ is the symmetrization of f_n as a function of all its $n+1$ (3.7), such that

$$\tilde{f}_n(x_1, \dots, x_{n+1}) = \frac{1}{n+1}\Big[f_n(\cdot, x_1) + \cdots + f_n(\cdot, x_{n+1})\Big].$$

Then

$$D_t\Big(\int_0^T u(s)\delta W(s)\Big) = (n+1)I_n[\tilde{f}_n(\cdot, t)] \tag{A.14}$$

and

$$\tilde{f}_n(\cdot, t) = \frac{1}{n+1}\Big[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n) + f_n(t, \cdot)\Big]. \tag{A.15}$$

By combining (A.14) with (A.15) gives us

$$\begin{aligned} D_t\Big(\int_0^T u(s)\delta W(s)\Big) &= I_n\Big[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n) + f_n(t, \cdot)\Big] \\ &= I_n\Big[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)\Big] + u(t). \end{aligned} \tag{A.16}$$

Note here that f_n is symmetric with respect to its first n variables, so we may choose t to be the first of them, in the first n terms on the right-hand side. Further, to compare (A.16) with the right-hand side of (3.24) we consider

$$\begin{aligned}\delta(D_t u) &= \int_0^T D_t u(s)\delta W(s) \\ &= \int_0^T nI_{n-1}[f_n(\cdot, t, s)]\delta W(s) \\ &= nI_n[\hat{f}_n(\cdot, t, \cdot)].\end{aligned} \tag{A.17}$$

Here
$$\hat{f}_n(x_1, \dots, x_{n-1}, t, x_n) = \frac{1}{n}\Big[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)\Big]$$

is the symmetrization of $f_n(x_1, \dots, x_{n-1}, t, x_n)$ with respect to $x_1, \dots, x_n$. From (A.17) we get

$$\int_0^T D_t u(s)\delta W(s) = I_n\Big[f_n(t, \cdot, x_1) + \cdots + f_n(t, \cdot, x_n)\Big]. \tag{A.18}$$

Then by comparing (A.16) and (A.18) we get (3.24).

As next step we consider the general case

$$u(s) = \sum_{n=0}^{\infty} I_n[f_n(\cdot, s)].$$

We define

$$u_m(s) = \sum_{n=0}^{m} I_n[f_n(\cdot, s)], \qquad m \in \mathbb{N}.$$

By (3.22) we get

$$\|u - u_m\|^2_{L^2(P\times\lambda)} \to_{m\to\infty} 0$$

then we have

$$D_t(\delta(u_m)) = \delta(D_t u_m) + u_m(t), \qquad \forall m. \tag{A.19}$$

Having (A.17) we can say that (3.23) is equivalent to

$$\begin{aligned}E\Big[\int_0^T (\delta(D_t u))^2 dt\Big] &= \sum_{n=0}^{\infty} n^2 n! \int_0^T \|\hat{f}_n(\cdot, t, \cdot)\|^2_{L^2([0,T]^n)} dt \\ &= \sum_{n=0}^{\infty} n^2 n! \|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty\end{aligned} \tag{A.20}$$

since $D_t u \in Dom(\delta)$. For $m \to \infty$

$$\|\delta(D_t u) - \delta(D_t u_m)\|^2_{L^2(P\times\lambda)} = \sum_{n=m+1}^{\infty} n^2 n! \|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} \to 0. \tag{A.21}$$

Therefore, by (A.19)

$$\delta(D_t u_m) \to \delta(D_t u) + u(t), \qquad m \to \infty \text{ in } L^2(P\times\lambda).$$

Note here that

$$(n+1)\tilde{f}_n(\cdot,t) = n\hat{f}_n(\cdot,t,\cdot) + f_n(\cdot,t)$$

and hence

$$(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} \le \frac{2n^2 n!}{n+1}\|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} + \frac{2n!}{n+1}\|f_n\|^2_{L^2([0,T]^{n+1})}.$$

Therefore,

$$\begin{aligned}
\|\delta(u)\|^2_{\mathbb{D}_{1,2}} &= \sum_{n=0}^{\infty}(n+1)(n+1)!\|\tilde{f}_n\|^2_{L^2([0,T]^{n+1})} \\
&\le \sum_{n=0}^{\infty}\Big[2n^2 n!\|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} + 2n!\|f_n\|^2_{L^2([0,T]^{n+1})}\Big] \\
&\le 2\|\delta(D_t u)\|^2_{L^2(P\times\lambda)} + 2\|u\|^2_{L^2(P\times\lambda)} < \infty,
\end{aligned}$$

by (A.20) and (3.22). with this we can conclude with that $\delta(u)$ is well-defined and belongs to $\mathbb{D}_{1,2}$. By adapting similar computations, we obtain

$$\begin{aligned}
&\|D_t\Big(\int_0^T u(s)\delta W(s)\Big) - D_t\Big(\int_0^T u_m(s)\delta W(s)\Big)\|^2_{L^2(P\times\lambda)} \\
&= \|\sum_{n=m+1}^{\infty}(n+1)I_n(\tilde{f}_n(\cdot,t))\|^2_{L^2(P\times\lambda)} \\
&= \int_0^T \sum_{n=m+1}^{\infty}(n+1)^2 n!\|\tilde{f}_n(\cdot,t))\|^2_{L^2([0,T]^n)}dt \\
&\le 2\sum_{n=m+1}^{\infty}\Big[n^2 n!\|\hat{f}_n\|^2_{L^2([0,T]^{n+1})} + n!\|f_n\|^2_{L^2([0,T]^{n+1})}\Big],
\end{aligned} \tag{A.22}$$

which tends to zero when $m \to \infty$. Having (A.21) and (A.22), we get (3.24)

$$D_t(\delta(u)) = \delta(D_t u) + u(t), \qquad \text{when in (A.19)} .$$

□

Proofs in Chapter 5

Proof of Theorem 5.3.5. First we observe the relation by Theorem 5.3.3:

$$\mathscr{D}_s X_T^0(\widetilde{\eta} + \lambda\xi\mu) = DX_T^0(\widetilde{\eta} + \lambda\xi\mu)[\lambda \mathscr{D}_s \xi\mu] \qquad (\omega, s) - a.e.$$

Then by multiplying with $\frac{\xi}{\mathscr{D}_s\xi}$ yields

$$\frac{\xi}{\mathscr{D}_s\xi}\mathscr{D}_s X_T^0(\widetilde{\eta} + \lambda\xi\mu) = DX_T^0(\widetilde{\eta} + \lambda\xi\mu)[h]\lambda\xi \qquad (\omega, s) - a.e. \tag{A.23}$$

Recall that $\mathscr{D}_s\xi \neq 0$ a.e., and the right-hand side in (A.23) is defined ω-wise. The evaluation at $\lambda = \frac{1}{\xi}$ yields $DX_T^0(\widetilde{\eta} + h)[h]$, thus

$$\begin{aligned}
\left\{\frac{\xi}{\mathscr{D}_s\xi}\mathscr{D}_s X_T^0(\widetilde{\eta} + \lambda\xi\mu)\right\}\Bigg|_{\lambda=\frac{1}{\xi}} &= DX_T^0(\widetilde{\eta} + \lambda\xi\mu)[h]\lambda\xi\Bigg|_{\lambda=\frac{1}{\xi}} \\
&= DX_T^0(\widetilde{\eta} + h)[h] \\
&= DX_T^0(\eta)[h].
\end{aligned}$$

By considering that $D\Phi(X_T^0(\eta))$ is defined path-wise, and multiplying with $1 = \int_{-r}^0 a(s)ds$ and applying the chain rule we obtain

$$\begin{aligned}
E[D(\Phi \circ X_T^0)(\eta)[h]] &= E[D\Phi(X_T^0(\eta))DX_T^0(\eta)[h]] \\
&= E\left[\int_{-r}^0 D\Phi(X_T^0(\eta))DX_T^0(\eta)[h]a(s)ds\right] \\
&= E\left[\left\{\int_{-r}^0 D\Phi(X_T^0(\widetilde{\eta} + \lambda\xi\mu))\mathscr{D}_s X_T^0(\widetilde{\eta} + \lambda\xi\mu)a(s)\frac{\xi}{\mathscr{D}_s\xi}ds\right\}\Big|_{\lambda=\frac{1}{\xi}}\right] \\
&= E\left[\left\{\int_{-r}^0 \mathscr{D}_s\{\Phi(X_T^0(\widetilde{\eta} + \lambda\xi\mu))\}a(s)\frac{\xi}{\mathscr{D}_s\xi}ds\right\}\Big|_{\lambda=\frac{1}{\xi}}\right].
\end{aligned}$$

Moreover, the partial formula for the Skorohod integral yields

$$\begin{aligned}
E[D(\Phi \circ X_T^0)(\eta)[h]] &= E\left[\left\{\Phi(X_T^0(\widetilde{\eta} + \lambda\xi\mu))\delta\Big(a(\cdot)\frac{\xi}{\mathscr{D}_s\xi}\Big) - \delta\Big(\Phi(X_T^0(\widetilde{\eta} + \lambda\xi\mu))a(\cdot)\frac{\xi}{\mathscr{D}_s\xi}\Big)\right\}\Big|_{\lambda=\frac{1}{\xi}}\right] \\
&= E\left[\Phi(X_T^0(\eta))\delta\Big(a(\cdot)\frac{\xi}{\mathscr{D}_s\xi}\Big) - \left\{\delta\Big(\Phi(X_T^0(\widetilde{\eta} + \lambda\xi\mu))a(\cdot)\frac{\xi}{\mathscr{D}_s\xi}\Big)\right\}\Big|_{\lambda=\frac{1}{\xi}}\right].
\end{aligned}$$

$\delta\Big(a(\cdot)\frac{\xi}{\mathscr{D}_s\xi}\Big)$ is $\mathcal{F}^{\mathbb{B}}$-measurable, $\Phi(X_T^0(\eta))$ is $\mathcal{F}^W$ and by independence the proof of Theorem 5.3.5 completes.

□

Appendix B

The R code

The R code used to solve, will be presented here:

B.1 Programs for Chapter 2

B.1.1 Programs for plots in Figure 2.1

R program to create the plot.

```
##########################################################################
#
#                         FIGURE: 2.1 IN CHAPTER 2
#
##########################################################################

#------------------------------------------------
#Path and plot of Brownian motion:
#------------------------------------------------
#Brownian motion with , T = time horizon and n = partition

n <- 1000                       #Number of time points
d <- 1000                       #Number of random variables to generate
delta_t <- 30/n                 #Subinterval width

#---------------------
#Simulate the path:
#---------------------
B <- cumsum(rnorm(d,0,1)*sqrt(delta_t))
B[1]<-0

#------
#Plot:
#------
t = seq (0 ,30 , length =d)
plot(t,B,type = 'l', main='Path of Brownian motion',xlab ='t'
       ,ylab ='', xlim =c (0 ,30))
```

B.2 Programs for Chapter 6

R program to create the simulation.

```
################################################################################
#
#                              SIMULATION IN CHAPTER 6
#
################################################################################


#--------------------------------------------------------------
#Simulate the derivative free representation:
#--------------------------------------------------------------

#Defining the parameters

n <- 1000                  #Number of time points
d <- 1000                  #Number of random variables to generate


p<- 20                     #Number of paths 20
r<-c(1,2,5,10,20,30,50,80)

for (j in 1:length(r)){
T<- r[j] + r[j]*0.20                #T>r
delta_t <- r[j]/n          #Subinterval width
print(r[j])


#-----------------------------------------------------------------
#Simulating the paths and plots of Brownian motion:
#-----------------------------------------------------------------


#Brownian Motion with 20 paths, t in [-r,0]:

B_mid<- matrix(0, nrow = p, ncol = d)
BM<- matrix(0, nrow = p, ncol = d)

for(rown in 1:p){
        B_mid[rown,] <- cumsum(rnorm(d,0,1)*sqrt(delta_t))
        B_mid[rown,1] <- 0
        for(i in 1:d-1){
                BM[rown,i] = B_mid[rown,d-i]
        }
}
```

```
#-------------------------------
#Approximation of delta(u):
#-------------------------------


delta_mid<- matrix(0, nrow = nrow(BM), ncol =  (ncol(BM)-1))
delta<-c()
#lambda <- exp(B[-r])

for (k in 1:nrow(BM)){
         for (i in 1:(ncol(BM)-1)){
         delta_mid[k,i] <-  (exp(BM[k,1])*exp(-1/2*(t[i+1]-t[i])^2)
                           *exp(BM[k,i]-BM[k,1])*exp(-BM[k,i+1])*exp(T)
                               *(1-exp(-r[j]))*(1/r[j]))*(BM[k,i+1]-BM[k,i])
         }

}

delta<-c()
for (col in  1:nrow(delta_mid)){
 delta[col] <- sum(delta_mid[col,])
}


#------------------------------------------------------------
#Approximation of the derivative free representation
#                  by Monte Carlo simulation:
#------------------------------------------------------------


partial_hP <- -1/N * sum(delta)
print(partial_hP)

}
```

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