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# Approximations of Fractional Brownian Motion.

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Master's Thesis, Spring 2018

This master's thesis is submitted under the master's programme *Modelling and Data Analysis*, with programme option *Finance, Insurance and Risk*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group $E_8$, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

# Abstract

This thesis is a study in approximation of the fractional Brownian motion. We first define weak convergence of continuous stochastic processes, and we define and prove the tools needed to show weak convergence. Then we use the representation of fractional Brownian motion due to Mandelbrot and Van Ness as an inspiration for a discrete stochastic process. We use linear interpolation to extend this process to a continuous process. As the time-intervals in the approximation becomes smaller, our processes converge weakly to the fractional Brownian motion.

We will also look at Difference Calculus. When we combine this with the earlier results we will get many interesting approximations, some will follow very elegantly without difficult proofs.

Lastly we will look at applications to finance. Here we will approximate known processes, and also approximate stochastic differential equations encountered in finance. We will see that the solutions also converge weakly to known results.

# Contents

# Preface

This thesis is the result of a year's work. My advisor Tom Lindstrøm and I discussed various topics I could write about. After a discussion with Fred Espen Benth, also a professor at UiO, we found out that fractional Brownian motion had been a hot topic in finance, so it seemed like a good topic to look deeper into.

I am also very interested in various types of convergence in probability theory. Lastly I find the elegance of discrete stochastic processes very intriguing. I was allowed to control the path of the thesis so it was natural to incorporate my interests. So the three aspects, fractional Brownian motion, convergence and lastly, discrete stochastic processes are the topics of the thesis. My advisor and I therefore formulated the problem statement to be to investigate how "simpler" processes could be used to approximate the fractional Brownian motion. I quickly decided that I wanted to use continuous processes that were made up of discrete processes and linear interpolation. This means that the process is uniquely determined by the value at each time-point that is a multiple of a positive number $\delta = \Delta t$. So strictly speaking these are continuous processes, but it is better to view them as discrete processes extended to $C[0, \infty)$ by linear interpolation. This also gives us the advantage that every process used to approximate the fractional Brownian motion is continuous, and this may be desirable since the fractional Brownian motion has continuous trajectories.

One of my pet peeves is that in a lot of mathematical texts many aspects of the proofs are omitted. For example, in stochastic analysis it is not always proved that processes are well-defined, that functions are measurable when they should be etc. So I have taken the approach that if we work with a function that should be measurable, or if we have a set that should be measurable, we show or prove that they indeed are. This is not as trivial as it might seem. For example, when working with a proof of this kind, see proposition B.2.15, I found that I needed separability of $C[0, \infty)$(equipped with a metric to be defined later) for the proof of measurability to go through. So measurability results are not as trivial as one might think. The mature mathematical reader should be able to quickly recognize these proofs and skip them if desired.

### Acknowledgements

I would like to thank my family for their support and love. My mother, for always believing I can do anything, my father for helping me with practical problems, and my little brother for our interesting talks.

Also thanks to my advisor Tom Lindstrøm. First for allowing me to pursue the topics that interested me. Secondly, for your support during this year of

writing. Thirdly, for our discussions of mathematics during the years. You have been as good an advisor as your were a lecturer in MAT1140 and MAT2400, where you opened my eyes to the wonderful world of mathematics.

Jan Josef Kristensen
Oslo, November 2017

# Chapter 1

# Introduction

This chapter is meant to be a guide to the rest of the thesis. While reading the thesis it is recommended to keep the problem statement in mind; we want approximate the fractional Brownian motion with simpler processes. As we will see later, the approximation processes will not be entirely discrete, as we will use linear interpolation to make them continuous.

## 1.1 Conventions

Let us quickly look at some conventions in this thesis. We let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, \ldots\}$. Notice that 0 is not included here.

If $a_i$ is a sequence and $b, c$ are natural numbers with $c < b$ we define

$$\sum_{i=b}^{c} a_i \doteq 0.$$

And from definition 3.2.1 we see that the same follows for the $\Sigma_\delta$ sums.

If $a$ is a real number we let $\lfloor a \rfloor$ be the floor function used on $a$. It is defined as

$$\lfloor a \rfloor \doteq \max \{z \in \mathbb{Z} : z \leq a\}.$$

Where $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

We have two different ways of writing stochastic processes. Let us take $X^{(\delta_n)}$ from chapter 3 as an example. Usually stochastic processes are written $(X_t^{(\delta_n)})_{t \in [0,\infty)}$, but we will often just write $X^{(\delta_n)}$. This is when we want to emphasize that we are working with random functions taking values in $C[0, \infty)$.

We will sometimes use the abbreviation fBm for the Fractional Brownian motion.

## 1.2 Outline of the thesis

This thesis is fairly long since we have many proofs. It is important not getting lost in the details, but also focus on the main ideas of the thesis. In this section I will try to explain the main goals of each chapter.

### Chapter 2

In this chapter we create the tools needed for weak convergence of continuous stochastic processes where the time parameter is in $[0, \infty)$. The chapter is based on the work in [Bil99] and [Bil68]. Billingsley works with process on $C[0, 1]$ or cádlág processes. We modify Billingsley's proofs to work for $C[0, \infty)$.

### Chapter 3

Here we introduce the fractional Brownian motion. And we use the representation given by Mandelbrot and Van Ness of a fractional Brownian motion as an inspiration to define a process $X^{(\delta)}$(definition 3.2.6), which we call the Mandelbrot and Van Ness sum. For $\{\delta_n\}$ a positive sequence converging to zero, we show weak convergence of $X^{(\delta_n)}$ to the fractional Brownian motion.

### Chapter 4

In chapter 4 we introduce difference calculus with arbitrary step size. The work is based on chapter one, and parts of chapter two in [GP15]. The crucial tool which allows us to generalize to an arbitrary step-size is the modified Gamma function, which we come up with in definition 4.2.1. We then use the modified Gamma function to define falling powers.

In section 4.3 - section 4.9 we generalize more of the concepts in [GP15], but they are not used later.

### Chapter 5

In this chapter we modify the process $X^{(\delta)}$ from chapter 3 to a process $Y^{(\delta)}$. We change some of the powers in $X^{(\delta)}$ to falling powers. The falling powers are those from section 4.2. We again show weak convergence to the Fractional Brownian Motion as $\delta$ goes to zero.

### Chapter 6

In chapter 6 we still want to approximate the fractional Brownian motion. We end up with two processes $Z^{(\delta)}$(definition 6.1.1) and $U^{(\delta)}$(definition 6.2.6). However, $Z^{(\delta)}$ and $U^{(\delta)}$ are in this case defined by their difference $\Delta Z_t^{(\delta)} \doteq Z_{t+\delta}^{(\delta)} - Z_t^{(\delta)}$, and likewise for $\Delta U_t^{(\delta)}$. This is because now it is the difference that has a nice representation. $Z^{(\delta)}$ is based on $Y^{(\delta)}$, and $U^{(\delta)}$ is based on $X^{(\delta)}$. We show weak convergence to the fractional Brownian motion as $\delta$ becomes smaller and smaller. The proof for the $Z^{(\delta)}$ is very elegant because differences of falling powers behave well.

### Chapter 7

The four processes we worked with earlier($X^{(\delta)}, Y^{(\delta)}, Z^{(\delta)}, U^{(\delta)}$) all contained infinite sums in their definitions. We want a process with only finite sum, so we use $U^{(\delta)}$ as an inspiration for $V^{(\delta)}$ which is described with only finite sums. We show that we still have weak convergence to the fractional Brownian motion.

### Chapter 8

Here we first show in theorem 8.1.2 that we can approximate the process $(S_0 e^{f(t)+\sigma B_{t,H}})_{t\in[0,\infty)}$, where $f$ is a continuous function. We also look at some examples where the geometric fractional Brownian motion is used in finance, and explain how we can approximate these functions. We model a risky asset by a stochastic difference equation. The difference equation is chosen so that it resembles the stochastic differential equation from stochastic analysis, used to model the risky asset. We see that we have to modify the difference equation for $H < 1/2$. We show that the solution converges weakly to the geometric fractional Brownian motion.

### Chapter 9

We conclude the thesis. Lastly we give some ideas for further research.

### The Appendices

The appendices are meant to be a place where we prove some of the statements needed in the text. These results are placed in the appendix so we can focus better on the results in the main text. The appendices are not meant as an introduction to the subjects discussed there. However, they are written in a way where they can be read as stand-alone chapters.

## 1.3 How to read the thesis

Many authors of mathematical texts recommend skipping the proofs the first read-through. This can also be done while reading this thesis in order to not get lost in the details.

Every chapter is based on the previous chapters, so one should understand chapter 2 before starting on chapter 3 etc. However, chapter 4 is an exception. Chapter 4 can be read as a stand-alone chapter, and it is also only section 4.1 and section 4.2 that are used later.

Chapter 3, chapter 5, chapter 6 and chapter 7 may seem long. But one should keep in mind that they are very simple in the sense that their only goal is to show weak convergence of a certain sequence of processes.

The appendices contain results needed in the main text. They are not meant to be introduction to the various topics, they only contain results needed in the main chapters.

# Chapter 2

# Weak convergence in $C[0,\infty)$

In this chapter we create the machinery needed for weak convergence of continuous stochastic processes on $[0,\infty)$. We will generalize the work of Patric Billingsley for $C[0,1]$ to $C[0,\infty)$. The results we generalize are from two editions of the same book, *Convergence of Probability Measures*, one edition from 1968, [Bil68] and the other from 1999, [Bil99]. We will also refer to some results given in a compendium written by Serik Sagitov, [Sag15], this compendium is also based on Billingsley's books.

## 2.1 Quick introduction to weak convergence

Later we will see that $C[0,\infty)$ equipped with the appropriate metric is a metric space. The notion of weak convergence can be formulated without stochastic processes, only in terms of probability spaces and metric spaces. So in this section we will give the definition of weak convergence in terms of general metric spaces. This can also be found in [Bil99, p. 7].

We let $S$ denote a general metric space, and $\mathcal{S}$ be the Borel $\sigma$-algebra on $S$. So $(S,\mathcal{S})$ will be a measurable space. We equip the real numbers $\mathbb{R}$ with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$. A continuous function

$$f : S \to \mathbb{R},$$

will be $\mathcal{S}/\mathcal{B}(\mathbb{R})$-measurable. So if $P_n$ is a probability-measure on $\mathcal{S}$ we have that $f$ will be a random variable on the probability space $(S,\mathcal{S},P_n)$. If $P$ is another probability measure on $\mathcal{S}$ we let $E_n[f]$ denote the expectation of $f$ using the probability space $(S,\mathcal{S},P_n)$, and we let $E[f]$ denote the expectation using $(S,\mathcal{S},P)$. We can now give the definition of weak convergence.

**Definition 2.1.1** (**Weak Convergence**). *Let $S$ denote a metric space, and let $\mathcal{S}$ denote the Borel $\sigma$-algebra on $S$. Let $\{P_n\}$ be a sequence of probability measures on $\mathcal{S}$, and let $P$ be another probability measure on $\mathcal{S}$. We say that $\{P_n\}$ converges weakly to $P$ if for every bounded continuous function:*

$$f : S \to \mathbb{R},$$

*we have*

$$E_n[f] \to E[f].$$

By [Bil99, Theorem 1.2, p. 8] a sequence of probability measures can only converge weakly to one probability measure, that is, the limit is unique.

## 2.2 Non-probabilistic results for the space $C[0,\infty)$

In this section we will develop some of the properties of $C[0,\infty)$ without concerning ourselves with probability theory, only the real analysis aspect of the theory. The first result shows that with the appropriate metric we have that $C[0,\infty)$ is a metric space.

**Theorem 2.2.1.** *The function $\rho : C[0,\infty) \times C[0,\infty) \to \mathbb{R}$, given by*

$$\rho(f,g) = \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - g(t)| : t \in [0,i]\}),$$

*is a bounded metric on $C[0,\infty)$.*

*Proof.* Obviously the value is bounded by 1, because of the geometric series $\sum_{i=1}^{\infty} 2^{-i} = 1$.

$\rho(f,f)$ is obviously 0.

Assume that $\rho(f,g) = 0$. We must show that $f(t) = g(t), \forall t$. Assume that $t$ is given, choose $N > t$. We have $\sup\{|f(t) - g(t)| : t \in [0,N]\} = 0$. So $f(t) = g(t)$.

We also obviously have $\rho(f,g) = \rho(g,f)$.

Lastly we must prove the triangle inequality. Assume that $f,g,h \in C[0,\infty)$. We get

$$\begin{aligned}
\rho(f,g) &= \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - g(t)| : t \in [0,i]\}) \\
&= \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - h(t) + h(t) - g(t)| : t \in [0,i]\}) \\
&\leq \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - h(t)| + |h(t) - g(t)| : t \in [0,i]\}) \\
&\leq \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - h(t)| : t \in [0,i]\} + \sup\{|h(t) - g(t)| : t \in [0,i]\}) \\
&\leq \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|f(t) - h(t)| : t \in [0,i]\}) \\
&\quad + \sum_{i=1}^{\infty} \min(2^{-i}, \sup\{|h(t) - g(t)| : t \in [0,i]\}) \\
&= \rho(f,h) + \rho(h,g),
\end{aligned}$$

in the second step we used the triangle inequality and lemma D.1.5, in the third we used the sub-additivity of the supremum and lemma D.1.5, and in the fourth step we used lemma D.1.4.

□

We will also need completeness and separability which we prove next.

**Theorem 2.2.2.** *The metric space $(C[0,\infty),\rho)$ is separable and complete.*

*Proof.* We first show show that $C[0,k]$ is separable with the metric $\rho_k(f,g) = \sup\{|f(t)-g(t)| : t \in [0,k]\}$. [Bil99, Example 1.3, p.11] tells us that the result holds for $C[0,1]$. We then extend it to $C[0,k]$ by change of variable. Let $\{g_n\}$ be the dense subset of $C[0,1]$. We claim that $\{f_n\}, f_n(t) \doteq g_n(t/k)$, is a dense subset of $C[0,k]$. Let $\epsilon > 0, f \in C[0,k]$ be given. Define $g(t) \doteq f(tk) \in C[0,1]$. Let $\sup\{|g(t)-g_n(t)| : t \in [0,1]\} < \epsilon$, for some $n$. Then $\sup\{|f(s)| - f_n(s)| : s \in [0,k]\} = \sup\{|g(s/k)| - g_n(s/k)| : s \in [0,k]\} = \sup\{|g(t)| - g_n(t)| : s \in [0,1]\} < \epsilon$. Hence $C[0,k]$ is separable.

Let us now show that $C[0,\infty)$ is separable. For every $k$, we have that $C[0,k]$ is separable. There is a countable collection of functions that is dense on $C[0,k]$, extend each function in this collection to a function on $C[0,\infty)$ by requiring that it is constant on $[k,\infty)$, denote this collection $\mathcal{F}_k$. We will show that $\mathcal{F} = \bigcup_{k\in\mathbb{N}} \mathcal{F}_k$ is a countable dense subset of $C[0,\infty)$. It is countable since a countable union of countable sets is countable. To show that it is dense, assume that $f \in C[0,\infty)$ and $\epsilon > 0$ is given. Let $N$ be such that $\sum_{k=N+1}^{\infty} 2^{-k} < \epsilon/2$. Choose $g \in \mathcal{F}$ such that $\sup\{|f(t)-g(t)| : t \in [0,N]\} < \frac{\epsilon}{2N}$, this is possible by the construction of $\mathcal{F}$. We now have

$$\begin{aligned} \rho(f,g) &= \sum_{k=1}^{\infty} \min(2^{-k}, \sup\{|f(t)-g(t)| : t \in [0,k]\}) \\ &\leq \sum_{k=1}^{N} \sup\{|f(t)-g(t)| : t \in [0,N]\} + \sum_{k=N+1}^{\infty} 2^{-k} \\ &< N\frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So $C[0,\infty)$ is separable.

Lastly let us now show that $C[0,\infty)$ with $\rho$ is complete. Assume that $\{f_n\} \in C[0,\infty)$ is a Cauchy sequence, let $t \in [0,\infty)$. We want to show that $f_n(t)$ converges pointwise. Let $K > t$ be a natural number. We have

$$|f_n(t) - f_m(t)| \leq \sup\{|f_n(s) - f_m(s)| : s \in [0,K]\}.$$

The last expression can be made arbitrarily small if $n$ and $m$ is increased since $\{f_n\}$ is a Cauchy-sequence with the $\rho$-norm. So $\{f_n(t)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete $f_n(t)$ converges pointwise, call the limit $f(t)$, and notice that the limit is independent of our choice of $K$. The function $f$ is continuous at each $t$. To see this first pick $K > t$ to be a natural number. Let $\epsilon > 0$, and let $M$ be such that if $n,m \geq M$ we have

$$\sup\{|f_n(t) - f_m(t)| : t \in [0,K]\} < \epsilon/2.$$

Now let $s \in [0,K]$ be arbitrary, and assume $n \geq M$, we get

$$|f(s) - f_n(s)| \leq |f(s) - f_m(s)| + |f_m(s) - f_n(s)|,$$

if we let $m \geq M$ we have that the second expression $|f_m(s) - f_n(s)|$ is smaller than $\epsilon/2$, and we can get the first expression as small as we want by increasing $m$ since we have pointwise convergence. This means that we have shown uniform convergence on $[0,K]$, and since uniform convergence of continuous functions on compact intervals give us a continuous function, we have that $f$ is continuous. We will now use that $f_n$ converges uniformly to $f$ if we restrict us to $[0,k]$ for every k. Let us now show convergence in the $\rho$-norm. Assume $\epsilon > 0$ is given. Let $N$ be such that $\Sigma_{j=N+1}^{\infty} 2^{-i} < \epsilon/2$. Choose $n^*$ such that $\sup\{|f(t) - f_n(t)| \ t \in [0,N]\} < \frac{\epsilon}{2N}$, if $n \geq n*$, this we can accomplish by the established uniform convergence on $[0,N]$. We then get if $n \geq n^*$

$$\begin{aligned}
\rho(f_n, f) &= \sum_{k=1}^{\infty} \min(2^{-k}, \sup\{|f_n(t) - f(t)| : t \in [0,k]\}) \\
&\leq \sum_{k=1}^{N} \sup\{|f_n(t) - f(t)| : t \in [0,k]\} + \sum_{k=N+1}^{N} 2^{-k} \\
&\leq \sum_{k=1}^{N} \sup\{|f_n(t) - f(t)| : t \in [0,N]\} + \frac{\epsilon}{2} \\
&\leq N\frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $C[0,\infty)$ is complete.

□

Usually the modulus of continuity is only defined on compact intervals. In order to generalize Billingsley's work we define a modified modulus of continuity, $w_x$. Note the similarity with the metric defined in theorem 2.2.1.

**Definition 2.2.3.** *For $x \in C[0,\infty)$ define the modulus of continuity by*

$$w_x(\xi) \doteq \sum_{k=1}^{\infty} \min\left(2^{-k}, \sup\{|x(s) - x(t)| : s,t \in [0,k], |s-t| \leq \xi\}\right),$$

*for $\xi \in (0,1]$.*

Next we prove a result which ensures continuity of the modulus of continuity.

**Theorem 2.2.4.** *If $\xi \in (0,1]$ we have $|w_x(\xi) - w_y(\xi)| \leq 2\rho(x,y)$.*

*Proof.* Assume without loss og generality that $w_x(\xi) \geq w_y(\xi)$. Because the series converges absolutely we have

$$\begin{aligned}
&w_x(\xi) - w_y(\xi) \\
&= \sum_{k=1}^{\infty} \Bigg[ \min\left(2^{-k}, \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|x(s) - x(t)|\}\right) \\
&\qquad - \min\left(2^{-k}, \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s) - y(t)|\}\right)\Bigg].
\end{aligned} \tag{2.1}$$

For $a, b, c$ are non-negative numbers we have by lemma D.1.6

$$\min(a,b) - \min(a,c) \leq \min(a, |b-c|).$$

Using this in eq. (2.1) we get

$$\begin{aligned}
&w_x(\xi) - w_y(\xi) \\
&\leq \sum_{k=1}^{\infty} \left[ \min\left(2^{-k}, \left| \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|x(s)-x(t)|\} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s)-y(t)|\} \right| \right) \right]
\end{aligned}$$

We will show that

$$\begin{aligned}
&\left| \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|x(s)-x(t)|\} - \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s)-y(t)|\} \right| \\
&\leq 2 \sup_{t\in[0,k]} \{|x(t)-y(t)|\}.
\end{aligned} \tag{2.2}$$

Assume without loss of generality that

$$\sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s)-y(t)|\} \geq \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|x(s)-x(t)|\}.$$

Assume that $\epsilon > 0$ is given. Choose $s', t' \in [0,k], |s'-t'| \leq \delta$ such that

$$y(s') - y(t') + \epsilon > \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s)-y(t)|\}$$

We then have

$$\begin{aligned}
&\sup_{s,t\in[0,k],|s-t|\leq\xi} \{|y(s)-y(t)|\} - \sup_{s,t\in[0,k],|s-t|\leq\xi} \{|x(s)-x(t)|\} \\
&\leq y(s') - y(t') + \epsilon - x(s') + x(t') \\
&= y(s') - x(s') + x(t') - y(t') + \epsilon \\
&\leq 2 \sup_{t\in[0,k]} \{|x(t)-y(t)|\} + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, eq. (2.2) must hold. Hence we have proved that

$$w_x(\xi) - w_y(\xi) \leq \sum_{k=1}^{\infty} \left[ \min\left(2^{-k}, 2 \sup_{t\in[0,k]} \{|x(t)-y(t)|\}\right) \right].$$

Since we by lemma D.1.7 have

$$\min\left(2^{-k}, 2 \sup_{t\in[0,k]} \{|x(t)-y(t)|\}\right) \leq 2 \min\left(2^{-k}, \sup_{t\in[0,k]} \{|x(t)-y(t)|\}\right),$$

the result follows. □

The next two results will be helpful when we later encounter finite-dimensional distributions.

**Theorem 2.2.5.** *On the space $\mathbb{R}^k, k \in \mathbb{N}$, $d_2(x,y) = \max_{1\leq i\leq k} |x_i - y_i|$ is a metric. Let $d_1(x,y) = |x-y|$ denote the standard-metric on $\mathbb{R}^k$. Then the identity maps $I_1 : (\mathbb{R}^k, d_1) \to (\mathbb{R}^k, d_2)$ and $I_2 : (\mathbb{R}^k, d_2) \to (\mathbb{R}^k, d_1)$ are continuous.*

*Proof.* This result is well known. We give the proof as an aid to the reader.

We have $d_2(x,x) = 0$. If $d_2(x,y) = 0$, then $x = y$. Obviously we have $d_2(x,y) = d_2(y,x)$. Let $x,y,z \in \mathbb{R}^k$. We have $d_2(x,y) = \max_{1\leq i\leq k} |x_i - y_i| \leq \max_{1\leq i\leq k} (|x_i - z_i| + |z_i - y_i|) \leq \max_{1\leq i\leq k} |x_i - z_i| + \max_{1\leq i\leq k} |z_i - y_i|$. So $d_2$ is a metric.

We have

$$\begin{aligned} d_2(x,y)^2 &= \left(\max_{1\leq i\leq k} |x_i - y_i|\right)^2 \\ &= \max_{1\leq i\leq k} |x_i - y_i|^2 \\ &\leq \sum_{i^1}^{k} |x_i - y_i|^2 \\ &= d_1(x,y)^2. \end{aligned}$$

So $d_2(x,y) \leq d_1(x,y)$. So $I_1$ is continuous.

We also have

$$\begin{aligned} d_1(x,y) &= \sqrt{\sum_{i=1}^{k} |x_i - y_i|^2} \\ &\leq \sqrt{k \max_{1\leq i\leq k} |x_i - y_i|^2} \\ &\leq \sqrt{k} d_2(x,y). \end{aligned}$$

Hence $I_2$ is also continuous. □

We prove a simple, but useful result next. The fact that the numbers are strictly increasing is not needed, but we include it because this is what we will impose later for simplicity.

**Theorem 2.2.6.** *Assume that you have $k$ non-negative real numbers $t_1$,$t_2$,. . .,$t_k$, such that $t_1 < t_2 < \cdots < t_k$. Let $x = (t_1, t_2, \ldots, t_k)$ The map $\pi_x$: $(C[0,\infty), \rho) \to (\mathbb{R}^k, d_1)$, given by $\pi_x(f) = (f(t_1), f(t_2), \ldots, f(t_k))$ is continuous.*

*Proof.* Because of theorem 2.2.5 we can consider the metric $d_2$ instead of $d_1$. Let $f \in C[0,\infty)$, assume $\epsilon > 0$ be given. Choose a natural number $N$ such that $N > \max_{1\leq i\leq k} t_i$. Assume that $\Delta = \min(2^{-(N+1)}, \epsilon)$. If $g \in C[0,\infty)$ and we require that $\rho(f,g) < \Delta$, we have

$$\begin{aligned}
d_2(\pi_x(f),\pi_x(g)) &= d_2\Big(\big(f(t_1), f(t_2),\dots,f(t_k)\big),\big(g(t_1),g(t_2),\dots,g(t_k)\big)\Big)\\
&= \max_{1\le i\le k} |f(t_i)-g(t_i)|\\
&\le \sup_{t\in[0,N]} \{|f(t)-g(t)|\}\\
&= \min\left(2^{-N}, \sup_{t\in[0,N]}\{|f(t)-g(t)|\}\right)\\
&\le \sum_{i=1}^{\infty} \min\left(2^{-i}, \sup_{t\in[0,i]}\{|f(t)-g(t)|\}\right)\\
&= \rho(f,g) < \Delta \le \epsilon.
\end{aligned}$$

Hence we have the required continuity. We used that $\sup\{|f(t)-g(t)| : t \in [0,N]\}$ must be smaller than $2^{-N}$, or else we would have a contradiction since $\Delta \le 2^{-(N+1)}$. □

In a metric space we define compactness of a set to be that every sequence in that set has a convergent subsequence with limit in the set. A set is defined to be relatively compact if if has a compact closure. It is easy to show that a set then is relatively compact if and only if every sequence in the set has a convergent subsequence(not necessarily converging to a point in the set). The classical Arzelá-Ascoli theorem deals with relative compactness on $C[a,b]$, we will in the next theorem generalize this to $C[0,\infty)$.

**Theorem 2.2.7 (Generalisation of the Arzelà-Ascoli Theorem, adaptation of [Bil99, Theorem 7.2, p 81]).** *The set $A \subset C[0,\infty)$ (equipped with $\rho$) is relatively compact if and only if*

$$\sup_{x\in A} |x(0)| < \infty, \tag{2.3}$$

*and*

$$\lim_{\xi\to 0} \sup_{x\in A} w_x(\xi) = 0. \tag{2.4}$$

*Proof.* We first prove that $\lim_{\xi\to 0}\sup_{x\in A} w_x(\xi) = 0$ if and only if $A$ is equicontinuous on each interval $[0,k]$, $k\in\mathbb{N}$. Assume first that $\lim_{\xi\to 0}\sup_{x\in A} w_x(\xi) = 0$ and that $k$ is fixed. Assume that $\epsilon > 0$ is given. Let $\Delta$ be such that if $\xi \le \Delta$

$$\sup_{x\in A} w_x(\xi) < \min(\epsilon, 2^{-(k+1)}).$$

If $t_1,t_2\in[0,k], |t_1,t_2| < \Delta$, $x\in A$, we get

$$\begin{aligned}
&|x(t_1)-x(t_2)|\\
&\le \sup_{s_1,s_2\in[0,k],|s_1-s_2|\le\Delta} \{|x(s_1)-x(s_2)|\}\\
&= \min(2^{-k}, \sup\{|x(s_1)-x(s_2)| : s_1,s_2\in[0,k], |s_1-s_2|\le\Delta\})\\
&\le w_x(\Delta)\\
&< \epsilon.
\end{aligned}$$

So $A$ is equicontinuous on $[0,k]$. Assume conversely that $A$ is equicontinuous on every compact interval. Let $\epsilon > 0$ be given. Choose $N$ such that $\sum_{j=N+1}^{\infty} 2^{-j} < \epsilon/2$. Choose $\Delta$ such that for every $x \in A$, and every $s_1, s_2 \in [0,N]$ with $|s_1 - s_2| \leq \Delta$ we have

$$|x(s_1) - x(s_2)| < \frac{\epsilon}{2N}.$$

For every $\xi \leq \Delta$ we then have

$$\begin{aligned}
&\sup_{x\in A} w_x(\xi) \\
&= \sup_{x\in A} \left\{ \sum_{k=1}^{\infty} \min\left(2^{-k}, \sup\{|x(s) - x(t)| : s,t \in [0,k], |s-t| \leq \xi\}\right) \right\} \\
&= \sup_{x\in A} \left\{ \quad \sum_{k=1}^{N} \min\left(2^{-k}, \sup\{|x(s) - x(t)| : s,t \in [0,k], |s-t| \leq \xi\}\right) \right. \\
&\qquad \left. + \sum_{k=N+1}^{\infty} \min\left(2^{-k}, \sup\{|x(s) - x(t)| : s,t \in [0,k], |s-t| \leq \xi\}\right) \right\} \\
&\leq \sup_{x\in A} \{N \frac{\epsilon}{2N} + \frac{\epsilon}{2}\} \\
&= \epsilon.
\end{aligned}$$

Hence $\lim_{\xi\to 0} \sup_{x\in A} w_x(\xi) = 0$.

Next we prove that if $A \in C[0,\infty)$ is equicontinuous then for a given $k$, the functions in $A$ are uniformly bounded on $[0,k]$ if and only if $\sup_{x\in A} |x(0)| < \infty$. The "only if" statement is trivial and follows directly. So assume that $\sup_{x\in A} |x(0)| < \infty$. Let $\epsilon = 1$. Choose a $\Delta$ such that if $x \in A$, $s_1, s_2 \in [0,k], |s_1 - s_2| < \Delta$, we have $|x(s_1) - x(s_2)| < 1$. It follows by repeated use of the triangle inequality that $x$ is bounded on $[0,k]$ by

$$|x(0)| + \frac{k}{\Delta} + 1.$$

So $A$ is uniformly bounded on $[0,k]$ by

$$\sup_{x\in A} |x(0)| + \frac{k}{\Delta} + 1.$$

To prove the theorem it now suffices to prove that $A \subset C[0,\infty)$ is relatively compact if and only if the functions in $A$ are equicontinuous and uniformly bounded on every compact interval. Assume first that $A$ is relatively compact. For a given $k$ let $A_k$ be the functions of $A$ restricted to $[0,k]$, obviously this is a collection of continuous functions. We will show that $A_k$ is uniformly bounded and equicontinuous. If we can show that $A_k$ is relatively compact under the sup-norm, the result will follow from the Arzela-Ascoli theorem, see [PP13, p. 39]. We let $\{f_n\}$ be a sequence in $A_k$. Let $\{f'_n\}$ be the corresponding sequence in $A$. A subtle point is that $\{f'_n\}$ may not be unique since for a given $f_n$ there may be two or more functions in $A$ that corresponds to this, however it doesn't matter, we only need one sequence $\{f'_n\}$. Since $A$ is relatively compact we have that there is a subsequence $\{f'_{n_m}\}$ that converges in the $\rho$-norm to $f' \in C[0,\infty)$, if we let $f$ be its restriction to $[0,k]$, we must show that $f_{n_m}$ converges to this element in the sup-norm. But this follows directly since we must have

$$\min\left(2^{-k}, \sup\{|f_{n_k}(t) - f(t)| : t \in [0,k]\}\right) \to 0,$$

as $m \to \infty$.

Lastly we assume that the functions in $A$ are equicontinuous and uniformly bounded on each compact interval. By the Arzela-Ascoli theorem, $A_k$ is relatively compact. We must show that $A$ is relatively compact. Let $\{f_n\}$ be a sequence in $A$. For each $k$ let $f_n^{(k)}$ be the restriction of $f_n$ to $[0,k]$. On $[0,1]$ there is a subsequence of $\{f_n\}$, $\{f_{n,1}\}$ such that $f_{n,1}^{(1)}$ converges in the sup-norm on $[0,1]$ to $f^{(1)} \in C[0,1]$. There is also a subsequence of $\{f_{n,1}\}$ call it $\{f_{n,2}\}$ that converges uniformly on $[0,2]$ to $f^{(2)} \in C[0,2]$. Continue like this recursively such that each $\{f_{n,k+1}\}$ is a subsequence of $\{f_{n,k}\}$. We claim that $f_{n,n}$ converges in the $\rho$-norm to a function $f \in C[0,\infty)$. Since for big enough $n$ $f_{n,n}$ will be a subsequence of $f_{n,k}$ we have that $f_{n,n}$ converges pointwise to a continuous function $g \in C[0,\infty)$. Since $f_{n,n}^{(k)}$ converges uniformly to $f^{(k)}$, we must have that $g^{(k)} = f^{(k)}$. Let $\epsilon > 0$ be given, choose $N$ such that $\sum_{i=N+1}^{\infty} 2^{-i} < \epsilon/2$. Also choose $n'$ such that if $n \geq n'$ we have

$$\sup\{|f_{n,n}^{(N)} - f^{(N)}| : t \in [0,n]\} < \frac{\epsilon}{2N}.$$

We then get if $n \geq n'$

$$\begin{aligned}
\rho(g, f_{n,n}) &= \sum_{k=1}^{\infty} \min(2^{-k}, \sup\{|g(t) - f_{n,n}(t)|, t \in [0,k]\}) \\
&\leq \sum_{k=1}^{N} \sup\{|g(t) - f_{n,n}(t)| : t \in [0,k]\} + \epsilon/2 \\
&\leq N\frac{\epsilon}{2N} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This completes the proof. □

Next we define the Borel $\sigma$-algebra on $C[0,\infty)$.

**Definition 2.2.8.** *Let $\mathcal{C}$ denote the Borel $\sigma$-algebra on $(C[0,\infty), \rho)$. Since $(C[0,\infty), \rho)$ is separable this coincides with the $\sigma$-algebra generated by the open balls.*

The proof that the $\sigma$-algebra generated by the open balls is the same as the $\sigma$-algebra generated by the open sets is the same for all separable metric spaces, and is proved in proposition B.2.10. The next lemma is not used in any proofs, but it ensures that the sets in eq. (2.5) are measurable.

**Lemma 2.2.9.** *Let $f \in C[0,\infty), k \in \mathbb{N}$ be given. The function*

$$\begin{aligned}
&F : (C[0,\infty), \mathcal{C}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ given by} \\
&F(g) = \sup_{k \in [0,k]} \{|g(t) - f(t)|\},
\end{aligned}$$

*is continuous.*

*Proof.* Let

$$\Delta = \min(2^{-(k+1)}, \epsilon).$$

Let $g, h \in C[0,\infty)$ with $\rho(g,h) < \Delta$. Assume without loss of generality that

$$\sup_{k\in[0,k]} \{|g(t) - f(t)|\} \geq \sup_{k\in[0,k]} \{|h(t) - f(t)|\}.$$

We also have because of continuity and the fact that $[0,k]$ is compact

$$\sup_{k\in[0,k]} \{|g(t) - f(t)|\} = |g(t') - f(t')|\}, t' \in [0,k].,$$

for a given $t'$. t Using the reverse triangle inequality we then get

$$\begin{aligned}
&\sup_{k\in[0,k]} \{|g(t) - f(t)|\} - \sup_{k\in[0,k]} \{|h(t) - f(t)|\} \\
&\leq |g(t') - f(t')| - |h(t') - f(t')| \\
&\leq |g(t') - h(t')| \\
&\leq \sup_{t\in[0,k]} \{|g(t) - h(t)|\} \\
&= \min\left(2^{-k}, \sup_{t\in[0,k]} \{|g(t) - h(t)|\}\right) \\
&\leq \rho(g,h) \\
&\leq \epsilon.
\end{aligned}$$

□

The next result is rather technical, but it is needed later. It gives an explicit characterisation of the closed balls in $(C[0,\infty), \rho)$.

**Theorem 2.2.10** (**Characterisation of closed balls.**). *Denote the closed ball around $f \in C[0,\infty)$*

$$\overline{B}(f,\epsilon) = \{g \in C[0,\infty) : \rho(f,g) \leq \epsilon\}.$$

*If $\epsilon < 1$ we have*

$$\overline{B}(f,\epsilon) = \bigcup_{k\in\mathbb{N}} \left\{g \in C[0,\infty) : \sum_{k_2=1}^{k} \sup_{t\in[0,k_2]} \{|f(t) - g(t)|\} \leq \epsilon - 2^{-k}\right\}, \quad (2.5)$$

*note that for some combinations of $\epsilon$ and $k$ the first sets in the union might be empty.*

*Proof.* Let

$$A(f,\epsilon,k) = \left\{g \in C[0,\infty) : \sum_{k_2=1}^{k} \sup_{t\in[0,k_2]} \{|f(t) - g(t)|\} \leq \epsilon - 2^{-k}\right\}.$$

We must prove

$$\overline{B}(f,\epsilon) = \bigcup_{k\in\mathbb{N}} A(f,\epsilon,k).$$

Assume first that $g \in A(f,\epsilon,k')$, we then have

$$\begin{aligned}
\rho(f,g) &= \sum_{k=1}^{\infty} \min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} \\
&= \sum_{k=1}^{k'} \min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} + \sum_{k=k'+1}^{\infty} \min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} \\
&\leq \sum_{k=1}^{k'} \sup_{t\in[0,k]}\{|f(t)-g(t)|\} + 2^{-k'} \\
&\leq \epsilon - 2^{-k'} + 2^{-k'} \\
&= \epsilon.
\end{aligned}$$

Hence

$$\bigcup_{k\in\mathbb{N}} A(f,\epsilon,k) \subset \overline{B}(f,\epsilon).$$

Assume conversely that $g \in \overline{B}(f,\epsilon), g \neq f$(if $g = f$, choose $k'$ big enough, then $f \in A(f,\epsilon,k')$). Since $\epsilon < 1$ there must exist a $k'$ such that for $k > k'$ we have

$$\min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} = 2^{-k},$$

and for $k \leq k'$

$$\min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} = \sup_{t\in[0,k]}\{|f(t)-g(t)|\}.$$

To see this note first that if for a $k$

$$\min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} = 2^{-k},$$

we must also have

$$\min\left\{2^{-(k+1)}, \sup_{t\in[0,k+1]}\{|f(t)-g(t)|\}\right\} = 2^{-(k+1)}.$$

We also can't have for all $k$ that

$$\min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} = 2^{-k},$$

because then $\rho(f,g) = 1 > \epsilon$. And if for $k \geq 2$ we have

$$\min\left\{2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right\} = \sup_{t\in[0,k]}\{|f(t)-g(t)|\},$$

we must also have

$$\min\left\{2^{-(k-1)}, \sup_{t\in[0,k-1]}\{|f(t)-g(t)|\}\right\} = \sup_{t\in[0,k-1]}\{|f(t)-g(t)|\}.$$

Hence if $g \in \overline{B}(f,\epsilon)$, there must be a $k'$ such that

$$\begin{aligned}\rho(f,g) &= \sum_{k=1}^{\infty}\min\left(2^{-k}, \sup_{t\in[0,k]}\{|f(t)-g(t)|\}\right)\\ &= \sum_{k=1}^{k'}\sup_{t\in[0,k]}\{|f(t)-g(t)|\} + \sum_{k=k'+1}^{\infty}2^{-k}.\end{aligned}$$

Since $\rho(f,g) \leq \epsilon$ we get

$$\sum_{k=1}^{k'}\sup_{t\in[0,k]}\{|f(t)-g(t)|\} \leq \epsilon - 2^{-k'}.$$

Hence $g \in A(f,\epsilon,k')$. And we have proven

$$\bigcup_{k\in\mathbb{N}} A(f,\epsilon,k) \supset \overline{B}(f,\epsilon).$$

□

## 2.3 Probabilistic results for $C[0,\infty)$

In this section we will add probability measures to the measurable space $(C[0,\infty),\mathcal{C})$. Note that if $x \in \mathbb{R}^k$, and the coordinates of $x$ are non-negative and distinct, and $(C[0,\infty),\mathcal{C},P)$ is a probability-space, then we also have that $(\mathbb{R}^k,\mathcal{B}(\mathbb{R}^k),P\pi_x^{-1})$ is a probability-space. The collection of all $P\pi_x^{-1}$ is called the finite dimensional distributions for $P$. The next result tells us that the finite-dimensional distributions uniquely determines the probability measure on $(C[0,\infty),\mathcal{C})$. The reason we prove the theorem for coordinates strictly increasing is that it makes things a little easier in the next chapter.

**Theorem 2.3.1** (**Adaptation of [Bil99, Example 1.3, p11]**). *Assume that $P$ and $Q$ are two probability-measures on $(C[0,\infty),\mathcal{C})$. For every positive integer $k$, let $\mathcal{F}_k = \{x : x \in \mathbb{R}^k, x_1, x_2, \ldots, x_k$ are non-negative and $t_1 < t_2 < \cdots < t_k\}$. If for every $x \in \bigcup_{k\in\mathbb{N}}\mathcal{F}_k$, $P\pi_x^{-1}$ and $Q\pi_x^{-1}$ are equal probability-measures on $(\mathbb{R}^k,\mathcal{B}(\mathbb{R}^k))$, where $k$ is such that $x \in \mathbb{R}^k$, then $P$ and $Q$ coincide on $(C[0,\infty),\mathcal{C})$. In other words, if the finite dimensional distributions of $P$ and $Q$ coincide, then $P$ and $Q$ coincide.*

*Proof.* The inspiration for this proof is from [Bil99]. Let

$$\mathcal{K} = \{\pi_x^{-1}(H) : x \in \bigcup_{k\in\mathbb{N}} \mathcal{F}_k, H \in \mathcal{B}(\mathbb{R}^k)\},$$

where $k$ is such that $x \in \mathbb{R}^k$. We have that $\mathcal{K} \subset \mathcal{C}$, because by theorem 2.2.6 $\pi_x$ is continuous. From the hypothesis we have that $P$ and $Q$ coincide on $\mathcal{K}$. If we can show that $\mathcal{K}$ is a $\pi$-system (closed under intersections), and that $\sigma(\mathcal{K}) = \mathcal{C}$, and that the sets $A \in \mathcal{C}$ where $P$ and $Q$ coincide must be a Dynkin-system, then the result will follow from Dynkin's $\pi - \delta$-theorem [Kle13, Theorem 1.19, p. 6].

First we prove that $\mathcal{K}$ is a $\pi$-system. Let $y, z \in \bigcup_{k\in\mathbb{N}} \mathcal{F}_k, y \in \mathbb{R}^{k_y}, z \in \mathbb{R}^{k_z}, H_y \in \mathcal{B}(\mathbb{R}^{k_y}), H_z \in \mathcal{B}(\mathbb{R}^{kz})$. Let $x$ be the vector of strictly increasing coordinates that contains all the coordinates of $y$ and $z$. Note that the length of $x$, $k_x$, may be less than $k_y + k_z$ if $y$ and $z$ have some common elements. Each coordinate in $y$ is mapped to a coordinate in $x$, and likewise each coordinate in $z$ is mapped to a coordinate in $x$. Let $n(y,i) \in \{1, \dots k_x\}$ be the coordinate that the $i - th$ coordinate of $y$ is mapped to, likewise for $z$. Let

$$\begin{aligned} H'_y &= \{a \in \mathbb{R}^{k_x} : (a_{n(y,1)}, a_{n(y,2)}, \dots, a_{n(y,k_y)}) \in H_y\} \\ H'_z &= \{a \in \mathbb{R}^{k_x} : (a_{n(z,1)}, a_{n(z,2)}, \dots, a_{n(z,k_z)}) \in H_z\}. \end{aligned}$$

The sets $H'_y, H'_z$ are Borel sets because the function

$$\begin{aligned} &f_y : \mathbb{R}^{k_x} \to \mathbb{R}^{k_y}, \\ &f_y(a) = (a_{n(y,1)}, a_{n(y,2)}, \dots, a_{n(y,k_y)}), \end{aligned}$$

is continuous, likewise for $z$. The reason that it is continuous is that

$$\max_{1\leq 1\leq k_y} |a_{n(y,i)} - b_{n(y,i)}| \leq \max_{1\leq i\leq k_x} |a_i - b_i|,$$

and using theorem 2.2.5 the result is immediate. In order to show that $\mathcal{K}$ is a $\pi$-system it suffices to show that

$$\pi_y^{-1}(H_y) \cap \pi_z^{-1}(H_z) = \pi_x^{-1}(H'_y \cap H'_z).$$

By elementary set-theory we have

$$\pi_x^{-1}(H'_y \cap H'_z) = \pi_x^{-1}(H'_y) \cap \pi_x^{-1}(H'_z).$$

Hence it suffices to show that

$$\pi_y^{-1}(H_y) = \pi_x^{-1}(H'_y),$$

and likewise for $z$. Assume that $f \in \pi_y^{-1}(H_y)$. Then $(f(y_1), f(y_2), \dots, f(y_{k_y})) \in H_y$. We also have that $\pi_x(f) \in H'_y$ because

$$\begin{aligned} &(\pi_x(f)_{n(y,1)}, \pi_x(f)_{n(y,2)}, \dots, \pi_x(f)_{n(y,k_y)}) \\ &= (f(x_{n(y,1)}), f(x_{n(y,2)}), \dots, f(x_{n(y,k_y)})) \\ &= (f(y_1), f(y_2), \dots, f(y_{k_y})). \end{aligned}$$

Conversely, assume that $f \in \pi_x^{-1}(H'_y)$. We want to show that $\pi_y(f) \in H_y$. Since $f \in \pi_x^{-1}(H'_y)$ we have

$$(\pi_x(f)_{n(y,1)}, \pi_x(f)_{n(y,2)}, \ldots, \pi_x(f)_{n(y,k_y)}) \in H_y,$$

however,

$$\begin{aligned}
&(\pi_x(f)_{n(y,1)}, \pi_x(f)_{n(y,2)}, \ldots, \pi_x(f)_{n(y,k_y)}) \\
&= (f(x_{n(y,1)}), f(x_{n(y,2)}), \ldots, f(x_{n(y,k_y)})) \\
&= (f(y_1), f(y_2), \ldots, f(y_{k_y}) \\
&= \pi_y(f).
\end{aligned}$$

We have proved that $\mathcal{K}$ is a $\pi$-system.

Denote the collection $\mathcal{R}$ by

$$\mathcal{R} = \{A \in \mathcal{C} : P(A) = Q(A)\}.$$

We will show that $\mathcal{R}$ is a Dynkin-system. Obviously $C[0,\infty) \in \mathcal{R}$ since both $P$ and $Q$ are probability-measures. Assume so that $A, B \in \mathcal{R}, B \subset A$, we then have $A \backslash B \in \mathcal{C}$ and

$$P(A\backslash B) = P(A) - B(B) = Q(A) - Q(B) = Q(A\backslash B).$$

Assume that $\{A_i\}_{i\in\mathbb{N}}$ is a disjoint sequence of pairwise disjoint sets in $\mathcal{R}$. Then $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{C}$ and

$$P\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} Q(A_i) = Q\left(\bigcup_{i\in\mathbb{N}} A_i\right).$$

Hence $\mathcal{K}$ is a Dynkin-system.

Lastly we will prove that $\sigma(\mathcal{K}) = \mathcal{C}$. We will do this by proving that each closed ball $\overline{B}(f,\epsilon), f \in C[0,\infty)$ is an element of $\sigma(\mathcal{K})$. If $\epsilon \geq 1$, then $\overline{B}(f,\epsilon) = C[0,\infty)$, so we may assume that $\epsilon < 1$. Let $f \in C[0,\infty), k \in \mathbb{N}$ be given, we will prove that the function

$$F_{f,k}(g) = \sup_{t\in[0,k]} \{|f(t) - g(t)|\},$$

is $\sigma(\mathcal{K})$-measurable. We have that for each $r \in \mathbb{R}, r \geq 0$

$$\begin{aligned}
F_{f,k}^{-1}([0,r]) &= \bigcap_{q\in[0,k]\cap\mathbb{Q}} \{g \in C[0,\infty) : |g(q) - f(q)| \leq r\} \\
&= \bigcap_{q\in[0,k]\cap\mathbb{Q}} \pi_q^{-1}([f(q) - r, f(q) + r]) \in \sigma(\mathcal{K}),
\end{aligned}$$

hence $F_{f,k}$ is $\sigma(\mathcal{K})$-measurable. This means that for given $f \in C[0,\infty), k'$, we have that

$$\begin{aligned}
G_{f,k}(g) &= \sum_{k=1}^{k'} F_{f,k}(g) \\
&= \sum_{k=1}^{k'} \sup_{t\in[0,k]} \{|f(t) - g(t)|\},
\end{aligned}$$

is also $\sigma(\mathcal{K})$-measurable. By theorem 2.2.10 we have

$$\overline{B}(f,\epsilon) = \bigcup_{k\in\mathbb{N}} \left\{ g \in C[0,\infty) : \sum_{k_2=1}^{k} \sup_{t\in[0,k_2]} \{|f(t)-g(t)|\} \leq \epsilon - 2^{-k} \right\}.$$

However we also have

$$\begin{aligned} &\left\{ g \in C[0,\infty) : \sum_{k_2=1}^{k} \sup_{t\in[0,k_2]} \{|f(t)-g(t)|\} \leq \epsilon - 2^{-k} \right\} \\ &= G_{f,k}^{-1}([0,\epsilon - 2^{-k}]) \\ &\in \sigma(\mathcal{K}), \end{aligned}$$

by the measurability just proved. Hence $\overline{B}(f,\epsilon) \in \sigma(\mathcal{K})$, since $(C[0,\infty),\rho)$ is separable (theorem 2.2.2) we have that $\mathcal{C}$ is generated by the open balls (proposition B.2.10), and hence also by the closed balls, so it follows that $\mathcal{C} = \sigma(\mathcal{K})$. ☐

From the proof of theorem 2.3.1 we get a very useful result.

**Corollary 2.3.2.** *The sets of the form*

$$\{f \in C[0,\infty) : f(t') \in B\}, t' \in [0,\infty), B \in \mathcal{B}(\mathbb{R})$$

*generate the sigma-algebra* $\mathcal{C}$.

*Proof.* In the proof of theorem 2.3.1 we showed that the sets of the form

$$\{f \in C[0,\infty) : (f(t_1),\ldots,f(t_k)) \in B\}, k \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_k, B \in \mathcal{B}(\mathbb{R}^k)$$

generate $\mathcal{C}$. So it is sufficient to show that we can generate these sets. Let the sigma-algebra generated by the sets

$$\{f \in C[0,\infty) : f(t') \in B\}, t' \in [0,\infty), B \in \mathcal{B}(\mathbb{R}),$$

be called $\mathcal{G}$. We must show that $\mathcal{G} = \mathcal{C}$. By the statement above we trivially have $\mathcal{G} \subset \mathcal{C}$. By taking intersections we also have that the sets of the form

$$\{f \in C[0,\infty) : f(t') \in B\}, t' \in [0,\infty), B \in \mathcal{B}(\mathbb{R})$$

generate

$$\{f \in C[0,\infty) : (f(t_1),\ldots,f(t_k)) \in B_1 \times B_2 \times \cdots \times B_k\}, t_i \in [0,\infty), B_i \in \mathcal{B}(\mathbb{R}).$$

However we also have by elementary set-theory that the collection

$$\mathcal{R} = \left\{ A \subset \mathbb{R}^k : \{f \in C[0,\infty) : (f(t_1), f(t_2),\ldots,f(t_k)) \in A\} \in \mathcal{G} \right\},$$

where the t's and k are kept fixed, is a sigma-algebra. So by using that the sets of the form $B_1 \times B_2 \times \cdots \times B_k$ generate $\mathcal{B}(\mathbb{R}^k)$, we get that the sets of the form

$$\{f \in C[0,\infty) : (f(t_1), f(t_2),\ldots,f(t_k)) \in B\}, 0 \leq t_1 < t_2 < \ldots < t_k, B \in \mathcal{B}(\mathbb{R}^k),$$

are contained in $\mathcal{G}$. So we have managed to generate the sets we mentioned in the start of the proof. Hence $\mathcal{C} \subset \mathcal{G}$.

☐

## 2.4 Tightness and weak convergence of probability measures in $C[0, \infty)$

In this section we will use the results earlier to create the machinery needed for weak convergence on $C[0, \infty)$. An important concept is tightness of a collection of probability measures. We first define tightness in terms of an arbitrary metric space.

**Definition 2.4.1** (**Tightness, also defined on page 59 in [Bil99]**). *Let $(S, \mathcal{S})$ be a metric space $S$ with its Borel sigma-algebra. We say that a collection of probability measures, $\Pi$, is tight if for every $\epsilon > 0$ there exists a compact set $K \subset S$ such that*

$$P(K) > 1 - \epsilon,$$

*for all $P \in \Pi$.*

In the proof of the next theorem we also need the notion of relative compactness of probability measures. If $(S, \mathcal{S})$ is as in definition 2.4.1 we say that a collection of probability measures, $\Pi$, is relatively compact if every sequence in $\Pi$ has a convergent subsequence converging to a measure $Q$ also defined on $(S, \mathcal{S})$, but $Q$ does not have to be in $\Pi$, this definition can also be found in [Bil99, p. 57]. The type of convergence is weak convergence. The famous Prophorov's theorem connects tightness and relative compactness of probability measures. It says that if a collection of probability measures is tight, it is also relatively compact. And conversely if the metric space is separable and complete we have that relative compactness implies tightness. We will only need the first part.

**Theorem 2.4.2** (**Adaptation of [Sag15, Theorem 4.14, p. 21]**). *Let $P_n$ be probability measures on $(C[0, \infty), \mathcal{C})$, such that $\{P_n\}$ is tight. Assume that their finite dimensional distributions $P_n \pi^{-1}_{(t_1, t_2, \dots, t_k)}$ converges weakly to probability measures $\mu_{(t_1, t_2, \dots, t_k)}$ on the relevant finite dimensional spaces. As before we can assume that $t_1 < t_2 < \dots < t_k$. Then there exists a probability measure $P$ on $(C[0, \infty), \mathcal{C})$ such that $P\pi^{-1}_{(t_1, t_2, \dots, t_k)} = \mu_{(t_1, t_2, \dots, t_k)}$, and also, $P_n$ converges weakly to $P$.*

*Proof.* By Prohorov's theorem ([Bil99, Theorem 5.1, p. 59]) we have that since the collection of probability measures is tight, it is also also relatively compact. By [Bil99, Theorem 2.6, p. 20] it is enough to show that there exists a probability measure $P$ such that for every subsequence $\{P_{n_k}\}$, there is a further subsequence $\{P_{(n_k)_m}\}$ that converges weakly to $P$. So let $\{P_{n_k}\}$ be an arbitrary subsequence. By relative compactness, this sequence contains a subsequence $\{P_{(n_k)_m}\}$ that converges weakly to a probability measure $P'$. Since $\pi_{(t_1, t_2, \dots, t_k)}$ is continuous, the mapping theorem, [Bil99, p. 21] tells us that $\{P_{(n_k)_m} \pi^{-1}_{(t_1, t_2, \dots, t_k)}\}$ converges weakly to $P' \pi^{-1}_{(t_1, t_2, \dots, t_k)}$. Since weak convergence is unique, see [Bil99, p. 14], we have that $P' \pi^{-1}_{(t_1, t_2, \dots, t_k)} = \mu_{(t_1, t_2, \dots, t_k)}$. So we have showed that for every subsequence $\{P_{n_k}\}$ there exists another subsequence $\{P_{(n_k)_m}\}$ that converges weakly to measure $P'$(depending on the subsequences), such that the finite dimensional distribution of $P'$ is given by $\mu$. However, by theorem 2.3.1 probability measures on $(C[0, \infty), \mathcal{C})$ are uniquely determined by their finite dimensional distributions. Hence $P'$ is the same for all the subsequences, and the theorem is proved. □

We end this section with a characterisation of tightness of probability measures on $(C[0,\infty), \mathcal{C})$.

**Theorem 2.4.3 (Adaptation of [Bil99, Theorem 7.3, p.82]).** *Let $\{P_n\}$ be a sequence of probability measures on $(C[0,\infty), \mathcal{C})$. $\{P_n\}$ is tight if and only if the following two conditions hold*

*(i) For each positive $\eta$, there exists an $a$ and an $n_0$ such that*

$$P_n\left(x : |x(0)| \geq a\right) \leq \eta, \qquad n \geq n_0. \tag{2.6}$$

*(ii) For each positive $\epsilon$ and $\eta$, there exists a $\xi, 0 < \xi < 1$, and an $n_0$ such that*

$$P_n\left(x : w_x(\xi) \geq \epsilon\right) \leq \eta, \qquad n \geq n_0. \tag{2.7}$$

*Proof.* Assume first that $\{P_n\}$ is tight. Given $\eta$ choose a compact set $K$ so that $P_n(K) > 1-\eta$ for all $n$. By theorem 2.2.7 we have that $K \subset \{x : |x(0)| < a\}$ for a big enough $a$. Again by theorem 2.2.7 must also be a $\xi \in (0,1)$ such that $\sup_{x\in K} w_x(\xi) < \epsilon$, so $K \subset \{x : w_x(\xi) < \epsilon\}$. So we get for each $n$

$$\begin{aligned} P_n\left(\{x : |x(0)| \geq a\}\right) &= 1 - P_n\left(\{x : |x(0)| < a\}\right) \\ &\leq 1 - P_n\left(K\right) \\ &< 1 - (1-\eta) = \eta. \end{aligned}$$

We also have

$$\begin{aligned} P_n\left(\{x : w_x(\xi) \geq \epsilon\}\right) &= 1 - P_n\left(\{x : w_x(\xi) < \epsilon\}\right) \\ &\leq 1 - P_n\left(K\right) \\ &< 1 - (1-\eta) = \eta. \end{aligned}$$

Hence the conditions must be satisfied with $n_0 = 1$. The fact that $\{x : w_x(\xi) \geq \epsilon\}$ is measurable is because it is closed in $(C[0,\infty), \mathcal{C})$. This can be shown with the aid of theorem 2.2.4.

Conversely assume that both conditions in the theorem hold. First we will prove that if the conditions hold for a given $n_0$, they also hold for $n_0 = 1$. Since $C[0,\infty)$ is separable and complete, it follows from [Bil99, Theorem 1.3, p.8] that a single probability-measure on $C[0,\infty)$ is tight. By what we proved in the previous paragraph condition (i) and (ii) must hold for all $n < n_0$, but they may hold with different $a_n$ and $\xi_n$. But by choosing $a' = \max\{a_1, a_2, \dots, a_{n_0-1}, a\}$, and $\xi' = \min\{\xi_1, \xi_2, \dots, \xi_{n_0-1}, \xi\}$ the conditions $(i), (ii)$ must hold for $n_0 = 1$. Note that if $\xi_1 \leq \xi_2$ we have $\{x : w_x(\xi_1) \geq \epsilon\} \subset \{x : w_x(\xi_2) \geq \epsilon\}$, since $w_x(\xi)$ increases with $\xi$. Hence we can assume that conditions (i) and (ii) hold with $n_0 = 1$. Given $\eta$ choose $a$ such that if

$$B = \{x : |x(0)| \leq a\},$$

then $P_n(B) \geq 1-\eta$ for all $n$. For each $k \in \mathbb{N}$ choose $\xi_k$ such that if

$$B_k = \{x : w_x(\xi_k) < 1/k\},$$

then $P_n(B_k) \geq 1 - \eta/2^k$ for all $n$. Let $K$ be the closure of

$$A = B \cap \bigcap_{k \in \mathbb{N}} B_k.$$

We have for all n

$$\begin{aligned} P_n(A) &= 1 - P_n(A^c) \\ &= 1 - P_n\left(B^c \cup \bigcup_{k\in\mathbb{N}} B_k^c\right) \\ &\geq 1 - \left[P_n(B^c) + \sum_{k=1}^{\infty} P_n(B_k^c)\right] \\ &\geq 1 - \eta - \sum_{k=1}^{\infty} \eta/2^k \\ &= 1 - 2\eta. \end{aligned}$$

We have that $A$ satisfies the conditions in theorem 2.2.7 by noting that if $\xi_1 \leq \xi_2$

$$\{x : w_x(\xi_2) < 1/k\} \subset \{x : w_x(\xi_1) < 1/k\}.$$

So $A$ is relatively compact, which means that $K$ is compact. Hence $\{P_n\}$ is tight.

□

## 2.5 More on tightness

In theorem 2.4.3 we have conditons for tightness on $C[0,\infty)$. However, to use these conditions in practice would be very hard. Later we will need to prove tightness of a sequence of probability measures. In this section we will end up with a more useful criterion for tightness of probability measures on $(C[0,\infty), \mathcal{C})$. First we define the usual known modulus of continuity $w_{x,k}$ on the compact set $[0,k]$.

**Definition 2.5.1.** *For $x \in C[0,\infty), k \in \mathbb{N}, 0 < \xi \leq 1$ we define*

$$w_{x,k}(\xi) \doteq \sup\{|x(t) - x(s)| : s, t, \in [0,k], |s-t| \leq \xi\}.$$

We now prove a technical theorem involcing $w_{x,k}$.

**Theorem 2.5.2** (**Adaptation of [Bil99, Theorem 7.4, p.83]**). *Let $k \in \mathbb{N}$, and suppose that $0 = t_0 < t_1 < \cdots < t_v = k$ and*

$$\min_{1<i<v} (t_i - t_{i-1}) \geq \xi.$$

*Then, for arbitrary $x \in C[0,\infty)$,*

$$w_{x,k}(\xi) \leq 3 \max_{1\leq i\leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\}.$$

*And for a probability measure $P$ on $(C[0,\infty), \mathcal{C})$ and $\epsilon > 0$ we have*

$$P\big(x : w_{x,k}(\xi) \geq 3\epsilon\big) \leq \sum_{i=1}^{v} P\big(x : \sup_{t_{i-1}\leq s\leq t_i} \{|x(s) - x(t_{i-1})| \geq \epsilon\}\big).$$

*Proof.* Assume that $s, t \in [t_{i-1}, t_i]$ we then have

$$|x(s) - x(t)| \leq |x(s) - x(t_{i-1})| + |x(t) - x(t_{i-1})|,$$

by the triangle inequality. This expression is bounded by

$$2 \max_{1 \leq i \leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\}.$$

Assume now that $s \in [t_{i-1}, t_i]$ and $t \in [t_i, t_{i+1}]$ we then get

$$\begin{aligned} |x(s) - x(t)| &\leq |x(s) - x(t_{i-1})| + |x(t_{i-1}) - x(t_i)| + |x_{t_i} - x(t)| \\ &\leq 3 \max_{1 \leq i \leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\}. \end{aligned}$$

If $s, t \in [0, k], |s - t| \leq \xi$ we must have that either $s, t$ is in the same interval $[t_{i-1}, t_i], 1 \leq i < v$, or it must be two different intervals $[t_{i-1}, t_i], [t_i, t_{i+1}], 1 \leq i < v$. Hence we have that

$$3 \max_{1 \leq i \leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\}, \tag{2.8}$$

is an upper bound for

$$\{|x(s) - x(t)| : s, t \in [0, k], |s - t| \leq \xi\}.$$

So (2.8) must be bigger than the *least* upper bound

$$\sup\{|x(s) - x(t)| : s, t \in [0, k], |s - t| \leq \xi\} = w_{x,k}(\xi).$$

For the last part we now get

$$\begin{aligned} &P\big(\{x : w_{x,k}(\xi) \geq 3\epsilon\}\big) \\ &\leq P\big(\{x : 3 \max_{1 \leq i \leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\} \geq 3\epsilon\}\big) \\ &= P\big(\{x : \max_{1 \leq i \leq v} \sup\{|x(s) - x(t_{i-1})| : s \in [t_{i-1}, t_i]\} \geq \epsilon\}\big) \\ &= P\left(\bigcup_{1 \leq i \leq v} \left\{x : \sup_{s \in [t_{i-1}, t_i]} \{|x(s) - x(t_{i-1})|\} \geq \epsilon\right\}\right) \\ &\leq \sum_{i=1}^{v} P\left(\left\{x : \sup_{s \in [t_{i-1}, t_i]} \{|x(s) - x(t_{i-1})|\} \geq \epsilon\right\}\right). \end{aligned}$$

This completes the proof. □

We have a corollary to theorem 2.5.2 that will be used in theorem 2.5.5. The inspiration for the corollary is also from [Bil68]. There a similar criterion as the one in corollary 2.5.3 is needed. However since we work with functions on $C[0, \infty)$ and not just $C[0, 1]$ we have to find a way to connect the modulus of continuity on finite sets, with our modulus of continuity for the entire non-negative real line.

**Corollary 2.5.3.** *Let $\{P_n\}$ be a sequence of probability measures on $(C[0, \infty)$, $\mathcal{C})$, such that $P_n(\{x : x(0) = 0\}) = 1$. Assume that for each $k \in \mathbb{N}, \epsilon > 0, \eta > 0$*

*there exists a $\xi$ such that $0 < \xi < 1$ and $k\xi^{-1}$ is an integer, and that the inequality*

$$\sum_{j=0}^{k\xi^{-1}-1} P_n\left(\left\{x : \sup_{j\xi\le s\le (j+1)\xi}\{|x(s)-x(j\xi)|\} \ge \epsilon\right\}\right) \le \eta,$$

*is satisfied for all $n$. Then $\{P_n\}$ is tight.*

*Proof.* We will verify both conditions of theorem 2.4.3. Condition (i) is trivially satisfied by choosing $a = 1$ and $n_0 = 1$.

Now let $\epsilon, \eta$ be given. If we can prove that there exists a $\xi$ such that $0 < \xi < 1$ and

$$P_n(\{x : w_x(\xi) \ge \epsilon\}) \le \eta,\ \forall n \in \mathbb{N},$$

then condition (ii) will be satisfied with $n_0 = 1$. Choose $k \in \mathbb{N}$ such that

$$\sum_{j=k+1}^{\infty} 2^{-j} < \epsilon/2.$$

By the hypothesis there exists a $\xi, 0 < \xi < 1$ such that $k\xi^{-1}$ is an integer and also

$$\sum_{j=0}^{k\xi^{-1}-1} P_n\left(\left\{x : \sup_{j\xi\le s\le (j+1)\xi}\{|x(s)-x(j\xi)|\} \ge \frac{\epsilon}{6k}\right\}\right) \le \eta. \tag{2.9}$$

We get for all $n$

$$\begin{aligned} P_n(\{x : w_x(\xi) \ge \epsilon\}) &= P_n\left(\left\{x : \sum_{j=1}^{\infty} \min(2^{-j}, w_{x,j}(\xi)) \ge \epsilon\right\}\right) \\ &\le P_n\left(\left\{x : \sum_{j=1}^{k} w_{x,j}(\xi) + \sum_{j=k+1}^{\infty} 2^{-j} \ge \epsilon\right\}\right) \\ &\le P_n\left(\left\{x : \sum_{j=1}^{k} w_{x,j}(\xi) \ge \frac{\epsilon}{2}\right\}\right). \end{aligned} \tag{2.10}$$

If $j \le k$ we have if $s, t \in [0, j], |s-t| \le \xi$

$$|x(t) - x(s)| \le w_{x,k}(\xi),$$

so

$$w_{x,j}(\xi) \le w_{x,k}(\xi).$$

This means that

$$\begin{aligned} P_n\left(\left\{x : \sum_{j=1}^{k} w_{x,j}(\xi) \ge \frac{\epsilon}{2}\right\}\right) &\le P_n\left(\left\{x : k \cdot w_{x,k}(\xi) \ge \frac{\epsilon}{2}\right\}\right) \\ &= P_n\left(\left\{x : w_{x,k}(\xi) \ge \frac{\epsilon}{2k}\right\}\right). \end{aligned} \tag{2.11}$$

If we combine eq. (2.10) and eq. (2.11) we get for all $n$

$$P_n(x : w_x(\xi) \geq \epsilon) \leq P_n\left(w_{x,k}(\xi) \geq \frac{\epsilon}{2k}\right). \tag{2.12}$$

By theorem 2.5.2 we have for all $n$

$$\begin{aligned} P_n\left(x : w_{x,k}(\xi) \geq \frac{\epsilon}{2k}\right) &\leq \sum_{j=1}^{k\xi^{-1}} P_n\left(x : \sup_{(j-1)\xi \leq s \leq j\xi} \{|x(s) - x((j-1)\xi)|\} \geq \frac{\epsilon}{6k}\right) \\ &= \sum_{j=0}^{k\xi^{-1}-1} P_n\left(x : \sup_{(j)\xi \leq s \leq (j+1)\xi} \{|x(s) - x((j)\xi)|\} \geq \frac{\epsilon}{6k}\right). \end{aligned} \tag{2.13}$$

If we combine eq. (2.9), eq. (2.12) and eq. (2.13) we get

$$P_n(x : w_x(\xi) \geq \epsilon) \leq \eta,$$

and the corollary is proved. □

We will now give a theorem from [Bil68] that we do not need to generalize, so the proof is omitted.

**Theorem 2.5.4** (**[Bil68, Theorem 12.2, p 94]**). *Let $\phi_1, \phi_2, \ldots, \phi_m$ be random variables on the same underlying probability space $(\Omega, \mathcal{A}, P)$, they need not be independent. Define $S_0 \doteq 0, S_k \doteq \phi_1 + \cdots + \phi_k$. Also define*

$$M_m \doteq \max_{0 \leq k \leq m} |S_k|.$$

*Let $\gamma \geq 0, \alpha > 1$ be given. Assume there exists nonnegative real numbers $u_1, u_2, \ldots, u_m$ such that for all positive $\lambda$ we have*

$$P(|S_j - S_i| \geq \lambda) \leq \frac{1}{\lambda^\gamma}\left(\sum_{l=i+1}^{j} u_l\right)^\alpha.$$

*We then have*

$$P(M_m \geq \lambda) \leq \frac{K'_{\gamma,\alpha}}{\lambda^\gamma}\left(\sum_{l=1}^{m} u_l\right)^\alpha,$$

*where $K'_{\gamma,\alpha}$ only depends on $\gamma$ and $\alpha$.*

The next theorem is the main result of this section.

**Theorem 2.5.5** (**Adaptation of [Bil68, Theorem 12.3, p. 95]**). *Let $\{P_n\}$ be a sequence of probability measures on $(C[0,\infty), \mathcal{C})$. Assume that the next two conditions are satisfied.*

*(i)* $P_n(x : x(0) = 0) = 1, \forall n.$

*(ii) There exists constants $\gamma \geq 0$ and $\alpha > 1$ and a nondecreasing, continuous function $F : [0,\infty) \to \mathbb{R}$ such that*

$$P_n(\{x : |x(t_1) - x(t_2)| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma}|F(t_2) - F(t_1)|^\alpha,$$

*holds for all $t_2, t_1, n$ and all positive $\lambda$.*

*Then $\{P_n\}$ is tight.*

*Proof.* By corollary 2.5.3 we must show that for each $k \in \mathbb{N}, \epsilon > 0, \eta > 0$ there exists a $\xi$ such that $0 < \xi < 1$ and $k\xi^{-1}$ is an integer, and that for all $n$ the inequality

$$\sum_{j=0}^{k\xi^{-1}-1} P_n\left(\left\{x : \sup_{j\xi \le s \le (j+1)\xi} \{|x(s) - x(t)|\} \ge \epsilon\right\}\right) \le \eta$$

holds. So assume $k, \epsilon, \eta$ is given.

We temporarily fix $j$ as zero or a natural number, and $\xi$ a real number. For $x \in C[0,\infty)$ and for $m$ a positive integer we define

$$\phi_i(x) = x\left(j\xi + \frac{i}{m}\xi\right) - x\left(j\xi + \frac{i-1}{m}\xi\right), i = 1, 2, \ldots, m.$$

Since the projection mapping is continuous we have that for each $n$, these are random variables on $(C[0,\infty), \mathcal{C}, P_n)$. Let $u_i = F(j\xi + i\xi m^{-1}) - F(j\xi + (i-1)\xi m^{-1})$. We have $\alpha$ and $\gamma$ from the hypothesis and assume that $S_i$ is as explained in theorem 2.5.4. Let $\lambda$ be a positive real number, we then have from the hypothesis if $i_1 \le i_2$

$$\begin{aligned} P_n(\{x : |S_{i_2}(x) - S_{i_1}(x)| \ge \lambda\}) &= P_n\left(\left\{x : \left|x\left(j\xi + \frac{i_2}{m}\xi\right) - x\left(j\xi + \frac{i_1 - 1}{m}\xi\right)\right| \ge \lambda\right\}\right) \\ &\le \frac{1}{\lambda^\gamma}\left|F\left(j\xi + \frac{i_2}{m}\xi\right) - F\left(j\xi + \frac{i_1-1}{m}\xi\right)\right|^\alpha \\ &= \frac{1}{\lambda^\gamma}\left(\sum_{l=i_1}^{i_2} u_i\right)^\alpha. \end{aligned}$$

This means that the hypothesis in theorem 2.5.4 is satisfied and hence we have with $\lambda = \epsilon$

$$\begin{aligned} P_n\left(\left\{x : \max_{0 \le i \le m}\left|x\left(j\xi + \frac{i}{m}\xi\right) - x(j\xi)\right| \ge \epsilon\right\}\right) &\le \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma}\left(\sum_{l=1}^{m} u_i\right)^\alpha \\ &= \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma}|F((j+1)\xi) - F(j\xi)|^\alpha. \end{aligned}$$

Define

$$A_m \doteq \left\{x \in C[0,\infty) : \max_{0 \le i \le m}\left|x\left(j\xi + \frac{i}{m}\xi\right) - x(j\xi)\right| \ge \epsilon\right\}.$$

If $b \in \mathbb{N}$ we have $A_{2^b} \subset A_{2^{b+1}}$. We also have because of continuity

$$\left\{x \in C[0,\infty) : \sup_{j\xi \le s \le (j+1)\xi} |x(s) - x(j\xi)| \ge \epsilon\right\} \subset \bigcup_{b \in \mathbb{N}} A_{2^b}.$$

So we get by the continuity of measures.

$$
\begin{aligned}
& P_n\left(\left\{x \in C[0,\infty) : \sup_{j\xi \le s \le (j+1)\xi} |x(s) - x(j\xi)| \ge \epsilon\right\}\right) \\
\le{}& P_n\left(\bigcup_{b\in\mathbb{N}} A_{2^b}\right) \\
={}& \lim_{b\to\infty} P_n(A_{2^b}) \\
\le{}& \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma} |F((j+1)\xi) - F(j\xi)|^\alpha .
\end{aligned}
$$

We then have if $k\xi^{-1}$ is an integer

$$
\begin{aligned}
& \sum_{j=0}^{k\xi^{-1}-1} P_n\left(\left\{x : \sup_{j\xi \le s \le (j+1)\xi} \{|x(s) - x(t)|\} \ge \epsilon\right\}\right) \\
\le{}& \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma} \sum_{j=0}^{k\xi^{-1}-1} \Big|F((j+1)\xi) - F(j\xi)\Big|^\alpha \\
={}& \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma} \sum_{j=0}^{k\xi^{-1}-1} \Big(F((j+1)\xi) - F(j\xi)\Big)\Big(F((j+1)\xi) - F(j\xi)\Big)^{\alpha-1} \\
\le{}& \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma} \Big[\max_{0\le j<k\xi^{-1}} |F((j+1)\xi) - F(j\xi)|\Big]^{\alpha-1} \sum_{j=0}^{k\xi^{-1}-1} \Big(F((j+1)\xi) - F(j\xi)\Big) \\
={}& \frac{K'_{\gamma,\alpha}}{\epsilon^\gamma} \Big[\max_{0\le j<k\xi^{-1}} |F((j+1)\xi) - F(j\xi)|\Big]^{\alpha-1} (F(k) - F(0)) .
\end{aligned}
$$

Since $\alpha > 1$ and by uniform continuity on $[0, k]$ we can get this expression as small as we want by choosing $\xi = k/N$ where $N$ is big enough. Hence the proof is complete. □

We give a corollary to theorem 2.5.5 where we use moment conditions in (ii) instead of probability conditions. These conditions are also explained in [Bil68, p. 95].

**Corollary 2.5.6.** *Let $\{P_n\}$ be a sequence of probability measures on $(C[0,\infty)$, $\mathcal{C})$. Assume that the next two conditions are satisfied.*

*(i)* $P_n(x : x(0) = 0) = 1, \forall n$.

*(ii) There exists constants $\gamma \ge 0$ and $\alpha > 1$ and a nondecreasing, continuous function $F$ $[0,\infty)$ such that*

$$E_n[|x(t_2) - x(t_1)|^\gamma] \le |F(t_2) - F(t_1)|^\alpha,$$

*holds for all $t_2, t_1, n$.*

*Then $\{P_n\}$ is tight.*

*Proof.* This follows directly because if $\lambda \geq 0$ we get

$$\begin{aligned}
P_n(\{x : |x(t_2) - x(t_1) \geq \lambda|\}) &= E_n[1_{\{x:|x(t_2)-x(t_1)|\geq\lambda\}}(x)] \\
&= E_n[1_{\{x:|x(t_2)-x(t_1)|^\gamma\geq\lambda^\gamma\}}(x)] \\
&= E_n[1_{\{x:|x(t_2)-x(t_1)|^\gamma\geq\lambda^\gamma\}}(x)\lambda^\gamma]\frac{1}{\lambda^\gamma} \\
&\leq \frac{1}{\lambda^\gamma}E_n[|x(t_2) - x(t_1)|^\gamma] \\
&\leq \frac{1}{\lambda^\gamma}|F(t_2) - F(t_1)|^\alpha.
\end{aligned}$$

□

# Chapter 3

# Weak convergence of the Mandelbrot and Van Ness sum

In this chapter we will show that a discrete stochastic process, where we have linear interpolation between the time points, converges weakly in $C[0,\infty)$ to the fractional brownian motion. I have decided to name this discrete stochastic process the Mandelbrot and Van Ness sum, why this name is fitting will become clear after we explain the work of Mandelbrot and Van Ness.

## 3.1 Introduction to the Fractional Brownian motion

The existence of the fractional Brownian motion has already been shown in other books, so we will not do it here. We will give a modified proposition from another book, for the description and existence of the fractional Brownian motion. First the definition of a Gaussian process.

**Definition 3.1.1 (modified from [Nou12, p. 7]).** *A stochastic process $X = (X_t)_{t\in[0,\infty)}$ is said to be Gaussian if, for all $d \geq 1$ and all $t_1, \ldots, t_d \in [0,\infty)$, $(X_{t_1}, \ldots, X_{t_d})$ is a Gaussian random vector. If the expectation of $(X_{t_1}, \ldots, X_{t_d})$ is $\vec{0}$ we say that $X$ is centered.*

A Gaussian random vector can be defined as follows.

**Definition 3.1.2.** *A real vector $(Y_1, Y_2, \ldots, Y_k)$ is Gaussian if there exists a real vector $\vec{m} \in \mathbb{R}^k$, and symmetric positive semi-definite matrix $A \in \mathbb{R}^k \times \mathbb{R}^k$ such that,*

$$E\left[\exp\left(\sum_{j=1}^{k} iu_jY_j\right)\right] = \exp\left(i\sum_{j=1}^{k} u_jm_j - \frac{1}{2}\sum_{j_1=1}^{k}\sum_{j_2=1}^{k} A_{j_1,j_2}u_{j_1}u_{j_2}\right).$$

*The distribution is called a multivariate normal distribution.*

**Remark.** *Sometimes it is required that the vector in definition 3.1.2 is positive definite, not positive semi-definite. We have to use the positive semi-definite definition, because at $t = 0$ the fBm is a.s. zero, so its distribution is degenerate, but we still want to call it Gaussian. It can be shown that $E[Y_j] = m_j$, and $cov(Y_{j_1}, Y_{j_2}) = A_{j_1,j_2}$. Note also that conversely if you have a real vector $\vec{m}$ and* *a symmetric positive semi-definite matrix $A$ then there exists a Gaussian vector with mean-vector $\vec{m}$ and coviariance matrix and $A$, see [Nou12, section 1.1].*

Now we give the description and existence of the Fractional Brownian motion.

**Proposition 3.1.3 (modified version of Proposition 1.6, p. 8 in [Nou12]).** *Let $H > 0$ be a real parameter. Then, there exists a continuous centered Gaussian process $B_H = (B_{t,H})_{t \geq 0}$ with covariance given by*

$$cov(B_{s,H}, B_{t,H}) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), s, t \geq 0, \tag{3.1}$$

*if and only if $H \leq 1$. We call this process the **Fractional Brownian motion**.*

**Remark.** *Proposition 3.1.3 tells us that the finite-dimensional distributions are Gaussian. So we do not have to check that the covariance matrix is positive semi-definite. That is, if a covariance-matrix is given by eq. (3.1) it must be positive semi-definite. Note also that by using projection mappings the proposition tells us that if we have $0 \leq t_1 < t_2 < \ldots < t_k$ there exists a Gaussian random vector $(Y_1, Y_2, \ldots, Y_k)$ with expectation zero for each component, and $cov(Y_j, Y_l) = \frac{1}{2}(t_j^{2H} + t_l^{2H} - |t_j - t_l|^{2H})$.*

As mentioned in [DOT02, p. 9] the case $H = 1$ corresponds to a straight line. So we will restrict ourselves to the case $H \in (0, 1)$. We give a simple proof as to why the case $H = 1$ is of no interest.

**Proposition 3.1.4.** *Assume that for $H = 1$ the stochastic process $B_1 = (B_{t,1})_{t \in [0,\infty)}$ is defined on the underlying probability space $(\Omega, \mathcal{A}, P)$. Then there is a set $A \in \mathcal{A}$ with $P(A) = 1$, such that for every $\omega \in A$ we have*

$$B_{t,1}(\omega) = tB_{1,1}(\omega).$$

*Proof.* For every rational number $q \in \mathbb{Q} \cap [0, \infty)$ we have by eq. (3.1)

$$\begin{aligned} E[(B_{q,1} - qB_{1,1})^2] &= E[B_{q,1}^2 - 2qB_{q,1}B_{1,1} + q^2 B_{1,1}^2] \\ &= q^2 - 2q^2 + q^2 \\ &= 0. \end{aligned}$$

So for each $q \in \mathbb{Q}$ there must exist a set $A_q$ with $P(A_q) = 1$ such that for $\omega \in A_q$

$$B_{q,1}(\omega) = qB_{1,1}(\omega).$$

Now we let

$$A \doteq \bigcap_{q \in \mathbb{Q}} A_q$$

We get

$$\begin{aligned} P(A) &= 1 - P(A^c) \\ &= 1 - P\left(\bigcup_{q\in\mathbb{Q}} A_q^c\right) \\ &\geq 1 - \sum_{q\in\mathbb{Q}} P(A_q^c) \\ &= 1. \end{aligned}$$

We also have that if $\omega \in A$ for every rational number $q \in \mathbb{Q} \cap [0, \infty)$

$$B_{q,1}(\omega) = qB_{1,1}(\omega).$$

However, because of continuity this must hold for every $t \in [0, \infty)$. To see this choose a sequence of rational numbers $(q_n) \to t$, with each $q_n \in [0, \infty)$. If we hold $\omega \in A$ fixed we have because of continuity of the process that

$$B_{q_n,1}(\omega) \to B_{t,1}(\omega).$$

And because of the convergence of rational numbers we have that

$$q_n B_{1,1}(\omega) \to tB_{1,1}(\omega).$$

Since limits in $\mathbb{R}$ are unique under the Euclidean metric, the result follows. □

Because of proposition 3.1.4 we will only consider $H \in (0, 1)$.

## 3.2 Historical perspective, and definition of the Mandelbrot and Van Ness sum

In this section we will present some representations of the fractional Brownian motion as stochastic integrals with respect to the Brownian motion. These stochastic integrals are not precisely the same stochastic integrals that are taught at most graduate courses today, where books like [Øks03] or [KS12] are used. The main difference is that the stochastic integrals we will look at are taken over an infinite interval. We will not go into the theory of these stochastic integrals as they will only serve as an inspiration to the discrete approximation that we will choose. We will instead prove that our discrete approximation with linear interpolation converges weakly in $C[0, \infty)$ without using the theory of stochastic integrals.

In the 1968 paper [MV68] Benoit B. Mandelbrot and John W. Van Ness defined their version of the Fractional Brownian motion as a stochastic integral with respect to a Brownian motion. First they present the process

$$\begin{aligned} B_H(0, \omega) &= b_0 \\ B_H(t, \omega) - B_H(0, \omega) &= \frac{1}{\Gamma(H + 1/2)}\left(\int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}]dB(s, \omega)\right. \\ &\qquad \left. + \int_{0}^{t} (t - s)^{H-1/2} dB(s, \omega)\right). \end{aligned} \tag{3.2}$$

As we mentioned earlier we will not go into the construction of this stochastic integral. However, it is worth noticing that we do not get the same Fractional Brownian Motion as in section 3.1 because the scaling factor $\Gamma(H+1/2)$ does not give us the correct variance. Mandelbrot and Van Ness later in the article calculates the variance, and gives us the scaling factor we need if we want the fractional Brownian motion to have variance as in section 3.1, that is $t^{2H}$. The name *Fractional Brownian motion* can be motivated because the representation above can be viewed as a fractional integral where the integrator is a Brownian motion.

In [ST94] there is another representation of Fractional Brownian motion in terms of stochastic integrals. Some notation is needed here, we will introduce the notation $u_+$. Let $r$ be a real number, we define

$$u_+^r \doteq \begin{cases} u^r & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases} \tag{3.3}$$

The representation in [ST94] for the fBm is given as

$$\frac{1}{C_H}\int_{-\infty}^{\infty}\left((t-x)_+^{H-1/2}-(-x)_+^{H-1/2}\right)M(dx), t\in\mathbb{R}$$
$$C_H=\left(\int_0^{\infty}\left((1+x)^{H-1/2}-x^{H-1/2}\right)^2 dx+\frac{1}{2H}\right)^{1/2}.$$

In this representation $t$ can be on the entire real line. We will again not go into the construction of this integral and the meaning of $M(dx)$ etc., the interested reader can read chapter 3 and 7 in [ST94]. However, it is worth noticing the similarities between the two stochastic integral representations of the fractional Brownian motion.

To approximate the Fractional Brownian motion we look at the two representations we have above. We work heuristically and make a guess of what is a good approximation of the fBm, and we will later prove that indeed this guess converges weakly in $C[0,\infty)$ to the fBm. We let $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ be a collection of independent random variables defined on a probability space $(\Omega, \mathcal{A}, P)$, each taking the values $\pm 1$ with probability $1/2$. The existence of these random variables and the probability space can be justified by the discussion in [Wil91, p. 42]. Let $\delta > 0$ be a real number. From the representations above we hypothesise that a good representation for the fBm is as following, if $t \geq 0$ is a multiple of $\delta$ we have

$$\frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}, \tag{3.4}$$

and we interpolate linearly between those values of $t$ that are not a multiple of $\delta$. The summation is meant to be such that $\tau$ takes a step-length size $\delta$ for each step, up to and including $t-\delta$. We make a formal definition of this summation.

**Definition 3.2.1.** *Assume $\delta > 0$, and that $a \leq b$ where both $a$ and $b$ are multiples of $\delta$. Let $a_\tau$ be a sequence indexed by $\tau$ where the $\tau$s are multiples of*

$\delta$.

$$\sum_{\tau=a}^{b}{}_{\delta}\, a_\tau \doteq \sum_{r=a/\delta}^{b/\delta} a_{r\delta}.$$

*If $b < a$ we define*

$$\sum_{\tau=a}^{b}{}_{\delta}\, a_\tau \doteq 0.$$

*And we define*

$$\sum_{\tau=-\infty}^{a}{}_{\delta} \doteq \sum_{r=-\infty}^{a/\delta} a_{r\delta}.$$

Equation (3.4) is a heuristic approximation of the stochastic integrals from [MV68] and [ST94] given above, where $\sqrt{\delta} w_{\tau/\delta}$ approximates $dB$. The first thing we have to check is that our sum is well-defined. We first check that $C_H$ is well-defined.

**Proposition 3.2.2.** *We have that*

$$C_H \doteq \left( \int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right)^{1/2},$$

*where $C_H > 0$ is well-defined.*

*Proof.* Recall that $H \in (0,1)$. We need to show that the integral,

$$\int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx,$$

is well-defined. That means that we need to show that it is finite, and hence that the integrand is integrable. We split the integral in the two parts

$$\int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx = \int_0^1 \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \int_1^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx.$$

We look at the first integral, recall that $(a-b)^2 \leq 2a^2 + 2b^2$.

$$\int_0^1 \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx \leq 2 \int_0^1 \left( (1+x)^{2H-1} + x^{2H-1} \right) dx.$$

We have that $(1+x)^{2H-1}$ is continuous on $[0,1]$, and hence bounded. We also have that

$$\int_0^1 x^{2H-1} dx < \infty,$$

because $1 - 2H < 1$. For the second integral

$$\int_1^\infty \left((1+x)^{H-1/2} - x^{H-1/2}\right)^2 dx,$$

we will compare it with the integral of the function $1/x^{3-2H}$ which converges because $3 - 2H > 1$. We calculate the limit of the square-roots

$$\lim_{x\to\infty} \frac{(1+x)^{H-1/2} - x^{H-1/2}}{1/x^{3/2-H}} = \lim_{x\to\infty} \frac{\left(\frac{1+x}{x}\right)^{H-1/2} - 1}{x^{-1}}.$$

Using l'Hôpital's rule we get

$$\begin{aligned}\lim_{x\to\infty} \frac{\left(\frac{1+x}{x}\right)^{H-1/2} - 1}{x^{-1}} &= \lim_{x\to\infty} \frac{(H-1/2)\left(\frac{1+x}{x}\right)^{H-3/2} \frac{x-(1+x)}{x^2}}{-x^{-2}} \\ &= H - \frac{1}{2}.\end{aligned}$$

This means that

$$\lim_{x\to\infty} \frac{((1+x)^{H-1/2} - x^{H-1/2})^2}{1/x^{3-2H}} = \left(H - \frac{1}{2}\right)^2.$$

This means that there exists an $M$ such that when $x \geq M$ we have that

$$((1+x)^{H-1/2} - x^{H-1/2})^2 \leq 2 \cdot \frac{(H-0.5)^2}{x^{3-2H}}.$$

Hence the result follows. □

So now we have established that $C_H$ is a well-defined real number. We now give the definition of the Mandelbrot and Van Ness sum, and then we prove that the sum converges with probability 1. Later we will redefine it on the set that has probability one.

**Definition 3.2.3 (first definition).** *Let $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Define the stochastic process $X^{(\delta)} = (X_t^{(\delta)})_{t\in[0,\infty)}$, which also depends on $H \in (0,1)$ by*

*(i) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$ we define*

$$X_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta}\, w_{\tau/\delta}(\omega),$$

*where*

$$C_H \doteq \left(\int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2}\right)^2 dx + \frac{1}{2H}\right)^{1/2}.$$

*(ii) We extend $X^\delta$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define*

$$X_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot X_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot X_{(N+1)\delta}^{(\delta)}(\omega).$$

*We call $X^\delta$ the Mandelbrot and Van Ness sum.*

What remains is proving that the sum in the definition 3.2.3 converges so that we know that the stochastic process is well-defined. We begin with a lemma.

**Lemma 3.2.4.** *Assume that $H \in (0,1)$, $\delta > 0$ and that $t = L\delta, L \in \mathbb{N} \cup \{0\}$. We then have*

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 < \infty.$$

*Proof.* If $t = 0$, the result is obvious, so we assume $t > 0$. We rewrite the expression

$$\begin{aligned}
&\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 \\
= &\sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 + \sum_{\tau=0}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 \\
\leq &\sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 + M \\
= &\sum_{k=1}^{\infty} \left[(t+k\delta)^{H-\frac{1}{2}} - (k\delta)^{H-\frac{1}{2}}\right]^2 + M.
\end{aligned}$$

We have that $t = L\delta, L \in \mathbb{N}$. So the last expression becomes

$$\delta^{2H-1} \sum_{k=1}^{\infty} \left[(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right]^2 + M.$$

So the result follows if we can show that for $L \in \mathbb{N}$ we have

$$\sum_{k=1}^{\infty} \left[(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right]^2 < \infty.$$

We will use the comparison test with the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{3-2H}},$$

which we know converges since $3 - 2H > 1$. We must show

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\left[(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right]^2}{\frac{1}{k^{3-2H}}} \\
= &\lim_{k\to\infty} \left[(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right]^2 k^{3-2H} < \infty.
\end{aligned}$$

Or equivalently

$$\lim_{k\to\infty} \left|(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right| k^{3/2-H} < \infty.$$

We have

$$\begin{aligned} &\left|(L+k)^{H-\frac{1}{2}} - (k)^{H-\frac{1}{2}}\right| k^{3/2-H} \\ = &\left|\left(\frac{L+k}{k}\right)^{H-\frac{1}{2}} - 1\right| k. \end{aligned}$$

So the result will follow if

$$\lim_{k\to\infty} \left(\left(\frac{L+k}{k}\right)^{H-\frac{1}{2}} - 1\right) k,$$

exists and do not take the values $\pm\infty$.

But this follows by rewriting the expression to

$$\lim_{k\to\infty} \frac{\left(\frac{L+k}{k}\right)^{H-\frac{1}{2}} - 1}{\frac{1}{k}},$$

and using l'Hôpital's rule. □

Now we prove that there is a set with probability one, where the series converges.

**Proposition 3.2.5.** *Let $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Assume also that $H \in (0,1)$ and that $\delta > 0$. Then there exists a set $A_\delta \in \mathcal{A}$(also depending on H) with*

$$P(A_\delta) = 1,$$

*such that if $t \geq 0$ is a real number, and $t$ is a multiple of $\delta$ we have that for $\omega \in A_\delta$*

$$\frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega)$$

*converges in $\mathbb{R}$.*

*Proof.* Assume first that $t$ is a multiple of $\delta$. From lemma 3.2.4 we have

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \frac{1}{C_H^2} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 \delta < \infty.$$

It then follows [MW13, Proposition 7.11, p. 260], that there is a set $A_{\delta,t} \in \mathcal{A}$ with $P(A_{\delta,t}) = 1$. Such that for $\omega \in A_{\delta,t}$ we have that

$$\frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega),$$

converges in $\mathbb{R}$. The fact that $A_{\delta,t} \in \mathcal{A}$ is not seen in the statement of [MW13, Proposition 7.11, p. 260] but its proof. There the set where we have convergence is in their notation $E^c$, it is seen from the proof that this is a measurable set. The same is the fact that the convergence is in $\mathbb{R}$ and not for example in the extended real numbers, this follows from the fact that the proof uses Cauchy-sequences in $\mathbb{R}$.

There are only a countable number of $t \geq 0$ that is a multiple of $\delta_n$, let

$$A_\delta = \bigcap_{r \in \mathbb{N} \cup \{0\}} A_{\delta, r\delta}.$$

Because of countability and elementary properties of measures we have that $P(A_\delta) = 1$. By construction $A_\delta$ has the required properties, and the proof is done. □

We now see that definition 3.2.3 is well-defined. We may redefine the process on a set of probability zero to ensure that we have convergence everywhere, we define it so that on the set $A_\delta^c$ in proposition 3.2.5 the process is identically equal to zero. That means that if $A_\delta$ is the set in proposition 3.2.5, and $t \geq 0$ is a multiple of $\delta$ we have

$$X_t^\delta(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega), \qquad (3.5)$$

and we interpolate linearly as before for other values of $t$. Notice that for a given $t$ $X_t^\delta$ will be a random variable on $(\Omega, \mathcal{A}, P)$. We require measurability for this to be the case. However it is measurable because the set $A_\delta$ is measurable and $X_t^{(\delta)}$ is a limit of measurable functions. We redefine the Mandelbrot and Van Ness sum to ensure that it is well-defined.

**Definition 3.2.6 (modified).** *Let $H \in (0,1), \delta > 0$ be given. Let $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $A_\delta$ be as in proposition 3.2.5. Define the stochastic process $X^{(\delta)} = (X_t^{(\delta)})_{t \in [0,\infty)}$, which also depends on $H$ like this*

*(i) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$ we define*

$$X_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega),$$

*where*

$$C_H \doteq \left( \int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right)^{1/2}.$$

*(ii) We extend $X^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a unique number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define*

$$X_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot X_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot X_{(N+1)\delta}^{(\delta)}(\omega).$$

*We call $X^{(\delta)}$ the Mandelbrot and Van Ness sum.*

We need an explicit formula for $X_t^{(\delta)}$ so we end this section by deriving this. It is advantageous to work with an ordinary sum and not the delta-sum $\sum_{\tau=-\infty}^{T}{}_{\delta}$, so we will try to get an expression ridding ourselves of this sum. We let $\lfloor x \rfloor$ be the floor function. This means that the if $t$ is not a multiple of $\delta$ the $N$ in definition 3.2.3 is $\lfloor t/\delta \rfloor$. So we can write assuming that $\omega \in A_\delta$ and we suppress the $\omega$ in our notation

$$\begin{aligned}
X_t^{(\delta)} &= \frac{1}{C_H}[(\lfloor t/\delta \rfloor + 1)\delta - t]/\delta \cdot X^{(\delta)}_{\lfloor t/\delta \rfloor \delta} \frac{1}{C_H}(t - \lfloor t/\delta \rfloor \delta)/\delta X^{(\delta)}_{\lfloor t/\delta \rfloor \delta + \delta} \\
&= \frac{1}{C_H}(1 + \lfloor t/\delta \rfloor - t/\delta) \cdot X^{(\delta)}_{\lfloor t/\delta \rfloor \delta} + \frac{1}{C_H}(t/\delta - \lfloor t/\delta \rfloor) X^{(\delta)}_{\lfloor t/\delta \rfloor \delta + \delta} \\
&= \frac{1}{C_H}(1 + \lfloor t/\delta \rfloor - t/\delta) \sum_{\tau=-\infty}^{\lfloor t/\delta \rfloor \delta - \delta}{}_{\delta} \left[ (\lfloor t/\delta \rfloor \delta - \tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_{\tau/\delta} \\
&\quad + \frac{1}{C_H}(t/\delta - \lfloor t/\delta \rfloor) \sum_{\tau=-\infty}^{\lfloor t/\delta \rfloor \delta}{}_{\delta} \left[ (\lfloor t/\delta \rfloor \delta + \delta - \tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_{\tau/\delta} \\
&= \frac{1}{C_H}(1 + \lfloor t/\delta \rfloor - t/\delta) \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \left[ (\lfloor t/\delta \rfloor \delta - r\delta)^{H-\frac{1}{2}} - (-r\delta)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_r \\
&\quad + \frac{1}{C_H}(t/\delta - \lfloor t/\delta \rfloor) \sum_{r=-\infty}^{\lfloor t/\delta \rfloor} \left[ (\lfloor t/\delta \rfloor \delta + \delta - r\delta)^{H-\frac{1}{2}} - (-r\delta)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_r \\
&= \frac{\delta^H}{C_H}(1 + \lfloor t/\delta \rfloor - t/\delta) \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \left[ (\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] w_r \\
&\quad + \frac{\delta^H}{C_H}(t/\delta - \lfloor t/\delta \rfloor) \sum_{r=-\infty}^{\lfloor t/\delta \rfloor} \left[ (\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \right] w_r,
\end{aligned}$$

where we have suppressed the dependence on $\omega$, however for the sum to make sense we must have $\omega \in A_\delta$. We define

**Definition 3.2.7.**

$$\begin{aligned}
a(t, r, \delta) = {} & (1 + \lfloor t/\delta \rfloor - t/\delta) \cdot (\lfloor t/\delta \rfloor - r)^{H-1/2} \\
& + (t/\delta - \lfloor t/\delta \rfloor) \cdot (\lfloor t/\delta \rfloor + 1 - r)^{H-1/2} \\
& - (-r)_+^{H-1/2}.
\end{aligned}$$

This means that by using definition 3.2.6 we get

$$X_t^{(\delta)} = \frac{\delta^H}{C_H} \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} a(t, r, \delta) w_r I_{A_\delta} + \frac{\delta^H}{C_H}(t/\delta - \lfloor t/\delta \rfloor) w_{\lfloor t/\delta \rfloor} I_{A_\delta}. \tag{3.6}$$

$a(t, r, \delta)$ will be difficult to work with, so we define $\underline{a}(t, r, \delta)$ and $\overline{a}(t, r, \delta)$ which are easier to work with.

**Definition 3.2.8.**

$$\underline{a}(t,r,\delta) \doteq (\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}$$
$$\overline{a}(t,r,\delta) \doteq (\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}.$$

Hence we have

$$a(t,r,\delta) = (1 - t/\delta + \lfloor t/\delta \rfloor)\underline{a}(t,r,\delta) + (t/\delta - \lfloor t/\delta \rfloor)\overline{a}(t,r,\delta).$$

We give a corollary to lemma 3.2.4 in the notation of $a(t,r,\delta)$.

**Corollary 3.2.9.** *Let $H \in (0,1), \delta > 0, t \in [0,\infty)$. We then have*

$$\sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \delta^{2H} a(t,r,\delta)^2 < \infty.$$

*Proof.* We have, using $(a+b)^2 \le 2a^2 + 2b^2$, that

$$\begin{aligned}
&\sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \delta^{2H} a(t,r,\delta)^2 \\
&= \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \delta^{2H} \left((1 - t/\delta + \lfloor t/\delta \rfloor)\underline{a}(t,r,\delta) + (t/\delta - \lfloor t/\delta \rfloor)\overline{a}(t,r,\delta)\right)^2 \\
&\le 2\delta^{2H} \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \left(\underline{a}(t,r,\delta)^2 + \overline{a}(t,r,\delta)^2\right).
\end{aligned}$$

We look at $\underline{a}$ and $\overline{a}$ separately. We have

$$\begin{aligned}
\delta^{2H} \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \underline{a}(t,r,\delta)^2 &= \delta^{2H} \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \left((\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right)^2 \\
&= \delta \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} \left((\lfloor t/\delta \rfloor \delta - r\delta)^{H-\frac{1}{2}} - (-r\delta)_+^{H-\frac{1}{2}}\right)^2 \\
&= \delta \sum_{\tau=-\infty}^{\lfloor t/\delta \rfloor \delta - \delta}{}_{\delta} \left((\lfloor t/\delta \rfloor \delta - \tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right)^2.
\end{aligned}$$

The last term is finite by lemma 3.2.4. For $\overline{a}$ we get similarly

$$\begin{aligned}
\delta^{2H}\sum_{r=-\infty}^{\lfloor t/\delta\rfloor-1}\overline{a}(t,r,\delta)^2 &= \delta^{2H}\sum_{r=-\infty}^{\lfloor t/\delta\rfloor-1}\left((\lfloor t/\delta\rfloor+1-r)^{H-\frac{1}{2}}-(-r)_+^{H-\frac{1}{2}}\right)^2\\
&= \delta\sum_{r=-\infty}^{\lfloor t/\delta\rfloor-1}\left((\lfloor t/\delta\rfloor\delta+\delta-r\delta)^{H-\frac{1}{2}}-(-r\delta)_+^{H-\frac{1}{2}}\right)^2\\
&= \delta\sum_{\tau=-\infty}^{\lfloor t/\delta\rfloor\delta-\delta}{}_{\delta}\left((\lfloor t/\delta\rfloor\delta+\delta-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right)^2\\
&= \delta\sum_{\tau=-\infty}^{\lfloor t/\delta\rfloor\delta-\delta}{}_{\delta}\left(((\lfloor t/\delta\rfloor+1)\delta-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right)^2\\
&\leq \delta\sum_{\tau=-\infty}^{(\lfloor t/\delta\rfloor+1)\delta-\delta}{}_{\delta}\left(((\lfloor t/\delta\rfloor+1)\delta-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right)^2.
\end{aligned}$$

The last term is again finite by lemma 3.2.4. □

## 3.3 Induced measures

In this section we will clarify what we mean when we say that sequence of stochastic processes converges weakly to the fBm. Let $\delta_n$ be given. We will then show that $X^{(\delta_n)}$ induces a probability measure on $(C[0,\infty),\mathcal{C})$, this measure will be called $P_n$. So if $\{\delta_n\}$ is a sequence of positive real numbers converging to zero, we will by the end of this chapter show that $\{P_n\}$ converges weakly to a measure $P$ on $(C[0,\infty),\mathcal{C})$ and this measure will have the distribution properties of the fBm. It is important to be aware of this terminology, when we say that a sequence of stochastic processes converges weakly to another stochastic process, we mean that the induced measures converges weakly to the induced measure of the process.

**Definition 3.3.1.** *Let $H\in(0,1),\delta_n>0$. Let $X^{(\delta_n)}$ be as in definition 3.2.6. We define the measure $P_n$ on $(C[0,\infty),\mathcal{C})$ as*

$$P_n(B)=P(X^{(\delta_n)})\in B), B\in\mathcal{C},$$

*here $X^{(\delta_n)}$ denotes the entire process on $[0,\infty)$.*

This definition is well-defined because for each $B\in\mathcal{C}$ we have

$$(X^{(\delta_n)})^{-1}(B)\in\mathcal{A},$$

where $(\Omega,\mathcal{A},P)$ is the underlying probability space. To see this, first keep $t$ fixed as $t'$, since $X_{t'}^{(\delta_n)}$ is pointwise limit of random variables we have that $X_{t'}^{\delta_n}$ also is measurable, remember that $A_{\delta_n}$ is measurable, see the discussion surrounding eq. (3.5) or definition 3.2.6. So for given $t', B'\in\mathcal{B}(\mathbb{R})$ we have that

$$\left(X_{t'}^{(\delta_n)}\right)^{-1}(B')\in\mathcal{A}.$$

The fact that $(C[0,\infty),\mathcal{C},P_n)$ is a probability space now follows by lemma 3.3.2 and theorem C.1.1.

**Lemma 3.3.2.** *Let $Z = (Z_t)_{t\in[0,\infty)}$ be a continuous stochastic process on the underlying probability space $(\Omega, \mathcal{A}, P)$. This means that for every $\omega \in \Omega$*

$$Z_{\cdot}(\omega) : [0, \infty) \to \mathbb{R},$$

*is a continuous function in $t$, and for every $t' \in [0, \infty), B \in \mathcal{B}(\mathbb{R})$ we have*

$$Z_{t'}^{-1}(B) \in \mathcal{A}.$$

*Let $\mathcal{C}$ be the Borel sigma-algebra on $C[0, \infty)$ defined in definition 2.2.8. We then have for every $C \in \mathcal{C}$*

$$Z^{-1}(C) \in \mathcal{A}, \tag{3.7}$$

*where we in eq. (3.7) view $Z$ as the function*

$$Z : \Omega \to C[0, \infty),$$

*where $Z(\omega)$ is the continuous trajectory in $t$.*

*Proof.* We first show that the collection

$$\mathcal{H} = \left\{C \in \mathcal{C} : Z^{-1}(C) \in \mathcal{A}\right\},$$

is a sigma-algebra. Since $Z^{-1}(\emptyset) = \emptyset$, we have that $\emptyset \in \mathcal{H}$. If $C \in \mathcal{H}$, we have that $C \in \mathcal{C}$, so $C^c \in \mathcal{C}$ and

$$Z^{-1}(C^c) = (Z^{-1}(C))^c =\in \mathcal{A},$$

by elementary set-theory. Hence $C^c \in \mathcal{H}$. Assume now that $\{C_n\}$ is a countable collection where each $C_n \in \mathcal{C}$. Then $\cup_n C_n \in \mathcal{C}$ and by elementary set-theory and the definition of a sigma-algebra we have

$$Z^{-1}(\cup_n C_n) = \cup_n Z^{-1}(C_n) \in \mathcal{A}.$$

So $\cup_n C_N \in \mathcal{H}$. We have therefore shown that $\mathcal{H}$ is a sigma-algebra. Let $\mathcal{E}$ be the collection

$$\mathcal{E} = \{\{f \in C[0, \infty) : f(t') \in B\} : t' \in [0, \infty), B \in \mathcal{B}(\mathbb{R})\}.$$

We will show that $\mathcal{E} \subset \mathcal{H}$. Let $A \in \mathcal{E}$, then there is $t' \in [0, \infty), B \in \mathcal{R}$ such that

$$A = \{f \in C[0, \infty) : f(t') \in B\}.$$

By corollary 2.3.2 $\sigma(\mathcal{E}) = \mathcal{C}$, so obviously $A \in \mathcal{C}$. We also have that

$$\begin{aligned} Z^{-1}(A) &= \{\omega \in \Omega : Z(\omega) \in A\} \\ &= \{\omega \in \Omega : Z_{t'}(\omega) \in B\} \\ &= Z_{t'}^{-1}(B) \\ &\in \mathcal{A}. \end{aligned}$$

Where the last line follows by the hypothesis of this lemma. To summarize we have that $\mathcal{H} \subset \mathcal{C}, \mathcal{E} \subset \mathcal{H}$ and $\mathcal{H}$ is a sigma-algebra. By corollary 2.3.2 $\sigma(\mathcal{E}) = \mathcal{C}$ so by proposition A.2.1 $\sigma(\mathcal{E}) \subset \mathcal{H} \subset \mathcal{C} = \sigma(\mathcal{E})$, hence $\mathcal{H} = \mathcal{C}$. This completes the proof.

□

We will also need the definition of convergence in distribution.

**Definition 3.3.3.** *Let $(Z_1^{(n)}, Z_2^{(n)}, \dots, Z_k^{(n)})$ be a sequence of random vectors on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$, each defined on an underlying probablity space $(\Omega_n, \mathcal{A}_n, P_n)$. Let $(Z_1, Z_2, \dots, Z_k)$ be a random vector on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ defined on an underlying probability space $(\Omega', \mathcal{A}', P')$. We say that $(Z_1^{(n)}, Z_2^{(n)}, \dots, Z_k^{(n)})$ converges to $(Z_1, Z_2, \dots, Z_k)$ in distribution if for every bounded continuous function*

$$f : \mathbb{R}^k \to \mathbb{R},$$

*we have*

$$E_n[f((Z_1^{(n)}, Z_2^{(n)}, \dots, Z_k^{(n)}))] \to E'[f((Z_1, Z_2, \dots, Z_k))].$$

From theorem 2.4.2 we see that the two main requirements we have to check is weak convergence of the finite-dimensional distributions and tightness. Let $0 \le t_1 < t_2 < \dots < t_k$. Later we will show that $(X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \dots, X_{t_k}^{(\delta_n)})$ as a random variable on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ converges in distribution to $(Y_1, Y_2, \dots, Y_k)$. $(Y_1, Y_2, \dots, Y_k)$ is a Gaussian vector on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ with expectation zero for all components and covariance such that

$$cov(Y_{t_j}, Y_{t_l}) = \frac{1}{2}\left(t_j^{2H} + t_l^{2H} - |t_j - t_l|^{2H}\right).$$

The existence of $\vec{Y}$ is justified by the remark after proposition 3.1.3. Now we will show that this implies that the finite-dimensional induced measures converges weakly.

**Proposition 3.3.4.** *Assume that $\{\delta_n\}$ is a collection of positive real numbers. Let $X^{(\delta_n)}$ be as in definition 3.2.6 and $P_n$ as in definition 3.3.1. Let $0 \le t_1 < t_1 < t_2 < \dots < t_k$. Assume that $(X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \dots, X_{t_k}^{(\delta_n)})$ converges in distribution to a random vector $(Y_1, Y_2, \dots, Y_k) = \vec{Y}$, defined on a probability space $(\Omega', \mathcal{A}', P')$. Then $P_n \pi_{t_1,t_2,\dots,t_k}^{-1}$ converges weakly to $\mu = P'\vec{Y}^{-1}$, where we have defined $\mu(A) \doteq P'(\vec{Y}^{-1}(A)$ and $P_n \pi_{t_1,t_2,\dots,t_k}^{-1}(A) = P_n(\pi_{t_1,t_2,\dots,t_k}^{-1} A)$. Notice that $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), P_n \pi_{t_1,t_2,\dots,t_k}^{-1})$ is a probability space for all $n$, and $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \mu)$ is a probability space.*

*Proof.* The fact that $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \mu)$ is a probability space follows from the fact $Y$ is a random vector, and $(\Omega', \mathcal{A}', P')$ is a probability space. From elementary properties of inverse images one sees directly that it is a probability space, by confirming that $\mu$ takes values on $\mathcal{B}(\mathbb{R}^k)$, $\mu(\mathbb{R}^k) = 1$, $\mu(A) = 1 - \mu(A^c)$ and countable additivity. The details are

$$\mu(\mathbb{R}^k) = P'\vec{Y}^{-1}(\mathbb{R}^k) = P'(\Omega') = 1.$$
$$\mu(A^c) = P'(\vec{Y}^{-1}(A^c)) = P'(\vec{Y}^{-1}(A)^c) = 1 - P'(\vec{Y}^{-1}(A)) = 1 - \mu(A),$$
$$\mu(\cup_i A_i) = P'(\vec{Y}^{-1}(\cup_i A_i)) = P'(\cup_i \vec{Y}^{-1} A_i) = \sum_i P'(\vec{Y}^{-1} A_i) = \sum_i \mu(A_i),$$

where we have assumed that $\{A_i\}$ is a countable mutually disjoint collection. We also have that $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), P_n \pi_{t_1,t_2,\dots,t_k}^{-1})$ is a probability space because

$\pi_{t_1,t_2,\ldots,t_k} : C[0,\infty) \to \mathbb{R}^k$ is continuous and, from the discussion after definition 3.3.1 we know that $(C[0,\infty), \mathcal{C}, P_n)$ is a probability space. We see that $P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}$ takes values on $\mathcal{B}(\mathbb{R}^k)$, and one can confirm the the three axioms of a probability measure directly as we did above:

$$\begin{aligned}
P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(\mathbb{R}^k) &= P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(\mathbb{R}^k)) = P_n(C[0,\infty)) = 1.\\
P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(A^c) &= P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(A^c)) = P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(A)^c)\\
&= 1 - P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(A)) = 1 - P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(A)\\
P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(\cup_i A_i) &= P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(\cup_i A_i)) = P_n(\cup_i \pi^{-1}_{t_1,t_2,\ldots,t_k}(A_i))\\
&= \sum_i P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}(A_i)) = \sum_i P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(A_i),
\end{aligned}$$

where $\{A_i\}$ is a countable mutually disjoint collection.

By the hypothesis we have that for every bounded continuous function

$$f : \mathbb{R}^k \to \mathbb{R},$$

$$E[f((X^{(\delta_n)}_{t_1}, X^{(\delta_n)}_{t_2}, \ldots, X^{(\delta_n)}_{t_k}))] \to E'[f((Y_1, Y_2, \ldots, Y_k))].$$

Let $E_{n,\vec{t}}$ denote the expectation of random variables on

$$(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}).$$

That $f$ is a random variable follows since it is measurable by continuity. It will suffice to prove that

$$E[f((X^{(\delta_n)}_{t_1}, X^{(\delta_n)}_{t_2}, \ldots, X^{(\delta_n)}_{t_k}))] = E_{n,\vec{t}}[f(x_1, x_2, \ldots, x_k)], \tag{3.8}$$

and that

$$E'[f((Y_1, Y_2, \ldots, Y_k))] = E_\mu[f(x_1, x_2, \ldots, x_k)]. \tag{3.9}$$

Both these statements are proved using what is called the bootstrap method in measure theory. This is a method where we first prove something for simple functions, and then use linearity and limit argument to prove the general case. We show the details. Assume that $A \in \mathcal{B}(\mathbb{R}^k)$. We then have

$$\begin{aligned}
E[I_A((X^{(\delta_n)}_{t_1}, X^{(\delta_n)}_{t_2}, \ldots, X^{(\delta_n)}_{t_k}))] &= P((X^{(\delta_n)}_{t_1}, X^{(\delta_n)}_{t_2}, \ldots, X^{(\delta_n)}_{t_k}) \in A)\\
&= P(X^{\delta_n} \in \pi^{-1}_{t_1,t_2,\ldots,t_k}A)\\
&= P_n(\pi^{-1}_{t_1,t_2,\ldots,t_k}A)\\
&= P_n\pi^{-1}_{t_1,t_2,\ldots,t_k}(A)\\
&= E_{n,\vec{t}}[I_A((x_1, x_2, \ldots, x_k))].
\end{aligned}$$

Similarly we have

$$\begin{aligned}
E'[I_A((Y_1, Y_2, \ldots, Y_k))] &= P'((Y_1, Y_2, \ldots, Y_k) \in A)\\
&= P'(\vec{Y}^{-1}(A))\\
&= \mu(A)\\
&= E_\mu(I_A(x_1, x_2, \ldots, x_k)).
\end{aligned}$$

By linearity of the expectation(it is an integral) we get the same equalities for functions of the form

$$\sum_{i=1}^{K} c_i I_{A_i}(x_1, x_2, \dots, x_k), c_i \in \mathbb{R}. \tag{3.10}$$

Since $f$ is continuous it is measurable. From measure theory we know that we can approximate $f$ by simpler functions as those above, where the simple functions converge pointwise to $f$. Since $|f|$ is bounded by $L \in R$, the absolute value of these functions are also bounded by $L$(by the way they are constructed). For instance we split $f$ in its positive and negative part, and remember that $f^{-1}([0, \infty))$ is a Borel-set and use [MW13, Propositiopn 5.7 a]. The result will now follow from the dominated convergence theorem. We let $\{s_i\}$ be a sequence of simple functions converging pointwise to $f$, with $|f| \le L, |s_i| \le L, \forall i$. We get

$$\begin{aligned}
E[f((X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \dots, X_{t_k}^{(\delta_n)}))] &= E\left[\lim_{i\to\infty} s_i((X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \dots, X_{t_k}^{(\delta_n)}))\right] \\
&= \lim_{i\to\infty} E\left[s_i((X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \dots, X_{t_k}^{(\delta_n)}))\right] \\
&= \lim_{i\to\infty} E_{n,\bar{t}}[s_i((x_1, x_2, \dots, x_k))] \\
&= E_{n,\bar{t}}[\lim_{i\to\infty} s_i((x_1, x_2, \dots, x_k))] \\
&= E[f(x_1, x_2, \dots, x_k)],
\end{aligned}$$

where we in the second and fourth equality have used the dominated convergence theorem. By the same calculations we also have

$$E'[f(Y_1, Y_2, \dots, Y_k)] = E_\mu[f(x_1, x_2, \dots, x_k)].$$

□

We have now laid some of the groundwork for later use of theorem 2.4.2 by looking at finite-dimensional convergence. Now we will look at tightness. We will prove a proposition that later will make it easier to use corollary 2.5.6. We need this because we will work with $X^{(\delta_n)}$ but our result in corollary 2.5.6 holds for $P_n$. In the proof we will also use the bootstrap method as in proposition 3.3.4 but now we will use the monotone convergence theorem, not the dominated convergence theorem.

**Proposition 3.3.5.** *Assume that $\{\delta_n\}$ is a collection of positive real numbers. Let $X^{(\delta_n)}$ be as in definition 3.2.6 and $P_n$ as in definition 3.3.1. We then have*

$$P_n(\{x \in C[0, \infty) : x(0) = 0\}) = 1, \forall n.$$

*And also for $t_1, t_2, \gamma \in [0, \infty)$*

$$E_n[|x(t_2) - x(t_1)|^\gamma] = E[|X_{t_1}^{(\delta_n)} - X_{t_2}^{(\delta_n)}|^\gamma], \forall n.$$

*The first expectation is taken over the space $(C[0, \infty), \mathcal{C}, P_n)$, the second is taken over $(\Omega, \mathcal{A}, P)$ where $X^{(\delta_n)}$ is defined.*

*Proof.* First notice that

$$\{x \in C[0,\infty) : x(0) = 0\} \in \mathcal{C},$$

because $\pi_0 : C[0,\infty) \to \mathbb{R}$ is continuous by theorem 2.2.6. We get

$$\begin{aligned} P_n(\{x : x(0) = 0\}) &= P(X^{(\delta_n)} \in \{x \in C[0,\infty) : x(0) = 0\}) \\ &= P(X_0^{(\delta_n)} = 0) \\ &= 1. \end{aligned}$$

We actually have a stronger result than $P(X_0^{(\delta_n)} = 0) = 1$, as we can see that $X_0^{(\delta_n)}(\omega) = 0$ for all $\omega \in \Omega$, this is seen directly from definition 3.2.6 point i).

Notice also that

$$|x(t_2) - x(t_1)|^\gamma : C[0,\infty) \to \mathbb{R},$$

is continuous by theorem 2.2.6(redefine $t_1, t_2$ to be increasing if necessary, if they are equal the value is identically zero and hence continuous) and from the fact that

$$|z_1 - z_2|^\gamma : \mathbb{R}^2 \to \mathbb{R},$$

is continuous. This means that $|x(t_2) - x(t_1)|^\gamma$ is a random variable on $(C[0,\infty), \mathcal{A}, P_n)$. We go through similar bootstrap argument as in proposition 3.3.4. What we will prove is that for a positive measurable function

$$f : C[0,\infty) \to \mathbb{R}$$

not necessarily integrable, we have

$$E_n[f(x)] = E[f(X^{(\delta_n)})]. \tag{3.11}$$

Since

$$|x(t_2) - x(t_1)|^\gamma : C[0,\infty) \to \mathbb{R},$$

is continuous and hence measurable, the result will follow if we can prove eq. (3.11). $X^{(\delta_n)}$ is an element of $C[0,\infty)$ for each $\omega$ so $f(X^{(\delta_n)})$ makes sense. Notice that if $B \in \mathcal{B}(\mathbb{R})$ we have $(f(X^{(\delta_n)}))^{-1}(B) = (X^{(\delta_n)})^{-1}(f^{-1}(B))$, by elementary set-theory. By the assumed measurability of $f$ we have that $f^{-1}(B) \in \mathcal{C}$, and by the discussion after definition 3.3.1 we have that $(X^{(\delta_n)})^{-1}(f^{-1}(B)) \in \mathcal{A}$, so $f(X^{(\delta_n)})$ is a random variable on $(\Omega, \mathcal{A}, P)$. This holds for all measurable functions $C[0,\infty) \to \mathbb{R}$, for instance $I_A, A \in \mathcal{C}$. We can now start the bootstrap argument. Let $A \in \mathcal{C}$, we get

$$\begin{aligned} E_n[I_A(x)] &= P_n(\{x \in C[0,\infty) : x \in A\}) \\ &= P(\{\omega : X^{(\delta_n)}(\omega) \in A\}) \\ &= E[I_A(X^{(\delta_n)}(\omega))]. \end{aligned}$$

For a simple function $s$ given by

$$\sum_{i=1}^{K} c_i I_{A_i}(x), c_i \in \mathbb{R}, A_i \in \mathcal{C},$$

we get

$$\begin{aligned} E_n\left[\sum_{i=1}^K c_i I_{A_i}(x)\right] &= \sum_{i=1}^K c_i E_n\left[I_{A_i}(x)\right] \\ &= \sum_{i=1}^K c_i E[I_{A_i}(X^{(\delta_n)}(\omega))] \\ &= E\left[\sum_{i=1}^K c_i I_{A_i}(X^{(\delta_n)}(\omega))\right]. \end{aligned}$$

By [MW13, Proposition 5.7 a)] there is a sequence of simple positive functions $\{s_i\}$ increasing monotonically to $f$, and then we get

$$\begin{aligned} E_n[f(x)] &= E_n\left[\lim_{i\to\infty} s_i(x)\right] \\ &= \lim_{i\to\infty} E_n\left[s_i(x)\right] \\ &= \lim_{i\to\infty} E\left[s_i(X^{(\delta_n)}(\omega))\right] \\ &= E\left[\lim_{i\to\infty} s_i(X^{(\delta_n)}(\omega))\right] \\ &= E[f(X^{(\delta_n)}(\omega))], \end{aligned}$$

where we have used the monotone convergence theorem in the second and fourth equality. As we said above, now the result follows by using $f(x) = |x(t_2) - x(t_1)|^\gamma$.

□

## 3.4 Some results in preparation for finite-dimensional weak convergence

In this section we prove some results that will be used in the next section where we prove weak convergence of the finite-dimensional distributions. The results in this section are rather technical. One may view these results as proving that some integrals and sums related to the representation of the fBm by Mandelbrot and Van Ness shown in section 3.2 behave well as $\delta_n$ goes to zero. However these results are not important by themselves, but they will help us in the next section. We first start with a simple lemma, notice the similarity with lemma 3.2.4

**Lemma 3.4.1.** *Let $H \in (0,1)$. For $t \geq 0$ we have that*

$$\int_{-\infty}^t \left((t-x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right)^2 < \infty.$$

*Proof.*

$$\begin{aligned}
&\int_{-\infty}^{t}\left((t-x)^{H-\frac{1}{2}}-(-x)_{+}^{H-\frac{1}{2}}\right)^2 dx\\
=&\int_{-\infty}^{-1}\left((t-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right)^2 dx+\int_{-1}^{0}\left((t-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right)^2 dx\\
&\qquad+\int_{0}^{t}(t-x)^{2H-1}dx.
\end{aligned}$$

The first is integral obviously finite if $H = 1/2$, because then the integrand is zero. From [ST94, p. 321], we have for $H \neq 1/2$

$$\lim_{x\to-\infty}\frac{\left((t-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right)^2}{\frac{(H-1/2)^2}{(-x)^{3-2H}}}=1.$$

So for $x$ small enough(large negative value) we have that

$$\left((t-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right)^2\leq\frac{2(H-1/2)^2}{(-x)^{3-2H}}.$$

Hence the first integral is finite by the comparison with the integral

$$\int_{-\infty}^{-1}\frac{(H-1/2)^2}{(-x^{3-2H})}dx,$$

which converges because $3-2H>1$. For the second integral we get by using that $(a-b)^2\leq 2a^2+2b^2$

$$\begin{aligned}
\int_{-1}^{0}\left((t-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right)^2 dx\leq 2&\int_{-1}^{0}(t-x)^{2H-1}dx\\
+2&\int_{-1}^{0}(-x)^{2H-1}dx,
\end{aligned}$$

both these integrals are finite because $1-2H<1$. The third integral

$$\int_{0}^{t}(t-x)^{2H-1}dx,$$

is also finite because because $1-2H<1$. □

The next lemma will be useful later when we will use the Lindeberg central limit theorem. In the theorem we will take the integral of a non-negative function, but we need to prove that the integral is strictly positive. It is an interesting result where we will need some linear algebra. The trick of using Vandermonde matrices is attributed to professor Tom Lindstrøm of UiO.

**Lemma 3.4.2.** *Assume $\vec{u}\in\mathbb{R}^k, \vec{u}\neq\vec{0}$. Let $0<t_1<t_2<\cdots<t_k$. Also let $H\in(0,1), H\neq 1/2$. We then have*

*(i)*

$$\int_{-\infty}^{0} \left( \sum_{j=1}^{k} u_j \left[ (t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}} \right] \right)^2 dx > 0,$$

*(ii)*

$$\int_{0}^{t_1} \left( \sum_{j=1}^{k} u_j (t_j - x)^{H-\frac{1}{2}} \right)^2 dx > 0,$$

*(iii) If $k \geq 2$ we have for $l \in \{2, \dots, k\}$ then $u_l^2 + u_{l+1}^2 + \cdots + u_k^2 \neq 0$ implies*

$$\int_{t_{l-1}}^{t_l} \left( \sum_{j=l}^{k} u_j (t_j - x)^{H-\frac{1}{2}} \right)^2 dx > 0,$$

*Proof.* For point (i) notice that by continuity it suffices to prove that if

$$\sum_{j=1}^{k} u_j \left[ (t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}} \right] = 0$$

for all $x \in (-\infty, 0)$, we have that $\vec{u} = \vec{0}$. The statement

$$\sum_{j=1}^{k} u_j \left[ (t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}} \right] = 0, \forall x \in (-\infty, 0),$$

is equivalent to

$$\sum_{j=1}^{k} u_j \left( \frac{t_j - x}{-x} \right)^{H-\frac{1}{2}} = \sum_{j=1}^{k} u_j, \qquad \forall x \in (-\infty, 0).$$

By differentiating both sides with respect to x we get

$$\begin{aligned} &\sum_{j=1}^{k} u_j \left( H - \frac{1}{2} \right) \left( \frac{t_j - x}{-x} \right)^{H-\frac{3}{2}} \frac{-x(-1) - (t_j - x)(-1)}{x^2} \\ &= \sum_{j=1}^{k} u_j \left( H - \frac{1}{2} \right) \left( \frac{t_j - x}{-x} \right)^{H-\frac{3}{2}} \frac{-t_j}{x^2} \\ &= 0, \qquad \forall x \in (-\infty, 0). \end{aligned}$$

Multiplying away common factors of $x$ we get

$$\sum_{j=1}^{k} u_j t_j \left( H - \frac{1}{2} \right) (t_j - x)^{H-\frac{3}{2}} = 0, \qquad \forall x \in (-\infty, 0).$$

Let $v_j = u_j t_j(H-1/2)$, since $t_j(H-1/2) \neq 0$ we see that the problem is reduced to showing that $\vec{v} = \vec{0}$. By noting that for odd number $N$ we have $H - N/2 \neq 0$, repeated differentiation tells us that

$$\sum_{j=1}^{k} v_j (t_j - x)^{H-\frac{N}{2}} = 0, \; \forall x \in (-\infty, 0), N \in \{3, 5, 7, \dots\}.$$

If we set $x = -1$, this gives us the matrix equation

$$\begin{bmatrix} (t_1+1)^{H-\frac{3}{2}} & (t_2+1)^{H-\frac{3}{2}} & \dots & (t_k+1)^{H-\frac{3}{2}} \\ (t_1+1)^{H-\frac{5}{2}} & (t_2+1)^{H-\frac{5}{2}} & \dots & (t_k+1)^{H-\frac{5}{2}} \\ \vdots & \vdots & \ddots & \\ (t_1+1)^{H-\frac{2k+1}{2}} & (t_2+1)^{H-\frac{2k+1}{2}} & \dots & (t_k+1)^{H-\frac{2k+1}{2}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

To show that $\vec{v} = \vec{0}$ it suffices to show that the determinant of the matrix is non-zero. By linear algebra we have that multiplying a column of a matrix with a non-zero number will only change the determinant by a non-zero factor, so if we multiply each column of the determinant with $1/(t_j+1)^{H-\frac{2k+1}{2}}$, the new determinant will be non-zero if and only if the old is. The new matrix is

$$\begin{bmatrix} (t_1+1)^{k-1} & (t_2+1)^{k-1} & \dots & (t_k+1)^{k-1} \\ (t_1+1)^{k-2} & (t_2+1)^{k-2} & \dots & (t_k+1)^{k-2} \\ \vdots & \vdots & \ddots & \\ (t_1+1)^{2} & (t_2+1)^{2} & \dots & (t_k+1)^{2} \\ (t_1+1)^{1} & (t_2+1)^{1} & \dots & (t_k+1)^{1} \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

By rearranging the rows, and taking the transpose we get a new matrix, and the old matrix will be non-singular if and only if new matrix is non-singular. The new matrix is

$$\begin{bmatrix} 1 & (t_1+1)^1 & (t_1+1)^2 & \dots & (t_1+1)^{k-1} \\ 1 & (t_2+1)^1 & (t_2+1)^2 & \dots & (t_2+1)^{k-1} \\ \vdots & \vdots & \ddots & & \\ 1 & (t_{k-1}+1)^1 & (t_{k-1}+1)^2 & \dots & (t_{k-1}+1)^{k-1} \\ 1 & (t_k+1)^1 & (t_k+1)^2 & \dots & (t_k+1)^{k-1} \end{bmatrix}.$$

The determinant of this matrix is non-zero since it is a Vandermonde matrix, and all the $t_j' s$ are different, see for instance Proposition 109, p. 209 in [Zip93] and the comment following immediately after.

We will prove point (ii) and (iii) simultaneously. We will prove a slightly stronger result from which (ii) and (iii) will follow immediately, namely we will prove that if $l \in \{1, 2, \dots, k\}$, $u_l^2 + u_{l+1}^2 + \dots + u_k^2 \neq 0$ and $(a, b)$ is an interval such that $0 \leq a < b \leq t_l$, we have

$$\int_a^b \left( \sum_{j=l}^{k} u_j (t_j - x)^{H-\frac{1}{2}} \right)^2 dx > 0.$$

If $l = k$ the result is obvious, hence we can assume that $l < k$. Because of continuity it suffices to prove that if

$$\sum_{j=l}^{k} u_j(t_j - x)^{H-\frac{1}{2}} = 0, \forall x \in (a, b),$$

it follows that $u_l = u_{l+1} = \cdots = u_k = 0$. By noticing that $H - N/2 \neq 0$ for $N \in \{1, 3, 5, 7, \ldots\}$ repeated differentiation gives us that

$$\sum_{j=l}^{k} u_j(t_j - x)^{H-\frac{N}{2}} = 0, \forall x \in (a, b), N \in \{1, 3, 5, \ldots\}.$$

Choosing $x$ to be $r = (a + b)/2$ we note that $t_l - r > 0$. We now get the matrix equation

$$\begin{bmatrix} (t_l - r)^{H-\frac{1}{2}} & (t_{l+1} - r)^{H-\frac{1}{2}} & \cdots & (t_k - r)^{H-\frac{1}{2}} \\ (t_l - r)^{H-\frac{3}{2}} & (t_{l+1} - r)^{H-\frac{3}{2}} & \cdots & (t_k - r)^{H-\frac{3}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ (t_l - r)^{H-\frac{2(k-l)+1}{2}} & (t_{l+1} - r)^{H-\frac{2(k-l)+1}{2}} & \cdots & (t_k - r)^{H-\frac{2(k-l)+1}{2}} \end{bmatrix} \begin{bmatrix} u_l \\ u_{l+1} \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We must prove that the matrix is non-singular. By multiplying each column with a non-zero constant, we get a new matrix which is non-singular if and only if the old one is, because the determinant of the matrix will only change by a non-zero constant. We multiply each column by the reciprocal of the last element in each column. The new matrix is

$$\begin{bmatrix} (t_l - r)^{k-l} & (t_{l+1} - r)^{k-l} & \cdots & (t_k - r)^{k-l} \\ (t_l - r)^{k-l-1} & (t_{l+1} - r)^{k-l-1} & \cdots & (t_k - r)^{k-l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

By re-arranging the rows and taking the transpose this matrix is non-singular if and only if

$$\begin{bmatrix} 1 & (t_l - r) & (t_l - r)^2 & \cdots & (t_l - r)^{k-l-1} & (t_l - r)^{k-l} \\ 1 & (t_{l+1} - r) & (t_{l+1} - r)^2 & \cdots & (t_{l+1} - r)^{k-l-1} & (t_{l+1} - r)^{k-l} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ 1 & (t_{k-1} - r) & (t_{k-1} - r)^2 & \cdots & (t_{k-1} - r)^{k-l-1} & (t_{k-1} - r)^{k-l} \\ 1 & (t_k - r) & (t_k - r)^2 & \cdots & (t_k - r)^{k-l-1} & (t_k - r)^{k-l}. \end{bmatrix},$$

is non-singular. But the last matrix is non-singular because it is a Vandermonde-matrix with different coefficients, see [Zip93, p. 209].

□

Next we have a lemma which gives us the explicit value of an integral.

**Lemma 3.4.3.** *Assume that $t_1, t_2 \in (0, \infty)$ with $t_1 \leq t_2$. We then have*

$$\begin{aligned} &\frac{1}{C_H^2} \int_{-\infty}^{t_1} \left( (t_1 - x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right) \left( (t_2 - x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right) dx \\ =& \frac{1}{2} \left( t_1^{2H} + t_2^{2H} - (t_2 - t_1)^{2H} \right). \end{aligned}$$

*Proof.* The proof is based on calculations from [ST94, pp. 321-322], and some of the calculations are from there. Integrability follows from Hölder's inequality and lemma 3.4.1.

We first show that it holds when $t_1 = t_2$, using the substitution $zt_1$ we get

$$\begin{aligned}
&\frac{1}{C_H^2}\int_{-\infty}^{t_1}\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)^2 dx\\
&=\frac{1}{C_H^2}\int_{-\infty}^{1}\left((t_1-zt_1)^{H-\frac{1}{2}}-(-zt_1)_+^{H-\frac{1}{2}}\right)^2 t_1 dz\\
&=\frac{t_1^{2H}}{C_H^2}\left[\int_{-\infty}^{0}\left((1-z)^{H-\frac{1}{2}}-(-z)^{H-\frac{1}{2}}\right)^2 dz+\int_0^1(1-z)^{2H-1}dz\right].
\end{aligned}$$

We use the substitution $u=-z$ on the first integral and $u = 1-z$ on the second

$$\begin{aligned}
&=\frac{t_1^{2H}}{C_H^2}\left[\int_{\infty}^{0}\left((1+u)^{H-\frac{1}{2}}-u^{H-\frac{1}{2}}\right)^2 du(-1)+\int_1^0 u^{2H-1}du(-1)\right]\\
&=\frac{t_1^{2H}}{C_H^2}\left[\int_{0}^{\infty}\left((1+u)^{H-\frac{1}{2}}-u^{H-\frac{1}{2}}\right)^2 du+\frac{1}{2H}\right]\\
&=t_1^{2H}.
\end{aligned}$$

For technical reasons that will become apparent soon we calculate the next integral

$$\begin{aligned}
&\frac{1}{C_H^2}\int_{-\infty}^{t_2}\left[\left((t_2-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)-I_{(-\infty,t_1)}(x)\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)\right]^2 dx\\
&=\frac{1}{C_H^2}\int_{-\infty}^{t_2}\left[(t_2-x)^{H-\frac{1}{2}}-I_{(-\infty,t_1)}(x)(t_1-x)^{H-\frac{1}{2}}\right]^2 dx.
\end{aligned}$$

Using the substitution $u = x - t_1$ this becomes

$$\frac{1}{C_H^2}\int_{-\infty}^{t_2-t_1}\left[(t_2-t_1-u)^{H-\frac{1}{2}}-(-u)_+^{H-\frac{1}{2}}\right] du.$$

By our previous calculations with $t_1' = t_2 - t_1$ this is equal to

$$(t_2-t_1)^{2H}.$$

However by expanding the square inside the integral we also get

$$\begin{aligned}
&\frac{1}{C_H^2}\int_{-\infty}^{t_2}\left[\left((t_2-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)-I_{(-\infty,t_1)}(x)\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)\right]^2 dx\\
&=\frac{1}{C_H^2}\int_{-\infty}^{t_2}\left((t_2-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)^2 dx\\
&\quad-\frac{2}{C_H^2}\int_{-\infty}^{t_1}\left((t_2-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right) dx\\
&\quad+\frac{1}{C_H^2}\int_{-\infty}^{t_1}\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)^2 dx\\
&=t_2^{2H}+t_1^{2H}-\frac{2}{C_H^2}\int_{-\infty}^{t_1}\left((t_2-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right)\left((t_1-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right) dx.
\end{aligned}$$

So we have proved

$$
\begin{aligned}
&(t_2 - t_1)^{2H} \\
= &\, t_2^{2H} + t_1^{2H} - \frac{2}{C_H^2} \int_{-\infty}^{t_1} \left( (t_2 - x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right) \left( (t_1 - x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right) dx.
\end{aligned}
$$

Hence the lemma follows. □

The next lemma shows that in some sense the tail of the Mandelbrot and Van Ness sum disappears in the limit. A concrete use of the lemma will be explained later.

**Lemma 3.4.4.** *Let $H \in (0,1)$, and $a(t,r,\delta)$ as in definition 3.2.7. We have that for every $t \in [0,\infty)$*

$$
\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} a(t,r,\delta)^2 \to 0,
$$

*as $\delta \to 0$ with $\delta > 0$.*

*Proof.* If $t = 0$ then $a(t,r,\delta) = 0$, so we can assume that $t > 0$. We first look at the part of $a(t,r,\delta)$ corresponding to $\underline{a}$(see definition 3.2.8)

$$
\begin{aligned}
&\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \left[ (\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \right]^2 \\
= &\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \left[ (\lfloor t/\delta \rfloor \delta - r\delta)^{H-\frac{1}{2}} - (-r\delta)^{H-\frac{1}{2}} \right]^2 \delta \\
= &\sum_{\tau=-\infty}^{-\lfloor 1/\delta^2 \rfloor \delta} {}_{\delta} \left[ (\lfloor t/\delta \rfloor \delta - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \right]^2 \delta.
\end{aligned}
$$

If we first assume that $H \geq 1/2$, we have that this is less than or equal to

$$
\sum_{\tau=-\infty}^{-\lfloor 1/\delta^2 \rfloor \delta} {}_{\delta} \left[ (t - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \right]^2 \delta
$$

Assume for convenience that $\delta < 0.1$. If we differentiate the expression inside the sum with respect to $\tau$ we get that it is non-negative, hence it is non-decreasing when $\tau$ increases. This means that this sum is bounded by

$$
\begin{aligned}
&\int_{-\infty}^{-\lfloor 1/\delta^2 \rfloor \delta + \delta} \left[ (t - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \right]^2 d\tau \\
\leq &\int_{-\infty}^{-1/\delta + 2\delta} \left[ (t - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \right]^2 d\tau.
\end{aligned}
$$

The last expression goes to zero because

$$\int_{\tau=-\infty}^{0} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right]^2 d\tau < \infty,$$

by lemma 3.4.1, so the convergence to zero follows by the Dominated convergence theorem. If $H < 1/2$ we have

$$\begin{aligned} &\sum_{\tau=-\infty}^{-\lfloor 1/\delta^2 \rfloor \delta}{}_{\delta} \left[(\lfloor t/\delta \rfloor \delta - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right]^2 \delta \\ \leq &\sum_{\tau=-\infty}^{-\lfloor 1/\delta^2 \rfloor \delta}{}_{\delta} \left[(-\tau)^{H-\frac{1}{2}} - (t-\tau)^{H-\frac{1}{2}}\right]^2 \delta. \end{aligned}$$

If we differentiate inside the sum with respect to $\tau$ we again get that it is non-negative, so the argument above means that this also converges to 0 as $\delta$ goes to zero.

The part corresponding to $\overline{a}$ is

$$\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \left[(\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right]^2.$$

This must also converge to 0 as $\delta$ goes to 0, this can be seen for instance by noting that

$$\begin{aligned} &\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \left[(\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right]^2 \\ \leq &\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \left[(\lfloor t'/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right]^2, \end{aligned}$$

where $t' = 2t$, and $\delta$ is small enough. It is easy to see that this holds if $H \geq 1/2$, it also holds if $H < 1/2$ by changing the order of the terms inside the square-bracket(which we can do since we square it). Hence the result follows from the earlier argument, but with $t'$ instead of $t$.

We end up with

$$
\begin{aligned}
&\sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} a(t,r,\delta)^2 \\
&= \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \Bigg( \quad [1+\lfloor t/\delta \rfloor - t/\delta] \cdot (\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} \\
&\qquad\qquad + [t/\delta - \lfloor t/\delta \rfloor](\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}} \Bigg)^2 \\
&= \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} \delta^{2H} \Bigg( \Big[1+\lfloor t/\delta \rfloor - t/\delta\Big] \Big[(\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big] \\
&\qquad\qquad + \Big[t/\delta - \lfloor t/\delta \rfloor\Big] \Big[(\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big] \Bigg)^2 \\
&\le \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} 2\delta^{2H} \Big[1+\lfloor t/\delta \rfloor - t/\delta\Big]^2 \Big[(\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]^2 \\
&\qquad + \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} 2\delta^{2H} \Big[t/\delta - \lfloor t/\delta \rfloor\Big]^2 \Big[(\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]^2 \\
&\le \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} 2\delta^{2H} \Big[(\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]^2 \\
&\qquad + \sum_{r=-\infty}^{-\lfloor 1/\delta^2 \rfloor} 2\delta^{2H} \Big[(\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]^2.
\end{aligned}
$$

This converges to 0 by what we proved above. □

The next lemma shows that as $\delta_n$ becomes small, we have that $\delta_n^H a(t,r,\delta_n)$ become small it a uniform way.

**Lemma 3.4.5.** *Let $H \in (0,1)$. Let $\{\delta_n\}$ be a positive sequence that converges to zero. Let $t$ be given. Define $A_{t,\delta_n} \doteq \{r \in \mathbb{Z} : r \le \lfloor t/\delta_n \rfloor - 1\}$. We then have*

$$\lim_{n\to\infty} \sup_{r \in A_{t,\delta_n}} \{\delta_n^H |a(t,r,\delta_n)|\} = 0.$$

*Proof.* For $H = 1/2$ we have

$$
\begin{aligned}
&\delta_n^H \, |a(t,r,\delta_n)| \\
&= \delta_n^H \Big| (1+\lfloor t/\delta \rfloor - t/\delta) \cdot (\lfloor t/\delta \rfloor - r)^0 \\
&\qquad + (t/\delta - \lfloor t/\delta \rfloor) \cdot (\lfloor t/\delta \rfloor + 1 - r)^0 \\
&\qquad - (-r)_+^0 \Big| \\
&= \delta_n^{\frac{1}{2}} \left|1 - (-r)_+^0\right| \\
&\le 2\delta_n^H.
\end{aligned}
$$

The last expression goes to zero as $n$ goes to infinity. So for the rest of the proof we can assume $H \neq 1/2$.

First note that we have

$$\begin{aligned}
&\sup_{r\in A_{t,\delta_n}} \{\delta_n^H |a(t,r,\delta_n)|\} \\
&= \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |(1-t/\delta_n+\lfloor t/\delta_n\rfloor)\underline{a}(t,r,\delta_n)+(t/\delta_n-\lfloor t/\delta_n\rfloor)\overline{a}(t,r,\delta_n)|\} \\
&\le \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t,r,\delta_n)+\overline{a}(t,r,\delta_n)|\} \\
&\le \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t,r,\delta_n)|\} + \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\overline{a}(t,r,\delta_n)|\}.
\end{aligned}$$

Hence, it suffices to show the lemma for $\underline{a}$ and $\overline{a}$. We start with $\underline{a}$. Assume also first that $H < 1/2$. For negative r we have that

$$\underline{a}(t,r,\delta_n) = (\lfloor t/\delta_n\rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}},$$

is negative. The derivative with respect to r is negative. This means that the biggest absolute value of this expression occurs when $r = -1$ and then we have

$$|\underline{a}(t,-1,\delta_n)| = |(\lfloor t/\delta_n\rfloor + 1)^{H-\frac{1}{2}} - 1| \le 2.$$

For non-negative $r$ we have that

$$|\underline{a}(t,r,\delta_n)| = (\lfloor t/\delta_n\rfloor - r)^{H-\frac{1}{2}}.$$

The largest value occurs when $r = \lfloor t/\delta_n\rfloor - 1$, and then we have

$$|\underline{a}(t,\lfloor t/\delta_n\rfloor - 1,\delta_n)| = |(1)^{H-\frac{1}{2}}| = 1.$$

So we have in this case proved that

$$\sup_{r\in A_{t,\delta_n}} \{\delta_n^H |a(t,r,\delta_n)|\} \le 2\delta_n^H,$$

hence the result follows. Now assume that $H > 1/2$. For negative $r$ we have that

$$\underline{a}(t,r,\delta_n) = (\lfloor t/\delta_n\rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}},$$

is positive. The derivative with respect to $r$ is also positive. So for negative $r$ the biggest absolute-value of $\underline{a}(t,r,\delta_n)$ occurs when $r = -1$. We then have

$$\begin{aligned}
|\underline{a}(t,-1,\delta_n)| &= |(\lfloor t/\delta_n\rfloor + 1)^{H-\frac{1}{2}} - 1| \\
&\le (\lfloor t/\delta_n\rfloor + 1)^{H-\frac{1}{2}} + 1 \\
&\le (t/\delta_n + 1)^{H-\frac{1}{2}} + 1.
\end{aligned}$$

For non-negative $r$ we have that

$$\underline{a}(t,r,\delta_n) = (\lfloor t/\delta_n\rfloor - r)^{H-\frac{1}{2}}$$

is positive and decreasing. So the largest value occurs when $r = 0$. We get

$$\underline{a}(t,0,\delta_n) = (\lfloor t/\delta_n\rfloor)^{H-\frac{1}{2}} \le (t/\delta_n)^{H-\frac{1}{2}}.$$

So we have for all $r \in A_{t,\delta_n}$

$$|\underline{a}(t,r,\delta_n)| \leq (t/\delta_n + 1)^{H-\frac{1}{2}} + 1 + (t/\delta_n)^{H-\frac{1}{2}}.$$

So we have

$$\begin{aligned}\sup_{r\in A_{t,\delta_n}} \{\delta_n^H |a(t,r,\delta_n)|\} &\leq \delta_n^H \left[(t/\delta_n + 1)^{H-\frac{1}{2}} + 1 + (t/\delta_n)^{H-\frac{1}{2}}\right] \\ &= \delta_n^{\frac{1}{2}} \left[(t+\delta_n)^{H-\frac{1}{2}} + \delta_n^{H-\frac{1}{2}} + t^{H-\frac{1}{2}}\right],\end{aligned}$$

where $K$ is a constant. The last expression converges to zero as n tends to infinity so the result is proved for $\underline{a}$.

We now prove the result for $\overline{a}$. Assume first that $H > 1/2$. Note first that for $\delta_n \leq 1$ we have

$$\left\lfloor \frac{t+2}{\delta_n} \right\rfloor \geq \frac{t+2}{\delta_n} - 1 \geq \frac{t}{\delta_n} + 1.$$

So for negative $r$ and big enough $n$ we have

$$\begin{aligned}|\overline{a}(t,r,\delta_n)| &= (\lfloor t/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \\ &\leq (t/\delta_n + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \\ &\leq \left(\left\lfloor \frac{t+2}{\delta_n} \right\rfloor - r\right)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \\ &= \underline{a}(t+2,r,\delta_n).\end{aligned}$$

For non-negative $r \in A_{t,\delta_n}$ we have that

$$|\overline{a}(t,r,\delta_n)| = (\lfloor t/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}},$$

this decreases with $r$ so it has its maximum-value when $r = 0$. For this value we have

$$\begin{aligned}|\overline{a}(t,0,\delta_n)| &= (\lfloor t/\delta_n \rfloor + 1)^{H-\frac{1}{2}} \\ &\leq (\lfloor (t+2)/\delta_n \rfloor)^{H-1/2} \\ &= \underline{a}(t+2,0,\delta_n).\end{aligned}$$

So we have proved that for $r \in A_{t,\delta_n}$

$$|\overline{a}(t,r,\delta_n)| \leq \underline{a}(t+2,r,\delta_n).$$

Since $A_{t,\delta_n} \subset A_{t+2,\delta_n}$ we get by what we already have proved for $\underline{a}$

$$\begin{aligned}\limsup_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\overline{a}(t,r,\delta_n)|\} &\leq \limsup_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} \\ &\leq \limsup_{n\to\infty} \sup_{r\in A_{t+2,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} \\ &= \lim_{n\to\infty} \sup_{r\in A_{t+2,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} \\ &= 0.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\overline{a}(t,r,\delta_n)|\} = 0.$$

Now assume that $H < 1/2$ and $\delta_n \leq 1$. For negative $r$ we have

$$\begin{aligned}
|\overline{a}(t,r,\delta_n)| &= (-r)^{H-\frac{1}{2}} - (\lfloor t/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}} \\
&\leq (-r)^{H-\frac{1}{2}} - (\lfloor (t+2)/\delta_n \rfloor - r)^{H-1/2} \\
&= |\underline{a}(t+2,r,\delta_n)|.
\end{aligned}$$

For non-negative $r$ we have

$$|\overline{a}(t,r,\delta_n)| = (\lfloor t/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}}.$$

This value increases when $r$ increases so its maximum value occurs when $r = \lfloor t/\delta_n \rfloor - 1$, and we get

$$\begin{aligned}
|\overline{a}(t, \lfloor t/\delta_n \rfloor - 1, \delta_n)| &= (\lfloor t/\delta_n \rfloor + 1 - (\lfloor t/\delta_n \rfloor - 1))^{H-\frac{1}{2}} \\
&= 2^{H-\frac{1}{2}}.
\end{aligned}$$

Hence we have proved for $r \in A_{t,\delta_n}$

$$|\overline{a}(t,r,\delta_n)| \leq |\underline{a}(t+2,r,\delta_n)| + 2^{H-\frac{1}{2}}.$$

So by using that we have proven the result for $\underline{a}$, and the fact that $A_{t,\delta_n} \subset A_{t+2,\delta_n}$, we get

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\overline{a}(t,r,\delta_n)|\} \\
&\leq \limsup_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)| + \delta_n^H 2^{H-\frac{1}{2}}\} \\
&\leq \limsup_{n\to\infty} \left[ \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} + \delta_n^H 2^{H-\frac{1}{2}} \right] \\
&\leq \limsup_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} + \limsup_{n\to\infty} \delta_n^H 2^{H-\frac{1}{2}} \\
&\leq \limsup_{n\to\infty} \sup_{r\in A_{t+2,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} + \limsup_{n\to\infty} \delta_n^H 2^{H-\frac{1}{2}} \\
&= \lim_{n\to\infty} \sup_{r\in A_{t+2,\delta_n}} \{\delta_n^H |\underline{a}(t+2,r,\delta_n)|\} + \lim_{n\to\infty} \delta_n^H 2^{H-\frac{1}{2}} \\
&= 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \sup_{r\in A_{t,\delta_n}} \{\delta_n^H |\overline{a}(t,r,\delta_n)|\} = 0,$$

and the lemma is proved.

□

In the next lemma we prove that a sum converges to an integral.

**Lemma 3.4.6.** *Let $t_1, t_2 \in (0, \infty)$, and let $\{\delta_n\}$ be a positive sequence that converges to zero. We then have*

$$\lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} a(t_1, r, \delta_n) \cdot a(t_2, r, \delta_n)$$
$$= \frac{1}{C_H^2} \int_{-\infty}^{0} \left[(t_1 - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right] \cdot \left[(t_2 - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right] dx.$$

*Proof.* The result is obvious if $H = 1/2$, because then both sides are zero. Also note that the integral is positive and exists from Hölder's inequality and lemma 3.4.1.

If $H > 1/2$ we have that $a(t_i, r, \delta_n), \underline{a}(t_i, r, \delta_n), \overline{a}(t_i, r, \delta_n) \geq 0$, for $i = 1, 2$, all $\delta_n$ and all the $r$ we sum over. We also have $\underline{a}(t_i, r\delta_n) \leq a(t_i, r\delta_n) \leq \underline{a}(t_i, r\delta_n)$. Therefore

$$\begin{aligned} 0 &\leq \underline{a}(t_1, r, \delta_n)\underline{a}(t_2, r, \delta_n) \\ &\leq a(t_1, r, \delta_n)a(t_2, r, \delta_n) \\ &\leq \overline{a}(t_1, r, \delta_n)\overline{a}(t_2, r, \delta_n). \end{aligned}$$

If $H < 1/2$ then $a(t_i, r, \delta_n), \underline{a}(t_i, r, \delta_n), \overline{a}(t_i, r, \delta_n) \leq 0$, for $t = 1, 2$, all $\delta_n$ and all the $r$ we sum over. As we have $-\underline{a}(t_i, r, \delta_n) \leq -a(t_i, r, \delta_n) \leq -\overline{a}(t_i, r, \delta_n)$, we again get

$$\begin{aligned} 0 \leq& \underline{a}(t_1, r, \delta_n)\underline{a}(t_2, r, \delta_n) \\ &\leq a(t_1, r, \delta_n)a(t_2, r, \delta_n) \\ &\leq \overline{a}(t_1, r, \delta_n)\overline{a}(t_2, r, \delta_n). \end{aligned}$$

By the Squeeze theorem we only need to prove the lemma for the two cases where we substitute $\underline{a}$ and $\overline{a}$ for $a$. We have

$$\begin{aligned} &\frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \underline{a}(t_1, r, \delta_n) \cdot \underline{a}(t_2, r, \delta_n) \\ =& \frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \left[(\lfloor t_1/\delta_n \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right] \left[(\lfloor t_2/\delta_n \rfloor - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right] \\ =& \frac{\delta_n}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \left[(\lfloor t_1/\delta_n \rfloor\delta_n - r\delta_n)^{H-\frac{1}{2}} - (-r\delta_n)^{H-\frac{1}{2}}\right] \left[(\lfloor t_2/\delta_n \rfloor\delta_n - r\delta_n)^{H-\frac{1}{2}} - (-r\delta_n)^{H-\frac{1}{2}}\right] \\ =& \frac{1}{C_H^2} \sum_{\tau=-\lfloor 1/\delta_n^2 \rfloor\delta_n+\delta_n}^{-\delta_n}{}_{\delta_n} \left[(\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right] \left[(\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right] \delta_n \\ =& \frac{1}{C_H^2} \int_{-\infty}^{0} \sum_{\tau=-\lfloor 1/\delta_n^2 \rfloor\delta_n+\delta_n}^{-\delta_n}{}_{\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \left[(\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right] \\ &\qquad \cdot \left[(\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\right] \Bigg) ds. \end{aligned}$$

We also have

$$\begin{aligned}&\left[(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\cdot\left[(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\\ \leq\ &[(t_1-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}][(t_2-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}],\end{aligned}$$

for $H\in(0,1), H\neq 1/2$ and negative $\tau$. We also have that the derivative of

$$[(t_1-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}][(t_2-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}],$$

with respect to $\tau$ is non-negative on $(-\infty,0)$. Together this means that

$$\begin{aligned}&\sum_{\tau=-\lfloor 1/\delta_n^2\rfloor\delta_n+\delta_n}^{-\delta_n}{}_{\delta_n}\Bigg(I_{[\tau,\tau+\delta_n)}(s)\left[(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\\ &\qquad\cdot\left[(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\Bigg)\\ \leq\ &\sum_{\tau=-\lfloor 1/\delta_n^2\rfloor\delta_n+\delta_n}^{-\delta_n}{}_{\delta_n}\Bigg(I_{[\tau,\tau+\delta_n)}(s)\left[(t_1-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\\ &\qquad\cdot\left[(t_2-\tau)^{H-\frac{1}{2}}-(-\tau)^{H-\frac{1}{2}}\right]\Bigg)\\ \leq\ &[(t_1-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}][(t_2-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}],\ s\in(-\infty,0).\end{aligned}$$

By noting that first function in the inequality converges to the last function pointwise by lemma D.3.1, the result follows by the Dominated Convergence Theorem.

Almost the exact same calculations that we did for $\underline{a}$ works for $\overline{a}$, but to bound the functions we now use

$$[(t_1+1-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}][(t_2+1-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}]$$

However, we will still get pointwise convergence to

$$[(t_1-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}][(t_2-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}].$$

We show the details

$$\frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \overline{a}(t_1, r, \delta_n) \cdot \overline{a}(t_2, r, \delta_n)$$

$$= \frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \Bigg( \Big[ (\lfloor t_1/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}} \Big] \Bigg)$$

$$= \frac{\delta_n}{C_H^2} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor+1}^{-1} \Bigg( \Big[ (\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - r\delta_n)^{H-\frac{1}{2}} - (-r\delta_n)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - r\delta_n)^{H-\frac{1}{2}} - (-r\delta_n)^{H-\frac{1}{2}} \Big] \Bigg)$$

$$= \frac{1}{C_H^2} \sum_{\delta}{}_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n+\delta_n}^{-\delta_n} \Bigg( \Big[ (\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \delta_n \Bigg)$$

$$= \frac{1}{C_H^2} \int_{-\infty}^{0} \sum_{\delta_n}{}_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n+\delta_n}^{-\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \Big[ (\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \Bigg) ds.$$

Assume now that $n$ is so big that $|\delta_n| < 1$. We have

$$\Big[ (\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big]$$
$$\leq \Big[ (t_1 + 1 - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (t_2 + 1 - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big].$$

Also note that the derivative of $[(t_1+1-\tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}] \cdot [(t_2+1-\tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}]$ with respect to $\tau$ is non-negative for $\tau \in (-\infty, 0)$. This means that

$$\sum_{\delta_n}{}_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n+\delta_n}^{-\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \Big[ (\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \Bigg)$$

$$\leq \sum_{\delta_n}{}_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n+\delta_n}^{-\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \Big[ (t_1 + 1 - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (t_2 + 1 - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \Bigg)$$

$$\leq [(t_1 + 1 - s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}][(t_2 + 1 - s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}].$$

The last function is integrable over $(-\infty, 0)$ by Hölder's inequality and by lemma 3.4.1. The result now follows by lemma D.3.2, and the Dominated convergence Theorem.

□

The next lemma is very similar to lemma 3.4.6.

**Lemma 3.4.7.** *Let $z_1, z_2$ be real numbers such that $0 \leq z_1 < z_2$. Let $t_1, t_2 \in [z_2, \infty)$, let $\{\delta_n\}$ be a positive sequence which converges to zero. We then have*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} a(t_1, r, \delta_n) a(t_2, r, \delta_n) \\
&= \frac{1}{C_H^2} \int_{z_1}^{z_2} (t_1 - x)^{H-\frac{1}{2}} (t_2 - x)^{H-\frac{1}{2}} dx.
\end{aligned}$$

*Proof.* Note first that by lemma D.1.2 there are $n_i, i = 1, 2$ such that if $n \geq n_i$

$$\left\lfloor \frac{z_1}{\delta_n} \right\rfloor < \left\lfloor \frac{z_2}{\delta_n} \right\rfloor - 1 < \left\lfloor \frac{t_i}{\delta_n} \right\rfloor. \tag{3.12}$$

So for the rest of the proof assume that $n$ is chosen so big that eq. (3.12) holds.
For $H = 1/2$ we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} a(t_1, r, \delta_n) a(t_2, r, \delta_n) \\
&= \lim_{n\to\infty} \frac{\delta_n}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} 1 \cdot 1 \\
&= \lim_{n\to\infty} \frac{\delta_n}{C_H^2} \left( \lfloor z_2/\delta_n \rfloor - \lfloor z_1/\delta_n \rfloor \right) \\
&= \lim_{n\to\infty} \frac{1}{C_H^2} \left( \lfloor z_2/\delta_n \rfloor \delta_n - \lfloor z_1/\delta_n \rfloor \delta_n \right) \\
&= \frac{z_2 - z_1}{C_H^2} \\
&= \frac{1}{C_H^2} \int_{z_1}^{z_2} (t_1 - x)^{H-\frac{1}{2}} (t_2 - x)^{H-\frac{1}{2}} dx.
\end{aligned}$$

So we have proven the case $H = 1/2$.
For $H > 1/2$ we have for $i = 1, 2$, $r \in \{\lfloor z_1/\delta_n \rfloor, \lfloor z_1/\delta_n \rfloor + 1, \ldots, \lfloor z_2/\delta_n \rfloor - 1\}$

$$\begin{aligned}
0 &\leq \left( \lfloor t_i/\delta_n \rfloor - r \right)^{H-\frac{1}{2}} = \underline{a}(t_i, r, \delta_n) \\
&\leq \left( \lfloor t_i/\delta_n \rfloor + 1 - r \right)^{H-\frac{1}{2}} = \overline{a}(t_i, r, \delta_n).
\end{aligned}$$

So

$$\begin{aligned}
0 &\leq \underline{a}(t_1, r, \delta_n) \underline{a}(t_2, r, \delta_n) \\
&\leq a(t_1, r, \delta_n) a(t_2, r, \delta_n) \\
&\leq \overline{a}(t_1, r, \delta_n) \overline{a}(t_2, r, \delta_n),
\end{aligned} \tag{3.13}$$

because $a$ is a linear interpolation where the linear function takes $\underline{a}$ and $\overline{a}$ at the endpoints. For $H < 1/2$ we have for $i = 1, 2$, $r \in \{\lfloor z_1/\delta_n \rfloor, \lfloor z_1/\delta_n \rfloor +$

$1, \dots, \lfloor z_2/\delta_n \rfloor - 1\}$

$$\begin{aligned} 0 &\le (\lfloor t_i/\delta_n \rfloor + 1 - r)^{H-\frac{1}{2}} = \overline{a}(t_i, r, \delta_n) \\ &\le (\lfloor t_i/\delta_n \rfloor - r)^{H-\frac{1}{2}} = \underline{a}(t_i, r, \delta_n). \end{aligned}$$

So

$$\begin{aligned} 0 &\le \overline{a}(t_1, r, \delta_n)\overline{a}(t_2, r, \delta_n) \\ &\le a(t_1, r, \delta_n)a(t_2, r, \delta_n) \\ &\le \underline{a}(t_1, r, \delta_n)\underline{a}(t_2, r, \delta_n), \end{aligned} \tag{3.14}$$

again because by definition $a$ is linearly interpolated between $\underline{a}$ and $\overline{a}$. Equation (3.13) and eq. (3.14) together with the Squeeze theorem for sequences tells us it suffices to prove the lemma for $\underline{a}$ and $\overline{a}$ instead of $a$.

We start with $\underline{a}$. We get

$$\begin{aligned} &\lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} \underline{a}(t_1, r, \delta_n)\underline{a}(t_2, r, \delta_n) \\ =& \lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} (\lfloor t_1/\delta_n \rfloor - r)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor - r)^{H-\frac{1}{2}} \\ =& \lim_{n\to\infty} \frac{\delta_n}{C_H^2} \sum_{r=\lfloor z_1/\delta_n \rfloor}^{\lfloor z_2/\delta_n \rfloor - 1} (\lfloor t_1/\delta_n \rfloor\delta_n - r\delta_n)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - r\delta_n)^{H-\frac{1}{2}} \\ =& \frac{1}{C_H^2} \lim_{n\to\infty} \sum_{\tau=\lfloor z_1/\delta_n \rfloor\delta_n}^{\lfloor z_2/\delta_n \rfloor\delta_n - \delta_n}{}_{\delta_n} (\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} \delta_n \\ =& \frac{1}{C_H^2} \lim_{n\to\infty} \int_{z_1/2}^{z_2} \sum_{\tau=\lfloor z_1/\delta_n \rfloor\delta_n}^{\lfloor z_2/\delta_n \rfloor\delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s) (\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} ds. \end{aligned}$$

The limits in the last integral are chosen because $\lfloor z_2/\delta_n \rfloor\delta_n \le z_2$, and for big enough $n$ we have that $\lfloor z_1/\delta_n \rfloor\delta_n \ge z_1/2$. This is because $\lfloor z_1/\delta_n \rfloor\delta_n$ converges to $z_1$ as n goes to infinity. If $H > 1/2$ there exists a constant K such that for all $s \in (z_1/2, z_2)$ and all $n$

$$\sum_{\tau=\lfloor z_1/\delta_n \rfloor\delta_n}^{\lfloor z_2/\delta_n \rfloor\delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s) (\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} < K.$$

This follows from the fact that

$$(\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} \le (t_1 - \tau)^{H-\frac{1}{2}} (t_2 - \tau)^{H-\frac{1}{2}}.$$

The last expression is continuous in $\tau$ on $[z_1/2, z_2]$(remember $H > 1/2$) and hence bounded on this interval. So we are able to bound

$$\sum_{\tau=\lfloor z_1/\delta_n \rfloor\delta_n}^{\lfloor z_2/\delta_n \rfloor\delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s) (\lfloor t_1/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}} (\lfloor t_2/\delta_n \rfloor\delta_n - \tau)^{H-\frac{1}{2}},$$

by an integrable function(the constant function $K$). Since we have pointwise convergence Lebesgue almost-everywhere on $(z_1/2, z_2)$ by lemma D.3.3, the lemma follows from the Dominated Convergence Theorem.

If $H < 1/2$ we must rewrite the expression like this

$$\begin{aligned}
&\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}ds\\
=&\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}ds\\
&+\frac{1}{C_H^2}\lim_{n\to\infty}(\lfloor t_1/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}\delta_n,
\end{aligned}$$

provided both limits on the right hand side exist. We will prove existence of these limits and show what they are. We have that

$$\begin{aligned}
&(\lfloor t_1/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}\delta_n\\
\leq&\ \delta_n^{H-\frac{1}{2}}\delta_n^{H-\frac{1}{2}}\delta_n\\
=&\ \delta_n^{2H}.
\end{aligned}$$

So we get

$$\frac{1}{C_H^2}\lim_{n\to\infty}(\lfloor t_1/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\lfloor z_2/\delta_n\rfloor+\delta_n)^{H-\frac{1}{2}}\delta_n=0.$$

We now look at the term

$$\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}ds.$$

Notice first that

$$\begin{aligned}
&\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}ds\\
=&\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n}{}_{\delta_n} I_{[\tau+\delta,\tau+2\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}ds,
\end{aligned}$$

because as we noted, for big enough $n$ we have $\lfloor z_1/\delta_n\rfloor\delta_n \geq z_1/2$, and we also have

$$\begin{aligned}
\tau+2\delta_n &\leq \lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n+2\delta_n\\
&\leq z_2.
\end{aligned}$$

For $\tau \in \{\lfloor z_1/\delta_n\rfloor\delta_n, \lfloor z_1/\delta_n\rfloor\delta_n+\delta_n, \ldots, \lfloor z_2/\delta_n\rfloor\delta_n-2\delta_n\}$ we have

$$\begin{aligned}
&(\lfloor t_1/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n-\tau)^{H-\frac{1}{2}}\\
\leq&(t_1-\delta_n-\tau)^{H-\frac{1}{2}}(t_2-\delta_n-\tau)^{H-\frac{1}{2}}.
\end{aligned}$$

This is because for $i = 1, 2$ we have $t_i - \delta_n \leq \lfloor t_i/\delta_n \rfloor \delta_n$, since $\lfloor t_i/\delta_n \rfloor > t_i/\delta_n - 1$. So we have

$$\begin{aligned}
&\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta_n,\tau+2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}} \\
&\leq \sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta_n,\tau+2\delta_n)}(s)(t_1 - \delta_n - \tau)^{H-\frac{1}{2}}(t_2 - \delta_n - \tau)^{H-\frac{1}{2}} \\
&= \sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta,\tau+2\delta_n)}(s)(t_1 - (\tau + \delta_n))^{H-\frac{1}{2}}(t_2 - (\tau + \delta_n))^{H-\frac{1}{2}} \\
&= \sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n + \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(t_1 - \tau)^{H-\frac{1}{2}}(t_2 - \tau)^{H-\frac{1}{2}}.
\end{aligned} \tag{3.15}$$

We also have

$$\begin{aligned}
&\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n + \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(t_1 - \tau)^{H-\frac{1}{2}}(t_2 - \tau)^{H-\frac{1}{2}} \\
&\leq (t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}},
\end{aligned} \tag{3.16}$$

for $s \in (z_1/2, z_2)$. This is because the right-hand side is non-zero, and if the left-hand side is non-zero, there must be a $\tau^*$ such that $s \in [\tau^*, \tau^* + \delta_n)$ and then $(t_1 - \tau^*)^{H-\frac{1}{2}}(t_2 - \tau^*)^{H-\frac{1}{2}} \leq (t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}}$, because $H < 1/2$. Combining eq. (3.15) and eq. (3.16) we get

$$\begin{aligned}
&\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta_n,\tau+2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}} \\
&\leq (t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}},
\end{aligned}$$

for $s \in (z_1/2, z_2)$. $(t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}}$ is integrable over $(z_1/2, z_2)$ by Hölder's inequality and lemma 3.4.1. Since

$$\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta_n,\tau+2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}}$$

converges pointwise to

$$I_{[z_1,z_2)}(s)(t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}},$$

Lebesgue almost everywhere on $(z_1/2, z_2)$ by lemma D.3.3, the dominated convergence theorem tells us that

$$\begin{aligned}
&\frac{1}{C_H^2} \lim_{n\to\infty} \int_{z_1/2}^{z_2} \sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} I_{[\tau+\delta,\tau+2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}} ds \\
=&\frac{1}{C_H^2} \int_{z_1}^{z_2} (t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}} ds.
\end{aligned}$$

So we are done with proving the result for $\underline{a}$.

We now prove the result for $\overline{a}$. The calculations will be very similar. Like earlier we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\delta_n^{2H}}{C_H^2}\sum_{r=\lfloor z_1/\delta_n\rfloor}^{\lfloor z_2/\delta_n\rfloor-1}\overline{a}(t_1,r,\delta_n)\overline{a}(t_2,r,\delta_n)\\
=&\lim_{n\to\infty}\frac{\delta_n^{2H}}{C_H^2}\sum_{r=\lfloor z_1/\delta_n\rfloor}^{\lfloor z_2/\delta_n\rfloor-1}(\lfloor t_1/\delta_n\rfloor+1-r)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor+1-r)^{H-\frac{1}{2}}\\
=&\lim_{n\to\infty}\frac{\delta_n}{C_H^2}\sum_{r=\lfloor z_1/\delta_n\rfloor}^{\lfloor z_2/\delta_n\rfloor-1}(\lfloor t_1/\delta_n\rfloor\delta_n+\delta_n-r\delta_n)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n+\delta_n-r\delta_n)^{H-\frac{1}{2}}\\
=&\frac{1}{C_H^2}\lim_{n\to\infty}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}(\lfloor t_1/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}\delta_n\\
=&\frac{1}{C_H^2}\lim_{n\to\infty}\int_{z_1/2}^{z_2}\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}ds.
\end{aligned}$$

If $H < 1/2$ we have

$$\begin{aligned}
&(\lfloor t_1/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}\\
&\leq(t_1-\tau)^{H-\frac{1}{2}}(t_2-\tau)^{H-\frac{1}{2}},
\end{aligned}$$

since $\lfloor t_i/\delta_n\rfloor\delta_n+\delta_n > t_i$ for $i = 1, 2$. So we have

$$\begin{aligned}
&\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-\frac{1}{2}}\\
\leq&\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}I_{[\tau,\tau+\delta_n)}(s)(t_1-\tau)^{H-\frac{1}{2}}(t_2-\tau)^{H-\frac{1}{2}}.
\end{aligned}$$

We also have for $s\in(z_1/2,z_2)$

$$\begin{aligned}
&\sum_{\tau=\lfloor z_1/\delta_n\rfloor\delta_n}^{\lfloor z_2/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}I_{[\tau,\tau+\delta_n)}(s)(t_1-\tau)^{H-\frac{1}{2}}(t_2-\tau)^{H-\frac{1}{2}}\\
&\leq(t_1-s)^{H-\frac{1}{2}}(t_2-s)^{H-\frac{1}{2}},
\end{aligned}$$

because the right-hand side is always non-negative, and if the left hand side is non-zero, we have that $s\in[\tau^*,\tau^*+\delta_n)$ and then the result follows because

$$\begin{aligned}
&(t_1-\tau^*)^{H-\frac{1}{2}}(t_2-\tau^*)^{H-\frac{1}{2}}\\
&\leq(t_1-s)^{H-\frac{1}{2}}(t_2-s)^{H-\frac{1}{2}}.
\end{aligned}$$

Since $(t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}}$ is integrable (Hölder and the fact that $t_i \geq z_2$ and $1 - 2H < 1$) we have that

$$\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} ds,$$

is bounded by an integrable function on $(z_1/2, z_2)$. Since it also converges pointwise Lebesgue almost everywhere on $(z_1/2, z_2)$ by lemma D.3.3, the result follows from the dominated convergence theorem.

Now let $H > 1/2$ , assume $n$ is so big that $\delta_n < 1$, we then have

$$\begin{aligned} &(\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} \\ \leq\ &(t_1 + 1)^{H-\frac{1}{2}}(t_2 + 1)^{H-\frac{1}{2}} = K. \end{aligned}$$

So again we have that

$$\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - \delta_n}{}_{\delta_n} I_{[\tau,\tau+\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}},$$

is bounded by an integrable function on $(z_1/2, z_2)$. Since we also have pointwise convergence Lebesgue almost everywhere by lemma D.3.3 the result follows by the dominated convergence theorem. □

## 3.5 Weak convergence of the finite-dimensional distributions of the Mandelbrot and Van Ness sum

In this section we will prove convergence in distribution of the finite-dimensional distributions of the Mandelbrot and Van Ness sum. This will be a big step in order to use the finite-dimensional requirement of theorem 2.4.2. We first give a lemma which takes care of the case that the first component might be identically zero.

**Lemma 3.5.1.** *Assume that $(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)})$ converges in distribution to a multivariate normal random variable with expectation $\vec{m} \in \mathbb{R}^k$, and covariance matrix $A \in \mathbb{R}^k \times \mathbb{R}^k$ which is positive semi-definite. Let $\vec{p'} \in \mathbb{R}^{k+1}$ be such that*

$$\begin{aligned} &p_1 = 0 \\ &p_i = m_{i-1}, \quad 2 \leq i \leq k + 1. \end{aligned}$$

*Let $B \in \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$ be such that*

$$B_{i,j} = \begin{cases} 0, & \text{for } i = 1 \text{ or } j = 1 \\ A_{i-1,j-1}, & \text{for } i \neq 1 \text{ and } j \neq 1. \end{cases}$$

*Then $B$ is positive semi-definite, and $(0, Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)})$ converges in distribution to a multivariate normal random variable with expectation $\vec{p}$ and covariance matrix $B$.*

*Proof.* We first show that $B$ is positive semi-definite. let $\vec{r} \in \mathbb{R}^{k+1}$ we then have

$$\begin{aligned}\vec{r}^T B \vec{r} =& \sum_{i=1}^{k+1}\sum_{j=1}^{k+1} r_i r_j B_{i,j} \\ =& \sum_{i=2}^{k+1}\sum_{j=2}^{k+1} r_i r_j B_{i,j} \\ =& \sum_{i=2}^{k+1}\sum_{j=2}^{k+1} r_i r_j A_{i-1,j-1} \\ \geq& 0,\end{aligned}$$

where we in the last inequality used that $A$ is positive semi-definite.

Let $\vec{u} \in \mathbb{R}^{k+1}$. We get

$$\begin{aligned}&\lim_{n\to\infty} E\left[\exp\left(iu_1 \cdot 0 + i\sum_{j=2}^{k+1} u_j Y_{j+1}^{(n)}\right)\right] \\ =& \lim_{n\to\infty} E\left[\exp\left(i\sum_{j=2}^{k+1} u_j Y_{j+1}^{(n)}\right)\right] \\ =& \exp\left(i\sum_{j=2}^{k+1} m_{j-1}u_j - \frac{1}{2}\sum_{j=2}^{k+1}\sum_{l=2}^{k+1} u_j u_l A_{j-1,l-1}\right) \\ =& \exp\left(i\sum_{j=1}^{k+1} p_j u_j - \frac{1}{2}\sum_{j=1}^{k+1}\sum_{l=1}^{k+1} u_j u_l B_{j,l}\right).\end{aligned}$$

We used the convergence in distribution of $(Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)})$. A small technical detail here is that $\exp(i\vec{u}^T\vec{x})$ is a bounded complex continuous function, not a bounded real continuous function as defined in definition 3.3.3. But the result will follow easily from Euler's formula and linearity of the expectation, and the fact that sin and cos are bounded continuous functions. We have shown that we have convergence of characteristic functions. By Glivenko's theorem ([App09, Theorem 1.1.14, p. 18]) we have convergence in distribution. □

Now we turn to the main result of this section.

**Theorem 3.5.2.** *Let $0 \leq t_1 < t_2 <, \cdots, < t_k$, $H \in (0,1)$. Assume that $\{\delta_n\}$ is a sequence of positive numbers which converges to zero. Let $X^{(\delta_n)}$ be as in definition 3.2.6. We then have that*

$$(X_{t_1}^{(\delta_n)}, X_{t_2}^{(\delta_n)}, \ldots, X_{t_k}^{(\delta_n)})$$

*converges in distribution to a multivariate normal random variable*

$$(Y_1, Y_2, \ldots, Y_k),$$

*where each component has expectation zero, and with covariance matrix such that*

$$cov(Y_i, Y_j) = \frac{1}{2}\left(|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}\right). \tag{3.17}$$

*Proof.* First note that by the remark after proposition 3.1.3 we know that $Y$ exist.

Note that if $t_1 = 0$ and $t_l > 0$ we have

$$\frac{1}{2}\left(|0|^{2H} + |t_l|^{2H} - |t_l - 0|^{2H}\right) = 0,$$

so by lemma 3.5.1 we can assume that $t_1 > 0$.

The result will follow from lemma C.3.1 if we can show that every linear combination

$$u_1 X_{t_1}^{(\delta_n)} + u_2 X_{t_2}^{(\delta_n)} + \cdots + u_k X_{t_k}^{(\delta_n)},$$

converges in distribution to a normal random variable, with expectation zero, and variance $\vec{u}^T A \vec{u}$, where $A$ is a $k \times k$-matrix where $A_{i,j}$ is given by eq. (3.17). By the alternative representation of $X^{(\delta_n)}$ in eq. (3.6) we have

$$\begin{aligned}
&u_1 X_{t_1}^{(\delta_n)} + u_2 X_{t_2}^{(\delta_n)} + \cdots + u_k X_{t_k}^{(\delta_n)} \\
=&\frac{\delta_n^H}{C_H} \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}} \\
&+ \frac{\delta_n^H}{C_H} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}} \\
&+ \frac{\delta_n^H}{C_H} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}} \\
&+ \frac{\delta_n^H}{C_H} \sum_{r=\lfloor t_1/\delta_n \rfloor}^{\lfloor t_2/\delta_n \rfloor - 1} \Big(u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}} \\
&+ \cdots \\
&+ \frac{\delta_n^H}{C_H} \sum_{r=\lfloor t_l/\delta_n \rfloor}^{\lfloor t_{l+1}/\delta_n \rfloor - 1} \Big(u_{l+1} a(t_{l+1}, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}} \\
&+ \cdots \\
&+ \frac{\delta_n^H}{C_H} \sum_{r=\lfloor t_{k-1}/\delta_n \rfloor}^{\lfloor t_k/\delta_n \rfloor - 1} u_k a(t_k, r, \delta_n) w_r I_{A_{\delta_n}} \\
&+ \frac{\delta_n^H}{C_H} \Big(u_1 (t_1/\delta_n - \lfloor t_1/\delta_n \rfloor) w_{\lfloor t_1/\delta_n \rfloor} + \cdots + u_k (t_k/\delta_n - \lfloor t_k/\delta_n \rfloor) w_{\lfloor t_k/\delta_n \rfloor}\Big) I_{A_{\delta_n}}.
\end{aligned}$$

We name the different terms of the sum

$$R_{-1}^n \doteq \frac{\delta_n^H}{C_H} \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}}$$

$$R_0^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r$$

$$R_1^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r$$

$$R_2^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=\lfloor t_1/\delta_n \rfloor}^{\lfloor t_2/\delta_n \rfloor - 1} \Big(u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r$$

$$R_l^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=\lfloor t_{l-1}/\delta_n \rfloor}^{\lfloor t_l/\delta_n \rfloor - 1} \Big(u_l a(t_l, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r$$

$$R_k^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=\lfloor t_{k-1}/\delta_n \rfloor}^{\lfloor t_k/\delta_n \rfloor - 1} u_k a(t_k, r, \delta_n) w_r$$

$$R_{k+1}^n \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \Big(u_1 (t_1/\delta_n - \lfloor t_1/\delta_n \rfloor) w_{\lfloor t_1/\delta_n \rfloor} + \cdots + u_k (t_k/\delta_n - \lfloor t_k/\delta_n \rfloor) w_{\lfloor t_k/\delta_n \rfloor}\Big).$$

We obviously have that $R_{k+1}^n$ converges to zero a.s., this means that it also converges to zero in probability. By [Bil99, Theorem 3.1, p. 27] it suffices to look at what $R_{-1}^n + R_0^2 + R_1^n + \cdots + R_k^n$ converges to in distribution, because $R_{-1}^n + R_0^2 + R_1^n + \cdots + R_k^n + R_{k+1}$ will converge to the same random variable if we have convergence in distribution.

We will prove that $R_{-1}^n$ also converge to zero in probability. First we will show that $R_{-1}^n$ is a square-integrable random variable, with zero expectation. We get with the help of Fatou's lemma

$$\begin{aligned}
&E\left[(R_{-1}^n)^2\right] \\
&= E\left[\left(\frac{\delta_n^H}{C_H} \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r I_{A_{\delta_n}}\right)^2\right] \\
&= \left(\frac{\delta_n^H}{C_H}\right)^2 E\left[\lim_{M\to\infty} \left(I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right] \\
&\le \left(\frac{\delta_n^H}{C_H}\right)^2 \liminf_{M\to\infty} E\left[\left(I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right].
\end{aligned}$$

We have by using that $P(A_{\delta_n^c}) = 0$,

$$E\left[\left(I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right]$$
$$= E\left[\left(\Big(I_{A_{\delta_n}} + I_{A^c_{\delta_n}}\Big) \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right]$$
$$= \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big)^2$$
$$\leq k^2 \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \left(u_1^2 a(t_1, r, \delta_n)^2 + u_2^2 a(t_2, r, \delta_n)^2 + \cdots + u_k^2 a(t_k, r, \delta_n)^2\right)$$
$$\leq k^2 \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \left(u_1^2 a(t_1, r, \delta_n)^2 + u_2^2 a(t_2, r, \delta_n)^2 + \cdots + u_k^2 a(t_k, r, \delta_n)^2\right).$$

We used that the integral of an integrable random variable over a set of probability zero, in our case the set is $A^c_{\delta_n}$, is zero. Hence we have that $E[(R^n_{-1})^2]$, and all the corresponding quantities where we sum down to $-M$ is square-integrable. With the second moment being bounded by

$$\left(\frac{\delta_n^H}{C_H}\right)^2 k^2 \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \left(u_1^2 a(t_1, r, \delta_n)^2 + u_2^2 a(t_2, r, \delta_n)^2 + \cdots + u_k^2 a(t_k, r, \delta_n)^2\right).$$

By lemma 3.4.4 this value is finite for large enough $n$. We will now show that

$$E[R^n_{-1}] = 0.$$

Notice that

$$E\left[\frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right]$$
$$= E\left[\frac{\delta_n^H}{C_H} \Big(I_{A_{\delta_n}} + I_{A^c_{\delta_n}}\Big) \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right]$$
$$= 0,$$

using that the expectation over $A^c_{\delta_n}$ of an integrable function is zero. And since for every $M$

$$\frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r,$$

is bounded in $L^2(\Omega)$ by a common constant, we have that the collection

$$\left\{\frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=-M}^{-\lfloor 1/\delta_n^2 \rfloor} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right\}_{M \in \mathbb{N}},$$

is uniformly integrable by [Wil91, Proposition 13.3(a)]. So when $M$ goes to infinity we do not only have a.s. convergence but also $L^1$-convergence, see [Wil91, Theorem 13.7, p. 131], hence $E[R^n_{-1}] = 0$. We are now ready to prove that $R^n_{-1}$ converges to zero in probability. By Chebychev's inequality we have have for every $\epsilon > 0$

$$P(|R^n_{-1}| \geq \epsilon) \leq \frac{\sigma^2_{R^n_{-1}}}{\epsilon^2}$$
$$\leq \frac{1}{\epsilon^2}\left(\frac{\delta_n^H}{C_H}\right)^2 k^2 \sum_{r=-\infty}^{-\lfloor 1/\delta_n^2 \rfloor} \left(u_1^2 a(t_1, r, \delta_n)^2 + u_2^2 a(t_2, r, \delta_n)^2 + \cdots + u_k^2 a(t_k, r, \delta_n)^2\right),$$

which converges to zero by lemma 3.4.4. Hence we do not need to look at $R^n_{-1}$ for exactly the same reason we were able to disregard $R^n_{k+1}$. We need to investigate the convergence in distribution of

$$R^n_0 + R^n_1 + R^n_2 + \cdots + R^n_k.$$

Next we prove that $R^n_0$ converges in distribution to a normal random variable with expectation zero and variance

$$\frac{1}{C_H^2}\int_{-\infty}^0 \left(\sum_{j=1}^k u_j \left[(t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right]\right)^2 dx.$$

We remember that

$$R^n_0 \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r.$$

Denote

$$Z_{r,\delta_n} \doteq \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r.$$

We get

$$E[Z_{r,\delta_n}]$$
$$= E\left[\frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right]$$
$$= E\left[\frac{\delta_n^H}{C_H} \left(I_{A_{\delta_n}} + I_{A^c_{\delta_n}}\right) \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right]$$
$$= 0,$$

Also we have

$$\begin{aligned}
&V[Z_{r,\delta_n}] \\
&= E[Z^2_{r,\delta_n}] \\
&= E\left[\left(\frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right] \\
&= E\left[\left(\frac{\delta_n^H}{C_H} \Big(I_{A_{\delta_n}} + I_{A^c_{\delta_n}}\Big) \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big) w_r\right)^2\right] \\
&= \frac{\delta_n^{2H}}{C_H^2} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big)^2.
\end{aligned}$$

We also have that for fixed $\delta_n$, $\{Z_{r,\delta_n}\}$ are independent, see proposition C.2.4.

$$s_n^2 \doteq \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} V[Z_{r,\delta_n}]$$

From lemma 3.4.6 we have that

$$\begin{aligned}
\lim_{n\to\infty} s_n^2 &= \lim_{n\to\infty} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} V[Z_{r,\delta_n}] \\
&= \lim_{n\to\infty} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \frac{\delta_n^{2H}}{C_H^2} \Big(u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)\Big)^2 \\
&= \lim_{n\to\infty} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \sum_{i=1}^{k} \sum_{j=1}^{k} u_i u_j \frac{\delta_n^{2H}}{C_H^2} a(t_i, r, \delta_n) a(t_j, r, \delta_n) \\
&= \frac{1}{C_H^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{-\infty}^{0} u_i u_j \left((t_i - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right) \left((t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right) dx \\
&= \frac{1}{C_H^2} \int_{-\infty}^{0} \left(\sum_{j=1}^{k} u_j \left[(t_j - x)^{H-\frac{1}{2}} - (-x)^{H-\frac{1}{2}}\right]\right)^2 dx \\
&\doteq s^2.
\end{aligned}$$

If $\vec{u} = \vec{0}$ the result follows directly, so we can assume that $\vec{u} \neq \vec{0}$. By lemma 3.4.2 $s^2 > 0$. We want to use the Lindeberg Central Limit Theorem, see [Bil95, Theorem 27.2, p. 359]. We need to check the Lindeberg condition, which in our notation is that for every $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \frac{1}{s_n^2} E\left[I_{|Z_{r,\delta_n}| \geq \epsilon s_n}(\omega) Z_{r,\delta_n}(\omega)^2\right] = 0.$$

We will in fact get that

$$\sum_{r=-\lfloor 1/\delta_n^2 \rfloor + 1}^{-1} \frac{1}{s_n^2} E\left[I_{|Z_{r,\delta_n}| \geq \epsilon s_n}(\omega) Z_{r,\delta_n}(\omega)^2\right] = 0,$$

when $n$ is big enough. Notice that $s_n^2$ is positive for large $n$ since it converges to $s$, so we do not divide by zero. Since

$$s_n \to s > 0,$$

we have for big enough n that

$$2\epsilon s > \epsilon s_n > \epsilon s/2 > 0. \tag{3.18}$$

Let $A_{t,\delta_n}$ be as in lemma 3.4.5. We have

$$\begin{aligned}
&\max_{r\in\{-\lfloor 1/\delta_n^2\rfloor+1,\dots,-1\}} |Z_{r,\delta_n}| \\
=&\max_{r\in\{-\lfloor 1/\delta_n^2\rfloor+1,\dots,-1\}} \left|\frac{\delta_n^H}{C_H} I_{A_{\delta_n}}\Big(u_1 a(t_1,r,\delta_n) + u_2 a(t_2,r,\delta_n) + \dots + u_k a(t_k,r,\delta_n)\Big) w_r\right| \\
\le&\frac{1}{C_H}\max_{r\in\{-\lfloor 1/\delta_n^2\rfloor+1,\dots,-1\}} \Big[|u_1\delta_n^H a(t_1,r,\delta_n)| + |u_2\delta_n^H a(t_2,r,\delta_n)| \\
&\qquad\qquad + \dots + |u_k\delta_n^H a(t_k,r,\delta_n)|\Big] \\
\le&\frac{1}{C_H}\Big[|u_1| \sup_{t\in A_{t_1,\delta_n}}\{\delta_n^H |a(t_1,r,\delta_n)|\} + |u_2| \sup_{t\in A_{t_2,\delta_n}}\{\delta_n^H |a(t_2,r,\delta_n)|\} \\
&\quad + \dots + |u_k| \sup_{t\in A_{t_k,\delta_n}}\{\delta_n^H |a(t_k,r,\delta_n)|\}\Big].
\end{aligned}$$

The last expression converges to zero by lemma 3.4.5. This means that for large enough $n$

$$I_{|Z_{r,\delta_n}|\ge \epsilon s/2}(\omega) = 0, \forall\omega\in\Omega,$$

for all the r's we sum over. This means that by eq. (3.18) we must also have

$$I_{|Z_{r,\delta_n}|\ge \epsilon s_n}(\omega) = 0, \forall\omega\in\Omega,$$

for all the relevant r when $n$ is large enough. Hence the Lindeberg condition will be satisfied. This means that by the Lindeberg Central Limit theorem we have that $\frac{R_0^n}{s_n}$ converges in distribution to a standard normal variable $Z$. Notice that $R_0^n/s_n$ will be well-defined for big $n$, because $s_n \to s > 0$, so it will be strictly positive for big $n$. Since $s_n \to s$, and s is a constant, it follows from Slutsky's theorem, see [Mit96, p. 248] that $R_0^n$ converges in distribution to $sZ$. In order to use Slutsky's theorem we use that $s_n$ converges in probability to $s$ but this follows easily by the convergence in $\mathbb{R}$. $sZ$ is by elementary statistics a normal random variable with expectation zero and variance $s^2$.

We now use the same techniques to show that $R_1^n$ converge in distribution to a normal random variable with expectation zero and variance

$$s^2 \doteq \frac{1}{C_H^2}\int_0^{t_1}\left(\sum_{j=1}^k u_j\left[(t_j-x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}}\right]\right)^2 dx.$$

If $\vec{u} = \vec{0}$ the result is immediate, so we may assume that $\vec{u} \neq \vec{0}$. Lemma 3.4.2 then tells us

$$s^2 > 0.$$

We remember that

$$R_1^n = \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \Big( u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n) \Big) w_r.$$

As before let

$$Z_{r,\delta_n} = \frac{\delta_n^H}{C_H} I_{A_{\delta_n}} \left( u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n) \right) w_r,$$

so for a given $\delta_n$, $\{Z_{r,\delta_n}\}$ is an independent collection by proposition C.2.4. Again we define

$$\begin{aligned} s_n^2 \doteq& \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} V(Z_{r,\delta_n}) \\ =& \frac{\delta_n^{2H}}{C_H^2} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \left( u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n) \right)^2 . \end{aligned}$$

By lemma 3.4.7

$$\begin{aligned} & \lim_{n\to\infty} s_n^2 \\ =& \lim_{n\to\infty} \frac{\delta_n^{2H}}{C_H^2} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \left( u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n) \right)^2 \\ =& \frac{1}{C_H^2} \int_0^{t_1} \left( \sum_{j=1}^{k} u_j \left[ (t_j - x)^{H-\frac{1}{2}} - (-x)_+^{H-\frac{1}{2}} \right] \right)^2 dx \\ & s^2 > 0. \end{aligned}$$

We have for all $r \in \{0, 1, \dots, \lfloor t_1/\delta_n \rfloor - 1\}$, with $A_{t,\delta_n}$ as in lemma 3.4.5

$$\begin{aligned} & \max_{r\in\{0,1,\dots,\lfloor t_1/\delta_n \rfloor - 1\}} |Z_{r,\delta_n}| \\ \leq & \max_{r\in\{0,1,\dots,\lfloor t_1/\delta_n \rfloor - 1\}} \frac{\delta_n^H}{C_H} |u_1 a(t_1, r, \delta_n) + u_2 a(t_2, r, \delta_n) + \cdots + u_k a(t_k, r, \delta_n)| \\ \leq & \frac{1}{C_H} \Bigg[ |u_1| \sup_{r\in A_{t_1,\delta_n}} \{\delta_n^H |a(t_1, r, \delta_n)|\} + |u_2| \sup_{r\in A_{t_2,\delta_n}} \{\delta_n^H |a(t_2, r, \delta_n)|\} \\ & \qquad + \cdots + |u_k| \sup_{r\in A_{t_k,\delta_n}} \{\delta_n^H |a(t_k, r, \delta_n)|\} \Bigg]. \end{aligned}$$

This goes to zero by lemma 3.4.5. We remember the Lindeberg condition is, for every $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{r=0}^{\lfloor t_1/\delta_n \rfloor - 1} \frac{1}{s_n^2} E\left[ I_{|Z_{r,\delta_n}| \geq \epsilon s_n}(\omega) Z_{r,\delta_n}(\omega)^2 \right] = 0.$$

Since for large enough $n$ we have

$$2\epsilon s > \epsilon s_n > \epsilon s/2 > 0,$$

we don't have any trouble with dividing by zero. Since we also have that $\max_{r\in\{0,1,\dots,\lfloor t_1/\delta_n\rfloor-1\}}|Z_{r,\delta_n}|$ will be smaller $\epsilon s/2$ for large $n$, we have that for large $n$

$$\sum_{r=0}^{\lfloor t_1/\delta_n\rfloor-1}\frac{1}{s_n^2}E\left[I_{|Z_{r,\delta_n}|\geq\epsilon s_n}(\omega)Z_{r,\delta_n}(\omega)^2\right]=0.$$

Hence the Lindeberg condition is satisfied. So by [Bil95, Theorem 27.2] we have that $R_1^n/s_n$ converges in distribution to a standard normal random variable Z. $R_1^n/s_n$ is again well-defined for big $n$ since $s_n\to s$, so $s_n>0$ for big $n$. Since $s_n\to s$ we again have by Slutsky's theorem([Mit96, p. 248]) that $R_1^n$ converges in distribution to $sZ$, where we have used that convergence of real numbers implies convergence in probability if we view the numbers as deterministic random variables. By elementary statistics $sZ$ is a normal random variable with expectation zero and variance $s^2$.

We will treat $R_2^n,R_3^n,\dots,R_k^n$ simultaneously because of their similar structure. We recall for $l\in\{2,3,\dots,k\}$

$$R_l^n\doteq\frac{\delta_n^H}{C_H}I_{A_{\delta_n}}\sum_{r=\lfloor t_{l-1}/\delta_n\rfloor}^{\lfloor t_l/\delta_n\rfloor-1}\Big(u_la(t_l,r,\delta_n)+\dots+u_ka(t_k,r,\delta_n)\Big)w_r$$

We will prove that $R_l^n$ converges to a normal random variable with expectation zero and variance

$$s^2\doteq\frac{1}{C_H^2}\int_{t_{l-1}}^{t_l}\left(\sum_{j=l}^{k}u_j\left[(t_j-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right]\right)^2dx.$$

If $u_l=u_{l+1},\dots,u_k=0$, the result is obvious, so we may assume $u_l^2+u_{l+1}^2+\dots+u_k^2\neq0$. It then follows by lemma 3.4.2 that

$$s^2>0.$$

As before we define

$$Z_{r,\delta_n}=\frac{\delta_n^H}{C_H}I_{A_{\delta_n}}\left(u_l(t_l,r,\delta_n)+\dots+u_ka(t_k,r,\delta_n)\right)w_r.$$

So for each $n$, $\{Z_{r,\delta_n}\}$ is a collection of independent random variables (proposition C.2.4) with expectation zero. We also define

$$\begin{aligned}s_n^2\doteq&\sum_{r=\lfloor t_{l-1}/\delta_n\rfloor}^{\lfloor t_l/\delta_n\rfloor-1}V(Z_{r,\delta_n})\\=&\frac{\delta_n^2}{C_H^2}\sum_{r=\lfloor t_{l-1}/\delta_n\rfloor}^{\lfloor t_l/\delta_n\rfloor-1}\left(u_l(t_l,r,\delta_n)+\dots+u_ka(t_k,r,\delta_n)\right)^2.\end{aligned}$$

By lemma 3.4.7 we have

$$s_n^2 \to s^2.$$

Again we examine the Lindeberg condition in appropriate notation. We must show that for every $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{r=\lfloor t_{l-1}/\delta_n\rfloor}^{\lfloor t_l/\delta_n\rfloor -1} \frac{1}{s_n^2} E\left[I_{|Z_{r,\delta_n}|\geq \epsilon s_n}(\omega) Z_{r,\delta_n}(\omega)^2\right] = 0.$$

As earlier we have, where $A_{t,\delta_n}$ is defined in lemma 3.4.5,

$$\begin{aligned}
&\max_{r\in\{\lfloor t_{l-1}/\delta_n\rfloor,\dots,\lfloor t_l/\delta_n\rfloor-1\}} |Z_{r,\delta_n}| \\
\leq &\max_{r\in\{\lfloor t_{l-1}/\delta_n\rfloor,\dots,\lfloor t_l/\delta_n\rfloor-1\}} \frac{\delta_n^H}{C_H}|u_l a(t_l,r,\delta_n) + \cdots + u_k a(t_k,r,\delta_n)| \\
\leq &\frac{1}{C_H}\left[|u_l| \sup_{r\in A_{t_l,\delta_n}} \{\delta_n^H |a(t_l,r,\delta_n)|\} + \cdots + |u_k| \sup_{r\in A_{t_k,\delta_n}} \{\delta_n^H |a(t_k,r,\delta_n)|\}\right].
\end{aligned}$$

This goes to zero by lemma 3.4.5, so for large enough $n$ this will be smaller than $\epsilon s/2$. Since $s_n$ converges to $s$, for large $n$ we will have $\epsilon s_n > \epsilon s/2$. Then we have

$$\sum_{r=\lfloor t_{l-1}/\delta_n\rfloor}^{\lfloor t_l/\delta_n\rfloor -1} \frac{1}{s_n^2} E\left[I_{|Z_{r,\delta_n}|\geq \epsilon s_n}(\omega) Z_{r,\delta_n}(\omega)^2\right] = 0.$$

So by the Lindeberg central limit theorem [Bil95, Theorem 27.2] we have that $R_l^n/s_n$ converges in distribution to a standard normal variable Z. As mentioned before this is a well-defined random variable for big $n$ because $s_n > 0$ for large $n$. Since $s$ is a constant, we have that by Slutsky's theorem([Mit96, p. 248]), using the trick that $s_n$ also converges in probability to $s$, that $R_l^n$ converges in distribution to $sZ$. $sZ$ is a normal random variable with expectation zero and variance $s^2$.

Since for each $n$, $R_0^n, R_1^n, \dots, R_k^n$ consist of different $w$'s, they are independent by proposition C.2.5. Denote the random variable sequence $(R_l^n)$ converges to in distribution by $R_l$ with variance $\sigma_l^2$. We then get for each $b \in \mathbb{R}$

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{E}\left[\exp(ib(R_0^n + R_1^2 + \cdots + R_k^n))\right] \\
=&\lim_{n\to\infty} \mathrm{E}\left[\exp(ib(R_0^n)\right]\mathrm{E}\left[\exp(ib(R_1^n)\right]\cdots\mathrm{E}\left[\exp(ib(R_k^n)\right] \\
=&\exp(-1/2\sigma_0^2 b^2)\exp(-1/2\sigma_1^2 b^2)\exp(-1/2\sigma_2^2 b^2)\cdots\exp(-1/2\sigma_k^2 b^2) \\
=&\exp(-1/2b^2(\sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2)).
\end{aligned}$$

Hence $R_0^n + R_1^2 + \cdots + R_k^n$ converges in distribution to a normal random variable with variance $\sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \cdots \sigma_k^2$, and expected value zero. However we also have

$$\sigma_0^2+\sigma_1^2+\sigma_2^2+\cdots+\sigma_k^2 = \frac{1}{C_H^2}\int_{-\infty}^{0}\left(\sum_{j=1}^{k}u_j\left[(t_j-x)^{H-\frac{1}{2}}-(-x)^{H-\frac{1}{2}}\right]\right)^2 dx$$

$$+\frac{1}{C_H^2}\int_{0}^{t_1}\left(\sum_{j=1}^{k}\left[u_j\left[(t_j-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right]\right]\right)^2 dx$$

$$+\sum_{l=2}^{k}\frac{1}{C_H^2}\int_{t_{l-1}}^{t_l}\left(\sum_{j=l}^{k}u_j\left[(t_j-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right]\right)^2 dx.$$

$$=\frac{1}{C_H^2}\sum_{j_1=1}^{k}\sum_{j_1=1}^{k}\int_{-\infty}^{\min(t_{j_1},t_{j_2})} u_{j_1}u_{j_2}\left[(t_{j_1}-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right]\left[(t_{j_2}-x)^{H-\frac{1}{2}}-(-x)_+^{H-\frac{1}{2}}\right]dx.$$

By lemma 3.4.3 this equals

$$\sum_{j_1=1}^{k}\sum_{j_2=1}^{k}u_{j_1}u_{j_2}\cdot\frac{1}{2}(t_{j_1}^{2H}+t_{j_2}^{2H}-|t_{j_1}-t_{j_2}|^{2H}) = \vec{u}^T A\vec{u}.$$

Where

$$A_{i,j}=\frac{1}{2}(t_{j_1}^{2H}+t_{j_2}^{2H}-|t_{j_1}-t_{j_2}|^{2H}).$$

This completes the proof. □

We will give a corollary to theorem 3.5.2 showing weak convergence of the induced measures.

**Corollary 3.5.3.** *Let $H\in(0,1)$ and $\{\delta_n\}$ be a positive sequence of real numbers converging to zero. Let $P_n$ be as in definition 3.3.1. Let $0\le t_1<t_2<\cdots<t_k$. Then $P_n\pi_{t_1,t_2,\ldots,t_k}^{-1}$ converges weakly to a measure $\mu$ on $(\mathbb{R}^k,\mathcal{B}(\mathbb{R}^k))$, where $\mu$ is uniquely determined by the fact that if $\vec{r}\in\mathbb{R}^k$ we have*

$$\int_{\mathbb{R}^k}e^{i\vec{r}^T\vec{x}}d\mu(x)=\exp\left(-\frac{1}{2}\vec{r}^T A\vec{r}\right),$$

*where*

$$A_{j_1,j_2}=\frac{1}{2}\left(t_{j_1}^{2H}+t_{j_2}^{2H}-|t_{j_1}-t_{j_2}|^{2H}\right).$$

*Proof.* Let $Y$ be the Gaussian random vector from theorem 3.5.2, assume it is a random vector on the probability space $(\Omega',\mathcal{A}',P')$. By proposition 3.3.4 and theorem 3.5.2 we have that $P_n\pi_{t_1,t_2,\ldots,t_k}^{-1}$ converges weakly to a measure $\mu$ such that $(\mathbb{R}^k,\mathcal{B}(\mathbb{R}^k),\mu)$ is a measure space, and

$$\mu(B)=P'(\vec{Y}^{-1}(B)), B\in\mathcal{B}(\mathbb{R}^k).$$

We have that for $\vec{r}\in\mathbb{R}^k$

$$E'\left[\exp(i\vec{r}^T\vec{Y})\right]=\exp\left(-\frac{1}{2}\vec{r}^T A\vec{r}\right),$$

where $A$ is given in the hypothesis of this corollary. In the proof of proposition 3.3.4 we proved that for bounded continuous real functions we have (see eq. (3.9))

$$E'[f((Y_1, Y_2, \ldots, Y_k))] = E_\mu[f(x_1, x_2, \ldots, x_k)].$$

We then have

$$\begin{aligned}
\int_{\mathbb{R}^k} e^{i\vec{r}^T x} d\mu(x) &= E_\mu\left[e^{i\vec{r}^T \vec{x}}\right] \\
&= E_\mu\left[\cos(\vec{r}^T\vec{x}) + i\sin(\vec{r}^T\vec{x})\right] \\
&= E_\mu\left[\cos(\vec{r}^T\vec{x})\right] + iE_\mu\left[\sin(\vec{r}^T\vec{x})\right] \\
&= E'\left[\cos(\vec{r}^T\vec{Y})\right] + iE'\left[\sin(\vec{r}^T\vec{Y})\right] \\
&= E'\left[\cos(\vec{r}^T\vec{Y}) + i\sin(\vec{r}^T\vec{Y})\right] \\
&= E'\left[\exp(i\vec{r}^T\vec{Y})\right] \\
&= \exp\left(-\frac{1}{2}\vec{r}^T A\vec{r}\right).
\end{aligned}$$

The fact that the measure $\mu$ is uniquely determined by the function

$$\vec{r} \to \int_{\mathbb{R}^k} e^{i\vec{r}^T\vec{x}} d\mu(x),$$

is explained in [App09, p. 16]. It is the same concept as that random variables are uniquely determined by their characteristic function. □

**Remark.** *In corollary 3.5.3 we describe the measure $\mu$ in terms of its characteristic function. However we also have the connection to the multivariate normal distribution, so one would think we would be able to give a better description like*

$$\mu(B) = \int_B (2\pi)^{-\frac{1}{2}k} \det(A)^{-0.5} \exp\left(-\frac{1}{2}\vec{x}^T A^{-1}\vec{x}\right) d\lambda^k(\vec{x}), B \in \mathcal{B}(\mathbb{R}^k),$$

*where $\lambda^k$ denotes the Lebesgue-measure on $(R^k, \mathbb{B}(\mathbb{R}^k))$. The reason we do not give this description is that the matrix $A$ may not be invertible, (it is not when $t_1 = 0$). In this case a probability density function does not exist. Hence we give the description in terms of the characteristic function.*

## 3.6 Tightness of $\{P_n\}$

In the last section we tackled the finite-dimensional part of theorem 2.4.2, and in this section we will tackle the tightness part of the same theorem. We start with a simple result

**Lemma 3.6.1.** *Let $\{y_r\}, r \in \mathbb{N}$ be a sequence of independent random variable defined on a probability space $(\Omega, \mathcal{A}, P)$. Assume that each $y_r$ takes the values*

$\pm 1$ *with equal probability. Assume that* $\{a_r\}, \{b_r\}$ *are real sequences such that*

$$\sum_{r=1}^{\infty} a_r^2 < \infty$$
$$\sum_{r=1}^{\infty} b_r^2 < \infty.$$

*Also assume that there is a set* $A \in \mathcal{A}$*, with* $P(A) = 1$*, such that the random variables* $Z_1, Z_2$ *given by*

$$Z_1 = \sum_{r=1}^{\infty} a_r y_r(\omega) I_A(\omega)$$
$$Z_2 = \sum_{r=1}^{\infty} b_r y_r(\omega) I_A(\omega),$$

*are well-defined (i.e. that when* $\omega \in A$ *the sums converge). We then have*

$$E[Z_1^2] = \sum_{r=1}^{\infty} a_r^2$$
$$E[Z_1^2] = \sum_{r=1}^{\infty} b_r^2$$
$$E[(Z_1 - Z_2)^2] = \sum_{r=1}^{\infty} (a_r - b_r)^2 < \infty.$$

*Proof.* We first show that $E[Z_1^2] = \sum\limits_{r=1}^{\infty} a_r^2$. By Fatou's lemma we have

$$\begin{aligned}
E[Z_1^2] =& E\left[\lim_{M\to\infty}\left(\sum_{t=1}^{M} a_r y_r I_A\right)^2\right] \\
\leq& \liminf_{M\to\infty} E\left[\left(\sum_{t=1}^{M} a_r y_r I_A\right)^2\right] \\
=& \liminf_{M\to\infty} E\left[\left(\sum_{t=1}^{M} a_r y_r\right)^2 I_A\right] \\
=& \liminf_{M\to\infty} E\left[\left(\sum_{t=1}^{M} a_r y_r\right)^2 (I_A + I_{A^c})\right] \\
=& \liminf_{M\to\infty} E\left[\left(\sum_{t=1}^{M} a_r y_r\right)^2\right] \\
=& \liminf_{M\to\infty} \sum_{r=1}^{M} a_r^2 \\
=& \sum_{r=1}^{\infty} a_r^2,
\end{aligned}$$

where we here used that the integral of an integrable function over $A^c$ must be zero. This tells us that $Z_1$ is a square-integrable random variable, and $E[Z_1^2] \leq \sum_{r=1}^{\infty} a_r^2$. Since we have that the every element in the collection $\{\sum_{r=1}^{M} a_r y_r I_a\}_M$ is bounded in $L^2(\Omega)$ by $\sum_{r=1}^{\infty} a_r^2$, we have that the collection is uniformly integrable by [Wil91, 13.3 (a), p 127]. Since sure convergence implies a.s. convergence which in turn implies convergence in probability, we have by [Wil91, 13.7, p. 131]

$$\sum_{r=1}^{M} a_r y_r I_A \xrightarrow[L^1(\Omega)]{} \sum_{r=1}^{\infty} a_r y_r I_A. \tag{3.19}$$

Hence we get

$$\begin{aligned}
|E[Z_1]| &= \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A\right]\right| \\
&= \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A + \sum_{r=1}^{M} a_r y_r I_A\right]\right| \\
&= \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A\right] + E\left[\sum_{r=1}^{M} a_r y_r I_A\right]\right| \\
&\leq \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A\right]\right| + \left| E\left[\sum_{r=1}^{M} a_r y_r I_A\right]\right| \\
&\leq \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A\right]\right| + \left| E\left[\sum_{r=1}^{M} a_r y_r (I_A + I_{A^c})\right]\right| \\
&= \left| E\left[\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A\right]\right| \\
&\leq E\left[\left|\sum_{r=1}^{\infty} a_r y_r I_A - \sum_{r=1}^{M} a_r y_r I_A\right|\right],
\end{aligned}$$

where we in the last inequality used Jensen's inequality. The last expression can be made as small as possible by the $L^1(\Omega)$ convergence in eq. (3.19). Hence

$$\mathrm{E}[Z_1] = 0.$$

We also have that $\sum_{r=1}^{M} a_r y_r I_A$ and $\sum_{r=M+1}^{\infty} a_r y_r I_A$ are independent by proposition C.2.6, so we get

$$\begin{aligned}
E[Z_1^2] &= E[Z_1^2] - E[Z_1]^2 \\
&= V(Z_1) \\
&= V\left(\sum_{r=1}^{\infty} a_r y_r I_A\right) \\
&= V\left(\sum_{r=1}^{M} a_r y_r I_A + \sum_{r=M+1}^{\infty} a_r y_r I_A\right) \\
&= V\left(\sum_{r=1}^{M} a_r y_r I_A\right) + V\left(\sum_{r=M+1}^{\infty} a_r y_r I_A\right) \\
&\geq V\left(\sum_{r=1}^{M} a_r y_r I_A\right) \\
&= E\left[\left(\sum_{r=1}^{M} a_r y_r I_A\right)^2\right] - E\left[\sum_{r=1}^{M} a_r y_r I_A\right]^2 \\
&= E\left[\left(\sum_{r=1}^{M} a_r y_r\right)^2\right] - E\left[\sum_{r=1}^{M} a_r y_r\right]^2 \\
&= \sum_{r=1}^{M} a_r^2,
\end{aligned}$$

where we used that the integral of an integrable function over $A^c$ must be zero since $P(A^c) = 0$. Hence we have shown that $E[Z_1^2] \geq \sum_{r=1}^{M} a_r^2$ for every $M$ so we must have that $E[Z_1^2] \geq \sum_{r=1}^{\infty} a_r^2$. Earlier we showed that $E[Z_1^2] \leq \sum_{r=1}^{\infty} a_r^2$, hence we have that $E[Z_1^2] = \sum_{r=1}^{\infty} a_r^2$.

The exact same argument with $b_r$ instead of $a_r$ tells us that $E[Z_2^2] = \sum_{r=1}^{\infty} b_r^2$.

We have that

$$Z_1 - Z_2 = \sum_{r=1}^{\infty} (a_r - b_r) y_r I_A,$$

by convergence for each $\omega$. Since $(a_r - b_r)^2 \leq 2a_r^2 + 2b_r^2$, we have that

$$\sum_{r=1}^{\infty} (a_r - b_r)^2 \leq 2\sum_{r=1}^{\infty} a_r^2 + 2\sum_{r=1}^{\infty} b_r^2 < \infty.$$

By what we proved above we then have that $E[(Z_1 - Z_2)^2] = \sum_{r=1}^{\infty} (a_r - b_r)^2$. □

Now we turn to the most important result in this section. In the proof we will use the representation of the Mandelbrot and Van Ness sum used in eq. (3.6)

$$X_t^{(\delta)} = \frac{\delta^H}{C_H} \sum_{r=-\infty}^{\lfloor t/\delta \rfloor - 1} a(t, r, \delta) w_r I_A + \frac{\delta^H}{C_H} (t/\delta - \lfloor t/\delta \rfloor) w_{\lfloor t/\delta \rfloor} I_{A_\delta}, \tag{3.20}$$

where

$$\begin{aligned} a(t,r,\delta) &= (1-t/\delta+\lfloor t/\delta \rfloor)\underline{a}(t,r,\delta)+(t/\delta-\lfloor t/\delta \rfloor)\overline{a}(t,r,\delta) \\ &= (1-t/\delta+\lfloor t/\delta \rfloor)\left((\lfloor t/\delta \rfloor - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right) \\ &\quad + (t/\delta-\lfloor t/\delta \rfloor)\left((\lfloor t/\delta \rfloor + 1 - r)^{H-\frac{1}{2}} - (-r)_+^{H-\frac{1}{2}}\right). \end{aligned}$$

**Theorem 3.6.2.** *Let $H \in (0,1), \delta_n > 0$. Let $X^{(\delta_n)}$ be as in definition 3.2.6. There exists a constant $K_H$ such that for all $n$ and all $t, s \in [0,\infty)$ we have*

$$E[(X_t^{(\delta_n)} - X_s^{(\delta_n)})^2] \leq K_H |t-s|^{2H},$$

*where $K_H$ only depends on $H$.*

*Proof.* Throughout this proof we will use an equality proved on page 322 in [ST94]. Namely that for $b \geq a$ we have

$$\frac{1}{C_H^2}\int_{-\infty}^{b} \left((b-x)^{H-0.5} - (a-x)^{H-0.5} 1_{(-\infty,a)}(x)\right)^2 dx = |b-a|^{2H}. \quad (3.21)$$

A direct consequence of this is

$$\frac{1}{C_H^2}\int_{-\infty}^{a} \left((b-x)^{H-0.5} - (a-x)^{H-0.5}\right)^2 dx \leq |b-a|^{2H}, \quad (3.22)$$

and also

$$\frac{1}{C_H^2}\int_{a}^{b} (b-x)^{2H-1} \leq |b-a|^{2H}. \quad (3.23)$$

Assume for notational convenience and without loss of generality that $t \geq s$. Throughout the proof $\delta_n$ is constant. We will have to check three different cases depending on how close $s$ and $t$ are, and we begin with the closest case

**1,$\lfloor s/\delta_n \rfloor = \lfloor t/\delta_n \rfloor$.** By lemma 3.6.1 and corollary 3.2.9 we have, by using the representation of $X^{(\delta_n)}$ in eq. (3.20)

$$\begin{aligned}
&E[(X_t^{(\delta_n)} - X_s^{(\delta_n)})^2]\\
=&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-1}\Bigg((1-t/\delta_n+\lfloor t/\delta_n\rfloor)\Big((\lfloor t/\delta_n\rfloor-1)^{H-\frac{1}{2}}-(-r)_+^{H-\frac{1}{2}}\Big)\\
&\qquad+(t/\delta_n-\lfloor t/\delta_n\rfloor)\Big((\lfloor t/\delta_n\rfloor+1-r)^{H-\frac{1}{2}}-(-r)_+^{H-\frac{1}{2}}\Big)\\
&\qquad-(1-s/\delta_n+\lfloor t/\delta_n\rfloor)\Big((\lfloor t/\delta_n\rfloor-1)^{H-\frac{1}{2}}-(-r)_+^{H-\frac{1}{2}}\Big)\\
&\qquad-(s/\delta_n-\lfloor t/\delta_n\rfloor)\Big((\lfloor t/\delta_n\rfloor+1-r)^{H-\frac{1}{2}}-(-r)_+^{H-\frac{1}{2}}\Big)\Bigg)^2\\
&+\frac{\delta_n^{2H}}{C_H^2}(t/\delta_n-s/\delta_n)^2\\
=&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-1}(s/\delta_n-t/\delta_n)^2\left((\lfloor t/\delta_n\rfloor-r)^{H-0.5}-(\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\right)^2\\
&\quad+\frac{\delta_n^{2H}}{C_H^2}(s/\delta_n-t/\delta_n)^2.
\end{aligned}$$

$$\begin{aligned}
=&\frac{1}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-1}(s/\delta_n-t/\delta_n)^2\left((\lfloor t/\delta_n\rfloor\delta_n-r\delta_n)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n+\delta_n-r\delta_n)^{H-0.5}\right)^2\delta_n\\
&\quad+\frac{\delta_n^{2H}}{C_H^2}(s/\delta_n-t/\delta_n)^2.\\
=&\frac{1}{C_H^2}\sum_{\tau=-\infty}^{\lfloor t/\delta_n\rfloor\delta_n-\delta_n}{}_{\delta_n}(s/\delta_n-t/\delta_n)^2\left((\lfloor t/\delta_n\rfloor\delta_n-\tau)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-0.5}\right)^2\delta_n\\
&\quad+\frac{\delta_n^{2H}}{C_H^2}(s/\delta_n-t/\delta_n)^2.
\end{aligned}$$

By differentiation we get that the expression inside the sum increases with $\tau$ so this expression is bounded by

$$\begin{aligned}
&\frac{1}{C_H^2}(s/\delta_n-t/\delta_n)^2\int_{-\infty}^{\lfloor t/\delta_n\rfloor\delta_n}\left((\lfloor t/\delta_n\rfloor\delta_n-x)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n+\delta_n-x)^{H-0.5}\right)^2dx\\
&\qquad+\frac{1}{C_H^2}\frac{|t-s|^{2-2H}}{\delta_n^{2-2H}}|t-s|^{2H}.
\end{aligned}$$

With the help of eq. (3.22) this expression is bounded by

$$\begin{aligned}
&(s/\delta_n-t/\delta_n)^2\delta_n^{2H}+\frac{1}{C_H^2}|t-s|^{2H}\\
=&|t-s|^{2H}(t/\delta_n-s/\delta_n)^{2-2H}+\frac{1}{C_H^2}|t-s|^{2H}\\
\le&K_1|t-s|^{2H},
\end{aligned}$$

where $K_1 = 1 + 1/C_H^2$ because

$$t/\delta_n - s/\delta_n \leq \lfloor t/\delta_n \rfloor + 1 - \lfloor s/\delta_n \rfloor = 1.$$

**2, $\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor = 1$.** By lemma 3.6.1 and corollary 3.2.9 we have, by using the representation of $X^{(\delta_n)}$ in eq. (3.20) that

$$\begin{aligned}
&E[(X_t^{(\delta_n)} - X_s^{(\delta_n)})^2] \\
&= \frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\infty}^{\lfloor t/\delta_n \rfloor - 2} \Big( [1 + \lfloor t/\delta_n \rfloor - t/\delta_n](\lfloor t/\delta_n \rfloor - r)^{H-0.5} \\
&\qquad + [t/\delta_n - \lfloor t/\delta_n \rfloor](\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \\
&\qquad - [\lfloor t/\delta_n \rfloor - s/\delta_n](\lfloor t/\delta_n \rfloor - 1 - r)^{H-0.5} \\
&\qquad - [s/\delta_n + 1 - \lfloor t/\delta_n \rfloor](\lfloor t/\delta_n \rfloor - r)^{H-0.5} \Big)^2 \\
&+ \frac{\delta_n^{2H}}{C_H^2} \Big( s/\delta_n - \lfloor t/\delta_n \rfloor + 1 - [1 + \lfloor t/\delta_n \rfloor - t/\delta_n] - [t/\delta_n - \lfloor t/\delta_n \rfloor] 2^{H-0.5} \Big)^2 \\
&+ \frac{\delta_n^{2H}}{C_H^2} \Big( t/\delta_n - \lfloor t/\delta_n \rfloor \Big)^2
\end{aligned}$$

We will look at the three parts separately. We get for the third part

$$\begin{aligned}
\frac{\delta_n^{2H}}{C_H^2} \Big( t/\delta_n - \lfloor t/\delta_n \rfloor \Big)^2 &= \frac{\delta_n^{2H}}{C_H^2} \Big( t/\delta_n - \lfloor t/\delta_n \rfloor \Big)^{2H} \Big( t/\delta_n - \lfloor t/\delta_n \rfloor \Big)^{2-2H} \\
&\leq \frac{\delta_n^{2H}}{C_H^2} \Big( t/\delta_n - \lfloor t/\delta_n \rfloor \Big)^{2H} \cdot 1 \\
&\leq \frac{\delta_n^{2H}}{C_H^2} (t/\delta_n - s/\delta_n)^{2H} \\
&= \frac{1}{C_H^2} (t-s)^{2H} \\
&= K_2 (t-s)^{2H}, K_2 = \frac{1}{C_H^2}.
\end{aligned}$$

Where we have used

$$s/\delta_n \leq \lfloor s/\delta_n \rfloor + 1 = \lfloor t/\delta_n \rfloor.$$

Let $k$ denote $2^{H-0.5}$. For the second part we then get

$$\begin{aligned}
&\frac{\delta_n^{2H}}{C_H^2} \Big( s/\delta_n - \lfloor t/\delta_n \rfloor + 1 - [1 + \lfloor t/\delta_n \rfloor - t/\delta_n] - [t/\delta_n - \lfloor t/\delta_n \rfloor] 2^{H-0.5} \Big)^2 \\
&= \frac{\delta_n^{2H}}{C_H^2} \Big( s/\delta_n - \lfloor t/\delta_n \rfloor - \lfloor t/\delta_n \rfloor + t/\delta_n - [t/\delta_n - \lfloor t/\delta_n \rfloor] k \Big)^2
\end{aligned}$$

By using that $(a+b+c)^2 \leq 9(a^2+b^2+c^2)$, we get that this is dominated by

$$\begin{aligned}
&\frac{9\delta_n^{2H}}{C_H^2}\Big((\lfloor t/\delta_n\rfloor - s/\delta_n)^2 + (t/\delta_n - \lfloor t/\delta_n\rfloor)^2 + k^2(t/\delta_n - \lfloor t/\delta_n\rfloor)^2\Big) \\
&= \frac{9\delta_n^{2H}}{C_H^2}\Big((\lfloor t/\delta_n\rfloor - s/\delta_n)^{2H}(\lfloor t/\delta_n\rfloor - s/\delta_n)^{2-2H} \\
&\quad + (t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2-2H} \\
&\quad + k^2(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2-2H}\Big).
\end{aligned}$$

We have that $\lfloor t/\delta_n\rfloor - s/\delta_n \leq 1$, and $t/\delta_n - \lfloor t/\delta_n\rfloor \leq 1$. We also have

$$\begin{aligned}
\lfloor t/\delta_n\rfloor - s/\delta_n &\geq \lfloor t/\delta_n\rfloor - \lfloor s/\delta_n\rfloor - 1 \\
&= 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{9\delta_n^{2H}}{C_H^2}\Big((\lfloor t/\delta_n\rfloor - s/\delta_n)^{2H}(\lfloor t/\delta_n\rfloor - s/\delta_n)^{2-2H} \\
&\quad + (t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2-2H} \\
&\quad + k^2(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2-2H}\Big) \\
&\leq \frac{9\delta_n^{2H}}{C_H^2}\Big((\lfloor t/\delta_n\rfloor - s/\delta_n)^{2H} \\
&\quad + (t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H} \\
&\quad + k^2(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}\Big)
\end{aligned}$$

Since $t/\delta_n - s/\delta_n \geq \lfloor t/\delta_n\rfloor - s/\delta_n$, and

$$\begin{aligned}
t/\delta_n - \lfloor t/\delta_n\rfloor &= t/\delta_n - \lfloor s/\delta_n\rfloor - 1 \\
&\leq t/\delta_n - s/\delta_n,
\end{aligned} \tag{3.24}$$

we get

$$\begin{aligned}
&\frac{9\delta_n^{2H}}{C_H^2}\Big((\lfloor t/\delta_n\rfloor - s/\delta_n)^{2H} \\
&\quad + (t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H} \\
&\quad + k^2(t/\delta_n - \lfloor t/\delta_n\rfloor)^{2H}\Big) \\
&\leq \frac{9}{C_H^2}(1+1+k^2)(t-s)^{2H} \\
&= K_3(t-s)^{2H}, K_3 \doteq \frac{9}{C_H^2}(2+k^2).
\end{aligned}$$

We now look at the first part

$$
\begin{aligned}
&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-2}\Bigg([1+\lfloor t/\delta_n\rfloor-t/\delta_n](\lfloor t/\delta_n\rfloor-r)^{H-0.5}\\
&\quad+[t/\delta_n-\lfloor t/\delta_n\rfloor](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\\
&\quad-[\lfloor t/\delta_n\rfloor-s/\delta_n](\lfloor t/\delta_n\rfloor-1-r)^{H-0.5}\\
&\quad-[s/\delta_n+1-\lfloor t/\delta_n\rfloor](\lfloor t/\delta_n\rfloor-r)^{H-0.5}\Bigg)^2\\
&=\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-2}\Bigg([2\lfloor t/\delta_n\rfloor-t/\delta_n-s/\delta_n](\lfloor t/\delta_n\rfloor-r)^{H-0.5}\\
&\quad+[t/\delta_n-\lfloor t/\delta_n\rfloor](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\\
&\quad-[\lfloor t/\delta_n\rfloor-s/\delta_n](\lfloor t/\delta_n\rfloor-1-r)^{H-0.5}\\
&\quad\pm[\lfloor t/\delta_n\rfloor-s/\delta_n](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\Bigg)^2\\
&=\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-2}\Bigg([2\lfloor t/\delta_n\rfloor-t/\delta_n-s/\delta_n](\lfloor t/\delta_n\rfloor-r)^{H-0.5}\\
&\quad-[2\lfloor t/\delta_n\rfloor-t/\delta_n-s/\delta_n](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\\
&\quad-[\lfloor t/\delta_n\rfloor-s/\delta_n](\lfloor t/\delta_n\rfloor-1-r)^{H-0.5}\\
&\quad+[\lfloor t/\delta_n\rfloor-s/\delta_n](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}\Bigg)^2
\end{aligned}
$$

By using that $(a+b)^2 \leq 2a^2 + 2b^2$ we get that this is dominated by

$$\frac{2\delta_n^{2H}}{C_H^2} \sum_{r=-\infty}^{\lfloor t/\delta_n \rfloor - 2} \Bigg[ [2\lfloor t/\delta_n \rfloor - t/\delta_n - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor - r)^{H-0.5} - (\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Bigg)^2 + [\lfloor t/\delta_n \rfloor - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} - (\lfloor t/\delta_n \rfloor - 1 - r)^{H-0.5} \Bigg)^2 \Bigg]$$

$$= \frac{2}{C_H^2} \sum_{r=-\infty}^{\lfloor t/\delta_n \rfloor - 2} \Bigg[ [2\lfloor t/\delta_n \rfloor - t/\delta_n - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor \delta_n - r\delta_n)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - r\delta_n)^{H-0.5} \Bigg)^2 \delta_n + [\lfloor t/\delta_n \rfloor - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - r\delta_n)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n - \delta_n - r\delta_n)^{H-0.5} \Bigg)^2 \delta_n \Bigg]$$

By switching notation this is equal to

$$\frac{2}{C_H^2} \sum_{\tau=-\infty}^{\lfloor t/\delta_n \rfloor \delta_n - 2\delta_n}{}_{\delta_n} \Bigg[ [2\lfloor t/\delta_n \rfloor - t/\delta_n - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor \delta_n - \tau)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-0.5} \Bigg)^2 \delta_n + [\lfloor t/\delta_n \rfloor - s/\delta_n]^2 \Bigg( (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n - \delta_n - \tau)^{H-0.5} \Bigg)^2 \delta_n \Bigg]$$

By differentiation we get that the expression inside the sum increases with $\tau$ so we get that this is dominated by

$$\frac{2}{C_H^2} [2\lfloor t/\delta_n \rfloor - t/\delta_n - s/\delta_n]^2 \int_{-\infty}^{\lfloor t/\delta_n \rfloor \delta_n - \delta_n} \Big( \lfloor t/\delta_n \rfloor \delta_n + \delta_n - x)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n - x)^{H-0.5} \Big)^2 dx + \frac{2}{C_H^2} [\lfloor t/\delta_n \rfloor - s/\delta_n]^2 \int_{-\infty}^{\lfloor t/\delta_n \rfloor \delta_n - \delta_n} \Big( (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - x)^{H-0.5} - (\lfloor t/\delta_n \rfloor \delta_n - \delta_n - x)^{H-0.5} \Big)^2 dx.$$

From eq. (3.22) we get that this is bounded by (again by using $((a+b)^2 \leq 2a^2 + 2b^2)$)

$$\begin{aligned}
&2[2\lfloor t/\delta_n \rfloor - t/\delta_n - s/\delta_n]^2 \delta_n^{2H} + 2[\lfloor t/\delta_n \rfloor - s/\delta_n]^2 (2\delta_n)^{2H} \\
&\leq 4(t/\delta_n - \lfloor t/\delta_n \rfloor)^2 \delta_n^{2H} + 4(\lfloor t/\delta_n \rfloor - s/\delta_n)^2 \delta_n^{2H} + 8(t/\delta_n - s/\delta_n)^2 \delta_n^{2H},
\end{aligned}$$

where we have used that $\lfloor t/\delta_n \rfloor - s/\delta_n \geq 0$. We remember from eq. (3.24) that $t/\delta_n - \lfloor t/\delta_n \rfloor \leq t/\delta_n - s/\delta_n$ so we have,

$$\begin{aligned}
&4(t/\delta_n - \lfloor t/\delta_n \rfloor)^2 \delta_n^{2H} + 4(\lfloor t/\delta_n \rfloor - s/\delta_n)^2 \delta_n^{2H} + 8(t/\delta_n - s/\delta_n)^2 \delta_n^{2H} \\
&\leq 16(t/\delta_n - s/\delta_n)^2 \delta_n^{2H},
\end{aligned}$$

where we also used that

$$\begin{aligned}
0 &= 1 - 1 \\
&= \lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor - 1 \\
&\leq \lfloor t/\delta_n \rfloor - s/\delta_n \\
&\leq t/\delta_n - s/\delta_n.
\end{aligned}$$

Since we have

$$\begin{aligned}
t - s &= \delta_n (t/\delta_n - s/\delta_n) \\
&\leq \delta_n (\lfloor t/\delta_n \rfloor + 1 - \lfloor s/\delta_n \rfloor) \\
&\leq 2\delta_n,
\end{aligned}$$

we get that

$$\begin{aligned}
16(t/\delta_n - s/\delta_n)^2 \delta_n^{2H} &= 16(t-s)^{2H} (t-s)^{2-2H} \delta_n^{2H-2} \\
&\leq 16(t-s)^{2H} (2\delta_n)^{2-2H} \delta_n^{2H-2} \\
&= K_4 (t-s)^{2H}, K_4 = 16 \cdot 2^{2-2H}.
\end{aligned}$$

**3, $\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor = N \geq 2$.** In this part it is important that our constants are independent of $N$. Again with the aid of lemma 3.6.1 and corollary 3.2.9, and the representation of $X^{(\delta_n)}$ in eq. (3.20), we have:

$$
\begin{aligned}
&E[(X_t^{\delta_n} - X_s^{\delta_n})^2] \\
&= \frac{\delta_n^{2H}}{C_H^2} \sum_{r=-\infty}^{\lfloor t/\delta_n \rfloor - N - 1} \Big( [1 + \lfloor t/\delta_n \rfloor - t/\delta_n](\lfloor t/\delta_n \rfloor - r)^{H-0.5} \\
&\qquad + [t/\delta_n - \lfloor t/\delta_n \rfloor](\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \\
&\qquad - [1 + \lfloor t/\delta_n \rfloor - N - s/\delta_n](\lfloor t/\delta_n \rfloor - N - r)^{H-0.5} \\
&\qquad - [s/\delta_n - \lfloor t/\delta_n \rfloor + N](\lfloor t/\delta_n \rfloor - N + 1 - r)^{H-0.5} \Big)^2 \\
&\quad + \frac{\delta_n^{2H}}{C_H^2} \Big( [1 + \lfloor t/\delta_n \rfloor - t/\delta_n] N^{H-0.5} + [t/\delta_n - \lfloor t/\delta_n \rfloor](N+1)^{H-0.5} \\
&\qquad - (s/\delta_n - \lfloor t/\delta_n \rfloor + N) \Big)^2 \\
&\quad + \frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N + 1}^{\lfloor t/\delta_n \rfloor - 1} \Big( [1 + \lfloor t/\delta_n \rfloor - t/\delta_n](\lfloor t/\delta_n \rfloor - r)^{H-0.5} \\
&\qquad + [t/\delta_n - \lfloor t/\delta_n \rfloor](\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Big)^2 \\
&\quad + \frac{\delta_n^{2H}}{C_H^2} (t/\delta_n - \lfloor t/\delta_n \rfloor)^2.
\end{aligned}
$$

We look at the different parts separately. For the fourth part we have

$$
\begin{aligned}
\frac{\delta_n^{2H}}{C_H^2}(t/\delta_n - \lfloor t/\delta_n \rfloor)^2 &\leq \frac{\delta_n^{2H}}{C_H^2}(1)^2 \\
&\leq K_5 (t-s)^{2H}, K_5 = \frac{1}{C_H^2},
\end{aligned}
$$

where we have used

$$
\begin{aligned}
t - s &= \delta_n (t/\delta_n - s/\delta_n) \\
&\geq \delta_n (\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor - 1) \\
&\geq \delta_n.
\end{aligned}
$$

For the third part we get, using $(a+b)^2 \leq 2a^2 + 2b^2$

$$
\begin{aligned}
&\frac{\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N + 1}^{\lfloor t/\delta_n \rfloor - 1} \Big( [1 + \lfloor t/\delta_n \rfloor - t/\delta_n](\lfloor t/\delta_n \rfloor - r)^{H-0.5} \\
&\qquad + [t/\delta_n - \lfloor t/\delta_n \rfloor](\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Big)^2 \\
&\leq \frac{2\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N + 1}^{\lfloor t/\delta_n \rfloor - 1} \left[ \Big( (\lfloor t/\delta_n \rfloor - r)^{H-0.5} \Big)^2 + \Big( (\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Big)^2 \right]
\end{aligned}
$$

If $H \geq 1/2$ we have

$$\begin{aligned}
&\frac{2\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N+1}^{\lfloor t/\delta_n \rfloor -1} \left[ \Big( (\lfloor t/\delta_n \rfloor - r)^{H-0.5} \Big)^2 + \Big( (\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Big)^2 \right] \\
&\leq \frac{2\delta_n^{2H}}{C_H^2} (N-1) \Big( (N-1)^{2H-1} + (N)^{2H-1} \Big) \\
&\leq \frac{4\delta_n^{2H}}{C_H^2} N^{2H} \\
&= \frac{4\delta_n^{2H}}{C_H^2} (\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor)^{2H1} \\
&= \frac{4}{C_H^2} (\lfloor t/\delta_n \rfloor \delta_n - \lfloor s/\delta_n \rfloor \delta_n)^{2H} \\
&\leq \frac{4}{C_H^2} (t - s + \delta_n)^{2H} \\
&\leq \frac{4 \cdot 2^{2H}}{C_H^2} (t-s)^{2H} \\
&= K_6 (t-s)^{2H}, K_6 = \frac{4 \cdot 2^{2H}}{C_H^2}.
\end{aligned}$$

If $H < 1/2$ we get by differentiating the expression below inside the sum by $\tau$, a non-negative derivative, therefore we can bound it by the integral

$$\begin{aligned}
&\frac{2\delta_n^{2H}}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N+1}^{\lfloor t/\delta_n \rfloor -1} \left[ \Big( (\lfloor t/\delta_n \rfloor - r)^{H-0.5} \Big)^2 + \Big( (\lfloor t/\delta_n \rfloor + 1 - r)^{H-0.5} \Big)^2 \right] \\
=&\frac{2}{C_H^2} \sum_{r=\lfloor t/\delta_n \rfloor - N+1}^{\lfloor t/\delta_n \rfloor -1} \left[ \Big( (\lfloor t/\delta_n \rfloor \delta_n - r\delta_n)^{H-0.5} \Big)^2 + \Big( (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - r\delta_n)^{H-0.5} \Big)^2 \right] \delta_n \\
=&\frac{2}{C_H^2} \sum_{\tau=\lfloor t/\delta_n \rfloor \delta_n - N\delta_n + \delta_n}^{\lfloor t/\delta_n \rfloor \delta_n - \delta_n}{}_{\delta_n} \left[ \Big( (\lfloor t/\delta_n \rfloor \delta_n - \tau)^{H-0.5} \Big)^2 + \Big( (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-0.5} \Big)^2 \right] \delta_n \\
\leq&\frac{2}{C_H^2} \int_{\lfloor t/\delta_n \rfloor \delta_n - N\delta_n + \delta_n}^{\lfloor t/\delta_n \rfloor \delta_n} (\lfloor t/\delta_n \rfloor \delta_n - x)^{2H-1} + (\lfloor t/\delta_n \rfloor \delta_n + \delta_n - x)^{2H-1} dx.
\end{aligned}$$

By eq. (3.23) we get that this is dominated by

$$\begin{aligned}
&\leq 2(N\delta_n - \delta_n)^{2H} + 2(N\delta_n)^{2H} \\
&\leq 4(N\delta_n)^{2H} \\
&= 4(\lfloor t/\delta_n \rfloor \delta_n - \lfloor s/\delta_n \rfloor \delta_n)^{2H} \\
&\leq 4(t - s + \delta_n)^{2H} \\
&\leq 4 \cdot 2^{2H} (t-s)^{2H} \\
&= K_7 (t-s)^{2H}, K_7 = 4 \cdot 2^{2H}.
\end{aligned}$$

Where we in the second to last inequality used

$$\begin{aligned} t - s + \delta_n &= \delta_n(t/\delta_n - (s/\delta_n - 1)) \\ &\geq \delta_n(\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor). \end{aligned} \tag{3.25}$$

We also used

$$\begin{aligned} t - s &= \delta_n(t/\delta_n - s/\delta_n) \\ &\geq \delta_n(\lfloor t/\delta_n \rfloor - \lfloor s/\delta_n \rfloor - 1) \\ &= \delta_n(N-1) \\ &\geq \delta_n. \end{aligned} \tag{3.26}$$

We now look at the second part. If we first assume that $H \geq 1/2$ we get

$$\begin{aligned} &\frac{\delta_n^{2H}}{C_H^2}\Big([1 + \lfloor t/\delta_n \rfloor - t/\delta_n]N^{H-0.5} + [t/\delta_n - \lfloor t/\delta_n \rfloor](N+1)^{H-0.5} \\ &\qquad - (s/\delta_n - \lfloor t/\delta_n \rfloor + N)\Big)^2 \\ &\leq \frac{9\delta_n^{2H}}{C_H^2}\Big(N^{2H-1} + (N+1)^{2H-1} + (s/\delta_n - \lfloor t/\delta_n \rfloor + N)^2\Big) \\ &\leq \frac{9\delta_n^{2H}}{C_H^2}\Big(2(N+1)^{2H-1} + (s/\delta_n - \lfloor s/\delta_n \rfloor)^2\Big) \\ &\leq \frac{18}{C_H^2}(\lfloor t/\delta_n \rfloor\delta_n - \lfloor s/\delta_n \rfloor\delta_n + \delta_n)^{2H} \cdot (N+1)^{-1} + \frac{9\delta_n^{2H}}{C_H^2} \\ &\leq \frac{18}{C_H^2}(t - s + 2\delta_n)^{2H} + \frac{9}{C_H^2}(t-s)^{2H} \\ &\leq \frac{18 \cdot 3^{2H}}{C_H^2}(t-s)^{2H} + \frac{9}{C_H^2}(t-s)^{2H} \\ &= K_8(t-s)^{2H}, K_8 = \frac{1}{C_H^2}\left(18 \cdot 3^{2H} + 9\right), \end{aligned}$$

where we again used eq. (3.25) and eq. (3.26).

If $H < 1/2$ we instead get for the second part

$$\begin{aligned} &\frac{\delta_n^{2H}}{C_H^2}\Big([1 + \lfloor t/\delta_n \rfloor - t/\delta_n]N^{H-0.5} + [t/\delta_n - \lfloor t/\delta_n \rfloor](N+1)^{H-0.5} \\ &\qquad - (s/\delta_n - \lfloor t/\delta_n \rfloor + N)\Big)^2 \\ &\leq \frac{9\delta_n^{2H}}{C_H^2}\Big(N^{2H-1} + (N+1)^{2H-1} + (s/\delta_n - \lfloor t/\delta_n \rfloor + N)^2\Big) \\ &\leq \frac{18\delta_n^{2H}}{C_H^2}N^{2H-1} + \frac{9\delta_n^{2H}}{C_H^2} \\ &\leq \frac{18}{C_H^2}(\lfloor t/\delta_n \rfloor\delta_n - \lfloor s/\delta_n \rfloor\delta_n)^{2H}N^{-1} + \frac{9}{C_H^2}(t-s)^{2H} \\ &\leq \frac{18 \cdot 2^{2H}}{C_H^2}(t-s)^{2H} + \frac{9}{C_H^2}(t-s)^{2H} \\ &= K_9(t-s)^{2H}, K_9 = \frac{1}{C_H^2}\left(18 \cdot 2^{2H} + 9\right). \end{aligned}$$

We now go to the first part

$$\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Bigg([1+\lfloor t/\delta_n\rfloor-t/\delta_n](\lfloor t/\delta_n\rfloor-r)^{H-0.5} + [t/\delta_n-\lfloor t/\delta_n\rfloor](\lfloor t/\delta_n\rfloor+1-r)^{H-0.5} - [1+\lfloor t/\delta_n\rfloor-N-s/\delta_n](\lfloor t/\delta_n\rfloor-N-r)^{H-0.5} - [s/\delta_n-\lfloor t/\delta_n\rfloor+N](\lfloor t/\delta_n\rfloor-N+1-r)^{H-0.5}\Bigg)^2.$$

We first assume that $H \geq 1/2$. By lemma D.1.3 we get that the last expression is bounded by

$$\begin{aligned}
&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Big((\lfloor t/\delta_n\rfloor+1-r)^{H-0.5}-(\lfloor t/\delta_n\rfloor-N-r)^{H-0.5}\Big)^2\\
=&\frac{1}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Big((\lfloor t/\delta_n\rfloor\delta_n+\delta_n-r\delta_n)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n-N\delta_n-r\delta_n)^{H-0.5}\Big)^2\delta_n\\
=&\frac{1}{C_H^2}\sum_{\tau=-\infty}^{\lfloor t/\delta_n\rfloor\delta_n-N\delta_n-\delta_n}{}_{\delta_n}\Big((\lfloor t/\delta_n\rfloor\delta_n+\delta_n-\tau)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n-N\delta_n-\tau)^{H-0.5}\Big)^2\delta_n\\
\leq&\frac{1}{C_H^2}\int_{-\infty}^{\lfloor t/\delta_n\rfloor\delta_n-N\delta_n}\Big((\lfloor t/\delta_n\rfloor\delta_n+\delta_n-x)^{H-0.5}-(\lfloor t/\delta_n\rfloor\delta_n-N\delta_n-x)^{H-0.5}\Big)^2dx.
\end{aligned}$$

We can bound the expression by the integral because if we differentiate the expression inside the sum with respect to $\tau$, the derivative is positive. We can therefore view the sum as a lower sum for the integral. From eq. (3.22) we get that this is bounded by

$$\begin{aligned}
\leq (N\delta_n+\delta_n)^{2H} &= (\lfloor t/\delta_n\rfloor\delta_n-\lfloor s/\delta_n\rfloor\delta_n+\delta_n)^{2H}\\
&\leq (t-s+2\delta_n)^{2H}\\
&\leq 3^{2H}(t-s)^{2H}\\
&= K_{10}(t-s)^{2H}, K_{10}\doteq 3^{2H}.
\end{aligned}$$

If $H < 1/2$ we get by lemma D.1.3

$$\begin{aligned}
&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Big([1+\lfloor t/\delta_n\rfloor - t/\delta_n](\lfloor t/\delta_n\rfloor - r)^{H-0.5}\\
&\qquad + [t/\delta_n - \lfloor t/\delta_n\rfloor](\lfloor t/\delta_n\rfloor + 1 - r)^{H-0.5}\\
&\qquad - [1+\lfloor t/\delta_n\rfloor - N - s/\delta_n](\lfloor t/\delta_n\rfloor - N - r)^{H-0.5}\\
&\qquad - [s/\delta_n - \lfloor t/\delta_n\rfloor + N](\lfloor t/\delta_n\rfloor - N + 1 - r)^{H-0.5}\Big)^2\\
\le&\frac{\delta_n^{2H}}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Big((\lfloor t/\delta_n\rfloor - N - r)^{H-0.5} - (\lfloor t/\delta_n\rfloor + 1 - r)^{H-0.5}\Big)^2\\
=&\frac{1}{C_H^2}\sum_{r=-\infty}^{\lfloor t/\delta_n\rfloor-N-1}\Big((\lfloor t/\delta_n\rfloor\delta_n - N\delta_n - r\delta_n)^{H-0.5} - (\lfloor t/\delta_n\rfloor\delta_n + \delta_n - r\delta_n)^{H-0.5}\Big)^2\delta_n\\
=&\frac{1}{C_H^2}\sum_{\tau=-\infty}^{\lfloor t/\delta_n\rfloor\delta_n-N\delta_n-\delta_n}{}_{\delta_n}\Big((\lfloor t/\delta_n\rfloor\delta_n - N\delta_n - \tau)^{H-0.5} - (\lfloor t/\delta_n\rfloor\delta_n + \delta_n - \tau)^{H-0.5}\Big)^2\delta_n\\
\le&\frac{1}{C_H^2}\int_{-\infty}^{\lfloor t/\delta_n\rfloor\delta_n-N\delta_n}\Big((\lfloor t/\delta_n\rfloor\delta_n - N\delta_n - x)^{H-0.5} - (\lfloor t/\delta_n\rfloor\delta_n + \delta_n - x)^{H-0.5}\Big)^2dx,
\end{aligned}$$

The reason we have the last inequality is because we can view the sum as a lower sum for the integral. To see this, note first that the expression inside the last sum is increasing, this is seen by the derivative with respect to $\tau$. So we have that we can bound it by the integral. From eq. (3.22) we again get that this is dominated by

$$\begin{aligned}
\le (N\delta_n + \delta_n)^{2H} &= (\lfloor t/\delta_n\rfloor\delta_n - \lfloor s/\delta_n\rfloor\delta_n + \delta_n)^{2H}\\
&\le (t - s + 2\delta_n)^{2H}\\
&\le 3^{2H}(t-s)^{2H}\\
&= K_{11}(t-s)^{2H}, K_{11} = 3^{2H}.
\end{aligned}$$

If we let

$$K_H = \sum_{l=1}^{11} K_l,$$

the result follows.

□

From theorem 3.6.2 we now have to prove tightness. Notice the role $\alpha$ has in corollary 2.5.6. In that corollary we have a nondecreasing continuous function $F$ and we see the term

$$E_n[|x(t_2) - x(t_1)|^\gamma] \le |F(t_2) - F(t_1)|^\alpha,$$

where $\alpha$ must be greater than 1. We have not connected theorem 3.6.2 to this result yet, but we see that there we have a term

$$K_H|t-s|^{2H}.$$

And in fact, if we only worked with $H > 1/2$ we could have chosen $\alpha = 2H$. This idea is found in [Sot01a], where Sottinen approximates the fractional brownian motion for $H > 1/2$. The approximation technique differs from ours, and he works on the space of Càdlàg function on $[0, \infty)$ not continuous functions as we do. In the paper [Par13], Peter Parczewski extends the result by Sottinen to the case $H \in (0, 1)$. In [Par13, eq 2.5, p. 331] there is a useful moment condition, which comes from a partition argument. We would like to also have a moment condition that allows us to go to the case $H \leq 1/2$, but we will use an inequality that is called Khintchine's inequality.

**Theorem 3.6.3** (**Khintchince's inequality, see [Ton84, p. 167]**). *Let $\{\epsilon_n\}$ be independent and identically distributed Bernoulli random variables such that $P(\epsilon_n = 1) = P(\epsilon_n = -1) = 1/2$. Then for every $p > 0$, there exists positive constants $L_p$ and $U_p$ such that*

$$L_p \cdot \left( \sum_{i=m}^{n} c_i^2 \right)^{p/2} \leq E \left[ \left| \sum_{i=m}^{n} c_i \epsilon_i \right|^p \right] \leq U_p \cdot \left( \sum_{i=m}^{n} c_i^2 \right)^{p/2},$$

*for all $n \geq m$ and all constants $c_i$.*

Now we expand theorem 3.6.2 to powers higher than $2H$.

**Theorem 3.6.4.** *Let $H \in (0, 1), \delta_n > 0$. Let $X^{(\delta_n)}$ be as in definition 3.2.6. There exists $\alpha > 1, \gamma \geq 0$ and $R_H$ such that for all $n$ and all $t, s \in [0, \infty)$ we have*

$$E[(X_t^{(\delta_n)} - X_s^{(\delta_n)})^\gamma] \leq R_H |t - s|^\alpha,$$

*where $\alpha, \gamma, R_H$ only depends on $H$.*

*Proof.* Choose $p$ such that it is the smallest natural number that satisfies

$$2Hp > 1,$$

hence $p$ only depends on $H$. Let $\alpha = 2Hp > 1$, and $\gamma = 2p$. We then have that $\alpha$ and $\gamma$ only depend on $H$. Assume without loss of generality that $t \geq s$. We will check different cases depending on how close $s$ and $t$ are. However the strategy is not different for the different cases, we just get extra terms in some cases.

Assume first that $\lfloor t/\delta_n \rfloor = \lfloor s/\delta_n \rfloor$. By eq. (3.6) we have

$$X_t^{(\delta_n)} - X_s^{(\delta_n)} = \frac{\delta_n^H}{C_H} \sum_{r=-\infty}^{\lfloor s/\delta_n \rfloor - 1} \left( a(t, r, \delta_n) - a(s, r, \delta_n) \right) w_r I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} \left( t/\delta_n - s/\delta_n \right) w_{\lfloor s/\delta_n \rfloor} I_{A_{\delta_n}}.$$

By Fatou's lemma we have (remember that $\gamma$ is an even integer)

$$\begin{aligned}
&E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^\gamma\right] \\
&= E\Bigg[\lim_{M\to\infty}\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} \\
&\qquad\qquad + \frac{\delta_n^H}{C_H}\left(t/\delta_n - s/\delta_n\right) w_{\lfloor s/\delta_n\rfloor} I_{A_{\delta_n}}\Bigg)^\gamma\Bigg] \\
&\le \liminf_{M\to\infty} E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} \\
&\qquad\qquad + \frac{\delta_n^H}{C_H}\left(t/\delta_n - s/\delta_n\right) w_{\lfloor s/\delta_n\rfloor} I_{A_{\delta_n}}\Bigg)^\gamma\Bigg] \\
&= \liminf_{M\to\infty} E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r \\
&\qquad\qquad + \frac{\delta_n^H}{C_H}\left(t/\delta_n - s/\delta_n\right) w_{\lfloor s/\delta_n\rfloor}\Bigg)^\gamma\Bigg],
\end{aligned}$$

the last equality follows because the expectation over $A_{\delta_n}^c$ is zero for a well-defined positive function since $P(A_{\delta_n}^c) = 0$. If we let $U_\gamma$ be as in Khintchine's inequality (theorem 3.6.3), we get from Khintchine's inequality

$$\begin{aligned}
&E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r \\
&\qquad + \frac{\delta_n^H}{C_H}\left(t/\delta_n - s/\delta_n\right) w_{\lfloor s/\delta_n\rfloor}\Bigg)^\gamma\Bigg] \\
&\le U_\gamma\left(\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left[\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right)^2\right] + \frac{\delta_n^{2H}}{C_H^2}\left(t/\delta_n - s/\delta_n\right)^2\right)^{\gamma/2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^\gamma\right] \\
&\le \liminf_{M\to\infty} U_\gamma\left(\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left[\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right)^2\right] + \frac{\delta_n^{2H}}{C_H^2}\left(t/\delta_n - s/\delta_n\right)^2\right)^{\gamma/2} \\
&= U_\gamma\left(\sum_{r=-\infty}^{\lfloor s/\delta_n\rfloor -1}\left[\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right)^2\right] + \frac{\delta_n^{2H}}{C_H^2}\left(t/\delta_n - s/\delta_n\right)^2\right)^{\gamma/2}.
\end{aligned}$$

By corollary 3.2.9 and lemma 3.6.1 this is equal to

$$U_\gamma E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^2\right]^{\gamma/2}.$$

By theorem 3.6.2 we then have

$$
\begin{aligned}
E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^\gamma\right] &\le U_\gamma E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^2\right]^{\gamma/2} \\
&\le U_\gamma \left(K_H |t-s|^{2H}\right)^{\gamma/2} \\
&= U_\gamma K_H^{\gamma/2} |t-s|^{2p/2 \cdot 2H} \\
&= U_\gamma K_H^{\gamma/2} |t-s|^{2Hp}.
\end{aligned}
$$

Hence the result follows with $R_H = U_\gamma K_H^{\gamma/2}$.

We now look at the case $\lfloor t/\delta_n \rfloor = \lfloor s/\delta_n \rfloor + 1$, the calculations are exactly the same, but the terms are a little different. We compactify the argument as it is identical as the one above

$$
\begin{aligned}
&E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^\gamma\right] \\
&= E\Bigg[\lim_{M\to\infty}\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} \\
&\qquad + \frac{\delta_n^H}{C_H}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n) - s/\delta_n + \lfloor s/\delta_n\rfloor\right) w_{\lfloor s/\delta_n\rfloor} I_{A_{\delta_n}} \\
&\qquad + \frac{\delta_n^H}{C_H}(t/\delta_n - \lfloor t/\delta_n\rfloor) w_{\lfloor t/\delta_n\rfloor} I_{A_{\delta_n}}\Bigg)^\gamma\Bigg], \text{ by eq. (3.6)} \\
&\le \liminf_{M\to\infty} E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} \\
&\qquad + \frac{\delta_n^H}{C_H}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n) - s/\delta_n + \lfloor s/\delta_n\rfloor\right) w_{\lfloor s/\delta_n\rfloor} I_{A_{\delta_n}} \\
&\qquad + \frac{\delta_n^H}{C_H}(t/\delta_n - \lfloor t/\delta_n\rfloor) w_{\lfloor t/\delta_n\rfloor} I_{A_{\delta_n}}\Bigg)^\gamma\Bigg], \text{ by Fatou's lemma} \\
&= \liminf_{M\to\infty} E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r \\
&\qquad + \frac{\delta_n^H}{C_H}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n) - s/\delta_n + \lfloor s/\delta_n\rfloor\right) w_{\lfloor s/\delta_n\rfloor} \\
&\qquad + \frac{\delta_n^H}{C_H}(t/\delta_n - \lfloor t/\delta_n\rfloor) w_{\lfloor t/\delta_n\rfloor}\Bigg)^\gamma\Bigg], \text{ since } P(A_{\delta_n}^c) = 0 \\
&\le \liminf_{M\to\infty} U_\gamma\Bigg(\sum_{r=-M}^{\lfloor s/\delta_n\rfloor -1}\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n) - a(s,r,\delta_n)\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n) - s/\delta_n + \lfloor s/\delta_n\rfloor\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}(t/\delta_n - \lfloor t/\delta_n\rfloor)^2\Bigg)^{\gamma/2}, \text{ by Khintchine's inequality}
\end{aligned}
$$

$$= U_\gamma \Bigg( \sum_{r=-\infty}^{\lfloor s/\delta_n \rfloor - 1} \frac{\delta_n^{2H}}{C_H^2} \left(a(t,r,\delta_n) - a(s,r,\delta_n)\right)^2 + \frac{\delta_n^{2H}}{C_H^2} (a(t, \lfloor s/\delta_n \rfloor, \delta_n) - s/\delta_n + \lfloor s/\delta_n \rfloor)^2 + \frac{\delta_n^{2H}}{C_H^2} (t/\delta_n - \lfloor t/\delta_n \rfloor)^2 \Bigg)^{\gamma/2}$$

$$= U_\gamma E\left[\left(X_t^{\delta_n} - X_s^{\delta_n}\right)^2\right]^{\gamma/2}, \text{ by corollary 3.2.9 and lemma 3.6.1}$$
$$\leq U_\gamma \left(K_H |t-s|^{2H}\right)^{\gamma/2}, \text{ by theorem 3.6.2}$$
$$= U_\gamma K_H^{\gamma/2} |t-s|^{2pH}.$$

Lastly we look at the case when $\lfloor t/\delta_n \rfloor \geq \lfloor s/\delta_n \rfloor + 2$. Again the argument is the same, but with different terms, so we show a compact version.

$$E\left[\left(X_t^{(\delta_n)} - X_s^{(\delta_n)}\right)^\gamma\right]$$
$$= E\Bigg[\lim_{M\to\infty} \Bigg( \frac{\delta_n^H}{C_H} \sum_{r=-M}^{\lfloor s/\delta_n \rfloor - 1} \left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} \left(a(t, \lfloor s/\delta_n \rfloor, \delta_n) - s/\delta_n + \lfloor s/\delta_n \rfloor\right) w_{\lfloor s/\delta_n \rfloor} I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} \sum_{r=\lfloor s/\delta_n \rfloor + 1}^{\lfloor t/\delta_n \rfloor - 1} a(t,r,\delta_n) w_r I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} (t/\delta_n - \lfloor t/\delta_n \rfloor) w_{\lfloor t/\delta_n \rfloor} I_{A_{\delta_n}} \Bigg)^\gamma\Bigg], \text{ by eq. (3.6)}$$
$$\leq \liminf_{M\to\infty} E\Bigg[\Bigg( \frac{\delta_n^H}{C_H} \sum_{r=-M}^{\lfloor s/\delta_n \rfloor - 1} \left(a(t,r,\delta_n) - a(s,r,\delta_n)\right) w_r I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} \left(a(t, \lfloor s/\delta_n \rfloor, \delta_n) - s/\delta_n + \lfloor s/\delta_n \rfloor\right) w_{\lfloor s/\delta_n \rfloor} I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} \sum_{r=\lfloor s/\delta_n \rfloor + 1}^{\lfloor t/\delta_n \rfloor - 1} a(t,r,\delta_n) w_r I_{A_{\delta_n}} + \frac{\delta_n^H}{C_H} (t/\delta_n - \lfloor t/\delta_n \rfloor) w_{\lfloor t/\delta_n \rfloor} I_{A_{\delta_n}} \Bigg)^\gamma\Bigg], \text{ by Fatou's lemma}$$

$$
\begin{aligned}
&= \liminf_{M\to\infty} E\Bigg[\Bigg(\frac{\delta_n^H}{C_H}\sum_{r=-M}^{\lfloor s/\delta_n\rfloor-1}\left(a(t,r,\delta_n)-a(s,r,\delta_n)\right)w_r \\
&\qquad + \frac{\delta_n^H}{C_H}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n)-s/\delta_n+\lfloor s/\delta_n\rfloor\right)w_{\lfloor s/\delta_n\rfloor} \\
&\qquad + \frac{\delta_n^H}{C_H}\sum_{r=\lfloor s/\delta_n\rfloor+1}^{\lfloor t/\delta_n\rfloor-1} a(t,r,\delta_n)w_r \\
&\qquad + \frac{\delta_n^H}{C_H}(t/\delta_n-\lfloor t/\delta_n\rfloor)w_{\lfloor t/\delta_n\rfloor}\Bigg)^{\gamma}\Bigg], \text{ since } P(A_{\delta_n}^c)=0 \\
&\le \liminf_{M\to\infty} U_\gamma\Bigg(\sum_{r=-M}^{\lfloor s/\delta_n\rfloor-1}\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n)-a(s,r,\delta_n)\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n)-s/\delta_n+\lfloor s/\delta_n\rfloor\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}\sum_{r=\lfloor s/\delta_n\rfloor+1}^{\lfloor t/\delta_n\rfloor-1} a(t,r,\delta_n)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}(t/\delta_n-\lfloor t/\delta_n\rfloor)^2\Bigg)^{\gamma/2}, \text{ by Khintchine's inequality} \\
&= U_\gamma\Bigg(\sum_{r=-\infty}^{\lfloor s/\delta_n\rfloor-1}\frac{\delta_n^{2H}}{C_H^2}\left(a(t,r,\delta_n)-a(s,r,\delta_n)\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}\left(a(t,\lfloor s/\delta_n\rfloor,\delta_n)-s/\delta_n+\lfloor s/\delta_n\rfloor\right)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}\sum_{r=\lfloor s/\delta_n\rfloor+1}^{\lfloor t/\delta_n\rfloor-1} a(t,r,\delta_n)^2 \\
&\qquad + \frac{\delta_n^{2H}}{C_H^2}(t/\delta_n-\lfloor t/\delta_n\rfloor)^2\Bigg)^{\gamma/2} \\
&= U_\gamma E\left[\left(X_t^{\delta_n}-X_s^{\delta_n}\right)^2\right]^{\gamma/2}, \text{ by corollary 3.2.9 and lemma 3.6.1} \\
&\le U_\gamma\left(K_H|t-s|^{2H}\right)^{\gamma/2}, \text{ by theorem 3.6.2} \\
&= U_\gamma K_H^{\gamma/2}|t-s|^{2pH}.
\end{aligned}
$$

This completes the proof.

□

We are now ready to prove tightness.

**Theorem 3.6.5.** *Let $H\in(0,1)$. Let $\{\delta_n\}$ be a sequence of positive real numbers. Let $P_n$ be as in definition 3.3.1. Then $\{P_n\}$ is a tight collection of probability measures on $(C[0,\infty),\mathcal{C})$.*

*Proof.* $X^{(\delta_n)}$ is as in definition 3.3.1. By corollary 2.5.6 there are two conditions that must be satisfied. The first condition is that

$$P_n(x : x(0) = 0) = 1, \forall n.$$

This condition is satisfied by proposition 3.3.5.

The second condition of corollary 2.5.6 is that we must show that there exists $\gamma \geq 0, \alpha > 1$ and a nondecreasing, continuous function $F : [0, \infty) \to \mathbb{R}$ such that

$$E_n[|x(t_2) - x(t_1)|^\gamma] \leq |F(t_2) - F(t_1)|^\alpha,$$

holds for all $t_2, t_1, n$. We will now show this. Let $\alpha, \gamma, R_H$ be as in theorem 3.6.4, then $\alpha > 1, \gamma \geq 0$. We then have for all $n$

$$\begin{aligned} &E_n[|x(t_2) - x(t_1)|^\gamma] \\ &= E[|X^{(\delta_n)}_{t_1} - X^{(\delta_n)}_{t_2}|^\gamma], \text{ by proposition 3.3.5} \\ &\leq R_H|t_2 - t_1|^\alpha, \text{ by theorem 3.6.4} \\ &= |R_H^{1/\alpha} t_2 - R_H^{1/\alpha} t_1|^\alpha. \end{aligned}$$

The result follows if we let $F : [0, \infty) \to \mathbb{R}$ be given by

$$F(z) = R_H^{1/\alpha} z,$$

this is a nondecreasing, continuous function. □

## 3.7 Weak convergence of the Mandelbrot and Van Ness sum

All our work has been leading up to this section. We will now show the weak convergence of the Mandelbrot and Van Ness sum, or more precisely the weak convergence of the probability measures induced by the Mandelbrot and Van Ness sum. These measures converge weakly to the induced by the Fractional Brownian motion. We start with inspecting this measure. We recall that the existence and the description of the Fractional Brownian motion can be found in proposition 3.1.3.

**Definition 3.7.1.** *Let $H \in (0, 1)$. Let $B_H$ be the Fractional Brownian motion defined in proposition 3.1.3, assume that it is defined on an underlying probability space $(\Omega^*, \mathcal{A}^*, P^*)$. The measure $P$ defined on $(C[0, \infty), \mathcal{C})$ by*

$$P(C) = P^*(B_H^{-1}(C)), C \in \mathcal{C},$$

*is the measure induces by the Fractional Brownian motion.*

We now show that this definition makes sense, and also a characterisation of this measure.

**Theorem 3.7.2.** *Assume that $H \in (0, 1)$. Let $P$ be the measure induced by the Fractional Brownian motion(see definition 3.7.1). Then $P$ is well-defined probability measure.*

*If $\vec{t} = (t_1, t_2, \dots, t_k)$ with $0 \le t_1 < t_2 < \cdots < t_k$ let $\mu_{\vec{t}}$ be the probability measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ given by*

$$\mu_{\vec{t}}(B) = P(\pi_{\vec{t}}^{-1}(B)), B \in \mathcal{B}(\mathbb{R}^k).$$

*Then $P$ is uniquely determined by the fact that if $\vec{t}, \vec{r} \in \mathbb{R}^k$ with $0 \le t_1 < t_2 < \cdots < t_k$ we have*

$$\int_{\mathbb{R}^k} e^{i\vec{r}^T\vec{x}} d\mu_{\vec{t}}(\vec{x}) = \exp\left(-\frac{1}{2}\vec{r}^T A \vec{r}\right),$$

*where $A \in \mathbb{R}^k \times \mathbb{R}^k$ with*

$$A_{j,l} = \frac{1}{2}\left(t_j^{2H} + t_l^{2H} - |t_j - t_l|^{2H}\right).$$

*Proof.* Denote the underlying probability space of $B_H$ by $(\Omega^*, \mathcal{A}^*, P^*)$. The fact that $P$ is a well-defined probability measure on $(C[0,\infty), \mathcal{C})$ follows directly from lemma 3.3.2 which tells us that $B_H^{-1}(C) \in \mathcal{A}^*, C \in \mathcal{C}$, and theorem C.1.1. So we can define $P(C) = P^*(B_H^{-1}(C))$.

The fact that $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), \mu_{\vec{t}})$ is a probability space follows by the same argument as in proposition 3.3.4 where we showed that $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), P_n\pi^{-1}_{t_1,t_2,\dots,t_k})$ was a probability space. Now let $\vec{t}, \vec{r} \in \mathbb{R}^k$ with $0 \le t_1 < t_2 < \cdots < t_k$. By definition 3.1.1, definition 3.1.2 and proposition 3.1.3 we have

$$E_*\left[\exp(i\vec{r}^T(B_{t_1,H}, B_{t_2,H}, \dots, B_{t_k,H}))\right] = \exp\left(-\frac{1}{2}\vec{r}^T A \vec{r}\right).$$

where $A$ is as described in this theorem. The exact same argument(the bootstrap method) as in proposition 3.3.4 where we ended up with eq. (3.8) tells us that

$$E_*\left[\exp(i\vec{r}^T(B_{t_1,H}, B_{t_2,H}, \dots, B_{t_k,H}))\right] = \int_{\mathbb{R}^k} e^{i\vec{r}^T\vec{x}} d\mu_{\vec{t}}(\vec{x}),$$

where we have used Euler's formula and linearity of the integral to go from the real to the complex case.

Now assume that $Q$ is another probability measure on $(C[0,\infty), \mathcal{C})$ with the same finite-dimensional properties described in this theorem. That is if $0 \le t_1 < t_2 < \ldots < t_k$, let $\nu_{\vec{t}}$ be such that

$$\nu_{\vec{t}}(B) = Q(\pi_{\vec{t}}^{-1}(B)), B \in \mathcal{B}(\mathbb{R}^k),$$

and assume that

$$\int_{\mathbb{R}^k} e^{i\vec{r}^T\vec{x}} d\mu_{\vec{t}}(\vec{x}) = \int_{\mathbb{R}^k} e^{i\vec{r}^T\vec{x}} d\nu_{\vec{t}}(\vec{x}).$$

By [App09, p. 16] characteristic functions determine the measure uniquely on $\mathbb{R}^k$. So the finite dimensional distributions of $P$ and $Q$ are equal. By theorem 2.3.1 $P$ and $Q$ are equal. □

We now turn to the main theorem of this chapter. All our work has been leading up to this. Most of the work is done, so the proof will only consist of connecting different results already proven.

**Theorem 3.7.3** (**Weak convergence of the Mandelbrot and Van Ness sum**). *Let $H \in (0,1)$, assume that $\{\delta_n\}$ is a sequence of positive real numbers converging to zero. For each $\delta_n$ let $P_n$ be the measure induced by the Mandelbrot and Van Ness sum, $X^{\delta_n}$ (see definition 3.3.1). Then $\{P_n\}$ converges weakly to the measure $P$ induced by the Fractional Brownian motion defined in definition 3.7.1.*

*Proof.* By corollary 3.5.3 and theorem 3.7.2 the finite dimensional distributions of $\{P_n\}$ converge weakly to the finite-dimensional distributions of $P$, the measure induced by the Fractional Brownian motion in definition 3.7.1

By theorem 3.6.5 $\{P_n\}$ is tight.

By theorem 2.4.2 $\{P_n\}$ converges weakly to a measure $Q$ where the finite-dimensional distributions are the same as $P$. By the uniqueness statement of theorem 3.7.2 (or theorem 2.3.1) the result follows.

□

# Chapter 4

# Difference calculus with arbitrary step size.

The Mandelbrot and Van Ness sum we worked with in chapter 3 was a discrete stochastic process where we used linear interpolation to get a continuous stochastic process. We also explained in section 3.2 that the word *fractional* in the name Fractional Brownian motion came from the similarity between the stochastic integral representation of the fBm(see eq. (3.2)) and ordinary fractional integrals. In order to approximate the fBm it may be of interest for us to further investigate these two concepts, discreteness in some sense, and fractional calculus. This leads us to the book *Discrete Fractional Calculus*, by Christopher Goodrich and Allan C. Petersen, see [GP15].

In chapter 1 of [GP15] we are introduced to difference calculus with the forward $\Delta$-operator. However the work is done on sets of the form

$$\mathbb{N}_a \doteq \{a, a+1, a+2, \dots, \}, \qquad a \in \mathbb{R}$$
$$\mathbb{N}_a^b \doteq \{a, a+1, a+2, \dots, b\}, \qquad a, b \in \mathbb{R}, b-a \in \mathbb{N}.$$

These sets are not suitable for our purposes because the smallest difference between two elements is one, but as we saw in theorem 3.7.3 we work with a sequence $\{\delta_n\}$ of positive real numbers converging to zero. So it seems that it might be useful to have a theory with an arbitrary(small) step size. So in this chapter we will try to generalize some concepts of the first chapter in [GP15] to an arbitrary step size. This is not as easy as it may seem, because it is not entirely obvious how $\delta$ should enter the results. We will also generalize the Laplace transform of chapter two of [GP15], and briefly look at fractional sums and differences which is found in section 2.3 of [GP15], which gives us a small introduction to discrete fractional calculus.

Note that there are some inconsistencies in chapter one of [GP15] regarding real versus complex numbers. For instance we have on [GP15, p. 6] that what is called the *regressive functions* are real, but on point (iii) and (iv) on Theorem 1.27 on page 16, it is implicitly assumed that complex numbers are allowed. We see the same in [GP15, Theorem 1.31, p.18] where complex roots are allowed. We will try to avoid these inconsistencies and only concern ourselves with the real case. These inconsistencies also lead us to be a little careful when defining the Laplace transform.

## 4.1 Introduction

As mentioned above [GP15] difference calculus is done on the set $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$, where $a$ is a real number. We will try to do the same on the set $\mathbb{D}_a \doteq \{a, a+\delta, a+2\delta, \ldots\}$, where a is a real number, and $\delta$ is a positive real number. We suppress the dependence on $\delta$ in the notation.

**Definition 4.1.1 (adaptation of [GP15, page 1]).** *If $f : \mathbb{D}_a \to \mathbb{R}$, we define $\Delta f(t) = f(t+\delta) - f(t)$.*

**Definition 4.1.2 (adaptation of [GP15, page 2]).** *$\Delta^0$ denotes the identity operator. If $n \in \mathbb{N}$ we define $\Delta^n = \Delta[\Delta^{n-1}]$.*

**Definition 4.1.3 (adaptation of [GP15, page 1]).** *We define the forward operator $\sigma$, by $\sigma(t) = t + \delta$.*

Here are some useful properties of these two operators.

**Theorem 4.1.4 (adaptation of [GP15, page 2]).** *Assume that $f, g : \mathbb{D}_a \to \mathbb{R}$, and $\alpha, \beta$ are real numbers. Then for $t \in \mathbb{D}_a$ :*

*(i)* $\Delta\alpha = 0$

*(ii)* $\Delta(\alpha f(t)) = \alpha \Delta f(t)$

*(iii)* $\Delta[f+g](t) = \Delta f(t) + \Delta g(t)$

*(iv)* $\Delta\alpha^{t+\beta} = (\alpha^\delta - 1)\alpha^{t+\beta}$

*(v)* $\Delta[fg](t) = f(\sigma(t))\Delta g(t) + \Delta f(t) g(t)$

*(vi)* $\Delta\left(\frac{f}{g}\right)(t) = \frac{g(t)\Delta f(t) - f(t)\Delta g(t)}{g(t)g(\sigma(t))}$, *if* $g(t), g(\sigma(t)) \neq 0$

*Proof.*

(i) $\Delta\alpha(t) = \alpha - \alpha = 0$

(ii) $\Delta(\alpha f(t)) = \alpha f(t+\delta) - \alpha f(t) = \alpha(f(t+\delta) - f(t)) = \alpha\Delta f(t)$

(iii) $\Delta[f+g](t)$
$= [f+g](t+\delta) - [f+g](t)$
$= f(t+\delta) + g(t+\delta) - (f(t) + g(t))$
$= f(t+\delta) - f(t) + g(t+\delta) - g(t)$
$= \Delta f(t) + \Delta g(t)$

(iv) $\Delta\alpha^{t+\beta}$
$= \alpha^{t+\delta+\beta} - \alpha^{t+\beta}$
$= (\alpha^\delta - 1)\alpha^{t+\beta}$

(v) $f(\sigma(t))\Delta g(t) + \Delta f(t) g(t)$
$= f(t+\delta)(g(t+\delta) - g(t)) + (f(t+\delta) - f(t))g(t)$
$= f(t+\delta)g(t+\delta) - f(t)g(t)$
$= \Delta[fg](t)$

(vi) $\Delta\left(\frac{f}{g}\right)(t)$
$= \frac{f(t+\delta)}{g(t+\delta)} - \frac{f(t)}{g(t)}$
$= \frac{f(t+\delta)g(t)-g(t+\delta)f(t)}{g(t+\delta)g(t)}$
$= \frac{f(t+\delta)g(t)-f(t)g(t)+f(t)g(t)-g(t+\delta)f(t)}{g(t+\delta)g(t)}$
$= \frac{\Delta f(t)g(t)-\Delta g(t)f(t)}{g(t)g(\sigma(t))}$

□

## 4.2 The Gamma and falling functions

The construction of the discrete fractional calculus depends heavily on the Gamma function, $\Gamma(z)$. If $z$ is a real positive number we define

$$\Gamma(z) \doteq \int_0^\infty e^{-t}t^{z-1}dt.$$

By [SS03, Proposition 1.1, p. 160] it can be extended to a holomorphic function on the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, where we here still have the integral representation. By [SS03, Theorem 1.3, p. 161] we can extend this function again to a holomorphic function on $z \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$, the function will have poles on $\{\ldots, -3, -2, -1, 0\}$. A very important property for the Gamma function function is that for $z \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$

$$\Gamma(z+1) = z\Gamma(z), \tag{4.1}$$

for a proof see [SS03, Lemma 1.2, p.161] and the remark on the bottom of 162 in [SS03] What makes the discrete calculus work in chapter 1 of [GP15], is the role that the number 1 plays in eq. (4.1). In order to work with arbitrary $\delta > 0$, we need another function. We define the *modified Gamma function*, $\Gamma_\delta$, by

**Definition 4.2.1.**

$$\Gamma_\delta(z) \doteq \delta^{\frac{z}{\delta}}\Gamma\left(\frac{z}{\delta}\right), z \in \mathbb{C}\backslash\{0, -\delta, -2\delta, \ldots\}.$$

The function is holomorpic on its domain. We call the set $\{0, -\delta, -2\delta, \ldots\}$ the poles of the modified Gamma function. For the modified Gamma function we have a very similar property to eq. (4.1).

**Theorem 4.2.2.** *For $z \in \mathbb{C}\backslash\{0, -\delta, -2\delta, \ldots\}$, we have*

$$\Gamma_\delta(z+\delta) = z\Gamma_\delta(z).$$

*Proof.*

$$\begin{aligned}\Gamma_\delta(z+\delta) &= \delta^{\frac{z+\delta}{\delta}}\Gamma\left(\frac{z+\delta}{\delta}\right)\\ &= \delta^{\frac{z}{\delta}}\delta\Gamma(\frac{z}{\delta}+1)\\ &= \delta^{\frac{z}{\delta}}\delta\frac{z}{\delta}\Gamma(\frac{z}{\delta})\\ &= z\delta^{\frac{z}{\delta}}\Gamma(\frac{z}{\delta})\\ &= z\Gamma_\delta(z).\end{aligned}$$

□

Before we start using the modified Gamma function, we look at a definition that motivates the Gamma functions usage.

**Definition 4.2.3** (**Falling function, adaptation of [GP15, page 2]**). *Let $n$ be a positive integer, we define*

$$t^{\underline{n}} = t(t-\delta)(t-2\delta)\cdots(t-(n-1)\delta).$$

*For $n = 0$, we define $t^{\underline{0}} = 1$.*

We now investigate $t^{\underline{n}}$ for the possibilities when $t+\delta$ or $t-(n-1)\delta$ are poles. First note that it is impossible for $t+\delta$ to be a pole, and $t-(n-1)\delta$ not to be a pole. If both $t+\delta$ and $t-(n-1)\delta$ are not poles, we get

$$\begin{aligned} t^{\underline{n}} &= t(t-\delta)(t-2\delta)\cdots(t-(n-1)\delta) \\ &= t(t-\delta)(t-2\delta)\cdots(t-(n-1)\delta)\frac{\Gamma_\delta(t-(n-1)\delta)}{\Gamma_\delta(t-(n-1)\delta)} \\ &= \frac{\Gamma_\delta(t+\delta)}{\Gamma_\delta(t-(n-1)\delta)}. \end{aligned}$$

If $t+\delta$ is not a pole, and $t-(n-1)\delta$ is a pole, then $t^{\underline{n}}$ is 0 because one of the factors in

$$t(t-\delta)(t-2\delta)\cdots(t-(n-1)\delta)$$

will be zero. As we want to use the expression

$$\frac{\Gamma_\delta(t+\delta)}{\Gamma_\delta(t-(n-1)\delta)} \tag{4.2}$$

in this case as well, we make the convention that this fraction is 0 when $t+\delta$ is not a pole, and $t-(n-1)\delta$ is a pole. This is a natural definition because when you get closer to a pole, the absolute value of the modified Gamma function increases.

Lastly we investigate the case when both $t+\delta$ and $t-(n-1)\delta$ are poles. In this case we have that there must exist two non-negative integers $k, K$ such that $t+\delta = -k\delta$ and $t-(n-1)\delta = -K\delta$ and $K \geq k$. We then get

$$\begin{aligned} t^{\underline{n}} =& [-(k+1)\delta][-(k+1)\delta-\delta]\cdots[-K\delta] \\ =& (-\delta)^n(k+1)(k+2)\cdots K \\ =& \frac{(-\delta)^{K-k}K!}{k!}. \end{aligned}$$

Using what we have shown above, we now define $t^{\underline{r}}$, where $r$ is a real number.

**Definition 4.2.4** (**adaptation of [GP15, page 4]**). *For $t \in \mathbb{D}_a$, $r \in \mathbb{R}$, we define*

$$t^{\underline{r}} \doteq \frac{\Gamma_\delta(t+\delta)}{\Gamma_\delta(t-(r-1)\delta)},$$

*for values of $t, r$ where the right-hand side makes sense.*

What "makes sense" in this definition we can explain with the help of the discussion above. If both $t+\delta$, and $t-(r-1)\delta$ are not poles then we evaluate the expression without problems. If the numerator and denominator are both poles then there exist non-negative integers $k_1, k_2$ such that $t+\delta = -k_1\delta$, $t-(r-1)\delta = -k_2\delta$. Then there is the definition

$$\frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-k_2\delta)} \doteq \frac{(-\delta)^{k_2-k_1}k_2!}{k_1!}. \tag{4.3}$$

Thirdly if $t-(r-1)\delta$ is a pole, but $t+\delta$ is not a pole, we define the fraction as zero. The fourth and last case is if $t+\delta$ is a pole, but $t-(r-1)\delta$ is not, then the fraction is undefined.

**Theorem 4.2.5 (adaptation of [GP15, Therem 1.8, page 5]).** *Assume that $\alpha \in \mathbb{R}, t+\alpha \in \mathbb{D}_a$. Then we have*

$$\Delta(t+\alpha)^{\underline{r}} = \delta r(t+\alpha)^{\underline{r-1}},$$

*whenever both sides of the expression are well-defined.*

*Proof.* This proof is long because we will check all the cases involving poles, this is not done in [GP15]. First note that

$$\begin{aligned}\Delta(t+\alpha)^{\underline{r}} &= (t+\delta+\alpha)^{\underline{r}} - (t+\alpha)^{\underline{r}} \\ &= \frac{\Gamma_\delta(t+2\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)} - \frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+\delta+\alpha-r\delta)}.\end{aligned} \tag{4.4}$$

$$\delta r(t+\alpha)^{\underline{r-1}} = \delta r\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)}. \tag{4.5}$$

There are many cases to consider, where different numerators and denominators have poles.

Assume first that $t+\delta+\alpha$ and $t+\delta+\alpha-r\delta$ are not poles. Then $t+2\delta+\alpha$ and $t+2\delta+\alpha-r\delta$ can not be poles either. We then get

$$\begin{aligned}&\frac{\Gamma_\delta(t+2\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)} - \frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+\delta+\alpha-r\delta)} \\ &= \frac{[t+\delta+\alpha-(t+\delta+\alpha-r\delta)]\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)} \\ &= r\delta\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)},\end{aligned}$$

where we here have used theorem 4.2.2. We see that eq. (4.4) and eq. (4.5) are equal in this case.

If $t+\delta+\alpha$ is a pole, but $t+\delta+\alpha-r\delta$ is not a pole, then both eq. (4.4) is undefined.

Assume now that $t+\delta+\alpha-r\delta$ is a pole, but $t+\delta+\alpha$ is not a pole. In this case $t+2\delta+\alpha$ can not be a pole either. But we do not know if $t+2\delta+\alpha-r\delta$ is a pole or not. So we are left with two possibilities to consider, which we will now do. First assume that $t+2\delta+\alpha-r\delta$ is a pole. We then get directly that both eq. (4.4) and eq. (4.5) are 0. If $t+2\delta+\alpha-r\delta$ is not a pole we must have that $t+\delta+\alpha-r\delta=0$, so $r\delta=t+\delta+\alpha$. We then get that $t+2\delta+\alpha=\delta+r\delta$. And since $\Gamma_\delta(\delta+r\delta)=r\delta\Gamma_\delta(r\delta)=r\delta\Gamma_\delta(t+\delta+\alpha)$, by theorem 4.2.2, we have that eq. (4.5) and eq. (4.4) are equal in this case also.

We also have to consider the case where both $t+\delta+\alpha$ and $t+\delta+\alpha-r\delta$ are poles. In this case we must also assume that $t+2\delta+\alpha-r\delta$ is a pole, or else eq. (4.5) will not be well-defined. But we do not know if $t+2\delta+\alpha$ is a pole or not, so in this case there are two possibilities that have to be considered. Assume first that $t+2\delta+\alpha$ is not a pole. Since $t+\delta+\alpha$ is a pole, we must have that $t+\delta+\alpha=0$. We then get

$$\begin{aligned}&\frac{\Gamma_\delta(t+2\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)}-\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+\delta+\alpha-r\delta)}\\&=0-\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+\delta+\alpha-r\delta)}=-\frac{\Gamma_\delta(0)}{\Gamma_\delta(-r\delta)}\end{aligned}.$$

Since $-r\delta$ must be a pole, we must have that $r$ is a non-negative integer. Then we get

$$-\frac{\Gamma_\delta(0)}{\Gamma_\delta(-r\delta)}=-(-\delta)^r r!,$$

by eq. (4.3). We also get

$$\delta r\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)}=\delta r\frac{\Gamma_\delta(0)}{\Gamma_\delta(\delta-r\delta)}=\delta r(-\delta)^{r-1}(r-1)!=-(-\delta)^r r!.$$

So we see that in this case we also get that eq. (4.4) and eq. (4.5) are equal. Secondly we now assume that $t+2\delta+\alpha$ is a pole. We are now in the case where all four cases in the last part of eq. (4.4) are poles. Denote $t+\delta+\alpha=-k_1\delta, t+\delta+\alpha-r\delta=-k_2\delta$. Together this gives that $r\delta=(k_2-k_1)\delta$, so $r$ must be an integer. We then get

$$\begin{aligned}&\frac{\Gamma_\delta(t+2\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)}-\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+\delta+\alpha-r\delta)}\\&=\frac{\Gamma_\delta(-k_1\delta+\delta)}{\Gamma_\delta(-k_1\delta+\delta-r\delta)}-\frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-k_1\delta-r\delta)}\\&=\frac{\Gamma_\delta(-(k_1-1)\delta)}{\Gamma_\delta(-(k_1-1+r)\delta)}-\frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-(k_1+r)\delta)}\\&=\frac{(-\delta)^r(k_1-1+r)!}{(k_1-1)!}-\frac{(-\delta)^r(k_1+r)!}{k_1!}\\&=(-\delta)^r\frac{1}{k_1!}\Big[k_1(k_1-1+r)!-(k_1+r)!\Big]\\&=(-\delta)^r\frac{(k_1-1+r)!}{k_1!}[k_1-k_1-r]\\&=\delta r\frac{(-\delta)^{r-1}(k_1-1+r)!}{k_1!}.\end{aligned}$$

Notice that for these calculations to make sense we must have that $k_1\geq 1$, but this is the case since $t+\delta+\alpha=-k_1\delta$ and $t+2\delta+\alpha=-k_3\delta$ . We also get

$$\delta r\frac{\Gamma_\delta(t+\delta+\alpha)}{\Gamma_\delta(t+2\delta+\alpha-r\delta)}=\delta r\frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-k_1\delta+\delta-r\delta)}=\delta r\frac{(-\delta)^{r-1}(k_1-1+r)!}{k_1!}.$$

Hence, in this case we also get that eq. (4.4) and eq. (4.5) are equal. □

**Theorem 4.2.6 (adaptation of [GP15, Therem 1.8, page 5]).** *Assume that $\alpha \in \mathbb{R}$, such that $\alpha - t - \delta \in \mathbb{D}_a$. Then*

$$\Delta(\alpha - t)^{\underline{r}} = -r\delta(\alpha - \sigma(t))^{\underline{r-1}},$$

*if both sides of the equation are well-defined.*

*Proof.* The proof will follow the same structure as the proof of theorem 4.2.5, it is also long because we will check all the cases involving poles. Note that

$$\begin{aligned}\Delta(\alpha - t)^{\underline{r}} &= (\alpha - t - \delta)^{\underline{r}} - (\alpha - t)^{\underline{r}} \\ &= \frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta)} - \frac{\Gamma_\delta(\alpha - t + \delta)}{\Gamma_\delta(\alpha - t - r\delta + \delta)}.\end{aligned} \tag{4.6}$$

$$-r\delta(\alpha - \sigma(t))^{\underline{r-1}} = -r\delta\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)}. \tag{4.7}$$

Assume first that $\alpha - t$ and $\alpha - t - r\delta$ are not poles. Then $\alpha - t + \delta$ and $\alpha - t - r\delta + \delta$ can not be poles either. We get

$$\begin{aligned}&\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta)} - \frac{\Gamma_\delta(\alpha - t + \delta)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= \frac{\left[\alpha - t - r\delta - (\alpha - t)\right]\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= -r\delta\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)}.\end{aligned}$$

So we see that in this case eq. (4.6) and eq. (4.7) are equal.

We can not have the case that $\alpha - t$ is a pole, but $\alpha - t - r\delta$ is not a pole, because then eq. (4.6) is undefined.

Assume now that $\alpha - t$ is not a pole, but $\alpha - t - r\delta$ is a pole. Then $\alpha - t + \delta$ can not be a pole either. But we do not know if $\alpha - t - r\delta + \delta$ is a pole or not. So there are two cases to check here. If $\alpha - t - r\delta + \delta$ is a pole, then both eq. (4.6) and eq. (4.7) are 0. If $\alpha - t - r\delta + \delta$ is not a pole, we must have that $\alpha - t - r\delta = 0$. We then get

$$\begin{aligned}&\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta)} - \frac{\Gamma_\delta(\alpha - t + \delta)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= 0 - (\alpha - t)\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= -r\delta\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)}.\end{aligned}$$

Hence we see that eq. (4.6) and eq. (4.7) are equal in this case also.

We now consider the case where both $\alpha - t$ and $\alpha - t - r\delta$ are poles. We must have that $\alpha - t - r\delta + \delta$ is also a pole, or else eq. (4.7) will not be well-defined. The two cases to check are if $\alpha - t + \delta$ is a pole or not. Assume first that

$\alpha - t + \delta$ is not a pole. We must then have that $\alpha - t = 0$. This gives that $\alpha - t - r\delta = -r\delta$ is a pole. So r must be a non-negative integer. We get

$$\begin{aligned} &\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta)} - \frac{\Gamma_\delta(\alpha - t + \delta)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= \frac{\Gamma_\delta(0)}{\Gamma_\delta(-r\delta)} - 0 \\ &= (-\delta)^r r!. \end{aligned}$$

We also get

$$-r\delta \frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} = -r\delta \frac{\Gamma_\delta(0)}{\Gamma_\delta(-(r-1)\delta)} = -r\delta(-\delta)^{r-1}(r-1)! = (-\delta)^r r!.$$

So eq. (4.6) and eq. (4.7) are equal. Secondly we assume that $\alpha - t + \delta$ is also a pole. Now all four terms in the last part of eq. (4.6) are poles. We denote $\alpha - t = -k_1\delta$, $\alpha - t - r\delta = -k_2\delta$. So $r\delta = (k_2 - k_1)\delta$, so $r$ must be an integer. We get

$$\begin{aligned} &\frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta)} - \frac{\Gamma_\delta(\alpha - t + \delta)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} \\ &= \frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-(k_1 + r)\delta)} - \frac{\Gamma_\delta(-(k_1 - 1)\delta)}{\Gamma_\delta(-(k_1 + r - 1)\delta)} \\ &= (-\delta)^r \frac{(k_1 + r)!}{k_1!} - (-\delta)^r \frac{(k_1 + r - 1)!}{(k_1 - 1)!} \\ &= (-\delta)^r \frac{1}{k_1!}\left[(k_1 + r)! - (k_1 + r - 1)! k_1\right] \\ &= (-\delta)^r \frac{(k_1 + r - 1)!}{k_1!}[k_1 + r - k_1] \\ &= -r\delta(-\delta)^{r-1} \frac{(k_1 + r - 1)!}{k_1!}. \end{aligned}$$

For these calculations to make sense we need to know that $k_1 \geq 1$, but this is the case since $a - t = -k_1\delta$ and $a - t + \delta = -k_3\delta$. We also get

$$-r\delta \frac{\Gamma_\delta(\alpha - t)}{\Gamma_\delta(\alpha - t - r\delta + \delta)} = -r\delta \frac{\Gamma_\delta(-k_1\delta)}{\Gamma_\delta(-(k_1 + r - 1)\delta)} = -r\delta(-\delta)^{r-1} \frac{(k_1 + r - 1)!}{k_1!}.$$

Hence we see that eq. (4.6) and eq. (4.7) are equal in this case also, and the proof is complete.

□

## 4.3 The Delta Exponential Function

In order to define the delta exponential function we first need to define what is called the regressive functions. These functions are originally defined on [GP15, p. 6]. We make a modified definition here.

**Definition 4.3.1.**

$$\mathcal{R} = \{p : \mathbb{D}_a \to \mathbb{R}, 1 + \delta p(t) \neq 0 \ for\ t \in \mathbb{D}_a\} \tag{4.8}$$

What is the motivation for putting the $\delta$ where we did? One way to look at it is that it insures that in the proof of the next theorem we do not divide by zero. The next theorem takes a big step towards defining the delta exponential function.

**Theorem 4.3.2.** *Assume that $p \in \mathcal{R}$. Then a solution to the initial value problem*

$$\frac{\Delta x(t)}{\Delta t} = \frac{\Delta x(t)}{\delta} = p(t)x(t) \tag{4.9}$$

$$x(s) = 1, \tag{4.10}$$

*is unique.*

*Proof.* Assume that both $x$ and $y$ solves the initial value problem, and that they differ. Assume first that they differ for a $t > s$, let $s' > s$, be the smallest value bigger than $s$ where they differ. This means, that they do not differ on $s' - \delta$. But we must have $x(s') = x(s' - \delta) + \delta p(s' - \delta)x(s' - \delta) = y(s' - \delta) + \delta p(s' - 1)y(s' - \delta) = y(s')$. Hence we have a contradiction.

Assume now that they differ on a $t < s$. Let $s' < s$ be the largest point smaller than $s$ where they differ. This means that they do not differ on $s' + \delta$. We have, since $p \in \mathcal{R}$

$$x(s' + \delta) - x(s') = \delta p(s')x(s') \implies x(s') = \frac{x(s' + \delta)}{1 + \delta p(s')}. \tag{4.11}$$

We then get

$$x(s') = \frac{x(s' + \delta)}{1 + \delta p(s')} = \frac{y(s' + \delta)}{1 + \delta p(s')} = y(s').$$

So we have a contradiction in this case as well. □

For the next result we need to explain what we mean with the product.

$$\prod_{\tau=s}^{t-\delta}{}_{\delta} f(\tau).$$

It is very similar to the corresponding sum that we encountered on page 32.

**Definition 4.3.3.** *Assume that $\delta > 0$ and that $f$ is a real function that is defined on $s, s + \delta, \ldots, t - \delta$, where we assume tat $t - s$ is a multiple of $\delta$ and $t > s$. Then we define*

$$\prod_{\tau=s}^{t-\delta}{}_{\delta} f(\tau) \doteq f(s)f(s + \delta)\cdots f(t - \delta).$$

**Theorem 4.3.4** ([**GP15, Theorem 1.11, p. 7**]). *Assume that $p \in \mathcal{R}$, $s, t \in \mathbb{D}_a$. A solution to the initial value problem 4.9 and 4.10 is*

$$e_p(t, s) = \begin{cases} 1, & t = s \\ \prod_{\tau=s}^{t-\delta}{}_{\delta}[1 + \delta p(\tau)], & t > s \\ \prod_{\tau=t}^{s-\delta}{}_{\delta}[1 + \delta p(\tau)]^{-1}, & t < s. \end{cases} \tag{4.12}$$

*Proof.* $e_p(s,s) = 1$ by definition. We also get

$$\begin{aligned}\frac{\Delta e_p(s,s)}{\delta} &= \frac{1+\delta p(s) - 1}{\delta} \\ &= p(s)e_p(s,s).\end{aligned}$$

If $t > s$ we have

$$\begin{aligned}\frac{\Delta e_p(t,s)}{\delta} &= \frac{\prod_{\tau=s}^{t}{}_{\delta}[1+\delta p(\tau)] - \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta p(\tau)]}{\delta} \\ &= \frac{e_p(t,s)\left(1+\delta p(t) - 1\right)}{\delta} \\ &= e_p(t,s)p(t).\end{aligned}$$

If $t = s - \delta$ we have

$$\begin{aligned}\frac{\Delta e_p(t,s)}{\delta} &= \frac{1-(1+\delta p(t))^{-1}}{\delta} \\ &= \frac{\frac{1+\delta p(t)-1}{1+\delta p(t)}}{\delta} \\ &= \frac{(1+\delta p(t))^{-1}\delta p(t)}{\delta} \\ &= e_p(t,s)p(t).\end{aligned}$$

Assume lastly that $t < s - \delta$. We get

$$\begin{aligned}\frac{\Delta e_p(t,s)}{\delta} &= \frac{\prod_{\tau=t+\delta}^{s-\delta}{}_{\delta}[1+\delta p(\tau)]^{-1} - \prod_{\tau=t}^{s-\delta}{}_{\delta}[1+\delta p(\tau)]^{-1}}{\delta} \\ &= \frac{\prod_{\tau=t+\delta}^{s-\delta}{}_{\delta}[1+\delta p(\tau)]^{-1}\left(1-[1+\delta p(t)]^{-1}\right)}{\delta} \\ &= \frac{\prod_{\tau=t+\delta}^{s-\delta}{}_{\delta}[1+\delta p(\tau)]^{-1}\left(\frac{1+\delta p(t)-1}{1+\delta p(t)}\right)}{\delta} \\ &= e_p(t,s)p(t).\end{aligned}$$

□

**Definition 4.3.5.** *If $p \in \mathcal{R}$ the delta exponential $e_p(t,s)$ is defined as the unique solution to the initial value problem 4.9, 4.10. Theorem 4.3.4 and theorem 4.3.2 guarantees the existence and the uniqueness of $e_p(\cdot,s)$.*

**Theorem 4.3.6** ([**GP15**, **p. 8**]). *If $p \in \mathcal{R}$ then a general solution of*

$$\frac{\Delta y(t)}{\delta} = p(t)y(t), t \in \mathbb{D}_a \tag{4.13}$$

*is given by*

$$y(t) = ce_p(t,a), t \in \mathbb{D}_a,$$

*where c is a constant.*

*Proof.* If $y(a) = 0$, then it is identically zero. So $c = 0$ in this case.

If $y(a) \neq 0$, define $x(t) \doteq y(t)/y(a)$. Then $x(a) = 1$, and $\frac{\Delta x(t)}{\delta} = y(t+\delta)/y(a) - y(t)/y(a) = p(t)y(t)/y(a) = p(t)x(t)$. So $x(t) = e_p(t,s)$, by the existence and uniqueness in theorem 4.3.2 and theorem 4.3.4, hence $y(t) = y(a)e_p(t,s)$.

□

We now generalize the definition made in [GP15, Theorem 1.16, p. 10]. The motivation for putting the $\delta$ where we did in this case is seen in the coming theorem 4.3.10 point (ix). In order for this property to hold, we have to define $\oplus$ like this.

**Definition 4.3.7.** *If* $p, g \in \mathcal{R}$*, we define* $p \oplus q = p + q + \delta pq$*.*

**Theorem 4.3.8** (**[GP15, Theorem 1.16, p.10]**)**.** $(\mathcal{R}, \oplus)$ *is an abelian group.*

*Proof.* We have closure because

$$1 + \delta(p + q + \delta pq) = (1 + \delta p)(1 + \delta q) \neq 0.$$

We have associativity because

$$\begin{aligned} &p \oplus (q \oplus r) = p \oplus (q + r + \delta qr) = p + q + r + \delta qr + \delta pq + \delta pr + \delta^2 pqr \\ =&(p + q + \delta pq) \oplus r = (p \oplus q) \oplus r. \end{aligned}$$

We have an identity element because the function identically equal to zero is in $\mathcal{R}$. And

$$\begin{aligned} 0 \oplus p &= 0 + p + \delta 0 \cdot p = p \\ p \oplus 0 &= p + 0 + \delta p \cdot 0 = p. \end{aligned}$$

We have well-defined inverse element

$$\ominus p = \frac{-p}{1 + \delta p},$$

because

$$\delta \frac{-p}{1 + \delta p} + 1 = \frac{1}{1 + \delta p} \neq 0.$$

and

$$\begin{aligned} p + \frac{-p}{1 + \delta p} + \delta p \cdot \frac{-p}{1 + \delta p} &= \frac{p + \delta p^2 - p - \delta p^2}{1 + \delta p} = 0, \\ \frac{-p}{1 + \delta p} + p + \delta \frac{-p}{1 + \delta p} \cdot p &= \frac{-p + p + \delta p^2 - \delta p^2}{1 + \delta p} = 0. \end{aligned}$$

The operation is also commutative because

$$p \oplus q = p + q + \delta pq = q \oplus p.$$

□

**Definition 4.3.9** ([**GP15, p.10**]). *We define circle minus subtraction on* $\mathcal{R}$ *by*

$$p \ominus q \doteq p \oplus [\ominus]q.$$

We now show some properties of the delta exponential function.

**Theorem 4.3.10** ([**GP15, Theorem 1.18, p.11**]). *Assume that* $p, q \in \mathcal{R}$ *and* $t, s, r \in \mathbb{D}_a$. *Then*

*(i)* $e_0(t,s) = 1, t \in \mathbb{D}_a$

*(ii)* $e_p(t,t) = 1$

*(iii)* $e_p(t,s) \neq 0, t \in \mathbb{D}_a$

*(iv) if* $1 + \delta p > 0$, *then* $e_p(t,s) > 0$

*(v)* $\Delta e_p(t,s) = \delta p(t) e_p(t,s)$

*(vi)* $e_p(\sigma(t), s) = [1 + \delta p(t)] e_p(t,s)$

*(vii)* $e_p(t,s) = \frac{1}{e_p(s,t)}$

*(viii)* $e_p(t,s) e_p(s,r) = e_p(t,r)$

*(ix)* $e_p(t,s) e_q(t,s) = e_{p \oplus q}(t,s)$

*(x)* $e_{\ominus p}(t,s) = \frac{1}{e_p(t,s)}$

*(xi)* $\frac{e_p(t,s)}{e_q(t,s)} = e_{p \ominus q}(t,s)$.

*Proof.* Point (i), (iii) and (iv) follows from theorem 4.3.4. (ii) is true since the delta exponential function by definition satisfies eq. (4.10). (v) is true since the delta exponential satisfies eq. (4.9).

By point $(v)$ we have that

$$\Delta e_p(t,s) = \delta p(t) e_p(t,s).$$

This tells us that

$$e_p(t + \delta, s) - e_p(t,s) = \delta p(t) e_p(t,s).$$

Hence (vi) follows.

For point (vii), if $t = s$ the result follows since both sides equals 1. If $t > s$ we get

$$e_p(t,s) = \prod_{\tau=s}^{t-\delta}{}_{\delta}[1 + \delta p(\tau)] = \frac{1}{\prod_{\delta \tau = s}^{t-\delta} [1 + \delta p(\tau)]^{-1}} = \frac{1}{e_p(s,t)}.$$

If $s > t$, then by what we just proved $e_p(s,t) = \frac{1}{e_p(t,s)}$, this implies that $e_p(t,s) = \frac{1}{e_p(s,t)}$ in this case also, hence point (vii) is proved.

In proving point (viii) we first look at the cases where at least two of $t, s, r$ are equal. If $t = s$ or $s = r$, the result follows from point (ii). If $t = r$, then by (ii) the the problem is reduced to showing that

$$e_p(t,s)e_p(s,t) = 1,$$

but this is true by (vii). Now we can assume that $s$, $r$ and $t$ are different. If $t > s > r$, we get

$$e_p(t,s)e_p(s,r) = \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta p(\tau)] \cdot \prod_{\tau=r}^{s-\delta}{}_{\delta}[1+\delta p(\tau)] = \prod_{\tau=r}^{t-\delta}{}_{\delta}[1+\delta p(\tau)] = e_p(t,r).$$

This result will be used for all the cases for the rest of this section. If $t > r > s$, by what we just proved we get $e_p(t,r)e_p(r,s) = e_p(t,s)$. Since $e_p(r,s) = \frac{1}{e_p(s,r)}$, by (vii), the result follows. If $r > t > s$ we have $e_p(r,t)e_p(t,s) = e_p(r,s)$. By using that $e_p(r,t) = \frac{1}{e_p(t,r)}$ and $e_p(r,s) = \frac{1}{e_p(s,r)}$, the result follows. If $r > s > t$ we get $e_p(r,s)e_p(s,t) = e_p(r,t)$, and using (vii) on all terms, the result follows. If $s > t > r$ we get $e_p(s,t)e_p(t,r) = e_p(s,r)$, and by using (vii) on $e_p(s,t)$ the result follows Lastly, if $s > r > t$, we have $e_p(s,r)e_p(r,t) = e_p(s,t)$. We use (vii) on $e_p(t,s)$ and $e_p(r,t)$ to get the result.

First note that $e_{p\oplus q}(t,s)$ is well-defined by theorem 4.3.8. If $t = s$ both sides are equal to 1. If $t > s$ we get

$$\begin{aligned} e_p(t,s)e_q(t,s) &= \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta p(\tau)] \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta q(\tau)] \\ &= \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta p(\tau)][1+\delta q(\tau)] \\ &= \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta p(\tau)+\delta q(\tau)+\delta^2 p(\tau)q(\tau)] \\ &= \prod_{\tau=s}^{t-\delta}{}_{\delta}[1+\delta[p\oplus q](\tau)] = e_{p\oplus q}(t,s). \end{aligned}$$

Similarly if $s > t$ we get

$$\begin{aligned} e_p(t,s)e_q(t,s) &= \prod_{\tau=t}^{s-\delta}{}_{\delta}[1+\delta p(\tau)]^{-1} \prod_{\tau=t}^{s-\delta}{}_{\delta}[1+\delta q(\tau)]^{-1} \\ &= \prod_{\tau=t}^{s-\delta}{}_{\delta}[1+\delta(p(\tau)+q(\tau)+p(\tau)q(\tau))]^{-1} \\ &= \prod_{\tau=t}^{s-\delta}{}_{\delta}[1+\delta[p\oplus q](\tau)]^{-1} = e_{p\oplus q}(t,s). \end{aligned}$$

We have from the proof of theorem 4.3.8 that $p \ominus p = 0$. And as from (i) we have that $e_0(t,s) = 1$, this gives us that

$$1 = e_0(t,s) = e_{p\ominus p}(t,s) = e_p(t,s)e_{\ominus p}(t,s),$$

where we in the last step used (ix). This shows that (x) holds.

Point (xi) follows from (ix) and (x). □

## 4.4 Delta Trigonometric Functions

**Definition 4.4.1** ([**GP15**, **p. 14**]). *Assume $\pm p \in \mathcal{R}$. Then the* ***delta hyperbolic cosine*** *and the* ***delta hyperbolic sine*** *are defined as follows*

$$\cosh_p(t,a) \doteq \frac{e_p(t,a) + e_{-p}(t,a)}{2},$$
$$\sinh_p(t,a) \doteq \frac{e_p(t,a) - e_{-p}(t,a)}{2}.$$

**Theorem 4.4.2** (**adaptation of** [**GP15**, **Theorem 1.25, p.15**]). *Assume $\pm p \in \mathcal{R}$. Then*

*(i)* $\cosh_p(a,a) = 1, \sinh_p(a,a) = 0$

*(ii)* $\cosh_p^2(t,a) - \sinh_p^2(t,a) = e_{-\delta p^2}(t,a)$

*(iii)* $\Delta \cosh_p(t,a) = \delta p(t) \sinh_p(t,a)$

*(iv)* $\Delta \sinh_p(t,a) = \delta p(t) \cosh_p(t,a)$

*(v)* $\cosh_{-p}(t,a) = \cosh_p(t,a)$

*(vi)* $\sinh_{-p}(t,a) = -\sinh_p(t,a)$

*(vii)* $e_p(t,a) = \cosh_p(t,a) + \sinh_p(t,a)$.

*Proof.* (i) follows from theorem 4.3.10 point (ii).

For (ii) we get

$$\begin{aligned}&\cosh_p^2(t,a) - \sinh_p^2(t,a)\\ =&\frac{e_p(t,a)^2 + 2e_p(t,a)e_{-p}(t,a) + e_{-p}(t,a)^2 - e_p(t,a)^2 + 2e_p(t,a)e_{-p}(t,a) - e_{-p}(t,a)^2}{4}\\ =&e_p(t,a)e_{-p}(t,a) = e_{p\oplus(-p)}(t,a) = e_{p+(-p)+\delta p(-p)}(t,a) = e_{-\delta p^2}(t,a).\end{aligned}$$

For (iii) we get

$$\begin{aligned}\Delta \cosh p(t,a) = \frac{\Delta e_p(t,a) + \Delta e_{-p}(t,a)}{2} &= \frac{\delta p(t) e_p(t,a) - \delta p(t) e_{-p}(t,a)}{2}\\ &= \delta p(t) \sinh_p(t,a).\end{aligned}$$

Similarly for (iv) we get

$$\begin{aligned}\Delta \sinh p(t,a) = \frac{\Delta e_p(t,a) - \Delta e_{-p}(t,a)}{2} &= \frac{\delta p(t) e_p(t,a) + \delta p(t) e_{-p}(t,a)}{2}\\ &= \delta p(t) \cosh_p(t,a).\end{aligned}$$

(v) and (vi) follow by simple calculation:

$$\begin{aligned}\cosh_{-p}(t,a) &= \frac{e_{-p}(t,a) + e_{-(-p)}(t,a)}{2} = \cosh_{-p}(t,a),\\ \sinh_{-p}(t,a) &= \frac{e_{-p}(t,a) - e_{-(-p)}(t,a)}{2} = -\frac{e_p(t,a) - e_{-p}(t,a)}{2}\\ &= -\sinh_p(t,a).\end{aligned}$$

For (vii) we also get by simple calculation

$$\begin{aligned}\cosh_p(t,a) + \sinh_p(t,a) &= \frac{e_p(t,a) + e_{-p}(t,a)}{2} + \frac{e_p(t,a) - e_{-p}(t,a)}{2} \\ &= e_p(t,a).\end{aligned}$$

□

**Definition 4.4.3** ([**GP15, p. 15**]). *For $\pm ip \in \mathcal{R}$, we define the **delta cosine function** and **delta sine function**:*

$$\cos_p(t,a) = \frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2}, \ \sin_p(t,a) = \frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i}.$$

**Theorem 4.4.4** ([**GP15, p.16**]). *Assume that $\pm p \in \mathcal{R}$. Then*

*(i)* $\sin_{ip}(t,a) = i \sinh_p(t,a)$

*(ii)* $\cos_{ip}(t,a) = \cosh_p(t,a)$.

*Proof.* Notice first that if $\pm p \in \mathcal{R}$, then we have that

$$\pm i(ip) = \pm(-p) = \pm p \in \mathcal{R}.$$

We get directly

$$\begin{aligned}\sin_{ip}(t,a) &= \frac{e_{i^2p}(t,a) - e_{-i^2p}(t,a)}{2i} \\ &= -i\frac{e_{-p}(t,a) - e_p(t,a)}{2} \\ &= i \sinh_p(t,a).\end{aligned}$$

We also get

$$\begin{aligned}\cos_{ip}(t,a) &= \frac{e_{i^2p}(t,a) + e_{-i^2p}(t,a)}{2} \\ &= \frac{e_{-p}(t,a) + e_p(t,a)}{2} \\ &= \cosh_p(t,a).\end{aligned}$$

□

**Theorem 4.4.5** ([**GP15, p.16**]). *Assume that $\pm ip \in \mathcal{R}$. Then*

*(i)* $\sinh_{ip}(t,a) = i \sin_p(t,a)$

*(ii)* $\cosh_{ip}(t,a) = \cos_p(t,a)$

*(iii)* $\cos_p(a,a) = 1$

*(iv)* $\sin_p(a,a) = 0$

*(v)* $\cos_p^2(t,a) + \sin_p^2(t,a) = e_{\delta p^2}(t,a)$

*(vi)* $\Delta \cos_p(t,a) = -\delta p(t) \sin_p(t,a)$

*(vii)* $\Delta \sin_p(t,a) = \delta p(t) \cos_p(t,a)$

*(viii)* $\cos_{-p}(t,a) = \cos_p(t,a)$

*(ix)* $\sin_{-p}(t,a) = -\sin_p(t,a)$

*(x)* $e_{ip}(t,a) = \cos_p(t,a) + i \sin_p(t,a)$.

*Proof.* (i)

$$\sinh_{ip}(t,a) = \frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2} \frac{i}{i} = i \sin_p(t,a).$$

(ii)

$$\cosh_{ip}(t,a) = \frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2} = \cos_p(t,a).$$

(iii)

$$\cos_p(a,a) = \frac{e_{ip}(a,a) + e_{-ip}(a,a)}{2} = 1.$$

(iv)

$$\sin_p(a,a) = \frac{e_{ip}(a,a) - e_{-ip}(a,a)}{2i} = 0.$$

(v)

$$\begin{aligned}
&\cos_p^2(t,a) + \sin_p^2(t,a) \\
=&\frac{e_{ip}(t,a)^2 + 2e_{ip}(t,a)e_{-ip}(t,a)e_{-ip}(t,a)^2}{4} \\
&- \frac{e_{ip}(t,a)^2 - 2e_{ip}(t,a)e_{-ip}(t,a) + e_{-ip}(t,a)^2}{4} \\
=&e_{ip}(t,a)e_{-ip}(t,a) = e_{ip+(-ip)+\delta ip(-ip)}(t,a) = e_{\delta p^2}(t,a).
\end{aligned}$$

(vi)

$$\begin{aligned}
&\Delta \cos_p(t,a) \\
=&\frac{\Delta e_{ip}(t,a) + \Delta e_{-ip}(t,a)}{2} \\
=&\frac{\delta ip(t)e_{ip}(t,a) + \delta(-ip(t))e_{-ip}(t,a)}{2} \\
=&\delta \frac{-p(t)e_{ip}(t,a) + (p(t))e_{-ip}(t,a)}{2i} \\
=&-\delta p(t) \frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i} \\
=&-\delta p(t) \sin_p(t,a).
\end{aligned}$$

(vii)

$$\begin{aligned}
&\Delta \sin_p(t,a)\\
&= \frac{\Delta e_{ip}(t,a) - \Delta e_{-ip}(t,a)}{2i}\\
&= \frac{\delta ip(t)e_{ip}(t,a) - \delta(-ip(t))e_{-ip}(t,a)}{2i}\\
&= \delta p(t)\frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2}\\
&= \delta p(t)\cos_p(t,a).
\end{aligned}$$

(viii)

$$\begin{aligned}
\cos_{-p}(t,a) &= \frac{e_{i(-p)}(t,a) + e_{-i(-p)}(t,a)}{2}\\
&= \frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2}\\
&= \cos_p(t,a).
\end{aligned}$$

(ix)

$$\begin{aligned}
\sin_{-p}(t,a) &= \frac{e_{i(-p)}(t,a) - e_{-i(-p)}(t,a)}{2i}\\
&= -\frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i}\\
&= -\sin_p(t,a).
\end{aligned}$$

(x)

$$\begin{aligned}
\cos_p(t,a) + i\sin_p(t,a) &= \frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2} + i\frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i}\\
&= e_{ip}(t,a).
\end{aligned}$$

□

## 4.5 The Delta Derivative

In this section we define a concept not found in chapter 1 of [GP15]. When the step size is 1 it does not matter if we divide the difference with the step size or not. However, when we have an arbitrary step size $\delta$ we get something that resembles the classical derivative from calculus when we divide the difference with the step size. We will explore this concept in this section.

**Definition 4.5.1.** *$f : \mathbb{D}_a \to \mathbb{R}$, we define $\mathrm{D}f(t) \doteq \frac{\Delta f(t)}{\Delta t} = \frac{f(t+\delta)-f(t)}{\delta}$.*

**Definition 4.5.2.** *We define $\mathrm{D}^0$ as the identity operator. And for $n \in \mathbb{N}$ we define $\mathrm{D}^n = \mathrm{D}[\mathrm{D}^{n-1}]$. We call $\mathrm{D}f(t)$ the delta derivative of $f$.*

**Theorem 4.5.3** (**adaptation of [GP15, page 2]**)**.** *Assume that $f, g : \mathbb{D}_a \to \mathbb{R}$, and $\alpha$ is a real number. Then for $t \in \mathbb{D}_a$ :*

*(i)* $\mathrm{D}\alpha = 0$

*(ii)* $\mathrm{D}(\alpha f(t)) = \alpha \mathrm{D}f(t)$

*(iii)* $\mathrm{D}[f+g](t) = \mathrm{D}f(t) + \mathrm{D}g(t)$

*(iv)* $\mathrm{D}[fg](t) = f(\sigma(t))\mathrm{D}g(t) + (\mathrm{D}f(t))g(t)$

*(v)* $\mathrm{D}\left(\frac{f}{g}\right)(t) = \frac{g(t)\mathrm{D}f(t) - f(t)\mathrm{D}g(t)}{g(t)g(\sigma(t))}$, *if* $g(t), g(\sigma(t)) \neq 0$

*Proof.* The result follows from from theorem 4.1.4 and the definition of $D$. We have

(i) $\mathrm{D}\alpha = \frac{\Delta\alpha}{\delta} = 0$ by theorem 4.1.4 (i).

(ii) $\mathrm{D}(\alpha f(t))$
$= \frac{\Delta(\alpha f(t))}{\delta}$
$= \frac{\alpha\Delta f(t)}{\delta}$ by theorem 4.1.4 (ii)
$= \alpha\mathrm{D}f(t)$.

(iii) $\mathrm{D}[f+g](t)$
$= \frac{\Delta(f+g)(t)}{\delta}$
$= \frac{\Delta f(t)+\Delta g(t)}{\delta}$ by theorem 4.1.4 (iii)
$= \mathrm{D}f(t) + \mathrm{D}g(t)$.

(iv) $\mathrm{D}[fg](t)$
$= \frac{\Delta[fg](t)}{\delta}$
$= \frac{f(\sigma(t))\Delta g(t)+\Delta f(t)g(t)}{\delta}$ by theorem 4.1.4 (v)
$= f(\sigma(t))\mathrm{D}g(t) + (\mathrm{D}f(t))g(t)$.

(v) $\mathrm{D}\left(\frac{f}{g}\right)(t)$
$= \frac{\Delta(f/g)(t)}{\delta}$
$= \frac{[g(t)\Delta f(t) - f(t)\Delta g(t)]/[g(t)g(\sigma(t))]}{\delta}$ by theorem 4.1.4 (vi)
$= \frac{g(t)\mathrm{D}f(t) - f(t)\mathrm{D}g(t)}{g(t)g(\sigma(t))}$.

□

Now we prove that $f$ is a constant if and only if the delta derivative is zero. We recognize the similarity between the similar result from ordinary calculus, but our proof is very simple. In ordinary calculus one needs the mean value theorem to prove the "only if" part.

**Proposition 4.5.4.** *Assume that* $f : \mathbb{D}_a \to \mathbb{R}$. $\mathrm{D}f(t) = 0$ *for all* $t$ *if and only if* $f$ *is a constant.*

*Proof.* If $f(t)$ is constant the result follows from theorem 4.5.3 (i).

Conversely, assume that $\mathrm{D}f(t) = 0$. Assume for contradiction that $f$ is not a constant. Then there must exist a $t'$ such that $f(t') \neq f(t'+\delta)$. But then we have

$$\mathrm{D}f(t') = \frac{f(t'+\delta) - f(t)}{\delta} \neq 0.$$

Hence we have a contradiction. □

Next we prove two results that are very similar to results in ordinary calculus where we differentiate polynomials. One may view these results as a motivation for introducing the falling functions.

**Proposition 4.5.5.** *Assume $\alpha \in \mathbb{R}$, and $t+\alpha \in \mathbb{D}_a$. Then*

$$\mathrm{D}[(t+\alpha)^{\underline{r}}] = r(t+\alpha)^{\underline{r-1}},$$

*whenever both sides are well-defined.*

*Proof.* We have

$$\mathrm{D}[(t+\alpha)^{\underline{r}}] \doteq \frac{\Delta(t+\alpha)^{\underline{r}}}{\delta} = \frac{\delta r(t+\alpha)^{\underline{r-1}}}{\delta} = r(t+\alpha)^{\underline{r-1}}.$$

Where we have used theorem 4.2.5. □

**Proposition 4.5.6.** *Assume that $\alpha \in \mathbb{R}$, and $\alpha - t - \delta \in \mathbb{D}_a$. We then have*

$$\mathrm{D}[(\alpha-t)^{\underline{r}}] = -r(\alpha-\sigma(t))^{\underline{r-1}},$$

*if both sides of the equation are well-defined.*

*Proof.*

$$\mathrm{D}[(\alpha-t)^{\underline{r}}] \doteq \frac{\Delta(\alpha-t)^{\underline{r}}}{\delta} = \frac{-r\delta(\alpha-\sigma(t))^{\underline{r-1}}}{\delta} = -r(\alpha-\sigma(t))^{\underline{r-1}},$$

where we have used theorem 4.2.6. □

We end this section by showing that discrete differentiation of the delta exponential function also behaves well and as one would expect if one has ordinary calculus in mind.

**Proposition 4.5.7.** *If $p \in \mathcal{R}$. Then*

$$\mathrm{D}e_p(t,s) = p(t)e_p(t,s).$$

*Proof.* We get

$$\begin{aligned}\mathrm{D}e_p(t,s) &= \frac{\Delta e_p(t,s)}{\delta} \\ &= \frac{\delta p(t)e_p(t,s)}{\delta} \\ &= p(t)e_p(t,s),\end{aligned}$$

using theorem 4.3.10 (v) for the second equality. □

## 4.6 The Delta Integral

One of the advantages with discrete calculus is that the sums behave like integrals, we will explore this concept in this section.

**Definition 4.6.1 (modification of [GP15, def 1.49]).** *Assume $f : \mathbb{D}_a \to \mathbb{R}$, and $c \leq d$ are in $\mathbb{D}_a$, then*

$$\int_c^d f(t)\Delta t \doteq \sum_{t=c}^{d-\delta}{}_{\delta} f(t)\delta.$$

We call this the delta integral. Next we prove some basic properties for this integral.

**Theorem 4.6.2 (adaptation of [GP15, Theorem 1.50, p.29]).** *Assume that $f, g : \mathbb{D}_a \to \mathbb{R}, b, c, d \in \mathbb{D}_a, b \leq c \leq d$ and $\alpha \in \mathbb{R}$. Then*

*(i)* $\int_b^c \alpha f(t)\Delta t = \alpha \int_b^c f(t)\Delta t$

*(ii)* $\int_b^c (f(t) + g(t))\Delta t = \int_b^c f(t)\Delta t + \int_b^c g(t)\Delta t$

*(iii)* $\int_b^b f(t)\Delta t = 0$

*(iv)* $\int_b^d f(t)\Delta t = \int_b^c f(t)\Delta t + \int_c^d f(t)\Delta t$

*(v)* $\left|\int_b^c f(t)\Delta t\right| \leq \int_b^c |f(t)| \Delta t$

*(vi) if* $F(t) \doteq \int_b^t f(s)\Delta s$, *for* $t \in \mathbb{D}_b$, *then* $\mathrm{D}F(t) = f(t)$

*(vii) if* $f(t) \geq g(t)$ *for* $t \in \mathbb{D}_b \backslash \mathbb{D}_c$, *and* $c \in \mathbb{D}_b$, *then* $\int_b^c f(t)\Delta t \geq \int_b^c g(t)\Delta t$.

*Proof.* (i)

$$\int_b^c \alpha f(t)\Delta t \doteq \sum_{t=b}^{c-\delta}{}_{\delta} \alpha f(t)\delta = \alpha \sum_{t=b}^{c-\delta}{}_{\delta} f(t)\delta = \alpha \int_b^c f(t)\Delta t$$

(ii)

$$\begin{aligned}\int_b^c (f(t) + g(t))\Delta t &= \sum_{t=b}^{c-\delta}{}_{\delta} (f(t) + g(t))\delta \\ &= \sum_{t=b}^{c-\delta}{}_{\delta} f(t)\delta + \sum_{t=b}^{c-\delta}{}_{\delta} g(t)\delta \\ &= \int_b^c f(t)\Delta t + \int_b^c g(t)\Delta t\end{aligned}$$

(iii)

$$\int_b^b f(t)\Delta t = \sum_{t=b}^{b-\delta}{}_{\delta} f(t)\delta = 0$$

(iv)

$$\begin{aligned}\int_b^d f(t)\Delta t &= \sum_{t=b}^{d-\delta}{}_{\delta} f(t)\delta \\ &= \sum_{t=b}^{c-\delta}{}_{\delta} f(t)\delta + \sum_{t=c}^{d-\delta}{}_{\delta} f(t)\delta \\ &= \int_b^c f(t)\Delta t + \int_c^d f(t)\Delta t\end{aligned}$$

(v)

$$\left|\int_b^c f(t)\Delta t\right| = \left|\sum_{t=b}^{c-\delta}{}_{\delta} f(t)\delta\right| \leq \sum_{t=b}^{c-\delta}{}_{\delta} |f(t)|\,\delta = \int_b^c |f(t)|\,\Delta t$$

(vi)

$$\mathrm{D}F(t) = \frac{F(t+\delta)-F(t)}{\delta} = \int_b^{t+\delta} \frac{f(s)}{\delta}\Delta s - \int_b^t \frac{f(s)}{\delta}\Delta s.$$

By (iv) this is

$$\int_t^{t+\delta} \frac{f(s)}{\delta}\Delta s = \sum_{s=t}^{t}{}_{\delta} \frac{f(s)}{\delta}\delta = f(t).$$

(vii)

$$\int_b^c f(t)\Delta t = \sum_{t=b}^{c-\delta}{}_{\delta} f(t)\delta \geq \sum_{t=b}^{c-\delta}{}_{\delta} g(t)\delta = \int_b^c g(t)\Delta t.$$

□

**Definition 4.6.3 (adaptation of [GP15, 1.51, p. 29]).** *If $f : \mathbb{D}_a \to \mathbb{R}$, we say that f is a discrete antiderivative of f if*

$$\mathrm{D}F(t) = f(t).$$

As we would expect discrete antiderivatives of the same function only differ by constants.

**Theorem 4.6.4 ([GP15, p. 29]).** *Let $f : \mathbb{D}_a \to \mathbb{R}$. Assume that $G(t)$ is discrete antiderivative of $f$. Then $F$ is a discrete antiderivative of $f$ if and only if $F$ is of the form $G(t) + C$, where $C$ is a constant.*

*Proof.* First note that $\mathrm{D}(G(t)+C) = \mathrm{D}(t) = f(t)$, by theorem 4.5.3. So $G(t)+C$ is discrete antiderivative.

Now assume that $\mathrm{D}F(t) = f(t)$. We then get

$$\mathrm{D}(F(t) - G(t)) = \mathrm{D}F(t) - \mathrm{D}G(t) = f(t) - f(t) = 0,$$

for all t. By proposition 4.5.4, we have that $F(t) - G(t) = C$. □

The next definition is also very natural.

**Definition 4.6.5** (**[GP15, Def. 1.54]**). *If $f : \mathbb{D}_a \to \mathbb{R}$, we define the discrete indefinite integral of by*

$$\int f(t)\Delta t \doteq F(t) + C,$$

*where $F$ is any discrete antiderivative of $f$, and $C$ is an arbitrary constant. Hence $\int f(t)\Delta t$ is only unique up to a constant.*

The indefinite integrals of the basic functions is shown in the next theorem.

**Theorem 4.6.6** (**adaptation of [GP15, Theorem 1.55]**). *Assume $p, r, \alpha$ are constants. Then the following hold.*

*(i)* $\int (t-\alpha)^{\underline{r}}\Delta t = \frac{1}{r+1}(t-\alpha)^{\underline{r+1}} + C$, *if $r \neq -1$, and both sides are well-defined;*

*(ii)* $\int (\alpha - \sigma(t))^{\underline{r}}\Delta t = \frac{-1}{r+1}(\alpha - t)^{\underline{r+1}} + C$, *if $r \neq -1$, and both sides are well-defined.*

*(iii)* $\int e_p(t,a)\Delta t = \frac{1}{p}e_p(t,a) + C, p \neq 0, -\delta^{-1}$.

*Proof.* The result follows directly from definition 4.6.5, proposition 4.5.5, proposition 4.5.6 and proposition 4.5.7. □

We also have the fundamental theorem in the discrete case. This theorem is very powerful.

**Theorem 4.6.7** (**Fundamental theorem of Difference Calculus. [GP15, p. 31]**). *Assume $f : \mathbb{D}_a \to \mathbb{R}$, and $F$ is any discrete antiderivative of $f$. If $c, d \in \mathbb{D}_a$, we have*

$$\int_c^d f(t)\Delta t = \int_c^d \Delta F(t)\Delta t = F(d) - F(c).$$

*Proof.* Assume that $F(t)$ is a discrete antiderivative of $f(t)$. Let

$$G(t) \doteq \int_a^t f(s)\Delta s,\ s \in \mathbb{D}_a.$$

By theorem 4.6.2(vi), $G(t)$ is a discrete antiderivative of $f(t)$. By theorem 4.6.4 $F(t) = G(t) + C$. We then have

$$\begin{aligned} F(d) - F(c) &= [(G(d) + C) - (G(c) + C)] \\ &= G(d) - G(c) \\ &= \int_c^d f(t)\Delta t. \end{aligned}$$

Where we in the last step used theorem 4.6.2(iv), so that $\int_a^d f(t)\Delta t - \int_a^c f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^d f(t)\Delta t - \int_a^c f(t)\Delta t = \int_c^d f(t)\Delta t$.

□

Integration by parts can also be done, but notice that in the discrete case the $\sigma$-operator enters.

**Theorem 4.6.8** (**Integration by parts.** [**GP15, p. 32**]). *Given two functions $f, g : \mathbb{D}_a \to \mathbb{R}$ and $b, c \in \mathbb{D}_a, b < c$, we have the integration by parts formulas*

$$\int_c^d f(t)\mathrm{D}g(t)\Delta t = f(d)g(d) - f(c)g(c) - \int_c^d g(\sigma(t))\mathrm{D}f(t)\Delta t. \tag{4.14}$$

$$\int_c^d f(\sigma(t))\mathrm{D}g(t)\Delta t = f(d)g(d) - f(c)g(c) - \int_c^d g(t)\mathrm{D}f(t)\Delta t. \tag{4.15}$$

*Proof.* (i) We get from theorem 4.5.3(iv)

$$\mathrm{D}[gf](t) = g(\sigma(t))\mathrm{D}f(t) + \mathrm{D}g(t)f(t),$$

the result then follows from theorem 4.6.7.

(ii) We also get from theorem 4.5.3

$$\mathrm{D}[fg](t) = f(\sigma(t))\mathrm{D}g(t) + \mathrm{D}f(t)g(t).$$

Again the result follows from theorem 4.6.7.

□

## 4.7 Discrete Taylor's Theorem

In this section we will obtain the discrete version of Taylor's theorem.

**Definition 4.7.1** ([**GP15, Def. 1.60**]). *We define the discrete* ***Taylor monomials*** *(based at $s \in \mathbb{D}_a$), $h_n(t, s)$, $n \in \{0\} \cup \mathbb{N}$ by*

$$h_n(t, s) = \frac{(t-s)^{\underline{n}}}{n!}.$$

**Remark.** *The Taylor monomials are well-defined, because we have*

$$(t-s)^{\underline{n}} = \frac{\Gamma_\delta(t-s+\delta)}{\Gamma_\delta(t-s-(n-1)\delta)},$$

*and it is impossible that the numerator is a pole while the denominator is not a pole.*

**Theorem 4.7.2** ([**GP15, page 33**]). *The Taylor monomials satisfy the following properties:*

*(i)* $h_0(t, a) = 1$

*(ii)* $h_n(t, t) = 0, n \in \mathbb{N}$

*(iii)* $\mathrm{D}h_{n+1}(t, a) = h_n(t, a)$

*(iv)* $\int h_n(t, a)\Delta t = h_{n+1}(t, a) + C$

*(v)* $\mathrm{D}_s h_{n+1}(t,s) = -h_n(t,\sigma(s))$

*(vi)* $\int h_n(t,\sigma(s))\Delta s = -h_{n+1}(t,s) + C$.

*Proof.* (i) This follows since

$$h_0(t,a) = (t-a)^{\underline{0}} = \frac{\Gamma_\delta(t-a+\delta)}{\Gamma_\delta(t-a+\delta)} = 1.$$

Notice that the expression is 1, both if $t-a+\delta$ is a pole or not.

(ii)

$$h_n(t,t) = \frac{0^{\underline{n}}}{n!} = \frac{\Gamma_\delta(\delta)}{n!\Gamma_\delta(\delta-n\delta)} = 0.$$

Since $n \geq 1$, so the denominator is a pole.

(iii)

$$\mathrm{D}h_{n+1}(t,a) = \frac{\mathrm{D}(t-a)^{\underline{n+1}}}{(n+1)!} = \frac{(t-a)^{\underline{n}}}{n!} = h_n(t,a),$$

where we used proposition 4.5.5.

(iv) This follows from definition 4.6.5 and the previous point.

(v)

$$\mathrm{D}_s h_{n+1}(t,s) = \frac{\mathrm{D}_s(t-s)^{\underline{n+1}}}{(n+1)!} = -\frac{(t-\sigma(s))^{\underline{n}}}{n!} = -h_n(t,\sigma(s)).$$

Where we have used proposition 4.5.6.

(vi) This follows from definition 4.6.5 and the previous point.

□

We are now ready for the main result of this section.

**Theorem 4.7.3** (**Taylor's Formula [GP15, page 33]** )**.** *Assume $f : \mathbb{D}_a \to \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$f(t) = p_n(t) + R_n(t), t \in \mathbb{D}_a,$$

*where the n-th degree Taylor-polynomial, $p_n(t)$ is given by*

$$p_n(t) \doteq \sum_{k=0}^{n} [\mathrm{D}^k f](a) \cdot \frac{(t-a)^{\underline{k}}}{k!} = \sum_{k=0}^{n} \mathrm{D}^k f(a) h_k(t,a).$$

*The Taylor-remainder, $R_n(t)$, is given by*

$$R_n(t) = \int_a^t \frac{(t-\sigma(s))^{\underline{n}}}{n!} \mathrm{D}^{n+1} f(s)\Delta s = \int_a^t h_n(t,\sigma(s))\mathrm{D}^{n+1} f(s)\Delta s.$$

*Proof.* The proof will follow from induction. Assume first that $n = 0$, we then get

$$p_0(t) + R_0(t) = f(a) + \int_a^t \mathrm{D}f(s)\Delta s = f(a) + f(t) - f(a) = f(t).$$

Now assume that the representation holds for $n$ we must show that it holds for $n+1$. We will use the integration by parts formula eq. (4.14) with a different notation:

$$\int_a^t h(s)\mathrm{D}g(s)\Delta s = h(t)g(t) - h(a)g(a) - \int_a^t g(\sigma(s))\mathrm{D}h(s)\Delta s.$$

Define

$$h(s) \doteq \mathrm{D}^{n+1}f(s)$$
$$g(s) \doteq -h_{n+1}(t,s).$$

From theorem 4.7.2(v), we get that $\mathrm{D}g(s) = h_n(t,\sigma(s))$. The integration by parts formula now gives us

$$\begin{aligned}
&\int_a^t h_n(t,\sigma(s))\mathrm{D}^{n+1}f(s)\Delta s \\
&= -h_{n+1}(t,t)\mathrm{D}^{n+1}f(t) + h_{n+1}(t,a)\mathrm{D}^{n+1}f(a) + \int_a^t h_{n+1}(t,\sigma(s))\mathrm{D}^{n+2}f(s)\Delta s \\
&= h_{n+1}(t,a)\mathrm{D}^{n+1}f(a) + \int_a^t h_{n+1}(t,\sigma(s))\mathrm{D}^{n+2}f(s)\Delta s.
\end{aligned} \tag{4.16}$$

By the induction hypothesis we have

$$f(t) = \sum_{k=0}^{n} \mathrm{D}^k f(a)h_k(t,a) + \int_a^t h_n(t,\sigma(s))\mathrm{D}^{n+1}f(s)\Delta s.$$

If we put eq. (4.16) into this expression we get

$$\begin{aligned}
f(t) &= \sum_{k=0}^{n} \mathrm{D}^k f(a)h_k(t,a) + h_{n+1}(t,a)\mathrm{D}^{n+1}f(a) + \int_a^t h_{n+1}(t,\sigma(s))\mathrm{D}^{n+2}f(s)\Delta s \\
&= \sum_{k=0}^{n+1} \mathrm{D}^k f(a)h_k(t,a) + \int_a^t h_{n+1}(t,\sigma(s))\mathrm{D}^{n+2}f(s)\Delta s.
\end{aligned}$$

So we see that the expression also holds for $n+1$, and the proof is done. □

## 4.8 The Delta Laplace Transform

In this section we will introduce and prove properties for the discrete Laplace Transform within our framework.

**Definition 4.8.1** ([**GP15**, **p. 88**] ). *Assume that $f : \mathbb{D}_a \to \mathbb{R}$. Then we define the (delta)* ***Laplace transform*** *of $f$ based at $a$ by*

$$\mathcal{L}_a\{f\}(s) \doteq \sum_{t=a}^{\infty}{}_{\delta} \frac{f(t)\delta}{(1+\delta s)^{\frac{t-a}{\delta}+1}},$$

*for all complex numbers $s \neq -\delta^{-1}$ such that this sum converges.*

**Remark.** *Note that in definition 4.8.1, we do not require absolute convergence of the sum. But we will see that absolute convergence is something which we sometimes will obtain, see theorem 4.8.4 or theorem 4.8.7. The definition of the Laplace transform may seem arbitrary, but Goodrich defined it as the integral of $e_{\ominus s}(\sigma(t), a)f(t)$, this is somewhat inconsistent since the delta exponential was only defined for real functions, and hence we can not use complex $s$. This can be solved by allowing complex functions from the start, or do as we do, just define it as what it would have been if complex functions were allowed.*

**Theorem 4.8.2** (**adaptation of** [**GP15**, **Theorem 2.2 p. 88**]). *Assume $f : \mathbb{D}_a \to \mathbb{R}$. Then*

$$\mathcal{L}_a\{f\}(s) = F_a(s) \doteq \sum_{t=0}^{\infty}{}_{\delta} \frac{f(a+t)\delta}{(1+\delta s)^{\left(\frac{t}{\delta}+1\right)}} = \sum_{k=0}^{\infty} \frac{f(a+k\delta)\delta}{(1+\delta s)^{k+1}},$$

*whenever the infinite series converges and $s \neq -\delta^{-1}$.*

*Proof.* This follows directly by first making the substitution $t' = t - a$, and then the substitution $k = t'/\delta$. We get

$$\begin{aligned}\mathcal{L}_a\{f\}(s) &= \sum_{t=a}^{\infty}{}_{\delta} \frac{f(t)\delta}{(1+\delta s)^{\left(\frac{t-a}{\delta}+1\right)}} \\ &= \sum_{t'=0}^{\infty}{}_{\delta} \frac{f(a+t')\delta}{(1+\delta s)^{\left(\frac{t'}{\delta}+1\right)}} \\ &= \sum_{k=0}^{\infty} \frac{f(a+k\delta)\delta}{(1+\delta s)^{k+1}}.\end{aligned}$$

□

.

**Definition 4.8.3** ([**GP15**, **Def. 2.3**]). *We say that $f : \mathbb{D}_a \to \mathbb{R}$ is of exponential order $r > 0$ if there exists a constant $A > 0$ such that*

$$|f(t)| \leq Ar^t,$$

*for $t$ sufficiently large.*

**Theorem 4.8.4** (**adaptation of** [**GP15**, **Theorem 2.4 p. 89**]). *Suppose $f : \mathbb{D}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then $\mathcal{L}_a\{f\}(s)$ converges absolutely for $|1 + \delta s| > r^{\delta}$.*

*Proof.* Since $f$ is of exponential order, there exists a $K$ so that if $k \geq K$, we have

$$|f(a+k\delta)| \leq Ar^{a+k\delta}.$$

If we in the proof of theorem 4.8.2 use absolute values instead, we get

$$\sum_{t=a}^{\infty}{}_{\delta} \left| \frac{f(t)\delta}{(1+\delta s)^{\frac{t-a}{\delta}+1}} \right| = \sum_{k=0}^{\infty} \frac{|f(a+k\delta)|\delta}{|1+\delta s|^{k+1}}.$$

We must show that the last sum converges for all $|1+\delta s| > r^{\delta}$. We get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{|f(a+k\delta)|\delta}{|1+\delta s|^{k+1}} \\
&= \sum_{k=0}^{K-1} \frac{|f(a+k\delta)|\delta}{|1+\delta s|^{k+1}} + \sum_{k=K}^{\infty} \frac{|f(a+k\delta)|\delta}{|1+\delta s|^{k+1}} \\
&\leq C + \sum_{k=K}^{\infty} \frac{Ar^{a+k\delta}\delta}{|1+\delta s|^{k+1}} = C + \frac{Ar^{a}\delta}{|1+\delta s|} \sum_{k=K}^{\infty} \left( \frac{r^{\delta}}{|1+\delta s|} \right)^{k} \\
&< \infty.
\end{aligned}$$

The last part is less than infinity since we end up with a geometric series and $\frac{r^{\delta}}{|1+\delta s|} < 1$. ☐

**Theorem 4.8.5** (**adaptation of [GP15, Example 2.6 p. 90]**). *Assume $p \neq \delta^{-1}$ is a real constant. Then $e_p(t,a)$ is of exponential order $|1+\delta p|^{\frac{1}{\delta}}$, and $\mathcal{L}_a\{e_p(t,a)\}(s) = \frac{1}{s-p}$ for $|1+\delta s| > |1+\delta p|$.*

*Proof.* By theorem 4.3.4

$$e_p(t,a) = (1+\delta p)^{\frac{t-a}{\delta}}.$$

So

$$|e_p(t,a)| = |(1+\delta p)|^{\frac{t-a}{\delta}} = |1+\delta p|^{\frac{-a}{\delta}} \left( |1+\delta p|^{\frac{1}{\delta}} \right)^{t}.$$

Hence it is of exponential order $|1+\delta p|^{\frac{1}{\delta}}$.

We also get with the aid of theorem 4.8.2

$$\begin{aligned}
\mathcal{L}_a\{e_p(t,a)\} &= \sum_{k=0}^{\infty} \frac{e_p(a+k\delta)\delta}{(1+\delta s)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{(1+\delta p)^k\delta}{(1+\delta s)^{k+1}} \\
&= \frac{\delta}{1+\delta s}\sum_{k=0}^{\infty}\left(\frac{1+\delta p}{1+\delta s}\right)^k \\
&= \frac{\delta}{1+\delta s}\left(\frac{1}{1-\frac{1+\delta p}{1+\delta s}}\right) \\
&= \frac{\delta}{1+\delta s-1-\delta p} \\
&= \frac{1}{s-p}.
\end{aligned}$$

Notice that we do not divide by zero because $s \neq p$ since $|1+\delta s| > |1+\delta p|$. □

**Theorem 4.8.6 (Linearity, [GP15, Theorem 2.6]).** *Suppose $f, g : \mathbb{D}_a \to \mathbb{R}$ and for a given $s \in \mathbb{C}$ the Laplace Transforms $\mathcal{L}_a\{f\}(s)$ and $\mathcal{L}_a\{g\}(s)$ exist. Let $c_1, c_2 \in \mathbb{R}$. Then $\mathcal{L}_a\{c_1 f + c_2 f_2\}(s)$ exist for this particular $s$, and*

$$\mathcal{L}_a\{c_1 f + c_2 f_2\}(s) = c_1\mathcal{L}_a\{f\}(s) + c_2\mathcal{L}_a\{g\}(s).$$

*Proof.* From elementary theory of series we have that since

$$\sum_{k=0}^{\infty} \frac{f(a+k\delta)\delta}{(1+\delta s)^{k+1}} \text{ and } \sum_{k=0}^{\infty} \frac{g(a+k\delta)\delta}{(1+\delta s)^{k+1}},$$

converges (conditionally), we get that

$$\sum_{k=0}^{\infty} \frac{c_1 f(a+k\delta) + c_2 c_1 g(a+k\delta)}{(1+\delta s)^{k+1}}\delta$$

converges (conditionally). And

$$\sum_{k=0}^{\infty} \frac{c_1 f(a+k\delta) + c_2 c_1 g(a+k\delta)}{(1+\delta s)^{k+1}}\delta = c_1\sum_{k=0}^{\infty} \frac{f(a+k\delta)\delta}{(1+\delta s)^{k+1}} + c_2\sum_{k=0}^{\infty} \frac{g(a+k\delta)\delta}{(1+\delta s)^{k+1}}.$$

The result follows by the representation in theorem 4.8.2. □

When working with infinite sums it is desirable that they converge absolutely, so we will now prove a theorem that guarantees absolute convergence in some cases. It will be proved with the help of the theory of (complex) power-series.

**Theorem 4.8.7.** *Assume that $f : \mathbb{D}_a \to \mathbb{R}$, and that there exists a real number $r > 0$ so that for all complex numbers $s$ satisfying $|1+\delta s| > r^{\delta}$, the Laplace Transform $\mathcal{L}_a\{f\}(s)$ exist. Then the Laplace Transform converges absolutely for these values of $s$ as well.*

*Proof.* We must show that

$$\sum_{k=0}^{\infty} \frac{|f(a+k\delta)\delta|}{|1+\delta s|^{k+1}} < \infty, |1+\delta s| > r^{\delta}.$$

Which is equivalent to showing that

$$\sum_{k=0}^{\infty} \frac{|f(a+k\delta)|}{|1+\delta s|^{k}} < \infty, |1+\delta s| > r^{\delta},$$

since multiplication by the factor $\frac{\delta}{|1+\delta s|}$ does not change the convergence properties. Define the power-series

$$h(z) \doteq \sum_{k=0}^{\infty} a_k z^k, a_k = f(a+k\delta), z \in \mathbb{C}.$$

We will first show that the series converges conditionally if $|z| < r^{-\delta}$. If $z = 0$ it obviously converges. If $z \neq 0$, we can solve the equation

$$z = \frac{1}{1+\delta s},$$

which gives

$$s = \frac{1}{\delta z} - \frac{1}{\delta}.$$

Since $|z| < r^{-\delta}$, we have that $|1+\delta s| > r^{\delta}$. From the hypothesis we have that

$$\sum_{k=0}^{\infty} \frac{f(a+k\delta)\delta}{(1+\delta s)^{k+1}},$$

converges conditionally. But that also means that

$$\sum_{k=0}^{\infty} \frac{f(a+k\delta)}{(1+\delta s)^{k}} = \sum_{k=0}^{\infty} a_k z^k.$$

converges conditionally.

From the general theory of complex power-series, see Theorem 2.5 p. 15 in [SS03], we get that h must have a radius of convergence $R$ such that $R \geq r^{-\delta}$. So for all $z < r^{-\delta}$, we have that $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely. So if we are given $s$, with $|1+\delta s| > r^{\delta}$, we set $z = \frac{1}{1+\delta s}$. Then $|z| < r^{-\delta}$, and hence

$$\sum_{k=0}^{\infty} \frac{|f(a+k\delta)|}{|1+\delta s|^{k}} = \sum_{k=0}^{\infty} |a_k||z|^k < \infty.$$

□

We now show a uniqueness result.

**Theorem 4.8.8 (Uniqueness, adaptation of [GP15, Theorem 2.7 p. 91]).** *Let $f, g : \mathbb{D}_a \to \mathbb{R}$ and let $r > 0$ be a real number. Assume that for all $s \in \mathbb{C}$, such that $|1 + \delta s| > r^{\delta}$ both $\mathcal{L}_a\{f\}(s)$ and $\mathcal{L}_a\{g\}(s)$ exist and*

$$\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s).$$

*We then have*

$$f(t) = g(t), \ \forall t \in \mathbb{D}_a.$$

*Proof.* By hypothesis we have that for all $|1 + \delta s| > r^{\delta}$

$$\sum_{k=0}^{\infty} \frac{f(a + \delta k)\delta}{(1 + \delta s)^{k+1}} = \sum_{k=0}^{\infty} \frac{g(a + \delta k)\delta}{(1 + \delta s)^{k+1}}. \tag{4.17}$$

From theorem 4.8.7 we have that these sums converge absolutely as well. We first show that $f(a) = g(a)$. By dividing eq. (4.17) with $\frac{\delta}{1+\delta s}$ we get

$$\sum_{k=0}^{\infty} \frac{f(a + \delta k)}{(1 + \delta s)^{k}} = \sum_{k=0}^{\infty} \frac{g(a + \delta k)}{(1 + \delta s)^{k}}, \tag{4.18}$$

and we still have absolute convergence for $|1+\delta s| > r^{\delta}$. Let $N \in \mathbb{N}, 1+\delta N > r^{\delta}$. If we can show that both

$$\sum_{k=1}^{\infty} \frac{|f(a + \delta k)|}{(1 + \delta N)^{k}} \text{ and } \sum_{k=1}^{\infty} \frac{|g(a + \delta k)|}{(1 + \delta N)^{k}},$$

goes to 0 as $N$ goes to infinity, we will be done. For each $N$ both sums are well-defined because of the absolute convergence, they both decrease when N increases, and they are bounded below by 0. By the completeness of the real numbers we must therefore show that they can become arbitrary close to 0. To show this for the first sum, let $\epsilon > 0$ be arbitrary, let $N'$ be such that $1 + \delta N' > r^{\delta}$. Since

$$\sum_{k=1}^{\infty} \frac{|f(a + \delta k)|}{(1 + \delta N')^{k}}$$

converges absolutely, there is an $M \in \mathbb{N}, M > 1$ so that

$$\sum_{k=M}^{\infty} \frac{|f(a + \delta k)|}{(1 + \delta N')^{k}} < \epsilon/2.$$

We now get that

$$\sum_{k=1}^{\infty} \frac{|f(a + \delta k)|}{(1 + \delta N)^{k}} = \sum_{k=1}^{M-1} \frac{|f(a + \delta k)|}{(1 + \delta N)^{k}} + \sum_{k=M}^{\infty} \frac{|f(a + \delta k)|}{(1 + \delta N)^{k}}.$$

We can get the first term less than $\epsilon/2$ by increasing $N$ sufficiently since it is a finite sum, the second sum will be less than $\epsilon/2$ if we choose $N > N'$. Hence $\sum_{k=1}^{\infty} \frac{|f(a+\delta k)|}{(1+\delta N)^{k}}$ goes to 0 as $N$ goes to infinity. The same with $\sum_{k=1}^{\infty} \frac{|g(a+\delta k)|}{(1+\delta N)^{k}}$. Hence $f(a) = g(a)$.

Now let $K \in \mathbb{N}_0$, and assume that for $k = 0, 1, \dots, K$ $f(a + k\delta) = g(a + k\delta)$. To prove the theorem by induction we must prove that $f(a + (K + 1)\delta) = g(a + (K + 1)\delta)$. By eq. (4.18)

$$\sum_{k=K+1}^{\infty} \frac{f(a+\delta k)}{(1+\delta s)^k} = \sum_{k=K+1}^{\infty} \frac{g(a+\delta k)}{(1+\delta s)^k}. \tag{4.19}$$

If we multiply by $(1 + \delta s)^{K+1}$, we get(while preserving the absolute convergence):

$$\sum_{k=K+1}^{\infty} \frac{f(a+\delta k)}{(1+\delta s)^{k-K-1}} = \sum_{k=K+1}^{\infty} \frac{g(a+\delta k)}{(1+\delta s)^{k-K-1}}. \tag{4.20}$$

By changing the summation limits this is equal to

$$\sum_{k=0}^{\infty} \frac{f(a+(K+1)\delta+\delta k)}{(1+\delta s)^k} = \sum_{k=0}^{\infty} \frac{g(a+(K+1)\delta+\delta k)}{(1+\delta s)^k}. \tag{4.21}$$

But if we let $a' = a + (K + 1)\delta$, the exact same argument as earlier in the proof shows that $f(a') = g(a')$, hence $f(a + (K + 1)\delta) = g(a + (K + 1)\delta)$.

□

## 4.9 Fractional Sums and Differences

We now generalize some concepts from section 2.3 in [GP15].

**Theorem 4.9.1** (**Adaptation of [GP15, Theorem 2.23 p. 99]**). *Let $f : \mathbb{D}_a \to \mathbb{R}$ be given, then*

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1 = \int_a^t h_{n-1}(t, \sigma(s))f(s)\Delta s. \tag{4.22}$$

*Proof.* The proof will follow by induction. If $n = 1$ the result follows since by theorem 4.7.2 point (i), $h_{n-1}(t, \sigma(s)) = 1$.

Assume now that the representation holds for $n$, we must show that it also holds for $n + 1$. We want to show that:

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_n} f(\tau_{n+1})\Delta\tau_{n+1} \cdots \Delta\tau_2\Delta\tau_1 = \int_a^t h_n(t, \sigma(s))f(s)\Delta s.$$

First let

$$g(\tau_n) = \int_a^{\tau_n} f(\tau_{n+1})\Delta\tau_{n+1}.$$

We then get

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_n} f(\tau_{n+1})\Delta\tau_{n+1} \cdots \Delta\tau_2\Delta\tau_1$$
$$= \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} g(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1.$$

By using the induction hypothesis we get

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} g(\tau_n)\Delta\tau_n \cdots \Delta\tau_2\Delta\tau_1 = \int_a^t h_{n-1}(t,\sigma(s))g(s)\Delta s.$$

We define $r(s) \doteq g(s)$, and $p(s) \doteq -h_n(t,s)$. By theorem 4.6.2 point (vi) we get that $\mathrm{D}r(s) = f(s)$, and by theorem 4.7.2 point (v) we have that $\mathrm{D}p(s) = h_{n-1}(t,\sigma(s))$. The integration by parts formula eq. (4.14) states (in different notation) that

$$\int_a^t r(s)\mathrm{D}p(s)\Delta s = r(t)p(t) - r(a)p(a) - \int_a^t p(\sigma(s))\mathrm{D}r(s)\Delta s.$$

Hence

$$\begin{aligned}
&\int_a^t h_{n-1}(t,\sigma(s))g(s)\Delta s \\
&= g(t)\cdot(-h_n(t,t)) - g(a)\cdot(-h_n(t,a)) - \int_a^t -h_n(t,\sigma(s))f(s)\Delta s \\
&= \int_a^t h_n(t,\sigma(s))f(s)\Delta s.
\end{aligned}$$

□

**Definition 4.9.2** (**[GP15, Def. 2.24]**). *Let $\nu \neq -1,-2,-3\dots$. Then the $\nu$-th fractional Taylor monomial based at $s$ is defined by*

$$h_\nu(t,s) = \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)},$$

*whenever the right-hand side is well-defined.*

**Remark.** *Since $\nu$ is not a negative integer, the right-hand side of definition 4.9.2 being well-defined only depends on $(t-s)^{\underline{\nu}}$ being well-defined.*

**Theorem 4.9.3** (**Adaptation of [GP15, Theorem 2.27 p. 101]**). *Let $t,s \in \mathbb{D}_a, \nu \in \mathbb{C}\backslash(\{0\} \cup -\mathbb{N})$. Then*

*(i)* $\mathrm{D}h_\nu(t,a) = h_{\nu-1}(t,a)$

*(ii)* $\mathrm{D}_s h_\nu(t,s) = -h_{\nu-1}(t,\sigma(s))$

*(iii)* $\int h_\nu(t,s)\Delta t = h_{\nu+1}(t,a) + C$

*(iv)* $\int h_\nu(t,\sigma(s))\Delta s = -h_{\nu+1}(t,s) + C$

*whenever both sides of the equations are well-defined.*

*Proof.* (i)

$$\mathrm{D}h_\nu(t,a) = \frac{\mathrm{D}(t-a)^{\underline{\nu}}}{\Gamma(\nu+1)} = \frac{(t-a)^{\underline{\nu-1}}}{\Gamma(\nu)} = h_{\nu-1}(t,a),$$

where we used proposition 4.5.5.

(ii)

$$\mathrm{D}_s h_\nu(t,s) = \frac{\mathrm{D}_s(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)} = -\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} = -h_{\nu-1}(t,\sigma(s)),$$

where we have used proposition 4.5.6.

(iii) This follows from definition 4.6.5 and point (i).

(iv) This follows from definition 4.6.5 and point (ii).

□

**Definition 4.9.4 (modification of [GP15, Def 2.25, p. 101]).** *Assume $f : \mathbb{D}_a \to \mathbb{R}$ and $\nu > 0$. Then the $\nu$-th fractional sum of $f$ (based at $a$) is defined by*

$$\mathrm{D}_a^{-\nu} f(t) \doteq \int_a^{t-\nu\delta+\delta} h_{\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau = \sum_{\tau=a}^{t-\nu\delta}{}_\delta h_{\nu-1}(t,\sigma(\tau))f(\tau)\delta,$$

*for $t \in \mathbb{D}_{a+\nu\delta}$. We define $\mathrm{D}_a^0 f(t) = f(t)$.*

**Remark.** *Note that definition 4.9.4 is well-defined. This is because since $t \in \mathbb{D}_{a+\nu\delta}$, we have $t = a + \nu\delta + N\delta, N \in \{0\} \cup \mathbb{N}$. So the integral limits are $a$ and $a + (N+1)\delta$. We have that the values of $\tau$ are $a, a+\delta, \ldots a+N\delta$, and we have that $h_{\nu-1}(t,\sigma(\tau))$ is well-defined for these values because*

$$h_{\nu-1}(t,\sigma(\tau)) = \frac{1}{\Gamma(\nu)} \frac{\Gamma_\delta(t-\tau)}{\Gamma_\delta(t-\tau-\nu\delta+\delta)},$$

*and $t-\tau$ is never a pole for these values of $\tau$.*

We define differentiation of a non-integer order.

**Definition 4.9.5 (modification of [GP15, Def 2.29, p. 103]).** *Assume $f : \mathbb{D}_a \to \mathbb{R}, \nu > 0$, choose a positive integer $N$ such that $N-1 < \nu \leq N$. Then e define the $\nu$-th fractional derivative by*

$$\mathrm{D}_a^\nu f(t) \doteq \mathrm{D}^N \mathrm{D}_a^{-(N-\nu)} f(t), t \in \mathbb{D}_{a+(N-\nu)\delta}.$$

**Lemma 4.9.6 (Leibniz Formulas, adaptation of [GP15, Lemma 2.32 p. 103]).** *Assume $\mu \in \mathbb{R}$, $f : \mathbb{D}_{a+\mu\delta} \times \mathbb{D}_a \to \mathbb{R}$. Then*

$$\mathrm{D}\left[\int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau\right] = \int_a^{t-\mu\delta+\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + f(t+\delta, t-\mu\delta+\delta). \quad (4.23)$$

*and*

$$\mathrm{D}\left[\int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau\right] = \int_a^{t-\mu\delta+2\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + f(t, t-\mu\delta+\delta). \quad (4.24)$$

*For $t \in \mathbb{D}_{a+\delta\mu}$.*

*Proof.*

$$\mathrm{D}\left[\int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau\right] = \frac{\int_a^{t+\delta-\mu\delta+\delta} f(t+\delta,\tau)\Delta\tau - \int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau}{\delta}$$

$$= \int_a^{t-\mu\delta+\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + \frac{\int\limits_{t-\mu\delta+\delta}^{t-\mu\delta+2\delta} f(t+\delta,\tau)\Delta\tau}{\delta}$$

$$= \int_a^{t-\mu\delta+\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + f(t+\delta, t-\mu\delta+\delta).$$

We also get

$$\mathrm{D}\left[\int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau\right] = \frac{\int_a^{t+\delta-\mu\delta+\delta} f(t+\delta,\tau)\Delta\tau - \int_a^{t-\mu\delta+\delta} f(t,\tau)\Delta\tau}{\delta}$$

$$= \int_a^{t-\mu\delta+2\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + \frac{\int\limits_{t-\mu\delta+\delta}^{t-\mu\delta+2\delta} f(t,\tau)\Delta\tau}{\delta}$$

$$= \int_a^{t-\mu\delta+2\delta} \mathrm{D}_t f(t,\tau)\Delta\tau + f(t, t-\mu\delta+\delta).$$

□

**Theorem 4.9.7** (**Adaptation of [GP15, Theorem 2.33 p. 104]**). *Let $f : \mathbb{D}_a \to \mathbb{R}, \nu > 0$, and let $N \in \mathbb{N}$ be such that $N-1 < \nu \leq N$. Then*

$$\mathrm{D}_a^\nu f(t) = \begin{cases} \int_a^{t+\nu\delta+\delta} h_{-\nu-1}(t,\sigma(\tau))f(\tau)\Delta\tau, & N-1<\nu<N \\ \mathrm{D}^N f(t), & \nu = N, \end{cases} \tag{4.25}$$

*for $t \in \mathbb{D}_{a+(N-\nu)\delta}$.*

*Proof.* If $\nu = N$ we have

$$\mathrm{D}_a^\nu f(t) = \mathrm{D}^N \mathrm{D}_a^0 f(t) = \mathrm{D}^N f(t).$$

If $\nu$ is not an integer, we will prove the statement by induction on $N$. Assume first that $0 < \nu < 1$. We then get

$$\mathrm{D}_a^\nu f(t) = \mathrm{D}\mathrm{D}_a^{-(1-\nu)} f(t) = D\left[\int_a^{t-(1-\nu)\delta+\delta} h_{-\nu}(t,\sigma(\tau))f(\tau)\Delta\tau\right].$$

With the help of Leibniz formula eq. (4.24) we get

$$D\left[\int_a^{t-(1-\nu)\delta+\delta} h_{-\nu}(t,\sigma(\tau))f(\tau)\Delta\tau\right]$$
$$= \int_a^{t-(1-\nu)\delta+2\delta} h_{-1-\nu}(t,\sigma(\tau))f(\tau)\Delta\tau + h_{-\nu}(t, t-(1-\nu)\delta+2\delta)f(t-(1-\nu)\delta+\delta)$$
$$= \int_a^{t+\nu\delta+\delta} h_{-1-\nu}(t,\sigma(\tau))f(\tau)\Delta\tau + h_{-\nu}(t, t+\nu\delta+\delta)f(t+\nu\delta).$$

We have that

$$h_{-\nu}(t, t+\nu\delta+\delta) = \frac{(-\nu\delta-\delta)^{\underline{-\nu}}}{\Gamma(1-\nu)} = \frac{1}{\Gamma(1-\nu)}\frac{\Gamma_\delta(-\nu\delta)}{\Gamma_\delta(0)} = 0.$$

We get that the last expression is zero, because we have a pole in the denominator, but not in the numerator(remember that $0 < \nu < 1$). Hence the result is true if $N = 1$.

Assume now that the statement is true if $N - 1 < \nu \leq N$. We must show that it is also true if $N < \nu \leq N + 1$. The case $\nu = N + 1$ has already been proved at the very start of the proof, so we assume that $N < \nu < N + 1$. Let $\nu' = \nu - 1$, then $N - 1 < \nu' < N$. We also get

$$\mathrm{D}_a^\nu f(t) = \mathrm{D}^{N+1}\mathrm{D}^{-(N+1-\nu)} f(t) = \mathrm{D}\mathrm{D}^N \mathrm{D}_a^{-(N-\nu')} f(t) = \mathrm{D}\mathrm{D}_a^{\nu'} f(t).$$

From the induction hypothesis we have

$$\mathrm{D}_a^{\nu'} f(t) = \int_a^{t+\nu'\delta+\delta} h_{-1-\nu'}(t, \sigma(\tau)) f(\tau)\Delta\tau.$$

So we get

$$\mathrm{D}_a^\nu f(t) = D\left[\int_a^{t-(1-\nu)\delta+\delta} h_{-\nu}(t, \sigma(\tau)) f(\tau)\Delta\tau\right].$$

We use Leibniz Formula eq. (4.24) again

$$D\left[\int_a^{t-(1-\nu)\delta+\delta} h_{-\nu}(t, \sigma(\tau)) f(\tau)\Delta\tau\right]$$
$$= \int_a^{t+\nu\delta+\delta} h_{-1-\nu}(t, \sigma(\tau)))\Delta\tau + h_{-\nu}(t, t-(1-\nu)\delta+2\delta) f(t-(1-\nu)\delta+\delta).$$

However, we have

$$\begin{aligned} h_{-\nu}(t, t-(1-\nu)\delta+2\delta) =& h_{-\nu}(t, t+\nu\delta+\delta) \\ =& \frac{(-\nu\delta-\delta)^{\underline{-\nu}}}{\Gamma(1-\nu)} \\ =& \frac{1}{\Gamma(1-\nu)}\frac{\Gamma_\delta(-\nu\delta)}{\Gamma_\delta(0)} = 0, \end{aligned}$$

since the denominator is a pole, while the numerator is not a pole. □

# Chapter 5

# The Falling Mandelbrot and Van Ness sum

In this chapter we will combine results from chapter 3 and chapter 4. However, we will only use the two first sections, section 4.1 and section 4.2 from chapter 4. A brief explanation of what we will do is that we will replace $H - 1/2$ in the Mandelbrot and Van Ness sum, see for instance eq. (3.5), with $\underline{H - 1/2}$. That is, we will work with the falling counterpart.

We also recall from chapter 4 that the set $\mathbb{D}_a$ denotes

$$\{a, a + \delta, a + 2\delta, \ldots\},$$

where the set also depends on an underlying $\delta > 0$ which is known. In this chapter we will require that $a = 0$. This requirement is parallel with the work in chapter 3 where even though the Mandelbrot and Van Ness sum was defined on $[0, \infty)$, we used linear interpolation between the points in the set

$$\{0, \delta, 2\delta, \ldots\}.$$

## 5.1 Definition of the falling Mandelbrot and Van Ness sum

We have to do some work resembling what we did in section 3.2. It is not as easy as to just replace $H - 1/2$ with $\underline{H - 1/2}$ in the Mandelbrot and Van Ness sum, we have to check that everything will be well-defined.

We start with a lemma which tells us that the falling power functions and regular power functions are close when the argument is large. Remember that the falling function is defined in definition 4.2.4.

**Lemma 5.1.1.** *Let $r \in \mathbb{R}, \delta > 0$. We then have*

$$\lim_{t\to\infty} \frac{t^{\underline{r}}}{t^r} = 1,$$

*where $t \in \mathbb{D}_0$.*

*Proof.* First note that since

$$t^{\underline{r}} \doteq \frac{\Gamma_\delta(t + \delta)}{\Gamma_\delta(t - (r - 1)\delta)},$$

there is no problems with poles. This is because when $t$ is large enough $t-(r-1)\delta$ will not be a pole, and $t+\delta$ is never a pole. So we have

$$\begin{aligned}\frac{t^{\underline{r}}}{t^r} &= \frac{\Gamma_\delta(t+\delta)}{\Gamma_\delta(t+\delta-r\delta)\cdot t^r} \\ &= \frac{\delta^{\frac{t}{\delta}+1}\Gamma(\frac{t}{\delta}+1)}{\delta^{\frac{t}{\delta}+1-r}\Gamma(\frac{t}{\delta}+1-r)\cdot t^r} \\ &= \frac{\delta^r}{t^r}\cdot\frac{\Gamma(\frac{t}{\delta}+1)}{\Gamma(\frac{t}{\delta}+1-r)}.\end{aligned} \tag{5.1}$$

Notice that the last in expression eq. (5.1) is well-defined for all large values of $t$. So when calculating the limit we do not need to assume that $t \in \mathbb{D}_0$, but we can use standard techniques. If we find that the limit of the last expression exist when $t \to \infty$, the limit will also exist when $t \to \infty$ and we restrict $t$ to $\mathbb{D}_a$.

We will also use Stirling's approximation for Gamma functions, see [Tem15, p. 65]. Stirling's formula tells us that

$$\lim_{t\to\infty}\frac{\Gamma(t)}{\sqrt{2\pi}t^{t-\frac{1}{2}}e^{-t}} = 1.$$

We get with the aid of eq. (5.1)

$$\begin{aligned}&\lim_{t\to\infty}\frac{t^{\underline{r}}}{t^r} \\ &= \lim_{t\to\infty}\frac{\delta^r}{t^r}\cdot\frac{\Gamma(\frac{t}{\delta}+1)}{\Gamma(\frac{t}{\delta}+1-r)} \\ &= \lim_{t\to\infty}\frac{\delta^r}{t^r}\cdot\frac{\frac{\Gamma(\frac{t}{\delta}+1)}{\sqrt{2\pi}(\frac{t}{\delta}+1)^{\frac{t}{\delta}+\frac{1}{2}}e^{-(\frac{t}{\delta}+1)}}\cdot\sqrt{2\pi}(\frac{t}{\delta}+1)^{\frac{t}{\delta}+\frac{1}{2}}e^{-(\frac{t}{\delta}+1)}}{\frac{\Gamma(\frac{t}{\delta}+1-r)}{\sqrt{2\pi}(\frac{t}{\delta}+1-r)^{\frac{t}{\delta}+\frac{1}{2}-r}e^{-(\frac{t}{\delta}+1-r)}}\cdot\sqrt{2\pi}(\frac{t}{\delta}+1-r)^{\frac{t}{\delta}+\frac{1}{2}-r}e^{-(\frac{t}{\delta}+1-r)}}.\end{aligned}$$

By Stirling's formula this limit exist if and only if the limit

$$\lim_{t\to\infty}\frac{\delta^r}{t^r}\cdot\frac{\sqrt{2\pi}(\frac{t}{\delta}+1)^{\frac{t}{\delta}+\frac{1}{2}}e^{-(\frac{t}{\delta}+1)}}{\sqrt{2\pi}(\frac{t}{\delta}+1-r)^{\frac{t}{\delta}+\frac{1}{2}-r}e^{-(\frac{t}{\delta}+1-r)}},$$

exist, and upon existence they must be equal. We get

$$\begin{aligned}&\lim_{t\to\infty}\frac{\delta^r}{t^r}\cdot\frac{\sqrt{2\pi}(\frac{t}{\delta}+1)^{\frac{t}{\delta}+\frac{1}{2}}e^{-(\frac{t}{\delta}+1)}}{\sqrt{2\pi}(\frac{t}{\delta}+1-r)^{\frac{t}{\delta}+\frac{1}{2}-r}e^{-(\frac{t}{\delta}+1-r)}}, \\ &= \left(\frac{\delta}{e}\right)^r\lim_{t\to\infty}\left[\left(\frac{\frac{t}{\delta}+1}{\frac{t}{\delta}+1-r}\right)^{\frac{t}{\delta}+\frac{1}{2}}\cdot\left(\frac{\frac{t}{\delta}+1-r}{t}\right)^r\right].\end{aligned} \tag{5.2}$$

Since we can calculate the limit of a product as the product of the limit, we can split the expression inside the square bracket. We get

$$\begin{aligned}\lim_{t\to\infty}\left(\frac{\frac{t}{\delta}+1}{\frac{t}{\delta}+1-r}\right)^{\frac{t}{\delta}+\frac{1}{2}} &= \lim_{t\to\infty}\left(1+\frac{r}{\frac{t}{\delta}+1-r}\right)^{\frac{t}{\delta}+\frac{1}{2}} \\ &= \lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{x-\frac{1}{2}+r}.\end{aligned}$$

The last equality is valid because the substitution $x = t/\delta + 1 - r$ is continuous. We then get

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{x-\frac{1}{2}+r}\\
&=\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{x}\cdot\left(1+\frac{r}{x}\right)^{r-\frac{1}{2}}.
\end{aligned}$$

We have

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{r-\frac{1}{2}}=1,$$

and

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{x}=e^{r}$$

where the last limit is a well known limit from calculus. We now turn to the second limit in the square bracket in eq. (5.2).

$$\begin{aligned}
\lim_{t\to\infty}\left(\frac{\frac{t}{\delta}+1-r}{t}\right)^{r} &= \lim_{t\to\infty}\left(\frac{\frac{t}{\delta}+1-r}{\frac{t}{\delta}\cdot\delta}\right)^{r}\\
&=\delta^{-r}\lim_{t\to\infty}\left(\frac{\frac{t}{\delta}+1-r}{\frac{t}{\delta}}\right)^{r}\\
&=\delta^{-r}.
\end{aligned}$$

Hence we end up with

$$\begin{aligned}
\lim_{t\to\infty}\frac{t^{\underline{r}}}{t^{r}} &= \left(\frac{\delta}{e}\right)^{r}\lim_{t\to\infty}\left[\left(\frac{\frac{t}{\delta}+1}{\frac{t}{\delta}+1-r}\right)^{\frac{t}{\delta}+\frac{1}{2}}\cdot\left(\frac{\frac{t}{\delta}+1-r}{t}\right)^{r}\right]\\
&=\left(\frac{\delta}{e}\right)^{r}\cdot e^{r}\cdot\delta^{-r}\\
&=1.
\end{aligned}$$

□

We now make the analogous definition of the one in eq. (3.3).

**Definition 5.1.2.** *Let $\delta > 0, r \in (-0.5, 0.5)$. Assume that*

$$u \in \{\ldots, -3\delta, -2\delta, -\delta, 0, \delta, 2\delta, 3\delta, \ldots\}.$$

*Then we define*

$$u_{+}^{\underline{r}} \doteq \begin{cases} u^{\underline{r}} & \text{if } u > 0\\ 0 & \text{if } u \le 0.\end{cases} \tag{5.3}$$

**Remark.** *Notice that this definition is well-defined, because if $u > 0$ we have*

$$u_{+}^{\underline{r}} = \frac{\Gamma_{\delta}(u+\delta)}{\Gamma_{\delta}(u-(r-1)\delta)}.$$

*Notice that* $u + \delta \geq 2\delta$, *and*

$$\begin{aligned} u - (r-1)\delta &\geq \delta - (r-1)\delta \\ &= (2-r)\delta \\ &> 1.5\delta. \end{aligned}$$

*So we do not have any problems with poles.*

Now we make the parallel definition of lemma 3.2.4.

**Lemma 5.1.3.** *Assume that* $H \in (0,1)$, $\delta > 0$ *and that* $t = L\delta, L \in \mathbb{N} \cup \{0\}$. *We then have*

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}} \right]^2 < \infty.$$

*Proof.* Notice from definition 5.1.2, its remark and the fact that $(t-\tau)^{\underline{H-\frac{1}{2}}} = (t-\tau)_{+}^{\underline{H-\frac{1}{2}}}$ for our values of $\tau$, the sum is well-defined. We need to prove that it is not infinity. If $t = 0$, the result is obvious. If $t > 0$, then $t = L\delta, L \in \mathbb{N}$. We then get

$$\begin{aligned} &\sum_{r=-\infty}^{t-\delta}{}_{\delta} \left[ (t-r)^{\underline{H-\frac{1}{2}}} - (-r)_{+}^{\underline{H-\frac{1}{2}}} \right]^2 \\ =&\sum_{r=-\infty}^{L\delta-\delta}{}_{\delta} \left[ (L\delta-r)^{\underline{H-\frac{1}{2}}} - (-r)_{+}^{\underline{H-\frac{1}{2}}} \right]^2 \\ =&\sum_{r=-\infty}^{-\delta}{}_{\delta} \left[ (L\delta-r)^{\underline{H-\frac{1}{2}}} - (-r)_{+}^{\underline{H-\frac{1}{2}}} \right]^2 + \sum_{r=0}^{L\delta-\delta}{}_{\delta} \left[ (L\delta-r)^{\underline{H-\frac{1}{2}}} - (-r)_{+}^{\underline{H-\frac{1}{2}}} \right]^2 \\ \leq&\sum_{k=1}^{\infty} \left[ (L\delta+k\delta)^{\underline{H-\frac{1}{2}}} - (k\delta)^{\underline{H-\frac{1}{2}}} \right]^2 + M. \end{aligned}$$

Hence it is enough to consider convergence properties of the last series. We have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left[(L\delta+k\delta)^{\underline{H-\frac{1}{2}}}-(k\delta)^{\underline{H-\frac{1}{2}}}\right]^2 \\
=&\sum_{k=1}^{\infty}\Big[(L\delta+k\delta)^{\underline{H-\frac{1}{2}}}-((L-1)\delta+k\delta)^{\underline{H-\frac{1}{2}}} \\
&+((L-1)\delta+k\delta)^{\underline{H-\frac{1}{2}}}-((L-2)\delta+k\delta)^{\underline{H-\frac{1}{2}}} \\
&+((L-2)\delta+k\delta)^{\underline{H-\frac{1}{2}}}-((L-3)\delta+k\delta)^{\underline{H-\frac{1}{2}}} \\
&\vdots \\
&+((L-(L-1))\delta+k\delta)^{\underline{H-\frac{1}{2}}}-(k\delta)^{\underline{H-\frac{1}{2}}}\Big]^2 \\
=&\sum_{k=1}^{\infty}\left[\sum_{j=1}^{L}((j+k)\delta)^{\underline{H-\frac{1}{2}}}-((j+k-1)\delta)^{\underline{H-\frac{1}{2}}}\right]^2 \\
=&\sum_{k=1}^{\infty}\left[\sum_{j=1}^{L}\left(H-\frac{1}{2}\right)\delta\cdot((j+k-1)\delta)^{\underline{H-\frac{3}{2}}}\right]^2 \\
\leq&\ \delta^2L^2\left(H-\frac{1}{2}\right)^2\sum_{k=1}^{\infty}\sum_{j=1}^{L}\left(((j+k-1)\delta)^{\underline{H-\frac{3}{2}}}\right)^2.
\end{aligned}
$$

Where we get the third equality by using theorem 4.2.5, and we get to the last step by using the rough inequality $(\sum_{i=1}^{n}a_i)^2\leq n^2\sum_{i=1}^{n}a_i^2$. Since the terms are positive, we can change the order of summation and we must show that

$$\sum_{j=1}^{L}\sum_{k=1}^{\infty}\left(((j+k-1)\delta)^{\underline{H-\frac{3}{2}}}\right)^2<\infty.$$

Since the outer sum is finite it suffices to show

$$\sum_{k=1}^{\infty}\left(((j+k-1)\delta)^{\underline{H-\frac{3}{2}}}\right)^2<\infty, j\in\mathbb{N}.$$

We will obtain this result by using the limit comparison test with the corresponding series form ordinary calculus $\sum_{k=1}^{\infty}((j+k-1)\delta)^{2H-3}=\sum_{i=j}^{\infty}1/i^{3-2H}<\infty$, we have the convergence since $3-2H>1$. By the limit comparison test it suffices to show that

$$\lim_{k\to\infty}\frac{\left(((j+k-1)\delta)^{\underline{H-\frac{3}{2}}}\right)^2}{((j+k-1)\delta)^{2H-3}},$$

exists and is not equal to $\infty$. Obviously all terms are positive as the the limit comparison test requires. We have

$$\lim_{k\to\infty}\frac{\left(((j+k-1)\delta)_{\underline{\quad}}^{H-\frac{3}{2}}\right)^2}{((j+k-1)\delta)^{2H-3}} = \left(\lim_{k\to\infty}\frac{((j+k-1)\delta)_{\underline{\quad}}^{H-\frac{3}{2}}}{((j+k-1)\delta)^{H-\frac{3}{2}}}\right)^2$$
$$= 1^2$$
$$= 1.$$

Where we in the first equality use the fact that the function $x \to x^2$ from $[0,\infty) \to [0,\infty)$ is continuous, and that

$$\frac{((j+k-1)\delta)_{\underline{\quad}}^{H-\frac{3}{2}}}{((j+k-1)\delta)^{H-\frac{3}{2}}},$$

will be positive for large $k$ by lemma 5.1.1. The second equality also follows from lemma 5.1.1.

□

We follow the procedure as in section 3.2 and prove the parallel statement of proposition 3.2.5. Remember that $C_H$ is defined in proposition 3.2.2 and is a well-defined real number.

**Proposition 5.1.4.** *Let $\mathcal{W} = \{\dots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \dots\}$ be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Assume also that $H \in (0,1)$ and that $\delta > 0$. Then there exists a set $B_\delta \in \mathcal{A}$(also depending on $H$) with*

$$P(B_\delta) = 1,$$

*such that if $t \geq 0$ is a real number, and $t$ is a multiple of $\delta$, we have that for $\omega \in B_\delta$*

$$\frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)_{\underline{\quad}}^{H-\frac{1}{2}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}}\right]\sqrt{\delta}w_{\tau/\delta}(\omega)$$

*converges in $\mathbb{R}$.*

*Proof.* The proof will be almost the same as the proof of proposition 3.2.5. Assume first that $t$ is a multiple of $\delta$. From lemma 5.1.3 we have that

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\frac{1}{C_H^2}\left[(t-\tau)_{\underline{\quad}}^{H-\frac{1}{2}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}}\right]^2\delta < \infty.$$

It then follows [MW13, Proposition 7.11, p. 260] and its proof, that there is a set $B_{\delta,t} \in \mathcal{A}$ with $P(B_{\delta,t}) = 1$, such that for $\omega \in B_{\delta,t}$ we have that

$$\frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)_{\underline{\quad}}^{H-\frac{1}{2}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}}\right]\sqrt{\delta}w_{\tau/\delta}(\omega),$$

converges in $\mathbb{R}$.

There are only a countable number of $t \geq 0$ that is a multiple of $\delta_n$, let

$$B_\delta = \bigcap_{r \in \mathbb{N} \cup \{0\}} B_{\delta, r\delta}.$$

Because of countability and elementary properties of measures we have that $P(B_\delta) = 1$. By construction $B_\delta$ has the required properties, and the proof is done. ☐

Now we are ready to define the falling Mandelbrot and Van Ness sum. It will be very similar to definition 3.2.6.

**Definition 5.1.5.** *Let $H \in (0,1), \delta > 0$ be given. Let*

$$\mathcal{W} = \{\dots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \dots\}$$

*be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $B_\delta$ be as in proposition 5.1.4. Define the stochastic process $Y^{(\delta)} = (Y_t^{(\delta)})_{t \in [0,\infty)}$, which also depends on $H$ as follows:*

*(i) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$ we define*

$$Y_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega),$$

*where*

$$C_H \doteq \left( \int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right)^{1/2}.$$

*(ii) We extend $Y^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define*

$$Y_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot Y_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot Y_{(N+1)\delta}^{(\delta)}(\omega).$$

*We call $Y^{(\delta)}$ the falling Mandelbrot and Van Ness sum. Notice that the falling powers depends on the modified Gamma function which again depends on an underlying $\delta$, this $\delta$ should of course be chosen as the same $\delta$ at as in the start of the definition.*

**Remark.** *We have that $Y^{(\delta)}$ is a well-defined stochastic process on $(\Omega, \mathcal{A}, P)$. By construction, for each $t \in [0,\infty), \omega \in \Omega$ we have that $Y_t^{(\delta)}(\omega)$ is a well-defined real number. And if we keep $t$ fixed we have that $Y_t^{(\delta)}$ is a random variable on $(\Omega, \mathcal{A}, P)$ because it is a linear combination of well-defined limits of elements in $\mathcal{W}$, which are random variables on $(\Omega, \mathcal{A}, P)$. Also notice that $B_\delta \in \mathcal{A}$ by proposition 5.1.4.*

## 5.2 Some helpful results

We now state some results that will be of use in later sections. First we give a technical result which shows that for big arguments the ratio of gamma functions will be close to power functions. We will need this result later because as we remember, falling functions are made up from ratio of Gamma functions.

**Lemma 5.2.1.** *Let $H \in (0, 1)$, then*

$$\sum_{r=1}^{\infty} \left( r^{H-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma\left(r + \frac{3}{2} - H\right)} \right)^2 < \infty.$$

*Proof.* We will prove by comparing our series with the series

$$\sum_{r=1}^{\infty} \frac{1}{r^{3-2H}},$$

which converges since $3 - 2H > 1$. We will need a result found in [Tem96, pp. 66-67], where he first defines the function

$$\Gamma^*(z) = \frac{\Gamma(z)}{\sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}}, \qquad \text{Re}(z) > 0.$$

It is stated that $\Gamma^*$ is of the form $1 + \mathcal{O}(1/z)$ as $z \to \infty$. It is also said that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b} \frac{\Gamma^*(z+a)}{\Gamma^*(z+b)} Q(z, a, b),$$

$$Q(z, a, b) = \left(1 + \frac{a}{z}\right)^{a-\frac{1}{2}} \left(1 + \frac{b}{z}\right)^{-b+\frac{1}{2}} e^{\left[\ln\left(1+\frac{a}{z}\right) - \frac{a}{z} - \ln\left(1+\frac{b}{x}+\frac{b}{z}\right)\right]},$$

lastly it is stated that $Q(z, a, b)$ is also of the form $1+\mathcal{O}(1/z)$ as $z \to \infty$. Because of the definition of $\Gamma^*$ we assume that both $\text{Re}(z + a) > 0$ and $\text{Re}(z + b) > 0$. Note that when we say that a function $f$ is $\mathcal{O}(g(z))$, we mean that there exists positive, real numbers $M, N$ such that if $|z| \geq M$ we have

$$|f(z)| \leq N|g(z)|.$$

We will work with $z = r$, $a = 1$, $b = 3/2 - H$. All of these quantities are positive, so there is no problem with the definitions above. We have that

$$\Gamma^*(r + a) = 1 + f_1(r), \qquad f_1(r) = \mathcal{O}\left(\frac{1}{r+a}\right), \text{ as } r \to \infty,$$

$$\Gamma^*(r + b) = 1 + f_2(r), \qquad f_2(r) = \mathcal{O}\left(\frac{1}{r+b}\right), \text{ as } r \to \infty,$$

$$Q(r, a, b) = 1 + f_3(r), \qquad f_3(r) = \mathcal{O}\left(\frac{1}{r}\right), \text{ as } r \to \infty.$$

Where we also have constants $M_1, N_1, M_2, N_2, M_3, N_3$ from the definition of $\mathcal{O}$ above. We then get

$$\begin{aligned}\frac{\Gamma(r+a)}{\Gamma(r+b)} &= r^{a-b} \frac{\Gamma^*(r+a)}{\Gamma^*(r+b)} Q(r, a, b), \\ &= r^{a-b} \left( \frac{1 + f_1(r)}{1 + f_2(r)} \cdot (1 + f_3(r)) \right).\end{aligned}$$

Define $h(r)$ such that

$$\frac{\Gamma(r+a)}{\Gamma(r+b)} = z^{a-b}\left(1+h(r)\right).$$

We will show that $h = \mathcal{O}(1/r)$ as $r \to \infty$. We have

$$\begin{aligned}
|h(r)| &= \left|\frac{1+f_1(r)}{1+f_2(r)} \cdot (1+f_3(r)) - 1\right| \\
&= \left|\frac{1+f_1(r)+f_3(r)+f_1(r)f_3(r)-1-f_2(r)}{1+f_2(r)}\right| \\
&= \left|\frac{f_1(r)+f_3(r)-f_2(r)+f_1(r)f_3(r)}{1+f_2(r)}\right|
\end{aligned}$$

Since $f_2(r) \to 0$, we have that there exist a constant $K_1$ such that $|1+f_2(r)| > \frac{1}{2}$, when $r \geq K_1$. There must also be a $K_2$ such that $|f_3(r)| < 1$, when $r \geq K_2$. Now if $r \geq \max\{K_1, K_2, M_1, M_2, M_3\}$ we have

$$\begin{aligned}
&\left|\frac{f_1(r)+f_3(r)-f_2(r)+f_1(r)f_3(r)}{1+f_2(r)}\right| \\
&\leq 2\left|f_1(r)+f_3(r)-f_2(r)+f_1(r)f_3(r)\right| \\
&\leq 2|f_1(r)|+2|f_2(r)|+2|f_3(r)|+2|f_1(r)||f_3(r)| \\
&\leq 4|f_1(r)|+2|f_2(r)|+2|f_3(r)| \\
&\leq 4N_1 \cdot \frac{1}{|r+a|} + 2N_2\frac{1}{|r+b|} + 2N_3\frac{1}{r}.
\end{aligned}$$

Since $r/|r+a|$ and $r/|r+b|$ converges to 1 as r goes to infinity, there is a $K_3$ such that both these quantities are less than 2 if $r \geq K_3$. Then if $r \geq \max\{K_1, K_2, K_3, M_1, M_2, M_3\}$ we have

$$\begin{aligned}
4N_1 \cdot \frac{1}{|r+a|} + 2N_2\frac{1}{|r+b|} + 2N_3\frac{1}{r} &= 4N_1 \cdot \frac{1}{|r+a|}\frac{r}{r} + 2N_2\frac{1}{|r+b|}\frac{r}{r} + 2N_3\frac{1}{r} \\
&\leq \left(8N_1+4N_2+2N_3\right)\frac{1}{r}.
\end{aligned}$$

Hence we have shown that

$$\frac{\Gamma(r+a)}{\Gamma(r+b)} = r^{a-b}\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right) \tag{5.4}$$

as $r \to \infty$. We now turn back to what we really want to show. We will use that

$$\begin{aligned}
\frac{\Gamma(r+1)}{\Gamma\left(r+\frac{3}{2}-H\right)} &= r^{1-3/2+H}(1+\epsilon(r)) \\
&= r^{H-\frac{1}{2}}(1+\epsilon(r)),
\end{aligned} \tag{5.5}$$

where $\epsilon(r) = \mathcal{O}(1/r)$. We get

$$\begin{aligned}\limsup_{r\to\infty} \frac{\left(r^{H-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma\left(r+\frac{3}{2}-H\right)}\right)^2}{\frac{1}{r^{3-2H}}} &= \limsup_{r\to\infty} \frac{\left(r^{H-\frac{1}{2}} - r^{H-\frac{1}{2}}(1+\epsilon(r))\right)^2}{r^{2H-3}} \\ &= \limsup_{r\to\infty} \frac{\left(r^{H-\frac{1}{2}}\epsilon(r)\right)^2}{r^{2H-3}} \\ &= \limsup_{r\to\infty} r^2\epsilon(r)^2.\end{aligned}$$

Since $e(r) = \mathcal{O}(1/r)$, there exists a constant $K$ such that

$$\limsup_{r\to\infty} r^2\epsilon(r)^2 \leq \limsup_{r\to\infty} r^2\frac{K^2}{r^2} = K^2 < \infty.$$

By a version of the limit comparison test, which uses the lim sup condition, see [Jun15, 6.2.7 (a), p. 171], the result follows.

□

The next result is an important step in showing that $X^\delta$ and $Y^\delta$ are close. Notice the similarity with theorem 3.6.2. An important part of the proposition is that $M_H$ is independent of $t$.

**Proposition 5.2.2.** *Let $H \in (0,1), \delta > 0$. Assume also that*

$$\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$$

*is a collection of independent random variables, each taking the values $\pm 1$ with equal probability, and defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $X^{(\delta)}$ be as in definition 3.2.6, and $Y^{(\delta)}$ be as in definition 5.1.5, where $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ is the same in both cases. We then have for every $t \in [0,\infty)$ that*

$$E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^2\right] \leq M_H\delta^{2H},$$

*where $M_H$ is a positive real number, only depending on $H$.*

*Proof.* Assume first that $t$ is of the form $N\delta, N \in \{0\} \cup \mathbb{N}$. We then have that $X_t^{(\delta)}, Y_t^{(\delta)}$ is of the form

$$\begin{aligned}X_t^{(\delta)}(\omega) &\doteq \frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}(\omega)\cdot I_{A_\delta}(\omega), \\ Y_t^{(\delta)}(\omega) &\doteq \frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}}\right]\sqrt{\delta}w_{\tau/\delta}(\omega)\cdot I_{B_\delta}(\omega).\end{aligned}$$

We have that $P(A_\delta) = 1, P(B_\delta) = 1$, so

$$P(A_\delta \cap B_\delta) = 1 - P(A_\delta^c \cup B_\delta^c) \geq 1 - P(A_\delta^c) - P(B_\delta^c) = 1 - 0 - 0 = 1.$$

Since integrals over sets with probability zero equal zero, we have that

$$E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^2\right] = E\left[\left(\left(X_t^{(\delta)} - Y_t^{(\delta)}\right) I_{A_\delta \cap B_\delta}\right)^2\right].$$

By defining

$$\tilde{X}_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta \cap B_\delta}(\omega),$$
$$\tilde{Y}_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta \cap B_\delta}(\omega),$$

we have that

$$E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^2\right] = E\left[\left(\tilde{X}_t^{(\delta)} - \tilde{Y}_t^{(\delta)}\right)^2\right].$$

Notice that $\tilde{X}_t^{(\delta)}$ and $\tilde{Y}_t^{(\delta)}$ are well-defined random variables. To see this, note that for $\omega \in A_\delta \cap B_\delta$ we have that $X_t^{(\delta)}(\omega) = \tilde{X}_t^{(\delta)}(\omega)$, on $(A_\delta \cap B_\delta)^c$ we have that $\tilde{X}_t^{(\delta)}(\omega) = 0$, and $A_\delta \cap B_\delta \in \mathcal{A}$, similarly for $\tilde{Y}_t^{(\delta)}$. So for the rest of the proof we will work with

$$E\left[\left(\tilde{X}_t^{(\delta)} - \tilde{Y}_t^{(\delta)}\right)^2\right].$$

By lemma 3.2.4

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \frac{\delta}{C_H^2} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]^2 < \infty,$$

and by lemma 5.1.3

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \frac{\delta}{C_H^2} \left[(t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}}\right]^2 < \infty,$$

so by lemma 3.6.1

$$\begin{aligned} & E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^2\right] \\ &= E\left[\left(\tilde{X}_t^{(\delta)} - \tilde{Y}_t^{(\delta)}\right)^2\right] \\ &= \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left((t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} - (t-\tau)^{\underline{H-\frac{1}{2}}} + (-\tau)_+^{\underline{H-\frac{1}{2}}}\right)^2 \\ &< \infty. \end{aligned} \tag{5.6}$$

Now we will get rid of the dependence of $t$. We have by using $(a-b)^2 \leq 2(a^2+b^2)$

$$\begin{aligned}
&\frac{\delta}{C_H^2}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left((t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}-(t-\tau)^{\underline{H-\frac{1}{2}}}+(-\tau)_+^{\underline{H-\frac{1}{2}}}\right)^2\\
&\leq \frac{2\delta}{C_H^2}\left[\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left((t-\tau)^{H-\frac{1}{2}}-(t-\tau)^{\underline{H-\frac{1}{2}}}\right)^2+\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left(-(-\tau)_+^{H-\frac{1}{2}}+(-\tau)_+^{\underline{H-\frac{1}{2}}}\right)^2\right]\\
&= \frac{2\delta}{C_H^2}\left[\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{1}{2}}-\tau^{\underline{H-\frac{1}{2}}}\right)^2+\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{1}{2}}-\tau^{\underline{H-\frac{1}{2}}}\right)^2\right]\\
&= \frac{4\delta}{C_H^2}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{1}{2}}-\tau^{\underline{H-\frac{1}{2}}}\right)^2.
\end{aligned}$$

We see that the last expression is independent of $t$, we will try to simplify it. Recall that we have

$$\begin{aligned}
&\tau^{\underline{H-\frac{1}{2}}}\\
&= \frac{\Gamma_\delta(\tau+\delta)}{\Gamma_\delta\left(\tau+\frac{3}{2}\delta-H\delta\right)}, \text{ by definition 4.2.4}\\
&= \frac{\delta^{\tau/\delta+1}\Gamma\left(\frac{\tau}{\delta}+1\right)}{\delta^{\tau/\delta+3/2-H}\Gamma\left(\frac{\tau}{\delta}+\frac{3}{2}-H\right)}, \text{ by definition 4.2.1}\\
&= \delta^{H-1/2}\frac{\Gamma\left(\frac{\tau}{\delta}+1\right)}{\Gamma\left(\frac{\tau}{\delta}+\frac{3}{2}-H\right)}.
\end{aligned}$$

Notice also that the second equality follows because we do not have any poles in the numerator and denominator as the only relevant values of $\tau$ in this expression are bigger than or equal to $\delta$. Hence we get, by using the substitution $r=\tau/\delta$

$$\begin{aligned}
\frac{4\delta}{C_H^2}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{1}{2}}-\tau^{\underline{H-\frac{1}{2}}}\right)^2 &= \frac{4\delta}{C_H^2}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{1}{2}}-\delta^{H-\frac{1}{2}}\frac{\Gamma\left(\frac{\tau}{\delta}+1\right)}{\Gamma\left(\frac{\tau}{\delta}+\frac{3}{2}-H\right)}\right)^2\\
&= \frac{4\delta}{C_H^2}\sum_{r=1}^{\infty}\left((r\delta)^{H-\frac{1}{2}}-\delta^{H-\frac{1}{2}}\frac{\Gamma(r+1)}{\Gamma\left(r+\frac{3}{2}-H\right)}\right)^2\\
&= \frac{4\delta^{2H}}{C_H^2}\sum_{r=1}^{\infty}\left(r^{H-\frac{1}{2}}-\frac{\Gamma(r+1)}{\Gamma\left(r+\frac{3}{2}-H\right)}\right)^2.
\end{aligned}$$

By lemma 5.2.1, the sum in the last expression is finite, call it $S_H$. So we have proved that if $t$ is a multiple of $\delta$ we have

$$E\left[\left(X_t^{(\delta)}-Y_t^{(\delta)}\right)^2\right]=E\left[\left(\tilde{X}_t^{(\delta)}-\tilde{Y}_t^{(\delta)}\right)^2\right]\leq\frac{4S_H}{C_H^2}\cdot\delta^{2H}.$$

If $t$ is not a multiple of $\delta$ we have that

$$t=q_1t_1+q_2t_2,$$

with $0<q_1<1, 0<q_2<1, t_2-t_1=\delta$ and both $t_1$ and $t_2$ are multiples of $\delta$. By the definition of $X_t^{(\delta)}$ and $Y_t^{(\delta)}$, see definition 3.2.6 and definition 5.1.5 we get

$$\begin{aligned}
E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^2\right] &= E\left[\left(q_1 X_{t_1}^{(\delta)} + q_2 X_{t_2}^{(\delta)} - q_1 Y_{t_1}^{(\delta)} - q_2 Y_{t_2}^{(\delta)}\right)^2\right] \\
&= E\left[\left(q_1 X_{t_1}^{(\delta)} - q_1 Y_{t_1}^{(\delta)} + q_2 X_{t_2}^{(\delta)} - q_2 Y_{t_2}^{(\delta)}\right)^2\right] \\
&\leq E\left[2\left(q_1 X_{t_1}^{(\delta)} - q_1 Y_{t_1}^{(\delta)}\right)^2 + 2\left(q_2 X_{t_2}^{(\delta)} - q_2 Y_{t_2}^{(\delta)}\right)^2\right] \\
&= 2q_1^2 E\left[\left(X_{t_1}^{(\delta)} - Y_{t_1}^{(\delta)}\right)^2\right] + 2q_2^2 E\left[\left(q_2 X_{t_2}^{(\delta)} - q_2 Y_{t_2}^{(\delta)}\right)^2\right] \\
&\leq 2q_1^2 \frac{4S_H}{C_H^2} \cdot \delta^{2H} + 2q_2^2 \frac{4S_H}{C_H^2} \cdot \delta^{2H} \\
&= \frac{8S_H}{C_H^2}\left(q_1^2 + q_2^2\right) \cdot \delta^{2H} \\
&\leq \frac{16S_H}{C_H^2} \cdot \delta^{2H}.
\end{aligned}$$

The result follows with $M_H = 16S_H/C_H^2$. □

Proposition 5.2.2 gives us a result with $\delta^{2H}$. We recall from section 3.6 that similar results with $\delta^{2H}$ were only useful if $H > 1/2$, because we needed an exponent bigger than one. We will see that a similar situation arises here, so we will again use Khintchine's inequality to obtain a result with exponent bigger than one, even if $H < 1/2$. The trade-off is that we may have to take a bigger power than two on the left-hand side. Notice the similarity between the next result and theorem 3.6.4.

**Proposition 5.2.3.** *Let $H \in (0,1), \delta > 0$. Assume also that*

$$\mathcal{W} = \{\dots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \dots\}$$

*is a collection of independent random variables, each taking the values $\pm 1$ with equal probability, defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $X^{(\delta)}$ be as in definition 3.2.6, and $Y^{(\delta)}$ be as in definition 5.1.5, where $\mathcal{W} = \{\dots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \dots\}$ is the same in both cases. Then there exists an even integer $k$ and a real number $\alpha > 1$, such that for every $t \in [0, \infty)$*

$$E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^k\right] \leq R_H \delta^\alpha,$$

*where $R_H$ is a positive real number. $R_H$, $k$ and $\alpha$ only depend on $H$.*

*Proof.* The proof will follow the same structure as the proof of theorem 3.6.4. Let $p$ be the smallest natural number such that

$$2Hp > 1,$$

hence $p$ only depends on $H$. Let $k = 2p$. We first assume that $t$ is a multiple of $\delta$. We then have

$$
\begin{aligned}
&E\left[\left(X_t^{(\delta)} - Y_t^{(\delta)}\right)^k\right] \\
&= E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{A_\delta} \\
&\qquad - \frac{1}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{B_\delta}\Bigg)^k\Bigg] \\
&= E\Bigg[\lim_{N\to\infty}\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{A_\delta} \\
&\qquad\qquad - \frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{B_\delta}\Bigg)^k\Bigg].
\end{aligned}
$$

By Fatou's lemma this is less than or equal to

$$
\begin{aligned}
&\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{A_\delta} \\
&\qquad\qquad - \frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{B_\delta}\Bigg)^k\Bigg].
\end{aligned}
$$

When the sum is finite we can remove the indicator functions and the random variables will still be well-defined. This fact, and the fact that $P(A_\delta) = P(B_\delta) =$

1 implies that

$$
\begin{aligned}
&\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{A_\delta}\\
&\qquad -\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{\underline{H-\frac{1}{2}}}-(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{B_\delta}\Bigg)^k\Bigg]\\
&=\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{A_\delta}\\
&\qquad -\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{\underline{H-\frac{1}{2}}}-(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\cdot I_{B_\delta}\Bigg)^k I_{A_\delta\cap B_\delta}\Bigg]\\
&=\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\\
&\qquad -\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{\underline{H-\frac{1}{2}}}-(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\Bigg)^k I_{A_\delta\cap B_\delta}\Bigg]\\
&=\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\\
&\qquad -\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[(t-\tau)^{\underline{H-\frac{1}{2}}}-(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]\sqrt{\delta}w_{\tau/\delta}\Bigg)^k\Bigg]\\
&=\liminf_{N\to\infty} E\Bigg[\Bigg(\frac{1}{C_H}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\Big[(t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}\\
&\qquad\qquad -(t-\tau)^{\underline{H-\frac{1}{2}}}+(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\Big]\sqrt{\delta}w_{\tau/\delta}\Bigg)^k\Bigg].
\end{aligned}
$$

By Khintchine's inequality, see theorem 3.6.3, the last expression is less than or equal to

$$
\begin{aligned}
&\liminf_{N\to\infty} U_k\left(\frac{\delta}{C_H^2}\sum_{\tau=-N\delta}^{t-\delta}\!{}_{\delta}\left[\left((t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}-(t-\tau)^{\underline{H-\frac{1}{2}}}+(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]^2\right)^{k/2}\\
&=U_k\left(\frac{\delta}{C_H^2}\sum_{\tau=-\infty}^{t-\delta}\!{}_{\delta}\left[\left((t-\tau)^{H-\frac{1}{2}}-(-\tau)_+^{H-\frac{1}{2}}-(t-\tau)^{\underline{H-\frac{1}{2}}}+(-\tau)_{\underline{+}}^{H-\frac{1}{2}}\right]^2\right)^{k/2}.
\end{aligned}
$$

From the proof of proposition 5.2.2, specifically eq. (5.6), this is equal to

$$
U_k\left(E\left[\left(X_t^{(\delta)}-Y_t^{(\delta)}\right)^2\right]\right)^{k/2}.
$$

The statement of proposition 5.2.2 tells us that

$$\begin{aligned}
U_k\left(E\left[\left(X_t^{(\delta)}-Y_t^{(\delta)}\right)^2\right]\right)^{k/2} &\leq U_k\left(M_H\delta^{2H}\right)^{k/2}\\
&= U_k M_H^p \delta^{kH}\\
&= U_k M_H^p \delta^{2Hp}.
\end{aligned}$$

Hence, we have proved that if $t$ is a multiple of $\delta$ we have

$$E\left[\left(X_t^{(\delta)}-Y_t^{(\delta)}\right)^k\right] \leq U_k M_H^p \delta^{2Hp}.$$

Assume now that $t$ is not a multiple of $\delta$. Then $t = q_1t_1+q_2t_2, t_2-t_1 = \delta, 0 < q_1 < 1, 0 < q_2 < 1$ and both $t_1$ and $t_2$ are multiples of $\delta$. From the definitions of $X_t^{(\delta)}, Y_t^{(\delta)}$ we obtain by using the crude inequality $(a+b)^k \leq 2^k(a^k+b^k)$(k is even)

$$\begin{aligned}
&E\left[\left(X_t^{(\delta)}-Y_t^{(\delta)}\right)^k\right]\\
&= E\left[\left(q_1X_{t_1}^{(\delta)}+q_2X_{t_2}^{(\delta)}-q_1Y_{t_1}^{(\delta)}-q_2Y_{t_2}^{(\delta)}\right)^k\right]\\
&\leq 2^kE\left[\left(q_1X_{t_1}^{(\delta)}-q_1Y_{t_1}^{(\delta)}\right)^k\right]+2^kE\left[\left(q_2X_{t_2}^{(\delta)}-q_2Y_{t_2}^{(\delta)}\right)^k\right]\\
&= 2^kq_1^kE\left[\left(X_{t_1}^{(\delta)}-Y_{t_1}^{(\delta)}\right)^k\right]+2^kq_2^kE\left[\left(X_{t_2}^{(\delta)}-Y_{t_2}^{(\delta)}\right)^k\right]\\
&\leq 2^kE\left[\left(X_{t_1}^{(\delta)}-Y_{t_1}^{(\delta)}\right)^k\right]+2^kE\left[\left(X_{t_2}^{(\delta)}-Y_{t_2}^{(\delta)}\right)^k\right].
\end{aligned}$$

By what we proved for values of $t$ that are multiples of $\delta$, this is less than or equal to

$$2^kU_kM_H^p\delta^{2Hp}+2^kU_kM_H^p\delta^{2Hp} = 2^{k+1}U_kM_H^p\delta^{2Hp}.$$

The result now follows with $R_H = 2^{k+1}U_kM_H^p$, which only depends on $H$ since $k$ and $p$ only depends on $H$. Let $\alpha \doteq 2Hp$, which we remember from the start of this proof is bigger than one by construction.

□

## 5.3 Closeness of $X^{(\delta)}$ and $Y^{(\delta)}$

Now we will show that $X^{(\delta)}$ and $Y^{(\delta)}$ are close in a probabilistic sense if $\delta$ is small. We first need a little lemma showing that some sets are measurable.

**Lemma 5.3.1.** *Let $H \in (0,1), \delta > 0, \epsilon > 0$. Let $X^{(\delta)}$ be as in definition 3.2.6 and $Y^{(\delta)}$ be as in definition 5.1.5, we assume that the underlying probability space $(\Omega, \mathcal{A}, P)$ in both definitions are the same. Let $\rho$ be the metric on $C[0,\infty)$ defined in theorem 2.2.1. We then have*

$$\left\{\omega\in\Omega : \rho\left(X^{(\delta)}(\omega), Y^{(\delta)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

$$\left\{\omega \in \Omega : \rho\left(X^{(\delta)}(\omega), Y^{(\delta)}(\omega)\right) \le \epsilon\right\} \in \mathcal{A}$$

*and*

$$\left\{\omega \in \Omega : \rho\left(X^{(\delta)}(\omega), Y^{(\delta)}(\omega)\right) < \epsilon\right\} \in \mathcal{A}$$

*Proof.* After definition 3.3.1 we showed that $X^{(\delta)}$ as a function

$$X^{(\delta)} : \Omega \to C[0, \infty),$$

is $\mathcal{A}/\mathcal{C}$-measurable, where $\mathcal{C}$ is the Borel sigma-algebra on $C[0, \infty)$. From the remark after the definition of $Y^{(\delta)}$ we know that $Y_t^{(\delta)}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable for each $t$. From lemma 3.3.2 we then have that $Y^{(\delta)}$ as a function

$$Y^{(\delta)} : \Omega \to C[0, \infty),$$

is $\mathcal{A}/\mathcal{C}$-measurable. By theorem 2.2.2, $(C[0, \infty), \rho)$ is separable. Our result now follows by proposition B.2.15 in appendix B.1, since $[\epsilon, \infty) \in \mathcal{B}(\mathbb{R})$, $(-\infty, \epsilon] \in \mathcal{B}(\mathbb{R})$ and $(-\infty, \epsilon) \in \mathcal{B}(\mathbb{R})$. □

Now we move to the important theorem of this section.

**Theorem 5.3.2.** *Let $H \in (0, 1), \epsilon > 0$ and $\{\delta_n\}$ be a sequence of positive numbers converging to zero. Let $X^{(\delta_n)}$ and $Y^{(\delta_n)}$ be the Mandelbrot and Van Ness sums from definition 3.2.6 and definition 5.1.5, where the underlying probability space $(\Omega, \mathcal{A}, P)$ and $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ are the same in both cases. Then*

$$P\left(\left\{\omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) \ge \epsilon\right\}\right) \to 0,$$

*as $n \to \infty$, where $\rho$ is the usual metric on $C[0, \infty)$.*

*Proof.* Notice first that by lemma 5.3.1 it makes sense to talk about the quantities

$$P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) \ge \epsilon\right\}\right)$$

and

$$P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right).$$

Also note that since

$$\begin{aligned} &P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) \ge \epsilon\right\}\right) \\ &= 1 - P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) \end{aligned}$$

it suffices to prove that

$$P\left(\left\{\omega : \rho\left(X^{(\delta_n)}, Y^{(\delta_n)}\right) < \epsilon\right\}\right) \to 1,$$

as $n \to \infty$. Choose $K \in \mathbb{N}$ such that $\Sigma_{k=K+1}^{\infty} 2^{-k} < \epsilon/2$.

Keep $\delta_n$ fixed for the moment. Define the set $F_j^{\delta_n}$ for $j \in \mathbb{N}$ by

$$F_j^{\delta_n} \doteq \left\{\omega \in \Omega : \left|X_{j\delta_n}^{(\delta_n)}(\omega) - Y_{j\delta_n}^{(\delta_n)}(\omega)\right| < \frac{\epsilon}{2K}\right\}.$$

Notice that $F_j^{\delta_n} \in \mathcal{A}$ since we have already established that $X^{(\delta_n)}$ and $Y^{(\delta_n)}$ are stochastic processes on $(\Omega, \mathcal{A}, P)$. Let

$$F^{\delta_n} \doteq \bigcap_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} F_j^{\delta_n}.$$

We will now show that

$$F^{\delta_n} \subset \left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}.$$

Assume that $\omega^* \in F^{\delta_n}$, and $t \in [0, K]$. If $t = j^*\delta, j^* \in \{0\} \cup \mathbb{N}$ we have by construction

$$\left|X_t^{(\delta_n)}(\omega^*) - Y_t^{(\delta_n)}(\omega^*)\right| < \frac{\epsilon}{2K},$$

where we also recall that $\left|X_0^{(\delta_n)}(\omega^*) - Y_0^{(\delta_n)}(\omega^*)\right| = 0$ by definition of $X^{(\delta_n)}$ and $Y^{(\delta_n)}$. If $t$ is not of the form $t = j^*\delta, j^* \in \mathbb{N}$, then there exists unique $j^* \in \mathbb{N}$, such that $(j^* - 1)\delta_n < t < j^*\delta_n$. There are unique real numbers $0 < r_1, r_2 < 1$ with $r_1 + r_2 = 1$ such that $t = r_1(j^* - 1)\delta_n + r_2 j^*\delta_n$. Notice that

$$\begin{aligned} j^*\delta_n &= j^*\delta_n - t + t \\ &< \delta_n + t \\ &\leq \delta_n + K. \end{aligned}$$

This means that we have

$$\begin{aligned} &\left|X_t^{(\delta_n)}(\omega^*) - Y_t^{(\delta_n)}(\omega^*)\right| \\ =& \left|r_1 X_{(j^*-1)\delta_n}^{(\delta_n)}(\omega^*) + r_2 X_{j^*\delta_n}^{(\delta_n)}(\omega^*) - r_1 Y_{(j^*-1)\delta_n}^{(\delta_n)}(\omega^*) - r_2 Y_{j^*\delta_n}^{(\delta_n)}(\omega^*)\right| \\ \leq& \left|r_1 X_{(j^*-1)\delta_n}^{(\delta_n)}(\omega^*) - r_1 Y_{(j^*-1)\delta_n}^{(\delta_n)}(\omega^*)\right| + \left|r_2 X_{j^*\delta_n}^{(\delta_n)}(\omega^*) - r_2 Y_{j^*\delta_n}^{(\delta_n)}(\omega^*)\right| \\ <& r_1\frac{\epsilon}{2K} + r_2\frac{\epsilon}{2K} \\ =& \frac{\epsilon}{2K}. \end{aligned}$$

So we have

$$\begin{aligned} &\rho\left(X^{(\delta_n)}(\omega^*), Y^{(\delta_n)}(\omega^*)\right) \\ =& \sum_{k=1}^{\infty} \min\left(2^{-k}, \sup\{|X_t^{(\delta_n)}(\omega^*) - Y_t^{(\delta_n)}(\omega^*)| : t \in [0,k]\}\right) \\ \leq& \sum_{k=1}^{K} \sup\{|X_t^{(\delta_n)}(\omega^*) - Y_t^{(\delta_n)}(\omega^*)| : t \in [0,k]\} + \sum_{k=K+1}^{\infty} 2^{-k} \\ <& K \cdot \frac{\epsilon}{2K} + \frac{\epsilon}{2} \\ =& \epsilon. \end{aligned}$$

Hence we have shown that

$$F^{\delta_n} \subset \left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}. \tag{5.7}$$

Let $R_H, \alpha$ and $k^*$ be as in proposition 5.2.3, where we use the notation $k^*$ instead of $k$ which is used in proposition 5.2.3. We recall that these three quantities only depend on $H$. For $j \in \mathbb{N}$ we get with the aid of the Chebyshev-Markov inequality, see [App09, p. 8]

$$\begin{aligned} P\left(F_j^{\delta_n}\right) &= P\left(\left\{\omega \in \Omega : \left|X_{j\delta_n}^{(\delta_n)}(\omega) - Y_{j\delta_n}^{(\delta_n)}(\omega)\right| < \frac{\epsilon}{2K}\right\}\right) \\ &= 1 - P\left(\left\{\omega \in \Omega : \left|X_{j\delta_n}^{(\delta_n)}(\omega) - Y_{j\delta_n}^{(\delta_n)}(\omega)\right| \geq \frac{\epsilon}{2K}\right\}\right) \\ &\geq 1 - \frac{E\left(\left|X_{j\delta_n}^{(\delta_n)} - Y_{j\delta_n}^{(\delta_n)}\right|^{k^*}\right)}{\left(\frac{\epsilon}{2K}\right)^{k^*}}. \end{aligned}$$

By proposition 5.2.3 this is bigger than or equal to

$$1 - \frac{R_H \delta_n^\alpha}{\left(\frac{\epsilon}{2K}\right)^{k^*}} = 1 - \frac{R_H \left(2K\right)^{k^*} \delta_n^\alpha}{\epsilon^{k^*}}.$$

This means that for every $j \in \mathbb{N}$

$$\begin{aligned} P\left(\left(F_j^{\delta_n}\right)^c\right) &= 1 - P\left(F_j^{\delta_n}\right) \\ &\leq 1 - 1 + \frac{R_H \left(2K\right)^{k^*} \delta_n^\alpha}{\epsilon^{k^*}} \\ &= \frac{R_H \left(2K\right)^{k^*} \delta_n^\alpha}{\epsilon^{k^*}}. \end{aligned}$$

We now get

$$\begin{aligned} P\left(F^{\delta_n}\right) &= P\left(\bigcap_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} F_j^{\delta_n}\right) \\ &= 1 - P\left(\left(\bigcap_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} F_j^{\delta_n}\right)^c\right) \\ &= 1 - P\left(\bigcup_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} \left(F_j^{\delta_n}\right)^c\right) \\ &\geq 1 - \sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} P\left(\left(F_j^{\delta_n}\right)^c\right) \\ &\geq 1 - \sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} \frac{R_H \left(2K\right)^{k^*} \delta_n^\alpha}{\epsilon^{k^*}}. \end{aligned}$$

The number of natural numbers $j$ such that $j\delta_n \leq K + \delta_n$ is

$$\left\lfloor \frac{K+\delta_n}{\delta_n} \right\rfloor \leq \frac{K+\delta_n}{\delta_n}.$$

Hence

$$\begin{aligned} P\left(F^{\delta_n}\right) &\geq 1 - \sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} \frac{R_H\,(2K)^{k^*}\,\delta_n^\alpha}{\epsilon^{k^*}} \\ &= 1 - \left\lfloor \frac{K+\delta_n}{\delta_n} \right\rfloor \cdot \frac{R_H\,(2K)^{k^*}\,\delta_n^\alpha}{\epsilon^{k^*}} \\ &\geq 1 - \frac{K+\delta_n}{\delta_n} \cdot \frac{R_H\,(2K)^{k^*}\,\delta_n^\alpha}{\epsilon^{k^*}} \\ &= 1 - (K\delta_n^{\alpha-1} + \delta_n^\alpha)\frac{R_H\,(2K)^{k^*}}{\epsilon^{k^*}}. \end{aligned} \tag{5.8}$$

We will now let $n$ go to infinity. We get with the aid of eq. (5.7) and eq. (5.8)

$$\begin{aligned} &P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) \\ &\geq P\left(F^{\delta_n}\right) \\ &\geq 1 - (K\delta_n^{\alpha-1} + \delta_n^\alpha)\frac{R_H\,(2K)^{k^*}}{\epsilon^{k^*}}. \end{aligned}$$

Since all probability measures are bounded by 1 we have

$$\begin{aligned} &\left|P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) - 1\right| \\ &= 1 - P\left(\left\{\omega \in \Omega : \rho\left(X^{(\delta_n)}(\omega), Y^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) \\ &\leq 1 - 1 + (K\delta_n^{\alpha-1} + \delta_n^\alpha)\frac{R_H\,(2K)^{k^*}}{\epsilon^{k^*}} \\ &= (K\delta_n^{\alpha-1} + \delta_n^\alpha)\frac{R_H\,(2K)^{k^*}}{\epsilon^{k^*}}. \end{aligned}$$

We remember from proposition 5.2.3 that $\alpha > 1$ so the last expression goes to zero as $n$ goes to infinity, because then $\delta_n$ goes to zero. This completes the proof.

□

## 5.4 Weak convergence of the falling Mandelbrot and Van Ness sum

In this section we will end up with the analogous of theorem 3.7.3, but for falling functions. We have not defined the measures induced by the falling Mandelbrot and Van Ness sum, so we do that first. It is the parallel definition to definition 3.3.1

**Definition 5.4.1.** *Let $H \in (0,1), \delta_n > 0$. Let $Y^{(\delta_n)}$ be as in definition 5.1.5. We define the measure $P_n^f$ on $(C[0,\infty), \mathcal{C})$ as*

$$P_n^f(B) = P(Y^{(\delta_n)} \in B), B \in \mathcal{C},$$

*here $Y^{\delta_n}$ denotes the entire process on $[0,\infty)$. The $f$ in $P_n^f$ is to highlight that this is a measure for the falling Mandelbrot and Van Ness sum.*

**Remark.** *The measures $P_n^f$ are well-defined. This follows from lemma 3.3.2 and theorem C.1.1.*

The next result will tell us that theorem 5.3.2 is sufficient for weak convergence of $Y^{\delta_n}$. We will state a more general result than is needed in this chapter, so that we can refer to it in later chapters. We will prove this result in detail. The idea of this work is from [Bil99, Theorem 3.1, p. 27]. We will use a slightly different notation, and fill in some details not done there, but the idea and structure of the proof are Billingsley's.

**Theorem 5.4.2.** *Let $(V_t^{(1,n)})_{t\in[0,\infty)}, (V_t^{(2,n)})_{t\in[0,\infty)}, n \in \mathbb{N}$, be two sequences of continuous stochastic processes on the same probability space $(\Omega, \mathcal{A}, P)$. Let $P_n^{(1)}$ be the induced probability measure of $(V_t^{(1,n)})_{t\in[0,\infty)}$ on $(C[0,\infty), \mathcal{C})$, likewise let $P_n^{(2)}$ be the induced probability measure of $(V_t^{(2,n)})_{t\in[0,\infty)}$. They are as usual defined by*

$$P_n^{(i)}(C) \doteq P\left(\left(V^{(i,n)}\right)^{-1}(C)\right), \qquad i \in \{1,2\}, C \in \mathcal{C}.$$

*Let $\rho$ be the metric on $C[0,\infty)$ defined in theorem 2.2.1. Assume that for every $\epsilon > 0$ we have*

$$P\left(\left\{\omega \in \Omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon\right\}\right) \to 0,$$

*as $n \to \infty$. Assume also that there is a measure $Q$ on $(C[0,\infty), \mathcal{C})$ such that $P_n^{(1)}$ converges weakly to $Q$. Then $P_n^{(2)}$ also converges weakly to $Q$.*

**Remark.** *Notice that $P_n^{(1)}$ and $P_n^{(2)}$ exist by lemma 3.3.2 and theorem C.1.1. We also have that*

$$\left\{\omega \in \Omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

*by proposition B.2.15 and theorem 2.2.2(separability).*

*Proof.* By the Portmanteau Theorem, see [Bil99, Theorem 2.1, p. 16], an equivalent condition for weak convergence is that for every $F \in \mathcal{C}$ with F closed we have

$$\limsup_{n\to\infty} P_n^{(2)}(F) \leq Q(F).$$

We will prove that this condition holds. If $F = \emptyset$ the result follows, because then both sides are zero. We can therefore assume that $F \neq \emptyset$. Let $(\epsilon_k)_{k\in\mathbb{N}}$

be a decreasing sequence of positive real numbers converging to zero. By lemma B.2.20 we have for every $k \in \mathbb{N}$

$$\begin{aligned}&\left\{\omega \in \Omega : V^{(2,n)}(\omega) \in F\right\}\\ \subset &\left\{\omega \in \Omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\} \cup \left\{\omega \in \Omega : V^{(1,n)}(\omega) \in F_{\epsilon_k}\right\},\end{aligned} \tag{5.9}$$

where $F_{\epsilon_k}$ is defined in proposition B.2.18. These three sets are elements of $\mathcal{A}$, the second by the remark for this theorem, the first and third by lemma 3.3.2. By eq. (5.9) we have

$$\begin{aligned}&P_n^{(2)}(F)\\ =\ &P\left(\left\{\omega : V^{(2,n)}(\omega) \in F\right\}\right)\\ \leq\ &P\left(\left\{\omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\} \cup \left\{\omega : V^{(1,n)}(\omega) \in F_{\epsilon_k}\right\}\right)\\ \leq\ &P\left(\left\{\omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\}\right) + P\left(\left\{\omega : V^{(1,n)}(\omega) \in F_{\epsilon_k}\right\}\right)\\ =\ &P\left(\left\{\omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\}\right) + P_n^{(1)}\left(F_{\epsilon_k}\right)\end{aligned}$$

Hence

$$\begin{aligned}&\limsup_{n\to\infty} P_n^{(2)}(F)\\ =\ &\lim_{n\to\infty} \sup_{n'\geq n} \{P_{n'}^{(2)}(F)\}\\ \leq\ &\lim_{n\to\infty} \sup_{n'\geq n} \left\{P\left(\left\{\omega : \rho\left(V^{(1,n')}(\omega), V^{(2,n')}(\omega)\right) \geq \epsilon_k\right\}\right) + P_{n'}^{(1)}\left(F_{\epsilon_k}\right)\right\}.\end{aligned}$$

By the subadditivity of $\limsup$ this is less than or equal to

$$\limsup_{n\to\infty} P\left(\left\{\omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\}\right) + \limsup_{n\to\infty} P_n^{(1)}(F_{\epsilon_k}).$$

By assumption we have

$$\limsup_{n\to\infty} P\left(\left\{\omega : \rho\left(V^{(1,n)}(\omega), V^{(2,n)}(\omega)\right) \geq \epsilon_k\right\}\right) = 0.$$

By assumption we also have that $P_n^{(1)}$ converges weakly to $Q$. Proposition B.2.18 tells us that $F_{\epsilon_k}$ is closed, so by the Portmanteau theorem we have

$$\limsup_{n\to\infty} P_n^{(1)}(F_{\epsilon_k}) \leq Q(F_{\epsilon_k}).$$

Hence we have

$$\begin{aligned}\limsup_{n\to\infty} P_n^{(2)}(F) \leq &\limsup_{n\to\infty} P_n^{(1)}(F_{\epsilon_k})\\ \leq &Q(F_{\epsilon_k}),\end{aligned}$$

this holds for all $k \in \mathbb{N}$. From proposition B.2.19 we have that $F_{\epsilon_{k+1}} \subset F_{\epsilon_k}$ and

$$\bigcap_{k\in\mathbb{N}} F_{\epsilon_k} = F.$$

So by continuity of measures, see [MW13, Theorem 5.1 c), p. 147] we have

$$\lim_{k\to\infty} Q(F_{\epsilon_k}) = Q(F). \tag{5.10}$$

From this it will follow that

$$\limsup_{n\to\infty} P_n^{(2)}(F) \leq Q(F).$$

To see this, assume for contradiction that

$$\limsup_{n\to\infty} P_n^{(2)}(F) > Q(F).$$

Set $\Delta = \limsup_{n\to\infty} P_n^{(2)}(F) - Q(F) > 0$. Because of eq. (5.10) we can choose a $k'$ such that

$$\left|Q(F_{\epsilon_{k'}}) - Q(F)\right| < \Delta/2.$$

Then we have

$$\begin{aligned}\limsup_{n\to\infty} P_n^{(2)}(F) \leq Q(F_{\epsilon_{k'}})& \\ &= Q(F_{\epsilon_{k'}}) - Q(F) + Q(F) \\ &\leq \Delta/2 + Q(F) \\ &=\Delta/2 - \Delta + \limsup_{n\to\infty} P_n^{(2)}(F).\end{aligned}$$

We can cancel $\limsup_{n\to\infty} P_n^{(2)}(F)$ on both sides since probability measures are finite, so we do not have any problems with infinity on both sides. It then follows that $0 \leq -\Delta/2$, but this is absurd, hence

$$\limsup_{n\to\infty} P_n^{(2)}(F) \leq P(F).$$

This completes the proof. □

We now state and prove the parallel result to theorem 3.7.3. Most of the work is already done, but we need to connect theorem 5.3.2 to weak convergence.

**Theorem 5.4.3** (**Weak convergence of the falling Mandelbrot and Van Ness sum**). *Let $H \in (0,1)$, assume that $\{\delta_n\}$ is a sequence of positive real numbers converging to zero. For each $\delta_n$ let $P_n^f$ be the measure induced by the falling Mandelbrot and Van Ness sum, $Y^{\delta_n}$ (see definition 5.4.1 for the definition of $P_n^f$). Then $\{P_n^f\}$ converges weakly to the measure $P$ induced by the Fractional Brownian motion, see definition 3.7.1 for details about $P$.*

*Proof.* This follows directly form theorem 5.3.2, theorem 5.4.2 and theorem 3.7.3. □

Let us briefly reflect about the work that has led us to theorem 5.4.3. Even though the proofs are long, we did profit from the work we did in chapter 3. It is worth noticing that even if we wanted to repeat all the work in chapter 3 it is not entirely clear how that would be done. In proving weak convergence in chapter 3 we relied on some values of integrals that were calculated in [ST94], especially in lemma 3.4.3 which was used for convergence of the finite-dimensional distributions, and in eq. (3.21) which was used to prove tightness of the measures. We do not have this integral identity when we substitute the powers with falling functions.

# Chapter 6

# Approximation processes described in terms of their difference

The inspiration for this chapter is the paper [Lin07], we will combine an idea from that paper with our previous work. In [Lin07] weak converge of a certain sequence of processes to the Fractional Brownian motion is described. In that paper it is not the stochastic processes that have nice properties, but the difference between time-points $t+\delta$ and $t$ have an elegant description. The weak convergence is proved on the space of cádlág functions, $D[0,\infty)$, not the space of continuous functions. The approximation processes used in [Lin07] are piecewise constant cádlág functions. In this chapter we will derive continuous approximation processes, where the difference will have a simple description. We will use the same tools as we did in chapter 3 and chapter 5, that is, we will use linear interpolation to get continuous functions, and we will use weak convergence on $C[0,\infty)$, as was introduced in chapter 2.

We started in chapter 3 with the ordinary Mandelbrot and Van Ness sum, and in chapter 5 we introduced the falling counterpart. Since differences of falling powers behave very well, we will in this chapter solve our challenge the opposite way, we will start with the processes which have falling powers. What we will find is that for the falling case the tools from chapter 4 will make the proofs very simple.

## 6.1 A process derived from the differences of $Y^{(\delta)}$

We recall the falling Mandelbrot and Van Ness sum from definition 5.1.5 where for $H \in (0,1), \delta > 0$, and $t \geq 0$ is a multiple of $\delta$ we have

$$Y_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega).$$

We also remember that we have shown that if $\omega \in B_\delta$ the sum converges, so it is well-defined. Let $t$ continue to be a multiple of $\delta$. Also define $\Delta Y_t^{(\delta)} \doteq$

$Y^{(\delta)}_{t+\delta} - Y^{(\delta)}_t$, we will investigate this quantity.

$$\begin{aligned}
&Y^{(\delta)}_{t+\delta}(\omega) - Y^{(\delta)}_t(\omega) \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{t}{}_{\delta} \left[ (t+\delta-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega) \\
&\qquad - \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_{+}^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega) \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t+\delta-\tau)^{\underline{H-\frac{1}{2}}} - (t-\tau)^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega) \\
&\qquad + \frac{1}{C_H} \delta^{\underline{H-\frac{1}{2}}} \sqrt{\delta} w_{t/\delta}(\omega) I_{B_\delta}(\omega)
\end{aligned}$$

Notice that this sum converges for every $\omega \in B_\delta$, this follows from elementary techniques from calculus which tells us that if two series converges (conditionally), then their difference converges (conditionally), see lemma D.3.4. For our values of $\tau$ we have that

$$(t+\delta-\tau)^{\underline{H-\frac{1}{2}}} - (t-\tau)^{\underline{H-\frac{1}{2}}},$$

is well defined, because we have

$$\begin{aligned}
t - \tau + \delta &\geq t - t + \delta + \delta \\
&= 2\delta.
\end{aligned}$$

So we see by definition 4.2.4 we don't have any problems where we divide a pole with a non-pole. Also notice that for our $\tau \in \{t-\delta, t-2\delta, t-3\delta, \ldots\}$ the expression

$$\delta(H-1/2)\,(t-\tau)^{\underline{H-\frac{3}{2}}}$$

is well-defined because again

$$\begin{aligned}
t - \tau + \delta &\geq t - t + \delta + \delta \\
&= 2\delta
\end{aligned}$$

So again we don't have any problems with poles. By theorem 4.2.5 we have for our values of $\tau$

$$(t+\delta-\tau)^{\underline{H-\frac{1}{2}}} - (t-\tau)^{\underline{H-\frac{1}{2}}} = \delta(H-1/2)\,(t-\tau)^{\underline{H-\frac{3}{2}}}.$$

This means that we have

$$\begin{aligned}
&Y_{t+\delta}^{(\delta)}(\omega) - Y_t^{(\delta)}(\omega)\\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t+\delta-\tau)^{\underline{H-\frac{1}{2}}} - (t-\tau)^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega)\\
&\qquad + \frac{1}{C_H} \delta^{\underline{H-\frac{1}{2}}} \sqrt{\delta} w_{t/\delta}(\omega) I_{B_\delta}(\omega)\\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ \delta (H-1/2)\,(t-\tau)^{\underline{H-\frac{3}{2}}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{B_\delta}(\omega)\\
&\qquad + \frac{1}{C_H} \delta^{\underline{H-\frac{1}{2}}} \sqrt{\delta} w_{t/\delta}(\omega) I_{B_\delta}(\omega)\\
&= \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{3}{2}}} w_{\tau/\delta}(\omega) I_{B_\delta}(\omega) \right]\\
&\quad + \frac{\delta^{\underline{H-\frac{1}{2}}} \sqrt{\delta} I_{B_\delta}(\omega)}{C_H} w_{t/\delta}(\omega).
\end{aligned}$$

We now define a new process by the difference properties we derived. It may seem unproductive to define the sum in terms of the difference, because as we will show, we will just end up getting the familiar falling Mandelbrot and Van Ness sum, $Y^{(\delta)}$, but we do it nevertheless because when we go to the non-falling case, the corresponding stochastic processes will not coincide.

**Definition 6.1.1.** *Let $H \in (0,1), \delta > 0$ be given. Let*

$$\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$$

*be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $B_\delta$ be as in proposition 5.1.4, then $P(B_\delta) = 1$ Define the stochastic process $Z^{(\delta)} = (Z_t^{(\delta)})_{t\in[0,\infty)}$, which also depends on $H$ like this*

*(i)*

$$Z_0^{(\delta)}(\omega) \doteq 0, \qquad \forall \omega \in \Omega$$

*(ii) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$, we define*

$$\begin{aligned}
\Delta Z_t^{(\delta)}(\omega) &\doteq Z_{t+\delta}^{(\delta)}(\omega) - Z_t^{(\delta)}(\omega)\\
&\doteq \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{\underline{H-\frac{3}{2}}} w_{\tau/\delta}(\omega) I_{B_\delta}(\omega) \right]\\
&\quad + \frac{\delta^{\underline{H-\frac{1}{2}}} \sqrt{\delta} I_{B_\delta}(\omega)}{C_H} w_{t/\delta}(\omega).
\end{aligned}$$

*where*

$$C_H \doteq \left( \int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right)^{1/2}.$$

*The falling powers depend on an underlying $\delta$, this $\delta$ is chosen to be the same as the one in the start of this definition.*

(iii) *We extend $Z^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define for all $\omega \in \Omega$*

$$Z_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot Z_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot Z_{(N+1)\delta}^{(\delta)}(\omega).$$

**Remark.** *Note that this definition is well-defined. First we have that the sums in point (ii) converge by our discussion above. Secondly we have that $\left(Z_t^{(\delta)}\right)_{t\in[0,\infty)}$ is a stochastic process on $(\Omega, \mathcal{A}, P)$. To see this note that if $t$ is a multiple of $\delta$ then $\Delta Z_t^{(\delta)}$ is a random variable on $(\Omega, \mathcal{A}, P)$ because it is a limit of other random variables, and if $t \geq 0$, $Z_t^{(\delta)}$ will be a finite linear combination of random variables described in point (ii). We recall that $B_\delta \in \mathcal{A}$.*

As before we define the measure induced by the process, it is identical to definition 5.4.1.

**Definition 6.1.2.** *Let $H \in (0,1), \delta_n > 0$. Let $Z^{(\delta_n)}$ be as in definition 6.1.1. We define the measure $P_n^{f,d}$ on $(C[0,\infty), \mathcal{C})$ as*

$$P_n^{f,d}(B) = P(Z^{(\delta_n)} \in B), B \in \mathcal{C},$$

*here $Z^{\delta_n}$ denotes the entire process on $[0,\infty)$. The $f$ and $d$ in $P_n^{f,d}$ is to highlight that this is a measure where we use differences, and it was derived from the falling Mandelbrot and Van Ness Sum.*

**Remark.** *Definition 6.1.2 is well-defined this follows because we established that $\left(Z_t^{\delta}\right)_{t\in[0,\infty)}$ is a stochastic process, so we can refer to lemma 3.3.2 and theorem C.1.1.*

We now get the next theorem without that much work. It tells us that $Z^{(\delta)}$ is a good process to approximate the fractional Brownian motion. The reason we are able to get this result so easy is because the difference of falling powers behave very nicely.

**Theorem 6.1.3.** *Let $H \in (0,1)$, assume that $\{\delta_n\}$ is a sequence of positive real numbers converging to zero. For each $\delta_n$ let $P_n^{f,d}$ be the measure induced by $Z_t^{(\delta_n)}$ (see definition 6.1.2). Then $\{P_n^{f,d}\}$ converges weakly to the measure $P$ induced by the Fractional Brownian motion, see definition 3.7.1 for details about $P$.*

*Proof.* The result will follow from theorem 5.4.3 if we can show that for each $n$ and each $C \in \mathcal{C}$

$$P_n^f(C) = P_n^{f,d}(C). \tag{6.1}$$

So keep $n$ fixed for the rest of the proof. Equation (6.1) will follow if we can show that for each $\omega \in \Omega$ and each $t \geq 0$ we have

$$Y_t^{(\delta_n)}(\omega) = Z_t^{(\delta_n)}(\omega),$$

because if the processes are equal, their induced measures are of course equal. For both $Z^{(\delta_n)}$ and $Y^{(\delta_n)}$ we interpolate linearly between points in time that are multiples of $\delta_n$. It therefore suffices to show that they are equal on these points in time. We will show this by induction. Let $\omega \in \Omega$ be arbitrary, keep this $\omega$ fixed. By definition we have

$$Z_0^{(\delta_n)}(\omega) = 0.$$

We also have

$$\begin{aligned} Y_0^{(\delta_n)}(\omega) &= \frac{1}{C_H} \sum_{\tau=-\infty}^{-\delta_n} {}_{\delta_n} \left[ (-\tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\ &= 0. \end{aligned}$$

Now assume that for $N \in \{0\} \cup \mathbb{N}$ we have $Z_{N\delta_n}^{(\delta_n)}(\omega) = Y_{N\delta_n}^{(\delta_n)}(\omega)$. We must show that this implies that $Z_{(N+1)\delta_n}^{(\delta_n)}(\omega) = Y_{(N+1)\delta_n}^{(\delta_n)}(\omega)$. By assumption we have

$$\begin{aligned} Z_{N\delta_n}^{(\delta_n)}(\omega) &= Y_{N\delta_n}^{(\delta_n)}(\omega) \\ &= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n-\delta_n} {}_{\delta_n} \left[ (N\delta_n - \tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega). \end{aligned}$$

We then get by using lemma D.3.4

$$\begin{aligned} & Z_{(N+1)\delta_n}^{(\delta_n)}(\omega) \\ &= Z_{N\delta_n}^{(\delta_n)}(\omega) + Z_{(N+1)\delta_n}^{(\delta_n)}(\omega) - Z_{N\delta_n}^{(\delta_n)}(\omega) \\ &= Z_{N\delta_n}^{(\delta_n)}(\omega) + \Delta Z_{N\delta_n}^{(\delta_n)}(\omega) \\ &= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n-\delta_n} {}_{\delta_n} \left[ (N\delta_n - \tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\ &\quad + \frac{\delta_n^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{N\delta_n-\delta_n} {}_{\delta_n} \left[ (N\delta_n - \tau)^{\underline{H-\frac{3}{2}}} w_{\tau/\delta_n}(\omega) I_{B_{\delta_n}}(\omega) \right] \\ &\quad + \frac{\delta_n^{\underline{H-\frac{1}{2}}} \sqrt{\delta_n} I_{B_{\delta_n}}(\omega)}{C_H} w_N(\omega). \\ &= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n-\delta_n} {}_{\delta_n} \Bigg[ (N\delta_n - \tau)^{\underline{H-\frac{1}{2}}} + \delta_n \left( H - \frac{1}{2} \right) (N\delta_n - \tau)^{\underline{H-\frac{3}{2}}} \\ &\qquad\qquad - (-\tau)_+^{\underline{H-\frac{1}{2}}} \Bigg] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\ &\quad + \frac{\delta_n^{\underline{H-\frac{1}{2}}} \sqrt{\delta_n} I_{B_{\delta_n}}(\omega)}{C_H} w_N(\omega). \end{aligned}$$

For our values of $\tau$ we have that $(N\delta_n - \tau)^{\underline{H-\frac{1}{2}}}$, $(N\delta_n - \tau)^{\underline{H-\frac{3}{2}}}$ and $((N+1)\delta_n - \tau)^{\underline{H-\frac{1}{2}}}$ are well-defined, because

$$N\delta_n - \tau + \delta_n \geq 2\delta_n,$$

so there are no problems with poles in the numerator and non-poles in the denominator. Then theorem 4.2.5 says that

$$((N+1)\delta_n - \tau)^{\underline{H-\frac{1}{2}}} = (N\delta_n - \tau)^{\underline{H-\frac{1}{2}}} + \delta_n \left(H - \frac{1}{2}\right) (N\delta_n - \tau)^{\underline{H-\frac{3}{2}}}.$$

Hence we have

$$\begin{aligned}
&Z^{(\delta_n)}_{(N+1)\delta_n}(\omega) \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n - \delta_n}{}_{\delta_n} \left[ (N\delta_n - \tau)^{\underline{H-\frac{1}{2}}} + \delta_n \left(H - \frac{1}{2}\right) (N\delta_n - \tau)^{\underline{H-\frac{3}{2}}} \right. \\
&\qquad\qquad \left. - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\
&\quad + \frac{\delta_n^{\underline{H-\frac{1}{2}}} \sqrt{\delta_n} I_{B_{\delta_n}}(\omega)}{C_H} w_N(\omega). \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n - \delta_n}{}_{\delta_n} \left[ ((N+1)\delta_n - \tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\
&\quad + \frac{\delta_n^{\underline{H-\frac{1}{2}}} \sqrt{\delta_n} I_{B_{\delta_n}}(\omega)}{C_H} w_N(\omega). \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{N\delta_n - \delta_n}{}_{\delta_n} \left[ ((N+1)\delta_n - \tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega) \\
&\quad + \frac{1}{C_H} \left[ ((N+1)\delta_n - N\delta_n)^{\underline{H-\frac{1}{2}}} - (-N\delta_n)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{N\delta_n/\delta_n}(\omega) I_{B_{\delta_n}}(\omega) \\
&= \frac{1}{C_H} \sum_{\tau=-\infty}^{(N+1)\delta_n - \delta_n}{}_{\delta_n} \left[ ((N+1)\delta_n - \tau)^{\underline{H-\frac{1}{2}}} - (-\tau)_+^{\underline{H-\frac{1}{2}}} \right] \sqrt{\delta_n} w_{\tau/\delta_n}(\omega) \cdot I_{B_{\delta_n}}(\omega).
\end{aligned}$$

By definition 5.1.5 this is equal to $Y^{(\delta_n)}_{(N+1)\delta_n}(\omega)$. This completes the proof. □

## 6.2 A process derived from the differences of $X^{(\delta)}$

In the section 6.1 we used $Y^{(\delta)}$ to derive the process $Z^{(\delta)}$, and as we saw in the proof of theorem 6.1.3 they turned out to have equal trajectories for each $\omega \in \Omega$. We will do something similar in this section, but with $X^{(\delta)}$. Since we have regular powers in this case, the proofs will not be so simple, and we trajectories won't necessarily be equal.

Let us recall from definition 3.2.6 that if $H \in (0,1), \delta > 0, \omega \in \Omega$ and $t \geq 0$ is a multiple of $\delta$ we have

$$X_t^{(\delta)}(\omega) \doteq \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega).$$

Now let us investigate the difference

$$\begin{aligned}\Delta X_t^{(\delta)}(\omega) &\doteq X_{t+\delta}^{(\delta)}(\omega) - X_t^{(\delta)}(\omega) \\ &= \frac{1}{C_H} \sum_{\tau=-\infty}^{t}{}_{\delta} \left[(t+\delta-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega) \\ &\quad - \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega) \\ &= \frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t+\delta-\tau)^{H-\frac{1}{2}} - (t-\tau)^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega) \\ &\quad + \frac{\delta^H w_{t/\delta} I_{A_\delta}(\omega)}{C_H}.\end{aligned}$$

From elementary calculus the approximation

$$(t+\delta-\tau)^{H-\frac{1}{2}} - (t-\tau)^{H-\frac{1}{2}} \approx \delta\left(H - \frac{1}{2}\right)(t-\tau)^{H-\frac{3}{2}},$$

is known. We will show that if we make this substitution in the sum in

$$\begin{aligned}&\frac{1}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t+\delta-\tau)^{H-\frac{1}{2}} - (t-\tau)^{H-\frac{1}{2}}\right] \sqrt{\delta} w_{\tau/\delta}(\omega) \cdot I_{A_\delta}(\omega) \\ &\quad + \frac{\delta^H w_{t/\delta} I_{A_\delta}(\omega)}{C_H},\end{aligned}$$

and modify the last term then define this to be $\Delta U_t^{(\delta)}$ of a new process $U^{(\delta)}$, we still get weak convergence to the fractional Brownian motion. Before we make a formal definition of $U^{(\delta)}$ we need a result to show that it is well-defined. The first lemma is closely related to lemma 3.2.4 and lemma 5.1.3. These lemmas were used in the aid of showing that $X^{(\delta)}$ and $Y^{(\delta)}$ were well-defined. We didn't need one for $Z^{(\delta)}$, because $Z^{(\delta)}$ was so close to $Y^{(\delta)}$ that it being well-defined followed from $Y^{(\delta)}$ being well-defined. We will end up not using $A_\delta$ from $X^{(\delta)}$, but we will find a new set with probability one, which we will use.

**Lemma 6.2.1.** *Let $H \in (0,1), \delta > 0$ and let $t \geq 0$ be a multiple of $\delta$. Then*

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{3}{2}}\right]^2 < \infty.$$

*Proof.* This proof is very simple. We must have $t = N\delta, N \in \{0\} \cup \mathbb{N}$. We first substitute $r\delta = \tau$, and then $N - r = r'$

$$\begin{aligned}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{3}{2}}\right]^2 &= \sum_{r=-\infty}^{N-1} \left[(N\delta - r\delta)^{H-\frac{3}{2}}\right]^2 \\ &= \delta^{2H-3} \sum_{r=-\infty}^{N-1} \frac{1}{(N-r)^{3-2H}} \\ &= \delta^{2H-3} \sum_{r'=1}^{\infty} \frac{1}{r'^{3-2H}}.\end{aligned}$$

Recalling that $3-2H>1$ the last series converges from a well-known result from calculus.

□

We now define a function in terms of Gamma functions. This is needed because we will prove that our process $U^{(\delta)}$ is close to $Z^{(\delta)}$.

**Definition 6.2.2.** *Let $H\in(0,1)$. Define the function*

$$\varepsilon_H:\mathbb{N}\to\mathbb{R},$$

*by*

$$\varepsilon_H(r)=\frac{r^{3/2-H}\Gamma(r+1)}{\Gamma(r-H+5/2)}-1.$$

**Remark.** *The definition is well-defined because we never divide by zero.*

Now we prove a result for $\varepsilon_H$ which we will need.

**Proposition 6.2.3.** *Let $H\in(0,1)$, let $\varepsilon_H$ be as in definition 6.2.2. Then there is a constant $K_{\varepsilon_H}$ such that for all $r\in\mathbb{N}$*

$$|\varepsilon_H(r)\cdot r|\leq K_{\varepsilon_H}.$$

*Proof.* In lemma 5.2.1 we showed in eq. (5.4) that

$$\frac{\Gamma(r+a)}{\Gamma(r+b)}=r^{a-b}\left(1+\mathcal{O}\left(\frac{1}{r}\right)\right),$$

as $r\to\infty$. Now let $a=1$ and $b=5/2-H$. We then have

$$\begin{aligned}r^{H-3/2}\left(1+\varepsilon_H(r)\right)&=r^{H-3/2}\left(1+\frac{r^{3/2-H}\Gamma(r+1)}{\Gamma(r-H+5/2)}-1\right)\\&=\frac{\Gamma(r+1)}{\Gamma(r-H+5/2)}.\end{aligned}$$

This means that $\varepsilon_H(r)$ is of the form $\mathcal{O}\left(\frac{1}{r}\right)$ as $r\to\infty$. Hence there exists $r^*,M^*$ such that if $r\geq r^*$ we have

$$|\varepsilon_H(r)|\leq M^*\cdot\frac{1}{r}.$$

So if $r\geq r^*$ we have

$$|\varepsilon_H(r)r|\leq M^*.$$

If $r^*=1$ choose, $K_{\varepsilon_H}=M^*$, if not, choose

$$K_{\varepsilon_H}=\max\left\{|\varepsilon_H(1)1|,|\varepsilon_H(2)2|,\ldots,|\varepsilon_H(r^*-1)(r^*-1)|,M^*\right\}.$$

Then $K_{\varepsilon_H}$ is well-defined because it is the maximum of a finite number of values, and by construction it satisfies the required properties. □

Now we define a constant to be used later.

**Definition 6.2.4.** *Let $H \in (0,1)$, define*

$$K_H^{(U)} \doteq \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-H\right)} + \left(H-\frac{1}{2}\right)\sum_{r=1}^{\infty} r^{H-\frac{3}{2}}\varepsilon_H(r),$$

*where $\varepsilon_H(t)$ is defined in definition 6.2.2.*

**Remark.** *$K_H^{(U)}$ is a well-defined real number because the sum converges absolutely. To see this, we utilize proposition 6.2.3 to get*

$$\begin{aligned}\sum_{r=1}^{\infty}\left|r^{H-\frac{3}{2}}\varepsilon_H(r)\right| &= \sum_{r=1}^{\infty}\left|r^{H-\frac{5}{2}}\varepsilon_H(r)r\right| \\ &\le K_{\varepsilon_H}\sum_{r=1}^{\infty}\left|r^{H-\frac{5}{2}}\right| \\ &< \infty,\end{aligned}$$

*since $H - 5/2 < -1$.*

We now prove that there is a set with probability one where we have convergence. The proof is very similar to proposition 3.2.5 and proposition 5.1.4.

**Proposition 6.2.5.** *Let $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Assume also that $H \in (0,1)$ and that $\delta > 0$. Then there exists a set $D_\delta \in \mathcal{A}$(also depending on H) with*

$$P(D_\delta) = 1,$$

*such that if $t \geq 0$ is a real number, and $t$ is a multiple of $\delta$, we have that for $\omega \in D_\delta$*

$$\frac{\delta^{3/2}(H-1/2)}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{3}{2}}w_{\tau/\delta}(\omega)\right]$$

*converges in $\mathbb{R}$.*

**Remark.** *In proposition 3.2.5 we used $A_\delta$ and in proposition 5.1.4 we used $B_\delta$, in section 6.1 we could still use $B_\delta$. The reason we do not name the set in this proposition $C_\delta$ is to not confuse it with the constant $C_H$, or with a set in $\mathcal{C}$.*

*Proof.* We follow the same outline as the proofs of proposition 3.2.5 and proposition 5.1.4. Assume first that $t$ is a multiple of $\delta$. From lemma 6.2.1 we have that

$$\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[\frac{\delta^{\frac{3}{2}}(H-1/2)}{C_H}(t-\tau)^{H-\frac{3}{2}}\right]^2 < \infty.$$

It then follows [MW13, Proposition 7.11, p. 260] and its proof, that there is a set $D_{\delta,t} \in \mathcal{A}$ with $P(D_{\delta,t}) = 1$. Such that for $\omega \in D_{\delta,t}$ we have that

$$\frac{\delta^{3/2}(H-1/2)}{C_H}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{3}{2}}w_{\tau/\delta}(\omega)\right],$$

converges in $\mathbb{R}$.

There are only a countable number of $t \geq 0$ that is a multiple of $\delta_n$, let

$$D_\delta = \bigcap_{r \in \mathbb{N} \cup \{0\}} D_{\delta, r\delta}.$$

Because of countability and elementary properties of measures we have that $P(D_\delta) = 1$. By construction $D_\delta$ has the required properties, and the proof is done. $\square$

We are now ready to define the main process for this section. It is similar to the process in definition 6.1.1 but we do not have falling powers.

**Definition 6.2.6.** *Let $H \in (0,1), \delta > 0$ be given. Let*

$$\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$$

*be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $D_\delta$ be as in proposition 6.2.5, then $P(D_\delta) = 1$. Define the stochastic process $U^{(\delta)} = (U_t^{(\delta)})_{t \in [0,\infty)}$, which also depends on $H$, as follows*

*(i)*

$$U_0^{(\delta)}(\omega) \doteq 0, \qquad\qquad \forall \omega \in \Omega$$

*(ii) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$ we define*

$$\begin{aligned} \Delta U_t^{(\delta)}(\omega) &\doteq U_{t+\delta}^{(\delta)}(\omega) - U_t^{(\delta)}(\omega) \\ &\doteq \frac{\delta^{3/2}(H - 1/2)}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} \left[ (t-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega) \right] \\ &\quad + \frac{K_H^{(U)} \delta^H}{C_H} w_{t/\delta}(\omega) I_{D_\delta}(\omega). \end{aligned}$$

*where*

$$C_H \doteq \left( \int_0^\infty \left( (1+x)^{H-1/2} - x^{H-1/2} \right)^2 dx + \frac{1}{2H} \right)^{1/2}.$$

*(iii) We extend $U^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define for all $\omega \in \Omega$*

$$U_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot U_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot U_{(N+1)\delta}^{(\delta)}(\omega).$$

**Remark.** *Note that this definition is well-defined. We have convergence of the sums in point (ii) by proposition 6.2.5. Secondly we have that $\left(U_t^{(\delta)}\right)_{t \in [0,\infty)}$ is a continuous stochastic process on $(\Omega, \mathcal{A}, P)$. To see this note that if $t$ is a multiple of $\delta$, then $\Delta U_t^{(\delta)}$ is a random variable on $(\Omega, \mathcal{A}, P)$ because it is a limit of other random variables, and if $t \geq 0$, $U_t^{(\delta)}$ will be a finite linear combination of random variables described in point (ii).*

As we did with $X^{(\delta)}, Y^{(\delta)}$ and $Z^{(\delta)}$ we define the measures induced by the process $U^{(\delta)}$.

**Definition 6.2.7.** *Let $H \in (0,1), \delta_n > 0$. Let $U^{(\delta_n)}$ be as in definition 6.2.6. We define the measure $P_n^d$ on $(C[0,\infty), \mathcal{C})$ as*

$$P_n^d(B) = P(U^{(\delta_n)} \in B), B \in \mathcal{C},$$

*The $d$ in $P_n^d$ is to denote that this is a measure induced by the process which is described by its difference.*

**Remark.** *As with $P_n, P_n^f, P_n^{f,d}$, definition 6.2.7 is well-defined because we established that $\left(U_t^\delta\right)_{t\in[0,\infty)}$ is a continuous stochastic process, so we can refer to lemma 3.3.2 and theorem C.1.1.*

## 6.3 Three helpful lemmas

This section contains three lemmas to be used in the next section. It can be read as it stands, or one can skip to the next section and look back to this section when the results are needed.

**Lemma 6.3.1.** *Let $H \in (0,1), 0 < \delta < 1$. Let $K \in \mathbb{N}$. Then there exists a constant $M_{K,H}^{(1)}$ such that for all $t$ such that $0 < t \leq K+1$ where $t$ is also a multiple of $\delta$, we have*

$$\frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left| \delta(H-1/2) \sum_{s=0}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{\underline{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] \right| \leq M_{K,H}^{(1)} \delta^{\alpha},$$

*where $M_{K,H}^{(1)}$ only depends on $H$ and $K$, and $\alpha = \min\{H, 1/2\}$.*

*Proof.* If $H = 1/2$ the values inside the outer sums are all zero, so the result then follows with $M_{K,H}^{(1)} = 1$. So we can assume that $H \neq 1/2$.

Keep $t$ fixed for the rest of the proof, we have by assumetion that $t = J\delta, J \in \mathbb{N}$. By definition of the falling powers we have

$$\begin{aligned}(s-\tau)^{\underline{H-\frac{3}{2}}} &= \frac{\Gamma_\delta(s-\tau+\delta)}{\Gamma_\delta(s-\tau-(H-3/2-1)\delta)} \\ &= \frac{\delta^{(s-\tau)/\delta+1}\Gamma\left(\frac{s-\tau}{\delta}+1\right)}{\delta^{(s-\tau)/\delta-H+5/2}\Gamma\left(\frac{s-\tau}{\delta}-H+\frac{5}{2}\right)} \\ &= \delta^{H-3/2}\frac{\Gamma\left(\frac{s-\tau}{\delta}+1\right)}{\Gamma\left(\frac{s-\tau}{\delta}-H+\frac{5}{2}\right)}.\end{aligned}$$

Using this we get

$$
\begin{aligned}
&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right|\\
&=\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\delta^{H-3/2}\frac{\Gamma\left(\frac{s-\tau}{\delta}+1\right)}{\Gamma\left(\frac{s-\tau}{\delta}-H+\frac{5}{2}\right)}-(s-\tau)^{H-\frac{3}{2}}\right]\right|\\
&=\frac{\sqrt{\delta}}{C_H}\sum_{r_1=-\infty}^{-1}\left|\delta(H-1/2)\sum_{r_2=0}^{J-1}\left[\delta^{H-3/2}\frac{\Gamma\left(r_2-r_1+1\right)}{\Gamma\left(r_2-r_1-H+\frac{5}{2}\right)}-(r_2\delta-r_1\delta)^{H-\frac{3}{2}}\right]\right|\\
&=\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\left|\sum_{r_2=0}^{J-1}\frac{\Gamma\left(r_2-r_1+1\right)}{\Gamma\left(r_2-r_1-H+\frac{5}{2}\right)}-(r_2-r_1)^{H-\frac{3}{2}}\right|\\
&\leq\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}\left|\frac{\Gamma\left(r_2-r_1+1\right)}{\Gamma\left(r_2-r_1-H+\frac{5}{2}\right)}-(r_2-r_1)^{H-\frac{3}{2}}\right|\\
&=\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{3}{2}}\left|\frac{(r_2-r_1)^{3/2-H}\Gamma\left(r_2-r_1+1\right)}{\Gamma\left(r_2-r_1-H+\frac{5}{2}\right)}-1\right|\\
&=\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{3}{2}}\left|\varepsilon_H(r_2-r_1)\right|,
\end{aligned}
$$

where $\varepsilon_H(r_2-r_1)$ is defined in definition 6.2.2. From proposition 6.2.3 we have

$$
\begin{aligned}
&\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{3}{2}}\left|\varepsilon_H(r_2-r_1)\right|\\
&=\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}}\left|\varepsilon_H(r_2-r_1)(r_2-r_1)\right|\\
&\leq\frac{\delta^H|H-1/2|}{C_H}\sum_{r_1=-\infty}^{-1}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}}K_{\varepsilon_H}\\
&=\frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{J-1}(r_2+1)^{H-\frac{5}{2}}+\sum_{r_1=-\infty}^{-2}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}}\right)\\
&\leq\frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}+\sum_{r_1=-\infty}^{-2}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}}\right).
\end{aligned}
$$

To summarize, we have now proved

$$
\begin{aligned}
&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right|\\
&\leq\frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}+\sum_{r_1=-\infty}^{-2}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}}\right).
\end{aligned}
\tag{6.2}
$$

Notice that for $r_1 \leq -2$ we have

$$\begin{aligned}\sum_{r_2=0}^{J-1}(r_2-r_1)^{H-\frac{5}{2}} &\leq \int_0^J (x-1-r_1)^{H-\frac{5}{2}}dx \\ &= \int_{-1-r_1}^{J-1-r_1} u^{H-\frac{5}{2}}du \\ &= \frac{1}{H-\frac{3}{2}} u^{H-\frac{3}{2}}\Big|_{-1-r_1}^{J-1-r_1} \\ &= \frac{1}{H-\frac{3}{2}}\left((J-1-r_1)^{H-\frac{3}{2}}-(-1-r_1)^{H-\frac{3}{2}}\right) \\ &= |H-3/2|^{-1}\left((-1-r_1)^{H-\frac{3}{2}}-(J-1-r_1)^{H-\frac{3}{2}}\right).\end{aligned} \tag{6.3}$$

In the first inequality we have used that for $r_2 \in \{0,\ldots,J-1\}$ and for $x \in [r_2, r_2+1]$, we have

$$(x-1-r_1)^{H-5/2} \geq (r_2-r_1)^{H-5/2}.$$

Combining eq. (6.2) and eq. (6.3) we get

$$\begin{aligned}&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right| \\ &\leq \frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right. \\ &\qquad\left.+\sum_{r_1=-\infty}^{-2}|H-3/2|^{-1}\left((-1-r_1)^{H-\frac{3}{2}}-(J-1-r_1)^{H-\frac{3}{2}}\right)\right) \\ &= \frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right. \\ &\qquad\left.+\sum_{r_1'=2}^{\infty}|H-3/2|^{-1}\left((-1+r_1')^{H-\frac{3}{2}}-(J-1+r_1')^{H-\frac{3}{2}}\right)\right) \\ &= \frac{\delta^H|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right. \\ &\qquad\left.+\sum_{r_3'=1}^{\infty}|H-3/2|^{-1}\left(r_3^{H-\frac{3}{2}}-(J+r_3)^{H-\frac{3}{2}}\right)\right).\end{aligned} \tag{6.4}$$

Assume first that $H < 1/2$, using eq. (6.4) we get

$$
\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left| \delta(H-1/2) \sum_{s=0}^{t-\delta}{}_{\delta} \left[ \underline{(s-\tau)^{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] \right| \\
&\leq \frac{\delta^H |H-1/2| K_{\varepsilon_H}}{C_H} \Bigg( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} \\
&\qquad\qquad + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} \left( r_3^{H-\frac{3}{2}} - (J+r_3)^{H-\frac{3}{2}} \right) \Bigg) \\
&\leq \frac{|H-1/2| K_{\varepsilon_H}}{C_H} \left( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} r_3^{H-\frac{3}{2}} \right) \cdot \delta^H.
\end{aligned}
$$

So the result follows with

$$
M_{K,H}^{(1)} = \frac{|H-1/2| K_{\varepsilon_H}}{C_H} \left( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} r_3^{H-\frac{3}{2}} \right).
$$

Assume now that $H > 1/2$. From eq. (6.4) we have

$$
\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left| \delta(H-1/2) \sum_{s=0}^{t-\delta}{}_{\delta} \left[ \underline{(s-\tau)^{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] \right| \\
&\leq \frac{\delta^H |H-1/2| K_{\varepsilon_H}}{C_H} \Bigg( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} \\
&\qquad\qquad + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} \left( r_3^{H-\frac{3}{2}} - (J+r_3)^{H-\frac{3}{2}} \right) \Bigg). \\
&= \frac{\sqrt{\delta} |H-1/2| K_{\varepsilon_H}}{C_H} \Bigg( \delta^{H-1/2} \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} \\
&\qquad\qquad + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} \left( (r_3\delta)^{H-\frac{3}{2}} - (J\delta+r_3\delta)^{H-\frac{3}{2}} \right) \delta \Bigg) \\
&\leq \frac{\sqrt{\delta} |H-1/2| K_{\varepsilon_H}}{C_H} \Bigg( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} \\
&\qquad\qquad + \sum_{r_3'=1}^{\infty} |H-3/2|^{-1} \left( (r_3\delta)^{H-\frac{3}{2}} - (J\delta+r_3\delta)^{H-\frac{3}{2}} \right) \delta \Bigg),
\end{aligned}
$$

where we in the last step remember that $\delta < 1$. The last expression is equal to

$$
\begin{aligned}
&\frac{\sqrt{\delta} |H-1/2| K_{\varepsilon_H}}{C_H} \Bigg( \sum_{r_2=0}^{\infty} (r_2+1)^{H-\frac{5}{2}} \\
&\qquad\qquad + |H-3/2|^{-1} \sum_{\tau=\delta}^{\infty}{}_{\delta} \left( \tau^{H-\frac{3}{2}} - (t+\tau)^{H-\frac{3}{2}} \right) \delta \Bigg).
\end{aligned}
$$

The last sum increases with $t$ so we have

$$\begin{aligned}
&\frac{\sqrt{\delta}|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right.\\
&\qquad\qquad\left.+\,|H-3/2|^{-1}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{3}{2}}-(t+\tau)^{H-\frac{3}{2}}\right)\delta\right)\\
&\le\frac{\sqrt{\delta}|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right.\\
&\qquad\qquad\left.+\,|H-3/2|^{-1}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{3}{2}}-(K+1+\tau)^{H-\frac{3}{2}}\right)\delta\right).
\end{aligned}$$

So we have that

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[(s-\tau)^{\underline{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right|\\
&\le\frac{\sqrt{\delta}|H-1/2|K_{\varepsilon_H}}{C_H}\left(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\right. \qquad (6.5)\\
&\qquad\qquad\left.+\,|H-3/2|^{-1}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{3}{2}}-(K+1+\tau)^{H-\frac{3}{2}}\right)\delta\right).
\end{aligned}$$

If we differentiate $x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}$ with respect to $x$, we get

$$(H-3/2)\left(x^{H-5/2}-(K+1+x)^{H-5/2}\right).$$

This expression is negative for positive values of $x$. This means that on $x\in(\tau-\delta,\tau]$ we have

$$x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}\ge\tau^{H-\frac{3}{2}}-(K+1+\tau)^{H-\frac{3}{2}}.$$

So by using that $x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}$ is positive for positive values of $x$, we have

$$\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{3}{2}}-(K+1+\tau)^{H-\frac{3}{2}}\right)\delta\le\int_0^{\infty}\left(x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}\right)dx.$$

If we can show that the last integral is finite, then we will be done because by eq. (6.5) we will have that

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[(s-\tau)^{\underline{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right|\\
&\leq \frac{\sqrt{\delta}|H-1/2|K_{\varepsilon_H}}{C_H}\Bigg(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\\
&\qquad\qquad +|H-3/2|^{-1}\sum_{\tau=\delta}^{\infty}{}_{\delta}\left(\tau^{H-\frac{3}{2}}-(K+1+\tau)^{H-\frac{3}{2}}\right)\delta\Bigg).\\
&\leq \frac{\sqrt{\delta}|H-1/2|K_{\varepsilon_H}}{C_H}\Bigg(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\\
&\qquad\qquad +|H-3/2|^{-1}\int_0^{\infty}\left(x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}\right)dx\Bigg)\\
&= M_{H,K}^{(1)}\sqrt{\delta},
\end{aligned}$$

where

$$\begin{aligned}
M_{H,K}^{(1)} = \frac{|H-1/2|K_{\varepsilon_H}}{C_H}&\Bigg(\sum_{r_2=0}^{\infty}(r_2+1)^{H-\frac{5}{2}}\\
&+|H-3/2|^{-1}\int_0^{\infty}\left(x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}\right)dx\Bigg).
\end{aligned}$$

So let us end this proof by showing that

$$\int_0^{\infty}\left(x^{H-\frac{3}{2}}-(K+1+x)^{H-\frac{3}{2}}\right)dx<\infty.$$

One can split the integral up and integrate first on $(0,1]$ then on $[1,\infty)$. On $(0,1]$ there is no problem because $H-3/2>-1$. We know that

$$\int_1^{\infty} x^{2H-3}<\infty,$$

since $2H-3<-1$. So on $(1,\infty)$ we use the limit-comparison test with $x^{2H-3}$.

By rewriting and using l'Hôpitals rule we get

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x^{H-\frac{3}{2}} - (K+1+x)^{H-\frac{3}{2}}}{x^{2H-3}} \\
&= \lim_{x\to\infty} \frac{1 - \left(\frac{K+1+x}{x}\right)^{H-\frac{3}{2}}}{x^{H-\frac{3}{2}}} \\
&= \lim_{x\to\infty} \frac{-(H-3/2)\left(\frac{K+1+x}{x}\right)^{H-\frac{5}{2}}\left(\frac{x-(K+1+x)}{x^2}\right)}{(H-3/2)x^{H-5/2}} \\
&= \lim_{x\to\infty} \frac{-\left(\frac{K+1+x}{x}\right)^{H-\frac{5}{2}}\left(\frac{-(K+1)}{x^2}\right)}{x^{H-5/2}} \\
&= \lim_{x\to\infty} (K+1)\left(\frac{K+1+x}{x}\right)^{H-\frac{5}{2}} x^{1/2-H} \\
&= 0.
\end{aligned}$$

We get 0 in the last equality because $1/2-H<0$. This completes the proof. □

**Lemma 6.3.2.** *Let $H\in(0,1), \delta>0$, and $K\in\mathbb{N}$. Then there exists a constant $M^{(2)}_{K,H}$ such that for all $t$ such that $2\delta\le t\le K+1$ where $t$ is also a multiple of $\delta$ we have*

$$\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[(s-\tau)^{\underline{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}}\right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)}\delta^{H-\frac{1}{2}}\right| \le M^{(2)}_{K,H}\delta^{\alpha}, \tag{6.6}$$

*where $M^{(2)}_{K,H}$ only depends on $H$ and $K$, and $\alpha=\min\{H,1/2\}$. $K_H^{(U)}$ is defined in definition 6.2.4.*

*Proof.* We will first show that the lemma holds for $H=1/2$. Notice that

$$\delta^{\underline{\frac{1}{2}-\frac{1}{2}}} = \frac{\Gamma_\delta(\delta+\delta)}{\Gamma_\delta(\delta-(-1)\delta)} = 1.$$

$$K_{1/2}^{(U)} = \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-\frac{1}{2}\right)} + \left(H-\frac{1}{2}\right)\sum_{r=1}^{\infty} r^{H-\frac{3}{2}}\varepsilon_H(r) = 1.$$

This means that the left hand-side in (6.6) is zero, so the result follows with $M^{(2)}_{K,H}=1$. So for the rest of the proof we can assume that $H\neq 1/2$.

For our combinations of $s$ and $\tau$ in (6.6) our values of $s-\tau$ are greater than $\delta$. So by the definition of falling powers, see definition 4.2.4, and by the definition of $\varepsilon_H(r)$, see definition 6.2.2, we get

$$\begin{aligned}
(s-\tau)^{\underline{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} &= \frac{\Gamma_\delta(s-\tau+\delta)}{\Gamma_\delta(s-\tau-(H-3/2-1)\delta)} - (s-\tau)^{H-\frac{3}{2}} \\
&= \delta^{H-\frac{3}{2}} \frac{\Gamma\left(\frac{s-\tau}{\delta}+1\right)}{\Gamma\left(\frac{s-\tau}{\delta}+5/2-H\right)} - (s-\tau)^{H-\frac{3}{2}} \\
&= (s-\tau)^{H-\frac{3}{2}} \left( \frac{\left(\frac{s-\tau}{\delta}\right)^{3/2-H} \Gamma\left(\frac{s-\tau}{\delta}+1\right)}{\Gamma\left(\frac{s-\tau}{\delta}+5/2-H\right)} - 1 \right) \\
&= (s-\tau)^{H-\frac{3}{2}} \varepsilon_H\left(\frac{s-\tau}{\delta}\right).
\end{aligned}$$

Hence we have by using that $t = J\delta$

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sideset{}{_\delta}\sum_{\tau=0}^{t-2\delta} \left| \delta(H-1/2) \sideset{}{_\delta}\sum_{s=\tau+\delta}^{t-\delta} \left[ (s-\tau)^{\underline{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
=& \frac{\sqrt{\delta}}{C_H} \sideset{}{_\delta}\sum_{\tau=0}^{t-2\delta} \left| \delta(H-1/2) \sideset{}{_\delta}\sum_{s=\tau+\delta}^{t-\delta} \left[ (s-\tau)^{H-\frac{3}{2}} \varepsilon_H\left(\frac{s-\tau}{\delta}\right) \right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
=& \frac{\sqrt{\delta}}{C_H} \sum_{r_1=0}^{J-2} \left| \delta(H-1/2) \sum_{r_2=r_1+1}^{J-1} \left[ (r_2\delta - r_1\delta)^{H-\frac{3}{2}} \varepsilon_H(r_2-r_1) \right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right|.
\end{aligned} \tag{6.7}$$

We need to rewrite $\delta^{\underline{H-1/2}}$. We have

$$\begin{aligned}
\delta^{\underline{H-\frac{1}{2}}} &= \frac{\Gamma_\delta(\delta+\delta)}{\Gamma_\delta(\delta-(H-1/2-1)\delta)} \\
&= \frac{\delta^2\Gamma(2)}{\delta^{5/2-H}\Gamma\left(\frac{5}{2}-H\right)} \\
&= \delta^{H-\frac{1}{2}} \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-H\right)}.
\end{aligned}$$

Combining this with eq. (6.7) we get

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sideset{}{_\delta}\sum_{\tau=0}^{t-2\delta} \left| \delta(H-1/2) \sideset{}{_\delta}\sum_{s=\tau+\delta}^{t-\delta} \left[ (s-\tau)^{\underline{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
=& \frac{\sqrt{\delta}}{C_H} \sum_{r_1=0}^{J-2} \left| \delta(H-1/2) \sum_{r_2=r_1+1}^{J-1} \left[ (r_2\delta - r_1\delta)^{H-\frac{3}{2}} \varepsilon_H(r_2-r_1) \right] + \delta^{\underline{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
=& \frac{\delta^H}{C_H} \sum_{r_1=0}^{J-2} \left| (H-1/2) \sum_{r_2=r_1+1}^{J-1} \left[ (r_2-r_1)^{H-\frac{3}{2}} \varepsilon_H(r_2-r_1) \right] + \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-H\right)} - K_H^{(U)} \right|.
\end{aligned} \tag{6.8}$$

We will now use a trick similar to what Tom Lindstrøm did in his article [Lin07] for $H < 1/2$ on page 6 where he wrote a sum as the difference of the infinite sums. Because of the presence of $\varepsilon_H(r)$ this trick will work for all $H \in (0,1)$

for us. We have

$$\begin{aligned}
&\sum_{r_2=r_1+1}^{J-1}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right] \\
&=\sum_{r_2=r_1+1}^{\infty}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right]-\sum_{r_2=J}^{\infty}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right].
\end{aligned} \tag{6.9}$$

This rewriting of the series is justified because we have absolute convergence of the infinite sum. Let us see why:

$$\begin{aligned}
&\sum_{r_2=r_1+1}^{\infty}\left|(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right| \\
&=\sum_{r_2=r_1+1}^{\infty}\left|(r_2-r_1)^{H-\frac{5}{2}}\varepsilon_H\left(r_2-r_1\right)\left(r_2-r_1\right)\right| \\
&\leq K_{\varepsilon_H}\sum_{r_2=r_1+1}^{\infty}(r_2-r_1)^{H-\frac{5}{2}} \\
&<\infty.
\end{aligned}$$

In the second inequality we used proposition 6.2.3, and the last inequality follows because $H-5/2<-1$. Combining eq. (6.8) and eq. (6.9) we get

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]+\underline{\delta^{H-\frac{1}{2}}}-K_H^{(U)}\delta^{H-\frac{1}{2}}\right| \\
&=\frac{\delta^H}{C_H}\sum_{r_1=0}^{J-2}\left|(H-1/2)\sum_{r_2=r_1+1}^{J-1}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right]+\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-H\right)}-K_H^{(U)}\right| \\
&=\frac{\delta^H}{C_H}\sum_{r_1=0}^{J-2}\Bigg|\;(H-1/2)\sum_{r_2=r_1+1}^{\infty}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right] \\
&\qquad\qquad -(H-1/2)\sum_{r_2=J}^{\infty}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right]+\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}-H\right)}-K_H^{(U)}\Bigg| \\
&=\frac{\delta^H\left|(H-1/2)\right|}{C_H}\sum_{r_1=0}^{J-2}\left|\sum_{r_2=J}^{\infty}\left[(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right]\right| \\
&\leq\frac{\delta^H\left|(H-1/2)\right|}{C_H}\sum_{r_1=0}^{J-2}\sum_{r_2=J}^{\infty}\left|(r_2-r_1)^{H-\frac{3}{2}}\varepsilon_H\left(r_2-r_1\right)\right| \\
&=\frac{\delta^H\left|(H-1/2)\right|}{C_H}\sum_{r_1=0}^{J-2}\sum_{r_2=J}^{\infty}\left|(r_2-r_1)^{H-\frac{5}{2}}\varepsilon_H\left(r_2-r_1\right)\left(r_2-r_1\right)\right| \\
&\leq\frac{\delta^H\left|(H-1/2)\right|K_{\varepsilon_H}}{C_H}\sum_{r_1=0}^{J-2}\sum_{r_2=J}^{\infty}(r_2-r_1)^{H-\frac{5}{2}},
\end{aligned} \tag{6.10}$$

where we in fourth equality used the definition of $K_H^{(U)}$, and in the last inequality we used proposition 6.2.3. We recall that since our values of $t$ is greater than or equal to $2\delta$, we have $J \geq 2$. Notice that for our values of $r_1$ and $r_2$ we have that $r_2 - r_1 \geq 2$. For $r_1 \in \{0, \dots, J-2\}$ we have that $(x-1-r_1)^{H-5/2}$ is decreasing on $[J, \infty)$. So for a given $r_2 \in \{J, J+1, \dots\}$ we have that for $x \in [r_2, r_2+1]$

$$(x-1-r_1)^{H-\frac{5}{2}} \geq (r_2 - r_1)^{H-\frac{5}{2}}.$$

This means that we have

$$\begin{aligned}
\sum_{r_2=J}^{\infty} (r_2 - r_1)^{H-\frac{5}{2}} &\leq \int_J^{\infty} (x-1-r_1)^{H-\frac{5}{2}} dx \\
&= \int_{J-1-r_1}^{\infty} u^{H-\frac{5}{2}} du \\
&= \frac{1}{\left|H - \frac{3}{2}\right|} (J-1-r_1)^{H-\frac{3}{2}}.
\end{aligned}$$

Combining this with eq. (6.10) we get

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left| \delta(H-1/2) \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ \underline{(s-\tau)^{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] + \underline{\delta^{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
&\leq \frac{\delta^H \left|(H-1/2)\right| K_{\varepsilon_H}}{C_H} \sum_{r_1=0}^{J-2} \sum_{r_2=J}^{\infty} (r_2 - r_1)^{H-\frac{5}{2}} \\
&\leq \frac{\delta^H \left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{r_1=0}^{J-2} (J-1-r_1)^{H-\frac{3}{2}} \\
&= \frac{\delta^H \left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{r_3=1}^{J-1} r_3^{H-\frac{3}{2}},
\end{aligned} \tag{6.11}$$

where we lastly used the substitution $r_3 = J - 1 - r_1$.

For $H < 1/2$ our result now follows directly from eq. (6.11) with

$$M_{H,K}^{(2)} = \frac{\left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{r_3=1}^{\infty} r_3^{H-\frac{3}{2}}.$$

Assume now that $H > 1/2$. From eq. (6.11) we get

$$\begin{aligned}
&\frac{\sqrt{\delta}}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left| \delta(H-1/2) \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ \underline{(s-\tau)^{H-\frac{3}{2}}} - (s-\tau)^{H-\frac{3}{2}} \right] + \underline{\delta^{H-\frac{1}{2}}} - K_H^{(U)} \delta^{H-\frac{1}{2}} \right| \\
&\leq \frac{\delta^H \left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{r_3=1}^{J-1} r_3^{H-\frac{3}{2}} \\
&= \frac{\sqrt{\delta} \left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{r_3=1}^{J-1} (r_3 \delta)^{H-\frac{3}{2}} \delta \\
&= \frac{\sqrt{\delta} \left|H-1/2\right| \left|H-3/2\right|^{-1} K_{\varepsilon_H}}{C_H} \sum_{\tau=\delta}^{t-\delta}{}_{\delta} \tau^{H-\frac{3}{2}} \delta.
\end{aligned} \tag{6.12}$$

We have that the function $x^{H-3/2}$ is decreasing on $(0,\infty)$. Assume that $\tau \geq \delta$ we then have for $x \in (\tau-\delta, \tau]$

$$x^{H-\frac{3}{2}} \geq \tau^{H-\frac{3}{2}}.$$

This means that

$$\begin{aligned} \sum_{\tau=\delta}^{t-\delta}{}_{\delta} \tau^{H-\frac{3}{2}}\delta &\leq \int_0^{t-\delta} x^{H-\frac{3}{2}} du \\ &= \frac{1}{H-\frac{1}{2}} x^{H-\frac{1}{2}} \Bigg|_0^{t-\delta} \\ &= (H-1/2)^{-1}(t-\delta)^{H-\frac{1}{2}} \\ &\leq (H-1/2)^{-1} t^{H-\frac{1}{2}} \\ &\leq (H-1/2)^{-1}(K+1)^{H-\frac{1}{2}}. \end{aligned} \tag{6.13}$$

Notice that we were able to evaluate the integral because $H > 1/2$. This is crucial for this argument. From eq. (6.12) and eq. (6.13) the result now follows for $H > 1/2$ with

$$M_{H,K}^{(2)} = \frac{|H-3/2|^{-1} K_{\varepsilon_H}(K+1)^{H-\frac{1}{2}}}{C_H}.$$

□

**Lemma 6.3.3.** *Let $H \in (0,1), 0 < \delta < 1$. Then there exists a non-negative constant $M_H^{(1)}$ such that*

$$\left| \frac{\delta^{H-\frac{1}{2}}\sqrt{\delta}}{C_H} - \frac{K_H^{(U)}\delta^H}{C_H} \right| \leq M_H^{(1)}\delta^\alpha$$

*where $M_H^{(1)}$ only depends on $H$ and $K$, and $\alpha = \min\{H, 1/2\}$. $K_H^{(U)}$ is defined in definition 6.2.4.*

*Proof.* A direct calculation gives us

$$\begin{aligned} \left| \frac{\delta^{H-\frac{1}{2}}\sqrt{\delta}}{C_H} - \frac{K_H^{(U)}\delta^H}{C_H} \right| &= \left| \frac{\Gamma_\delta(\delta+\delta)\sqrt{\delta}}{\Gamma_\delta(\delta-(H-1/2-1)\delta)C_H} - \frac{K_H^{(U)}\delta^H}{C_H} \right| \\ &= \left| \frac{\delta^2\Gamma(2)\sqrt{\delta}}{\delta^{5/2-H}\Gamma(5/2-H)C_H} - \frac{K_H^{(U)}\delta^H}{C_H} \right| \\ &= \left| \frac{\Gamma(2)}{\Gamma(5/2-H)C_H} - \frac{K_H^{(U)}}{C_H} \right| \delta^H. \end{aligned}$$

Hence for $H \leq 1/2$ the results holds with

$$M_H^{(1)} = \left| \frac{\Gamma(2)}{\Gamma(5/2-H)C_H} - \frac{K_H^{(U)}}{C_H} \right|.$$

For $H > 1/2$ we get by noting that $H - 1/2 > 0$, and $0 < \delta < 1$

$$\delta^H = \sqrt{\delta}\delta^{H-1/2} \leq \sqrt{\delta},$$

so the result holds with the same $M_H^{(1)}$ for $H > 1/2$.

□

## 6.4 Closeness of $Z^{(\delta)}$ and $U^{(\delta)}$

Now we will prove that $Z^{(\delta)}$ and $U^{(\delta)}$ are "close" if $\delta$ is small. We will end up with similar results from when we proved that $X^{(\delta)}$ and $Y^{(\delta)}$ were close.

**Proposition 6.4.1.** *Let $H \in (0,1), 0 < \delta < 1$. Assume also that*

$$\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$$

*is a collection of independent random variables, each taking the values $\pm 1$ with equal probability, assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$. Let $Z^{(\delta)}$ be as in definition 6.1.1, and $U^{(\delta)}$ be as in definition 6.2.6, where $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ is the same in both cases.*

*Then there exists a set $E_{(\delta,H)} \in \mathcal{A}$ only depending on $\delta$ and $H$, with $P(E_{(\delta,H)}) = 1$, such that for every every $K \in \mathbb{N}$, there exists a constant $M_{K,H}$ only depending on $K$ and $H$ such that for $t \in [0,K], \omega \in E_{(\delta,H)}$ we have*

$$\left| Z_t^{(\delta)}(\omega) - U_t^{(\delta)}(\omega) \right| \leq M_{K,H}\delta^\alpha,$$

*where $\alpha$ is a real number such that $\alpha = \min\{H, 1/2\}$.*

*Proof.* First we assume that $t \in [0, K+1]$ and that $t$ is a multiple of $\delta$. By definition

$$\left| Z_t^{(\delta)}(\omega) - U_t^{(\delta)}(\omega) \right| = 0,$$

if $t = 0$ so we assume that $t > 0$. We then recall that

$$Z_t^{(\delta)}(\omega) = \sum_{s=0}^{t-\delta}{}_{\delta} \left\{ \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{B_\delta}(\omega) \right], \right.$$
$$\left. + \frac{\delta^{H-1/2}\sqrt{\delta}}{C_H} w_{s/\delta}(\omega) I_{B_\delta}(\omega) \right\}.$$

and

$$U_t^{(\delta)}(\omega) = \sum_{s=0}^{t-\delta}{}_{\delta} \left\{ \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega) \right] \right.$$
$$\left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) I_{D_\delta}(\omega) \right\}.$$

Define $E_{(\delta,H)} \doteq B_\delta \cap D_\delta$, since $A_\delta$ and $D_\delta$ only depend on $\delta$ and $H$ so does $E_{(\delta,H)}$. We also have that $P(E_{(\delta,H)}) = 1$. From now on assume that $\omega \in E_{(\delta,H)}$.

Then the sums in both expression above for $Z_t^{(\delta)}(\omega)$ and $U_t^{(\delta)}(\omega)$ converge by the discussion before definition 6.1.1 and by proposition 6.2.5. Since the outer sums are finite it follows by a simple argument that we can interchange the sums, see lemma D.3.4. We get

$$
\begin{aligned}
&U_t^{(\delta)}(\omega) \\
&= \sum_{s=0}^{t-\delta}{}_{\delta} \left\{ \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) \right] \right. \\
&\qquad \left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) \right\}. \\
&= \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left( \frac{\delta^{3/2}(H-1/2)}{C_H} \left\{ \sum_{s=0}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} \right] \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left( \frac{\delta^{3/2}(H-1/2)}{C_H} \left\{ \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} \right] \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \sum_{s=0}^{t-\delta}{}_{\delta} \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) \\
&= \frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left( \left\{ \sum_{s=0}^{t-\delta}{}_{\delta} \left[ \delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}} \right] \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \frac{\sqrt{\delta}}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left( \left\{ \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ \delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}} \right] \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \sum_{s=0}^{t-\delta}{}_{\delta} \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) \\
&= \frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left( \left\{ \sum_{s=0}^{t-\delta}{}_{\delta} \left[ \delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}} \right] \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \frac{\sqrt{\delta}}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left( \left\{ \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ \delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}} \right] + K_H^{(U)}\delta^{H-\frac{1}{2}} \right\} w_{\tau/\delta}(\omega) \right) \\
&\quad + \frac{K_H^{(U)}\delta^H}{C_H} w_{(t-\delta)/\delta}(\omega),
\end{aligned}
$$

notice that if $t=\delta$ the sum from $\tau=0$ to $\tau=t-2\delta$ does not appear. We now manipulate the expression for $Z_t^{(\delta)}(\omega)$ to resemble the manipulated expression for $U_t^{(\delta)}(\omega)$, remembering that we assume $\omega \in E_{(\delta,H)}$ and $t>0$ and $t$ is a multiple of $\delta$, we get

$$\begin{aligned}
Z_t^{(\delta)}(\omega) =& \sum_{s=0}^{t-\delta}{}_{\delta} \left( \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{\underline{H-\frac{3}{2}}} w_{\tau/\delta}(\omega) \right] \right. \\
& \left. + \frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H} w_{s/\delta}(\omega) \right) \\
=& \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left( \frac{\delta^{3/2}(H-1/2)}{C_H} \left\{ \sum_{s=0}^{t-\delta}{}_{\delta} (s-\tau)^{\underline{H-\frac{3}{2}}} \right\} w_{\tau/\delta}(\omega) \right) \\
& + \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left( \frac{\delta^{3/2}(H-1/2)}{C_H} \left\{ \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} (s-\tau)^{\underline{H-\frac{3}{2}}} \right\} w_{\tau/\delta}(\omega) \right) \\
& + \sum_{s=0}^{t-\delta}{}_{\delta} \frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H} w_{s/\delta}(\omega) \\
=& \frac{\sqrt{\delta}}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \left( \left\{ \sum_{s=0}^{t-\delta}{}_{\delta} \delta(H-1/2)\,(s-\tau)^{\underline{H-\frac{3}{2}}} \right\} w_{\tau/\delta}(\omega) \right) \\
& + \frac{\sqrt{\delta}}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \left( \left\{ \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ \delta(H-1/2)\,(s-\tau)^{\underline{H-\frac{3}{2}}} \right] + \delta^{\underline{H-\frac{1}{2}}} \right\} w_{\tau/\delta}(\omega) \right) \\
& + \frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H} w_{(t-\delta)/\delta}(\omega).
\end{aligned}$$

Again if $t = \delta$ the sum from $\tau = 0$ to $\tau = t - 2\delta$ does not appear. This means that

$$
\begin{aligned}
&\left|Z_t^{(\delta)}(\omega) - U_t^{(\delta)}(\omega)\right| \\
= &\left|\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left(\left\{\sum_{s=0}^{t-\delta}{}_{\delta}\,\delta(H-1/2)\,\underline{(s-\tau)^{H-\frac{3}{2}}}\right\}w_{\tau/\delta}(\omega)\right)\right. \\
&+\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left(\left\{\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\delta(H-1/2)\,\underline{(s-\tau)^{H-\frac{3}{2}}}\right]+\delta^{\underline{H-\frac{1}{2}}}\right\}w_{\tau/\delta}(\omega)\right) \\
&+\frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H}w_{(t-\delta)/\delta}(\omega) \\
&-\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left(\left\{\sum_{s=0}^{t-\delta}{}_{\delta}\left[\delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}}\right]\right\}w_{\tau/\delta}(\omega)\right) \\
&-\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left(\left\{\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\delta(H-1/2)\,(s-\tau)^{H-\frac{3}{2}}\right]+K_H^{(U)}\delta^{H-\frac{1}{2}}\right\}w_{\tau/\delta}(\omega)\right) \\
&\left.-\frac{K_H^{(U)}\delta^H}{C_H}w_{(t-\delta)/\delta}(\omega)\right| \\
= &\left|\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left(\left\{\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right\}w_{\tau/\delta}(\omega)\right)\right. \\
&+\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left(\left\{\delta(H-1/2)\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]+\delta^{\underline{H-\frac{1}{2}}}-K_H^{(U)}\delta^{H-\frac{1}{2}}\right\}w_{\tau/\delta}(\omega)\right) \\
&\left.+\left(\frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H}-\frac{K_H^{(U)}\delta^H}{C_H}\right)w_{(t-\delta)/\delta}(\omega)\right| \\
\le\ &\frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right| \\
&+\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]+\delta^{\underline{H-\frac{1}{2}}}-K_H^{(U)}\delta^{H-\frac{1}{2}}\right| \\
&+\left|\frac{\delta^{\underline{H-\frac{1}{2}}}\sqrt{\delta}}{C_H}-\frac{K_H^{(U)}\delta^H}{C_H}\right|,
\end{aligned}
\tag{6.14}
$$

where we have used that for a real series $\Sigma_{i=1}^{\infty}a_i$ which converges (conditionally) $|\Sigma_{i=1}^{\infty}a_i| \le \Sigma_{i=1}^{\infty}|a_i|$. This is just a simple convergence argument utilizing the triangle inequality and the proof is omitted. If $t=\delta$ the sum from $\tau=0$ to $\tau=t-2\delta$ does not appear. Notice also that the dependence of $\omega$ is now gone.

Assume first that $t=\delta$, then from eq. (6.14), lemma 6.3.1 and lemma 6.3.3

$$\begin{aligned}
&\left|Z_t^{(\delta)}(\omega)-U_t^{(\delta)}(\omega)\right| \\
\leq\; & \frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right| \\
&+\left|\frac{\underline{\delta^{H-\frac{1}{2}}}\sqrt{\delta}}{C_H}-\frac{K_H^{(U)}\delta^H}{C_H}\right| \\
\leq\; & M_{K,H}^{(1)}\delta^\alpha+M_H^{(1)}\delta^\alpha.
\end{aligned} \tag{6.15}$$

If $t\geq 2\delta$ we have by eq. (6.14), lemma 6.3.1, lemma 6.3.2 and lemma 6.3.3

$$\begin{aligned}
&\left|Z_t^{(\delta)}(\omega)-U_t^{(\delta)}(\omega)\right| \\
\leq\; & \frac{\sqrt{\delta}}{C_H}\sum_{\tau=-\infty}^{-\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=0}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]\right| \\
&+\frac{\sqrt{\delta}}{C_H}\sum_{\tau=0}^{t-2\delta}{}_{\delta}\left|\delta(H-1/2)\sum_{s=\tau+\delta}^{t-\delta}{}_{\delta}\left[\underline{(s-\tau)^{H-\frac{3}{2}}}-(s-\tau)^{H-\frac{3}{2}}\right]+\underline{\delta^{H-\frac{1}{2}}}-K_H^{(U)}\delta^{H-\frac{1}{2}}\right| \\
&+\left|\frac{\underline{\delta^{H-\frac{1}{2}}}\sqrt{\delta}}{C_H}-\frac{K_H^{(U)}\delta^H}{C_H}\right| \\
\leq\; & M_{K,H}^{(1)}\delta^\alpha+M_{K,H}^{(2)}\delta^\alpha+M_H^{(1)}\delta^\alpha.
\end{aligned} \tag{6.16}$$

To summarize, we have now proved that if $t\in[0,K+1]$, $t$ is a multiple of $\delta$ and $\omega\in E_{(\delta,H)}$ we have

$$\begin{aligned}
&\left|Z_t^{(\delta)}(\omega)-U_t^{(\delta)}(\omega)\right| \\
&\leq M_{K,H}^{(1)}\delta^\alpha+M_{K,H}^{(2)}\delta^\alpha+M_H^{(1)}\delta^\alpha.
\end{aligned} \tag{6.17}$$

Assume now that $t\in[0,K]$ but $t$ is not a multiple of $\delta$. We then have that there exists $N\in\mathbb{N}, r_1,r_2\in(0,1)$ such that

$$\begin{aligned}
(N-1)\delta<t<N\delta \\
r_1(N-1)\delta+r_2N\delta=t \\
r_1+r_2=1.
\end{aligned}$$

Notice also that

$$N\delta=N\delta-t+t\leq\delta+t\leq K+\delta\leq K+1.$$

So by the definition of $Z^{(\delta)}, U^{(\delta)}$ and eq. (6.17) we get for $\omega \in E_{(\delta,H)}$

$$\begin{aligned}
&\left|Z_t^{(\delta)}(\omega) - U_t^{(\delta)}(\omega)\right| \\
&= \left|r_1 Z_{(N-1)\delta}^{(\delta)}(\omega) + r_2 Z_{N\delta}^{(\delta)}(\omega) - r_1 U_{(N-1)\delta}^{(\delta)}(\omega) - r_2 U_{N\delta}^{(\delta)}(\omega)\right| \\
&\leq r_1 \left|Z_{(N-1)\delta}^{(\delta)}(\omega) - U_{(N-1)\delta}^{(\delta)}(\omega)\right| + r_2 \left|Z_{N\delta}^{(\delta)}(\omega) - U_{N\delta}^{(\delta)}(\omega)\right| \\
&\leq r_1 \left(M_{K,H}^{(1)} + M_{K,H}^{(2)} + M_H^{(1)}\right) \delta^\alpha + r_2 \left(M_{K,H}^{(1)} + M_{K,H}^{(2)} + M_H^{(1)}\right) \delta^\alpha \\
&= \left(M_{K,H}^{(1)} + M_{K,H}^{(2)} + M_H^{(1)}\right) \delta^\alpha.
\end{aligned} \tag{6.18}$$

So the result follows with

$$M_{K,H} = M_{K,H}^{(1)} + M_{K,H}^{(2)} + M_H^{(1)}.$$

□

We will now prove the corresponding result to lemma 5.3.1.

**Lemma 6.4.2.** *Let $H \in (0,1), \delta > 0, \epsilon > 0$. Let $Z^{(\delta)}$ be as in definition 6.1.1 and $U^{(\delta)}$ be as in definition 6.2.6, we assume that the underlying probability space $(\Omega, \mathcal{A}, P)$ in both definitions are the same. Let $\rho$ be the metric on $C[0,\infty)$ defined in theorem 2.2.1. We then have*

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) \leq \epsilon\right\} \in \mathcal{A}$$

*and*

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) < \epsilon\right\} \in \mathcal{A}$$

*Proof.* After the definitions of $P_n^{f,d}$ and $P_n^d$, see definition 6.1.2 and definition 6.2.7, we explained that these measures were well-defined. Which among other things means that for each $C \in \mathcal{C}$ we have $\left(Z^{(\delta)}\right)^{-1}(C) \in \mathcal{A}$ and $\left(U^{(\delta)}\right)^{-1}(C) \in \mathcal{A}$. We also recall from theorem 2.2.2 that $C[0,\infty)$ is separable. The result then follows from proposition B.2.15. □

We now give a result very similar to theorem 5.3.2, but our proof will be much simpler this time.

**Theorem 6.4.3.** *Let $H \in (0,1), \epsilon > 0$ and $\{\delta_n\}$ be a sequence of positive numbers converging to zero. Let $Z^{(\delta)}$ be as in definition 6.1.1 and $U^{(\delta)}$ be as in definition 6.2.6, where the underlying probability space $(\Omega, \mathcal{A}, P)$ and $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ are the same in both cases. Then*

$$P\left(\left\{\omega : \rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \to 0,$$

*as $n \to \infty$, where $\rho$ is the usual metric on $C[0,\infty)$.*

*Proof.* Note that by lemma 6.4.2 we have

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) \leq \epsilon\right\} \in \mathcal{A}$$

and

$$\left\{\omega \in \Omega : \rho\left(Z^{(\delta)}(\omega), U^{(\delta)}(\omega)\right) < \epsilon\right\} \in \mathcal{A},$$

so we can take the probability of these sets. Let $K$ be such that

$$\sum_{k=K+1}^{\infty} 2^{-k} < \frac{\epsilon}{2}.$$

Let $M_{K,H}$ and $\alpha$ be as in proposition 6.4.1. Since $\{\delta_n\}$ is a sequence of positive numbers converging to zero, we can choose an $n^*$ such that for $n \geq n^*$ we have

$$\delta_n^* < 1,$$

and

$$M_{H,K}\delta_n^\alpha \leq \frac{\epsilon}{2K}.$$

Assume now that $n \geq n^*$, $\omega \in E_{\delta_n,H}$, where $E_{\delta_n,H}$ is defined in proposition 6.4.1. We then have by proposition 6.4.1

$$\begin{aligned}
&\rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_n)}(\omega)\right) \\
&= \sum_{k=1}^{\infty} \min(2^{-k}, \sup\{|Z_t^{(\delta_n)}(\omega) - Z_t^{(\delta_n)}(\omega)| : t \in [0,k]\}) \\
&\leq \sum_{k=1}^{K} \sup\{|Z_t^{(\delta_n)}(\omega) - Z_t^{(\delta_n)}(\omega)| : t \in [0,k]\} + \sum_{k=1}^{\infty} 2^{-k} \\
&< K \cdot M_{H,K}\delta_n^\alpha + \frac{\epsilon}{2} \\
&\leq K \cdot \frac{\epsilon}{2K} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

So

$$E_{\delta_n,H} \subset \left\{\omega \in \Omega : \rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_N)}(\omega)\right) < \epsilon\right\}.$$

Since $P(E_{\delta_n,H}) = 1$ by proposition 6.4.1 we have that

$$P\left(\left\{\omega \in \Omega : \rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) = 1.$$

So

$$\begin{aligned}
&P\left(\left\{\omega \in \Omega : \rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \\
&= P\left(\left\{\omega \in \Omega : \rho\left(Z^{(\delta_n)}(\omega), U^{(\delta_n)}(\omega)\right) < \epsilon\right\}^c\right) \\
&= 0.
\end{aligned}$$

This completes the proof. □

## 6.5 Weak convergence of $U^{(\delta)}$

As expected we have that $U^{(\delta)}$ will be a good approximation to the Fractional Brownian motion when $\delta$ is small. We will formalize this in the next theorem.

**Theorem 6.5.1.** *Let $H \in (0,1)$, assume that $\{\delta_n\}$ is a sequence of positive real numbers converging to zero. For each $\delta_n$ let $P_n^d$ be the measure induced by $U_t^{(\delta_n)}$ (see definition 6.2.7). Then $\{P_n^d\}$ converges weakly to the measure $P$ induced by the Fractional Brownian motion, see definition 3.7.1 for details about $P$.*

*Proof.* This follows directly from theorem 6.4.3, theorem 5.4.2 and theorem 6.1.3. □

# Chapter 7

# An approximation process with finite summation

Up to now we have constructed four processes which can be used to approximate the fractional Brownian motion. These processes are $X^{(\delta)}$(definition 3.2.6), $Y^{(\delta)}$(definition 5.1.5), $Z^{(\delta)}$(definition 6.1.1) and $U^{(\delta)}$ defined in definition 6.2.6. In all of these processes we have to deal with infinite sums. It may be desirable to have a process with a finite sum, because finite sums are easier to deal with. So the goal of this chapter will be to create a similar process to the four processes already created, but where the tail is a finite sum.

## 7.1 Definition of the process

Based on $U^{(\delta)}$ we define a new process $V^{(\delta)}$ where the tail is finite, i.e. the sum is finite. That we choose to base our new process on $U^{(\delta)}$ and not $X^{(\delta)}, Y^{(\delta)}$ or $Z^{(\delta)}$ is mostly an arbitrary choice, and we could have chosen to show it with one of the other three processes.

When cutting off the tail, we have to make sure we do it in a satisfactory way. Since our processes are based on the Mandelbrot and Van Ness representation, see eq. (3.2) it is clear that we have to take account of more and more of the tail as $\delta$ gets smaller. Here we have to be careful, because it is not just enough to increase the number of elements as $\delta$ decreases as we may risk getting "stuck". Getting "stuck" here would mean that we do not a have representation that goes to infinity, but for instance only to $-T$, where $T$ is a positive real number. In the Mandelbrot and Van Ness representation this would mean that we end up integrating to $-T$, not to $-\infty$.

We will in fact define our process so that how much of the tail we take into account will depend on $H$. This is a technical detail to ensure weak convergence later. We are now ready to define our process.

**Definition 7.1.1.** *Let $H \in (0,1), \delta > 0$ be given. Let*

$$\mathcal{W} = \{\dots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \dots\}$$

*be a collection of independent random variables, each taking the values $\pm 1$ with equal probability. Assume that they are defined on a probability space $(\Omega, \mathcal{A}, P)$.*

*Let $k$ be the smallest natural number such that*

$$2k(1-H) > 1. \tag{7.1}$$

*Let $K_H^{(U)}$ be as in definition 6.2.4. Define the stochastic process $V^{(\delta)} = (V_t^{(\delta)})_{t\in[0,\infty)}$, which also depends on $H$ as follows*

*(i)*

$$V_0^{(\delta)}(\omega) \doteq 0, \qquad \forall \omega \in \Omega$$

*(ii) If $t \geq 0$ and there exists an $N \in \mathbb{N} \cup \{0\}$ such that $t = N\delta$, we define*

$$\begin{aligned}\Delta V_t^{(\delta)}(\omega) &\doteq V_{t+\delta}^{(\delta)}(\omega) - V_t^{(\delta)}(\omega) \\ &\doteq \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor \delta^{-(k+1)}\rfloor\delta}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{3}{2}}\, w_{\tau/\delta}(\omega)\right] \\ &\quad + \frac{K_H^{(U)}\delta^H}{C_H} w_{t/\delta}(\omega).\end{aligned}$$

*where*

$$C_H \doteq \left(\int_0^\infty \left((1+x)^{H-1/2} - x^{H-1/2}\right)^2 dx + \frac{1}{2H}\right)^{1/2}.$$

*(iii) We extend $V^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N+1)\delta$ and we define for all $\omega \in \Omega$*

$$V_t^{(\delta)}(\omega) = ((N+1)\delta - t)/\delta \cdot V_{N\delta}^{(\delta)}(\omega) + (t-N\delta)/\delta \cdot V_{(N+1)\delta}^{(\delta)}(\omega).$$

**Remark.** *We have that $V^{(\delta)}$ is a well-defined continuous stochastic process on $(\Omega, \mathcal{A}, P)$. For each $t$, $V_t^{(\delta)}$ is measurable since it is a linear combination of measurable random variables. And it is continuous by construction.*

Let us now compare this definition with the definition of $U^{(\delta)}$ in definition 6.2.6. There are two main differences. One that has already been pointed out is that we now have a finite tail. Another is that we now have omitted the set $D_\delta$ which had probability one. This set was to ensure that the infinite sums in definition 6.2.6 converge, but since we now only have finite sums, this is no longer a concern for us, so we get an even simpler representation.

## 7.2 Closeness of $V^{(\delta)}$ and $U^{(\delta)}$

It is of no surprise that $U^{(\delta)}$ and $V^{(\delta)}$ are "close" in a probabilistic sense. The next lemma is a technical result which will be used later.

**Lemma 7.2.1.** *Let $H \in (0,1), H \neq 1/2, \delta \in (0, 0.5]$. Let $k$ and $K$ be natural numbers. We then have*

$$\frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left((K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2 \leq R_{H,K} \cdot \delta^{2k(1-H)},$$

*where $R_{H,K}$ only depends on $H$ and $K$.*

*Proof.* Notice first that our values of $-\tau$ are bigger than 1 because

$$\begin{aligned} -\lfloor \delta^{-(k+1)} \rfloor \delta - \delta <& -\left(\delta^{-(k+1)} - 1\right)\delta - \delta \\ =& -\delta^{-k} \\ \leq& -2. \end{aligned}$$

So $-\tau - 1$ is always positive. If $H > 1/2$ we get

$$\begin{aligned} &\left|(K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right| \\ &= (K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}} \\ &\leq (K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}} \\ &= \left|(K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}}\right|. \end{aligned}$$

Similarly for $H < 1/2$ we get

$$\begin{aligned} &\left|(K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right| \\ &= (-\tau-\delta)^{H-\frac{1}{2}} - (K+1-\tau-\delta)^{H-\frac{1}{2}} \\ &\leq (-\tau-1)^{H-\frac{1}{2}} - (K+1-\tau)^{H-\frac{1}{2}} \\ &= \left|(K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}}\right|. \end{aligned}$$

This means that

$$\begin{aligned} &\frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta - \delta}{}_{\delta} \left((K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2 \\ &\leq \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta - \delta}{}_{\delta} \left((K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}}\right)^2. \end{aligned} \tag{7.2}$$

The derivative of $((K+1-x)^{H-\frac{1}{2}} - (-x-1)^{H-\frac{1}{2}})^2$ is positive when $x < -1$. We then get

$$\begin{aligned} &\frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta - \delta}{}_{\delta} \left((K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}}\right)^2 \\ &\leq \frac{1}{C_H^2} \int_{-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta} \left((K+1-x)^{H-\frac{1}{2}} - (-x-1)^{H-\frac{1}{2}}\right)^2 dx, \end{aligned} \tag{7.3}$$

because by the limits in the integral

$$\begin{aligned} x \leq& -\lfloor \delta^{-(k+1)} \rfloor \delta \\ <& -\delta^{-k} + \delta \\ \leq& -2 + \frac{1}{2} \\ =& -1.5. \end{aligned}$$

Notice that

$$\lim_{x\to-\infty}\frac{(K+1-x)^{H-\frac{1}{2}}-(-x-1)^{H-\frac{1}{2}}}{(-x)^{H-\frac{3}{2}}}$$
$$=\lim_{x\to-\infty}\frac{\left(\frac{K+1-x}{-x}\right)^{H-\frac{1}{2}}-\left(\frac{-x-1}{-x}\right)^{H-\frac{1}{2}}}{(-x)^{-1}}$$
$$=\lim_{x\to-\infty}\left(H-\frac{1}{2}\right)\frac{1}{-(-x)^{-2}(-1)}\left[\left(\frac{K+1-x}{-x}\right)^{H-\frac{3}{2}}\frac{-1(-x)-(-1)(K+1-x)}{x^2}\right.$$
$$\left.-\left(\frac{-x-1}{-x}\right)^{H-\frac{3}{2}}\frac{-(-x)-(-1)(-x-1)}{x^2}\right]$$

$$=\lim_{x\to-\infty}\left(H-\frac{1}{2}\right)\frac{\left(\frac{K+1-x}{-x}\right)^{H-\frac{3}{2}}\frac{K+1}{x^2}-\left(\frac{-x-1}{-x}\right)^{H-\frac{3}{2}}\frac{-1}{x^2}}{x^{-2}}$$
$$=\left(H-\frac{1}{2}\right)(K+2),$$

where we have used l'Hôpital's rule in the second step. This means that there exists a $K'$ such that if $x\leq K'$ we have

$$\left((K+1-x)^{H-\frac{1}{2}}-(-x-1)^{H-\frac{1}{2}}\right)^2\leq 2(K+2)^2\left(H-\frac{1}{2}\right)^2(-x)^{2H-3}.$$

$K'$ only depends on $H$ and $K$. We recall again that

$$-\lfloor\delta^{-(k+1)}\rfloor\delta<-1.5. \tag{7.4}$$

If $K'>-1.5$ we have that

$$\left((K+1-x)^{H-\frac{1}{2}}-(-x-1)^{H-\frac{1}{2}}\right)^2\leq 2(K+2)^2\left(H-\frac{1}{2}\right)^2(-x)^{2H-3}, \tag{7.5}$$

on $(-\infty,-1.5]$. Let us investigate what happens if $K'\leq-1.5$. On the compact interval $[K',-1.5]$ the continuous function $(-x)^{2H-3}$ is positive. Because of continuity and compactness it has a well-defined positive minimum, call this minimum $a$. There is at least one place in the interval where this value is taken, and all the other values are greater or equal to this value. The positive function $((K+1-x)^{H-\frac{1}{2}}-(-x-1)^{H-\frac{1}{2}})^2$ is also continuous on the interval, and has a well-defined maximum, call this maximum $b$. For $x\in[K',-1.5]$ we therefore have

$$\left((K+1-x)^{H-\frac{1}{2}}-(-x-1)^{H-\frac{1}{2}}\right)^2\leq\frac{2b}{a}(-x)^{2H-3}.$$

On $(-\infty,K')$ eq. (7.5) still holds. So if we now define

$$K''\doteq\max\left\{2(K+2)^2\left(H-\frac{1}{2}\right)^2,\frac{2b}{a}\right\},$$

we have that

$$\left((K+1-x)^{H-\frac{1}{2}} - (-x-1)^{H-\frac{1}{2}}\right)^2 \leq K''(-x)^{2H-3},$$

on $(-\infty, -1.5]$ no matter if $K' \leq -1.5$ or $K' > -1.5$. Using this with eq. (7.3) and recalling eq. (7.4) we have

$$\begin{aligned}
&\frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left((K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}}\right)^2 \\
&\leq \frac{1}{C_H^2} \int_{-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta} \left((K+1-x)^{H-\frac{1}{2}} - (-x-1)^{H-\frac{1}{2}}\right)^2 dx \\
&\leq \frac{K''}{C_H^2} \int_{-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta} (-x)^{2H-3} dx \qquad (7.6) \\
&= \frac{K''}{C_H^2} \int_{\lfloor\delta^{-(k+1)}\rfloor\delta}^{\infty} y^{2H-3} dy \\
&= \frac{K''}{C_H^2} \cdot \frac{1}{2H-2} \cdot (-1) \cdot \left(\lfloor\delta^{-(k+1)}\rfloor\delta\right)^{2H-2} \\
&= \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \cdot \left(\lfloor\delta^{-(k+1)}\rfloor\delta\right)^{2H-2}.
\end{aligned}$$

Since $\lfloor\delta^{-(k+1)}\rfloor > \delta^{-(k+1)} - 1$ we get

$$\begin{aligned}
&\frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \cdot \left(\lfloor\delta^{-(k+1)}\rfloor\delta\right)^{2H-2} \\
&< \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \left((\delta^{-(k+1)} - 1)\delta\right)^{2H-2} \qquad (7.7) \\
&= \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \left(\delta^{-k} - \delta\right)^{2H-2}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\delta^{-k} - \delta &= \frac{1}{2}\delta^{-k} + \frac{1}{2}\delta^{-k} - \delta \\
&\geq \frac{1}{2}\delta^{-k} + \delta\delta^{-k} - \delta \\
&= \frac{1}{2}\delta^{-k} + \delta(\delta^{-k} - 1) \\
&\geq \frac{1}{2}\delta^{-k},
\end{aligned}$$

since by assumption $\delta \in (0, 0.5]$. Combining this with eq. (7.2), eq. (7.6) and eq. (7.7) we have

$$\begin{aligned}
&\frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta - \delta}{}_{\delta} \left( (K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}} \right)^2 \\
&\leq \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor \delta^{-(k+1)} \rfloor \delta - \delta}{}_{\delta} \left( (K+1-\tau)^{H-\frac{1}{2}} - (-\tau-1)^{H-\frac{1}{2}} \right)^2 \\
&\leq \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \cdot \left( \lfloor \delta^{-(k+1)} \rfloor \delta \right)^{2H-2} \\
&< \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \left( \delta^{-k} - \delta \right)^{2H-2} \\
&\leq \frac{K''}{C_H^2} \cdot \frac{1}{2-2H} \left( \frac{1}{2} \delta^{-k} \right)^{2H-2} \\
&= \frac{K''}{C_H^2} \cdot \frac{2^{2-2H}}{2-2H} \delta^{2k(1-H)}.
\end{aligned}$$

Defining $R_{H,K}$ to be

$$R_{H,K} \doteq \frac{K''}{C_H^2} \frac{2^{2-2H}}{2-2H},$$

completes the proof. Notice that $R_{H,K}$ only depends on $H$ and $K$, specifically $K''$ did only depend on $H$, $K$ and $K'$. And $K'$ did only depend on $H$ and $K$. □

With the next result we move a step further towards our goal of weak convergence.

**Lemma 7.2.2.** *Let $H \in (0,1), \delta \in (0, 0.5]$. Let $U^{(\delta)}$ be as in definition 6.2.6 and $V^{(\delta)}$ be as in definition 7.1.1. We assume that they are defined on the same probability space $(\Omega, \mathcal{A}, P)$ and that $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ is the same for both processes. For every $K \in \mathbb{N}$ there exists a constant $M_{H,K}$ such that for every $t \in [0, K]$ we have*

$$E\left[ \left( U_t^{(\delta)} - V_t^{(\delta)} \right)^2 \right] \leq M_{H,K} \cdot \delta^{\alpha},$$

*where $\alpha > 1$. $M_{H,K}$ only depends on $H$ and $K$.*

*Proof.* Assume first that $t$ is a multiple of $\delta$ and that $t \in [0, K+1]$. If $t = 0$ we have

$$E\left[ \left( U_t^{(\delta)} - V_t^{(\delta)} \right)^2 \right] = 0,$$

so we assume that $t > 0$. We get

$$\begin{aligned} U_t^{(\delta)}(\omega) &= \sum_{s=0}^{t-\delta}{}_{\delta} \Delta U_s^{(\delta)}(\omega) \\ &= \sum_{s=0}^{t-\delta}{}_{\delta} \left\{ \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega) \right] \right. \\ &\qquad \left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) I_{D_\delta}(\omega) \right\} \\ &= \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{-\delta}{}_{\delta} \sum_{s=0}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega) \right] \\ &\quad + \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega) \right] \\ &\quad + \frac{K_H^{(U)}\delta^H}{C_H} \sum_{s=0}^{t-\delta}{}_{\delta} w_{s/\delta}(\omega) I_{D_\delta}(\omega). \end{aligned} \tag{7.8}$$

We here used that if we have a finite outer sum, and an infinite inner sum, and the infinite inner sum always converges, we can interchange the limits. This is proven in lemma D.3.4. For $V^{(\delta)}$ we similarly get

$$\begin{aligned} V_t^{(\delta)}(\omega) &= \sum_{s=0}^{t-\delta}{}_{\delta} \Delta V_s^{(\delta)}(\omega) \\ &= \sum_{s=0}^{t-\delta}{}_{\delta} \left\{ \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor \delta^{-(k+1)} \rfloor \delta}^{s-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) \right] \right. \\ &\qquad \left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{s/\delta}(\omega) \right\} \\ &= \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor \delta^{-(k+1)} \rfloor \delta}^{-\delta}{}_{\delta} \sum_{s=0}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) \right] \\ &\quad + \frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=0}^{t-2\delta}{}_{\delta} \sum_{s=\tau+\delta}^{t-\delta}{}_{\delta} \left[ (s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) \right] \\ &\quad + \frac{K_H^{(U)}\delta^H}{C_H} \sum_{s=0}^{t-\delta}{}_{\delta} w_{s/\delta}(\omega). \end{aligned} \tag{7.9}$$

Notice that since $P(D_\delta) = 1$ we have

$$\begin{aligned} E\left[ \left( U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega) \right)^2 \right] &= E\left[ \left( U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega) \right)^2 I_{D_\delta}(\omega) \right] \\ &= E\left[ \left( U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega) I_{D_\delta}(\omega) \right)^2 \right]. \end{aligned} \tag{7.10}$$

Combining eq. (7.8), eq. (7.9) and eq. (7.10) we get

$$
\begin{aligned}
&E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega)\right)^2\right] \\
&= E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega) I_{D_\delta}(\omega)\right)^2\right] \\
&= E\left[\left(\frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega) I_{D_\delta}(\omega)\right]\right)^2\right].
\end{aligned}
$$

Notice that this expression is zero if $H = 1/2$, so from now on we can assume that $H \neq 1/2$, because whichever $R_{H,K}$ and $\alpha$ we find for $H \neq 1/2$ will work for $H = 1/2$. We continue, and the expression above equals

$$
\begin{aligned}
&= \frac{\delta^3(H-1/2)^2}{C_H^2} E\left[\left(\sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}} I_{D_\delta}(\omega)\right]\right) w_{\tau/\delta}(\omega)\right)^2\right] \\
&= \frac{\delta^3(H-1/2)^2}{C_H^2} E\left[\lim_{N\to\infty} \left(\sum_{\tau=-N\delta}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}} I_{D_\delta}(\omega)\right]\right) w_{\tau/\delta}(\omega)\right)^2\right] \\
&\leq \frac{\delta^3(H-1/2)^2}{C_H^2} \liminf_{N\to\infty} E\left[\left(\sum_{\tau=-N\delta}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}} I_{D_\delta}(\omega)\right]\right) w_{\tau/\delta}(\omega)\right)^2\right],
\end{aligned}
$$

where we have used Fatou's lemma in the last step. The sums inside the expectation are now finite, this combined with the fact that $P(D_\delta) = 1$ tells us that the expression stays the same if remove $I_{D_\delta}(\omega)$. Hence we have

$$
\begin{aligned}
&E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega)\right)^2\right] \\
&\leq \frac{\delta^3(H-1/2)^2}{C_H^2} \liminf_{N\to\infty} E\left[\left(\sum_{\tau=-N\delta}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}}\right]\right) w_{\tau/\delta}(\omega)\right)^2\right] \\
&= \frac{\delta^3(H-1/2)^2}{C_H^2} \liminf_{N\to\infty} \sum_{\tau=-N\delta}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}}\right]\right)^2 \\
&= \frac{\delta^3(H-1/2)^2}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} \left[(s-\tau)^{H-\frac{3}{2}}\right]\right)^2 \\
&= \frac{\delta(H-1/2)^2}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} (s-\tau)^{H-\frac{3}{2}} \delta\right)^2.
\end{aligned}
\tag{7.11}
$$

Notice that since $\delta \in (0, 0.5]$ we have that $\delta^{k+1} < 1$, so $\lfloor\delta^{-(k+1)}\rfloor \geq 1$. This means that for our values of $\tau$ we have $-\tau \geq 2\delta$. We also have that the function

$$(x-\tau)^{H-\frac{3}{2}},$$

is positive and decreasing for non-negative $x$. This means that

$$\sum_{s=0}^{t-\delta}{}_{\delta} (s-\tau)^{H-\frac{3}{2}}\,\delta \le \int_0^t (x-\tau-\delta)^{H-\frac{3}{2}}dx, \tag{7.12}$$

because for $x \in (s, s+\delta)$ we have

$$(s-\tau)^{H-\frac{3}{2}} \le (x-\tau-\delta)^{H-\frac{3}{2}}.$$

From standard calculus techniques we get

$$\int_0^t (x-\tau-\delta)^{H-\frac{3}{2}}dx = \frac{1}{H-\frac{1}{2}}\left((t-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right), \tag{7.13}$$

(remember that we have already taken care of the case $H = 1/2$). By eq. (7.11), eq. (7.12) and eq. (7.13) we have

$$\begin{aligned}
&E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega)\right)^2\right] \\
&\le \frac{\delta(H-1/2)^2}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\sum_{s=0}^{t-\delta}{}_{\delta} (s-\tau)^{H-\frac{3}{2}}\,\delta\right)^2 \\
&\le \frac{\delta(H-1/2)^2}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left(\int_0^t (x-\tau-\delta)^{H-\frac{3}{2}}dx\right)^2 \qquad (7.14) \\
&= \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left((t-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2 \\
&\le \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left((K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2.
\end{aligned}$$

The last step follows because

$$\begin{aligned}
&\left((t-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2 \\
&\le \left((K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2,
\end{aligned}$$

since $t \le K+1$. It is easiest to see this for $H > 1/2$, but it also holds for $H < 1/2$. By lemma 7.2.1 and eq. (7.14) we get

$$\begin{aligned}
&E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega)\right)^2\right] \\
&\le \frac{\delta}{C_H^2} \sum_{\tau=-\infty}^{-\lfloor\delta^{-(k+1)}\rfloor\delta-\delta}{}_{\delta} \left((K+1-\tau-\delta)^{H-\frac{1}{2}} - (-\tau-\delta)^{H-\frac{1}{2}}\right)^2 \\
&\le R_{H,K}\cdot\delta^{2k(1-H)}.
\end{aligned}$$

From lemma 7.2.1 we know that $R_{H,K}$ only depends on $H$ and $K$. By definition 7.1.1 we have that $2k(1-H) > 1$, and we have now taken care of the case when $t$ is a multiple of $\delta$.

Assume now that $t \in [0, K]$ and $t$ is not a multiple of $\delta$. Then there is a unique $N \in \{0\} \cup \mathbb{N}$ such that

$$N\delta < t < (N+1)\delta.$$

We also have $a, b$ with $a, b \in (0, 1)$ and $a + b = 1$ such that

$$aN\delta + b(N+1)\delta = t.$$

Notice that

$$\begin{aligned}(N+1)\delta &= (N+1)\delta - t + t \\ &\leq (N+1)\delta - N\delta + t \\ &= \delta + t \\ &< 1 + K.\end{aligned}$$

So by the definitions of $U^{(\delta)}$, $V^{(\delta)}$ and what we proved above for $t$ being a multiple of $\delta$, we have

$$\begin{aligned}&E\left[\left(U_t^{(\delta)}(\omega) - V_t^{(\delta)}(\omega)\right)^2\right] \\ &= E\left[\left(aU_{N\delta}^{(\delta)}(\omega) + bU_{(N+1)\delta}^{(\delta)}(\omega) - aV_{N\delta}^{(\delta)}(\omega) - bV_{(N+1)\delta}^{(\delta)}(\omega)\right)^2\right] \\ &\leq E\left[2\left(aU_{N\delta}^{(\delta)}(\omega) - aV_{N\delta}^{(\delta)}(\omega)\right)^2 + 2\left(bU_{(N+1)\delta}^{(\delta)}(\omega) - bV_{(N+1)\delta}^{(\delta)}(\omega)\right)^2\right] \\ &= 2a^2E\left[\left(U_{N\delta}^{(\delta)}(\omega) - V_{N\delta}^{(\delta)}(\omega)\right)^2\right] + 2b^2E\left[\left(U_{(N+1)\delta}^{(\delta)}(\omega) - V_{(N+1)\delta}^{(\delta)}(\omega)\right)^2\right] \\ &\leq 2E\left[\left(U_{N\delta}^{(\delta)}(\omega) - V_{N\delta}^{(\delta)}(\omega)\right)^2\right] + 2E\left[\left(U_{(N+1)\delta}^{(\delta)}(\omega) - V_{(N+1)\delta}^{(\delta)}(\omega)\right)^2\right] \\ &\leq 2R_{H,K} \cdot \delta^{2k(1-H)} + 2R_{H,K} \cdot \delta^{2k(1-H)} \\ &= 4R_{H,K} \cdot \delta^{2k(1-H)}.\end{aligned}$$

Letting $M_{H,k} \doteq 4R_{H,K}$ completes the proof. □

Now we have enough tools to prove the main result of this section.

**Theorem 7.2.3.** *Let $H \in (0,1), \epsilon > 0$ and $\{\delta_n\}$ be a sequence of positive real numbers converging to zero. Let $U^{(\delta)}$ be as in definition 6.2.6 and $V^{(\delta)}$ be as in definition 7.1.1, where the underlying probability space $(\Omega, \mathcal{A}, P)$ and $\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\}$ are the same in both cases. Then*

$$P\left(\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \to 0,$$

*as $n \to \infty$, where $\rho$ is the usual metric on $C[0, \infty)$.*

*Proof.* We first recall that by lemma 3.3.2, proposition B.2.15 and the seperability of $C[0,\infty)$(theorem 2.2.2)

$$\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

so the theorem statement makes sense. We proceed by using the ideas from the proof of theorem 5.3.2.

Let $K \in \mathbb{N}$ be such that

$$\sum_{r=K+1}^{\infty} 2^{-r} < \frac{\epsilon}{2}. \tag{7.15}$$

Let $\delta_n$ be fixed for now. For $j \in \mathbb{N}$ define the set $F_j^{\delta_n}$ by

$$F_j^{\delta_n} \doteq \left\{\omega \in \Omega : \left|U_{j\delta_n}^{(\delta_n)}(\omega) - V_{j\delta_n}^{(\delta_n)}(\omega)\right| < \frac{\epsilon}{2K}\right\}.$$

Also define

$$F^{\delta_n} \doteq \bigcap_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} F_j^{\delta_n}.$$

By the properties of the real numbers this is a finite intersection, hence it is measurable by elementary sigma-algebra properties. We will show that if $\omega \in F^{\delta_n}$ and $t \in [0, K]$ we have

$$\left|U_t^{(\delta_n)}(\omega) - V_t^{(\delta_n)}(\omega)\right| < \frac{\epsilon}{2K}. \tag{7.16}$$

By definition of $U^{(\delta_n)}$ and $V^{(\delta_n)}$ we have that it holds for $t = 0$, and by construction of $F^{(\delta_n)}$ it also holds for all $t$ that are multiples of $\delta_n$. If $t$ is not a multiple of $\delta_n$ there is a unique $N \in \{0\} \cup \mathbb{N}$ such that

$$N\delta_n < t < (N+1)\delta_n,$$

and $a, b \in (0,1)$ such that $a + b = 1$ and

$$aN\delta_n + b(N+1)\delta_n = t.$$

Also notice that

$$\begin{aligned}(N+1)\delta_n &= (N+1)\delta_n - t + t \\ &\leq \delta_n + K.\end{aligned}$$

Hence, by the definition of $U^{(\delta_n)}$ and $V^{(\delta_n)}$ we get

$$\begin{aligned}&\left|U_t^{(\delta_n)}(\omega) - V_t^{(\delta_n)}(\omega)\right| \\ &= \left|aU_{N\delta_n}^{(\delta_n)}(\omega) - aV_{N\delta_n}^{(\delta_n)}(\omega) + bU_{(N+1)\delta_n}^{(\delta_n)}(\omega) - bV_{(N+1)\delta_n}^{(\delta_n)}(\omega)\right| \\ &\leq a\left|U_{N\delta_n}^{(\delta_n)}(\omega) - V_{N\delta_n}^{(\delta_n)}(\omega)\right| + b\left|U_{(N+1)\delta_n}^{(\delta_n)}(\omega) - V_{(N+1)\delta_n}^{(\delta_n)}(\omega)\right| \\ &\leq a \cdot \frac{\epsilon}{2K} + b \cdot \frac{\epsilon}{2K} \\ &= \frac{\epsilon}{2K}.\end{aligned}$$

Now we will show that

$$F^{\delta_n} \subset \left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) < \epsilon\right\}.$$

So assume that $\omega \in F^{(\delta_n)}$. We get

$$\begin{aligned}
&\rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right)\\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|f(t) - g(t)| : t \in [0, r]\})\\
&\leq \sum_{r=1}^{K} \sup\{|f(t) - g(t)| : t \in [0, r]\} + \sum_{r=K+1}^{\infty} 2^{-r}\\
&< \sum_{r=1}^{K} \frac{\epsilon}{2K} + \frac{\epsilon}{2}\\
&= \epsilon.
\end{aligned}$$

Where we in the second to last step have used eq. (7.15) and eq. (7.16). Hence

$$F^{\delta_n} \subset \left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) < \epsilon\right\}. \tag{7.17}$$

We are now ready for the probability calculations. We get using eq. (7.17) and the monotonicity of measures

$$\begin{aligned}
&P\left(\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right)\\
&= 1 - P\left(\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right)\\
&\leq 1 - P\left(F^{\delta_n}\right)\\
&= 1 - P\left(\bigcap_{\substack{j\in\mathbb{N}\\ j\delta_n \leq K+\delta_n}} F_j^{\delta_n}\right)\\
&= P\left(\bigcup_{\substack{j\in\mathbb{N}\\ j\delta_n \leq K+\delta_n}} \left(F_j^{\delta_n}\right)^c\right)\\
&\leq \sum_{\substack{j\in\mathbb{N}\\ j\delta_n \leq K+\delta_n}} P\left(\left(F_j^{\delta_n}\right)^c\right)\\
&\leq \sum_{\substack{j\in\mathbb{N}\\ j\delta_n \leq K+\delta_n}} P\left(\left\{\omega \in \Omega : \left|U_{j\delta_n}^{(\delta_n)}(\omega) - V_{j\delta_n}^{(\delta_n)}(\omega)\right| \geq \frac{\epsilon}{2K}\right\}\right).
\end{aligned} \tag{7.18}$$

From eq. (7.18) and Markov's inequality we get

$$\begin{aligned} &P\left(\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \\ \leq &\sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} P\left(\left\{\omega \in \Omega : \left|U_{j\delta_n}^{(\delta_n)}(\omega) - V_{j\delta_n}^{(\delta_n)}(\omega)\right| \geq \frac{\epsilon}{2K}\right\}\right) \\ \leq &\sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} E\left[\left|U_{j\delta_n}^{(\delta_n)}(\omega) - V_{j\delta_n}^{(\delta_n)}(\omega)\right|^2\right]\left(\frac{2K}{\epsilon}\right)^2 . \end{aligned} \tag{7.19}$$

Now it is time to see what happens when $\delta_n$ goes to zero. First there is an $n^*$ such that if $n \geq n^*$ we have that $\delta_n \leq 1/2$. Let us assume that our $n$ is always greater than $n^*$. From lemma 7.2.2 and eq. (7.19) we have

$$\begin{aligned} &P\left(\left\{\omega : \rho\left(U^{(\delta_n)}(\omega), V^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \\ \leq &\sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} E\left[\left|U_{j\delta_n}^{(\delta_n)}(\omega) - V_{j\delta_n}^{(\delta_n)}(\omega)\right|^2\right]\left(\frac{2K}{\epsilon}\right)^2 \\ \leq &\sum_{\substack{j\in\mathbb{N} \\ j\delta_n \leq K+\delta_n}} M_{H,K}\delta_n^{\alpha} \cdot \left(\frac{2K}{\epsilon}\right)^2 \\ \leq &\frac{K+\delta_n}{\delta_n} \cdot M_{H,K}\delta_n^{\alpha} \cdot \left(\frac{2K}{\epsilon}\right)^2 \\ = &(K+\delta_n)M_{H,K}\delta^{\alpha-1}\left(\frac{2K}{\epsilon}\right)^2 . \end{aligned}$$

This expression converges to zero since $\alpha > 1$ from lemma 7.2.2. □

## 7.3 Weak convergence of $V^{(\delta)}$

We are now ready for the result we have been working towards in this chapter. As before we define the induced measure.

**Definition 7.3.1.** *Let $H \in (0,1), \delta_n > 0$. Let $V^{(\delta_n)}$ be as in definition 7.1.1. We define the measure $P_n^V$ on $(C[0,\infty), \mathcal{C})$ as*

$$P_n^V(B) = P(V^{(\delta_n)} \in B), B \in \mathcal{C},$$

**Remark.** *As with the other induced measures encountered in the thesis, this measure is well-defined by lemma 3.3.2 and theorem C.1.1.*

**Theorem 7.3.2.** *Let $H \in (0,1)$, assume that $\{\delta_n\}$ is a sequence of positive real numbers converging to zero. For each $\delta_n$ let $P_n^V$ be the measure induced by $V_t^{(\delta_n)}$ (see definition 7.3.1). Then $\{P_n^V\}$ converges weakly to the measure $P$ induced by the Fractional Brownian motion, see definition 3.7.1 for details about $P$.*

*Proof.* Follows from theorem 7.2.3, theorem 5.4.2 and theorem 6.5.1. □

## 7.4 Useful results for later use

Here we will prove some results regarding $V^{(\delta)}$ used in chapter 8.

**Lemma 7.4.1.** *Let $H \in (0,1), \delta > 0$. Let $V^{(\delta)}$ be as in definition 7.1.1. Then there exists $R_H > 0$ such that for every $t$ that is a multiple of $\delta$ we have*

$$E\left[\left(\Delta V_t^{(\delta)}\right)^2\right] \leq R_H \delta^{2H}.$$

*Proof.* We get directly from definition 7.1.1

$$\begin{aligned}
E\left[\left(\Delta V_t^{(\delta)}\right)^2\right] &= \frac{\delta^3 (H-1/2)^2}{C_H^2} \sum_{\tau=-\lfloor \delta^{-(k+1)} \rfloor \delta}^{t-\delta}{}_{\delta} (t-\tau)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}}{C_H}\right)^2 \cdot \delta^{2H} \\
&\leq \frac{\delta^3 (H-1/2)^2}{C_H^2} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} (t-\tau)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}}{C_H}\right)^2 \cdot \delta^{2H} \\
&= \frac{\delta^3 (H-1/2)^2}{C_H^2} \sum_{r=1}^{\infty} (r\delta)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}}{C_H}\right)^2 \cdot \delta^{2H}.
\end{aligned}$$

The result follows with

$$\begin{aligned}
R_H &\doteq \frac{(H-1/2)^2}{C_H^2} \sum_{r=1}^{\infty} r^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}}{C_H}\right)^2.
\end{aligned}$$

The series converges because $2H - 3 < -1$. □

**Lemma 7.4.2.** *Let $H \in (0,1), \delta > 0, k' \in \mathbb{N}$. Let $V^{(\delta)}$ be as in definition 7.1.1. Then there exists $R_H^{(k')} > 0$ such that for every $t$ that is a multiple of $\delta$ we have*

$$E\left[\left|\Delta V_t^{(\delta)}\right|^{k'}\right] \leq R_H^{(k')} \delta^{Hk'}.$$

*Proof.* By Khintchine's inequality, see theorem 3.6.3 and lemma 7.4.1 we have

$$\begin{aligned}
E\left[\left|\Delta V_t^{(\delta)}\right|^{k'}\right] &\leq U_{k'} \left(E\left[\left(\Delta V_t^{(\delta)}\right)^2\right]\right)^{k'/2} \\
&\leq U_{k'} \left(R_H \delta^{2H}\right)^{k'/2} \\
&= U_{k'} R_H^{k'/2} \delta^{Hk'}.
\end{aligned}$$

The result follows with $R_H^{(k')} \doteq U_{k'} R_H^{k'/2}$. $\square$

**Lemma 7.4.3.** *Let $H \in (0,1), K \in \mathbb{N}, \sigma > 0$, $\{\delta_n\}$ a sequence of positive real numbers converging to zero. Assume $\epsilon, \epsilon_2 > 0$. Let $k^*$ be the smallest natural number such that $H(k^*+1) > 1$. For any $k \geq k^*+1$ there exists an $n_{k^*}$ such that if $n \geq n_{k^*}$ we have*

$$P\left(\left\{\omega \in \Omega : \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\sigma \Delta V_t^{\delta_n}(\omega)\right|^k \geq \epsilon\right\}\right) \leq \epsilon_2.$$

*Proof.* By Markov's inequality and lemma 7.4.2 we get

$$\begin{aligned}
&P\left(\left\{\omega \in \Omega : \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\sigma \Delta V_t^{\delta_n}\right|^k \geq \epsilon\right\}\right) \\
&\leq E\left[\sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\sigma \Delta V_t^{\delta_n}(\omega)\right|^k\right] \cdot \frac{1}{\epsilon} \\
&= \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} E\left[\left|\sigma \Delta V_t^{\delta_n}(\omega)\right|^k\right] \cdot \frac{1}{\epsilon} \\
&\leq \sigma^k \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} R_H^{(k)} \delta_n^{Hk} \cdot \frac{1}{\epsilon}.
\end{aligned}$$

The number of $t$ with $t \leq K+1$ and $t$ also is a multiple of $\delta_n$ is bounded by $(K+1)/\delta_n + 1$, hence the expression above is bounded by

$$\begin{aligned}
&\sigma^k \left(\frac{K+1}{\delta_n} + 1\right) R_H^{(k)} \delta_n^{Hk} \cdot \frac{1}{\epsilon} \\
&= \sigma^k \left(K+1+\delta_n\right) R_H^{(k)} \delta_n^{Hk-1} \cdot \frac{1}{\epsilon}.
\end{aligned}$$

By assumption $Hk - 1 > 0$, so by increasing $n$ we can get this expression as small as we want, and hence smaller than $\epsilon_2$. $\square$

**Lemma 7.4.4.** *Let $H \in (0,1), K \in \mathbb{N}, \mu \in \mathbb{R}, \sigma > 0$, $\{\delta_n\}$ a sequence of positive real numbers converging to zero. Assume $\epsilon, \epsilon_2 > 0, k \in \mathbb{N}$. Then there exists an $n^*$ such that if $n \geq n^*$ we have*

$$P\left(\left\{\omega \in \Omega : \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\mu\delta_n \left(\sigma \Delta V_t^{\delta_n}(\omega)\right)^k\right| \geq \epsilon\right\}\right) \leq \epsilon_2.$$

*Proof.* We will mimic the proof of lemma 7.4.3. We get from Markov's inequality and lemma 7.4.2

$$
\begin{aligned}
&P\left(\left\{\omega\in\Omega : \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}}^{\delta_n} \left|\mu\delta_n\left(\sigma\Delta V_t^{\delta_n}(\omega)\right)^k\right| \geq \epsilon\right\}\right)\\
&\le E\left[\sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}}^{\delta_n} \left|\mu\delta_n\left(\sigma\Delta V_t^{\delta_n}(\omega)\right)^k\right|\right]\cdot\frac{1}{\epsilon}\\
&=\frac{|\mu|\,\sigma^k\delta_n}{\epsilon}\sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}}^{\delta_n} E\left[\left|\Delta V_t^{\delta_n}(\omega)\right|^k\right]\\
&\le\frac{|\mu|\,\sigma^k\delta_n}{\epsilon}\sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}}^{\delta_n} R_H^{(k)}\delta_n^{Hk}.
\end{aligned}
$$

The number of t's we sum over is bounded by $(K+1)/\delta_n+1$. So the expression above is bounded by

$$
\begin{aligned}
&\frac{|\mu|\,\sigma^k\delta_n}{\epsilon}\left(\frac{K+1}{\delta_n}+1\right)R_H^{(k)}\delta_n^{Hk}\\
&=\frac{|\mu|\,\sigma^k}{\epsilon}\left(K+1+\delta_n\right)R_H^{(k)}\delta_n^{Hk}
\end{aligned}
$$

The last expression goes to zero as $\delta_n$ goes to zero, so we can bound it by $\epsilon_2$. □

**Corollary 7.4.5.** *Let $H\in(0,1), K\in\mathbb{N}, \mu\in\mathbb{R}, \sigma>0$, $\{\delta_n\}$ a sequence of positive real numbers converging to zero. Assume $\epsilon,\epsilon_2>0$. Then there exists an $n^*$ such that if $n\geq n^*$ we have*

$$
P\left(\left\{\omega\in\Omega : \sum\nolimits_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}}^{\delta_n} \left|\mu\delta_n-\mu\delta_n e^{-\sigma\Delta V_t^{(\delta_n)}(\omega)}\right| < \epsilon\right\}\right) > 1-\epsilon_2.
$$

*Proof.* If $\mu=0$ the result is obvious, so assume $|\mu|\neq 0$. Define

$$
A_1^{(n)} \doteq \bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\le K+1}} \left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| < 0.5\right\}.
$$

Let $k'$ be such that $Hk' > 1$. We get by Markov's inequality and lemma 7.4.2

$$\begin{aligned}
&P\left(A_1^{(n)}\right) \\
&= P\left(\bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| < 0.5\right\}\right) \\
&= 1 - P\left(\bigcup_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| \geq 0.5\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} P\left(\left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| \geq 0.5\right\}\right) \\
&= 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} P\left(\left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right|^{k'} \geq 0.5^{k'}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \frac{R_H^{(k')}\delta_n^{Hk'}}{0.5^{k'}} \\
&\geq 1 - \left(\frac{K+1}{\delta_n} + 1\right)\frac{R_H^{(k')}\delta_n^{Hk'}}{0.5^{k'}} \\
&= 1 - (K+1+\delta_n)\frac{R_H^{(k')}\delta_n^{Hk'-1}}{0.5^{k'}}.
\end{aligned}$$

There is an $n_1$ such that if $n \geq n_1$, this expression is strictly greater than $1 - \epsilon_2/2$. Define

$$A_2^{(n)} \doteq \left\{\omega\in\Omega : \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\mu\delta_n\left(\sigma\Delta V_t^{\delta_n}(\omega)\right)\right| < \frac{\epsilon}{2}\right\}.$$

From lemma 7.4.4 there is an $n_2$ such that if $n \geq n_2$ $P(A_2^{(n)})$ is strictly greater than $1 - \epsilon_2/2$. We now have for $n \geq \max(n_1, n_2)$

$$P\left(A_1^{(\delta_n)} \cap A_2^{(\delta_n)}\right) > 1 - \epsilon_2.$$

If we can show

$$\begin{aligned}
&A_1^{(\delta_n)} \cap A_2^{(\delta_n)} \\
&\subset \left\{\omega\in\Omega : \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\mu\delta_n - \mu\delta_n e^{-\sigma\Delta V_t^{(\delta_n)}(\omega)}\right| < \epsilon\right\},
\end{aligned}$$

the proof will be complete. So assume $\omega \in A_1^{(\delta_n)} \cap A_2^{(\delta_n)}$. Because of the definition of $A_1^{(n)}$ and lemma D.2.1 we have

$$\begin{aligned}
&\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}^{\delta_n} |\mu|\delta_n \left|1 - e^{-\sigma\Delta V_t^{(\delta_n)}(\omega)}\right| \\
= &\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}^{\delta_n} |\mu|\delta_n \left|r_1(-\sigma\Delta V_t^{(\delta_n)}(\omega))\sigma\Delta V_t^{(\delta_n)}(\omega)\right| \\
\leq &\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}^{\delta_n} 2|\mu|\delta_n \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right|.
\end{aligned}$$

Because also $\omega \in A_2^{(n)}$ we have

$$\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}^{\delta_n} 2|\mu|\delta_n \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| < \epsilon.$$

Hence the proof is complete. $\square$

# Chapter 8

# Applications to finance

It is time to apply our previous results to finance. First we will prove a result from which we easily will be able to approximate later examples. We will then refer to different cases where the Fractional Brownian motion is used in finance. We will also prove results about discrete stochastic differential equations, or stochastic difference equations if you will. In continuous-time financial markets the price process of the risky asset is usually modelled as a solution to a differential equation. Instead of just approximating the solution it is interesting to see what happens if we instead approximate the differential equation. Will the solution to the approximated differential equation then approximate the original solution to the differential equation? The answer to this question will turn out to be yes, but this is provided that we approximate our stochastic differential equation in a certain way. We will see that for $H < 1/2$ the approximation can be very interesting. Lastly we will analyse the solution to the stochastic difference equation. There we will see that for $H > 1/2$ it is possible to obtain negative values with positive probability. We will also compare the result for $H = 1/2$ with known results from continuous-time stochastic analysis.

## 8.1 Weak convergence to the Geometric fBM

In the subsequent section we will encounter processes of the form

$$S_0 e^{f(t)+\sigma B_{t,H}},$$

where $S_0 > 0, \sigma \in \mathbb{R}$,

$$f : [0, \infty) \to \mathbb{R}$$

is continuous and $(B_{t,H})_{t\in[0,\infty)}$ is a fractional Brownian motion. In this section we will see how we can approximate this process. First we need a result showing that the process is well-defined.

**Proposition 8.1.1.** *Let $(K_t)_{t\in[0,\infty)}$ be a continuous stochastic process on the probability space $(\Omega^*, \mathcal{A}^*, P^*)$. Assume that $S_0 > 0, \sigma \in \mathbb{R}$ and that*

$$f : [0, \infty) \to \mathbb{R},$$

*is continuous. Then $(S_t)_{t\in[0,\infty)}$ given by*

$$S_t(\omega) = S_0 e^{f(t)+\sigma K_t(\omega)},$$

*is a continuous stochastic process on* $(\Omega^*, \mathcal{A}^*, P^*)$*. And the measure defined by*

$$P(C) = P^* \left(S^{-1}(C)\right), C \in \mathcal{C},$$

*is a well-defined probability measure on* $(C[0,\infty), \mathcal{C})$*. Here we view* $S$ *as a function*

$$S : \Omega^* \to C[0,\infty).$$

*Proof.* We have continuity for each $\omega$ (trajectory) from elementary calculus, because sums, products and compositions of continuous functions are continuous. We need to show that for fixed t

$$\left(S_0 e^{f(t)+\sigma K_t}\right)^{-1}(B) \in \mathcal{A}^*, \qquad\qquad B \in \mathcal{B}(\mathbb{R}).$$

Notice first that for each $t$ the function $f_2 : \mathbb{R} \to \mathbb{R}$ given by

$$f_2(y) = S_0 e^{f(t)+\sigma y},$$

is continuous. So assume that $O \subset \mathbb{R}$ is open. Then $f_2^{-1}(O)$ is open, and $K_t^{-1}(f_2^{-1}(O)) \in \mathcal{A}^*$, since elementary measure theory tells us that open sets are Borel-measurable. Since the collection

$$\left\{B \in \mathcal{B}(\mathbb{R}) : \left(S_0 e^{f(t)+\sigma K_t}\right)^{-1}(B) \in \mathcal{A}^*\right\},$$

is a $\sigma$-algebra which contains the open sets, the result follows because $\mathcal{B}(\mathbb{R})$ is generated by the open sets.

The fact that $P$ is a well-defined probability measure on $(C[0,\infty), \mathcal{C})$ follows from the previous paragraph, lemma 3.3.2 and theorem C.1.1. □

The next result gives us a method for approximating the Geometric Fractional Brownian Motion. In the previous chapters we have worked with five processes, $X^{(\delta)}, Y^{(\delta)}, Z^{(\delta)}, U^{(\delta)}$ and $V^{(\delta)}$. Any one of them could be used for the following work, but we will work with $V^{(\delta)}$ since this is the last process we worked with, and it may be desirable only contains finite sums. The benefit of working with $V^{(\delta)}$ is also that we do not have to drag around the set of probability one which ensures convergence.

**Theorem 8.1.2.** *Let* $H \in (0,1)$ *and* $\{\delta_n\}$ *be a sequence of positive real numbers converging to zero. Assume that* $S_0 > 0, \delta \in \mathbb{R}$ *and* $f : [0,\infty) \to \mathbb{R}$ *is a continuous function. For each* $n$ *let* $V^{(\delta_n)}$ *be as in definition 7.1.1, we assume that all the w's are defined on the space* $(\Omega^*, \mathcal{A}^*, P^*)$*. Let* $P_n$ *be the measure induced on* $(C[0,\infty), \mathcal{C})$ *by*

$$S_0 e^{f(t)+\sigma V_t^{(\delta_n)}},$$

*by proposition 8.1.1 this measure exists. Let* $(B_{t,H})_{t\in[0,\infty)}$ *be the Fractional Brownian Motion from proposition 3.1.3, and denote the probability space it is defined on by* $(\Omega^{**}, \mathcal{A}^{**}, P^{**})$*. Define* $P$ *to be the measure on* $(C[0,\infty), \mathcal{C})$ *induced by*

$$S_0 e^{f(t)+\sigma B_{t,H}},$$

*again we refer to proposition 8.1.1 for its existence.*

*Then* $\{P_n\}$ *converges weakly to* $P$*.*

*Proof.* Let $O \subset C[0,\infty)$ be an open set. Let $R$ be the mapping

$$R : C[0,\infty) \to C[0,\infty),$$

given by

$$[R(x)](t) = S_0 e^{f(t)+\sigma x(t)}.$$

This mapping is continuous. To see this first use proposition B.3.2 on $\sigma x$, then proposition B.3.3 on $f + \sigma x$, then proposition B.3.4 on $\exp(f + \sigma x)$ and lastly proposition B.3.2 again on $S_0 \exp(f + \sigma x)$. From elementary theory of metric spaces we know that compositions of continuous functions are continuous, hence $R$ is continuous. This means that $R^{-1}(O)$ is open.

If we view $S_0 \exp(f(t) + \sigma V^{(\delta_n)})$ and $S_0 \exp(f(t) + \sigma B_{\cdot,H})$ as functions from $\Omega^*$ and $\Omega^{**}$ to $C[0,\infty)$ we have

$$\begin{aligned} \left[S_0 e^{f(t)+\sigma V^{(\delta_n)}}\right]^{-1}(O) &= \left(V^{(\delta_n)}\right)^{-1}\left(R^{-1}(O)\right), \\ \left[S_0 e^{f(t)+\sigma B_{\cdot,H}}\right]^{-1}(O) &= \left(B_{\cdot,H}^{(\delta_n)}\right)^{-1}\left(R^{-1}(O)\right) \end{aligned} \tag{8.1}$$

from elementary set-theory.

Recalling that $R^{-1}(O)$ is open, we get from theorem 7.3.2, eq. (8.1) and the Portmanteau Theorem[Bil99, Theorem 2.1, p. 16]

$$\begin{aligned} \liminf_{n\to\infty} P_n(O) &= \liminf_{n\to\infty} P^*\left(\left[S_0 e^{f(t)+\sigma V^{(\delta_n)}}\right]^{-1}(O)\right) \\ &= \liminf_{n\to\infty} P^*\left(\left(V^{(\delta_n)}\right)^{-1}\left(R^{-1}(O)\right)\right) \\ &\geq \liminf_{n\to\infty} P^{**}\left(\left(B_{\cdot,H}^{(\delta_n)}\right)^{-1}\left(R^{-1}(O)\right)\right) \\ &= \liminf_{n\to\infty} P^{**}\left(\left[S_0 e^{f(t)+\sigma B_{\cdot,H}}\right]^{-1}(O)\right) \\ &= \liminf_{n\to\infty} P(O). \end{aligned}$$

Hence, by using the Portmanteau Theorem again we have that $\{P_n\}$ converges weakly to $P$. □

## 8.2 Some words about approximation of processes

We will soon look at various uses of the fBm in finance. And we will use theorem 8.1.2 to approximate these processes. However, let us take a moment to discuss the essence of what we are doing. Traditionally when modelling a risky asset, one way to work is that you make a model, you use data to estimate the parameters in the model, and statistical methods or simulate new virtual data to judge if the model is good. A good example for how this work is done in the basic Black-Scholes market can be found in chapter 2 of [Ben04]. This work can be done with the models we have introduced, and also models we will introduce later. However, let us also notice that as one of our goals has been to model the fractional Brownian motion with Rademacher variables, we are in a sense "modelling a model".

This concept of approximating a stochastic process with another stochastic process can be viewed as a purely theoretical exercise, without concerning one's self with its use in finance. On the other hand, it may be a practical exercise through selection of realistic parameters, and then simulation to see if the processes agree. What makes an approximation good does not have only one correct answer. Even though statistical analysis and simulation are powerful tools, we will not use them in this thesis. We will use weak convergence as a way to argue that we can approximate the processes. That is, we will as we have done previously approximate a continuous stochastic process with another simpler stochastic process. The simpler process will depend on a positive real number $\delta_n$, and we will say that we can approximate the original process if the simpler process converges weakly to the original when $\delta_n$ converges to zero. The weak convergence is of course in $(C[0,\infty),\mathcal{C})$, described in chapter 2.

The benefit of using weak convergence is that it will give the same result regardless of the numerical values of the parameters. It is a clearly defined concept. And even though it may be disagreement about how well it captures the approximation, the preciseness of weak convergence gives a common understanding to all mathematicians and statisticians interested in approximation. So if one has other preferences when measuring approximations, weak convergence can still be a useful tool. The disadvantage of using only weak convergence is that it does not tell us how small $\delta_n$ must be. In practice the approximations will be better for some values of the parameters than others, in our case the $H$ in the fBm. This can be solved through statistical methods, simulations, testing etc. Even though these practical methods are outside the scope of this thesis, it is important to be aware of them.

## 8.3 Financial applications

Let us now delve into various applications.

### Traditional Black-Scholes model

The theory of the traditional Black-Scholes market is found in many books, we will use [Ben04] as a reference. In section 2.1 of [Ben04] it is stated that the stock in the Black-Scholes market is modelled by

$$S_t = S_0 e^{\mu t + \sigma B_t}, \tag{8.2}$$

where $\mu$ is the drift and $\sigma$ is called the volatility. After introducing stochastic analysis Benth shows that a similar process is a solution of a stochastic differential equation

$$dS(t) = \alpha S(t) + \sigma S(t) dB(t). \tag{8.3}$$

The process now takes the form

$$S_t = S_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma B_t}.$$

We will not go into the theory of stochastic differential equations here. Notice that the two models for $S_t$ are equivalent, they are just different parameterizations. Even though it is not explicitly stated in [Ben04] we assume $\mu \in \mathbb{R}, \sigma > 0$.

Since $kt$, $k \in \mathbb{R}$ is a continuous function, we can use theorem 8.1.2 to approximate this process. Notice also that there is a small technical point here. In the Black-Scholes model one only defines the process up to a time $T$. We have the tools for processes on entire non-negative real line $[0, \infty)$. When we in chapter 2 expanded Billingsley's work from $C[0,1]$ to $C[0,\infty)$, we could instead have generalized it to $C[0,T]$. We will not go any deeper into this points, they can easily be solved by doing what we have done earlier, with the sup-metric on $C[0,T]$. We will just take the position that we want to approximate the model presented in the Black-Scholes model, but we do not want to be constrained by a final time $T$.

Since we in this case is working with a Brownian Motion we have to use $H = 1/2$ in the model. In this case, the result will simply significantly, so it is worth writing it out. We will do it for the case $f(t) = \mu t$. By definition 6.2.4, we have

$$K_{1/2}^{(U)} = \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2} - \frac{1}{2}\right)} = \frac{\Gamma(2)}{\Gamma(2)} = 1.$$

We also have by the definition of $C_{\frac{1}{2}}$, see proposition 3.2.2

$$C_{\frac{1}{2}} \doteq \left( \int_0^\infty \left( (1+x)^0 - x^0 \right)^2 dx + \frac{1}{2 \cdot \frac{1}{2}} \right)^{1/2} = 1.$$

So if $t$ is a multiple of $\delta_n$ we get from definition 7.1.1

$$\Delta V_t^{(\delta_n)} = \sqrt{\delta} w_{t/\delta_n}.$$

Hence we get if $t$ is a multiple of $\delta_n$

$$S_0 e^{\mu t + \sigma V_t^{(\delta_n)}} = S_0, \qquad\qquad t = 0,$$

and if $t > 0$ we have

$$\begin{aligned} &S_0 e^{\mu t + \sigma V_t^{(\delta_n)}} \\ &= S_0 \exp\left( \mu t + \sum_{s=0}^{t-\delta}{}_{\delta} \Delta V_s^{(\delta_n)} \right) \\ &= S_0 \exp\left( \mu t + \sum_{s=0}^{t-\delta}{}_{\delta} \sqrt{\delta} w_{t/\delta_n} \right). \end{aligned}$$

Intuitively this approximation should not come as a surprise. It is well known that a Brownian motion can be constructed as the limit of a scaled random walk, see [Ros10, p. 631].

### Fractional Black-Scholes model

Before we define this new model, let us investigate a property of the ordinary Black-Scholes Model. On page 16 in [Ben04] it is stated that the log-returns are independent. Let us repeat this work here. If we have the times $t_0, t_1, \ldots, t_n$ and the stock prices $s_{t_0}, s_{t_1}, \ldots, s_{t_N}$ we define the log-returns

$$x_i \doteq \ln\left(\frac{s_{t_i}}{s_{t_{i-1}}}\right), i \in \{1, \ldots, N\}.$$

If we assume the model in eq. (8.2) we get that the corresponding stochastic variables become

$$X_i = \ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right), i \in \{1, \dots, N\}.$$

Notice that in the model we have that $S_t$ is always positive, so there is no problem dividing and taking the logarithm. We now get

$$\begin{aligned}\ln\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) &= \ln\left(\frac{S_0 e^{\mu t_i + \sigma B_{t_i}}}{S_0 e^{\mu t_{i-1} + \sigma B_{t_i}}}\right) \\ &= \ln\left(e^{\mu(t_i - t_{i-1}) + \sigma\left(B_{t_i} - B_{t_{i-1}}\right)}\right) \\ &= \mu\left(t_i - t_{i-1}\right) + \sigma\left(B_{t_i} - B_{t_{i-1}}\right).\end{aligned}$$

By the properties of the Brownian motion we know that the collection

$$\left\{B_{t_1} - B_{t_0}, \dots, B_{t_N} - B_{t_{N-1}}\right\}$$

is a collection of independent random variables. Since for each $i$ the function $f_i(x) = \mu(t_i - t_{i-1}) + \sigma x$ is Borel-measurable, it follows from [MW13, Proposition 7.6, p. 242], that the collection

$$\left\{\mu\left(t_i - t_{i-1}\right) + \sigma\left(B_{t_i} - B_{t_{i-1}}\right) : i \in \{1, \dots, N\}\right\}$$

is independent. This means that if we use the model eq. (8.2) for the stock price, we expected the observed log-returns $\{\ln(s_i/s_{i-1})\}$ to exhibit independence.

It is noted in [DØ11, pp. 75-76] that there are empirical studies where the log-returns are not independent. It is suggested that a more realistic way to model the stock price is using the fractional Brownian motion, instead of just the Brownian Motion. We can then model the independence through $H$. The Fractional Black-Scholes model is defined in [DØ11, p. 77] by

$$S_t = S_0 e^{\mu t + \sigma B_{t,H}}, H \neq \frac{1}{2},$$

$\mu \in \mathbb{R}, \sigma > 0$. By theorem 8.1.2 we can again approximate this process with simpler random variables.

### A model via Wick calculus

We will not go into the details of Wick calculus in this thesis. But we will present some results which are relevant to us. In chapter 3 of [Bia+08] a wick integral for fBm with $H > 1/2$ is introduced. In [Bia+08, p. 65] the solution to the stochastic differential equation is presented. If we change the symbols to what we have used in this chapter the equation is

$$\frac{dS_t}{dt} = (\mu + \sigma B_{t,H}) \diamond S_t, \tag{8.4}$$

where $\diamond$ is the wick product. In [TNY11, p. 22] a multidimensional Fractional Brownian Motion is used to model a finance market. It is also shown how to find a price of a certain exchange option. However, the stochastic differential equation for the first coordinate is the same as the one above. With the tools of [TNY11] it is now valid for $H > 1/4$. The price process is in both cases

$$S_t = S_0 e^{\mu t - \frac{\sigma^2}{2} t^{2H} + \sigma B_{t,H}}.$$

Notice the similarity with the ordinary Black-Scholes model, the part $-\sigma^2/2 \cdot t^{2H}$ corresponds to $-\sigma^2/2 \cdot t^2$ in the standard Black-Scholes model. This model is also mentioned by Sottinen in [Sot01b], where pricing of options using this model is also briefly mentioned. Sottinen assumes $H > 1/2$. Since the function $f(t) = \mu t - \sigma^2/2t^{2H}$ is continuous we can again refer to theorem 8.1.2 for an approximation of this process.

### Usage for weather derivatives

The fractional Brownian motion also has some use for pricing weather derivatives. In [BB12, p. 148] a model for temperature variations is presented. This model is

$$T(t) = \Lambda(t) + X^H(t),$$

where $\Lambda(t)$ is a seasonal trend, and $X^H$ satisfies

$$dX^H(t) = -\alpha X^H(t)dt + \sigma dB^H(t).$$

This stochastic differential equation makes sense in the Wick framework, and Benth solves it for $H > 1/2$.

### The value of H in financial modelling, long-range dependence

We have seen that most of the use of the fractional Brownian motion in finance is for large $H$, usually $H > 1/2$. This is because for $H > 1/2$ the fBm has long-range dependence, and this property has been seen in real data, see for instance [BB12, p. 147]. In [Sot01b] Sottinen states that in financial modelling it is assumed that $1/2 < H < 1$. However in [GJR14] they show that the log-volatility behaves as a fBm with $H = 0.1$, so a small $H$ can also be interesting in practice. What we mean by saying that the fBm is long range dependent for $H > 1/2$, is that in this case we have

$$\sum_{r=1}^{\infty} E\left[B_{1,H}(B_{r+1} - B_{r,H})\right] = \infty,$$

see [BB12, p. 148] or [GJR14, p. 3].

### Interpolation can cause arbitrage

In this thesis we give different ways of approximating the price process. We do not go into the details of the financial markets; the trading strategies, filtration etc. However there is one thing we have to be aware of when using theorem 8.1.2. If we allow continuous-time trading then it is easy to see that we will have arbitrage. This is because we know that $V_t^{(\delta)}$ is linearly interpolated, so if $t$

is a multiple of $\delta$ and we know $S_t^{(\delta)}(\omega)$ and $S_{t+\delta/2}^{(\delta)}(\omega)$, we can easily calculate $S_{t+\delta}^{(\delta)}(\omega)$.

So if we decide to use this price process we should only allow trading at time-points that are multiples of $\delta$.

## 8.4 Stochastic difference equations

In the previous section we saw that the price of the risky asset could be taken as a solution to a stochastic differential equation. Let us fix $H$ and $\delta$ in our discrete model. If we use eq. (8.3) and eq. (8.4) as inspiration we can create a similar model for the discrete case. If $S_0 > 0, \mu \in \mathbb{R}, \sigma > 0, \delta > 0$ we may model the risky asset as

$$S_t^{(\delta)}(\omega) = S_0, \text{ if } t = 0,$$

and

$$\begin{aligned}\Delta S_t^{(\delta)}(\omega) &\doteq S_{t+\delta}^{(\delta)}(\omega) - S_t^{(\delta)}(\omega) \\ &= \mu S_t^{(\delta)}(\omega)\Delta t + \sigma S_t^{(\delta)}(\omega)\Delta V_t^{(\delta)}(\omega) \\ &= \mu S_t^{(\delta)}(\omega)\delta + \sigma S_t^{(\delta)}(\omega)\Delta V_t^{(\delta)}(\omega),\end{aligned} \tag{8.5}$$

if $t \geq 0$ is a multiple of $\delta$. Notice that the difference equation is defined pointwise for each $\omega$. Sottinen worked with a simplified version of this process in [Sot01a]. Here Sottinen assumes $H > 1/2$ and models a cádlág process, not a continuous process. We will work with all $H \in (0,1)$ and work with functions on $C[0,\infty)$ as we have done throughout the thesis, which means we will use linear interpolation between time-points. As we also allow a drift component we have an extra challenge in the proof.

The model in eq. (8.5) will work in our framework for $H > 1/2$. However, for $H < 1/2$ we have to modify the the difference equation. A natural number $k^*$ will decide how the difference equation looks. Let $k^* \in \mathbb{N}$ be the smallest natural number such that $H(k^*+1) > 1$, notice that $k^*$ is indeed well-defined because it can also be chosen as the smallest natural number such that $k^* > 1/H - 1$. We still assume that $S_t = S_0$, for $t = 0$, but we now assume that the difference equation is

$$\begin{aligned}&\Delta S_t^{(\delta)}(\omega) \\ &= \mu S_t^{(\delta)}(\omega)\Delta t + S_t^{(\delta)}(\omega)\left(\sigma\Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right) \\ &= \mu S_t^{(\delta)}(\omega)\delta + S_t^{(\delta)}(\omega)\left(\sigma\Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right).\end{aligned} \tag{8.6}$$

By induction it is easy to see that if $t > 0$ is a multiple of $\delta$ we have

$$S_t^{(\delta)}(\omega) = S_0 \prod_{s=0}^{t-\delta}{}_\delta \left[1 + \mu\delta + \left(\sigma\Delta V_s^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_s^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right)\right].$$

The reason for having to modify the model with $k^*$ is technical, we will use lemma 7.4.3 in order to prove weak convergence. The choice of $k^*$ comes from trying to prove the result for $H < 1/2$ with $k^* = 1$. As lemma 7.4.3 does not hold in that case, the current proof of weak convergence did not work. That is why we had to increase $k^*$ for small $H$. We make a formal definition of the process.

**Definition 8.4.1.** *Let $H \in (0,1), \delta > 0$. Using these two parameters let $V^{(\delta)}$ be as in definition 7.1.1, with $(\Omega, \mathcal{A}, P)$ also from definition 7.1.1. Let $\mu \in \mathbb{R}, \sigma > 0, S_0 > 0$. Let $k^*$ be the smallest natural number such that $H(k^* + 1) > 1$. We define the stochastic process $S^{(\delta)} = (S_t^{(\delta)})_{t\in[0,\infty)}$ like this*

*(i)*

$$S_0^{(\delta)} \doteq S_0, \qquad \forall \omega \in \Omega.$$

*(ii) If $t > 0$ and there exists an $N \in \mathbb{N}$ such that $t = N\delta$ we define*

$$S_t^{(\delta)} \doteq S_0 \prod_{s=0}^{t-\delta}{}_\delta \left[1 + \mu\delta + \left(\sigma\Delta V_s^{(\delta)} + \frac{\left(\sigma\Delta V_s^{(\delta)}\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta)}\right)^{k^*}}{k^*!}\right)\right].$$

*(iii) We extend $S^{(\delta)}$ to all of $[0,\infty)$ by linear interpolation. Specifically if $t$ is not a multiple of $\delta$, there must exist a number $N \in \mathbb{N} \cup \{0\}$ such that $N\delta < t < (N + 1)\delta$ and we define for all $\omega \in \Omega$*

$$S_t^{(\delta)}(\omega) = ((N + 1)\delta - t)/\delta \cdot S_{N\delta}^{(\delta)}(\omega) + (t - N\delta)/\delta \cdot S_{(N+1)\delta}^{(\delta)}(\omega).$$

**Remark.** *It is clear that $S^{(\delta)}$ is a continuous stochastic process on $(\Omega, \mathcal{A}, P)$. From definition 7.1.1 we have that $\Delta V_t^{(\delta)}$ is measurable for each $t$, and for each $t$ we have that $S_t^{(\delta)}$ is a Borel-measurable function of different $\Delta V_s^{(\delta)}$.*

### Weak convergence

The goal of this subsection is to prove that the measure induced by $S_t^{(\delta)}$ in definition 8.4.1 converges weakly to the measure induced by the stochastic process

$$S_t = S_0 e^{\mu t + \sigma B_{t,H}}.$$

We will use a trick that Sottinen used in [Sot01a] where he uses that if $x$ is positive, then $x = e^{\ln(x)}$, but for this to make sense the positivity is crucial, or else the logarithm is not defined.

**Lemma 8.4.2.** *Let $H \in (0,1), \sigma > 0, \epsilon > 0$. Let $k^*$ be the smallest natural number such that $H(k^*+1) > 1$. Then there exists $\delta^* > 0, R > 0, \alpha > 1$ such that if we define $V_t^{(\delta)}$ by definition 7.1.1, where we use the parameters $H, \delta$, where $\delta$ is any number such that $0 < \delta \leq \delta^*$, we have for any $t$ being a multiple of $\delta$*

$$P\left(\left\{\omega \in \Omega : \left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!}\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right| \geq \epsilon\right\}\right) \leq R\delta^\alpha.$$

*Proof.* Assume first that $H = 1/2$, then $k^* = 2$. We have by definition 7.1.1

$$\Delta V_t^{(\delta)}(\omega) = \frac{\delta^H}{C_H} w_{t/\delta}(\omega).$$

We then have

$$\left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2}\right| = \frac{\sigma\delta^H}{C_H} + \frac{\sigma^2\delta^{2H}}{2C_H^2}. \tag{8.7}$$

As $\delta$ get small we can get this as small as we like, hence there must be a $\delta^*$ such that the expression is smaller than $\epsilon$ for $\delta \leq \delta^*$. We then get

$$P\left(\left\{\omega \in \Omega : \left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta))}(\omega)\right)^2}{2}\right| \geq \epsilon\right\}\right) = 0,$$

for $\delta \leq \delta^*$, so we can trivially choose $R = 1, \alpha = 2$.

Assume now that $H < 1/2$. We have

$$\begin{aligned}
\left|\Delta V_t^{(\delta)}(\omega)\right| &= \left|\frac{\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{t-\delta}{}_{\delta} \left[(t-\tau)^{H-\frac{3}{2}} w_{\tau/\delta}(\omega)\right]\right. \\
&\quad \left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{t/\delta}(\omega)\right| \\
&\le \frac{\delta^{3/2}(1/2-H)}{C_H} \sum_{\tau=-\infty}^{t-\delta}{}_{\delta} (t-\tau)^{H-\frac{3}{2}} \\
&\quad + \frac{\left|K_H^{(U)}\right|\delta^H}{C_H} \\
&= \frac{\delta^{3/2}(1/2-H)}{C_H} \sum_{\tau=\delta}^{\infty}{}_{\delta} \tau^{H-\frac{3}{2}} \\
&\quad + \frac{\left|K_H^{(U)}\right|\delta^H}{C_H} \\
&= \frac{\delta^{3/2}(1/2-H)}{C_H} \sum_{r=1}^{\infty} (r\delta)^{H-\frac{3}{2}} \\
&\quad + \frac{\left|K_H^{(U)}\right|\delta^H}{C_H} \\
&= \frac{\delta^{H}(1/2-H)}{C_H} \sum_{r=1}^{\infty} r^{H-\frac{3}{2}} \\
&\quad + \frac{\left|K_H^{(U)}\right|\delta^H}{C_H} \\
&= R_1\delta^H.
\end{aligned}$$

Where we have defined

$$R_1 \doteq \frac{(1/2-H)}{C_H} \sum_{r=1}^{\infty} r^{H-\frac{3}{2}} + \frac{\left|K_H^{(U)}\right|}{C_H}.$$

Notice that we have convergence of the sum because $H < 1/2$. So we now get

$$\begin{aligned}
&\left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta))}(\omega)\right)^2}{2!}\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right| \\
&\le \sigma R_1\delta^H + \frac{\sigma^2}{2!}\cdot\left(R_1\delta^H\right)^2 + \cdots + \frac{\sigma^{k^*}}{k^*!}\cdot\left(R_1\delta^H\right)^{k^*}.
\end{aligned} \tag{8.8}$$

This expression goes to zero as $\delta$ goes to zero. So there exists a $\delta$ such that if $\delta \le \delta$ this expression is less than $\epsilon$. We then have for $\delta \le \delta^*$

$$P\left(\left\{\omega\in\Omega : \left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta))}(\omega)\right)^2}{2!}\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right| \ge \epsilon\right\}\right) = 0.$$

So again we can trivially choose $R = 1, \alpha = 2$.

Lastly we assume $H > 1/2$. Then $k^* = 1$. We get

$$\begin{aligned}
&E\left[\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2\right] \\
&= \sigma^2 E\left[\left(\frac{\delta^{3/2}(H-1/2)}{C_H}\sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{t-\delta}{}_{\delta}\left[(t-\tau)^{H-\frac{3}{2}}\, w_{\tau/\delta}(\omega)\right]\right.\right. \\
&\quad \left.\left. + \frac{K_H^{(U)}\delta^H}{C_H} w_{t/\delta}(\omega)\right)^2\right] \\
&= \frac{\delta^3(H-1/2)^2\sigma^2}{C_H^2}\sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{t-\delta}{}_{\delta}(t-\tau)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}\sigma}{C_H}\right)^2\cdot\delta^{2H} \\
&\le \frac{\delta^3(H-1/2)^2\sigma^2}{C_H^2}\sum_{\tau=-\infty}^{t-\delta}{}_{\delta}(t-\tau)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}\sigma}{C_H}\right)^2\cdot\delta^{2H} \\
&= \frac{\delta^3(H-1/2)^2\sigma^2}{C_H^2}\sum_{\tau=\delta}^{\infty}{}_{\delta}\tau^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}\sigma}{C_H}\right)^2\cdot\delta^{2H} \\
&= \frac{\delta^3(H-1/2)^2\sigma^2}{C_H^2}\sum_{r=1}^{\infty}(r\delta)^{2H-3} \\
&\quad + \left(\frac{K_H^{(U)}\sigma}{C_H}\right)^2\cdot\delta^{2H} \\
&= R_2\delta^{2H}.
\end{aligned}$$

Here we define $R_2$ by

$$R_2 \doteq \frac{(H-1/2)^2\sigma^2}{C_H^2}\sum_{r=1}^{\infty} r^{2H-3} + \left(\frac{K_H^{(U)}\sigma}{C_H}\right)^2.$$

The series converges because $2H - 3 < 1$. By Markov's inequality we get

$$\begin{aligned}
&P\left(\left\{\omega \in \Omega : \left|\sigma \Delta V_t^{(\delta)}(\omega)\right| \geq \epsilon\right\}\right) \\
&= P\left(\left\{\omega \in \Omega : \left(\sigma \Delta V_t^{(\delta)}(\omega)\right)^2 \geq \epsilon^2\right\}\right) \\
&\leq \frac{E\left[\left(\sigma \Delta V_t^{(\delta)}(\omega)\right)^2\right]}{\epsilon^2} \\
&\leq \frac{R_2}{\epsilon^2} \cdot \delta^{2H}.
\end{aligned}$$

In this part we did not need to bound $\delta$ by a $\delta^*$ so we can trivially choose $\delta^* = 1$, the result follows with $R = R_2/\epsilon^2$, and $\alpha = 2H$. Notice that $\alpha > 1$ since $H > 1/2$. □

We also need a result telling us that in a certain sequence of of stochastic processes are bounded by a common constant $R$ on a compact interval with high probability.

**Lemma 8.4.3.** *Let $\mu \in \mathbb{R}, \sigma > 0, S_0 > 0, H \in (0,1)$. Let $\{\delta_n\}$ be a sequence of positive numbers converging to zero. For each $n$ let $V^{(\delta_n)}$ be as in definition 7.1.1. We look at the processes*

$$S_0 e^{\mu t + \sigma V_t^{(\delta_n)}}.$$

*From proposition 8.1.1 we know that these are stochastic process on a probability space $(\Omega, \mathcal{A}, P)$ where $\{V^{(\delta_n)}\}$ are stochastic processes. For each $\epsilon > 0, K \in \mathbb{N}$ there exists $R > 0, n^* \in \mathbb{N}$ such that if $n \geq n^*$ we have*

$$P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < R \;\; \forall t \in [0,K]\right\}\right) > 1 - \epsilon.$$

*Proof.* Notice first that

$$\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < R \;\; \forall t \in [0,K]\right\} \in \mathcal{A},$$

this follows from the fact that $\{f \in C[0,\infty) : |f(t)| < R \;\; \forall t \in [0,K]\}$ is open by lemma B.2.21 so it is contained in $\mathcal{C}$, and we can therefore refer to proposition 8.1.1.

Let $(B_{t,H})_{t \in [0,\infty)}$ be the fractional Brownian motion, we assume that it is a stochastic process on a space $(\Omega^*, \mathcal{A}^*, P^*)$. By proposition 8.1.1

$$\left(S_0 e^{\mu t + \sigma B_{t,H}}\right)_{t \in [0,\infty)}$$

is a stochastic process on $(\Omega^*, \mathcal{A}^*, P^*)$. Again by lemma B.2.21 and proposition 8.1.1 we have that for each $j \in \mathbb{N}$

$$\left\{\omega^* \in \Omega^* : \left|S_0 e^{\mu t + \sigma B_{t,H}(\omega^*)}\right| < j \;\; \forall t \in [0,K]\right\} \in \mathcal{A}^*.$$

Since continuous functions are bounded on compact intervals, we have

$$\bigcup_{j \in \mathbb{N}} \left\{\omega^* \in \Omega^* : \left|S_0 e^{\mu t + \sigma B_{t,H}(\omega^*)}\right| < j \;\; \forall t \in [0,K]\right\} = \Omega^*.$$

If we denote $A_j \doteq \{\omega^* \in \Omega^* : |S_0 e^{\mu t + \sigma B_{t,h}(\omega^*)}| < j \quad \forall t \in [0,K]\}$ we have $A_j \subset A_{j+1}$. So since $P^*(\Omega^*) = 1$ we get by the continuity of measures(see [MW13, p. 148]) that there exists a $J$ such that

$$P^*\left(\left\{\omega^* \in \Omega^* : \left|S_0 e^{\mu t + \sigma B_{t,h}(\omega^*)}\right| < J \ \ \forall t \in [0,K]\right\}\right) > 1 - \frac{\epsilon}{2}.$$

Let $P_n^{(1)}$ be the measure on $(C[0,\infty), \mathcal{C})$ induced by $S_0 e^{\mu t + \sigma V_t^{(\delta_n)}}$ and let $P^{(1)}$ be the measure induced on $(C[0,\infty), \mathcal{C})$ by $S_0 e^{\mu t + \sigma B_{t,H}^{(\delta_n)}}$. From theorem 8.1.2 $\{P_n^{(1)}\}$ converges weakly to $P^{(1)}$. So by the Portmanteau Theorem[Bil99, Theorem 2.1, p. 16] and lemma B.2.21 we have

$$\begin{aligned}
&\liminf_{n\to\infty} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right) \\
&= \liminf_{n\to\infty} P_n^{(1)}\left(\{f \in C[0,\infty) : |f(t)| < J \ \ \forall t \in [0,K]\}\right) \\
&\geq P^{(1)}\left(\{f \in C[0,\infty) : |f(t)| < J \ \ \forall t \in [0,K]\}\right) \\
&= P^*\left(\left\{\omega^* \in \Omega^* : \left|S_0 e^{\mu t + \sigma B_{t,h}(\omega^*)}\right| < J \ \ \forall t \in [0,K]\right\}\right) \\
&> 1 - \frac{\epsilon}{2}.
\end{aligned} \tag{8.9}$$

Since probability measures always are in the interval $[0,1]$ the $\liminf$ of a sequence of probability measures also must lie in this interval, hence we do not have any problems with $\pm\infty$. There must exists an $n^*$ such that if $n \geq n^*$ we have

$$\begin{aligned}
&\left|\liminf_{n\to\infty} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right)\right. \\
&\quad\left. - \inf_{k \geq n^*} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_k)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right)\right| \\
&\leq \frac{\epsilon}{2}.
\end{aligned} \tag{8.10}$$

The triangle inequality tells us that $|b| = |b - a + a| \leq |b - a| + |a|$, so $|a| \geq |b| - |b - a|$. If $a, b \geq 0$ we have $a \geq b - |b - a|$. Using this fact, eq. (8.9) and eq. (8.10) we have

$$\begin{aligned}
&\inf_{k \geq n^*} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_k)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right) \\
&\geq \liminf_{n\to\infty} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right) \\
&\quad - \left|\liminf_{n\to\infty} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right)\right. \\
&\quad\left. - \inf_{k \geq n^*} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_k)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right)\right| \\
&\geq \liminf_{n\to\infty} P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\right| < J \ \ \forall t \in [0,K]\right\}\right) - \frac{\epsilon}{2} \\
&> 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2}.
\end{aligned}$$

This completes the proof. □

The next lemma is a technical result needed later. It is needed because for $S_0 \exp(\mu t + \sigma V_t^{(\delta_n)})$ it is the logarithm and not the process itself which is linearly interpolated. This interpolation creates a small technical challenge in theorem 8.4.5.

**Lemma 8.4.4.** *Let $\mu \in \mathbb{R}, \sigma > 0, S_0 > 0, H \in (0,1)$. Let $\{\delta_n\}$ be a sequence of positive numbers converging to zero. For each $n$ let $V^{(\delta_n)}$ be as in definition 7.1.1. We look at the processes*

$$\tilde{S}_t^{(\delta_n)} = S_0 e^{\mu t + \sigma V_t^{(\delta_n)}}.$$

*From proposition 8.1.1 we know that these are stochastic process on a probability space $(\Omega, \mathcal{A}, P)$ where $\{V^{(\delta_n)}\}$ are stochastic processes. For $t = N\delta_n, N \in \{0\} \cup \mathbb{N}$ we define*

$$\Delta \tilde{S}_t^{(\delta_n)} \doteq \tilde{S}_{t+\delta_n}^{(\delta_n)} - \tilde{S}_t^{(\delta_n)}.$$

*For every $K \in \mathbb{N}, \epsilon > 0, \epsilon_2 > 0$ there exists a $n^*$ such that if $n \geq n^*$ we have*

$$P\left(\bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega \in \Omega : \left|\Delta \tilde{S}_t^{(\delta_n)}(\omega)\right| < \epsilon\right\}\right) > 1 - \epsilon_2.$$

*Proof.* From lemma 8.4.3 we have that there exists $R > 0, n_1$, such that

$$P\left(\left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\}\right) > 1 - \frac{\epsilon_2}{2},$$

for $n \geq n_1$. Define $\epsilon^* \doteq \min(0.5, \epsilon/R)$, and $e^{**} \doteq [\min(|\ln(1-\epsilon^*)|, \ln(1+\epsilon^*))]/2$. Also define

$$A_1^{(n)} \doteq \bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega \in \Omega : \left|\sigma \Delta V_t^{(\delta_n)}(\omega)\right| < \epsilon^{**}\right\}.$$

Let $R^*, \delta^*, \alpha$ be from lemma 8.4.2, we get

$$\begin{aligned}
&P\left(A_1^{(n)}\right) \\
&= 1 - P\left(\left(A_1^{(n)}\right)^c\right) \\
&= 1 - P\left(\bigcup_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega \in \Omega : \left|\sigma \Delta V_t^{(\delta_n)}(\omega)\right| \geq \epsilon^{**}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} P\left(\left\{\omega \in \Omega : \left|\sigma \Delta V_t^{(\delta_n)}(\omega)\right| \geq \epsilon^{**}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} P\left(\left\{\omega \in \Omega : \left|\sigma \Delta V_t^{(\delta_n)}(\omega)\right| + \cdots + \left|\frac{\left(\sigma \Delta V_t^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right| \geq \epsilon^{**}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} R^* \delta_n^{\alpha},
\end{aligned}$$

if $n \geq n_2$, where $n_2$ comes from lemma 8.4.2 by making $\delta_n$ smaller than $\delta^*$ if $n \geq n_2$. The number of $t$'s in the sum is bounded by $(K+1)/\delta_n + 1$ so we have

$$\begin{aligned}
&1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} R^*\delta_n^\alpha \\
&\geq 1 - \left(\frac{K+1}{\delta_n} + 1\right) R^*\delta_n^\alpha \\
&= 1 - (K+1+\delta_n)\, R^*\delta_n^{\alpha-1}.
\end{aligned}$$

Since by lemma 8.4.2, $\alpha > 1$ there is a $n_3$ such that this expression is larger than $1 - \epsilon_2/2$ if $n \geq n_3$. If $n \geq \max(n_1, n_2, n_3)$ we then have

$$\begin{aligned}
&P\left(A_1^{(n)} \cap \left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\}\right) \\
&= 1 - P\left(\left(A_1^{(n)}\right)^c \cup \left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\}^c\right) \\
&\geq 1 - P\left(\left(A_1^{(n)}\right)^c\right) - P\left(\left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\}^c\right) \\
&= P\left(A_1^{(n)}\right) + P\left(\left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\}\right) - 1 \\
&> 1 - \frac{\epsilon_2}{2} + 1 - \frac{\epsilon_2}{2} - 1 \\
&= 1 - \epsilon_2.
\end{aligned}$$

Choose $n_4$ such that $|\mu\delta_n| < \epsilon^{**}$ for $n \geq n_4$. To complete the proof it now suffices to show that if $n \geq \max(n_1, n_2, n_3, n_4)$ we have

$$\begin{aligned}
&A_1^{(n)} \cap \left\{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R \ \ \forall t \in [0, K+1]\right\} \\
&\subset \bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega \in \Omega : \left|\Delta\tilde{S}_t^{(\delta_n)}(\omega)\right| < \epsilon\right\}.
\end{aligned}$$

So for the remainder of the proof assume $t \in [0, K+1]$, $n \geq \max(n_1, n_2, n_3, n_4)$, $t$ is a multiple of $\delta_n$ and $\omega \in A_1^{(n)} \cap \{\omega \in \Omega : \left|\tilde{S}_t^{(\delta_n)}(\omega)\right| < R\, \forall t \in [0, K+1]\}$. We get

$$\begin{aligned}
&\left|\Delta\tilde{S}_t^{(\delta_n)}(\omega)\right| \\
&= \left|S_0 e^{\mu(t+\delta_n)+\sigma(V_t^{(\delta_n)}(\omega)+\Delta V_t^{(\delta_n)}(\omega))} - S_0 e^{\mu t+\sigma V_t^{(\delta_n)}(\omega)}\right| \\
&= S_0 e^{\mu t+\sigma V_t^{(\delta_n)}(\omega)} \left|e^{\mu\delta_n+\sigma\Delta V_t^{(\delta_n)}(\omega)} - 1\right| \\
&< R\left|e^{\mu\delta_n+\sigma\Delta V_t^{(\delta_n)}(\omega)} - 1\right|.
\end{aligned} \tag{8.11}$$

We also have

$$\begin{aligned}
&\left|\mu\delta_n + \sigma\Delta V_t^{(\delta_n)}(\omega)\right| \\
&\leq |\mu\delta_n| + \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| \\
&\leq \epsilon^{**} + \epsilon^{**} \\
&= \min(|\ln(1-\epsilon^*)|, \ln(1+\epsilon^*)).
\end{aligned} \tag{8.12}$$

Combining eq. (8.12) and lemma D.1.1 we have

$$\begin{aligned} & R\left|e^{\mu\delta_n+\sigma\Delta V_t^{(\delta_n)}(\omega)}-1\right| \\ & \leq Re^* \\ & \leq R\frac{\epsilon}{R} \\ & = \epsilon. \end{aligned}$$

This completes the proof. □

The next result is what is needed to show weak convergence of the processes in definition 8.4.1. It is a long proof because there are many quantities we need to bound in order for the mathematical operations to be legal and in order to get the desired result.

**Theorem 8.4.5.** *Let $H \in (0,1), \epsilon > 0, S_0 > 0, \sigma > 0, \mu \in \mathbb{R}$, and let $\{\delta_n\}$ be a sequence of positive real numbers converging to zero. Let $V^{(\delta)}$ be as in definition 7.1.1, defined on the probability space $(\Omega, \mathcal{A}, P)$. Let $S_t^{(\delta_n)}$ be as in definition 8.4.1, and let*

$$\tilde{S}_t^{(\delta_n)} = S_0 e^{\mu t+\sigma V_t^{(\delta_n)}}, \qquad\qquad t \in [0,\infty).$$

*Then*

$$P\left(\left\{\omega : \rho\left(S^{(\delta_n)}(\omega), \tilde{S}^{(\delta_n)}(\omega)\right) \geq \epsilon\right\}\right) \to 0,$$

*as $n \to \infty$, where $\rho$ is the usual metric on $C[0,\infty)$.*

*Proof.* As we have remarked in similar proofs, we have

$$\left\{\omega : \rho\left(S^{(\delta_n)}(\omega), \tilde{S}^{(\delta_n)}(\omega)\right) \geq \epsilon\right\} \in \mathcal{A},$$

by proposition B.2.15, lemma 3.3.2 and theorem 2.2.2.

Let $\epsilon_2 > 0$, it suffices to prove that there exists $n^*$ such that for $n \geq n^*$ we have

$$P\left(\left\{\omega : \rho\left(S^{(\delta_n)}(\omega), \tilde{S}^{(\delta_n)}(\omega)\right) < \epsilon\right\}\right) > 1-\epsilon_2.$$

Let $K \in \mathbb{N}$ be such that $\Sigma_{r=K+1}^{\infty} 2^{-r} < \epsilon/2$.

By lemma 8.4.3 there is an $R > 0$ and $n_1$ such that for $n \geq n_1$

$$P\left(\left\{\omega \in \Omega : \left|S_0 e^{\mu t+\sigma V_t^{(\delta_n)}(\omega)}\right| < R \;\; \forall t \in [0, K+1]\right\}\right) > 1-\frac{\epsilon_2}{6}. \tag{8.13}$$

Define

$$A_1^{(n)} \doteq \left\{\omega \in \Omega : \left|S_0 e^{\mu t+\sigma V_t^{(\delta_n)}(\omega)}\right| < R \;\; \forall t \in [0, K+1]\right\}.$$

Also define $\epsilon^* \doteq \min(0.5, \epsilon/(100RK))$, let

$$\epsilon^{**} \doteq \min\left(\left|\ln(1-\epsilon^*)\right|, \ln(1+\epsilon^*)\right).$$

Chose $n_2$ such that if $n \geq n_2$ we have

$$|\mu\delta_n| < \frac{1}{100}. \tag{8.14}$$

Set

$$A_2^{(n)} \doteq \bigcap_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \Bigg\{\omega \in \Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right| < \frac{1}{100}\Bigg\},$$

where $k^*$ is from definition 8.4.1. With the aid of lemma 8.4.2 we have

$$\begin{aligned}
P\left(A_2^{(n)}\right) &= 1 - P\left(\left(A_2^{(n)}\right)^c\right) \\
&= 1 - P\left(\bigcup_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} \left\{\omega \in \Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right| \geq \frac{1}{100}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} P\left(\left\{\omega \in \Omega : \left|\sigma\Delta V_t^{(\delta_n)}(\omega)\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right| \geq \frac{1}{100}\right\}\right) \\
&\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} R^*\delta_n^{\alpha},
\end{aligned}$$

if $\delta_n \leq \delta^*$ ($\delta^*$ and $R^*$ is from lemma 8.4.2). The number of $t$'s in the sum is bounded by $(K+1)/\delta_n + 1$ so we get

$$\begin{aligned}
P\left(A_2^{(n)}\right) &\geq 1 - \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}} R^*\delta_n^{\alpha} \\
&\geq 1 - \left(\frac{K+1}{\delta_n} + 1\right) R^*\delta_n^{\alpha} \\
&= 1 - (K + 1 + \delta_n) R^*\delta_n^{\alpha-1}.
\end{aligned}$$

Since $\alpha > 1$ there is an $n_3$ such that if $n \geq n_3$, we have $\delta_n \leq \delta^*$ and

$$P\left(A_2^{(n)}\right) > 1 - \frac{\epsilon_2}{6}. \tag{8.15}$$

Define

$$A_3^{(n)} \doteq \left\{\omega \in \Omega : \sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\sigma\Delta V_t^{\delta_n}(\omega)\right|^{k^*+1} < \frac{\epsilon^{**}}{24}\right\}.$$

From lemma 7.4.3 there is an $n_4$ so that if $n \geq n_4$ we have

$$\begin{aligned} P\left(A_3^n\right) &= 1 - P\left(\left(A_3^n\right)^c\right) \\ &\geq 1 - \frac{\epsilon_2}{12} \\ &> 1 - \frac{\epsilon_2}{6}. \end{aligned}$$

Let

$$A_4^{(n)} \doteq \left\{ \omega \in \Omega : \sum_{\substack{t = N\delta_n, N \in \{0\} \cup \mathbb{N} \\ t \leq K+1}}^{\delta_n} \left| \mu\delta_n - \mu\delta_n e^{-\sigma \Delta V_t^{(\delta_n)}(\omega)} \right| < \frac{\epsilon^{**}}{4} \right\}.$$

From corollary 7.4.5 there is an $n_5$ such that if $n \geq n_5$

$$\begin{aligned} P\left(A_4^{(n)}\right) &= 1 - P\left(\left(A_4^{(n)}\right)^c\right) \\ &\geq 1 - \frac{\epsilon_2}{12} \\ &> 1 - \frac{\epsilon_2}{6}. \end{aligned}$$

Let $n_6$ be such that if $n \geq n_6$ we have

$$\delta_n < 1. \tag{8.16}$$

Define

$$A_5^{(n)} \doteq \left\{ \omega \in \Omega : \sum_{\substack{t = N\delta_n, N \in \{0\} \cup \mathbb{N} \\ t \leq K+1}}^{\delta_n} \left| \sigma \Delta V_t^{\delta_n}(\omega) \right|^{2k^*+2} < \frac{\epsilon^{**}}{192} \right\}.$$

From lemma 7.4.3 there is an $n_7$ so that if $n \geq n_7$ we have

$$\begin{aligned} P\left(A_5^n\right) &= 1 - P\left(\left(A_5^n\right)^c\right) \\ &\geq 1 - \frac{\epsilon_2}{12} \\ &> 1 - \frac{\epsilon_2}{6}. \end{aligned}$$

By lemma D.3.5 there is an $n_8$ such that if $n \geq n_8$

$$\sum_{\substack{t = N\delta_n, N \in \{0\} \cup \mathbb{N} \\ t \leq K+1}}^{\delta_n} \left| \mu^2 \delta_n^2 \right| < \frac{\epsilon^{**}}{48}. \tag{8.17}$$

Define

$$A_6^{(n)} \doteq \bigcap_{\substack{t = N\delta_n, N \in \{0\} \cup \mathbb{N} \\ t \leq K+1}} \left\{ \omega \in \Omega : \left| \Delta \tilde{S}_t^{(\delta_n)}(\omega) \right| < \frac{\epsilon}{100K} \right\}.$$

By lemma 8.4.4 there is an $n_9$ so that if $n \geq n_9$

$$P\left(A_6^{(n)}\right) > 1 - \frac{\epsilon_2}{6}.$$

Define $n^*$

$$n^* \doteq \max(\{n_i : i \in \{1, \dots, 9\}\})$$

Notice that for $n \geq n^*$

$$\begin{aligned}
&P\left(\bigcap_{i\in\{1,\dots,6\}} A_i^{(n)}\right) \\
&= 1 - P\left(\bigcup_{i\in\{1,\dots,6\}} \left(A_i^{(n)}\right)^c\right) \\
&\geq 1 - \sum_{i=1}^{6} P\left(\left(A_i^{(n)}\right)^c\right) \\
&= \sum_{i=1}^{6} P\left(A_i^{(n)}\right) - 5 \\
&> 6 \cdot \left(1 - \frac{\epsilon_2}{6}\right) - 5 \\
&= 1 - \epsilon_2.
\end{aligned}$$

To finish the proof it suffices to prove

$$\begin{aligned}
&A_1^{(n)} \cap A_2^{(n)} \cap A_3^{(n)} \cap A_4^{(n)} \cap A_5^{(n)} \cap A_6^{(n)} \\
&\subset \left\{\omega : \rho\left(S^{(\delta_n)}(\omega), \tilde{S}^{(\delta_n)}(\omega)\right) < \epsilon\right\},
\end{aligned}$$

for $n \geq n^*$. Assume first that $\omega \in A_1^{(n)} \cap A_2^{(n)} \cap A_3^{(n)} \cap A_4^{(n)} \cap A_5^{(n)} \cap A_6^{(n)}$ and $t \in (0, K+1]$ is a multiple of $\delta_n$. We have

$$\begin{aligned}
&\left|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)\right| \\
&= S_0 \left| e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)} - \prod_{s=0}^{t-\delta_n}{}_{\delta_n} \left[1 + \mu\delta_n + \sigma\Delta V_s^{(\delta_n)}(\omega) + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right]\right| \\
&= S_0 \Bigg| e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)} \\
&\qquad - \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + \mu\delta_n + \sigma\Delta V_s^{(\delta_n)}(\omega) + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right)\right)\Bigg| \\
&= S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)} \Bigg| 1 \\
&\quad - \exp\Bigg(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + \mu\delta_n + \sigma\Delta V_s^{(\delta_n)}(\omega) + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right) \\
&\quad - \mu\delta - \sigma\Delta V_t^{(\delta_n)}(\omega)\Bigg)\Bigg|.
\end{aligned}$$

The rewriting of the product using the exponential-logarithm trick is legal because every term in the product is positive by the bounds in definition of $A_2^{(n)}$ and eq. (8.14). Because of the bound in $A_2^{(n)}$ we can use lemma D.2.1 to write

$$\begin{aligned}
&1 + \sigma\Delta V_s^{(\delta_n)}(\omega) + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!} \\
&= e^{\sigma\Delta V_s^{(\delta_n)}(\omega)} + r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\left|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)\right| \\
&= S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\Bigg|1 \\
&- \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + \mu\delta_n + \sigma\Delta V_s^{(\delta_n)}(\omega) + \cdots + \frac{\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*}}{k^*!}\right)\right. \\
&\left. - \mu\delta_n - \sigma\Delta V_t^{(\delta_n)}(\omega)\right)\Bigg| \\
&= S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\Bigg|1 \\
&- \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(e^{\sigma\Delta V_s^{(\delta_n)}(\omega)} + r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1} + \mu\delta_n\right)\right. \\
&\left. - \mu\delta_n - \sigma\Delta V_t^{(\delta_n)}(\omega)\right)\Bigg| \\
&= S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\Bigg|1 \\
&- \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1} + \mu\delta_n\right]\right)\right. \\
&\left. - \mu\delta_n\right)\Bigg|.
\end{aligned} \tag{8.18}$$

Notice that

$$\begin{aligned}
&\left|e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1} + \mu\delta_n\right]\right| \\
&\le e^{1/100}\left(\frac{2}{100} + \frac{1}{100}\right) \\
&< 3 \cdot \frac{3}{100} \\
&< 0.5.
\end{aligned}$$

So by lemma D.2.2

$$\begin{aligned}
&\ln\left(1+e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)\\
&=e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\\
&\quad+r_2^*\cdot\left(e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)^2,
\end{aligned}$$

where

$$r_2^*\doteq r_2\left(e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right),$$

$r_2$ is from lemma D.2.2. Using this we get

$$\begin{aligned}
&\left|\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\ln\left(1+e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)-\mu\delta_n\right|\\
&\le\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\left|\ln\left(1+e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)-\mu\delta_n\right|\\
&=\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\Bigg|\ e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\\
&\qquad+r_2^*\cdot\left(e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)^2\\
&\qquad-\mu\delta_n\Bigg|\\
&\le\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\left|e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}\right|\\
&+\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\left|e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\mu\delta_n-\mu\delta_n\right|\\
&+\sum_{s=0}^{t-\delta_n}{}_{\delta_n}2\left|r_2^*e^{-2\sigma\Delta V_s^{(\delta_n)}(\omega)}r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))^2\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{2k^*+2}\right|\\
&+\sum_{s=0}^{t-\delta_n}{}_{\delta_n}2\left|r_2^*e^{-2\sigma\Delta V_s^{(\delta_n)}(\omega)}\mu^2\delta_n^2\right|
\end{aligned}\tag{8.19}$$

Since $|\sigma\Delta V_t^{(\delta_n)}(\omega)|<1/100$, a rough upper bound for $|\exp(-\sigma\Delta V_t^{(\delta_n)}(\omega))|$ and $|\exp(-2\sigma\Delta V_t^{(\delta_n)}(\omega))|$ is 3. We also recall that $|r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))|$ and $|r_2^*|$

are bounded by 2. So by eq. (8.19) we have

$$\begin{aligned}
&\left|\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1+e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)-\mu\delta_n\right| \\
\leq\ & \sum_{s=0}^{t-\delta_n}{}_{\delta_n} 6\left|\sigma\Delta V_s^{(\delta_n)}(\omega)\right|^{k^*+1}+\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\left|e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\mu\delta_n-\mu\delta_n\right| \\
&+\sum_{s=0}^{t-\delta_n}{}_{\delta_n} 48\left|\sigma\Delta V_s^{(\delta_n)}(\omega)\right|^{2k^*+2}+\sum_{s=0}^{t-\delta_n}{}_{\delta_n} 12\left|\mu^2\delta_n^2\right|.
\end{aligned}$$

By the definitions of $A_3^{(n)}, A_4^{(n)}, A_5^{(n)}$ and eq. (8.17)

$$\begin{aligned}
&\sum_{s=0}^{t-\delta_n}{}_{\delta_n} 6\left|\sigma\Delta V_s^{(\delta_n)}(\omega)\right|^{k^*+1}+\sum_{s=0}^{t-\delta_n}{}_{\delta_n}\left|e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\mu\delta_n-\mu\delta_n\right| \\
&\quad+\sum_{s=0}^{t-\delta_n}{}_{\delta_n} 48\left|\sigma\Delta V_s^{(\delta_n)}(\omega)\right|^{2k^*+2}+\sum_{s=0}^{t-\delta_n}{}_{\delta_n} 12\left|\mu^2\delta_n^2\right| \\
&\leq 6\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}{}_{\delta_n}\left|\sigma\Delta V_t^{\delta_n}(\omega)\right|^{k^*+1}+\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}{}_{\delta_n}\left|\mu\delta_n-\mu\delta_n e^{-\sigma\Delta V_t^{(\delta_n)}(\omega)}\right| \\
&\quad+48\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}{}_{\delta_n}\left|\sigma\Delta V_t^{\delta_n}(\omega)\right|^{2k^*+2}+12\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N}\\ t\leq K+1}}{}_{\delta_n}\left|\mu^2\delta_n^2\right| \\
&< 6\cdot\frac{\epsilon^{**}}{24}+\frac{\epsilon^{**}}{4}+48\cdot\frac{\epsilon^{**}}{192}+12\cdot\frac{\epsilon^{**}}{48} \\
&= \epsilon^{**}.
\end{aligned}$$

Hence we have shown

$$\begin{aligned}
&\left|\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1+e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1}+\mu\delta_n\right]\right)-\mu\delta_n\right| \\
<\ & \epsilon^{**} \\
=\ & \min\left(\left|\ln(1-\epsilon^*)\right|, \ln(1+\epsilon^*)\right).
\end{aligned} \tag{8.20}$$

From eq. (8.18) and the definition of $A_1^{(n)}$ we have

$$\begin{aligned}
&\left|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)\right| \\
= S_0 e^{\mu t + \sigma V_t^{(\delta_n)}(\omega)}\Bigg| 1 \\
&- \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1} + \mu\delta_n\right]\right)\right. \\
&\left. - \mu\delta_n\right)\Bigg|. \\
< R\Bigg| 1 \\
&- \exp\left(\sum_{s=0}^{t-\delta_n}{}_{\delta_n} \ln\left(1 + e^{-\sigma\Delta V_s^{(\delta_n)}(\omega)}\left[r_1(\sigma\Delta V_s^{(\delta_n)}(\omega))\left(\sigma\Delta V_s^{(\delta_n)}(\omega)\right)^{k^*+1} + \mu\delta_n\right]\right)\right. \\
&\left. - \mu\delta_n\right)\Bigg|.
\end{aligned} \tag{8.21}$$

Combining eq. (8.20) and eq. (8.21) with lemma D.1.1 we get

$$\begin{aligned}
&\left|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)\right| \\
&< R\epsilon^* \\
&\le R \cdot \frac{\epsilon}{100RK} \\
&= \frac{\epsilon}{100K}.
\end{aligned}$$

We note if $t = 0$ then by definition $|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)| = 0$.

To summarize, we have proven that if $\omega \in A_1^{(n)} \cap A_2^{(n)} \cap A_3^{(n)} \cap A_4^{(n)} \cap A_5^{(n)} \cap A_6^{(n)}, n \ge n^*$ and $t \in [0, K+1]$ where $t$ is a multiple of $\delta_n$ we have

$$\left|\tilde{S}_t^{(\delta_n)}(\omega) - S_t^{(\delta_n)}(\omega)\right| < \frac{\epsilon}{100K}.$$

Consider now the same assumptions for $\omega$ and $n$, but instead we have $t \in [0, K]$ and $t$ is not a multiple of $\delta_n$. There must be an $N \in \{0\} \cup \mathbb{N}$ such that $N\delta_n < (N+1)\delta_n$. By eq. (8.16) we have

$$(N+1)\delta_n = (N+1)\delta_n - t + t < \delta_n + K < K + 1.$$

From what we have proven already we have

$$\begin{aligned}
\left|\tilde{S}_{N\delta_n}^{(\delta_n)}(\omega) - S_{N\delta_n}^{(\delta_n)}(\omega)\right| &< \frac{\epsilon}{100K}, \\
\left|\tilde{S}_{(N+1)\delta_n}^{(\delta_n)}(\omega) - S_{(N+1)\delta_n}^{(\delta_n)}(\omega)\right| &< \frac{\epsilon}{100K}.
\end{aligned} \tag{8.22}$$

From the definition of $A_6^{(n)}$ we have

$$\begin{aligned} \left|\Delta\tilde{S}^{(\delta_n)}_{N\delta_n}(\omega)\right| &= \left|\tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &< \frac{\epsilon}{100K}. \end{aligned} \tag{8.23}$$

This means that we also have

$$\begin{aligned} \left|S^{(\delta_n)}_{(N+1)\delta_n}(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)\right| &\leq \left|S^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega)\right| \\ &\quad + \left|\tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &\quad + \left|\tilde{S}^{(\delta_n)}_{N\delta_n}(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &< \frac{3\epsilon}{100K}. \end{aligned} \tag{8.24}$$

Because we use linear interpolation of the logarithm of $\tilde{S}^{(\delta_n)}_t$ we must have that $|\tilde{S}_{(N+1)\delta_n}(\omega) - \tilde{S}_t(\omega)|$ is bounded by $|\tilde{S}_{(N+1)\delta_n}(\omega) - \tilde{S}_{N\delta_n}(\omega)|$, because the exponential function is monotone. Since we have used linear interpolation in the definition of $S^{(\delta_n}_t$ we must in the same way have that $|S^{(\delta_n)}_t(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)|$ is bounded by $|S^{(\delta_n)}_{(N+1)\delta_n}(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)|$. Using these two facts, together with eq. (8.22), eq. (8.23) and eq. (8.24) we have

$$\begin{aligned} \left|S^{(\delta_n)}_t(\omega) - \tilde{S}^{(\delta_n)}_t(\omega)\right| &\leq \left|S^{(\delta_n)}_t(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &\quad + \left|S^{(\delta_n)}_{N\delta_n}(\omega) - S^{(\delta_n)}_{(N+1)\delta_n}(\omega)\right| \\ &\quad + \left|S^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega)\right| \\ &\quad + \left|\tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_t(\omega)\right| \\ &< \left|S^{(\delta_n)}_{(N+1)\delta_n}(\omega) - S^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &\quad + \frac{3\epsilon}{100K} + \frac{\epsilon}{100K} \\ &\quad + \left|\tilde{S}^{(\delta_n)}_{(N+1)\delta_n}(\omega) - \tilde{S}^{(\delta_n)}_{N\delta_n}(\omega)\right| \\ &\leq \frac{3\epsilon}{100K} + \frac{4\epsilon}{100K} + \frac{\epsilon}{100K} \\ &= \frac{8\epsilon}{100K}. \end{aligned} \tag{8.25}$$

So if $\omega \in A_1^{(n)} \cap A_2^{(n)} \cap A_3^{(n)} \cap A_4^{(n)} \cap A_5^{(n)} \cap A_6^{(n)}$ we have

$$\begin{aligned}
&\rho\left(S^{(\delta_n)}(\omega), \tilde{S}^{(\delta_n)}(\omega)\right) \\
&= \sum_{i=1}^{\infty} \min\left(2^{-i}, \sup\left\{\left|S_t^{(\delta_n)}(\omega) - \tilde{S}_t^{(\delta_n)}(\omega)\right| : t \in [0,i]\right\}\right) \\
&\leq \sum_{i=1}^{K} \sup\left\{\left|S_t^{(\delta_n)}(\omega) - \tilde{S}^{(\delta_n)_t}(\omega)\right| : t \in [0,i]\right\} + \sum_{i=K+1}^{\infty} 2^{-i} \\
&\leq \sum_{i=1}^{K} \frac{8\epsilon}{100K} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

This completes the proof of theorem 8.4.5. □

We are now ready for the proof showing that our solution to the difference equation in eq. (8.6) approximates the geometric Fractional Brownian Motion. The proof is trivial because the hard work was done in theorem 8.4.5 and the lemmas leading up to theorem 8.4.5.

**Theorem 8.4.6.** *Let $H \in (0,1)$, $\{\delta_n\}$ a sequence of positive numbers converging to zero. Assume $S_0 > 0, \sigma > 0, \mu \in \mathbb{R}$, using these quantities let $S^{(\delta_n)}$ be as in definition 8.4.1. Denote $P_n$ to be the measure on $(C[0,\infty), \mathcal{C})$ induced by $S^{(\delta_n)}$. Define $P$ the measure on $(C[0,\infty), \mathcal{C})$ induced by*

$$S_0 e^{\mu t + \sigma B_{t,H}}.$$

*Then $\{P_n\}$ converges weakly to $P$.*

**Remark.** *The measure $P$ exist by proposition 8.1.1, and the measures $P_n$ exist by lemma 3.3.2 and theorem C.1.1.*

*Proof.* Theorem 8.4.6 follows directly from theorem 8.4.5, theorem 8.1.2 and theorem 5.4.2. □

## 8.5 Analysis of the solution to the difference equation

In this section we will further analyse some aspects of the price process $S^{(\delta)}$ from definition 8.4.1. A common theme in section 8.3 for the models of the risky assets was that they were all positive. We will first see that this may not be the case for $H > 1/2$ in definition 8.4.1.

### A weakness in the model for $H > 1/2$

Let $S^{(\delta)}$ be as in definition 8.4.1, we will see that no matter how small we choose $\delta$, for $H > 1/2$ it will still be possible to obtain non-positive values. This is not desirable when modelling a risky asset. However we will first see that for $H \leq 1/2$ we do not have this problem.

**Proposition 8.5.1.** *Let $H \in (0, 0.5], S_0 > 0, \sigma > 0, \mu \in \mathbb{R}$. There exists a $\delta^* > 0$ such that if $0 < \delta \leq \delta^*$ and $S^{(\delta)}$ as in definition 8.4.1 with $S_0, \sigma, \mu, \delta$, we have for all $\omega \in \Omega, t \in [0, \infty)$*

$$S_t^{(\delta)}(\omega) > 0.$$

*Proof.* By definition 8.4.1 it suffices to show that there is a $\delta^*$ such that if $\delta \leq \delta^*$ and $t$ is a multiple of $\delta$ we have

$$|\mu\delta| + \left|\sigma\Delta V_s^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_s^{(\delta)}(\omega)\right)^2}{2!}\right| + \cdots + \left|\frac{\left(\sigma\Delta V_s^{(\delta)}\right)^{k^*}}{k^*!}\right| < 1, \qquad (8.26)$$

for $\omega \in \Omega$.

Assume first that $H = 1/2$. First choose a $\delta^{(1)}$ such that for $\delta \leq \delta^{(1)}$ we have

$$|\mu\delta| < \frac{1}{2}.$$

When $H = 1/2$ we have $k^* = 2$, and from eq. (8.7)

$$\left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2}\right| = \frac{\sigma\delta^H}{C_H} + \frac{\sigma^2\delta^{2H}}{2C_H^2}.$$

There is a $\delta^{(2)}$ such that if $\delta \leq \delta^{(2)}$ this expression is bounded by $1/2$. Letting $\delta^* \doteq \min(\delta^{(1)}, \delta^{(2)})$ completes the case for $H = 1/2$.

Assume now that $H < 1/2$. Again choose $\delta^{(1)}$ such that if $\delta \leq \delta^{(1)}$ we have $|\mu\delta| < 1/2$. From eq. (8.8) we know that there is an $R_1 > 0$ such that

$$\left|\sigma\Delta V_t^{(\delta)}(\omega)\right| + \left|\frac{\left(\sigma\Delta V_t^{(\delta))}(\omega)\right)^2}{2!}\right| + \cdots + \left|\frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right|$$
$$\leq \sigma R_1\delta^H + \frac{\sigma^2}{2!}\cdot\left(R_1\delta^H\right)^2 + \cdots + \frac{\sigma^{k^*}}{k^*!}\cdot\left(R_1\delta^H\right)^{k^*}.$$

There is a $\delta^{(2)}$ such that this expression is also less than $1/2$ for $\delta \leq \delta^{(2)}$. Choosing $\delta^* = \min(\delta^{(1)}, \delta^{(2)})$ completes the the case $H \in (0, 0.5)$. □

Next we show that for $H > 1/2$ we have a positive probability of non-positive values.

**Proposition 8.5.2.** *Let $H \in (0.5, 1), \delta > 0, S_0 > 0, \sigma > 0, \mu \in \mathbb{R}$. Then*

$$P\left(\left\{\omega \in \Omega : S_t^{(\delta)}(\omega) \leq 0 \text{ for some } t \in [0, \infty)\right\}\right) > 0.$$

*Where we have defined $S^{(\delta)}$ by the quantities $H, S_0, \sigma, \mu, \delta$ and definition 8.4.1.*

**Remark.** *Note that $\{\omega \in \Omega : S_t^{(\delta)}(\omega) \leq 0 \text{ for some } t \in [0, \infty)\} \in \mathcal{A}$ by proposition C.1.3.*

*Proof.* Let $k$ be as in definition 7.1.1, the smallest natural number such that $2k(1-H) > 1$. For $N \in \mathbb{N}$ we have

$$\begin{aligned} &\frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor \delta^{-(k+1)} \rfloor \delta}^{N\delta-\delta}{}_{\delta} (N\delta-\tau)^{H-\frac{3}{2}} \\ &= \frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=\delta}^{N\delta+\lfloor \delta^{-(k+1)} \rfloor \delta}{}_{\delta} \tau^{H-\frac{3}{2}} \\ &= \frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{r=1}^{N+\lfloor \delta^{-(k+1)} \rfloor} (r\delta)^{H-\frac{3}{2}} \\ &= \frac{\sigma\delta^{H}(H-1/2)}{C_H} \sum_{r=1}^{N+\lfloor \delta^{-(k+1)} \rfloor} r^{H-\frac{3}{2}}. \end{aligned} \tag{8.27}$$

The series $\sum_{r=1}^{\infty} r^{H-3/2}$ diverges because $H > 1/2$. So by eq. (8.27) there is an $N' \in \mathbb{N}$ such that

$$\begin{aligned} &\frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor \delta^{-(k+1)} \rfloor \delta}^{N'\delta-\delta}{}_{\delta} (N'\delta-\tau)^{H-\frac{3}{2}} \\ &> 1 + |\mu\delta| + \left| \frac{\sigma K_H^{(U)} \delta^H}{C_H} \right|, \end{aligned} \tag{8.28}$$

where $K_H^{(U)}$ is from definition 6.2.4.

We recall that $S_t^{(\delta)}$ is defined on the same probability space as $\Delta V_t^{(\delta)}$. We denoted this space $(\Omega, \mathcal{A}, P)$, and we recall from definition 7.1.1 that on this space that the collection

$$\mathcal{W} = \{\ldots, w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, \ldots\},$$

is a collection of independent random variables. Define the set

$$A_{N'} \doteq \bigcap_{\substack{r \in \mathbb{Z} \\ -\lfloor \delta^{-(k+1)} \rfloor \leq r \leq N'-1}} \{\omega \in \Omega : w_r(\omega) = -1\}.$$

By independence we have

$$P(A_{N'}) = \left(\frac{1}{2}\right)^{N'+\lfloor \delta^{-(k+1)} \rfloor} > 0. \tag{8.29}$$

We will complete the proof if we can show

$$A_{N'} \subset \left\{\omega \in \Omega : S_t^{(\delta)}(\omega) \leq 0 \text{ for some } t \in [0, \infty)\right\}.$$

Assume $\omega \in A_{N'}$. Since $H \in (0.5, 1)$ we have that $k^* = 1$ from definition 8.4.1. We have

$$\begin{aligned}
&1 + \mu\delta + \sigma\Delta V^{(\delta)}_{N'\delta}(\omega) \\
&= 1 + \mu\delta + \frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{N'\delta-\delta} {}_{\delta} \left[(N'\delta-\tau)^{H-\frac{3}{2}}\, w_{\tau/\delta}(\omega)\right] \\
&\qquad + \frac{\sigma K^{(U)}_H \delta^H}{C_H} w_{t/\delta}(\omega) \\
&\leq 1 + |\mu\delta| + \frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{N'\delta-\delta} {}_{\delta} \left[(N'\delta-\tau)^{H-\frac{3}{2}}\, w_{\tau/\delta}(\omega)\right] \\
&\qquad + \left|\frac{\sigma K^{(U)}_H \delta^H}{C_H}\right| \\
&= 1 + |\mu\delta| - \frac{\sigma\delta^{3/2}(H-1/2)}{C_H} \sum_{\tau=-\lfloor\delta^{-(k+1)}\rfloor\delta}^{N'\delta-\delta} {}_{\delta} (N'\delta-\tau)^{H-\frac{3}{2}} \\
&\qquad + \left|\frac{\sigma K^{(U)}_H \delta^H}{C_H}\right| \\
&< 0,
\end{aligned} \tag{8.30}$$

where we in the last step used eq. (8.28). If one of the values in

$$\left\{S^{(\delta)}_{\delta}(\omega), S^{(\delta)}_{2\delta}(\omega), \ldots, S^{(\delta)}_{N'\delta}(\omega)\right\}$$

is non-positive we have the result we need, and we are done. So assume that all of them are positive. Then we have

$$\begin{aligned}
S^{(\delta)}_{(N'+1)\delta}(\omega) &= S^{(\delta)}_{N'\delta}(\omega)\left(1 + \mu\delta + \sigma\Delta V^{(\delta)}_{N'\delta}(\omega)\right) \\
&< 0,
\end{aligned}$$

in the last step we used eq. (8.30). This completes the proof. □

### H=1/2, what happened to $-\sigma^2 t/2$?

Given $S_0 > 0, \mu \in \mathbb{R}, \sigma > 0, \delta > 0$, $(S^{(\delta)}_t)_{t\in[0,\infty)}$ in definition 8.4.1 was the solution to the stochastic difference equation $S^{(\delta)}_0 = S_0$ , and if $t$ is a multiple of $\delta$

$$\begin{aligned}
&S^{(\delta)}_{t+\delta}(\omega) - S^{(\delta)}_t(\omega) \\
&= \mu S^{(\delta)}_t(\omega)\delta + S^{(\delta)}_t(\omega)\left(\sigma\Delta V^{(\delta)}_t(\omega) + \frac{\left(\sigma\Delta V^{(\delta)}_t(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V^{(\delta)}_t(\omega)\right)^{k^*}}{k^*!}\right).
\end{aligned}$$

Here $k^*$ was the smallest natural number such that $H(k^*+1) > 1$. By theorem 8.4.6 we have that as $\delta$ tends to zero the measure induced by this process on $(C[0,\infty), \mathcal{C})$ converges weakly to the measure induced by $S_0 \exp(\mu t + \sigma B_{t,H})$.

Let us investigate the model for $H = 1/2$. We are now only working with the Brownian motion. As we remarked in section 8.3, in continuous time models the risky asset can be modelled by the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

The solution to this SDE is

$$S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_{t,1/2}},$$

for instance by Itô's lemma. However our price process converges weakly to

$$S_0 e^{\mu t + \sigma B_{t,1/2}}.$$

What happened to $-\sigma^2/2$? To see why we get a different result than what we might have expected, we have to investigate our difference equation. For $H = 1/2$ $k^* = 2$. This means that the difference equation is

$$\begin{aligned} & S_{t+\delta}^{(\delta)}(\omega) - S_t^{(\delta)}(\omega) \\ &= \mu S_t^{(\delta)}(\omega)\delta + S_t^{(\delta)}(\omega)\left(\sigma \Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma \Delta V_t^{(\delta)}(\omega)\right)^2}{2!}\right). \end{aligned}$$

Let us try to solve this stochastic difference equation by deleting the term $(\sigma^2 \Delta V_t^{(\delta)})/2$. Denote this process $\bar{S}^{(\delta)}$. As before we assume linear interpolation between the the time points which are multiples of $\delta$. By induction we have if $t$ is a multiple of $\delta$

$$\bar{S}_t^{(\delta)}(\omega) = S_0 \prod_{s=0}^{t-\delta}{}_{\delta} \left[1 + \mu\delta + \sigma \Delta V_s^{(\delta)}(\omega)\right]. \tag{8.31}$$

From definition 6.2.4 $K_{1/2}^{(U)} = 1$, and from definition 7.1.1 $C_{1/2} = 1$ so again by definition 7.1.1

$$\Delta V_t^{(\delta)}(\omega) = \sqrt{\delta} w_{t/\delta}(\omega).$$

This means that

$$\begin{aligned} \bar{S}_t^{(\delta)}(\omega) &= S_0 \prod_{s=0}^{t-\delta}{}_{\delta} \left[1 + \mu\delta + \sigma \Delta V_s^{(\delta)}(\omega)\right] \\ &= S_0 \prod_{s=0}^{t-\delta}{}_{\delta} \left[1 + \mu\delta + \sigma \Delta V_s^{(\delta)}(\omega) + \frac{\sigma^2\delta}{2} - \frac{\sigma^2\delta}{2}\right] \\ &= S_0 \prod_{s=0}^{t-\delta}{}_{\delta} \left[1 + \left(\mu - \frac{\sigma^2}{2}\right)\delta + \sigma \Delta V_s^{(\delta)}(\omega) + \frac{\left(\sigma\sqrt{\delta} w_{s/\delta}\right)^2}{2}\right] \\ &= S_0 \prod_{s=0}^{t-\delta}{}_{\delta} \left[1 + \left(\mu - \frac{\sigma^2}{2}\right)\delta + \sigma \Delta V_s^{(\delta)}(\omega) + \frac{\left(\sigma \Delta V_s^{(\delta)}\right)^2}{2}\right]. \end{aligned}$$

This process is the same as the one in definition 8.4.1 but with $\mu' = \mu - \sigma^2/2$. So if $\{\delta_n\}$ is positive and converges to zero we have by theorem 8.4.6 that the measures induces by $(\bar{S}_t^{(\delta_n)})_{t\in[0,\infty)}$ on $(C[0,\infty)\mathcal{C})$ converges weakly to the measure induced by

$$S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_{t,1/2}}.$$

## An alternative difference equation

In the difference equation

$$\begin{aligned}&\Delta S_t^{(\delta)}(\omega)\\ &= \mu S_t^{(\delta)}(\omega)\delta + S_t^{(\delta)}(\omega)\left(\sigma\Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right),\end{aligned} \tag{8.32}$$

we recognize a part of the Taylor series for $\exp(\sigma V_t^{(\delta)}(\omega))$. If we assume $\mu S_t^{(\delta)}(\omega)\delta$ and $\sigma\Delta V_t^{(\delta)}(\omega)$ are small we may try to use the approximation $e^x - 1 \approx x$ to get

$$\begin{aligned}&\mu S_t^{(\delta)}(\omega)\delta + S_t^{(\delta)}(\omega)\left(\sigma\Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right)\\ &\approx S_t^{(\delta)}(\omega)\left[\exp\left(\mu\delta + \sigma\Delta V_t^{(\delta)}(\omega) + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^2}{2!} + \cdots + \frac{\left(\sigma\Delta V_t^{(\delta)}(\omega)\right)^{k^*}}{k^*!}\right) - 1\right]\\ &\approx S_t^{(\delta)}(\omega)\left[\exp\left(\mu\delta + \sigma\Delta V_t^{(\delta)}(\omega)\right) - 1\right].\end{aligned} \tag{8.33}$$

We have in fact already seen a process with these dynamics. For $S_0 > 0, \mu \in \mathbb{R}, \sigma > 0, \delta > 0$ consider the stochastic process

$$\tilde{S}_t^{(\delta)}(\omega) = S_0 e^{\mu t + \sigma V_t^{(\delta)}(\omega)},$$

which we used in section 8.4 as a tool for proving weak convergence for the process in definition 8.4.1. If $t$ is a multiple of $\delta$ we get

$$\begin{aligned}\Delta\tilde{S}_t^{(\delta)}(\omega) &\doteq \tilde{S}_{t+\delta}^{(\delta)}(\omega) - \tilde{S}_t^{(\delta)}(\omega)\\ &= S_0 e^{\mu(t+\delta)+\sigma(V_t^{(\delta)}(\omega)+\Delta V_t^{(\delta)}(\omega))} - S_0 e^{\mu t + \sigma V_t^{(\delta)}(\omega)}\\ &= S_0 e^{\mu t + \sigma V_t^{(\delta)}(\omega)}\left(e^{\mu\delta + \sigma\Delta V_t^{(\delta)}(\omega)} - 1\right)\\ &= \tilde{S}_t^{(\delta)}(\omega)\left(e^{\mu\delta + \sigma\Delta V_t^{(\delta)}(\omega)} - 1\right).\end{aligned}$$

So the two processes $S_t^{(\delta)}$ and $\tilde{S}_t^{(\delta)}$ does not only both converge weakly to the same process as $\delta$ becomes small. If we accept the approximations in eq. (8.33) their dynamics are also very similar. We will not go into a detailed analysis of how good the approximations are, but we note that $e^x - 1 \approx x$ for small $x$. There is a small technical aspect we have to be aware of; we do not use linear interpolation in $\tilde{S}_t^{(\delta)}$. The logarithm will be linearly interpolated since $V_t^{(\delta)}$ is linearly interpolated by definition. We could have redefined it to be linear between the time points of $t = N\delta N, \in \{0\} \cup \mathbb{N}$ and used techniques from section 8.4 to show weak convergence. However, as that would be purely a mathematical exercise and we only gain a different interpolation, it is omitted.

# Chapter 9

# Final Remarks

## 9.1 Conclusion

With the main theorems; theorem 3.7.3, 5.4.3, 6.1.3, 6.5.1 and theorem 7.3.2, we have shown that the induced measures of five processes $X^{(\delta)}, Y^{(\delta)}, Z^{(\delta)}, U^{(\delta)}$, $V^{(\delta)}$ all converge weakly to the measure induced by the fBm as $\delta$ tends to zero.

In chapter 8 we used $V^{(\delta)}$ for financial applications. We saw that we could approximate processes derived elsewhere. We also saw that we could use the stochastic differential equation from the Black-Scholes market as an inspiration for a stochastic difference equation. The solution to the difference equation converged weakly to the geometric fractional Brownian motion.

## 9.2 Further research

There are some natural generalisations to explore if one want to work further with the ideas in this thesis. In all of the five processes $X^{(\delta)}, Y^{(\delta)}, Z^{(\delta)}, U^{(\delta)}$ and $V^{(\delta)}$ we used independent random variables with the Rademacher distribution, that is, they took the values $\pm 1$ each with probability $1/2$ . Instead of Rademacher distributions we could have assumed as in Donsker's theorem([Bil99, p. 90]) that they are still independent, but we only assume that the expectation is zero, and the variance is one. This would create some new challenges, the Lindeberg condition in theorem 3.5.2 would not follow so easily.

We chose to work with continuous functions, and use linear interpolation because we wanted to use continuous functions when approximating the fBm which is continuous. We could instead have used piecewise constant cádlág functions. Some aspects would have been simpler then, because we would not have to bother with interpolation. It is also more natural to use cádlág functions in finance, because the price is constant on each time interval.

In regards to financial applications in chapter 8, we did not set up a financial market. This could also be done, and then investigate arbitrage opportunities. In setting up the market we have to be careful since we have used linear interpolation. This was discussed at the end of section 8.3.

# Appendices

# Appendix A

# $\sigma$-Algebras

## A.1 Definition of $\sigma$-algebras

**Definition A.1.1.** *Let $\Omega$ be a set. A collection $\mathcal{A}$ of subsets of $\Omega$ is called a $\sigma$-algebra, if*

*(i)* $\emptyset \in \mathcal{A}$

*(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.*

*(iii) If $\mathcal{B} = \{A_n\}$ is a countable collection where each $A_n \in \mathcal{A}$, then*

$$\bigcup_{A_n \in \mathcal{B}} A_n \in \mathcal{A}.$$

## A.2 Properties of $\sigma$-algebras

**Proposition A.2.1** ([**MW13, Proposition 1.15, p.24**]). *Assume $\mathcal{E}$ is a nonempty collection of subsets of $\Omega$. Then there is a smallest $\sigma$-algebra of subsets of $\Omega$ containing $\mathcal{E}$. This $\sigma$-algebra is sometimes denoted $\sigma(\mathcal{E})$.*

**Remark.** *That $\sigma(\mathcal{E})$ is the smallest $\sigma$-algebra means that if $\mathcal{H}$ is another sigma-algebra on $\Omega$ and $\mathcal{E} \subset \mathcal{H}$ we have*

$$\sigma(\mathcal{E}) \subset \mathcal{H}.$$

**Definition A.2.2.** *Let $\mathcal{A}$ be a $\sigma$-algebra on $\Omega$. Let*

$$\mathcal{A} \otimes \mathcal{A},$$

*denote the smallest $\sigma$-algebra on $\Omega \times \Omega$ containing all elements of the form*

$$A_1 \times A_2,$$

*where $A_1$ and $A_2$ are elements of $\mathcal{A}$.*

**Proposition A.2.3.** *Let $\mathcal{E}$ be a collection of subsets of $\Omega$. Assume there there is an increasing sequence of sets $E_n \subset E_{n+1}$ in $\mathcal{E}$ such that*

$$\bigcup_{n \in \mathbb{N}} E_n = \Omega.$$

*Let*

$$\mathcal{D} = \{E_1 \times E_2, E_1, E_2 \in \mathcal{E}\}.$$

*Then*

$$\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}) = \sigma(\mathcal{D}).$$

*Proof.* The inclusion $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}) \supset \sigma(\mathcal{D})$ is simple because if $E_1, E_2 \in \mathcal{E}$ we have $E_1 \in \sigma(\mathcal{E})$ and $E_2 \in \sigma(\mathcal{E})$, so $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E})$ is a $\sigma$-algebra containing all sets of the form $E_1 \times E_2 \in \mathcal{E}$, it must therefore contain the smallest $\sigma$-algebra with this property which is $\sigma(\mathcal{D})$.

What remains to prove is

$$\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}) \subset \sigma(\mathcal{D}).$$

We will use a technique found in various books in measure-theory. Let $B \in \mathcal{E}$ be given. Denote the collection

$$\mathcal{J}(B) = \{A \in \sigma(\mathcal{E}) : A \times B \in \sigma(\mathcal{D})\}.$$

By construction $\mathcal{J}(B) \subset \sigma(\mathcal{E})$, we will show the opposite inclusion so $\mathcal{J}(B) = \sigma(\mathcal{E})$. First note that $\mathcal{J}(B)$ contains $\mathcal{E}$ since $\sigma(\mathcal{D})$ contains all sets of the form $E \times B, E \in \mathcal{E}$. Next note that $\mathcal{J}(B)$ is a $\sigma$-algebra because $\emptyset \in \mathcal{J}(B)$. If $A \in \mathcal{J}(B)$ then $A \times B \in \sigma(\mathcal{D})$, we also have that

$$\begin{aligned} A^c \times B =& (A \times B)^c \cap (\Omega \times B) \\ =& (A \times B)^c \cap ((\cup_n E_n) \times B) \\ =& \cup_n \left[(A \times B)^c \cap (E_n \times B)\right] \in \sigma(\mathcal{D}), \end{aligned}$$

Lastly if $\{A_n\}$ is a countable collection where each $A_n \in \mathcal{J}(b)$ we have

$$(\cup A_n) \times B = \cup(A_n \times B) \in \sigma(\mathcal{D}).$$

Hence $\mathcal{J}(B)$ is a $\sigma$-algebra containing $\mathcal{E}$, so $\sigma(\mathcal{E}) = \mathcal{J}(B)$. We have proven that for any $B \in \mathcal{E}$ and any $A \in \sigma(\mathcal{E})$ we have $A \times B \in \sigma(\mathcal{D})$.

Now let $K \in \sigma(\mathcal{E})$, denote

$$\mathcal{F}(K) = \{A \in \sigma(\mathcal{E}) : K \times A \in \sigma(\mathcal{D})\}.$$

By construction $\mathcal{F}(K) \subset \sigma(\mathcal{E})$, we will show that $\mathcal{F}(K) = \sigma(\mathcal{E})$. Note that by what we proved in the previous paragraph $\mathcal{E} \subset \mathcal{F}(K)$. We now show that $\mathcal{F}$ is a $\sigma$-algebra. We have directly that $\emptyset \in \mathcal{F}(K)$. If $A \in \mathcal{F}(K)$ we have

$$\begin{aligned} K \times A^c =& (K \times A)^c \cap (K \times \Omega) \\ =& (K \times A)^c \cap (K \times (\cup_n E_n)) \\ =& \cup_n \left[(K \times A)^c \cap (K \times E_n)\right] \in \sigma(\mathcal{D}). \end{aligned}$$

since $\sigma$-algebras are closed under intersection, and $K \times E_n \in \sigma(\mathcal{D})$ by the previous paragraph. Lastly assume that $\{A_n\}$ is a countable collection where each $A_n \in \mathcal{J}(K)$. Then we have

$$K \times (\cup_n A_n) = \cup_n (K \times A_n) \in \sigma(\mathcal{D}).$$

Hence we have that $\mathcal{F}(K)$ is a $\sigma$-algebra containing $\mathcal{E}$, so $\mathcal{F}(K) = \sigma(\mathcal{E})$. What we have now proven is that if $K_1, K_2 \in \sigma(\mathcal{E})$ we have that $K_1 \times K_2 \in \sigma(\mathcal{D})$. So this means that $\sigma(\mathcal{D})$ is a $\sigma$-algebra containing all the sets that generates $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E})$, hence

$$\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}) \subset \sigma(\mathcal{D}).$$

This completes the proof.

□

# Appendix B

# Metric Spaces

## B.1 Definition of metric spaces

The pair $(S, \rho)$ is called a metric space if $S$ is a non-empty set, and $\rho$ is a function

$$\rho : S \times S \to [0, \infty),$$

which satisfies

(i) $\rho(x, y) = 0$ if and only if $x = y$,

(ii) $\rho(x, y) = \rho(y, x)$,

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, x)$.

**Remark.** *Some definitions allow $S = \emptyset$. We will exclude this case for simplicity.*

## B.2 Elementary concepts related to metric spaces

**Definition B.2.1** (Open ball)**.** *Let $(S, \rho)$ be a metric space. If $x \in S$ we define the open ball $B(x, d)$, where $d \in \mathbb{R}$ by*

$$B(x, d) \doteq \{y \in S : \rho(x, y) < d\}.$$

**Definition B.2.2** (Open set)**.** *Let $(S, \rho)$ be a metric space. A set $A \subset S$ is called open if for every $x$ in $A$ there exists a $d > 0$ such that*

$$B(x, d) \subset A.$$

**Definition B.2.3** (convergence)**.** *Let $(S, \rho)$ be a metric space. A sequence $\{x_n\}$ of elements in $S$ converges to an element $x \in S$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ we have*

$$\rho(x, x_n) < \epsilon.$$

**Proposition B.2.4.** *Let $(S, \rho)$ be a metric space. Every open ball is an open set.*

*Proof.* Let $B(x,d)$ be our open ball. If $d \leq 0$ then it is empty be definition, and the empty-set is open by definition(vacuously). We can therefore assume that $d > 0$. Assume that $y \in B(x,d)$, we must show that there exists a $d_2$ such that

$$B(y,d_2) \subset B(x,d).$$

If $y = x$ choose $d_2 = d$ and the result follows. If $y \neq x$ then $0 < \rho(x,y) < d$, let $d_2 = d - \rho(x,y) > 0$, we want to show that

$$B(y,d_2) \subset B(x,d).$$

Assume that $y^* \in B(y,d_2)$, then we have that $\rho(y,y^*) < d_2$, we get

$$\begin{aligned}\rho(x,y^*) \leq& \rho(x,y) + \rho(y,y^*)\\ <& \rho(x,y) + d_2\\ =& \rho(x,y) + d - \rho(x,y)\\ =& d.\end{aligned}$$

Hence $y^* \in B(x,d)$, and we have proved that

$$B(y,d_2) \subset B(x,d).$$

□

**Definition B.2.5** (Dense subset)**.** *Let $(S,\rho)$ be a metric space. A set $D \subset S$ is called dense if for every $x \in S$ there exists a sequence in $D$ converging to $x$.*

**Proposition B.2.6.** *Let $(S,\rho)$ be a metric space. A set $D \subset S$ is dense if and only if for each $x \in S$ and each rational number $q \geq 0$ we have*

$$B(x,q) \cap D \neq \emptyset.$$

*Proof.* Assume first that $D$ is dense, $x \in S$ and $q > 0$ a rational number. Let $\{x_n\}$ be a sequence in $D$ converging to $x$. Let $\epsilon = q/2$, then there is an $N$ such that if $n \geq N$ we have

$$\rho(x,x_n) < q/2.$$

But this means that $x_N \in B(x,q)$, the result follows since $x_N \in D$.

Assume conversely that for every $x \in S$ and every rational number $q > 0$ we have

$$B(x,q) \cap D \neq \emptyset.$$

We must show that there exists a sequence in $D$ converging to $X$. For every $n \in \mathbb{N}$ we have that

$$B(x,1/n) \cap D \neq \emptyset.$$

For each $n$ pick an arbitrary $x_n$ in $B(x,1/n) \cap D$. Then the sequence $\{x_n\}$ is in $D$ and it converges to $x$ because given $\epsilon > 0$ choose $N$ such that $1/N < \epsilon$, then if $n \geq N$ we have

$$x_n \in B(x,1/n),$$

so

$$\rho(x,x_n) < 1/n \leq 1/N < \epsilon.$$

□

**Definition B.2.7** (separable)**.** *A metric space $(S, \rho)$ is called separable if it has a countable dense subset.*

**Definition B.2.8.** *Let $(S, \rho)$ be a metric space. We define the Borel sigma-algebra, $\mathcal{B}(S) = \mathcal{S}$, as the sigma-algebra generated by the open sets.*

**Definition B.2.9.** *Let $(S, \rho)$ be a metric space. We define the ball sigma-algebra, $\boldsymbol{Ball}(S) = \mathcal{S}$, as the sigma-algebra generated by the open balls.*

**Proposition B.2.10.** *Let $(S, \rho)$ be a metric space, assume also that the metric space is separable. Then $\boldsymbol{Ball}(S) = \mathcal{B}(S)$.*

*Proof.* By proposition B.2.4 every open ball is open, so we have

$$\text{Ball}(S) \subset \mathcal{B}(S).$$

Assume now that the that $A \subset S$ is open. If we can show that $A$ is a countable union of open balls, we will be done. Let $D \subset S$ be the dense subset. Define the collection $\mathcal{H}$ like this

$$\mathcal{H} = \{B(x, q) : x \in D, q \in \mathbb{Q}, q > 0, B(x, q) \subset A\}.$$

Note that $\mathcal{H}$ is at most a countable set, because $D$ is at most countable, and there are countably many positive rational number, and a countable union of countable sets is a most countable, see [MW13, Proposition 1.10, p. 20]. Our proof will be complete if we can show

$$\bigcup_{H \in \mathcal{H}} H = A.$$

By construction of $\mathcal{H}$ we have

$$\bigcup_{H \in \mathcal{H}} H \subset A.$$

So it remains to prove

$$\bigcup_{H \in \mathcal{H}} H \supset A.$$

Let $a \in A$. We must show that there exists a $x \in D$ and a rational number $q > 0$ such that

$$a \in B(x, d),$$

and

$$B(x, d) \subset A.$$

Since $A$ is open there exists a real number $d > 0$ such that

$$B(a, d) \subset A.$$

If $a$ should happen to be in $D$, choose a rational number $q$ such that $0 < q < d$, this is possible since the rational numbers are dense in $\mathbb{R}$, then

$$B(a, q) \subset A.$$

Assume so that $a \in D^c$. Let $q$ be a positive rational number such that $q < d/2$. By proposition B.2.6 we have

$$B(a,q) \cap D \neq \emptyset,$$

So choose an $x$ in this non-empty intersection, then $\rho(x,a) < q$. We have that

$$B(a,q) \subset B(a,d) \subset A,$$

so it will suffice to prove that

$$B(x,q) \subset B(a,d).$$

This is the case, because if $y \in B(x,q)$ we have

$$\begin{aligned} \rho(y,a) &\leq \rho(y,x) + \rho(x,a) \\ &< q + q \\ &< \frac{d}{2} + \frac{d}{2} \\ &= d. \end{aligned}$$

This completes the proof. □

**Proposition B.2.11.** *Let $(S,\rho)$ be a metric space. Define the function $\rho'$*

$$(S \times S) \times (S \times S) \to [0,\infty),$$

*by*

$$\rho'((x_1,x_2),(y_1,y_2)) = \max\left(\rho(x_1,y_1), \rho(x_2,y_2)\right).$$

*Then $(S \times S, \rho')$ is a metric space.*

*Proof.* We check the conditions of a metric space are met. First we have

$$\begin{aligned} \rho'((x_1,x_2),(x_1,x_2)) &= \max\left(\rho(x_1,x_1), \rho(x_2,x_2)\right) \\ &= \max(0,0) = 0. \end{aligned}$$

Assume so that $\rho'((x_1,x_2),(y_1,y_2)) = 0$. We then have

$$\begin{aligned} \max\left(\rho(x_1,y_1), \rho(x_2,y_2)\right) =& \rho'((x_1,x_2),(y_1,y_2)) \\ =& 0. \end{aligned}$$

This means that both $\rho(x_1,y_1)$ and $\rho(x_2,y_2)$ are zero. So $x_1 = y_1$, and $x_2 = y_2$, hence

$$(x_1,x_2) = (y_1,y_2).$$

We also have

$$\begin{aligned} \rho'((x_1,x_2),(y_1,y_2)) =& \max\left(\rho(x_1,y_1), \rho(x_2,y_2)\right) \\ =& \max\left(\rho(y_1,x_1), \rho(y_2,x_2)\right) \\ =& \rho'((y_1,y_2),(x_1,x_2)). \end{aligned}$$

It remains to prove the last property. Since we have

$$\rho'((x_1,x_2),(y_1,y_2)) = \max\left(\rho(x_1,y_1),\rho(x_2,y_2)\right),$$

it will suffice to prove that

$$\rho(x_1,y_1) \le \rho'((x_1,x_2),(z_1,z_2)) + \rho'((z_1,z_2),(y_1,y_2)),$$

and

$$\rho(x_2,y_2) \le \rho'((x_1,x_2),(z_1,z_2)) + \rho'((z_1,z_2),(y_1,y_2)).$$

First we get

$$\begin{aligned}\rho(x_1,y_1) \le& \rho(x_1,z_1) + \rho(z_1,y_1)\\ \le& \max\left(\rho(x_1,z_1),\rho(x_2,z_2)\right) + \max\left(\rho(z_1,y_1),\rho(z_2,y_2)\right)\\ =& \rho'((x_1,x_2),(z_1,z_2)) + \rho'((z_1,z_2),(y_1,y_2)).\end{aligned}$$

Lastly we have

$$\begin{aligned}\rho(x_2,y_2) \le& \rho(x_2,z_2) + \rho(z_2,y_2)\\ \le& \max\left(\rho(x_1,z_1),\rho(x_2,z_2)\right) + \max\left(\rho(z_1,y_1),\rho(z_2,y_2)\right)\\ =& \rho'((x_1,x_2),(z_1,z_2)) + \rho'((z_1,z_2),(y_1,y_2)).\end{aligned}$$

This completes the proof. □

**Proposition B.2.12.** *Let $(S,\rho)$ be a separable metric space. Then the metric space $(S\times S,\rho')$, defined in proposition B.2.11, is a separable metric space.*

*Proof.* Let $D$ be the countable dense subset in $S$. We will show that

$$D\times D,$$

is a countable dense subset of $S\times S$. Note first that $D\times D$ is countable because by [MW13, Proposition 1.11, p. 20] the Cartesian product of two countable sets is countable. Let $(x,y)\in S\times S$, let $\{x_n\}$ be a sequence in $D$ converging to $x$, and $\{y_n\}$ a sequence in $D$ converging to $y$. Then $(x_n,y_n)$ is a sequence in $D\times D$, we will show that it converges to $(x,y)$. Let $\epsilon>0$, then there is $N_x$ such that if $n\ge N_x$

$$\rho(x,x_n)<\epsilon.$$

There is also an $N_y$ such that if $n\ge N_y$

$$\rho(y,y_n)<\epsilon.$$

Now if $n\ge\max\left(N_x,N_y\right)$, we then have

$$\begin{aligned}\rho'((x,y),(x_n,y_n)) =& \max\left(\rho(x,x_n),\rho(y,y_n)\right)\\ <&\epsilon.\end{aligned}$$

The proof is now done. □

**Proposition B.2.13.** *Let $(S, \rho)$ be a metric space, assume that it is separable. Let $(S \times S, \rho')$ be the metric space in proposition B.2.11. Then*

$$\mathcal{B}(S) \otimes \mathcal{B}(S) = \mathcal{B}(S \times S).$$

*Proof.* First note that by proposition B.2.10 $\text{Ball}(S) = \mathcal{B}(S)$, also by proposition B.2.12, $(S \times S, \rho')$ is separable, therefore we have by proposition B.2.10 again that $\text{Ball}(S \times S) = \mathcal{B}(S \times S)$. Hence it suffices to prove that

$$\text{Ball}(S) \otimes \text{Ball}(S) = \text{Ball}(S \times S).$$

Let

$$\mathcal{H} = \{B(x, d) : x \in S, d \in \mathbb{R}\}.$$

By definition we have $\sigma(\mathcal{H}) = \text{Ball}(S)$. Let

$$\mathcal{D} = \{B(x_1, d_1) \times B(x_2, d_2) : B(x_1, d_1), B(x_1, d_2) \in \mathcal{H}\}.$$

Pick an arbitrary element of $y \in S$, this is possible because we defined metric spaces to be non-empty. Notice that

$$\bigcup_{n \in \mathbb{N}} B(y, n) = S,$$

and

$$B(y, n) \subset B(y, n+1), n \in \mathbb{N}.$$

By proposition A.2.3

$$\text{Ball}(S) \otimes \text{Ball}(S) = \sigma(\mathcal{D}).$$

Our problem is therefore reduced to showing that

$$\sigma(\mathcal{D}) = \text{Ball}(S \times S).$$

To see that this is the case note first that for arbitrary $d$

$$\begin{aligned}
B((x, y), d) &= \{(s_1, s_2) \in S \times S : \rho'((x, y), (s_1, s_2)) < d)\} \\
&= \{(s_1, s_2) \in S \times S : \max(\rho(x, s_1), \rho(y, s_2)) < d)\} \\
&= \{(s_1, s_2) \in S \times S : \rho(x, s_1) < d, \rho(y, s_2) < d\} \\
&= \{s_1 \in S : \rho(x, s_1) < d\} \times \{s_2 \in S : \rho(y, s_2) < d\} \\
&= B(x, d) \times B(y, d) \in \mathcal{D}.
\end{aligned}$$

Hence

$$\sigma(\mathcal{D}) \supset \text{Ball}(S \times S).$$

Lastly assume that

$$B(x_1, d_1) \times B(x_2, d_2) \in \mathcal{D}.$$

We first show that $B(x_1, d_1) \times B(x_2, d_2)$ is an open set in $(S \times S, \rho')$. Since the empty-set is open we can assume that $B(x_1, d_1) \times B(x_2, d_2)$ is non-empty, this means that $d_1, d_2 > 0$. Assume

$$(y_1, y_2) \in B(x_1, d_1) \times B(x_2, d_2).$$

We must show that there exists a $\Delta > 0$ such that

$$B((y_1, y_2), \Delta) \subset B(x_1, d_1) \times B(x_2, d_2).$$

Let $\Delta = \min(d_1 - \rho(y_1, x_1), d_2 - \rho(y_2, x_2))$, and notice that $\Delta > 0$. We will show that

$$B((y_1, y_2), \Delta) \subset B(x_1, d_1) \times B(x_2, d_2).$$

Assume that $(z_1, z_2) \in B((y_1, y_2), \Delta)$ we then have

$$\begin{aligned}
\rho(x_1, z_1) &\leq \rho(x_1, y_1) + \rho(y_1, z_1) \\
&\leq \rho(x_1, y_1) + \max(\rho(y_1, z_1), \rho(y_2, z_2)) \\
&= \rho(x_1, y_1) + \rho'((y_1, y_2), (z_1, z_2)) \\
&< \rho(x_1, y_1) + \Delta \\
&\leq \rho(x_1, y_1) + d_1 - \rho(y_1, x_1) \\
&= d_1.
\end{aligned}$$

We also have

$$\begin{aligned}
\rho(x_2, z_2) &\leq \rho(x_2, y_2) + \rho(y_2, z_2) \\
&\leq \rho(x_2, y_2) + \max(\rho(y_1, z_1), \rho(y_2, z_2)) \\
&= \rho(x_2, y_2) + \rho'((y_1, y_2), (z_1, z_2)) \\
&< \rho(x_2, y_2) + \Delta \\
&\leq \rho(x_2, y_2) + d_2 - \rho(y_2, x_2) \\
&= d_2.
\end{aligned}$$

So $(z_1, z_2) \in B(x_1, d_1) \times B(x_2, d_2)$, hence

$$B((y_1, y_2), \Delta) \subset B(x_1, d_1) \times B(x_2, d_2).$$

We have now proven that $B(x_1, d_1) \times B(x_2, d_2)$ is open in $(S \times S, \rho')$. This means that

$$B(x_1, d_1) \times B(x_2, d_2) \in \mathcal{B}(S \times S).$$

But as we noted in the start of the proof

$$\text{Ball}(S \times S) = \mathcal{B}(S \times S).$$

So together we have proven that

$$\mathcal{D} \subset \text{Ball}(S \times S),$$

which means that

$$\sigma(\mathcal{D}) \subset \text{Ball}(S \times S).$$

This completes the proof. □

**Lemma B.2.14.** *Let $(S, \rho)$ be a metric space, let $(S \times S, \rho')$ be the metric space in proposition B.2.11. Then for every $A \in \mathcal{B}(\mathbb{R})$ we have*

$$\rho^{-1}(A) \in \mathcal{B}(S \times S).$$

*Proof.* We first show that for any $a \in \mathbb{R}$, we have

$$\rho^{-1}((-\infty, a)) \in \mathcal{B}(S \times S).$$

We will do this by showing that $\rho^{-1}((-\infty, a))$ is an open set in the metric space $(S \times S, \rho')$. If $a \leq 0$, then

$$\rho^{-1}((-\infty, a)) = \emptyset.$$

Since $\emptyset$ is open, we can assume that $a > 0$. Assume that $(x, y) \in \rho^{-1}((-\infty, a))$, we must show that there exists a $\Delta > 0$ such that

$$B((x, y), \Delta) \subset \rho^{-1}((-\infty, a)).$$

Let $\Delta = (a - \rho(x, y))/2$, notice that $\Delta > 0$. Assume that $(z_1, z_2) \in B((x, y), \Delta)$, we then have

$$\begin{aligned}
\rho(z_1, z_2) &\leq \rho(z_1, x) + \rho(x, y) + \rho(y, z_2) \\
&\leq \max(\rho(z_1, x), \rho(y, z_2)) + \rho(x, y) + \max(\rho(z_1, x), \rho(y, z_2)) \\
&= 2\max(\rho(z_1, x), \rho(y, z_2)) + \rho(x, y) \\
&= 2\max(\rho(z_1, x), \rho(z_2, y)) + \rho(x, y) \\
&= 2\rho'((z_1, z_2), (x, y)) + \rho(x, y) \\
&< 2(a - \rho(x, y))/2 + \rho(x, y) \\
&= a.
\end{aligned}$$

Hence $(z_1, z_2) \in \rho^{-1}(-\infty, a)$. So we have

$$B((x, y), \Delta) \subset \rho^{-1}((-\infty, a)),$$

and we have shown that $\rho^{-1}((-\infty, a))$ is an open set in $(S \times S, \rho')$. Hence

$$\rho^{-1}((-\infty, a)) \in \mathcal{B}(S \times S).$$

Now look at the collection

$$\mathcal{H} = \left\{ A \in \mathcal{B}(\mathbb{R}) : \rho^{-1}(A) \in \mathcal{B}(S \times S) \right\}.$$

We showed in the previous paragraph that $(-\infty, a) \in \mathcal{H}$ for every $a \in \mathbb{R}$. We now show that $\mathcal{H}$ is a sigma-algebra. We obviously have $\emptyset \in \mathcal{H}$ because $\emptyset \in \mathcal{B}(\mathbb{R})$ and

$$\rho^{-1}(\emptyset) = \emptyset \in \mathcal{B}(S \times S).$$

So $\emptyset \in \mathcal{H}$. If $A \in \mathcal{H}$, we have $A^c \in \mathcal{B}(\mathbb{R})$ and

$$\rho^{-1}(A^c) = (\rho^{-1}(A))^c \in B(S \times S),$$

so $A^c \in \mathcal{H}$. Assume lastly that $\{A_n\}$ is a countable collection where each $A_n \in \mathcal{H}$, we then have that $\cup_n A_n \in \mathcal{B}(\mathbb{R})$ and

$$\rho^{-1}(\cup_n A_n) = \cup_n \rho^{-1}(A_n) \in \mathcal{B}(S \times S).$$

So we have shown that $\mathcal{H}$ is a sigma-algebra containing all sets of the form $(-\infty, a)$ where $a \in \mathbb{R}$. By [Fol84, Proposition 1.2 (d), p. 21], we have that

$$\sigma\left(\{(-\infty, a) : a \in \mathbb{R}\}\right) = \mathcal{B}(\mathbb{R}).$$

Hence we must have

$$\mathcal{B}(\mathbb{R}) = \mathcal{H},$$

this completes the proof. □

**Proposition B.2.15.** *Let $(S, \rho)$ be a metric space with Borel sigma-algebra $\mathcal{B}(S) = \mathcal{S}$, assume that the metric space is separable. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Assume that you have two functions*

$$X : \Omega \to S$$
$$Y : \Omega \to S,$$

*where for each $A \in \mathcal{S}$ we have $X^{-1}(A) \in \mathcal{A}$ and $Y^{-1}(A) \in \mathcal{A}$. Then the function $h$ given by*

$$h(\omega) = \rho(X(\omega), Y(\omega)),$$

*is a well-defined random variable on $(\Omega, \mathcal{A}, P)$. Which means that*

$$h : \Omega \to \mathbb{R},$$

*and for every $B \in \mathcal{B}(\mathbb{R})$ we have*

$$h^{-1}(B) \in \mathcal{A}.$$

*Proof.* We first show that the mapping

$$V : \Omega \to S \times S,$$

given by

$$V(\omega) = (X(\omega), Y(\omega)),$$

is $\mathcal{A}/(\mathcal{S} \otimes \mathcal{S})$-measurable. First note that if $A_1, A_2 \in \mathcal{S}$ we have

$$V^{-1}(S_1 \times S_2) = (X, Y)^{-1}(S_1 \times S_2) = X^{-1}(S_1) \cap Y^{-1}(S_2) \in \mathcal{A}.$$

Since the sets of the form $S_1 \times S_1, S_1, S_2 \in \mathcal{S}$ generate $\mathcal{S} \otimes \mathcal{S}$ it follows by [Fol84, Proposition 2.1, p.42] that

$$V^{-1}(R) \in \mathcal{A}, R \in \mathcal{S} \otimes \mathcal{S}.$$

Notice that

$$h(\omega) = \rho(V(\omega)).$$

Now let $B \in \mathcal{B}(\mathbb{R})$. By lemma B.2.14 we have that

$$\rho^{-1}(B) \in \mathcal{B}(S \times S).$$

By proposition B.2.13 we have that $\mathcal{B}(S \times S) = \mathcal{S} \otimes \mathcal{S}$. So

$$\rho^{-1}(B) \in \mathcal{S} \otimes \mathcal{S}.$$

By what was proved above we then have that

$$V^{-1}(\rho^{-1}(B)) \in \mathcal{A}.$$

However, from elementary set-theory we have

$$h^{-1}(B) = V^{-1}(\rho^{-1}(B)).$$

This completes the proof. □

**Definition B.2.16** (Closed set)**.** *Let $(S, \rho)$ be a metric space. A set $A \subset S$ is closed if $A^c$ is open.*

**Definition B.2.17** (Distance from a set)**.** *Let $(S, \rho)$ be a metric space. Assume that $x \in S$ and $F \subset S$, we define*

$$\rho(x, F) = \inf \{\rho(x, y) : y \in F\}.$$

**Remark.** *If $F = \emptyset$, then $\rho(x, F) = \infty$. This is because $\inf \emptyset = \infty$ since vacuously every real number is a lower bound for every element in $\emptyset$, because there are no elements in $\emptyset$. So the biggest lower bound is infinity.*

**Proposition B.2.18 (Statement from [Bil99, p. 28].).** *Let $(S, \rho)$ be a metric space, $\epsilon > 0$. Let $F \subset S$. Define*

$$F_\epsilon = \{x \in S : \rho(x, F) \leq \epsilon\}.$$

*Then $F_\epsilon$ is closed.*

*Proof.* If $F = \emptyset$ then $F_\epsilon = \emptyset$, which is closed, so we can assume that $F \neq \emptyset$.

We must show that $F_\epsilon^c$ is open. If $F_\epsilon^c = \emptyset$, it is open. if $F_\epsilon^c \neq \emptyset$ assume that $x^* \in F_\epsilon^c$. We must show that there is a $\delta > 0$ such that

$$B(x^*, \delta) \subset F_\epsilon^c.$$

Since $x^*$ is not in $F_\epsilon$, we have that $\rho(x^*, F) > \epsilon$. Define $\delta \doteq (\rho(x^*, F) - \epsilon)/2 > 0$. Assume that $y^* \in B(x^*, \delta)$, if $y \in F$ we have

$$\rho(x^*, y) \leq \rho(x^*, y^*) + \rho(y^*, y),$$

so

$$\rho(y^*, y) \geq \rho(x^*, y) - \rho(x^*, y^*).$$

From this we get

$$\begin{aligned}\rho(y^*, y) &\geq \rho(x^*, y) - \rho(x^*, y^*) \\ &\geq \rho(x^*, F) - \rho(x^*, y^*) \\ &> \rho(x^*, F) - \delta \\ &= 2\delta + \epsilon - \delta \\ &= \epsilon + \delta,\end{aligned}$$

where we in the second last equality used that $\rho(x^*, F) = 2\delta + \epsilon$. This means that $\rho(y^*, F) \geq \epsilon + \delta > \epsilon$ because $\epsilon + \delta$ is a lower bound for

$$\{\rho(y^*, y) : y \in F\}.$$

Hence we have shown that

$$B(x^*, \delta) \subset F_\epsilon^c.$$

The proof is complete. □

**Proposition B.2.19.** *Assume that $(S, \rho)$ is a metric space, let $F \subset S$ be a closed set. Let $(\epsilon_k), k \in \mathbb{N}$ be a sequence of positive real numbers, converging to zero, with*

$$\epsilon_{k+1} \leq \epsilon_k.$$

*Then we have*

$$F_{\epsilon_{k+1}} \subset F_{\epsilon_k},$$

*and*

$$\bigcap_{k \in \mathbb{N}} F_{\epsilon_k} = F.$$

*Proof.* If $F = \emptyset$ then $\rho(x, F) = \infty$, $x \in S$, so $F_\epsilon = \emptyset$ for all $\epsilon > 0$. So the result follows in this case.

Assume now that $F \neq \emptyset$. If $x \in F_{\epsilon_{k+1}}$ then

$$\rho(x, F) \leq \epsilon_{k+1} \leq \epsilon_k,$$

so $x \in F_{\epsilon_k}$, hence we have proved

$$F_{\epsilon_{k+1}} \subset F_{\epsilon_k}.$$

Assume now that $x \in F$, since $\rho(x, x) = 0$ we must have that

$$\rho(x, F) = 0,$$

because the infimum of non-negative numbers is non-negative. So for any $k \in \mathbb{N}$

$$F \subset F_{\epsilon_k}.$$

Hence

$$F \subset \bigcap_{k \in \mathbb{N}} F_{\epsilon_k}.$$

Conversely, assume that

$$x \in \bigcap_{k \in \mathbb{N}} F_{\epsilon_k}.$$

Assume for contradiction that $x \in F^c$. Since $F$ is closed there must exist a $\delta > 0$ such that

$$B(x, \delta) \subset F^c.$$

So if $y \in F$ we have

$$\rho(x, y) \geq \delta.$$

Then $\delta$ is a lower bound for

$$\{\rho(x, y) : y \in F\},$$

so $\rho(x, F) \geq \delta$. But since $(\epsilon_k)$ converges to zero, we have that there is an $e_{k'}$ such that $\epsilon_{k'} < \delta$. Then we have $\rho(x, F) \leq \epsilon_{k'}$, since $x \in F_{e_{k'}}$. And we have $\rho(x, F) \geq \delta$. However this is a contradiction because then

$$\rho(x, F) \leq \epsilon_{k'} < \delta \leq \rho(x, F).$$

□

**Lemma B.2.20.** *Let $(S, \rho)$ be a metric space. Let $x, y \in S, \epsilon > 0$. Also let $F \subset S$. If $y \in F$ we either have that $\rho(x, y) \geq \epsilon$ or $x \in F_\epsilon$.*

*Proof.* Assume that $\rho(x, y) < \epsilon$, we must show that $x \in F_\epsilon$. Note that by definition

$$\rho(x, F) = \inf \{\rho(x, y) : y \in F\},$$

this means that $\rho(x, F) < \epsilon$. Hence $x \in F_\epsilon$. □

The next lemma is used in section 8.4.

**Lemma B.2.21.** *Let $K \in \mathbb{N}, R > 0$. Define the set $G \subset C[0, \infty)$ such that*

$$G \doteq \{f \in C[0, \infty) : |f(t)| < R \, \forall t \in [0, K]\}.$$

*Then $G$ is open in the metric space $(C[0, \infty), \rho)$.*

*Proof.* Assume that $f \in G$. We must show that there exists an $\epsilon > 0$ such that $B(f, \epsilon) \subset G$. Since $f$ is continuous and $[0, K]$ is compact, there is $t_1, t_2$ such that $f(t_1)$ is a maximum on $[0, K]$ and $f(t_2)$ is a minimum on $[0, K]$. Let $\Delta \doteq R - \max(|f(t_1)|, |f(t_2)|)$. Define $\epsilon = \min\left(2^{-(K+1)}, \Delta\right)$. Our goal is to show that

$$B(f, \epsilon) \subset G.$$

So assume that $g \in B(f,\epsilon)$. Notice first that

$$\min\left(2^{-K}, \sup\{|f(t)-g(t)| : t \in [0,K]\}\right) = \sup\{|f(t)-g(t)| : t \in [0,K]\} \tag{B.1}$$

To see this note that

$$\begin{aligned}
2^{-(K+1)} &\geq \epsilon \\
&> \rho(f,g) \\
&= \sum_{r=1}^{\infty} \min\left(2^{-r}, \sup\{|f(t)-g(t)| : t \in [0,r]\}\right) \\
&\geq \min\left(2^{-K}, \sup\{|f(t)-g(t)| : t \in [0,K]\}\right).
\end{aligned}$$

So if eq. (B.1) doesn't hold we have $2^{-(K+1)} > 2^{-K}$ which is absurd. Let $s \in [0,K]$ By the triangle inequality we get

$$\begin{aligned}
|g(s)| &= |g(s) - f(s) + f(s)| \\
&\leq |g(s) - f(s)| + |f(s)| \\
&\leq \sup\{|f(t)-g(t)| : t \in [0,K]\} + |f(s)| \\
&= \min\left(2^{-K}, \sup\{|f(t)-g(t)| : t \in [0,K]\}\right) + |f(s)| \\
&\leq \rho(f,g) + |f(s)| \\
&< \epsilon + |f(s)| \\
&\leq \Delta + |f(s)| \\
&= R - \max(|f(t_1)|, |f(t_2)|) + |f(s)| \\
&\leq R.
\end{aligned}$$

So $g \in G$ and hence $B(f,\epsilon) \subset G$. This completes the proof. □

## B.3 Mappings from $C[0,\infty)$ to $C[0,\infty)$

We will in this section prove that various mappings from $C[0,\infty)$ to $C[0,\infty)$ are continuous. These results are used in We will assume that the metric is the one in theorem 2.2.1.

**Definition B.3.1.** *Let $(S,\rho)$ be a metric space. A mapping*

$$H : S \to S,$$

*is continuous if $H^{-1}(O)$ is open in $S$ for every open set $O \in S$.*

**Proposition B.3.2.** *Let $\sigma \in \mathbb{R}$. Then the mapping*

$$H : C[0,\infty) \to C[0,\infty),$$

*given by*

$$[H(g)](t) = \sigma g(t),$$

*is continuous.*

*Proof.* From elementary calculus we have that $\sigma g \in C[0,\infty)$.

Assume first that $\sigma = 0$. Let $O \subset C[0,\infty)$ be an open set. There are two possibilities for $H^{-1}(O)$. If the function which is constant zero is in $O$ then $H^{-1}(O) = C[0,\infty)$. If the constant zero function is not in $O$ then $H^{-1}(O) = \emptyset$. Both $C[0,\infty)$ and $\emptyset$ are open in $(C[0,\infty),\rho)$.

So let us assume that $\sigma \neq 0$. Let $O \subset C[0,\infty)$ be an open set. We know that $\emptyset$ is an open set, so we can assume that $H^{-1}(O) \neq \emptyset$. Let $x \in H^{-1}(O)$, we must show that there exists an $\epsilon > 0$ such that

$$B(x,\epsilon) \subset H^{-1}(O).$$

Since $\sigma x \in O$ and $O$ is open, there is an $\epsilon_2 > 0$ such that

$$B(\sigma x,\epsilon_2) \subset O. \tag{B.2}$$

Let $K$ be such that $\sum_{r=K+1}^{\infty} 2^{-r} < \epsilon_2/2$. Define $\epsilon \doteq \min(\epsilon_2/(2|\sigma|K), 2^{-(K+1)})$. Let $y \in B(x,\epsilon)$, we must show that $\sigma y \in O$. Notice first that

$$\min(2^{-K}, \sup\{|x(t)-y(t)| : t \in [0,K]\}) = \sup\{|x(t)-y(t)| : t \in [0,K]\},$$

because

$$\begin{aligned} 2^{-(K+1)} &\geq \epsilon \\ &> \rho(x,y) \\ &= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|x(t)-y(t)| : t \in [0,r]\}) \\ &\geq \min(2^{-K}, \sup\{|x(t)-y(t)| : t \in [0,K]\}). \end{aligned}$$

This means that for $t \in [0,K]$ we have

$$\begin{aligned} |\sigma x(t) - \sigma y(t)| &= |\sigma||x(t)-y(t)| \\ &\leq |\sigma||x(t)-y(t)| \\ &\leq |\sigma| \sup\{|x(t)-y(t)| : t \in [0,K]\} \\ &= |\sigma| \min(2^{-K}, \sup\{|x(t)-y(t)| : t \in [0,K]\}) \\ &\leq |\sigma|\rho(x,y) \\ &< |\sigma|\epsilon \\ &\leq |\sigma|\frac{\epsilon_2}{2|\sigma|K} \\ &= \frac{\epsilon_2}{2K}, \end{aligned}$$

hence

$$\sup\{|\sigma x(t) - \sigma y(t)| : t \in [0,K]\} \leq \frac{\epsilon_2}{2K}.$$

So this means that

$$\begin{aligned}
&\rho(\sigma x, \sigma y) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|\sigma x(t) - \sigma y(t)| : t \in [0,r]\}) \\
&\leq \sum_{r=1}^{K} \sup\{|\sigma x(t) - \sigma y(t)| : t \in [0,r]\} \\
&\quad + \sum_{r=K+1}^{\infty} 2^{-r} \\
&< K \cdot \frac{\epsilon_2}{2K} + \frac{\epsilon_2}{2} \\
&= \epsilon.
\end{aligned}$$

This means that $\sigma y \in O$, by eq. (B.2), so

$$B(x,\epsilon) \subset H^{-1}(O).$$

This completes the proof.

□

**Proposition B.3.3.** *Let $g \in C[0,\infty)$. Then the mapping*

$$H : C[0,\infty) \to C[0,\infty),$$

*given by*

$$[H(f)](t) = g(t) + f(t),$$

*is continuous.*

*Proof.* Let $O \subset C[0,\infty)$ be an open set. If $H^{-1}(O) = \emptyset$ then it is open, so assume $H^{-1}(O) \neq \emptyset$. Let $x \in H^{-1}(O)$, we must show that there exists an $\epsilon > 0$ such that

$$B(x,\epsilon) \subset H^{-1}(O).$$

We have that $H(x) = x + g \in O$. Since $O$ is open, there exists an $\epsilon_2 > 0$ such that

$$B(x+g,\epsilon_2) \subset O.$$

Define $\epsilon \doteq \epsilon_2$. Assume that $y \in B(x,\epsilon)$. We then get

$$\begin{aligned}
&\rho(H(y), x+g) \\
&= \rho(y+g, x+g) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|y(t) + g(t) - (x(t) + g(t))| : t \in [0,r]\}) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|y(t) - x(t)| : t \in [0,r]\}) \\
&= \rho(y,x) \\
&< \epsilon \\
&= \epsilon_2.
\end{aligned}$$

This means that $H(y) \in B(x+g, \epsilon_2)$, and since $B(x+g, \epsilon_2) \subset O$ we have shown that

$$B(x, \epsilon) \subset H^{-1}(O).$$

This completes the proof. □

**Proposition B.3.4.** *The mapping*

$$H : C[0, \infty) \to C[0, \infty),$$

*given by*

$$[H(f)](t) = e^{f(t)},$$

*is continuous.*

*Proof.* Let $O \subset C[0, \infty)$ be an open set. If $H^{-1}(O) = \emptyset$ then it is open, so assume that $H^{-1}(O) \neq \emptyset$. Assume that $x \in H^{-1}(O)$, we must show that there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) \subset H^{-1}(O).$$

Since $H(x) = \exp(x) \in O$ we have that there is an $\epsilon_2 > 0$ such that

$$B(\exp(x), \epsilon_2) \subset O.$$

Let $K$ be such that $\sum_{r=K+1}^{\infty} 2^{-r} < \epsilon_2/2$. Because of continuity the function $x(t)$ has a well-defined maximum $M_1$ and minimum $M_2$ on $[0, K]$. Define $M = \max(|M_1|, |M_2|)$.

Assume first that $M = 0$. Then $x(t)$ is zero for $t \in [0, K]$. The function $e^z, z \in \mathbb{R}$ is continuous at zero, this means that there exists a $\delta$ such that if $|z| < \delta$

$$|1 - e^z| < \frac{\epsilon_2}{2K}. \tag{B.3}$$

Define $\epsilon \doteq \min(2^{-(K+1)}, \delta)$. Assume that $y \in B(x, \epsilon)$. We then have

$$\begin{aligned} 2^{-(K+1)} &\geq \epsilon \\ &> \rho(x, y) \\ &= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|x(t) - y(t)| : t \in [0, r]\}) \\ &\geq \min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0, K]\}), \end{aligned}$$

hence

$$\min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0, K]\}) = \sup\{|x(t) - y(t)| : t \in [0, K]\}.$$

So since $\rho(x, y) < \epsilon$, we get

$$\begin{aligned} &\sup\{|x(t) - y(t)| : t \in [0, K]\} \\ =\ &\min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0, K]\}) \\ \leq\ &\sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|x(t) - y(t)| : t \in [0, r]\}) \\ =\ &\rho(x, y) \\ <\ &\delta. \end{aligned}$$

Hence by eq. (B.3) we have

$$\sup\{|\exp(x(t)) - \exp(y(t))| : t \in [0,K]\} \leq \frac{\epsilon_2}{2K}.$$

So we get

$$\begin{aligned}
&\rho(e^x, e^y) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|\exp(x(t)) - \exp(y(t))| : t \in [0,r]\}) \\
&\leq \sum_{r=1}^{K} \sup\{|\exp(x(t)) - \exp(y(t))| : t \in [0,r]\} \\
&\quad + \sum_{r=K+1}^{\infty} 2^{-r} \\
&< K\frac{\epsilon_2}{2K} + \frac{\epsilon_2}{2K} \\
&= \epsilon_2.
\end{aligned}$$

Hence $\exp(y) \in B(\exp(x), \epsilon_2) \subset O$, so

$$B(x,\epsilon) \subset H^{-1}(O).$$

Assume now that $M \neq 0$. The function $e^z$ is uniformly continuous on $[-2M, 2M]$ because it is a compact interval. This means that there is a $\delta$ such that if $z_1, z_2, \in [-2M, 2M], |z_1 - z_2| < \delta$ we have

$$|\exp(z_1) - \exp(z_2)| < \frac{\epsilon_2}{2K}. \tag{B.4}$$

We define

$$\epsilon \doteq \min\left(2^{-(K+1)}, \delta, M\right).$$

Assume that $y \in B(x,\epsilon)$. We have

$$\begin{aligned}
2^{-(K+1)} &\geq \epsilon \\
&> \rho(x,y) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|x(t) - y(t)| : t \in [0,r]\}) \\
&\geq \min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0,K]\},
\end{aligned}$$

so

$$\begin{aligned}
&\min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0,K]\}) \\
&= \sup\{|x(t) - y(t)| : t \in [0,K]\}.
\end{aligned}$$

Hence we similarly get

$$\begin{aligned}
\min(\delta, M) &\geq \epsilon \\
&> \rho(x, y) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|x(t) - y(t)| : t \in [0, r]\}) \qquad \text{(B.5)} \\
&\geq \min(2^{-K}, \sup\{|x(t) - y(t)| : t \in [0, K]\}) \\
&= \sup\{|x(t) - y(t)| : t \in [0, K]\}.
\end{aligned}$$

We recall that for $t \in [0, K]$ we have $|x(t)| \leq M$. This combined with eq. (B.5) means that for $t \in [0, K]$

$$\begin{aligned}
|y(t)| &= |y(t) - x(t) + x(t)| \\
&\leq |y(t) - x(t)| + |x(t)| \\
&\leq \sup\{|x(t) - y(t)| : t \in [0, K]\} + M \\
&\leq M + M \\
&= 2M.
\end{aligned}$$

So for $t \in [0, K]$ we have that $x(t), y(t) \in [-2M, 2M]$, and from eq. (B.5) we have that $|x(t) - y(t)| < \delta$. So by eq. (B.4)

$$|\exp(x(t)) - \exp(y(t))| \leq \frac{\epsilon_2}{2K},$$

for $t \in [0, K]$. Hence

$$\begin{aligned}
&\rho(\exp(x), \exp(y)) \\
&= \sum_{r=1}^{\infty} \min(2^{-r}, \sup\{|\exp(x(t)) - \exp(y(t))| : t \in [0, r]\}) \\
&\leq \sum_{r=1}^{K} \sup\{|\exp(x(t)) - \exp(y(t))| : t \in [0, r]\} \\
&\quad + \sum_{r=K+1}^{\infty} 2^{-r} \\
&\leq K \cdot \frac{\epsilon_2}{2K} + \frac{\epsilon_2}{2K} \\
&= \epsilon_2.
\end{aligned}$$

Hence $\exp(y) \in B(\exp(x), \epsilon_2) \subset O$, so

$$B(x, \epsilon) \subset H^{-1}(O).$$

We have now shown that $H^{-1}(O)$ is open, and the proof is complete. □

# Appendix C

# Results from probability theory

## C.1 Elementary concepts

A probability space $(\Omega, \mathcal{A}, P)$ is a measure space where

$$P(\Omega) = 1.$$

We assume that measure theory is known to the reader. Chapter three, four and five in [MW13] contains the necessary information. Random variables are real-valued measurable functions on the probability space, for details see chapter seven of [MW13].

**Theorem C.1.1.** *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $(\Omega', \mathcal{A})$ be a measurable space. If*

$$f : \Omega \to \Omega',$$

*is $\mathcal{A}/\mathcal{A}'$-measurable, then $(\Omega', \mathcal{A}', P')$ where*

$$P'(A') = P(f^{-1}(A')), A' \in \mathcal{A},$$

*is a probability space.*

*Proof.*

$$P'(\emptyset) = P(f^{-1}(\emptyset)) = P(\emptyset) = 0$$
$$P'(\Omega') = P(f^{-1}(\Omega')) = P(\Omega) = 1.$$

Assume that $\{A_n\}$ is a mutually disjoint collection of sets in $\mathcal{A}'$. Then by elementary set-theory $\{f^{-1}(A_n')\}$ is a mutually disjoint collection of sets in $\mathcal{A}$, we get

$$\begin{aligned} P'(\cup_n A_n') =& P(f^{-1}(\cup_n A_n)) \\ =& P(\cup_n f^{-1}(A_n)) \\ =& \sum_n P(f^{-1}(A_n)) \\ =& \sum_n P'(A_n). \end{aligned}$$

□

We end this section by defining a stochastic process and proving a useful result regarding continuous processes.

**Definition C.1.2.** *Let $(\Omega, \mathcal{A}, P)$ be a probability space. A stochastic process on this space is a collection*

$$\{X_t : t \in [0, \infty)\},$$

*such that for each $t$, $X_t$ is a function satisfying*

*(i)*

$$X_t : \Omega \to \mathbb{R},$$

*(ii)*

$$X_t^{-1}(B) \in \mathcal{A},$$

*for all $B \in \mathcal{B}(\mathbb{R})$.*

**Proposition C.1.3.** *Let $(Y_t)_{t\in[0,\infty)}$ be a continuous stochastic process on $(\Omega, \mathcal{A}, P)$. Then*

$$\{\omega \in \Omega : Y_t(\omega) \leq 0 \textit{ for some } t \in [0, \infty)\} \in \mathcal{A}.$$

*Proof.* Let $n, k \in \mathbb{N}$, and define the set

$$A_{n,k} \doteq \left\{ \omega \in \Omega : Y_t(\omega) \geq \frac{1}{n} \text{ for all } t \in [0, k] \right\}.$$

We first show that these sets are measurable. Let $k, n$ be fixed. Since $(Y_t)_{t\in[0,\infty)}$ is a stochastic process, and $\sigma$-algebras are closed under countable intersections, it suffices to show

$$A_{n,k} = \bigcap_{q\in\mathbb{Q}\cap[0,k]} Y_q^{-1}\left([1/n, \infty)\right)$$

Assume $\omega \in A_{n,k}$. For all $t \in [0, k]$ we have that $Y_t(\omega) \geq 1/n$, since this must also hold for rational time-points we have $\omega \in \cap_{q\in\mathbb{Q}\cap[0,k]} Y_q^{-1}\left([1/n, \infty)\right)$. Conversely, assume $\omega \in \cap_{q\in\mathbb{Q}\cap[0,k]} Y_q^{-1}\left([1/n, \infty)\right)$. We then have that for every rational q number in $[0, k]$

$$Y_q(\omega) \geq \frac{1}{n}.$$

Assume for contradiction that $\omega \neq A_{n,k}$. Then there is a $t \in [0, k]$ such that $Y_t(\omega) < 1/n$. If $t$ is rational we already have a contradiction, so assume $t$ is irrational. Let $\epsilon = 1/n - Y_t(\omega)$. Because $Y_\cdot(\omega)$ is a continuous trajectory there exist a $\delta$ such that if $|t - t^*| < \delta$ we have $|Y_t(\omega) - Y_{t^*}(\omega)| < \epsilon$. Now choose a rational number $q^*$ in $\mathbb{Q} \cap [0, k] \cap (t - \delta, t + \delta)$. Then

$$\begin{aligned}
Y_{q^*}(\omega) &= Y_{q^*}(\omega) - Y_t(\omega) + Y_t(\omega) \\
&\leq |Y_{q^*}(\omega) - Y_t(\omega)| + Y_t(\omega) \\
&< \epsilon + Y_t(\omega) \\
&= \frac{1}{n}.
\end{aligned}$$

Hence we have a contradiction since $\omega \in Y_{q^*}^{-1}([1/n, \infty))$. So we have proven that for every $n, k \in \mathbb{N}$ we have that $A_{n,k} \in \mathcal{A}$.

Define the set

$$B \doteq \bigcap_{k \in \mathbb{N}} \left( \bigcup_{\substack{n \in \mathbb{N} \\ n \geq k}} A_{n,k} \right).$$

Since $\sigma$-algebras are closed under countable unions and intersections we also have that $B \in \mathcal{A}$.

Our next goal is to show

$$B = \{\omega \in \Omega : Y_t(\omega) > 0 \text{ for all } t \in [0, \infty)\}.$$

Assume $\omega \in B$. Let $t \in [0, \infty)$ be arbitrary. Choose $k^* \in \mathbb{N}$ such that $t < k^*$. We have that

$$\omega \in \bigcup_{\substack{n \in \mathbb{N} \\ n \geq k^*}} A_{n,k^*}.$$

So there must exist an $n^*$ such that $\omega \in A_{n^*,k^*}$. But then $Y_t(\omega) \geq 1/n^* > 0$. Hence

$$B \subset \{\omega \in \Omega : Y_t(\omega) > 0 \text{ for all } t \in [0, \infty)\}.$$

Assume conversely that $\omega \in \{\omega \in \Omega : Y_t(\omega) > 0 \text{ for all } t \in [0, \infty)\}$. Let $k^* \in \mathbb{N}$ be arbitrary. On the compact interval $[0, k^*]$ the continuous function $Y_\cdot(\omega)$ must have a well-defined positive minimum $a$. Choose $n^*$ such that $n^* \geq k^*$ and $1/n^* < a$. Then $\omega \in A_{n^*,k^*}$. Hence

$$B \supset \{\omega \in \Omega : Y_t(\omega) > 0 \text{ for all } t \in [0, \infty)\}.$$

Since $\sigma$-algebras are closed under complements, we get

$$\begin{aligned}
&\{\omega \in \Omega : Y_t(\omega) \leq 0 \text{ for some } t \in [0, \infty)\} \\
&= \left(\{\omega \in \Omega : Y_t(\omega) \leq 0 \text{ for some } t \in [0, \infty)\}^c\right)^c \\
&= \left(\{\omega \in \Omega : Y_t(\omega) > 0 \text{ for all } t \in [0, \infty)\}\right)^c \\
&= B^c \in \mathcal{A}.
\end{aligned}$$

□

## C.2 Independence

The concept of independent random variables is important in this thesis. Independence of random variables is presented in different ways in different texts, but they can all be shown to be equal. One way is in the form of generated $\sigma$-algebras of the random variables. We will use a simpler definition, but it is of course logically equivalent to the more complicated definitions using $\sigma$-algebras.

**Definition C.2.1** (slightly modified from [**PP13, p. 36**])**.** *Let $(\Omega, \mathcal{A}, P)$ be a probability space, assume that $\{X_j : 1 \leq j \leq n\}$ is a finite collection of random variables on this space. This collection is said to be independent if for any $n$ Borel sets $A_1, A_2, \ldots, A_n$ on the line*

$$P\left(\bigcap_{1\leq j\leq n} X_j^{-1}(A_j)\right) = \prod_{1\leq j\leq n} P\left(X_j^{-1}(A_j)\right).$$

**Definition C.2.2** ([**PP13, p. 36**])**.** *An infinite collection of random variables is said to be independent if every finite subcollection is independent*

The next theorem is due to Kac. We will present a modified version of the one in [App09], as Applebaum allows random variables to be vectors, but we are only interested in the case then they are real numbers.

**Theorem C.2.3** (**Kac's Theorem, modified from** [**App09, p. 18**] )**.** *Let $i$ be the imaginary unit. The random variables $X_1, X_2, \ldots, X_n$ are independent if and only if*

$$\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j X_j\right)\right] = \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(iu_j X_j\right)\right],$$

*for all $u_1, u_2, \ldots, u_n \in \mathbb{R}$.*

**Proposition C.2.4.** *Let $(X_r)_{r\in\mathbb{Z}}$ be a collection of independent random variables on the probability space $(\Omega, \mathcal{A}, P)$. Let $(a_r)_{r\in\mathbb{Z}}$ be a collection of real numbers. Assume that $A \in \mathcal{A}$ is a set such that*

$$P(A) = 1.$$

*Define*

$$Y_r \doteq a_r I_A(\omega) X_r(\omega).$$

*Then $(Y_r)_{r\in\mathbb{Z}}$ is a collection of independent random variables.*

**Remark.** *Note that for each $r$, $Y_r$ is measurable since $A$ is a measurable set.*

*Proof.* Let $n$ be a natural number. Assume that you have $n$ other natural numbers $i_1, i_2, \ldots, i_n$ and $n$ real numbers $u_1, u_2, \ldots, u_n$. We have

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j Y_{i_j}\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j a_{i_j} I_A X_{i_j}\right)\right] \\
&= \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j a_{i_j} I_A X_{i_j}\right) I_A + I_A^c\right] \\
&= \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j a_{i_j} I_A X_{i_j}\right) I_A + \exp\left(i\sum_{j=1}^{n} u_j a_{i_j} I_A^c X_{i_j}\right) I_A^c\right] \\
&= \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j a_{i_j} X_{i_j}\right)\right],
\end{aligned}$$

where we have used that the integral of an integrable function over a set of measure zero, is zero. By the independence of $\{x_n\}$ and theorem C.2.3 we have

$$\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} u_j a_{i_j} X_{i_j}\right)\right] = \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j a_{i_j} X_{i_j}\right)\right].$$

On the other hand we have

$$\begin{aligned}
&\prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j Y_{i_j}\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j a_{i_j} I_A X_{i_j}\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j a_{i_j} I_A X_{i_j}\right) + I_{A^c}\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j a_{i_j} I_A X_{i_j}\right) + \exp\left(i u_j a_{i_j} I_A^c X_{i_j}\right) I_{A^c}\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i u_j a_{i_j} X_{i_j}\right)\right],
\end{aligned}$$

where we again used that the integral of an integrable function over a set of measure zero is zero. The result now follows by another application of theorem C.2.3. □

**Proposition C.2.5.** *Let $(X_r)_{r\in\mathbb{Z}}$ be a sequence of independent random variables on the probability space $(\Omega, \mathcal{A}, P)$. Let $n$ be a natural number. Assume that for each $1 \le j \le n$ you have a natural number $N_j$, $N_j$ real numbers*

*$a_1^{(j)}, a_2^{(j)}, \dots, a_{N_j}^{(j)}$ and $N_j$ integers $z_1^{(j)}, z_2^{(j)}, \dots, z_{N_j}^{(j)}$. Also assume that if $j_1 \neq j_2$ or $k_1 \neq k_2$ we have*

$$z_{k_1}^{(j_1)} \neq z_{j_2}^{(j_2)}. \tag{C.1}$$

*Specifically this means that all the $z_k^{(j)}$ are different.*

*Let $A \subset \mathcal{A}$ be such that*

$$P(A) = 1.$$

*For each $1 \leq j \leq n$ define the random variable*

$$Y_j = \sum_{k=1}^{N_j} a_k^{(j)} I_A(\omega) X_{z_k^{(j)}}(\omega).$$

*Then $(Y_j)_{1 \leq j \leq n}$ is a collection of independent random variables.*

*Proof.* Let $i$ denote the imaginary unit. Notice that if $V$ is any random variable on $(\Omega, \mathcal{A}, P)$ and $a$ is a real number we have

$$\mathrm{E}[\exp(iaI_A V)] = \mathrm{E}[\exp(iaV)],$$

because the integral of an integrable function does not change if the function is modified on a set of measure zero.

Assume that $u_1, u_2, \dots, u_n \in \mathbb{R}$, we get

$$\begin{aligned}
&\prod_{j=1}^{n} \mathrm{E}\left[\exp\left(iu_j Y_j\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(iu_j \sum_{k=1}^{N_j} a_k^{(j)} I_A(\omega) X_{z_k^{(j)}}(\omega)\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(iu_j \sum_{k=1}^{N_j} a_k^{(j)} X_{z_k^{(j)}}(\omega)\right)\right] \\
&= \prod_{j=1}^{n} \mathrm{E}\left[\exp\left(i \sum_{k=1}^{N_j} u_j a_k^{(j)} X_{z_k^{(j)}}(\omega)\right)\right] \\
&= \prod_{j=1}^{n} \prod_{k=1}^{N_j} \mathrm{E}\left[\exp\left(iu_j a_k^{(j)} X_{z_k^{(j)}}(\omega)\right)\right].
\end{aligned}$$

In the last step we used theorem C.2.3, the independence of $(X_r)_{r \in \mathbb{Z}}$ and eq. (C.1).

We also get

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n}u_jY_j\right)\right]\\
&=\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n}u_j\sum_{k=1}^{N_j}a_k^{(j)}I_A(\omega)X_{z_k^{(j)}}(\omega)\right)\right]\\
&=\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n}u_j\sum_{k=1}^{N_j}a_k^{(j)}X_{z_k^{(j)}}(\omega)\right)\right]\\
&=\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n}\sum_{k=1}^{N_j}u_ja_k^{(j)}X_{z_k^{(j)}}(\omega)\right)\right]\\
&=\prod_{j=1}^{n}\prod_{k=1}^{N_j}\mathrm{E}\left[\exp\left(iu_ja_k^{(j)}X_{z_k^{(j)}}(\omega)\right)\right].
\end{aligned}$$

In the last step we again used theorem C.2.3, the independence of $(X_r)_{r\in\mathbb{Z}}$ and eq. (C.1). Since we have shown that

$$\prod_{j=1}^{n}\mathrm{E}\left[\exp\left(iu_jY_j\right)\right]=\mathrm{E}\left[\exp\left(i\sum_{j=1}^{n}u_jY_j\right)\right],$$

the result follows from theorem C.2.3. □

**Proposition C.2.6.** *Let $(y_r)_{r\in\mathbb{N}}$ be a sequence of independent random variables defined on a probability space $(\Omega,\mathcal{A},P)$. Assume that $(a_r)_{r\in\mathbb{N}}$ is a sequence of real numbers and $M$ a natural number. Also assume $A\in\mathcal{A}$ is a set such that*

$$P\left(A\right)=1,$$

*and that for each $\omega\in\mathcal{A}$ the sum*

$$\sum_{r=M+1}^{\infty}a_ry_r(\omega),$$

*converges. Then the random variables $z_1,z_2$, defined by*

$$\begin{aligned}
z_1(\omega)&\doteq\sum_{r=1}^{M}a_rI_A(\omega)y_r(\omega)\\
z_2(\omega)&\doteq\sum_{r=M+1}^{\infty}a_rI_A(\omega)y_r(\omega),
\end{aligned}$$

*are independent.*

**Remark.** *$z_2$ is a well-defined random variable because it is the pointwise limit of random variables.*

*Proof.* Let $u_1, u_2 \in \mathbb{R}$ and $i$ be the imaginary unit. We have

$$\begin{aligned} &\mathrm{E}\left[\exp i\left(u_1 z_1(\omega) + u_2 z_2(\omega)\right)\right] \\ &= \mathrm{E}\left[\exp\left(i \cdot \left[u_1 \cdot \sum_{r=1}^{M} a_r I_A(\omega) y_r(\omega) + u_2 \cdot \sum_{r=M+1}^{\infty} a_r I_A(\omega) y_r(\omega)\right]\right)\right]. \end{aligned}$$

Since the exponential function is continuous this is equal to

$$\mathrm{E}\left[\lim_{N\to\infty} \exp\left(i \cdot \left[u_1 \cdot \sum_{r=1}^{M} a_r I_A(\omega) y_r(\omega) + u_2 \cdot \sum_{r=M+1}^{N} a_r I_A(\omega) y_r(\omega)\right]\right)\right].$$

Since $|\exp(ib)| = 1$,if $b \in \mathbb{R}$, the dominated convergence theorem tells us that this is equal to

$$\lim_{N\to\infty} \mathrm{E}\left[\exp\left(i \cdot \left[u_1 \cdot \sum_{r=1}^{M} a_r I_A(\omega) y_r(\omega) + u_2 \cdot \sum_{r=M+1}^{N} a_r I_A(\omega) y_r(\omega)\right]\right)\right].$$

Since $P(A) = 1$ and we are taking the expectation of an integrable function we get that this is equal to

$$\lim_{N\to\infty} \mathrm{E}\left[\exp\left(i \cdot \left[\sum_{r=1}^{M} u_1 a_r y_r(\omega) + \sum_{r=M+1}^{N} u_2 a_r y_r(\omega)\right]\right)\right].$$

By the assumed independence of $\{y_r\}$ and theorem C.2.3 this is equal to

$$\lim_{N\to\infty} \left(\prod_{r=1}^{M} \mathrm{E}\left[\exp\left(i u_1 a_r y_r(\omega)\right)\right]\right)\left(\prod_{r=M+1}^{N} \mathrm{E}\left[\exp\left(i u_2 a_r y_r(\omega)\right)\right]\right).$$

Again by the assumed independence of $\{y_r\}$ and theorem C.2.3 this is equal to

$$\mathrm{E}\left[\exp\left(i u_1 \sum_{r=1}^{M} a_r y_r(\omega)\right)\right] \lim_{N\to\infty} \mathrm{E}\left[\exp\left(i u_2 \sum_{r=M+1}^{N} a_r y_r(\omega)\right)\right].$$

Since $P(A) = 1$ and we are dealing with bounded functions this is equal to

$$\mathrm{E}\left[\exp\left(i u_1 \sum_{r=1}^{M} a_r I_A(\omega) y_r(\omega)\right)\right] \lim_{N\to\infty} \mathrm{E}\left[\exp\left(i u_2 \sum_{r=M+1}^{N} a_r I_A(\omega) y_r(\omega)\right)\right].$$

By another application of the dominated convergence theorem, this is equal to

$$\mathrm{E}\left[\exp\left(i u_1 \sum_{r=1}^{M} a_r I_A(\omega) y_r(\omega)\right)\right] \mathrm{E}\left[\exp\left(i u_2 \sum_{r=M+1}^{\infty} a_r I_A(\omega) y_r(\omega)\right)\right].$$

This is by definition

$$\mathrm{E}\left[\exp\left(i u_1 z_1(\omega)\right)\right] \mathrm{E}\left[\exp\left(i u_2 z_2(\omega)\right)\right].$$

Since we have shown

$$\mathrm{E}\left[\exp i\left(u_1 z_1(\omega) + u_2 z_2(\omega)\right)\right] = \mathrm{E}\left[\exp\left(i u_1 z_1(\omega)\right)\right] \mathrm{E}\left[\exp\left(i u_2 z_2(\omega)\right)\right],$$

the result follows from theorem C.2.3. □

## C.3 Convergence in distribution

**Lemma C.3.1.** *Assume that* $\{(X_1^{(n)}, X_2^{(n)}, \dots, X_k^{(n)})\}_{n\in\mathbb{N}}$ *is a sequence of random vectors, and there exists a positive semi-definite matrix* $A$ *such that for every* $\vec{u} \in \mathbb{R}^k$ *we have that*

$$u_1 X_1^{(n)} + u_2 X_2^{(n)} + \cdots + u_k X_k^{(n)}$$

*converges in distribution to a normal random variable with expectation zero and variance* $\vec{u}^T Au$. *Then*

$$(X_1^{(n)}, X_2^{(n)}, \dots, X_k^{(n)})$$

*converges in distribution to a multivariate normal vector with expectation zero, and covariance matrix* $A$.

*Proof.* We get for $u \in \mathbb{R}^k$.

$$\begin{aligned} & E\left[\exp(i \cdot u^T \vec{X}^n)\right] \\ = & E\left[\exp(i \cdot (u_1 X_1^{(n)} + u_2 X_2^{(n)} + u_k X_k^{(n)}))\right] \\ = & \exp(-0.5 \cdot \vec{u}^T Au \cdot 1^2), \end{aligned}$$

since convergence in distribution implies convergence of characteristic functions. The result now follows from Glivenko's theorem. □

# Appendix D

# Useful results from Calculus and Real Analysis

In this appendix we will prove simple results from Calculus and Real analysis which are used.

## D.1 Inequalities

**Lemma D.1.1.** *Let* $0 < \epsilon < 1$*. Assume that* $|x| \leq \min(|\ln(1-\epsilon)|, \ln(1+\epsilon))$*. Then*

$$|1 - e^x| \leq \epsilon.$$

*Proof.* Since $|x| \leq |\ln(1-\epsilon)|$ we have that $-x \leq |\ln(1-\epsilon)|$. This implies that

$$\begin{aligned} x &\geq -|\ln(1-\epsilon)| \\ &= \ln(1-\epsilon). \end{aligned}$$

Since the exponential function increases monotonically we get

$$e^x \geq 1 - \epsilon,$$

this implies

$$1 - e^x \leq \epsilon. \tag{D.1}$$

Since $|x| \leq \ln(1+\epsilon)$, we have that $x \leq \ln(1+\epsilon)$. Again since the exponential function increases monotonically we have

$$e^x \leq 1 + \epsilon,$$

so

$$e^x - 1 \leq \epsilon. \tag{D.2}$$

The result now follows from eq. (D.1) and eq. (D.2). □

**Remark.** *It can actually be shown that for* $0 < \epsilon < 1$ *we have*

$$\min(|\ln(1-\epsilon)|, \ln(1+\epsilon)) = \ln(1+\epsilon),$$

*but this is of no concern for our arguments.*

**Lemma D.1.2.** *Let $\lfloor \cdot \rfloor$ denote the floor function. Assume $0 \leq z_1 < z_2 \leq t$, and let $\{\delta_n\}$ be a sequence of positive real numbers converging to zero. Then there exists an $n^*$ such that if $n \geq n^*$ we have*

$$\left\lfloor \frac{z_1}{\delta_n} \right\rfloor < \left\lfloor \frac{z_2}{\delta_n} \right\rfloor - 1 < \left\lfloor \frac{t}{\delta_n} \right\rfloor.$$

*Proof.* Since $z_2 \leq t$, we must have $\lfloor z_2/\delta_n \rfloor \leq \lfloor t/\delta_n \rfloor$. So for all $n$ we have

$$\left\lfloor \frac{z_2}{\delta_n} \right\rfloor - 1 < \left\lfloor \frac{t}{\delta_n} \right\rfloor.$$

Choose $n^*$ such that if $n \geq n^*$ we have $\delta_n < (z_2 - z_1)/2$, this means that if $n \geq n^*$ we have $-(z_2 - z_1)/\delta_n < -2$. So if $n \geq n^*$ we then have

$$\begin{aligned}
\left\lfloor \frac{z_1}{\delta_n} \right\rfloor &\leq \frac{z_1}{\delta_n} \\
&= \frac{z_1}{\delta_n} + \frac{z_2 - z_1}{\delta_n} - \frac{z_2 - z_1}{\delta_n} \\
&= \frac{z_2}{\delta_n} - \frac{z_2 - z_1}{\delta_n} \\
&< \frac{z_2}{\delta_n} - 2 \\
&< \left\lfloor \frac{z_2}{\delta_n} \right\rfloor - 1.
\end{aligned}$$

In the last inequality we used that for a positive real number $a$, we have $a - 1 < \lfloor a \rfloor$. □

**Lemma D.1.3.** *Assume $a_1, a_2, b_1, b_2 \geq 0$. Also assume*

$$\min(a_1, a_2) \geq \max(b_1, b_2).$$

*Let $q_1, q_2, q_3, q_4 \in [0, 1]$ with $q_1 + q_2 = 1$ and $q_3 + q_4 = 1$. Then*

$$(q_1 a_1 + q_2 a_2 - q_3 b_1 - q_4 b_2)^2 \leq (\max(a_1, a_2) - \min(b_1, b_2))^2$$

*Proof.* Notice first that

$$\begin{aligned}
& q_1 a_1 + q_2 a_2 - q_3 b_1 - q_4 b_2 \\
\geq & \; q_1 \min(a_1, a_2) + q_2 \min(a_1, a_2) - q_3 \max(b_1, b_2) - q_4 \max(b_1, b_2) \\
= & \; \min(a_1, a_2) - \max(b_2, b_2) \\
\geq & 0.
\end{aligned}$$

So we get

$$\begin{aligned}
& |q_1 a_1 + q_2 a_2 - q_3 b_1 - q_4 b_2| \\
= & \; q_1 a_1 + q_2 a_2 - q_3 b_1 - q_4 b_2 \\
\leq & \; q_1 \max(a_1, a_2) + q_2 \max(a_1, a_2) - q_3 \min(b_1, b_2) - q_4 \min(b_1, b_2) \\
= & \; \max(a_1, a_2) - \min(b_1, b_2).
\end{aligned}$$

□

**Lemma D.1.4.** *Assume that $a, b, c \in \mathbb{R}$, with $a, b, c \geq 0$. We then have*

$$\min(a, b + c) \leq \min(a, b) + \min(a, c).$$

*Proof.* Assume first that $a \leq b$. We then have

$$\begin{aligned}\min(a, b + c) &= a \\ &= \min(a, b) \\ &\leq \min(a, b) + \min(a, c).\end{aligned}$$

Assume now that $a \leq c$, again we get

$$\begin{aligned}\min(a, b + c) &= a \\ &= \min(a, c) \\ &\leq \min(a, b) + \min(a, c).\end{aligned}$$

Lastly we assume $a > b$ and $a > c$. We get

$$\begin{aligned}\min(a, b + c) &\leq b + c \\ &= \min(a, b) + \min(a, c).\end{aligned}$$

□

**Lemma D.1.5.** *Assume that $a, b, c \in \mathbb{R}$, with $a, b, c \geq 0$ and $b \leq c$. We then have*

$$\min(a, b) \leq \min(a, c).$$

*Proof.* Assume first that $a \leq b$. We then get

$$\begin{aligned}\min(a, b) &= a \\ &= \min(a, c).\end{aligned}$$

Where we in the last step used $a \leq b \leq c$. Assume now that $a > b$. We then have

$$\begin{aligned}\min(a, b) &= b \\ &\leq \min(a, c).\end{aligned}$$

Where we in the last step used that $a > b$ and $c \geq b$. □

**Lemma D.1.6.** *Assume that $a, b, c \in \mathbb{R}$, with $a, b, c \geq 0$. We then have*

$$\min(a, b) - \min(a, c) \leq \min(a, |b - c|). \tag{D.3}$$

*Proof.* We will check six different cases. We start with $a \leq b \leq c$. Then the left-hand side of eq. (D.3) is zero. This is also the case for $a \leq c \leq b$.

If $b \leq a \leq c$ the left-hand side of eq. (D.3) is $b - a$. This value is non-positive.

If $b \leq c \leq a$ the left-hand side of eq. (D.3) is $b - c$. This value is also non-positive.

Now if $c \leq a \leq b$, the left-hand side of eq. (D.3) is $a - c$. This value is both less than or equal $a$ and $b - c$. Notice that by assumption $b - c = |b - c|$.

Lastly assume that $c \leq b \leq a$. The left-hand side of eq. (D.3) is now $b - c$. This value is less than $a$ since $b \leq a$. We also have that $b - c$ is less than or equal $|b - c|$. □

**Lemma D.1.7.** *Assume $k \in \mathbb{N}, a \geq 0$. Then*

$$\min(2^{-k}, 2a) \leq 2\min(2^{-k}, a).$$

*Proof.* Assume first that $\min(2^{-k}, 2a) = 2a$. We then have

$$2a \leq 2^{-k} \leq 2 \cdot 2^{-k}.$$

Since we also have $2a \leq 2a$ the result follows for this case.

Assume now that $\min(2^{-k}, 2a) = 2^{-k}$. We have $2^{-k} \leq 2 \cdot 2^{-k}$. And we also have $2^{-k} \leq 2 \cdot a$. This completes the proof. □

## D.2 Taylor polynomials

**Lemma D.2.1.** *Let $x \in [-0.5, 0.5]$ and $k \in \{0\} \cup \mathbb{N}$. Then*

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + r_1(x)x^{k+1},$$

*where*

$$|r_1(x)| \leq 2.$$

*Proof.* Since the exponential function and its derivatives are continuous we can look at the Taylor polynomial. We look at the Taylor polynomial around zero with the Lagrange-remainder, see [Lin06, p. 591]. If $x = 0$ we can choose $r(x) = 0$. Assume $x \neq 0$ we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \frac{e^c x^{k+1}}{(k+1)!},$$

for $c \in (0, x)$ or $c \in (x, 0)$. Since the maximum value for $e^c$ is $e^{0.5} \approx 1.65$, and $(k+1)! \geq 1$, the result follows. □

**Lemma D.2.2.** *Let $x \in [-0.5, 0.5]$ and $k \in \mathbb{N}$. Then*

$$\ln(1 + x) = x + r_2(x)x^2,$$

*where*

$$|r_2(x)| \leq 2.$$

*Proof.* The derivative of $\ln(1 + x) = 1/(1 + x)$, the second derivative is $-1/(1 + x)^2$. Both of these are continuous in $[-0.5, 0.5]$ so we can look at the Taylor polynomial. We get using the Lagrange remainder, see [Lin06, p. 591]

$$\ln(1 + x) = 0 + \frac{1/(1 + 0)}{1}x + \frac{-1/(1 + c)^2}{2!}x^2.$$

We have

$$\begin{aligned}\left|\frac{-1/(1 + c)^2}{2!}\right| &= \frac{1}{2} \cdot \left|\frac{1}{(1 + c)^2}\right| \\ &\leq \frac{1}{2}\left|\frac{1}{(1 - 0.5)^2}\right| \\ &= 2.\end{aligned}$$

This completes the proof. □

## D.3 Convergence results

**Lemma D.3.1.** *Let* $t_1, t_2 \in (0, \infty), H \in (0, 1)$*. Also let* $\{\delta_n\}$ *be a sequence of positive real numbers converging to zero. Assume that* $s \in (-\infty, 0)$ *then*

$$\sum_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n + \delta_n}^{-\delta_n}{}_{\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \Big[ (\lfloor t_1/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \cdot \Big[ (\lfloor t_2/\delta_n \rfloor \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}} \Big] \Bigg) \to \Big( (t_1 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5} \Big) \Big( (t_2 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5} \Big),$$

*as* $n \to \infty$*.*

*Proof.* First note that

$$-\lfloor 1/\delta_n^2 \rfloor \delta_n + \delta_n < -\left(\frac{1}{\delta_n^2} - 1\right)\delta_n + \delta_n = \frac{-1}{\delta_n} + 2\delta_n.$$

The last expression goes to $-\infty$ as $n$ goes to infinity. So for big enough $n$ there is one and only one $\tau_n$ such that $s \in [\tau_n, \tau_n + \delta_n)$. Assume that $n$ is big enough for this to be the case. What we need to show is

$$\Big( (\lfloor t_1/\delta_n \rfloor \delta_n - \tau_n)^{H-\frac{1}{2}} - (-\tau_n)^{H-\frac{1}{2}} \Big) \cdot \Big( (\lfloor t_2/\delta_n \rfloor \delta_n - \tau_n)^{H-\frac{1}{2}} - (-\tau_n)^{H-\frac{1}{2}} \Big) \to \Big( (t_1 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5} \Big) \Big( (t_2 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5} \Big),$$

as $n$ goes to infinity. Note first that $\tau_n$ converges to $s$ since $|\tau_n - s| < \delta_n$. Also note that for $i = 1, 2$, $\lfloor t_i/\delta_n \rfloor \delta_n$ converges to $t_i$ since

$$|\lfloor t_i/\delta_n \rfloor \delta_n - t_i| = \delta_n \, |\lfloor t_i/\delta_n \rfloor - t_i/\delta_n| \leq \delta_n.$$

Since sums of sequences converges to the sums of the limits we have for $i = 1, 2$

$$\lfloor t_1/\delta_n \rfloor \delta_n - \tau_n \to t_i - s,$$

as $n$ goes to infinity. The function $x^{H-1/2}$ from $(0, \infty)$ to $\mathbb{R}$ is continuous. So we have that

$$(\lfloor t_1/\delta_n \rfloor \delta_n - \tau_n)^{H-\frac{1}{2}} \to (t_i - s)^{H-\frac{1}{2}},$$

and

$$(-\tau_n)^{H-\frac{1}{2}} \to (-s)^{H-\frac{1}{2}},$$

as $n$ goes to infinity. So again since limits behave well under sums we have for $i = 1, 2$

$$\begin{aligned}&\Big((\lfloor t_i/\delta_n \rfloor \delta_n - \tau_n)^{H-\frac{1}{2}} - (-\tau_n)^{H-\frac{1}{2}}\Big)\\ \to &\Big((t_i - s)^{H-\frac{1}{2}} - (-s)^{H-0.5}\Big).\end{aligned}$$

The result now follows by the fact that products of sequences in $\mathbb{R}$ converges to the product of the limits. ☐

**Lemma D.3.2.** *Let $t_1, t_2 \in (0, \infty), H \in (0, 1)$. Also let $\{\delta_n\}$ be a sequence of positive real numbers converging to zero. Assume that $s \in (-\infty, 0)$ then*

$$\begin{aligned}&\sum_{\tau=-\lfloor 1/\delta_n^2 \rfloor \delta_n + \delta_n}^{-\delta_n}{}_{\delta_n} \Bigg( I_{[\tau,\tau+\delta_n)}(s) \Big[(\lfloor t_1/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\Big]\\ &\qquad \cdot \Big[(\lfloor t_2/\delta_n \rfloor \delta_n + \delta_n - \tau)^{H-\frac{1}{2}} - (-\tau)^{H-\frac{1}{2}}\Big]\Bigg)\\ &\to \Big((t_1 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5}\Big)\Big((t_2 - s)^{H-\frac{1}{2}} - (-s)^{H-0.5}\Big),\end{aligned}$$

*as $n \to \infty$.*

*Proof.* The proof is almost identical to the proof of lemma D.3.1 but we in this case work with $\lfloor t_i/\delta_n \rfloor \delta_n + \delta_n$ instead of $\lfloor t_i/\delta_n \rfloor \delta_n$, we can do this since $\delta_n$ converges to zero. ☐

**Lemma D.3.3.** *Let $0 \le z_1 < z_2, H \in (0, 1)$ and assume that $t_1, t_2 \in [z_1, \infty)$. Let $\{\delta_n\}$ be a sequence of positive real numbers converging to zero. And let $k_1 \in \{1, 2\}, k_2 \in \{0, 1\}, k_3 \in \{0, 1\}$. If $s \in (z_1, z_2)$ then*

$$\begin{aligned}&\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - k_1\delta_n}{}_{\delta_n} I_{[\tau+k_2\delta_n,\tau+\delta_n+k_2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}}\\ &\qquad \cdot (\lfloor t_2/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}}\\ &\to (t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}},\end{aligned}$$

*as $n$ goes to infinity.*

*If $s \in (z_1/2, z_2)$ then*

$$\begin{aligned}&\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - k_1\delta_n}{}_{\delta_n} I_{[\tau+k_2\delta_n,\tau+\delta_n+k_2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}} \cdot\\ &\qquad (\lfloor t_2/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}}\\ &\to 0,\end{aligned}$$

*Proof.* We first assume that $s \in (z_1, z_2)$ Note that for any $a \in [0, \infty)$ we have that $\lfloor a/\delta_n \rfloor \delta_n$ converges to $a$ since $|\lfloor a/\delta_n \rfloor \delta_n - a| = |\lfloor a/\delta_n \rfloor - a/\delta_n|\delta_n$ and $|\lfloor a/\delta_n \rfloor - a/\delta_n|\delta_n \le \delta_n$. The smallest value $\tau + k_2\delta_n$ has is $\lfloor z_1/\delta_n \rfloor \delta_n + k_2\delta_n$ this values converges to $z_1$, the largest value $\tau + \delta_n + k_2\delta_n$ has is $\lfloor z_2/\delta_n \rfloor \delta_n - k_1\delta_n + \delta_n + k_2\delta_n$, this value converges to $z_2$. So for big enough $n$ there is always $\tau_n$ such that

$$s \in [\tau_n + k_2\delta_n, \tau_n + \delta_n + k_2\delta_n).$$

Since the length of the half-open interval is $\delta$ we have that $\tau_n$ converges to $s$. We also have that

$$(\lfloor t_1/\delta_n \rfloor \delta_n + k_3\delta_n - \tau_n)^{H-\frac{1}{2}}(\lfloor t_2/\delta_n \rfloor \delta_n + k_3\delta_n - \tau_n)^{H-\frac{1}{2}}$$

converges to $(t_1 - s)^{H-\frac{1}{2}}(t_2 - s)^{H-\frac{1}{2}}$ since limits behave well under sums and products and the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^{H-1/2}$ is continuous.

Assume now that $s \in (z_1/2, z_1)$. For this to make sense $z_1 > 0$. We have already established that $\lfloor z_1/\delta_n \rfloor \delta_n$ converges to $z_1$, so for large $n$ we have that $s < \lfloor z_1/\delta_n \rfloor \delta_n$. And then we have

$$\begin{aligned}&\sideset{}{_{\delta_n}}\sum_{\tau=\lfloor z_1/\delta_n \rfloor \delta_n}^{\lfloor z_2/\delta_n \rfloor \delta_n - k_1\delta_n} I_{[\tau+k_2\delta_n, \tau+\delta_n+k_2\delta_n)}(s)(\lfloor t_1/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}} \cdot \\ &\qquad\qquad (\lfloor t_2/\delta_n \rfloor \delta_n + k_3\delta_n - \tau)^{H-\frac{1}{2}} \\ &= 0\end{aligned}$$

□

**Lemma D.3.4.** *Let $N$ be a natural number. Assume that for $j \in \{1, \ldots, N\}$ we have a sequence of real numbers $(a_{i,j})_{i\in\mathbb{N}}$ such that*

$$\sum_{i=1}^{\infty} a_{i,j}$$

*converges. Then*

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{N} a_{i,j}\right)$$

*converges, and*

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{N} a_{i,j}\right) = \sum_{j=1}^{N}\left(\sum_{i=1}^{\infty} a_{i,j}\right)$$

*Proof.* Given $\epsilon > 0$ for each $j \in \{1, \ldots, N\}$ there is an $M_j$ such that if $K \geq M_j$

$$\left|\sum_{i=1}^{K} a_{i,j} - \sum_{i=1}^{\infty} a_{i,j}\right| < \frac{\epsilon}{N}.$$

Define $M \doteq \max(\{M_1, \ldots, M_N\})$. Assume now that $K \geq M$, we get

$$\begin{aligned}\left|\sum_{i=1}^{K}\left(\sum_{j=1}^{N} a_{i,j}\right) - \sum_{j=1}^{N}\left(\sum_{i=1}^{\infty} a_{i,j}\right)\right| &= \left|\sum_{j=1}^{N}\left(\sum_{i=1}^{K} a_{i,j}\right) - \sum_{j=1}^{N}\left(\sum_{i=1}^{\infty} a_{i,j}\right)\right| \\ &\leq \sum_{j=1}^{N}\left|\sum_{i=1}^{K} a_{i,j} - \sum_{i=1}^{\infty} a_{i,j}\right| \\ &< \sum_{j=1}^{N}\frac{\epsilon}{N} \\ &= \epsilon.\end{aligned}$$

□

**Remark.** *Note that in lemma D.3.4 it is crucial that we have convergence for each $j$ in the start of the statement. If we knew only that $\Sigma_{i=1}^{\infty}\Sigma_{j=1}^{N} a_{i,j}$ converged we could not exchange the sums. This is seen by the example $N = 2$, $a_{i,1} = 1, a_{i,2} = -1$.*

**Lemma D.3.5.** *Let $K \in \mathbb{N}, \mu \in \mathbb{R}$ and $\{\delta_n\}$ a sequence of positive real numbers converging to zero. Assume $\epsilon$. Then there exists an $n^*$ such that if $n \geq n^*$ we have*

$$\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\mu^2\delta_n^2\right| < \epsilon.$$

*Proof.*

$$\begin{aligned} &\sum_{\substack{t=N\delta_n, N\in\{0\}\cup\mathbb{N} \\ t\leq K+1}}^{\delta_n} \left|\mu^2\delta_n^2\right| \\ \leq\ & \left(\frac{K+1}{\delta_n}+1\right)\left|\mu^2\delta_n^2\right| \\ =\ & (K+1+\delta_n)\left|\mu^2\delta_n\right|. \end{aligned}$$

We can get this expression as small as we want by decreasing $\delta_n$. □

# List of symbols

$(S, \mathcal{S})$ metric space S, with Borel $\sigma$-algebra $\mathcal{S}$.

$B_H, (B_{t,H})_{t\in[0,\infty)}$ Fractional Brownian motion with parameter H.

$C[0,\infty)$ The space of real-valued continuous functions on $[0,\infty)$.

$C[a,b]$ The space of real-valued continuous functions on $[a,b]$.

$E_n[f]$ the expectation of $f$ using $(S, \mathcal{S}, P_n)$.

$P$ probability measure.

$P_n$ numbered probability measure.

$\doteq$ defined as.

$\mathbb{R}$ the real numbers.

$\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra on the real numbers.

$\mathcal{C}$ The Borel sigma-algebra on $C[0,\infty)$.

$\pi_x$ Projection-mapping, x is a real vector.

$w_x$ modulus of continuity for x on $[0,\infty)$.

$w_{x,k}$ modulus of continuity on the set $[0,k]$.

$C_H$ constant, defined in proposition 3.2.2.

$I_A$ Indicator function that takes the value 1 when the argument is an element of $A$, and 0 if it is an element of $A^c$.

$\sum_{\tau=-\infty}^{T}{}_{\delta}$ summation with step-length $\delta$ up to and including $T$, $T$ must be a multiple of $\delta$.

$\forall$ for all.

$\mathbb{D}_a$ The set $\{a, a+\delta, a+2\delta, \ldots\}$.

$\mathbb{N}$ The natural numbers, $\{1, 2, \ldots\}$.

**fBm** fractional Brownian motion.

# Bibliography


[App09] David Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[BB12] Fred Espen Benth and Jūratė Šaltytė Benth. *Modeling and pricing in financial markets for weather derivatives*. Vol. 17. World Scientific, 2012.

[Ben04] Fred Espen Benth. *Option theory with stochastic analysis: an introduction to mathematical finance*. Springer, 2004.

[Bia+08] Francesca Biagini et al. *Stochastic calculus for fractional Brownian motion and applications*. Springer, 2008.

[Bil68] Patrick Billingsley. *Convergence of probability measures*. Wiley, 1968.

[Bil95] Patrick Billingsley. *Probability and Measure*. 3rd ed. Wiley-Interscience, 1995.

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley, 1999.

[DØ11] Giulia Di Nunno and Bernt Øksendal, eds. *Advanced Mathematical Methods for finance*. Springer, 2011.

[DOT02] Paul Doukhan, George Oppenheim, and Murad Taqqu. *Theory and applications of long-range dependence*. Springer Science & Business Media, 2002.

[Fol84] Gerald B Folland. *Real analysis: modern techniques and their applications*. John Wiley & Sons, 1984.

[GJR14] Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. "Volatility is rough". In: *arXiv preprint arXiv:1410.3394* (2014).

[GP15] Christopher Goodrich and Allan C Peterson. *Discrete fractional calculus*. Springer, 2015.

[Jun15] Hugo D Junghenn. *A course in real analysis*. CRC Press, 2015.

[Kle13] Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.

[KS12] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*. Vol. 113. Springer Science & Business Media, 2012.

[Lin06] Tom Lindstrøm. *Kalkulus*. Universitetsforlaget, 2006.

[Lin07] Tom Lindstrøm. "A random walk approximation to fractional Brownian motion". In: *arXiv preprint arXiv:0708.1905* (2007).

[Mit96] Ron C Mittelhammer. *Mathematical statistics for economics and business*. Springer, 1996.

[MV68] Benoit B Mandelbrot and John W Van Ness. “Fractional Brownian motions, fractional noises and applications”. In: *SIAM review* 10.4 (1968), pp. 422–437.

[MW13] John N McDonald and Neil A Weiss. *A course in real analysis*. 2nd ed. Elsevier, 2013.

[Nou12] Ivan Nourdin. *Selected aspects of fractional Brownian motion*. Springer, 2012.

[Øks03] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer Science & Business Media, 2003.

[Par13] Peter Parczewski. “A fractional Donsker theorem”. In: *Stochastic Analysis and Applications* 32 (2013), pp. 328–347.

[PP13] Seshadev Padhi and Smita Pati. *Theory of Third-Order Differential Equations*. Springer Science & Business Media, 2013.

[Ros10] Sheldon M Ross. *Introduction to probability models*. Academic press, 2010.

[Sag15] Serik Sagitov. *Weak Convergence of Probability Measures*. Chalmers University of Technology and Gothenburg University, Aug. 2015. URL: http://www.math.chalmers.se/~serik/C-space.pdf.

[Sot01a] Tommi Sottinen. “Fractional Brownian motion, random walks and binary market models”. In: *Finance and Stochastics* 5.3 (2001), pp. 343–355.

[Sot01b] Tommy Sottinen. *Fractional Brownian Motion as a Model in Finance*. 2001. URL: http://lipas.uwasa.fi/~tsottine/talks/Vaasa2001.pdf.

[SS03] Elias M. Stein and Rami Shakarchi. *Princeton Lectures in Analysis, No. 2 : Complex Analysis*. Princeton University Press, 2003.

[ST94] Gennady Samorodnitsky and Murad S Taqqu. *Stable non-Gaussian random processes: stochastic models with infinite variance*. Vol. 1. CRC press, 1994.

[Tem15] Nico M Temme. *Asymptotic methods for integrals*. World Scientific, 2015.

[Tem96] Nico M Temme. *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*. John Wiley & Sons, 1996.

[TNY11] Allanus Tsoi, David Nualart, and George Yin, eds. *Stochastic analysis, stochastic systems, and applications to finance*. World Scientific, 2011.

[Ton84] Y. L. Tong, ed. *Inequalities in Statistics and Probability (Institute of Mathematical Statistics, Lecture Notes-Monograph Series, Vol. 5)*. Michigan State Univ, 1984.

[Wil91] David Williams. *Probability with martingales*. Cambridge university press, 1991.

[Zip93] Richard Zippel. *Effective Polynomial Computation*. Kluwer Academic Publishers, 1993.