Analysis of Contract-for-Difference with Vasicek Process and Cointegrated System

in Stock Market and Nordic Electricity Market

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Abstract

This thesis aims at pricing Contract-for-Difference (CFD) in stock market, and Electricity Price Area Differential (EPAD) in the Nord Pool. For CFD, our model consists of Geometric Brownian Motion and Vasicek process. For EPAD, we apply cointegrated system to build a model, with and without Vasicek process. Both parts claim calculable results. And they are given by a series of analysis- and calculation-work. Reader will also encounter some figures of simulation by R.

Acknowledgement

Thanks to Qifen Yuan, my only love. You encouraged me and gave me imperative support.

Thanks to my supervisor, Prof. Fred Espen Benth. I have been very lucky to have a supervisor who offered me patient guidance and meaningful advice.

Oslo, December 2017

Bo Huang

Introduction

This master thesis consists of two parts, modelling a Contract for Difference (CfD) in stock market, and, modelling an Electricity Price Area Differential (EPAD) in the Nordic electricity market.

I suggested the topic of CfD out of my personal trading experience. The stock market is always attractive to speculating and even gambling people, but the threshold is not low. Prices of some hottest stocks are high enough to scare off people who only want to make a try. For example, Amazon costed 1161$, Tesla costed 306$ and Apple costed 171$ on 1.Dec.2017. If a person want to implement a private trading strategy, or construct a portfolio consisting of a dozen of stocks, the start-up capital will not be smaller than a down payment for buying a small apartment.

Fortunately, CfD, which is a contract written on the difference between current and future stock price, offers a shortcut to participate in the stock market. In the recent years, there are more and more online CfD brokers covering the world's major markets, with higher gearing. Their margin requirements are only 5%, even 2%. Two years ago, I started trading on a famous online-CfD-platform with only 150$. Even if there was a little money, I was able to test my strategies and gain experience in the real market. Honestly, most of my strategies are strange and even weird. No one would be willing to test them by a lot of money.

After a short talk with my supervisor, Fred Espen Benth, the CfD's suggestion was confirmed by him. And Fred brought the topic of EPAD. He did not told a reason, I took that as a good point intuitively. EPAD is a forward contract with reference to the difference between the area price and the Nordic Spot system price. Compared with the CfD written on difference between future and current, EPAD written on two simultaneous prices' difference. Because EPAD is a relatively new financial derivative in electricity market (introduced 17.11.2000 at Nord Pool), there are not many available papers on its pricing problem.

We turn our attention to modelling work of these two financial products. Let's see the part of CfD briefly. Firstly, we model the risk-neutral stock price, S_t, by Geometric Brownian Motion, then the CfD's risk-neutral price can be modeled by $S_T - S_t$.

We assume the risk-free interest rate is stochastic and given by the Vasicek model. This gives two challenges. One is to prove a corresponding risk-neutral probability, $\mathcal{Q}$, exists. We used a little tricky method to complete the

proof. The other one is to get a calculable expression of the discounting effect, $\mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s\,ds}\Big|\mathcal{F}_t\right]$.

Then, we got the arbitrage-free price of the CfD with maturity T at time t as $S_t \cdot \left(1 - \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s\,ds}\Big|\mathcal{F}_t\right]\right)$. Finally, we linked the CfD's price to the stock futures' price.

Now let's move on to the EPAD's model. We model the two log-electricity-prices, $logS_1$ and $logS_2$, by a common Drifted Brownian Motion adding Ornstein-Uhlenbeck process respectively. Then the EPAD's price can be modeled as $logS_1 - logS_2$, which can be proved that it is a stationary process. This coincides the idea of cointegrated system.

We assume the Drifted Brownian Motion and the two OU-processes are pairwise related. This gives difficulty to get a calculable expression of the EPAD's no arbitrage price. After we solved it, we linked the EPAD's price to the price of futures on system price. And we figured out a static hedging strategy without EPAD, only futures.

Until now, the risk-free interest rate is assumed to be in the Vasicek model, is independent of the electricity prices. However it could be not. We got another calculable expression of EPAD's arbitrage-free price considering the related interest rate.

Both parts are accompanied by some simulation generated by R-coding.

Contents

Chapter 1

Pricing of CFD in The Stock Market with Vasicek-Model

1.1 CFD's Payoff and Risk-Neutral Probability's Existence

The main purpose of this chapter is to price a flexible option in the stock market, CFD (contract for difference), which is a contract written on the difference between current and future stock price. Before entering the topic, let's recall a few fundamental concepts informally.

For measure the potential probability in real market, we assume there exists a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a standard filtration $\{\mathcal{F}_t\}$, and a Brownian motion B.

In modelling a stock price at time t, S_t, we use the well-known Geometric Brownian Motion, that is

$$dS_t = \alpha S_t dt + \sigma S_t dB_t \tag{1.1}$$
$$where\ B_t\ is\ a\ Brownian\ motion\ w.r.t.\ \mathcal{P}$$
$$\alpha\ \&\ \sigma\ are\ strictly\ positive\ real\ numbers$$

On a stock price, the corresponding CFD's payoff should be defined as

Definition 1. *X_T is payoff of a CFD with maturity T, written on the difference between current and future price of a corresponding stock.*

$$X_T := S_T - S_t \tag{1.2}$$
$$where\ S_T := the\ future\ stock\ price$$
$$S_t := the\ current\ stock\ price$$

To get the CFD's arbitrage-free price, we need a Brownian motion, $\{W_t\}$, with respect to a risk-neutral probability, $\mathcal{Q}$. To capture the hidden probability

better, we let $\{W_t\}$ include a dynamic interest rate given by Vasicek process. We build such a W_t in the following definition. Then, we will introduce the Vasicek interest rate formally, and prove existence of the $\mathcal{Q}$.

Definition 2. *With respect to a risk-neutral probability $\mathcal{Q}$, a Brownian motion is*

$$dW_t = \frac{\alpha - r_t}{\sigma}dt + dB_t \qquad (1.3)$$

where $B_t :=$ Brownian motion w.r.t. $\mathcal{P}$ (the real world's probability)

$\qquad r_t :=$ *interest rate in Vasicek Model*

$\qquad \alpha$ & σ *are strictly positive constants*

Now we come to see the r_t given by Vasicek process.

Lemma 1. *Considering the risk-neutral probability Q , if the Vasicek model's SDE is constructed as (see Rogemar Mamon [1])*

$$dr_s = a(b - r_s)ds + \theta dW_s \ , \ s \in [t, T] \qquad (1.4)$$

where $a > 0$, $b > 0$, $\theta > 0$

then the solution should be

$$r_s = e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)} \, dW_u \qquad (1.5)$$

Proof. We apply the Itô formula (see Bernt Øksendal [2]) to $f(s, r_s) = e^{as}r_s$

$$df(s, r_s) = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial r}dr_s + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(dr_s)^2$$

$$where \ \frac{\partial f}{\partial s} = a \cdot e^{as} \cdot r_s \ , \ \frac{\partial f}{\partial r} = e^{as} \ , \ \frac{\partial^2 f}{\partial r^2} = 0$$

$$\Rightarrow d(e^{as}r_s) = a \cdot e^{as} \cdot r_s \cdot ds + e^{as} \cdot dr_s$$

$$Substitute \ dr_s = a \cdot (b - r_s)ds + \theta \cdot dW_s \ , \ it \ gives$$

$$d(e^{as}r_s) = a \cdot e^{as} \cdot r_s \cdot ds + e^{as} \cdot a \cdot b \cdot ds - a \cdot e^{as} \cdot r_s \cdot ds + \theta \cdot e^{as} \cdot dW_s$$

$$= a \cdot b \cdot e^{as} \cdot ds + \theta \cdot e^{as} \cdot dW_s$$

$$\Rightarrow e^{as}r_s - e^{at}r_t = \int_t^s a \cdot b \cdot e^{au} \, du + \theta \int_t^s e^{au}dW_u$$

$$\Rightarrow e^{as}r_s = e^{at}r_t + \int_t^s a \cdot b \cdot e^{au} \, du + \theta \int_t^s e^{au}dW_u$$

$$\Rightarrow r_s = e^{-as} \cdot \left[e^{at}r_t + \int_t^s a \cdot b \cdot e^{au} \, du + \theta \int_t^s e^{au}dW_u \right]$$

$$\Rightarrow r_s = e^{-a(s-t)}r_t + b \cdot (1 - e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)} \, dW_u$$

$\square$

Lemma 2. *The expectation and variance of the Vasicek model should be*

$$\mathbb{E}[r_s] = e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)}) \tag{1.6}$$

$$Var[r_s] = \theta^2 \cdot \frac{1 - e^{-2a(s-t)}}{2a} \tag{1.7}$$

Proof. We know that r_s, where $s \in (t, T)$, is Gaussian distributed, and we need to calculate its mean and variance w.r.t $\mathcal{Q}$

$$\mathbb{E}[r_s] = \mathbb{E}\left[e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)}\,dB_u\right]$$

$$= \underbrace{\mathbb{E}\left[e^{-a(s-t)}r_t\right]}_{e^{-a(s-t)}r_t} + \underbrace{\mathbb{E}\left[b(1 - e^{-a(s-t)})\right]}_{b(1 - e^{-a(s-t)})} + \underbrace{\mathbb{E}\left[\theta \cdot \int_t^s e^{-a(s-u)}\,dB_u\right]}_{0}$$

Both $e^{-a(s-t)}r_t$ and $b(1 - e^{-a(s-t)})$ are deterministic, so means of them are themselves. In the third expectation, it integrates a Brownian motion with respect to a deterministic. Brownian motion's independent increment makes the integral's mean be zero. It gets

$$\mathbb{E}[r_s] = e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)})$$

$$Var[r_s] = Var\left[e^{-a(s-t)}r_t + b\left(1 - e^{-a(s-t)}\right) + \theta \cdot \int_t^s e^{-a(s-u)}\,dB_u\right]$$

$$= Var\left[e^{-a(s-t)}r_t\right] + Var\left[b\left(1 - e^{-a(s-t)}\right)\right] + Var\left[\theta \cdot \int_t^s e^{-a(s-u)}\,dB_u\right]$$

Variance of deterministic function must be zero, so we ignore the first two terms. For the third term, it can be solved by the Itô-isometry (see Bernta Øksendal [3]) and the mean has been known as zero.

$$Var[r_s] = Var\left[\theta \cdot \int_t^s e^{-a(s-u)}\,dB_u\right]$$

$$= \theta^2 \cdot Var\left[\int_t^s e^{-a(s-u)}\,dB_u\right]$$

$$= \theta^2 \cdot \mathbb{E}\left[(\int_t^s e^{-a(s-u)}\,dB_u)^2\right]$$

$$= \theta^2 \cdot \mathbb{E}\left[\int_t^s e^{-2a(s-u)}\,du\right]$$

$$= \theta^2 \cdot \frac{1 - e^{-2a(s-t)}}{2a}$$

$\square$

Let's look at a group of figures illustrating the risk-neutral Brownian motion, W_s, the Vasicek interest rate, and the combination of them.

(a)

(b)

(c)

(d)
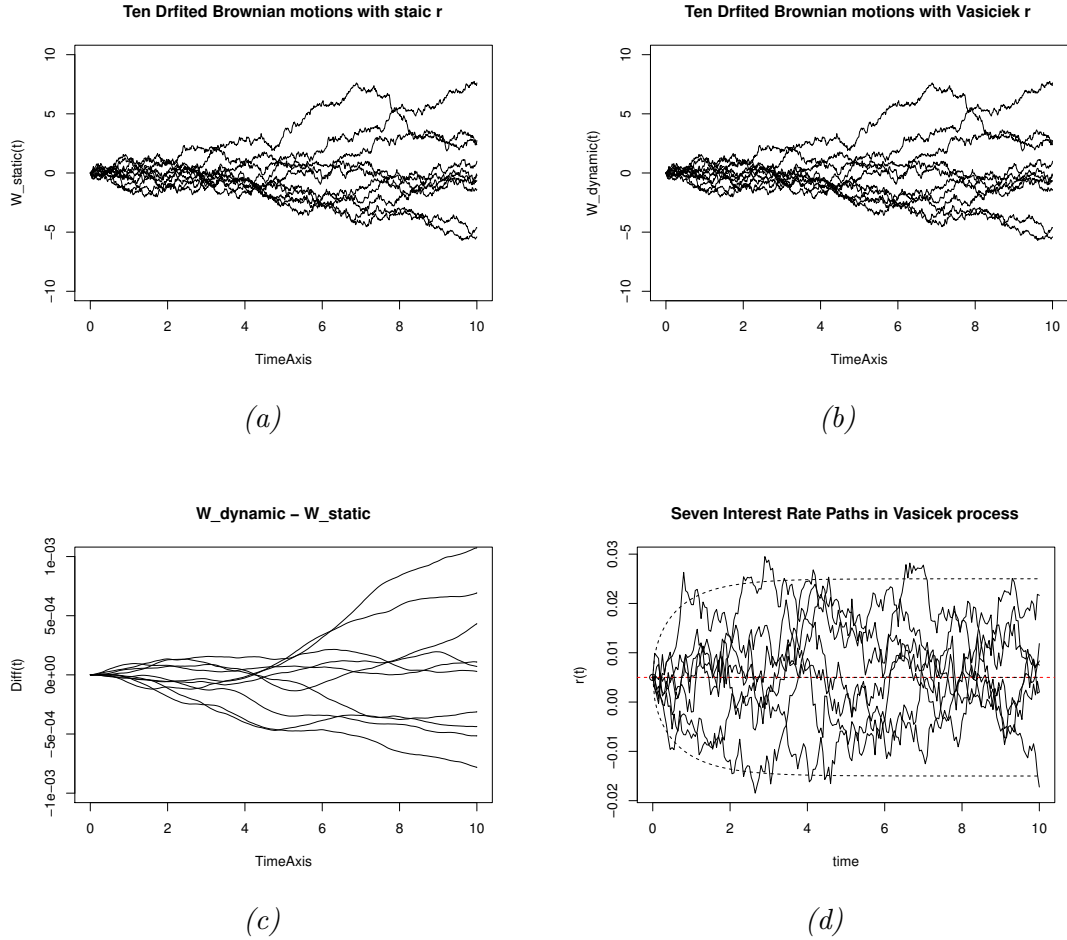
Figure 1.1: (a). Illustrate equation (1.3) with $\alpha = 0.1$, $\sigma = 100$, $r_t = 0.5\%$ for all t, $t = 0.001,\ 0.002,\ \cdots,\ 10$.

(b). Use the same equation and parameters as (a), but r_t illustrates Vasicek process in equation (1.4), with parameters $r_0 = 0.5\%$ (initial interest rate), $b = 0.5\%$ (object of mean-reverting), $a = 0.5$ (control the tendency of reverting), $\theta = 0.01$ (adjust the variation of Brownian motion).

(c). Illustrate the difference between the Brownian motions with and without the dynamic interest rate under Vasicek process, i.e. the lines in (a) minus the lines in figure (b).

(d). Illustrate the interest rate, r_t, under the Vasicek process in equation (1.4), i.e. the r_t was applied in (b).

By comparing the figures (a) and (b), the effect of dynamic interest rate is not visible. But the figure (c) shows the effect does exist. And we can expect the effect will be visible if the volatility is huge.

We come to prove the $\mathcal{Q}$ exists. The Girsanov theorem (See Bernt Øksendal [4]) offers an equation saying the $\mathcal{Q}$ does exist and looks like

$$dQ(D) = M_t \cdot dP(D) \ , \ D \in \Omega \tag{1.8}$$
$$in \ this \ case \ , \ M_t = e^{-\int_0^T \frac{\alpha - r_t}{\sigma} dB_t - \frac{1}{2} \int_0^T (\frac{\alpha - r_t}{\sigma})^2 \, dt}$$

However, the Girsanov theorem holds when M_t is a martingale w.r.t. $\mathcal{P}$. This can be satisfied by the Novikon Sufficient Condition (See the 'Remarks (2)' in Bernt Øksendal [4]) :

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T (\frac{\alpha - r_t}{\sigma})^2 \, dt}\right] < \infty \tag{1.9}$$
$$where \ \mathbb{E} = \mathbb{E}_\mathcal{P} \ is \ the \ expectation \ w.r.t \ to \ \mathcal{P}.$$

So it is sufficient to prove the Novikov Sufficient Condition holds. As a preparation to this tough proof, we look at a lemma giving r_s under the probability $\mathcal{P}$.

Lemma 3. *Under the probability $\mathcal{P}$, the Vasicek interest rate model's SDE and solution should be*

$$dr_s = \hat{a}(\hat{b} - r_s)ds + \theta dB_s \ , \ s \in [t, T] \tag{1.10}$$
$$r_s = e^{-\hat{a}(s-t)}r_t + \hat{b}(1 - e^{-\hat{a}(s-t)}) + \theta \cdot \int_t^s e^{-\hat{a}(s-u)} \, dB_u \tag{1.11}$$
$$where \ , \ \hat{a} = a + \frac{\theta}{\sigma} \ , \ \hat{b} = \frac{ab + \frac{\theta \alpha}{\sigma}}{a + \frac{\theta}{\sigma}}$$

Proof. Plug equation(1.3) in equation(1.4), it says

$$dr_s = a(b - r_s)ds + \theta \cdot \frac{\alpha - r_s}{\sigma}ds + \theta \cdot dB_s$$
$$= a \cdot b \cdot ds - a \cdot r_s \cdot ds + \frac{\theta}{\sigma} \cdot \alpha \cdot ds - \frac{\theta}{\sigma} \cdot r_s \cdot ds + \theta \cdot dB_s$$
$$= \left[\left(a \cdot b + \frac{\theta}{\sigma} \cdot \alpha\right) - \left(a + \frac{\theta}{\sigma}\right) \cdot r_s\right] \cdot ds + \theta \cdot dB_s$$
$$= \left(a + \frac{\theta}{\sigma}\right)\left[\frac{\left(a \cdot b + \frac{\theta}{\sigma} \cdot \alpha\right)}{\left(a + \frac{\theta}{\sigma}\right)} - r_s\right] \cdot ds + \theta \cdot dB_s$$
$$= equation(1.10)$$

Repeat the proof of Lemma 1 with $\hat{a}$, $\hat{b}$ and B instead of a, b and W respectively, it gives equation(1.11). $\qquad \qquad \blacksquare$

Our strategy is to use Corollary 5.13 and Corollary 5.14 in Karatzas and Shreve [5], which says that the Novikov sufficient condition will be satisfied with $0 < T < \infty$ if there exists a sequence $\{t_n\}_{n=0}^{\infty}$ of real numbers with $0 = t_0 < t_1 < t_2 < \cdots < t_n \uparrow \infty$, such that

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{t_{n-1}}^{t_n}(\frac{\alpha-r_s}{\sigma})^2\,ds}\right] < \infty \tag{1.12}$$

where $\mathbb{E} = \mathbb{E}_{\mathcal{P}}$ is the expectation w.r.t to $\mathcal{P}$.

The additional benefit is " $0 < T < \infty$ ". This means our hedging can cover a very long period.

Lemma 4. *We claim that if*

$$0 < t_n - t_{n-1} < \frac{2\hat{a}\sigma^2}{\theta^2} \tag{1.13}$$

then the equation(1.12) will hold.

Proof. We start with a simple calculation as

$$e^{\frac{1}{2}\int_{t_{n-1}}^{t_n}(\frac{\alpha-r_s}{\sigma})^2\,ds} = e^{\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2\,ds}$$

Since exponential functions are convex, and $\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2$ is positive Riemann-integrable, we use the Jensen's inequality (see Niculescu & Persson [6]) and get

$$e^{\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2\,dt} \leq \frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}e^{\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2}\,ds$$

The above inequality holds when it takes expectations on both sides

$$\mathbb{E}\left[e^{\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2\,ds}\right] \leq \mathbb{E}\left[\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}e^{\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2}\,ds\right]$$

We pay attention to the above inequality's right side. Since expectation is a type of integration, we would use Tonelli-Fubini's Theorem (see McDonald & Weiss [7]) to exchange the integral and expectation

$$\mathbb{E}\left[\frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}e^{\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2}\,ds\right] = \frac{1}{t_n-t_{n-1}}\int_{t_{n-1}}^{t_n}\mathbb{E}\left[e^{\frac{t_n-t_{n-1}}{2}(\frac{\alpha-r_s}{\sigma})^2}\right]\,ds$$

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It is sufficient to find a value of $t_n - t_{n-1}$ making $\mathbb{E}\left[e^{\frac{t_n - t_{n-1}}{2}(\frac{\alpha - r_s}{\sigma})^2}\right]$ be Riemann integrable.

Applying Lemma 2 to time interval $[t_{n-1} \ , \ t_n]$, gives the corresponding mean and variance as $e^{-\hat{a}(s-t_{n-1})}r_{t_{n-1}} + \hat{b}(1 - e^{-\hat{a}(s-t_{n-1})})$ and $\theta^2 \cdot \frac{1 - e^{-2\hat{a}(s-t_{n-1})}}{2\hat{a}}$. In the form of Gaussian distribution, r_s can be written as

$$r_s \sim \mathcal{N}\left[e^{-\hat{a}(s-t_{n-1})}r_{t_{n-1}} + \hat{b}(1 - e^{-\hat{a}(s-t_{n-1})}) \ , \ \theta^2 \cdot \frac{1 - e^{-2\hat{a}(s-t_{n-1})}}{2\hat{a}}\right]$$

$$\Rightarrow \frac{\alpha - r_s}{\sigma} \sim \mathcal{N}\left[\frac{\alpha - e^{-\hat{a}(s-t_{n-1})}r_{t_{n-1}} - \hat{b}(1 - e^{-\hat{a}(s-t_{n-1})})}{\sigma} \ , \ \frac{\theta^2}{\sigma^2}\frac{1 - e^{-2\hat{a}(s-t_{n-1})}}{2\hat{a}}\right]$$

$$Set \ \mu := \frac{\alpha - e^{-\hat{a}(s-t_{n-1})}r_{t_{n-1}} - \hat{b}(1 - e^{-\hat{a}(s-t_{n-1})})}{\sigma}$$

$$\nu^2 := \frac{\theta^2}{\sigma^2}\frac{1 - e^{-2\hat{a}(s-t_{n-1})}}{2\hat{a}}$$

$$then \ , \ \frac{\alpha - r_s}{\sigma} = \mu + \nu Z \ , \ Z \sim \mathcal{N}[0 \ , \ 1]$$

$$\Rightarrow \mathbb{E}\left[e^{\frac{t_n - t_{n-1}}{2}(\frac{\alpha - r_s}{\sigma})^2}\right] = \mathbb{E}\left[e^{\frac{t_n - t_{n-1}}{2}(\mu + \nu Z)^2}\right]$$

$$= \int_{\mathbb{R}} e^{\frac{t_n - t_{n-1}}{2}(\mu + \nu Z)^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}} \, dZ$$

$$= e^{\frac{t_n - t_{n-1}}{2}\cdot \mu^2} \cdot \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}(1 - (t_n - t_{n-1})\nu^2) + \mu\nu(t_n - t_{n-1})Z} \, dZ$$

It is sufficient to find a value of $t_n - t_{n-1}$ making the integral be convergent. By observation, it is sufficient to have

$$1 - (t_n - t_{n-1})\nu^2 > 0$$

$$\Rightarrow t_n - t_{n-1} < \frac{1}{\nu^2}$$

$$\Rightarrow t_n - t_{n-1} < (\frac{\sigma}{\theta})^2 \frac{2\hat{a}}{1 - e^{-2\hat{a}(s-t_{n-1})}}$$

We observe that when t_{n-1} is fixed and t_n goes to huge, $e^{-2\hat{a}(s-t_{n-1})}$ will converge to 0 as $s \to t_n$, and $\frac{1}{1 - e^{-2\hat{a}(s-t_{n-1})}}$ will converge to 1. It says that $t_n - t_{n-1} < \frac{2\hat{a}\sigma^2}{\theta^2}$ guarantees the Norvikov Sufficient condition.

The discussion under the link[1] gave me the idea to do this proof.

❑

[1]https://math.stackexchange.com/questions/133691/can-i-apply-the-girsanov-theorem-to-an-ornstein-uhlenbeck-process

The above lemma is a sufficient proof for Novikov sufficient condition holds. So we propose it without proof.

Proposition 1. *The Novikov sufficient condition holds for all finite time interval, i.e.*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T (\frac{\alpha - r_t}{\sigma})^2 \, dt}\right] < \infty \; ; \; 0 < T < \infty \tag{1.14}$$

where $\mathbb{E} = \mathbb{E}_{\mathcal{P}}$ is the expectation w.r.t to $\mathcal{P}$.

1.2 Pricing the CFD and Zero-coupon

Now that the risk-neutral probability $\mathcal{Q}$ does exist, it should be used to price the CFD. For simplicity, we assume the market is complete, and ignore all service-charge. After discounting, the CFD's expected payoff should be equal to its arbitrage-free price.

Definition 3. *Arbitrage-free price of the CFD at time t with maturity T, is denoted by P_t^T*

$$P_t^T = \mathbb{E}_{\mathcal{Q}}\left[\left(S_T - S_t \right) \cdot e^{-\int_t^T r_s \, ds} \middle| \mathcal{F}_t \right] \tag{1.15}$$

$S_T - S_t$ is the payoff defined in Definition 1, r_s is the interest rate given by Vasicek model, its mathematical introduction is given by Lemma 1. $\{\mathcal{F}_t\}$ is a filtration on the risk-neutral probability space $(\Omega, \mathcal{F}, \mathcal{Q})$. $\mathcal{F}_t$ includes all information up to time t.

To calculate the P_t^T, it is necessary to model the stock price under the risk-neutral probability $\mathcal{Q}$. We choose the Geometric Brownian Motion(see Benth [8]) to model stock price.

Lemma 5. *The stock price under $\mathcal{Q}$ is modeled as the following SDE with its solution*

$$dS_t = r_t S_t dt + \sigma S_t dW_t \tag{1.16}$$

$$S_t = S_0 \cdot e^{\int_0^t r_s \, ds - \frac{\sigma^2}{2} t + \sigma W_t} \tag{1.17}$$

where σ is a strictly positive constant.

Proof. Before we showing the bridge from the SDE to its solution, we must explain how the SDE, equation (1.16), was constructed. Substitute equation (1.3) in equation (1.1), we will see

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t (dW_t - \frac{\alpha - r_t}{\sigma} dt) \\ &= \alpha S_t dt + \sigma S_t dW_t - \alpha S_t dt + r_t S_t dt \\ &= r_t S_t dt + \sigma S_t dW_t \end{aligned}$$

We verify the solution by the Itô-formula with $S_t = f(t, W_t)$:

$$df(t, W_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial w} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial w^2} (dW_t)^2$$

$$where, \quad \frac{\partial f}{\partial t} = S_t \left(r_t - \frac{\sigma^2}{2} \right) , \quad \frac{\partial f}{\partial w} = \sigma S_t , \quad \frac{\partial^2 f}{\partial w^2} = \sigma^2 S_t , \quad (dW_t)^2 = dt$$

$$\Rightarrow dS_t = S_t r_t dt - \frac{\sigma^2}{2} S_t dt + \sigma S_t dW_t + \frac{\sigma^2}{2} S_t dt$$

$$\Rightarrow dS_t = r_t S_t dt + \sigma S_t dW_t$$

❑

We embed a dynamic interest rate instead of a constant parameter in the Geometric Brownian motion, this is closer to the real market. The following 4 figures simulate how the dynamic interest rate improve stock price.

(a)

(b)

(c)

(d)
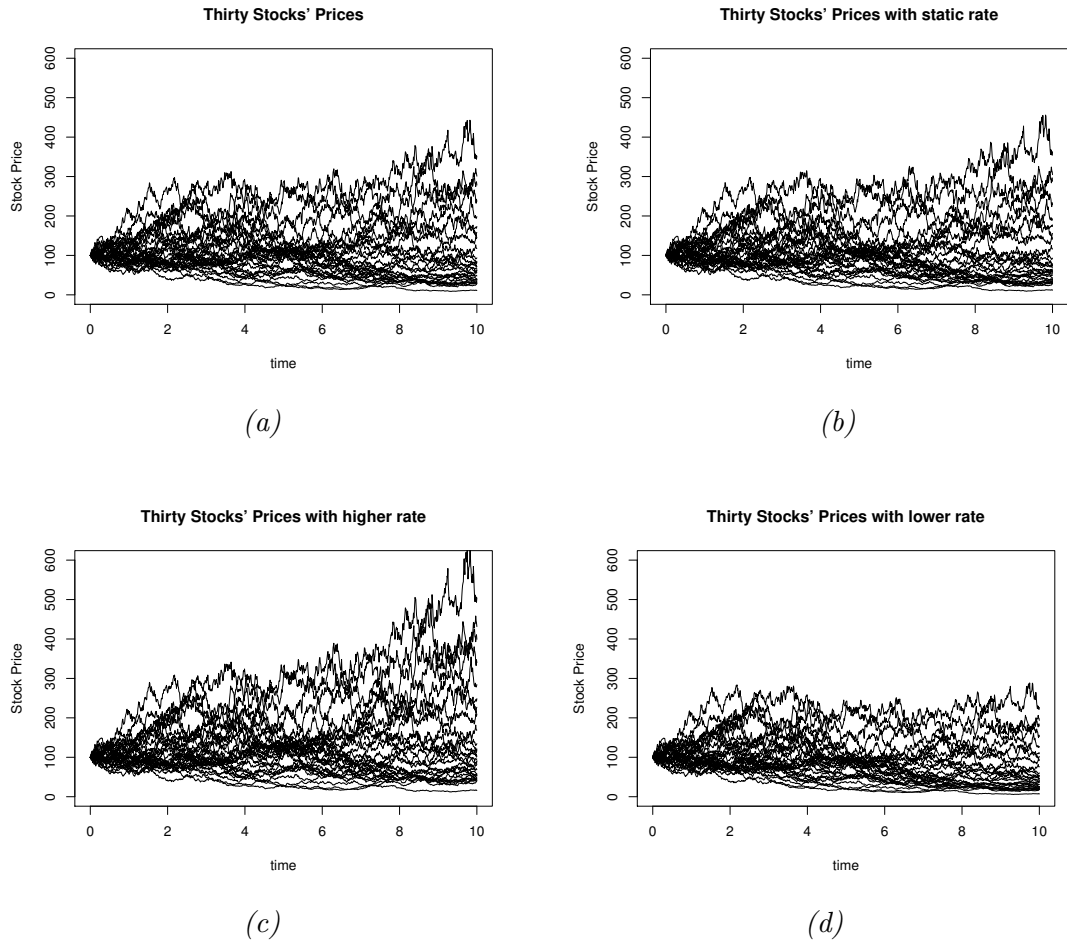
Figure 1.2: (a). Simulate the equation (1.16) with $S_0 = 100$ and $\sigma = 0.3$, where r_s is same as the setting in Figure 1.1-(b)
(b). Same as (a), except for a constant $r_s = 0.5\%$
(c). Same as (a), but the mean-reverting of Vasicek's interest rate, $b = 5\%$
(d). Same as (a), but the mean-reverting of Vasicek's interest rate, $b = -5\%$

Notice that $b = 0.5\%$ in (a). It is reasonable that (a) and (b) look quite similar, because r_s is only around 0.5%. (c) shows a better market quotation with a path of interest rate converging to 5%. (d) shows a peaceful stock market when the interest rate goes to -5%.

Proposition 2. *The arbitrage-free price of the CFD with maturity T at time t,* P_t^T, *can be rewritten as*

$$P_t^T = S_t \cdot \left(1 - \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]\right) \tag{1.18}$$

Proof. By Lemma 5, $S_T = S_t \cdot e^{\int_t^T r_s \, ds - \frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)}$. Substitute this in equation(1.15), it gets

$$
\begin{aligned}
P_t^T &= \mathbb{E}_{\mathcal{Q}}\left[\left(S_t \cdot e^{\int_t^T r_s \, ds - \frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)} - S_t\right) \cdot e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathcal{Q}}\left[S_t \cdot e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)} - S_t \cdot e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathcal{Q}}\left[S_t \cdot e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)} \Big| \mathcal{F}_t\right] - \mathbb{E}_{\mathcal{Q}}\left[S_t \cdot e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right] \\
&= S_t \cdot e^{-\frac{\sigma^2}{2}(T-t)} \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\sigma(W_T - W_t)} \Big| \mathcal{F}_t\right] - S_t \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]
\end{aligned}
$$

where,
$$
\begin{aligned}
\mathbb{E}_{\mathcal{Q}}\left[e^{\sigma(W_T - W_t)} \Big| \mathcal{F}_t\right] &= \mathbb{E}_{\mathcal{Q}}\left[e^{\sigma(W_T - W_t)}\right] \ , \ W_T - W_t \sim \mathcal{N}(0, T-t) \\
&= e^{0 + \frac{1}{2}\sigma^2(T-t)} \\
&= e^{\frac{\sigma^2}{2}(T-t)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow P_t^T &= S_t \cdot e^{-\frac{\sigma^2}{2}(T-t)} \cdot e^{\frac{\sigma^2}{2}(T-t)} - S_t \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right] \\
&= S_t \cdot \left(1 - \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]\right)
\end{aligned}
$$

❑

To know the discounting effect, we have to find a solvable expression for

$$\mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]$$

which is a price of the zero-coupon with maturity T at time t, with respect to probability $\mathcal{Q}$. For simplicity, we rename it as $Z(t,T)$:

$$Z(t,T) := \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]$$

We solve it in the last proposition of this section. Before that, we will build two lemmas which support the last proposition's proof.

Lemma 6. *We use the equation (1.5) of the Vasicek interest rate to construct a new equation*

$$\tau_s = r_s - b$$
$$= e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)}\, dW_u - b$$
$$= e^{-a(s-t)}r_t - be^{-a(s-t)} + \theta \cdot \int_t^s e^{-a(s-u)}\, dW_u$$
$$\Rightarrow \tau_s = e^{-as} \cdot (e^{at}\tau_t + \theta \int_t^s e^{au}dW_u) \qquad (1.19)$$
$$where, \ \tau_t = r_t - b$$

We claim this is the solution of the Ornstein-Uhlenbeck (see BARNDORFF-NIELSEN[9]) equation

$$d\tau_s = -a \cdot \tau_s ds + \theta dW_s$$
$$where, \ a > 0 \ and \ \theta > 0$$

Proof. We verify the solution by the Itô-formula with $f(s, \tau_s) = e^{as}\tau_s$

$$df(s, \tau_s) = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial \tau}d\tau_s + \frac{1}{2}\frac{\partial^2 f}{\partial \tau^2}(d\tau_s)^2$$
$$= ae^{as}\tau_s ds - e^{as}(-a\tau_s d_s + \theta dW_s) + 0$$
$$\Rightarrow df(s, \tau_s) = \theta e^{as} dW_s$$
$$\Rightarrow e^{as}\tau_s - e^{at}\tau_t = \theta \int_t^s e^{au}\, dW_u$$
$$\Rightarrow e^{as}\tau_s = e^{at}\tau_t + \theta \int_t^s e^{au}\, dW_u$$
$$\Rightarrow \tau_s = e^{-as} \cdot (e^{at}\tau_t + \theta \int_t^s e^{au}\, dW_u), \ same \ as \ (1.19)$$

$\square$

Lemma 7. *Assume $\{\tau_t\}$ is Ornstein-Uhlenbeck process, t is the present moment, u and s are two future time points. Then, on the risk-neutral probability space, τ_s and τ_u are Gaussian distributed with means and covariance as*

$$\mathbb{E}_{\mathcal{Q}}[\tau_s] = e^{-a(s-t)} \cdot \tau_t \qquad (1.20)$$
$$\mathbb{E}_{\mathcal{Q}}[\tau_u] = e^{-a(u-t)} \cdot \tau_t$$
$$Cov_{\mathcal{Q}}[\tau_s, \tau_u] = \frac{\theta^2}{2a}e^{-a(s+u)} \cdot (e^{2a(s\wedge u)} - e^{2at}) \qquad (1.21)$$

Proof.

$$\mathbb{E}_Q[\tau_s] = \mathbb{E}[e^{-a(s-t)}\tau_t + \theta \int_t^s e^{-a(s-u)}\,dW_u]$$

$$= e^{-a(s-t)} \cdot \tau_t$$

For $\mathbb{E}_Q[\tau_u]$, *the calculation is same.*

$$Cov_Q[\tau_s, \tau_u]$$

$$= \mathbb{E}_Q[(e^{-a(s-t)}\tau_t) + \theta \int_t^s e^{-a(s-v)}\,dW_v - e^{-a(s-t)}\tau_t) \cdot$$

$$(e^{-a(u-t)}\tau_t) + \theta \int_t^u e^{-a(u-v)}\,dW_v - e^{-a(u-t)}\tau_t)]$$

$$= \mathbb{E}_Q[\theta^2 \int_t^s e^{-as}e^{av}\,dW_v \cdot \int_t^u e^{-au}e^{av}\,dW_v]$$

$$= \theta^2 e^{-a(s+u)}\mathbb{E}_Q[(\int_t^{s\wedge u} e^{av}\,dW_v)^2]$$

$$= \theta^2 e^{-a(s+u)}\mathbb{E}_Q[\int_t^{s\wedge u} e^{2av}\,dv]$$

$$= \frac{\theta^2}{2a}e^{-a(s+u)} \cdot [e^{2a(s\wedge u)} - e^{2at}]$$

$\square$

Proposition 3. *Price of the zero-coupon with maturity T at time t, is*

$$Z(t,T) = exp\{-A_t^T \cdot r_t + B_t^T\} \tag{1.22}$$

$$where, \ A_t^T = \frac{1 - e^{-a(T-t)}}{a}$$

$$B_t^T = \left(A_t^T - (T-t)\right) \cdot \left(b - \frac{\theta^2}{4a^2}\right) - \frac{\theta^2}{8a} \cdot A_t^{T\,2}$$

Proof. The current interest rate r_t is known, and it includes all information up to the present moment, i.e. $\mathbb{E}_Q\left[e^{-\int_t^T r_s\,ds}\middle|\mathcal{F}_t\right] = \mathbb{E}_Q\left[e^{-\int_t^T r_s\,ds}\middle|r_t\right]$, and r_t is an initial value to calculate the r_s. So we see r_s as a function of r_t, i.e. $r_s = r_s(r_t)$. This leads to $\mathbb{E}_Q\left[e^{-\int_t^T r_s\,ds}\middle|r_t\right] = \mathbb{E}_Q\left[e^{-\int_t^T r_s(r_t)\,ds}\right]$. This intuitive inference can be verified mathematically:

$$\mathbb{E}_Q\left[e^{-\int_t^T r_s\,ds}\middle|\mathcal{F}_t\right] = \mathbb{E}_Q\left[e^{-\int_t^T [e^{-a(s-t)}r_t + b(1-e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)}\,dW_u]\,ds}\middle|\mathcal{F}_t\right]$$

$$= e^{-\int_t^T e^{-a(s-t)}r_t + b(1-e^{-a(s-t)})\,ds} \cdot \mathbb{E}_Q\left[e^{-\int_t^T [\theta \cdot \int_t^s e^{-a(s-u)}\,dW_u]\,ds}\middle|\mathcal{F}_t\right]$$

obviously , $e^{-a(s-u)}$ *is independent of* $\mathcal{F}_t$

$$= e^{-\int_t^T e^{-a(s-t)}r_t + b(1-e^{-a(s-t)})\,ds} \cdot \mathbb{E}_Q\left[e^{-\int_t^T [\theta \cdot \int_t^s e^{-a(s-u)}\,dW_u]\,ds}\right]$$

$$= \mathbb{E}_Q\left[e^{-\int_t^T r_s(r_t)\,ds}\right]$$

$$Z(t,T) := \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s \, ds} \Big| \mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s(r_t) \, ds}\right]$$

$$= e^{\mathbb{E}_{\mathcal{Q}}\left[-\int_t^T r_s(r_t) \, ds\right] + \frac{1}{2} Var_{\mathcal{Q}}\left[-\int_t^T r_s(r_t) \, ds\right]}$$

Firstly, we find the value of $\mathbb{E}_{\mathcal{Q}}[-\int_t^T r_s(r_t) \, ds] = -\mathbb{E}_{\mathcal{Q}}[\int_t^T r_s(r_t) \, ds]$

$$\mathbb{E}_{\mathcal{Q}}[\int_t^T r_s(r_t) \, ds] = \int_t^T \mathbb{E}_{\mathcal{Q}}[r_s(r_t)] \, ds$$

$$= \int_t^T \mathbb{E}_{\mathcal{Q}}[\tau_s(\tau_t) + b] \, ds$$

$$= \int_t^T \mathbb{E}_{\mathcal{Q}}[\tau_s(\tau_t)] \, ds + \int_t^T b \, ds$$

$$= \int_t^T e^{-a(s-t)} \tau_t \, ds + b(T-t), \text{ by equation (1.20)}$$

$$= \tau_t \cdot \int_t^T e^{-a(s-t)} \, ds + b(T-t)$$

$$= \tau_t \cdot (-\frac{1}{a})[e^{-a(T-t)} - e^{-a(t-t)}] + b(T-t)$$

$$\Rightarrow \mathbb{E}_{\mathcal{Q}}[-\int_t^T r_s(r_t) \, ds] = -\tau_t \cdot (-\frac{1}{a})[e^{-a(T-t)} - e^{-a(t-t)}] - b(T-t)$$

$$= (r_t - b) \cdot \frac{e^{-a(T-t)} - 1}{a} - b(T-t)$$

Next, we are on mission to find the value of $Var_{\mathcal{Q}}[-\int_t^T r_s(r_t) \, ds] = Var_{\mathcal{Q}}[\int_t^T r_s(r_t) \, ds]$. Because the b in the $r_s = \tau_s + b$ is just a constant, it gives

$$Var_{\mathcal{Q}}\left[\int_t^T r_s(r_t) \, ds\right] = Var_{\mathcal{Q}}\left[\int_t^T \tau_s(\tau_t) \, ds\right]$$

The trick is that variance is seen as a special case of covariance, i.e.

$$Var_{\mathcal{Q}}\left[\int_t^T \tau_s(\tau_t) \, ds\right] = Cov_{\mathcal{Q}}\left[\int_t^T \tau_s(\tau_t) \, ds, \int_t^T \tau_u(\tau_t) \, du\right]$$

Then we only need to calculate the covariance under risk-neutral probability.

$$Cov_Q\left[\int_t^T \tau_s(\tau_t)\,ds, \int_t^T \tau_u(\tau_t)\,du\right]$$

$$=\mathbb{E}_Q\left[\int_t^T \tau_s(\tau_t)\,ds - \mathbb{E}_Q\left[\int_t^T \tau_s(\tau_t)\,ds\right]\right]\cdot\mathbb{E}_Q\left[\int_t^T \tau_u(\tau_t)\,du - \mathbb{E}_Q\left[\int_t^T \tau_u(\tau_t)\,du\right]\right]$$

$$=\mathbb{E}_Q\left[\int_t^T \left(\tau_s - \mathbb{E}_Q[\tau_s]\right)ds \cdot \int_t^T \left(\tau_u - \mathbb{E}_Q[\tau_u]\right)du\right]$$

$$=\int_t^T \int_t^T \mathbb{E}_Q\left[\left(\tau_s - \mathbb{E}_Q[\tau_s]\right)\cdot\left(\tau_u - \mathbb{E}_Q[\tau_u]\right)\right]du\,ds$$

$$=\int_t^T \int_t^T Cov_Q[\tau_s, \tau_u]\,du\,ds$$

$$=\int_t^T \int_t^T \frac{\theta^2}{2a}\cdot e^{-2a(s+u)}\cdot\left[e^{2a(s\wedge u)} - e^{2at}\right]du\,ds$$

It is not hard to see that the result does not depend on $u\wedge s = s$ or $u\wedge s = u$. For convenience, we choose $u\wedge s = u$.

$$Var_Q\left[\int_t^T r_s(r_t)ds\right] = \frac{\theta^2}{2a}\int_t^T \int_t^s e^{-a(s+u)}\cdot\left(e^{2au} - e^{2at}\right)du\,ds$$

$$= \frac{\theta^2}{2a}\int_t^T \int_t^s e^{-as}e^{au} - e^{-as}e^{-au}e^{2at}\,du\,ds$$

$$= \frac{\theta^2}{2a}\left[\int_t^T \int_t^s e^{-as}e^{au}\,du\,ds - \int_t^T \int_t^s e^{-as}e^{-au}e^{2at}\,du\,ds\right]$$

$$= \frac{\theta^2}{2a}\left[\frac{a(T-t+e^{-a(T-t)}-1)}{a^2} - \frac{-1}{a^2}\cdot\left(\frac{1-e^{-2a(T-t)}}{2} + e^{-a(T-t)} - 1\right)\right]$$

$$= \frac{\theta^2}{4a^3}\cdot\left[2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}\right]$$

Substituting the expectation and variance, it gets

$$Z(t,T) = e^{-A_t^T r_t + B_t^T}\quad,\quad A_t^T = \frac{1-e^{-a(T-t)}}{a}$$

$$B_t^T = b\left(A_t^T - (T-t)\right) + \frac{\theta^2}{8a^3}\left(2a(T-t) - 3 + 4(1-aA_t^T) - (1-aA_t^T)^2\right)$$

$$= b\left(A_t^T - (T-t)\right) + \frac{\theta^2}{8a^3}\left(2a(T-t) - 3 + 4 - 4aA_t^T - 1 + 2aA_t^{T\,2} - a^2 A_t^{T\,2}\right)$$

$$= b\left(A_t^T - (T-t)\right) + \frac{\theta^2}{4a^2}(T-t) - \frac{\theta^2}{4a^2}A_t^T - \frac{\theta^2}{8a}A_t^{T\,2}$$

$$= \left(A_t^T - (T-t)\right)\cdot\left(b - \frac{\theta^2}{4a^2}\right) - \frac{\theta^2}{8a}A_t^{T\,2}$$

This proof is inspired by the second chapter of Rogemar S. Mamon [1]

❑

We draw a figure for giving a straight feeling above zero-coupon's price. We can expect that the expected average yield of a zero-coupon will converge to the corresponding expected interest rate, i.e. the parameter b. By comparison, the initial values' effect is ignorable in the long term.

Seven ZeroCoupons' Yield Curves

Figure 1.3: The dash line shows where the expected interest rate stays. The given original interest rates are -0.03, 0, 0.03, $\cdots$, 0.15, which are starting points of the seven solid lines. Clearly, a zero-coupon's long-term average yield always converge to the expected interest rate.

1.3 Link the futures price to the CFD

We have known how to calculate $Z(t,T)$, then we would like to investigate the futures price in the stock market, $F(t,T)$, with maturity T at time t. Because of the arbitrage-free assumption, the discounted futures price is expected to be same as the discounted stock price at time T :

$$\mathbb{E}_{\mathcal{Q}}[F(t,T) \cdot e^{-\int_t^T r_s\, ds}|\mathcal{F}_t] = \mathbb{E}_{\mathcal{Q}}[S_T \cdot e^{-\int_t^T r_s\, ds}|\mathcal{F}_t] \tag{1.23}$$

Proposition 4. *The futures price, $F(t,T)$ can be decided by the current stock price, S_t, and the zero-coupon price, $Z(t,T)$*

$$F(t,T) = S_t \cdot Z(t,T)^{-1} \tag{1.24}$$

Proof. Substitute equation (1.17) into equation (1.23)

$$\mathbb{E}_{\mathcal{Q}}\left[S_t \cdot e^{\int_t^T r_s\, ds - \frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)} \cdot e^{-\int_t^T r_s\, ds}\Big|\mathcal{F}_t\right] = \mathbb{E}_{\mathcal{Q}}\left[F(t,T) \cdot e^{-\int_t^T r_s\, ds}\Big|\mathcal{F}_t\right]$$

$$\Rightarrow \mathbb{E}_{\mathcal{Q}}\left[S_t \cdot e^{-\frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t)}\Big|\mathcal{F}_t\right] = F(t,T) \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_s\, ds}\Big|\mathcal{F}_t\right]$$

$$\Rightarrow S_t \cdot e^{-\frac{\sigma^2}{2}(T-t)} \cdot e^{\frac{\sigma^2}{2}(T-t)} = F(t,T) \cdot Z(t,T)$$

$$\Rightarrow S_t = F(t,T) \cdot Z(t,T)$$

$$\Rightarrow F(t,T) = S_t \cdot Z(t,T)^{-1}$$

❏

We are also interested in the connection between the CFD's price and the futures price

Proposition 5. *The CFD's price, P_t^T, can be decided by the futures price $F(t,T)$ and the zero-coupon price, $Z(t,T)$*

$$P_t^T = F(t,T) \cdot Z(t,T) \cdot \big(1 - Z(t,T)\big) \tag{1.25}$$

Proof. Recall the equation (1.18) and the definition of $Z(t,T)$, and use the equation (1.24), then

$$P_t^T = S_t \cdot \left(1 - \mathbb{E}_{\mathcal{Q}}\big[e^{-\int_t^T r_s\, ds}\big|\mathcal{F}_t\big]\right)$$

$$= S_t \cdot \big[1 - Z(t,T)\big]$$

$$= F(t,T) \cdot Z(t,T) \cdot \big[1 - Z(t,T)\big]$$

❏

We end this chapter with some simulation figures in the next page.

Let's do some simulation of the CFD's pricing model and figure out its features. Firstly, since $Z(t,T)$ always starts from 1, i.e. $Z(t,t) = 1$, it says $1 - Z(t,T)$ starts from 0. Then, the price-path of CFD will goes from 0, i.e. $P_t^t = S_t \cdot (1 - Z(t,t)) = 0$. This is not reasonable. Another point is that CFD's price, P_t^T, is impacted deeply by $1 - Z(t,T)$ which has an apparent increasing. This makes the CFD does not replicate the stock well.
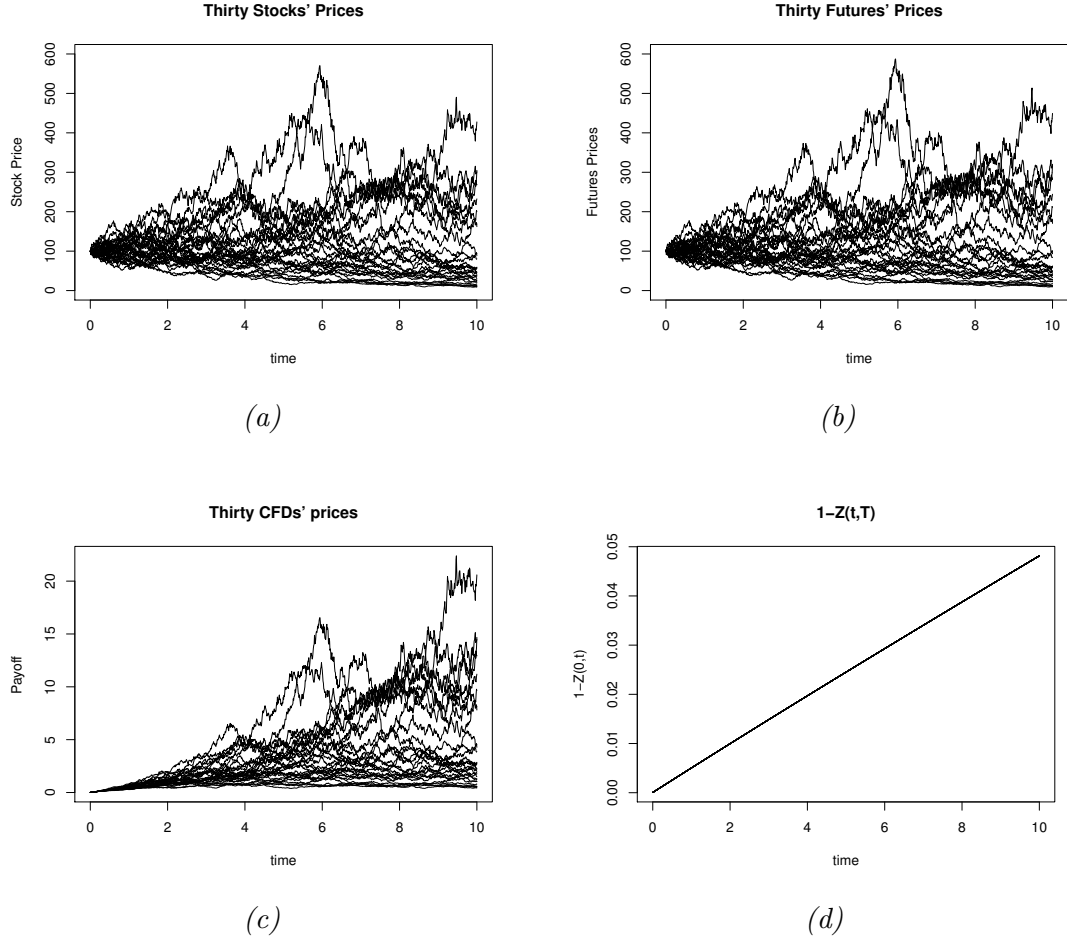
Figure 1.4: (a). Simulate 30 stock prices by the equation (1.16) with $S_0 = 100$ and $\sigma = 0.3$, where r_s is same as the setting in Figure 1.1-(b)

(b). Simulate the 30 futures price by the equation (1.24). For the zero-coupon effect, use the same parameters as in Figure 1.3, but $r_0 = 0.5\%$

(c).Simulate the 30 CFD-prices by the equation (1.25), with the same parameters as plot (a) and plot (b)

(d).Shows the path of $1 - Z(t,T)$ with the same parameters of Zero-Coupon in (b)

Paths of stocks and futures are similar to each other, well replicating. CFD's prices are different from them, because of the increasing effect of $1 - Z(t,T)$ being showed in (d). Maybe the relation between (c) and (d) is not obvious enough. It will be more persuasive with logarithmic coordinates. Please check figures in the following page.
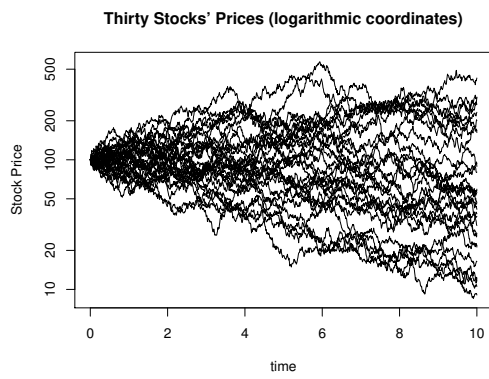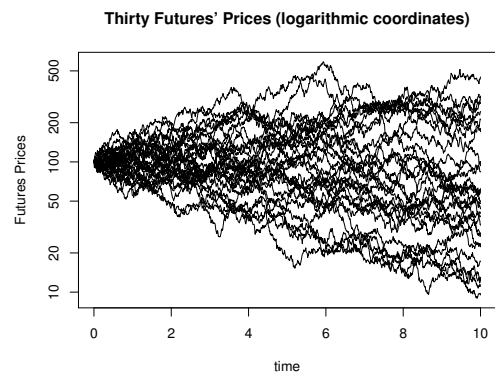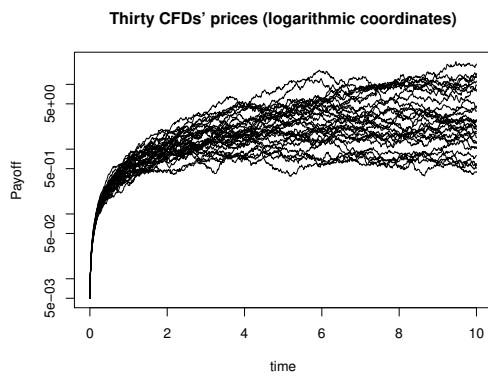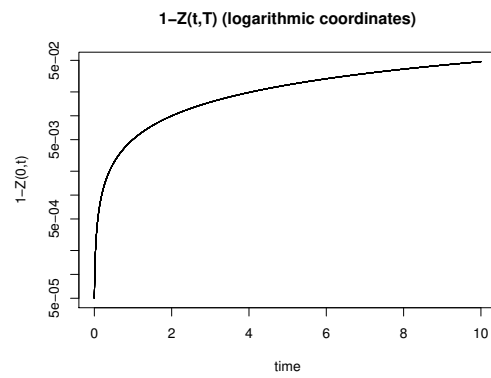
Figure 1.5: *Same simulation as Figure 1.4, but with logarithmic coordinates. The (d)'s impact on (c) is obvious now.*

Chapter 2

Pricing of EPAD in the electricity market

In this chapter, we will focus on pricing of the Electricity Price Area Differentials (EPAD). We quote from the nasdaqomx.com, EPAD is "a forward contract with reference to the difference between the area price and the Nord Pool Nordic Spot system price". We plan to model an area price and a system price with non-stationary process respectively, and subtraction between them will build a stationary process modeling the EPAD's price.

Like the CFD in Chapter 1, the modeling result should be combined with discounting effect driven by Vasicek model. In this chapter, the discounting effect is assumed to be independent of the electricity prices. However, they are relevant in the real market. we will discuss the same pricing with relevant discounting effect in the next chapter.

2.1   EPAD's payoff

Generally speaking, we consider difference between two synchronous prices S_1 and S_2. And payoff of the option written on the difference which is defined by the subtraction, i.e. $S_1 - S_2$.

Definition 4. *Electricity prices of an area and Nord pool spot system at time t are noted as $S_1(t)$ and $S_2(t)$ respectively, the corresponding EPAD's payoff is*

$$S_1(t) - S_2(t) \tag{2.1}$$

2.2   Cointegrated system

For mathematical convenience, we do not model the prices directly, but model the log-prices. An usual component of modelling electricity price is Ornstein-Uhlenbeck process who has some good properties. We take this nice process adding an Drifted Brownian Motion to build log-price model.

Definition 5. *An area electricity log-price, $logS_1(s)$, and system electricity log-price, $logS_2(s)$, are defined as*

$$\begin{cases} logS_1(s) = X(s) + Y_1(s) \\ logS_2(s) = X(s) + Y_2(s) \end{cases} \quad s \in [t, T] \tag{2.2}$$

where, $X(s)$ is a Drifted Brownian motion, both $Y_1(s)$ and $Y_2(s)$ are Ornstein-Uhlenbeck processes.

Remark 1. *The SDE's for $X(s)$, $Y_1(s)$ and $Y_2(s)$ are listed below*

$$\begin{cases} dX(s) = \mu ds + \eta dV(s), \mu \text{ and } \eta \text{ are constants} \\ dY_1(s) = -\mu_1 Y_1(s)ds + \eta_1 dV_1(s), \ \mu_1 > 0 \ \eta_1 > 0 \\ dY_2(s) = -\mu_2 Y_2(s)ds + \eta_2 dV_2(s), \ \mu_2 > 0 \ \eta_2 > 0 \end{cases} \tag{2.3}$$

And the corresponding solutions should be

$$\begin{cases} X(s) = X(t) + \mu \cdot (s - t) + \eta \cdot (V(s) - V(t)) \\ Y_1(s) = e^{-\mu_1 s}(e^{\mu_1 t}Y_1(t) + \eta_1 \int_t^s e^{\mu_1 k} dV_1(k)) \\ Y_2(s) = e^{-\mu_2 s}(e^{\mu_2 t}Y_2(t) + \eta_2 \int_t^s e^{\mu_2 k} dV_2(k)) \end{cases} \tag{2.4}$$

$X(s)$ is easy to get, while $Y_1(s)$ and $Y_2(s)$ can be gotten by Lemma 3 (let $\hat{b} = 0$). And $V(s)$, $V_1(s)$ and $V_2(s)$ are correlated Brownian Motions w.r.t. respective risk-neutral probabilities, with the same building method like

$$dV(s) = \frac{\alpha - \mu}{\eta}ds + dB(s), \ \alpha \text{ is a constant}$$

As a Brownian motion, variance of $X(s)$ goes to infinity as s goes to infinity. It results in neither $logS_1(s)$ nor $logS_2(s)$ is stationary. Fortunately, what we are interested in, is $logS_1(s) - logS_2(s) = Y_1(s) - Y_2(s)$. We hope this is a stationary process, i.e. $Var[Y_1(s) - Y_2(s)] < \infty$ when s goes to infinity. It is equal to say that equation(2.2) is a cointegrated system.

Intuitively, we believe $Y_1(s) - Y_2(s)$ is stationary, because Ornstein-Uhlenbeck process has stationarity. The rest of this section will present a formal proposition and mathematical proof.

Lemma 8. *Let covariance between $Y_1(s)$ and $Y_2(s)$ be denoted by ρ_{12}, which is assumed to be known. Then the variance of $Y_1(s) - Y_2(s)$ is*

$$Var\Big[Y_1(s) - Y_2(s)\Big] = \eta_1^2 \frac{1 - e^{-2\mu_1(s-t)}}{2\mu_1} + \eta_2^2 \frac{1 - e^{-2\mu_2(s-t)}}{2\mu_2}$$
$$- 2\eta_1 \eta_2 \rho_{12} \cdot (1 - e^{-(s-t)(\mu_1 + \mu_2)}) \tag{2.5}$$

Proof. By equation (2.4), it says

$$Y_1(s) = e^{-\mu_1(s-t)}Y_1(t) + \eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)$$

$$Y_2(s) = e^{-\mu_2(s-t)}Y_2(t) + \eta_2 \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)$$

where, $e^{-\mu_1(s-t)}Y_1(t)$ and $e^{-\mu_2(s-t)}Y_2(t)$ are constants, $V_1(k)$ and $V_2(k)$ are related.

$$
\begin{aligned}
Var\Big[Y_1(s) - Y_2(s)\Big] =& Var\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)} - \eta_2 \int_t^s e^{-\mu_2(s-k)}\Big] \\
=& Var\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\Big] + Var\Big[\eta_2 \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)\Big] \\
& - 2Cov\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\ ,\ \eta_2 \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)\Big]
\end{aligned}
$$

We calculate the two variance and the covariance with Itô-isometry

$$
\begin{aligned}
Var\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\Big] =& \eta_1^2 \cdot \mathbb{E}\Big[\Big(\int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\Big)^2\Big] \\
=& \eta_1^2 \cdot \mathbb{E}\Big[\int_t^s e^{-2\mu_1(s-k)}\, dk\Big] \\
=& \eta_1^2 \cdot e^{-2\mu_1 s} \int_t^s e^{2\mu_1 k}\, dk \\
=& \eta_1^2 \cdot e^{-2\mu_1 s} \cdot \frac{e^{2\mu_1 s} - e^{2\mu_1 t}}{2\mu_1} \\
=& \eta_1^2 \cdot \frac{1 - e^{-2\mu_1(s-t)}}{2\mu_1}
\end{aligned}
$$

Similarly ,

$$Var\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\Big] = \eta_2^2 \cdot \frac{1 - e^{-2\mu_2(s-t)}}{2\mu_2}$$

$$
\begin{aligned}
& 2 \cdot Cov\Big[\eta_1 \int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\ ,\ \eta_2 \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)\Big] \\
=& 2 \cdot \eta_1 \cdot \eta_2 \cdot Cov\Big[\int_t^s e^{-\mu_1(s-k)}\, dV_1(k)\ ,\ \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)\Big] \\
=& 2 \cdot \eta_1 \cdot \eta_2 \cdot \mathbb{E}\Big[\int_t^s e^{-\mu_1(s-k)}\, dV_1(k) \cdot \int_t^s e^{-\mu_2(s-k)}\, dV_2(k)\Big]
\end{aligned}
$$

Now we introduce an new Brownian motion, $V_{ind}(k)$, which is independent of $V_1(k)$ and $V_2(k)$. Then $V_2(k)$ can be represented by $V_1(k)$ and $V_{ind}(k)$ as

$$V_2(k) = \rho_{12}V_1(k) + \sqrt{1 - \rho_{12}}V_{ind}(k)$$

This equation can be verified as

$$Cov\Big[V_1(k), V_2(k)\Big] = Cov\Big[V_1(k)\ ,\ \rho_{12}V_1(k) + \sqrt{1-\rho_{12}}\cdot V_{ind}(k)\Big]$$

$$= Cov\Big[V_1(k)\ ,\ \rho_{12}V_1(k)\Big] + Cov\Big[V_1(k),\ \sqrt{1-\rho_{12}}\cdot V_{ind}(k)\Big]$$

$$= \rho_{12}\cdot Var\Big[V_1(k)\Big] + 0$$

$$= \rho_{12}\cdot k$$

Then the correlation between $V_1(k)$ and $V_2(k)$ is

$$Corr\Big[V_1(k), V_2(k)\Big] = \rho_{12}$$

We continue the above covariance's equation with the new form of $V_2(k)$

$$\mathbb{E}\Big[\int_t^s e^{-\mu_1(s-k)}\,dV_1(k)\cdot\int_t^s e^{-\mu_2(s-k)}\,d\Big(\rho_{12}V_1(k) + \sqrt{1-\rho_{12}}V_{ind}(k)\Big)\Big]$$

$$=\mathbb{E}\Big[\int_t^s e^{-\mu_1(s-k)}\,dV_1(k)\cdot\int_t^s e^{-\mu_2(s-k)}\,\rho_{12}dV_1(k)\Big]+$$

$$\mathbb{E}\Big[\int_t^s e^{-\mu_1(s-k)}\,dV_1(k)\cdot\int_t^s e^{-\mu_2(s-k)}\,\sqrt{1-\rho_{12}}dV_{ind}(k)\Big]$$

$$=\int_t^s e^{-\mu_1(s-k)}e^{-\mu_2(s-k)}\rho_{12}\,dk + 0$$

$$=\rho_{12}\cdot e^{-s(\mu_1+\mu_2)}\cdot\int_t^s e^{k(\mu_1+\mu_2)}\rho_{12}\,dk$$

$$=\frac{\rho_{12}}{\mu_1+\mu_2}\cdot e^{-s(\mu_1+\mu_2)}\cdot\big(e^{s(\mu_1+\mu_2)} - e^{t(\mu_1+\mu_2)}\big)$$

$$=\frac{\rho_{12}}{\mu_1+\mu_2}\big(1 - e^{-(s-t)(\mu_1+\mu_2)}\big)$$

Then we get the result of covariance as

$$Cov\Big[\eta_1\int_t^s e^{-\mu_1(s-k)}\,dV_1(k), \eta_2\int_t^s e^{-\mu_2(s-k)}\,dV_2(k)\Big]$$

$$=\eta_1\eta_2\frac{\rho_{12}}{\mu_1+\mu_2}\big(1 - e^{-(s-t)(\mu_1+\mu_2)}\big)$$

Combine the above calculations, the variance is

$$Var\Big[Y_1(s) - Y_2(s)\Big] = \eta_1^2\frac{1 - e^{-2\mu_1(s-t)}}{2\mu_1} + \eta_2^2\frac{1 - e^{-2\mu_2(s-t)}}{2\mu_2}$$

$$- 2\eta_1\eta_2\frac{\rho_{12}}{\mu_1+\mu_2}\big(1 - e^{-(s-t)(\mu_1+\mu_2)}\big)$$

This is same as what we claimed.

❑

Proposition 6. $Y_1(s) - Y_2(s)$ *is a stationary process.*

Proof. It is sufficient to show the right side of equation in Lemma 8 are bounded when $s \to \infty$.

Since $e^{-(s-t)} \to 0$ as $s \to \infty$, it says that when s converges to ∞ we get

$$\eta_1^2 \frac{1 - e^{-2\mu_1(s-t)}}{2\mu_1} \to \eta_1^2 \frac{1 - 0}{2\mu_1} = \frac{\eta_1^2}{2\mu_1}$$

$$\eta_2^2 \frac{1 - e^{-2\mu_2(s-t)}}{2\mu_2} \to \eta_2^2 \frac{1 - 0}{2\mu_2} = \frac{\eta_2^2}{2\mu_2}$$

$$2\eta_1\eta_2 \frac{\rho_{12}}{\mu_1 + \mu_2}(1 - e^{-(s-t)(\mu_1+\mu_2)}) \to 2\eta_1\eta_2 \frac{\rho_{12}}{\mu_1 + \mu_2}(1 - 0) = \frac{2\eta_1\eta_2\rho_{12}}{\mu_1 + \mu_2}$$

It is followed by

$$Var[Y_1(t) - Y_2(t)] \to \frac{\eta_1^2}{2\mu_1} + \frac{\eta_2^2}{2\mu_2} - \frac{2\eta_1\eta_2\rho_{12}}{\mu_1 + \mu_2}$$

Since each component of the above right side is a parameter assumed known, the variance is a constant. This completes the proof.

❏

2.3 Pricing of the EPAD

Thanks to the stationarity of difference between area log-price and system log-price, we have a base to discuss EPAD's no-arbitrage price in this section. We keep the 'complete market assumption' and 'no-service cost' in the electricity market, the discounted expected payoff is equal to the no-arbitrage price. In addition, discounting effect is still given by the Vasicek process.

Definition 6. *The EPAD's no-arbitrage price with maturity T at present time t, $\mathfrak{E}(t,T)$, is defined as*

$$\mathfrak{E}(t,T) = \mathbb{E}_{\mathcal{Q}}\left[\left(S_1(T) - S_2(T)\right) \cdot e^{-\int_t^T r_s\,ds} \big| \mathcal{F}_t\right] \tag{2.6}$$

where S_1 and S_2 coincide Definition 4 and Definition 5 in the previous section, r_s is the Vasicek interest rate (see Lemma 1). $\{\mathcal{F}_t\}$ is a filtration on the risk-neutral probability space. $\mathcal{F}_t$ includes all information up to time t.

Remark 2. *According to equation (2.2), S_1 and S_2 are in natural exponential form as*

$$\begin{cases} S_1(T) = e^{X(T)+Y_1(T)} = e^{X(T)}e^{Y_1(T)} \\ S_2(T) = e^{X(T)+Y_2(T)} = e^{X(T)}e^{Y_2(T)} \end{cases} \tag{2.7}$$

Because $X(T)$, $Y_1(T)$ and $Y_2(T)$ are initialized to the present values as $X(t)$, $Y_1(t)$ and $Y_2(t)$ respectively, $\{\mathcal{F}_t\}$ is not needed anymore. Plug in equations (2.4) and do some calculation, EPAD's price would be rewritten as a more calculable form.

Lemma 9.

$$\begin{aligned} \mathfrak{E}(t,T) =& \mathcal{C}_1 \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)}\,dV_1(k)} \cdot e^{-\int_t^T r_k\,dk}\right] \\ &- \mathcal{C}_2 \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-k)}\,dV_2(k)} \cdot e^{-\int_t^T r_k\,dk}\right] \\ where,\ \mathcal{C}_1 :=& e^{X(t)+\mu(T-t)+e^{-\mu_1(T-t)}Y_1(t)} \\ \mathcal{C}_2 :=& e^{X(t)+\mu(T-t)+e^{-\mu_2(T-t)}Y_2(t)} \end{aligned} \tag{2.8}$$

Proof. We go through the calculation bridging equation (2.6) to equation (2.8).

$$\begin{aligned} \mathfrak{E}(t,T) &= \mathbb{E}_{\mathcal{Q}}\left[\left(S_1(T) - S_2(T)\right) \cdot e^{-\int_t^T r_s\,ds} \big| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot e^{-\int_t^T r_s\,ds} \big| \mathcal{F}_t\right] - \mathbb{E}_{\mathcal{Q}}\left[S_2(T) \cdot e^{-\int_t^T r_s\,ds} \big| \mathcal{F}_t\right] \\ &= E_1 - E_2 \end{aligned}$$

$$E_1 = \mathbb{E}_{\mathcal{Q}}\left[e^{X(T)+Y_1(T)} \cdot e^{-\int_t^T r_s\, ds}\Big|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[e^{X(t)+\mu(T-t)+\eta(V(T)-V(t))}e^{Y_1(t)e^{-\mu_1(T-t)}+\eta_1\int_t^T e^{-\mu_1(T-k)}\, dV_2(k)} \cdot e^{-\int_t^T r_s\, ds}\Big|\mathcal{F}_t\right]$$

by the argument in proof of Proposition 3, $\mathcal{F}_t$ can be ignored.

$$= e^{X(t)+\mu(T-t)-Y_1(t)\mu_1(T-t)}\mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1\int_t^T e^{-\mu_1(T-k)}\, dV_1(k)} \cdot e^{-\int_t^T r_k\, dk}\right]$$

$$= \mathcal{C}_1 \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1\int_t^T e^{-\mu_1(T-k)}\, dV_1(k)} \cdot e^{-\int_t^T r_k\, dk}\right] \quad in\ equation(2.8)$$

For E_2, the calculation is almost same.

❑

To calculate the two $\mathbb{E}_{\mathcal{Q}}\left[\cdots\right]$'s in Lemma 9, we need three independent Brownian Motions and a correlation-matrix instead of $V(\cdot), V_1(\cdot)$ and $V_2(\cdot)$.

Lemma 10. *It assumes that pairwise correlations among $V(\cdot), V_1(\cdot)$ and $V_2(\cdot)$ are known as*

$$\rho_1 = Corr[V(\cdot), V_1(\cdot)], \quad \rho_2 = Corr[V(\cdot), V_2(\cdot)], \quad \rho_{12} = Corr[V_1(\cdot), V_2(\cdot)]$$

And the vectorised $[V(\cdot), V_1(\cdot), V_2(\cdot)]$ can be built as

$$\begin{bmatrix} V(t) \\ V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_4 & a_5 & a_6 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{U(t)} \\ \mathrm{U_1(t)} \\ \mathrm{U_2(t)} \end{bmatrix}$$

where $\mathrm{U}(\cdot)$, $\mathrm{U}_1(\cdot)$ and $\mathrm{U}_2(\cdot)$ are three independent $\mathcal{Q}$-Brownian Motions, while $a_1, a_2, \cdots, a_6$ are constants. Then,

$$a_1^2 = 1$$
$$a_2 = \rho_1$$
$$a_3^2 = 1 - \rho_1^2$$
$$a_4 = \rho_2$$
$$a_5^2 = \frac{(\rho_{12} - \rho_1\rho_2)^2}{1 - \rho_1^2}$$
$$a_6^2 = 1 - \rho_2^2 - \frac{(\rho_{12} - \rho_1\rho_2)^2}{1 - \rho_1^2}$$

Proof. (1) For a_1^2,

$$V(t) = a_1\mathrm{U(t)} \Rightarrow Var[V(t)] = Var[a_1\mathrm{U(t)}]$$
$$\Rightarrow t = a_1^2 t$$
$$\Rightarrow a_1^2 = 1$$

(2) For a_2

$$\rho_1 t = \mathbb{E}\left[(V(t) - \mathbb{E}[V(t)])(V_1(t) - \mathbb{E}[V_1(t)])\right]$$
$$= \mathbb{E}[V(t) \cdot V_1(t)]$$
$$= \mathbb{E}[a_1 U(t) \cdot (a_2 U(t) + a_3 U_1(t))]$$
$$= a_1 a_2 \mathbb{E}[U(t)^2] + a_1 a_3 \mathbb{E}[U(t)U_1(t)]$$
$$= a_1 \cdot a_2 \cdot t$$
$$\Rightarrow a_2 = \rho_1 \ or \ -\rho_1$$

Because ρ_1 can be positive or negative, we take

$$a_2 = \rho_1$$

(3) For a_3^2

$$V_1(t) = a_2 U(t) + a_3 U_1(t) \Rightarrow Var[V_1(t)] = Var[a_2 U(t) + a_3 U_1(t)]$$
$$\Rightarrow t = a_2^2 t + a_3^2 t$$
$$\Rightarrow 1 = a_2^2 + a_3^2$$
$$\Rightarrow a_3^2 = 1 - \rho_1^2$$

(4) For a_4

$$\rho_2 t = \mathbb{E}\left[(V(t) - \mathbb{E}[V(t)]) \cdot (V_2(t) - \mathbb{E}[V_2(t)])\right]$$
$$= \mathbb{E}\left[V(t) \cdot V_2(t)\right]$$
$$= \mathbb{E}\left[U(t)(a_4 U(t) + a_5 U_1(t) + a_6 U_2(t))\right]$$
$$= a_4 \mathbb{E}[U(t)^2] + a_5 \mathbb{E}[U(t) \cdot U_1(t)] + a_6 \mathbb{E}[U(t) \cdot U_2(t)]$$
$$= a_4 \cdot t + a_5 \cdot 0 + a_6 \cdot 0$$
$$= a_4 \cdot t$$
$$\Rightarrow a_4 = \rho_2$$

(5) For a_5

$$\rho_{12} t = \mathbb{E}\left[(V_1(t) - \mathbb{E}[V_1(t)]) \cdot (V_2(t) - \mathbb{E}[V_2(t)])\right]$$
$$= \mathbb{E}\left[V_1(t) \cdot V_2(t)\right]$$
$$= \mathbb{E}\left[(a_2 U(t) + a_3 U_1(t))(a_4 U(t) + a_5 U_1(t) + a_6 U_2(t))\right]$$
$$= \mathbb{E}[a_2 a_4 U(t)^2 + a_2 a_5 U(t)U_1(t) + a_2 a_6 U(t)U_2(t) + a_3 a_4 U(t)U_1(t)$$
$$+ a_3 a_5 U_1(t)^2 + a_3 a_6 U_1(t)U_2(t)]$$
$$= a_2 a_4 t + a_3 a_5 t$$
$$\Rightarrow \rho_{12} = a_2 a_4 + a_3 a_5$$
$$= \rho_1 \cdot \rho_2 + \left(\pm\sqrt{1 - \rho_1^2}\right) a_5$$
$$\Rightarrow a_5^2 = \frac{(\rho_{12} - \rho_1 \rho_2)^2}{1 - \rho_1^2}$$

(6) For a_6

$$Var[V_2(t)] = Var[a_4U(t) + a_5U_1(t) + a_6U_2(t)]$$
$$t = a_4^2 t + a_5^2 t + a_6^2 t$$
$$1 = a_4^2 + a_5^2 + a_6^2$$
$$a_6^2 = 1 - \rho_2^2 - \frac{(\rho_{12}^2 - \rho_1 \rho_2)^2}{1 - \rho_1^2}$$

❑

We end this section with the EPAD's no-arbitrage price consisting of known parameters, and a group figures of simulation-work.

Proposition 7. *The no-arbitrage price of the EPAD's can be calculated by*

$$\mathfrak{E}(t,T) = Z(t,T) \cdot \left(\mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)} - \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)} \right) \tag{2.9}$$

where, $Z(t,T)$ is the zero-coupon's price, $\mathcal{C}_1$ and $\mathcal{C}_2$ can be found in the Lemma 9. The other marks are

$$\omega_1^2 = \eta^2(T-t) + 2a_2\eta\eta_1 \frac{1 - e^{-\mu_1(T-t)}}{\mu_1} + a_2^2\eta_1^2 \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1}$$

$$\omega_2^2 = a_3^2\eta_1^2 \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1}$$

$$\omega_3^2 = \eta^2(T-t) + 2a_4\eta\eta_2 \frac{1 - e^{-\mu_2(T-t)}}{\mu_2} + a_4^2\eta_2^2 \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$\omega_4^2 = a_5^2\eta_2^2 \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$\omega_5^2 = a_6^2\eta_2^2 \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

Proof. Thanks to the Lemma 6. and Lemma 7., we need only to do the calculation of the two '$\mathbb{E}[\cdot]$'s in equation(2.8). The first one is

$$\mathbb{E}_{\mathcal{Q}} \left[e^{\eta(V(T) - V(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} dV_1(k)} \cdot e^{-\int_t^T r_k \, dk} \right]$$

$$= \mathbb{E}_{\mathcal{Q}} \left[e^{\eta(U(T) - U(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} d(a_2 U(k) + a_3 U_1(k))} \cdot e^{-\int_t^T r_k \, dk} \right]$$

$$= \mathbb{E}_{\mathcal{Q}} \left[e^{\int_t^T \eta \, dU(k)} e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} a_2 dU(k)} e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} a_3 dU_1(k)} e^{-\int_t^T r_k \, dk} \right]$$

$$= \mathbb{E}_{\mathcal{Q}} \left[e^{\int_t^T \eta + a_2\eta_1 e^{-\mu_1(T-k)} dU(k)} \right] \cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\eta_1 \int_t^T a_3 e^{-\mu_1(T-k)} dU_1(k)} \right] \cdot \mathbb{E}_{\mathcal{Q}} \left[e^{-\int_t^T r_k \, dk} \right]$$

Since $\int_t^T \eta + a_2\eta_1 e^{-\mu_1(T-k)}\, d\mathrm{U}(k)$ is normal distributed with mean as 0, and the variance is

$$
\begin{aligned}
\omega_1^2 =& \mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^T \eta + a_2\eta_1 e^{-\mu_1(T-k)}\, d\mathrm{U}(k)\right)^2\right] \\
=& \mathbb{E}_{\mathcal{Q}}\left[\int_t^T \left(\eta + a_2\eta_1 e^{-\mu_1(T-k)}\right)^2 dk\right] \\
=& \int_t^T \eta^2\, dk + 2a_2\eta\eta_1 \int_t^T e^{-\mu_1(T-k)}\, dk + a_2^2\eta_1^2 \int_t^T e^{-2\mu_1(T-k)}\, dk \\
=& \eta^2(T-t) + 2a_2\eta\eta_1 \frac{1 - e^{-\mu_1(T-t)}}{\mu_1} + a_2^2\eta_1^2 \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1}
\end{aligned}
$$

By the mean-formula of log-normal, it says

$$
\mathbb{E}_{\mathcal{Q}}\left[e^{\int_t^T \eta + a_2\eta_1 e^{-\mu_1(T-k)}\, d\mathrm{U}(k)}\right] = e^{\frac{1}{2}\omega_1^2}
$$

With the similar theorem, we calculate the variance of $\eta_1 \int_t^T a_3 e^{-\mu_1(T-k)}\, d\mathrm{U}_1(k)$,

$$
\begin{aligned}
\omega_2^2 =& \mathbb{E}_{\mathcal{Q}}\left[\left(\eta_1 \int_t^T a_3 e^{-\mu_1(T-k)}\, d\mathrm{U}_1(k)\right)^2\right] \\
=& \mathbb{E}_{\mathcal{Q}}\left[\left(\eta_1^2 a_3^2 \int_t^T e^{-2\mu_1(T-k)}\, dk\right)^2\right] \\
=& \eta_1^2 a_3^2 \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1} \\
\Rightarrow& \mathbb{E}_{\mathcal{Q}}\left[e^{\eta_1 \int_t^T a_3 e^{-\mu_1(T-k)}\, d\mathrm{U}_1(k)}\right] = e^{\frac{1}{2}\omega_2^2}
\end{aligned}
$$

and $\mathbb{E}_{\mathcal{Q}}\left[e^{-\int_t^T r_k\, dk}\right]$ is the zero-coupon's price $Z(t,T)$. So,

$$
\mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)}\, d V_1(k)} \cdot e^{-\int_t^T r_k\, dk}\right] = e^{\frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2} Z(t,T)
$$

We come to the second one

$$
\begin{aligned}
& \mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-k)}\, d V_2(k)} \cdot e^{-\int_t^T r_k\, dk}\right] \\
=& \mathbb{E}_{\mathcal{Q}}\left[e^{\eta(\mathrm{U}(T)-\mathrm{U}(t))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-k)}\, d(a_4\mathrm{U}(k)+a_5\mathrm{U}_1(k)+a_6\mathrm{U}_2(k))} \cdot e^{-\int_t^T r_k\, dk}\right] \\
=& \mathbb{E}_{\mathcal{Q}}\left[e^{\int_t^T \eta + a_4\eta_2 e^{-\mu_2(T-k)}\, d\mathrm{U}(k)}\right] \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\eta_2 a_5 \int_t^T e^{-\mu_2(T-k)}\, d\mathrm{U}_1(k)}\right] \\
& \cdot \mathbb{E}_{\mathcal{Q}}\left[e^{\eta_2 a_6 \int_t^T e^{-\mu_2(T-k)}\, d\mathrm{U}_2(k)}\right] \cdot Z(t,T)
\end{aligned}
$$

Similar to the above calculation, the key point is to get variances and then mean

of log-normal

$$
\begin{aligned}
\omega_3^2 &= \mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^T \eta + a_4\eta_2 e^{-\mu_2(T-k)}\,d\mathrm{U}(\mathrm{k})\right)^2\right] \\
&= \eta^2(T-t) + 2a_4\eta\eta_2\frac{1 - e^{-\mu_2(T-t)}}{\mu_2} + a_4^2\eta_2^2\frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2} \\
\omega_4^2 &= \mathbb{E}_{\mathcal{Q}}\left[\left(\eta_2 a_5\int_t^T e^{-\mu_2(T-k)}\,d\mathrm{U}_1(\mathrm{k})\right)^2\right] \\
&= \eta_2^2 a_5^2\frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2} \\
\omega_5^2 &= \mathbb{E}_{\mathcal{Q}}\left[\left(\eta_2 a_6\int_t^T e^{-\mu_2(T-k)}\,d\mathrm{U}_2(\mathrm{k})\right)^2\right] \\
&= \eta_2^2 a_6^2\frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}
\end{aligned}
$$

Then we get the second expectation as

$$
\mathbb{E}_{\mathcal{Q}}\left[e^{\eta(V(T)-V(t))}\cdot e^{\eta_2\int_t^T e^{-\mu_2(T-k)}\,dV_2(k)}\cdot e^{-\int_t^T r_k\,dk}\right] = e^{\frac{1}{2}\omega_3^2 + \frac{1}{2}\omega_4^2 + \frac{1}{2}\omega_5^2}\cdot Z(t,T)
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ❑

Let us look a group of figures of simulation which shows the impacting of correlation between the drifted Brownian motion, $X(\cdot)$, and the Ornstein-Uhlenbeck processes, $Y_1(\cdot)$ and $Y_2(\cdot)$, on the EPAD's price.

(a) $\rho_1 = 0.05$, $\rho_2 = 0.95$

(b) $\rho_1 = 0.95$, $\rho_2 = 0.05$

(c) $\rho_1 = 0.45$, $\rho_2 = 0.55$
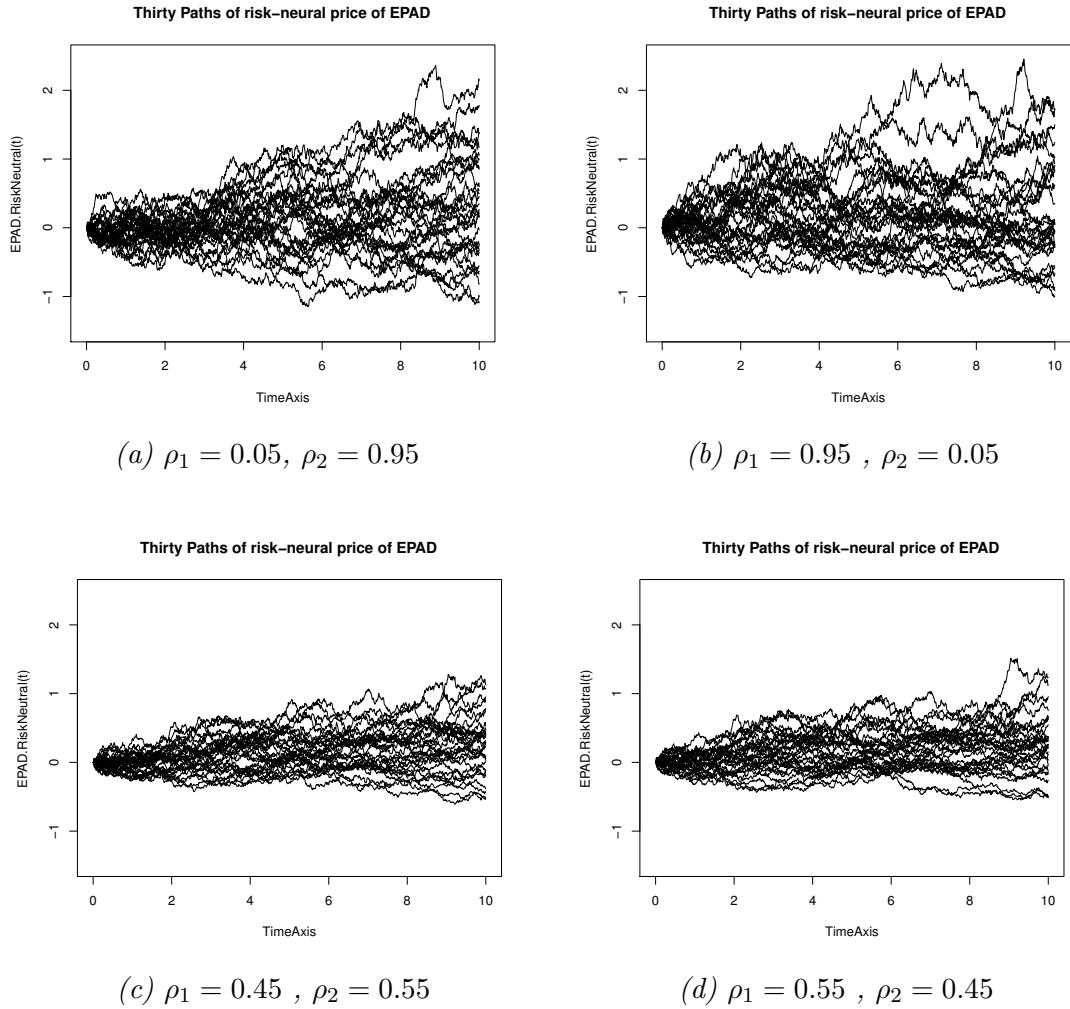

(d) $\rho_1 = 0.55$, $\rho_2 = 0.45$

Figure 2.1: It is easy to observe that the EPAD's variance is positive related to the difference between ρ_1 and ρ_2.

2.4 Link the EPAD to the futures written on the system spot price

People may have to hedge the EPAD with another financial derivative on the electricity market. We recommend the futures written on the system price. The futures price is supposed to replicate the system price. This guides us to price the futures by the pricing of the system electricity price, S_2.

Lemma 11. *The price of the futures, $\Lambda(t,T)$ at time t with maturity T, written on the system electricity price, $S_2(t)$, is*

$$\Lambda(t,T) = \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)} \tag{2.10}$$

where, $\mathcal{C}_2$ can be found in Lemma 9, while ω_3^2, ω_4^2 and ω_5^2 can be found in Proposition 7.

Proof. By the arbitrage-free principle, under the risk-neutral probability, the expectation of the futures price is equal to the expectation of the system price, i.e.

$$\mathbb{E}_{\mathcal{Q}}[\Lambda(t,T) \cdot exp(-\int_t^T r_s \, ds)|\mathcal{F}_t] = \mathbb{E}_{\mathcal{Q}}[S_2(T) \cdot exp(-\int_t^T r_s \, ds)|\mathcal{F}_t]$$

Since the $\Lambda(t,T)$ is an unknown constant, the left side should be

$$\Lambda(t,T) \cdot \mathbb{E}_{\mathcal{Q}}[e^{-\int_t^T r_s \, ds}|\mathcal{F}_t] = \Lambda(t,T) \cdot Z(t,T)$$

recall $Z(t,T)$ is the zero-coupon's price. The arguments in the proofs of Lemma 9 and Proposition 7 show that the right side is

$$Z(t,T) \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)}$$

Then, it is clear that $\Lambda(t,T) = \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)}$.

❑

We end this section by the link between the price of EPAD, $\mathfrak{E}(t,T)$, and the futures price, $\Lambda(t,T)$.

Proposition 8. *The price of the future written on the system electricity can be calculated as*

$$\Lambda(t,T) = \mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)} - \mathfrak{E}(t,T) \cdot Z(t,T)^{-1} \tag{2.11}$$

where, $\mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)}$ is the risk-neutral expectation of the local electricity price $S_1(T)$, without discounting. The constant $\mathcal{C}_1$ can be found in Lemma 9, while ω_1^2 and ω_2^2 can be found in Proposition 7.

Proof.

$$\Lambda(t,T) = \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)}$$
$$= \mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)} - \mathfrak{E}(t,T) \cdot Z(t,T)^{-1}$$

❑

2.5 Static Hedging Strategy without EPAD, Only Futures

At the momoent, Nasdaq offers EPADs covering only Århus, Copenhagen, Helsinki, Luleå, Malmø, Oslo, Riga, Sudsvall, Tallinn and Tromsø[1]. It is still meaningful to discuss how we should hedge local price risk effectively. Mathematically speaking, we hope that our hedging portfolio has a lowest possible variance for a given expectation.

Assume we live in an area without EPAD, and have to open a position in local price, $S_1(t)$. To hedge the potential risk up to time T, we open a pair of opposite positions in system price, $S_2(t)$, and its futures, $\Lambda(t,T)$. Then we build the portfolio at time t with maturity T, denoted by $\mathcal{H}(t,T)$

$$\mathcal{H}(t,T) = S_1(T) - h \cdot \Big(\Lambda(t,T) - S_2(T)\Big) \tag{2.12}$$

where h is a static scale. We aim to find a h minimizing $Var_{\mathcal{Q}}\big[\mathcal{H}(t,T)\big|\mathcal{F}_t\big]$ for a given $\mathbb{E}_{\mathcal{Q}}\big[\mathcal{H}(t,T)\big|\mathcal{F}_t\big]$. This looks complicated. Fortunately, it can be transferred to a much easier problem.

Lemma 12. *To minimize $Var_{\mathcal{Q}}\big[\mathcal{H}(t,T)\big|\mathcal{F}_t\big]$ for a given $\mathbb{E}_{\mathcal{Q}}\big[\mathcal{H}(t,T)\big|\mathcal{F}_t\big]$, is equal to minimize*

$$f(h) = \mathbb{E}_{\mathcal{Q}}\bigg[\mathcal{H}(t,T)^2\Big|\mathcal{F}_t\bigg]$$
$$= \mathbb{E}_{\mathcal{Q}}\bigg[\bigg(S_1(T) - h \cdot \Big(\Lambda(t,T) - S_2(T)\Big)\bigg)^2\Big|\mathcal{F}_t\bigg] \tag{2.13}$$

Proof. Apply the formula for conditional variance to $Var_{\mathcal{Q}}\big[\mathcal{H}(t,T)\big|\mathcal{F}_t\big]$, it says

---

[1]http://www.nasdaqomx.com/transactions/markets/commodities/markets/power/nordic-power

$$Var_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[\left(\mathcal{H}(t,T) - \mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]\right)^2\Big|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)^2 - 2\cdot\mathcal{H}(t,T)\cdot\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right] + \mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]^2\Big|\mathcal{F}_t\right]$$

$$= \underbrace{\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)^2\big|\mathcal{F}_t\right]}_{f(h)} - \underbrace{\mathbb{E}_{\mathcal{Q}}\left[2\cdot\mathcal{H}(t,T)\cdot\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]\Big|\mathcal{F}_t\right]}_{2\cdot\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]\cdot\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]} + \underbrace{\mathbb{E}_{\mathcal{Q}}\left[\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]^2\Big|\mathcal{F}_t\right]}_{\mathbb{E}_{\mathcal{Q}}\left[\mathcal{H}(t,T)\big|\mathcal{F}_t\right]^2}$$

The last two terms are given, so the minimization work depends only on $f(h)$.

$\square$

All we need to do is to find a static scale, h, minimizing the value of $f(h)$. We end this section with the following proposition and its proof consisting of mass calculation.

Proposition 9. *The following static portfolio, h, gives the minimum $f(h)$, i.e. the hedging with highest efficiency.*

$$h = \frac{\mathcal{C}_1}{\mathcal{C}_2}\cdot\frac{e^{\frac{1}{2}\cdot\sum\limits_{i=1}^{5}\omega_i^2}\cdot\left(1 - e^{\frac{1}{2}\xi}\right)}{e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2}} \tag{2.14}$$

where, $\{\omega_i\}_{i=1,\cdots,5}$ have been mentioned in Proposition 7. And the ξ is

$$\xi = 2\cdot\eta^2(T-t) + 2\cdot a_2\cdot\eta\cdot\eta_1\cdot\frac{1 - e^{-\mu_1(T-t)}}{\mu_1} + 2\cdot a_4\cdot\eta\cdot\eta_2\cdot\frac{1 - e^{-\mu_2(T-t)}}{\mu_2}$$

$$+ 2\cdot a_3\cdot a_5\cdot\eta_1\cdot\eta_2\cdot\frac{1 - e^{-2\mu_2(T-t)}}{\mu_1 + \mu_2} + 2\cdot a_2\cdot a_4\cdot\eta_1\cdot\eta_2\cdot\frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

where the constants, μ's, η's and a's, can be found in Remark 1 and Lemma 10 respectively.

Proof. We start calculating from the equation in Lemma 12

$$f(h) = \mathbb{E}_{\mathcal{Q}}\left[\left(S_1(T) - h \cdot \left(\Lambda(t,T) - S_2(T)\right)\right)^2 \bigg| \mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[S_1(T)^2 - 2 \cdot S_1(T) \cdot h \cdot \left(\Lambda(t,T) - S_2(T)\right) + h^2\left(\Lambda(t,T) - S_2(T)\right)^2 \bigg| \mathcal{F}_t\right]$$

$$= \underbrace{\mathbb{E}_{\mathcal{Q}}\left[S_1(T)^2 \bigg| \mathcal{F}_t\right]}_{I} - 2 \cdot \underbrace{\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot h \cdot \left(\Lambda(t,T) - S_2(T)\right) \bigg| \mathcal{F}_t\right]}_{II}$$

$$+ \underbrace{\mathbb{E}_{\mathcal{Q}}\left[h^2\left(\Lambda(t,T) - S_2(T)\right)^2 \bigg| \mathcal{F}_t\right]}_{III}$$

For the part I :

Recall equation (2.7), we know

$$S_1(T)^2 = (e^{X(T)+Y_1(T)})^2 = e^{2X(T)+2Y_1(T)}$$

$$= e^{2X(t)+2\mu(T-t)+2\eta[V(T)-V(t)]+2Y_1(t)e^{-\mu_1(T-t)}+2\eta_1\int_t^T e^{-\mu_1(T-s)}\,dV_1(s)}$$

$$= \underbrace{e^{2\cdot[X(t)+\mu(T-t)]+2\cdot Y_1(t)e^{-\mu_1(T-t)}}}_{\mathcal{C}_1^2} \cdot \underbrace{e^{2\eta[V(T)-V(t)]}e^{2\eta_1\int_t^T e^{-\mu_1(T-s)}\,dV_1(s)}}_{(i)\,:\ independent\ of\ \mathcal{F}_t}$$

To calculate (i), recall Lemma 10, which gives $U(t) = V(t)$ and $V_1(s) = a_2 U(s) + a_3 U_1(s)$, then

$$(i) = e^{2\eta[U(T)-U(t)]} \cdot e^{2\eta_1\int_t^T e^{-\mu_1(T-s)}\,d(a_2 U(s)+a_3 U_1(s))}$$

$$= e^{2\int_t^T \eta\,dU(s)} \cdot e^{2\eta_1\int_t^T a_2\cdot e^{-\mu_1(T-s)}\,dU(s)}e^{2\eta_1\int_t^T a_3\cdot e^{-\mu_1(T-s)}\,dU_1(s)}$$

$$= e^{2\int_t^T \eta+a_2\cdot\eta_1\cdot e^{-\mu_1(T-s)}\,dU(s)} \cdot e^{2\cdot\eta_1\int_t^T a_3\cdot e^{-\mu_1(T-t)}\,dU_1(s)}$$

Recall the proof for Proposition 7, it gets

$$2 \cdot \int_t^T \eta + a_2 \cdot \eta_1 \cdot e^{-\mu_1(T-s)}\,dU(s) \sim \mathcal{N}(0, 4\omega_1^2)$$

$$2 \cdot \eta_1 \int_t^T a_3 \cdot e^{-\mu_1(T-t)}\,dU_1(s) \sim \mathcal{N}(0, 4\omega_2^2)$$

Using the formula of log-normal distribution, it gets

$$\mathbb{E}_{\mathcal{Q}}\left[e^{2\cdot\int_t^T \eta+a_2\cdot\eta_1\cdot e^{-\mu_1(T-s)}\,dU(s)}\right] = e^{\frac{1}{2}\cdot 4\cdot\omega_1^2} = e^{2\omega_1^2}$$

$$\mathbb{E}_{\mathcal{Q}}\left[e^{2\cdot\eta_1\int_t^T a_3\cdot e^{-\mu_1(T-t)}\,dU_1(s)}\right] = e^{\frac{1}{2}4\omega_2^2} = e^{2\omega_2^2}$$

$$\Rightarrow \mathbb{E}_{\mathcal{Q}}\left[(i)\right] = e^{2\omega_1^2} \cdot e^{2\omega_2^2}$$

$$I = \mathbb{E}_{\mathcal{Q}}\left[S_1(T)^2\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathcal{Q}}\left[\mathcal{C}_1^2 \cdot (i)\middle|\mathcal{F}_t\right] = \mathcal{C}_1^2\mathbb{E}_{\mathcal{Q}}\left[(i)\right] = \mathcal{C}_1^2 \cdot e^{2\omega_1^2 + 2\omega_2^2}$$

For the part II :

$$II = \mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot h \cdot \left(\Lambda(t,T) - S_2(T)\right)\middle|\mathcal{F}_t\right]$$

$$= \underbrace{\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot h \cdot \Lambda(t,T)\middle|\mathcal{F}_t\right]}_{h \cdot \Lambda(t,T) \cdot \mathbb{E}_{\mathcal{Q}}\left[S_1(T)\middle|\mathcal{F}_t\right]} - \underbrace{\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot h \cdot S_2(T)\middle|\mathcal{F}_t\right]}_{h \cdot \mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot S_2(T)\middle|\mathcal{F}_t\right]}$$

The Lemma 11 gives us $\Lambda(t,T) = \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)}$, and the proofs for Lemma 9 and Proposition 7 give $\mathbb{E}_{\mathcal{Q}}\left[S_1(T)\middle|\mathcal{F}_t\right] = \mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)}$. We only need to calculate $\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot S_2(T)\middle|\mathcal{F}_t\right]$:

$$\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot S_2(T)\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathcal{Q}}\left[e^{X(T)} \cdot e^{Y_1(T)} \cdot e^{X(T)} \cdot e^{Y_2(T)}\middle|\mathcal{F}_t\right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[\underbrace{e^{2X(T)} \cdot e^{Y_1(T) + Y_2(T)}}_{\square}\middle|\mathcal{F}_t\right]$$

$$\square = e^{2 \cdot X(t) + 2 \cdot \mu \cdot (T-t) + 2 \cdot \eta \cdot [V(T) - V(t)]}$$

$$\cdot e^{Y_1(t) \cdot e^{-\mu_1 \cdot (T-t)} + \int_t^T \eta_1 e^{-\mu_1 \cdot (T-s)} dV_1(s) + Y_2(t) \cdot e^{-\mu_2 \cdot (T-t)} + \int_t^T \eta_2 e^{-\mu_2 \cdot (T-s)} dV_2(s)}$$

$$= \underbrace{e^{2 \cdot X(t) + 2 \cdot \mu \cdot (T-t) + Y_1(t) \cdot e^{-\mu_1 \cdot (T-t)} + Y_2(t) \cdot e^{-\mu_2 \cdot (T-t)}}}_{\mathcal{C}_1 \cdot \mathcal{C}_2}$$

$$\cdot \underbrace{e^{2 \cdot \eta \cdot [V(T) - V(t)]} \cdot e^{\int_t^T \eta_1 e^{-\mu_1 (T-s)} dV_1(s)} \cdot e^{\int_t^T \eta_2 e^{-\mu_2 \cdot (T-s)} dV_2(s)}}_{e^{\int_t^T 2\eta \, dU(s)} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-s)} d(a_2 U(s) + a_3 U_1(s))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-s)} d(a_4 U(s) + a_5 U_1(s) + a_6 U_2(s))}}$$

$$= \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\int_t^T 2\eta + a_2 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_4 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \, dU(s)}$$

$$\cdot e^{\int_t^T a_3 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_5 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \, dU_1(s)} \cdot e^{\int_t^T a_6 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \, dU_2(s)}$$

apparently, the $\square$ is independent of $\mathcal{F}_t$. And recall $U(s)$, $U_1(s)$ and $U_2(s)$ are independent Brownian motions. So,

$$\mathbb{E}_{\mathcal{Q}}\Big[S_1(T) \cdot S_2(T)\Big|\mathcal{F}_t\Big]$$

$$= \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot \underbrace{\mathbb{E}_{\mathcal{Q}}\Big[e^{\int_t^T 2\eta + a_2 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_4 \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\, dU(s)}\Big]}_{\triangle = e^{\frac{1}{2}\nu^2}}$$

$$\cdot \underbrace{\mathbb{E}_{\mathcal{Q}}\Big[e^{\int_t^T a_3 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_5 \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\, dU_1(s)}\Big]}_{\triangle_1 = e^{\frac{1}{2}\nu_1^2}} \cdot \underbrace{\mathbb{E}_{\mathcal{Q}}\Big[e^{\int_t^T a_6 \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\, dU_2(s)}\Big]}_{\triangle_2 = e^{\frac{1}{2}\nu_2^2}}$$

The $\triangle_1$, $\triangle_2$ and $\triangle_3$ are expectations of three independent log-normal variables respectively. So we only need to do the three powers' variances, ν^2, ν_1^2 and ν_2^2 :

$$\nu^2 = \mathbb{E}_{\mathcal{Q}}\Bigg[\bigg(\int_t^T 2\eta + a_2 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_4 \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\, dU(s)\bigg)^2\Bigg]$$

$$= \mathbb{E}_{\mathcal{Q}}\Bigg[\int_t^T \underbrace{\Big(2\eta + a_2 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_4 \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\Big)^2}\, ds\Bigg]$$

$$4\eta^2 + a_2^2 \cdot \eta_1^2 \cdot e^{-2\mu_1(T-s)} + a_4^2 \cdot \eta_2^2 \cdot e^{-2\mu_2(T-s)} + 4a_2 \cdot \eta \cdot \eta_1 \cdot e^{-\mu_1(T-s)}$$

$$+ 4a_4 \cdot \eta \cdot \eta_2 \cdot e^{-\mu_2(T-s)} + 2a_2 \cdot a_4 \cdot \eta_1 \cdot \eta_2 \cdot e^{-\mu_1(T-s)} \cdot e^{-\mu_2(T-s)}$$

We calculate integral for each term

$$\int_t^T 4\eta^2\, ds = 4\eta^2 \cdot (T-t)$$

$$\int_t^T a_2^2 \cdot \eta_1^2 \cdot e^{-2\mu_1(T-s)}\, ds = a_2^2 \cdot \eta_1^2 \cdot e^{-2\mu_1 T} \cdot \int_t^T e^{2\mu_1 s}\, ds$$

$$= a_2^2 \cdot \eta_1^2 \cdot \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1}$$

$$\int_t^T a_4^2 \cdot \eta_2^2 \cdot e^{-2\mu_2(T-s)}\, ds = a_4^2 \cdot \eta_2^2 \cdot \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$\int_t^T 4 \cdot a_2 \cdot \eta \cdot \eta_1 e^{-\mu_1(T-s)}\, ds = 4 \cdot a_2 \cdot \eta \cdot \eta_1 \cdot \frac{1 - e^{-\mu_1(T-t)}}{\mu_1}$$

$$\int_t^T 4 \cdot a_4 \cdot \eta \cdot \eta_2 \cdot e^{-\mu_2(T-s)}\, ds = 4 \cdot a_4 \cdot \eta \cdot \eta_2 \cdot \frac{1 - e^{-\mu_2(T-t)}}{\mu_2}$$

$$\int_t^T 2 \cdot a_2 \cdot a_4 \cdot \eta_1 \cdot \eta_2 \cdot e^{-\mu_1(T-s)-\mu_2(T-s)}\, ds = 2 \cdot a_2 \cdot a_4 \cdot \eta_1 \cdot \eta_2 \cdot \frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

Since all of the above values are constants, we don't need $\mathbb{E}_{\mathcal{Q}}[\cdot]$ any more.

$$\Rightarrow \nu^2 = 4\eta^2 \cdot (T-t) + a_2^2 \cdot \eta_1^2 \cdot \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1} + a_4^2 \cdot \eta_2^2 \cdot \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$+ 4 \cdot a_2 \cdot \eta \cdot \eta_1 \cdot \frac{1 - e^{-\mu_1(T-t)}}{\mu_1} + 4 \cdot a_4 \cdot \eta \cdot \eta_2 \cdot \frac{1 - e^{-\mu_2(T-t)}}{\mu_2}$$

$$+ 2 \cdot a_2 \cdot a_4 \cdot \eta_1 \cdot \eta_2 \cdot \frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

The above terms include ω_1^2 and ω_3^2, so we make it a little simpler

$$\Rightarrow \nu^2 = \omega_1^2 + \omega_3^2 + 2\eta^2 \cdot (T-t) + 2 \cdot a_2 \cdot \eta \cdot \eta_1 \cdot \frac{1 - e^{-\mu_1(T-t)}}{\mu_1}$$

$$+ 2 \cdot a_4 \cdot \eta \cdot \eta_2 \cdot \frac{1 - e^{-\mu_2(T-t)}}{\mu_2} + 2 \cdot a_2 \cdot a_4 \cdot \eta_1 \cdot \eta_2 \cdot \frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

By the similar method, we calculate ν_1^2 and ν_2^2 :

$$\nu_1^2 = \mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^T a_3 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_5 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \, dU_1(s) \right)^2 \right]$$

$$= \mathbb{E}_{\mathcal{Q}}\left[\int_t^T \underbrace{\left(a_3 \cdot \eta_1 \cdot e^{-\mu_1(T-s)} + a_5 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \right)^2}_{a_3^2 \cdot \eta_1^2 \cdot e^{-2 \cdot \mu_1(T-s)} + 2 \cdot a_3 \cdot a_5 \cdot \eta_1 \cdot \eta_2 \cdot e^{-\mu_1(T-s)-\mu_2(T-s)} + a_5^2 \cdot \eta_2^2 \cdot e^{-2\mu_2(T-s)}} \, ds \right]$$

$$= a_3^2 \cdot \eta_1^2 \cdot \frac{1 - e^{-2\mu_1(T-t)}}{2\mu_1} + 2 \cdot a_3 \cdot a_5 \cdot \eta_1 \cdot \eta_2 \cdot \frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

$$+ a_5^2 \cdot \eta_2^2 \cdot \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$\Rightarrow \nu_1^2 = \omega_2^2 + \omega_4^2 + 2 \cdot a_3 \cdot a_5 \cdot \eta_1 \cdot \eta_2 \cdot \frac{1 - e^{-(\mu_1+\mu_2)(T-t)}}{\mu_1 + \mu_2}$$

$$\nu_2^2 = \mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^T a_6 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \, dU_2(s) \right)^2 \right] = \mathbb{E}_{\mathcal{Q}}\left[\int_t^T \left(a_6 \cdot \eta_2 \cdot e^{-\mu_2(T-s)} \right)^2 \, ds \right]$$

$$= a_6^2 \cdot \eta_2^2 \cdot \frac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$$

$$\Rightarrow \nu_2^2 = \omega_5^2$$

We come back to the $\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot S_2(T) \Big| \mathcal{F}_t \right]$

$$\mathbb{E}_{\mathcal{Q}}\left[S_1(T) \cdot S_2(T) \Big| \mathcal{F}_t \right] = \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot (\nu^2 + \nu_1^2 + \nu_2^2)}$$

$$= \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot (\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2)} e^{\frac{1}{2}\xi}$$

where, ξ is same as we claimed.

Now we can get the value of the part II

$$II = h \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2}(\omega_3^2 + \omega_4^2 + \omega_5^2)} \cdot \mathcal{C}_1 \cdot e^{\frac{1}{2}(\omega_1^2 + \omega_2^2)} - h \cdot \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot (\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2)} \cdot e^{\frac{1}{2}\xi}$$

$$= h \cdot \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot \sum_{i=1}^{5} \omega_i^2} \cdot (1 - e^{\frac{1}{2} \cdot \xi})$$

For the part III :

$$\mathbb{E}_{\mathcal{Q}}\left[h^2 \Big(\Lambda(t, T) - S_2(T)\Big)^2 \Big| \mathcal{F}_t \right]$$

$$= h^2 \cdot \mathbb{E}_{\mathcal{Q}}\left[\Lambda(t, T)^2 - 2 \cdot \Lambda(t, T) \cdot S_2(T) + S_2(T)^2 \Big| \mathcal{F}_t \right]$$

$$= h^2 \cdot \underbrace{\Lambda(t, T)^2}_{\mathcal{C}_2^2 e^{\omega_3^2 + \omega_4^2 + \omega_5^2}} - 2 \cdot h^2 \cdot \underbrace{\Lambda(t, T) \mathbb{E}_{\mathcal{Q}}\left[S_2(T) \Big| \mathcal{F}_t \right]}_{\mathcal{C}_2^2 e^{\omega_3^2 + \omega_4^2 + \omega_5^2}} + h^2 \cdot \underbrace{\mathbb{E}_{\mathcal{Q}}\left[S_2(T)^2 \Big| \mathcal{F}_t \right]}_{(iii)}$$

By the calculation for $\mathbb{E}_{\mathcal{Q}}\left[S_1(T)^2 \Big| \mathcal{F}_t \right]$ in I, we know that $(iii) = \mathcal{C}_2^2 \cdot e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2}$

$$III = h^2 \cdot \left\{ \mathcal{C}_2^2 e^{\omega_3^2 + \omega_4^2 + \omega_5^2} - 2 \cdot \mathcal{C}_2^2 e^{\omega_3^2 + \omega_4^2 + \omega_5^2} + \mathcal{C}_2^2 \cdot e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} \right\}$$

$$= h^2 \cdot \mathcal{C}_2^2 \cdot \left(e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2} \right)$$

Now we use the results of I, II and III to calculate $f(h)$:

$$f(h) = I - 2 \cdot II + III$$

$$= \mathcal{C}_1^2 \cdot e^{2\omega_1^2 + 2\omega_2^2} - 2 \cdot h \cdot \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot \sum_{i=1}^{5} \omega_i^2} \cdot (1 - e^{\frac{1}{2} \cdot \xi})$$

$$+ h^2 \cdot \mathcal{C}_2^2 \cdot \left(e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2} \right)$$

$$\frac{\partial^2 f}{\partial h^2} = 2 \cdot \mathcal{C}_2^2 \cdot \left(e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2} \right) > 0$$

$$\Rightarrow f(h) \ is \ concave \ upward$$

Since this is a concave upward function, the h making first order derivative of $f(\cdot)$ be zero gives the minimum value.

$$\frac{\partial f}{\partial h} = -2 \cdot \mathcal{C}_1 \cdot \mathcal{C}_2 \cdot e^{\frac{1}{2} \cdot \sum_{i=1}^{5} \omega_i^2} \cdot (1 - e^{\frac{1}{2} \cdot \xi}) + 2 \cdot h \cdot \mathcal{C}_2^2 \cdot \left(e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2} \right) = 0$$

$$\Rightarrow h = \frac{\mathcal{C}_1}{\mathcal{C}_2} \cdot \frac{e^{\frac{1}{2} \cdot \sum_{i=1}^{5} \omega_i^2} \cdot (1 - e^{\frac{1}{2}\xi})}{e^{2\omega_3^2 + 2\omega_4^2 + 2\omega_5^2} - e^{\omega_3^2 + \omega_4^2 + \omega_5^2}}$$

❑

Chapter 3

Pricing the EPAD with Relevant interest rate

3.1 Introduce One More Related Brownian Motion

Recall the Definition 5 and the Definition 6, the EPAD's no-arbitrage price with maturity T at present time t, is

$$\mathfrak{E}(t,T) = \mathbb{E}_{\mathcal{Q}} \left[(S_1(T) - S_2(T)) \cdot exp \left(-\int_t^T r_s \, ds \right) | \mathcal{F}_t \right]$$

where, the $S_1(\cdot)$ and $S_2(\cdot)$ are

$$\begin{cases} logS_1(s) = X(s) + Y_1(s) \\ logS_2(s) = X(s) + Y_2(s) \end{cases} \quad s \in [t,T]$$

one step closer, the $X(\cdot)$, $Y_1(\cdot)$ and $Y_2(\cdot)$ are

$$\begin{cases} X(s) = X(t) + \mu \cdot (s-t) + \eta \cdot (V(s) - V(t)) \\ Y_1(s) = e^{-\mu_1 s}(e^{\mu_1 t}Y_1(t) + \eta_1 \int_t^s e^{\mu_1 k} \, dV_1(k)) \\ Y_2(s) = e^{-\mu_2 s}(e^{\mu_2 t}Y_2(t) + \eta_2 \int_t^s e^{\mu_2 k} \, dV_2(k)) \end{cases}$$

and the model of r_s can be found in Lemma 1

$$r_s = e^{-a(s-t)}r_t + b(1 - e^{-a(s-t)}) + \theta \cdot \int_t^s e^{-a(s-u)} \, dV_3(k)$$

where $a > 0$, $b > 0$, $\theta > 0$

Now we let the discounting effect $e^{-\int_t^T r_s \, ds}$ is relevant to $S_1(T)$ and $S_2(T)$. It means the four $\mathcal{Q}$–Brownian motions, V, V_1, V_2 and V_3 are pairwise relevant. And we set the correlations as

$$\rho_1 = Corr[V(\cdot), V_1(\cdot)], \ \ \rho_2 = Corr[V(\cdot), V_2(\cdot)], \ \ \rho_3 = Corr[V(\cdot), V_3(\cdot)],$$
$$\rho_{12} = Corr[V_1(\cdot), V_2(\cdot)], \ \ \rho_{13} = Corr[V_1(\cdot), V_3(\cdot)], \ \ \rho_{23} = Corr[V_2(\cdot), V_3(\cdot)]$$

And the vectorised $[V(\cdot), V_1(\cdot), V_2(\cdot), V_3(\cdot)]$ can be built as

$$\begin{bmatrix} V(k) \\ V_1(k) \\ V_2(k) \\ V_3(k) \end{bmatrix} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 \\ a_6 & a_7 & a_8 & a_9 \end{bmatrix} \cdot \begin{bmatrix} U(k) \\ U_1(k) \\ U_2(k) \\ U_3(k) \end{bmatrix}$$

where $U(\cdot)$, $U_1(\cdot)$, $U_2(\cdot)$ and $U_3(\cdot)$ are four independent $\mathcal{Q}$-Brownian Motions, while a_0, a_1, $\cdots$, a_9 are constants. Then, by some calculation, we can get the values of $a_0, \cdots, a_9$. We skip the calculation, because it is similar to the arguments in the proof of Lemma 10.

$$a_0^2 = 1$$

$$a_1 = \rho_1$$

$$a_2^2 = 1 - \rho_1^2$$

$$a_3 = \rho_2$$

$$a_4^2 = \frac{(\rho_{12} - \rho_1 \rho_2)^2}{1 - \rho_1^2}$$

$$a_6^2 = 1 - \rho_2^2 - \frac{(\rho_{12} - \rho_1 \rho_2)^2}{1 - \rho_1^2}$$

$$a_7^2 = \frac{(\rho_{13} - \rho_1 \rho_3)^2}{1 - \rho_1^2}$$

$$a_8^2 = \frac{\left(\rho_{23} - \rho_2 \rho_3 - \frac{(\rho_{12} - \rho_1 \rho_2)(\rho_{13} - \rho_1 \rho_3)}{1 - \rho_1^2} \right)^2}{1 - \rho_2^2 - \frac{(\rho_{12} - \rho_1 \rho_2)^2}{1 - \rho_1^2}}$$

$$a_9^2 = 1 - \rho_3^2 - \frac{(\rho_{13} - \rho_1 \rho_3)^2}{1 - \rho_1^2} - \frac{\left(\rho_{23} - \rho_2 \rho_3 - \frac{(\rho_{12} - \rho_1 \rho_2)(\rho_{13} - \rho_1 \rho_3)}{1 - \rho_1^2} \right)^2}{1 - \rho_2^2 - \frac{(\rho_{12} - \rho_1 \rho_2)^2}{1 - \rho_1^2}}$$

3.2 Price the EPAD again

The following result is corresponding to the Proposition 7, but with relevant interest. The major different point is that the independent zero-coupon, $Z(t, T)$, will be dismissed, while more parameters will be taken in.

Proposition 10.

$$\mathfrak{E}(t, T) = \mathcal{C}_3 \cdot \left(\mathcal{C}_1 \cdot e^{\frac{1}{2}[\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2]} - \mathcal{C}_2 \cdot e^{\frac{1}{2}[\zeta_4^2 + \zeta_5^2 + \zeta_6^2 + \zeta_7^2]} \right)$$

where, $\mathcal{C}_1 := e^{X(t) + \mu(T-t) + e^{-\mu_1(T-t)} Y_1(t)}$

$\mathcal{C}_2 := e^{X(t) + \mu(T-t) + e^{-\mu_2(T-t)} Y_2(t)}$

$\mathcal{C}_3 := e^{\frac{r_t}{a}(e^{-a(T-t)}-1)} \cdot e^{-\frac{b}{a}(e^{-a(T-t)}-1)} \cdot e^{-b(T-t)}$

$\zeta_0^2 = \eta^2(T-t) + 2 \cdot \eta \cdot \eta_1 \cdot a_1 \cdot \dfrac{1 - e^{-\mu_1(T-t)}}{\mu_1} + \eta_1^2 \cdot a_1^2 \cdot \dfrac{1 - e^{2\mu_1(T-t)}}{2\mu_1}$

$\qquad + \theta^2 \cdot \dfrac{a_6^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_1^2 = \eta_1^2 \cdot a_2^2 \cdot \dfrac{1 - e^{-2\mu_1(T-t)}}{2\mu_1}$

$\qquad + \theta^2 \cdot \dfrac{a_7^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_2^2 = \theta^2 \cdot \dfrac{a_8^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_3^2 = \theta^2 \cdot \dfrac{a_9^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_4^2 = \eta^2(T-t) + 2 \cdot \eta \cdot \eta_2 \cdot a_3 \cdot \dfrac{1 - e^{-\mu_2(T-t)}}{\mu_2} + \eta_1^2 \cdot a_3^2 \cdot \dfrac{1 - e^{2\mu_2(T-t)}}{2\mu_2}$

$\qquad + \theta^2 \cdot \dfrac{a_6^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_5^2 = \eta_2^2 \cdot a_4^2 \cdot \dfrac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$

$\qquad + \theta^2 \cdot \dfrac{a_7^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_6^2 = \eta_2^2 \cdot a_5^2 \cdot \dfrac{1 - e^{-2\mu_2(T-t)}}{2\mu_2}$

$\qquad + \theta^2 \cdot \dfrac{a_8^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

$\zeta_7^2 = \theta^2 \cdot \dfrac{a_9^2}{4a^3} \cdot [2a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}]$

Proof. According to Lemma 9, $\mathfrak{E}(t, T)$ can be rewritten as

$$\mathfrak{E}(t,T) = \mathcal{C}_1 \cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} dV_1(k)} \cdot e^{-\int_t^T r_k \, dk} \right]$$

$$- \mathcal{C}_2 \cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-k)} dV_2(k)} \cdot e^{-\int_t^T r_k \, dk} \right]$$

$$where, \ \mathcal{C}_1 := e^{X(t)+\mu(T-t)+e^{-\mu_1(T-t)}Y_1(t)}$$

$$\mathcal{C}_2 := e^{X(t)+\mu(T-t)+e^{-\mu_2(T-t)}Y_2(t)}$$

It is sufficient to calculate the two expectations. For simplicity, we name them as

$$\mathcal{E}_1 = \mathbb{E}_{\mathcal{Q}} \left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} dV_1(k)} \cdot e^{-\int_t^T r_k \, dk} \right]$$

$$\mathcal{E}_2 = \mathcal{C}_2 \cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\eta(V(T)-V(t))} \cdot e^{\eta_2 \int_t^T e^{-\mu_2(T-k)} dV_2(k)} \cdot e^{-\int_t^T r_k \, dk} \right]$$

For $\mathcal{E}_1$,

$$e^{\eta(V(T)-V(t))} = e^{\int_t^T \eta \, dU(k)}$$

$$e^{\eta_1 \int_t^T e^{-\mu_1(T-k)} dV_1(k)} = e^{\eta_1 \int_t^T a_1 \cdot e^{-\mu_1(T-k)} dU(k)} \cdot e^{\eta_1 \int_t^T a_2 \cdot e^{-\mu_1(T-k)} dU_1(k)}$$

$$e^{-\int_t^T r_s \, ds} = e^{-\int_t^T \left(e^{-a(k-t)} r_t + b(1-e^{-a(k-t)}) + \theta \cdot \int_t^k e^{-a(k-s)} dV_3(s) \right) dk}$$

$$= e^{-r_t \int_t^T e^{-a(k-t)} dk} \cdot e^{\int_t^T b(e^{-a(k-t)}-1) dk} \cdot e^{-\theta \int_t^T \left(\int_t^k e^{-a(k-s)} dV_3(s) \right) dk}$$

$$= e^{\frac{r_t}{a} \cdot (e^{-a(T-t)}-1)} \cdot e^{-\frac{b}{a}(e^{-a(T-t)}-1)-b(T-t)}$$

$$\cdot e^{-\theta \int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} dU(s) \, dk} \cdot e^{-\theta \int_t^T \int_t^k a_7 \cdot e^{-a(k-s)} dU_1(s) \, dk}$$

$$\cdot e^{-\theta \int_t^T \int_t^k a_8 \cdot e^{-a(k-s)} dU_2(s) \, dk} \cdot e^{-\theta \int_t^T \int_t^k a_9 \cdot e^{-a(k-s)} dU_3(s) \, dk}$$

Draw the deterministic terms out of the expectation, and make organization in line with the independent Brownian Motions. Then

$$\mathcal{E}_1 = e^{\frac{r_t}{a} \cdot (e^{-a(T-t)}-1)} \cdot e^{-\frac{b}{a}(e^{-a(T-t)}-1)-b(T-t)}$$

$$\cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} dU(k) - \theta \int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} dU(s) \, dk} \right]$$

$$\cdot \mathbb{E}_{\mathcal{Q}} \left[e^{\int_t^T \eta_1 a_2 e^{-\mu_1(T-k)} dU_1(k) - \theta \int_t^T \int_t^k a_7 \cdot e^{-a(k-s)} dU_1(s) \, dk} \right]$$

$$\cdot \mathbb{E}_{\mathcal{Q}} \left[e^{-\theta \int_t^T \int_t^k a_8 \cdot e^{-a(k-s)} dU_2(s) \, dk} \right]$$

$$\cdot \mathbb{E}_{\mathcal{Q}} \left[e^{-\theta \int_t^T \int_t^k a_9 \cdot e^{-a(k-s)} dU_3(s) \, dk} \right]$$

So we have to calculate the four expectations one by one. All of them can be calculated by the formula of log-normal's mean. So it is equal to calculate the four powers' means and variances. We look at the first one. Since $a_6 \cdot e^{-a(k-s)}$ is Riemann-integrable, by the Fubini-Tonelli's theorem, the integral and expectation can be exchanged

$$\mathbb{E}_{\mathcal{Q}}\left[\int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right] = \int_t^T \mathbb{E}_{\mathcal{Q}}\left[\int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s)\right] dk$$

$$= \int_t^T 0 \, dk = 0$$

$$\mathbb{E}_{\mathcal{Q}}\left[\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k) - \theta \int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right]$$

$$=\mathbb{E}_{\mathcal{Q}}\left[\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k)\right] - \theta \cdot \mathbb{E}_{\mathcal{Q}}\left[\int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right]$$

$$=0 - \theta \cdot 0 = 0$$

Then we come to calculate the corresponding variance

$$\mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k) - \theta \int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right]$$

$$=\mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k)\right] + \theta^2 \cdot \mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right]$$

The first term can be calculated by the Ito-isometry and its zero-mean

$$\mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k)\right]$$

$$=\mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^T \eta + \eta_1 a_1 e^{-\mu_1(T-k)} \, dU(k)\right)^2\right]$$

$$=\mathbb{E}_{\mathcal{Q}}\left[\int_t^T (\eta + \eta_1 a_1 e^{-\mu_1(T-k)})^2 \, dk\right]$$

$$= \int_t^T \eta^2 \, dk + \int_t^T 2 \cdot \eta \cdot \eta_1 \cdot a_1 \cdot e^{-\mu_1(T-k)} \, dk + \int_t^T \eta_1^2 \cdot a_1^2 e^{-2\mu_1(T-k)} \, dk$$

$$=\eta^2(T-t) + 2 \cdot \eta \cdot \eta_1 \cdot a_1 \cdot \frac{1 - e^{-\mu_1(T-t)}}{\mu_1} + \eta_1^2 \cdot a_1^2 \cdot \frac{1 - e^{2\mu_1(T-t)}}{2\mu_1}$$

To calculate the second term, we need to use the trick in the proof of Proposition 3, that treats variance as a special case of covariance,

$$\mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk\right]$$

$$=\mathcal{C}ov_{\mathcal{Q}}\left[\int_t^T \int_t^k a_6 \cdot e^{-a(k-s)} \, dU(s) \, dk \ , \ \int_t^T \int_t^j a_6 \cdot e^{-a(j-s)} \, dU(s) \, dj\right]$$

Then it is sufficient to calculate the covariance,

$$\mathcal{C}ov_{\mathcal{Q}}\left[\int_t^T\int_t^k a_6 \cdot e^{-a(k-s)}\,dU(s)\,dk\ ,\ \int_t^T\int_t^j a_6 \cdot e^{-a(j-s)}\,dU(s)\,dj\right]$$

$$=\mathbb{E}_{\mathcal{Q}}\left[\int_t^T\int_t^k a_6 \cdot e^{-a(k-s)}\,dU(s)\,dk\ \cdot\ \int_t^T\int_t^j a_6 \cdot e^{-a(j-s)}\,dU(s)\,dj\right]$$

$$=\int_t^T\int_t^T \mathbb{E}_{\mathcal{Q}}\left[\int_t^k a_6 \cdot e^{-a(k-s)}\,dU(s)\ \cdot\ \int_t^j a_6 \cdot e^{-a(j-s)}\,dU(s)\right]dk\,dj$$

$$=\int_t^T\int_t^T \mathbb{E}_{\mathcal{Q}}\left[\int_t^k a_6 \cdot e^{-a(k-s)}\,dU(s)\ \cdot\ \int_t^j a_6 \cdot e^{-a(j-s)}\,dU(s)\right]dk\,dj$$

$$=\int_t^T\int_t^T a_6^2 \cdot \mathbb{E}_{\mathcal{Q}}\left[\left(\int_t^{k\wedge j} e^{-a(k-s)}\,dU(s)\right)^2\right]dk\,dj$$

$$=a_6^2 \cdot \int_t^T\int_t^T\left(\int_t^{k\wedge j} e^{-2a(k-s)}\,ds\right)dk\,dj$$

$$=a_6^2 \cdot \int_t^T\int_t^T \frac{e^{-a(k+j)}}{2a}\cdot\left(e^{2a(k\wedge j)}-e^{2at}\right)dk\,dj$$

$$=\frac{a_6^2}{4a^3}\cdot\left[2a(T-t)-3+4e^{-a(T-t)}-e^{-2a(T-t)}\right]$$

$$\Rightarrow \mathcal{V}ar_{\mathcal{Q}}\left[\int_t^T \eta+\eta_1 a_1 e^{-\mu_1(T-k)}\,dU(k)-\theta\int_t^T\int_t^k a_6 \cdot e^{-a(k-s)}\,dU(s)\,dk\right]$$

$$=\eta^2(T-t)+2\cdot\eta\cdot\eta_1\cdot a_1\cdot\frac{1-e^{-\mu_1(T-t)}}{\mu_1}+\eta_1^2\cdot a_1^2\cdot\frac{1-e^{2\mu_1(T-t)}}{2\mu_1}$$

$$+\theta^2\cdot\frac{a_6^2}{4a^3}\cdot\left[2a(T-t)-3+4e^{-a(T-t)}-e^{-2a(T-t)}\right]$$

This is equal to the ζ_0^2 we claimed. The calculations for other ζ's follows the above method, but with different parameters, η's, μ's and a's.

$\square$

Conclusion and Outlook

In chapter 1, we worked for pricing a CFD on a stock. A stock price was modeled by Geometric Brownian Motion with a stochastic risk-free interested rate given by Vasicek model. We proved the existence of a risk-neutral probability and found a calculable expression of discounting effect. They were followed by a pricing formula of CFD. Then, we established a no-arbitrage connection between the CFD and futures.

In chapter 2, we worked for pricing a EPAD on the Nordic electricity market. Firstly, we modeled a local log-price and a system log-price as a two-dimensional stochastic process, with a common Drifted Brownian Motion adding respective Ornstein-Uhlenbeck process. It was proved as a cointegrated system. Then we derived EPAD's no-arbitrage price with pairwise related drifted Brownian motion and OU-processes, and an independent interest rate in Vasicek model. Next to it, a connection between EPAD and futures on system-price was constructed. In addition, a most efficient static hedging strategy without EPAD was found.

In chapter 3, we assumed the dynamical interest rate was related to the common Drifted Brownian Motion and the OU-processes, and derived the EPAD's price again.

Obviously, this thesis has many restrictions which are to be broken. It's time to have some outlook.

First of all, a Gaussian variance hidden in Brownian motion cannot capture the real world's volatile changing. If we go through stock charts, we will encounter countless gaps showing changes of 10% , 20% and even more. In the electricity market, dramatically price jumping may be the most serious problem. In this context, we should use Levy-process instead of Brownian motion.

Secondly, it is not enough that a model contains only one dynamical parameter. The real market does not hold any static driving factors. To capture the potential probability well, we should derive more parameters as dynamical process. In chapter 1, the simulation figures exposed how the Vasicek interest rate impacted the CFD's result. Maybe we should treat the α in Definiton 2 as a stochastic process which hedges the interest rate's influence to some extent.

Finally, as Fred said, in topics of multidimensional stochastic process, unstationarity deserves study.

Appendix A

R-code for Graphs

The discussion under the link[1] inspired me to do these coding-works.

A.1 Code Generates Figure 1.1

```
# Simulate Drifted Brownian motion under risk neutral probability
# dW(s) = (alpha - r)/sigma * ds +  dB(s)

# Pamameters
alpha = 0.1
sigma = 100 # scaling the variance of Gaussian
r_static = 0.005 # static interest rate

## define model parameters for constructing dynamic r(t)
r0 <- 0.005  # the initial interest rate
b <- 0.005   # objective of mean-reverting
a <- 0.5    # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian


n <- 10    # amounts of simulation trials
T <- 10    # total time-interval
m <- 1000   # amount of sub-time-intervals
dt <- T/m  # difference in time for each subinterval

W0 = 0     # Initial value
# matrix to hold Drifted Brownian motion's path
W_static <- matrix(0,m+1,n)
W_static[1,] <- W0  # put initial values into the first line

W_dynamic <- matrix(0,m+1,n)  # with dynamic r
W_dynamic[1,] <- W0         # with dynamic r
# matrix to hold short rate paths in Vasicek model
r_dynamic <- matrix(0,m+1,n)
r_dynamic[1,] <- r0  # put initial values into the first line
```

[1]https://www.r-bloggers.com/fun-with-the-vasicek-interest-rate-model/

```r
# the n paths of Brownian motion for X
set.seed(1000)
B = matrix(rnorm(m*n,0,1), ncol = n, byrow = T)
R = matrix(rnorm(m*n,0,1), ncol = n, byrow = T)

# Generate the paths of Vasicek interest rate
for(j in 1:n){
  for(i in 2:(m+1)){
    dr_dynamic <- a*(b-r_dynamic[i-1,j])*dt + theta*sqrt(dt)*R[i-1,j]
    r_dynamic[i,j] <- r_dynamic[i-1,j] + dr_dynamic
  }
}

# Generate the Brownian motion paths with static r
for(j in 1:n){
  for(i in 2:(m+1)){
    dW_static <- (alpha - r_static)/sigma*dt + sqrt(dt)*B[i-1,j]
    W_static[i,j] <- W_static[i-1,j] + dW_static
  }
}

# Generate the Brownian motion paths with dynamic r under Vasicek model
for(j in 1:n){
  for(i in 2:(m+1)){
    dW_dynamic <- (alpha - r_dynamic[i-1,j])/sigma*dt + sqrt(dt)*B[i-1,j]
    W_dynamic[i,j] <- W_dynamic[i-1,j] + dW_dynamic
  }
}

DiffOfWs=W_dynamic-W_static

t <- seq(0, T, dt)
matplot(t, W_static[,1:n], type="l", lty=1,
        main="Ten Drfited Brownian motions with staic r",
        ylab="W_static(t)", xlab="TimeAxis", col=1, ylim = c(-10, 10))

matplot(t, W_dynamic[,1:n], type="l", lty=1,
        main="Ten Drfited Brownian motions with Vasiciek r",
        ylab="W_dynamic(t)", xlab="TimeAxis", col=1, ylim = c(-10, 10))
matplot(t, DiffOfWs[,1:n], type="l", lty=1,
        main="W_dynamic - W_static",
        ylab="Diff(t)", xlab="TimeAxis", col=1, ylim = c(-0.001, 0.001))

## Simulate the 7 Sample Paths of Vasicek Interest Rate##

## define model parameters
r0 <- 0.005  # the initial interest rate  (Norges Bank's current interest rate)
b <- 0.005   # objective of mean-reverting (Norges Bank's current interest rate)
a <- 0.5     # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian
```

```
## simulate short rate paths
n <- 7     # amounts of simulation trials
T <- 10     # total time-interval
m <- 200    # amount of sub-time-intervals
dt <- T/m  # difference in time for each subinterval

r <- matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] <- r0  # put initial values into the first line

set.seed(1001)
for(j in 1:n){
  for(i in 2:(m+1)){
    dr <- a*(b-r[i-1,j])*dt + theta*sqrt(dt)*rnorm(1,0,1)
    r[i,j] <- r[i-1,j] + dr
  }
}


## plot paths
t <- seq(0, T, dt)
rT.expected <- b + (r0-b)*exp(-a*t) # mean of the Vasicek process
rT.stdev <- sqrt( theta^2/(2*a)*(1-exp(-2*a*t))) # standard deviation of the Vasicek
matplot(t, r[,1:n], type="l", lty=1,
        main="Seven Interest Rate Paths in Vasicek process",
        xlab="time",ylab="r(t)",col=1)
abline(h=b, col="red", lty=2)
lines(t, rT.expected, lty=2)
lines(t, rT.expected + 2*rT.stdev, lty=2)
lines(t, rT.expected - 2*rT.stdev, lty=2)
points(0,r0)
```

A.2   Code Generates Figure 1.2

```
# Simulate the stock price with GBM and Vasicek interest rate

# parameters in the Vasicek model
r0 <- 0.005  # the initial interest rate
b <- 0.005   # objective of mean-reverting
b1 <- 0.05
b2 <- -0.05
a <- 0.5     # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian

# parameter in the Geometric Brownian motion
sigma <- 0.3  # scaling the variance of the Brownian motion

S0 = 100         # the initial stock price

T=10   # the total time interval of our simulation
```

```
n=30    # the amount of simulation trials
m=1000 # amount of sub-time-intervals
dt <- T/m  # difference in time for each subinterval

S = matrix(0,m+1,n) # matrix to hold stock price paths
S[1,] = S0

S1 = matrix(0,m+1,n) # matrix to hold stock price paths, higher b
S1[1,] = S0

S2 = matrix(0,m+1,n) # matrix to hold stock price paths, lower b
S2[1,] = S0

S3 = matrix(0,m+1,n) # matrix to hold stock price paths, static r
S3[1,] = S0


# In our simulation, the interest rate parameter is dynamic.
# So we must simulate the interest rate first.
# And apply this sequence to simulate the stock price

r <- matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] <- r0  # put initial values into the first line

# matrix to hold high short rate paths, higher b
r1 <- matrix(0,m+1,n)
r1[1,] <- r0  # put initial values into the first line

# matrix to hold high short rate paths, lower b
r2 <- matrix(0,m+1,n)
r2[1,] <- r0  # put initial values into the first line

r3<- matrix(r0,m+1,n)  # matrix to hold short rate paths

set.seed(1029)
# Brownian motions for r
Br = matrix(rnorm(m*n,0,1), ncol = n, byrow = T)
# Brownian motions for stock price
Bs = matrix(rnorm(m*n,0,1), ncol = n, byrow = T)

# generate the sequence of dynamic interest rates
for(j in 1:n){
  for(i in 2:(m+1)){
    dr <- a*(b-r[i-1,j])*dt + theta*sqrt(dt)*Br[i-1,j]
    r[i,j] <- r[i-1,j] + dr
  }
}

# generate the sequence of dynamic interest rates, higher b
for(j in 1:n){
  for(i in 2:(m+1)){
```

```
    dr1 <- a*(b1-r1[i-1,j])*dt + theta*sqrt(dt)*Br[i-1,j]
    r1[i,j] <- r1[i-1,j] + dr1
  }
}

# generate the sequence of dynamic interest rates, lower b
for(j in 1:n){
  for(i in 2:(m+1)){
    dr2 <- a*(b2-r2[i-1,j])*dt + theta*sqrt(dt)*Br[i-1,j]
    r2[i,j] <- r2[i-1,j] + dr2
  }
}

# Simulation of the stock price with GBM considering Vasicek model
for(j in 1:n){
  for(i in 2:(m+1)){
    dS <- r[i-1,j]*S[i-1,j]*dt + sigma*sqrt(dt)*S[i-1,j]*Bs[i-1,j]
    S[i,j] <- S[i-1,j] + dS
  }
}

# Same simulation with higher b
for(j in 1:n){
  for(i in 2:(m+1)){
    dS1 <- r1[i-1,j]*S1[i-1,j]*dt + sigma*sqrt(dt)*S1[i-1,j]*Bs[i-1,j]
    S1[i,j] <- S1[i-1,j] + dS1
  }
}

# Same simulation with lower b
for(j in 1:n){
  for(i in 2:(m+1)){
    dS2 <- r2[i-1,j]*S2[i-1,j]*dt + sigma*sqrt(dt)*S2[i-1,j]*Bs[i-1,j]
    S2[i,j] <- S2[i-1,j] + dS2
  }
}

# Same simulation with static interest rate
for(j in 1:n){
  for(i in 2:(m+1)){
    dS3 <- r3[i-1,j]*S3[i-1,j]*dt + sigma*sqrt(dt)*S3[i-1,j]*Bs[i-1,j]
    S3[i,j] <- S3[i-1,j] + dS3
  }
}
TimeAxis <- seq(0, T, dt)
matplot(TimeAxis, S, type="l",lty=1,
        main="Thirty Stocks' Prices", ylab="Stock Price",
        xlab="time", col=1,
        ylim=c(0,600))
matplot(TimeAxis, S3, type="l",lty=1,
        main="Thirty Stocks' Prices with static rate",
```

```
        ylab="Stock Price", xlab="time", col=1,
        ylim=c(0,600))
matplot(TimeAxis, S1, type="l",lty=1,
        main="Thirty Stocks' Prices with higher rate",
        ylab="Stock Price", xlab="time", col=1,
        ylim=c(0,600))
matplot(TimeAxis, S2, type="l",lty=1,
        main="Thirty Stocks' Prices with lower rate",
        ylab="Stock Price", xlab="time", col=1,
        ylim=c(0,600))
```

A.3   Code Generates Figure 1.3

```
ZeroCoupon <-   # run the Z(t,T), t=0 in our simulation
  function(r0, a, b, theta, T){
    A = (1 - exp(-a*T))/a
    B = (A-T) * ( b - theta^2/(4*a^2) ) - theta^2/(8*a) * A^2
    return(exp(-A*r0 + B))
  }

## define model parameters for plotting yield curves

b <- 0.005   # objective of mean-reverting
a <- 0.5     # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian

T = 10
m = 1000
dt = T/m

r0 <- seq(-0.03, 0.15, 0.03) # 7 initial interest rates
n <- length(r0)
yield <- matrix(0, m+1, n)  # sequence for yield
Z <- matrix(0, m+1, n)      # sequence for Zero-Coupon's price
for(i in 1:n){
  for(j in 1:(m+1)){
    Z[j,i] <- ZeroCoupon(r0[i], a, b, theta, j*dt)
    yield[j,i] <- -log(ZeroCoupon(r0[i], a, b, theta, j*dt))/(j*dt)
  }
}

t <- seq(0, T, dt)
matplot(t, yield, type="l", col="black", lty=1,
        main="Seven ZeroCoupons' Yield Curves", xlab="time")
abline(h=b, col="black", lty=2)
```

A.4 Code Generates Figure 1.4 and 1.5

```
 # parameters in the Vasicek model
r0 <- 0.005  # the initial interest rate  (Norges Bank's current interest rate)
b <- 0.005   # objective of mean-reverting (Norges Bank's current interest rate)
a <- 0.5     # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian

ZeroCoupon <-   # run the Z(t,T), t=0 in our simulation
  function(r0, a, b, theta, T){
    A = (1 - exp(-a*T))/a
    B = (A-T) * ( b - theta^2/(4*a^2) ) - theta^2/(8*a) * A^2
    return(exp(-A*r0 + B))
  }



# parameter in the Geometric Brownian motion
sigma <- 0.3  # scaling the variance of the Brownian motion

S0 = 100         # the initial stock price

T=10   # the total time interval of our simulation
n=30    # the amount of simulation trials
m=1000 # amount of sub-time-intervals
dt <- T/m  # difference in time for each subinterval

ZeroCouponPrices <- matrix(0, m+1, n)       # sequence for Zero-Coupon's price
for(i in 1:n){
  for(j in 1:(m+1)){
    ZeroCouponPrices[j,i] <- ZeroCoupon(r0, a, b, theta, j*dt)
  }
}

S = matrix(0,m+1,n) # matrix to hold stock price paths
S[1,] = S0

# In our simulation, the interest rate parameter is dynamic. So we must simulate
# the interest rate first. And apply this sequence to simulate the stock price

r <- matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] <- r0  # put initial values into the first line

set.seed(1016)
for(j in 1:n){ # generate the sequence of dynamic interest rates
  for(i in 2:(m+1)){
    dr <- a*(b-r[i-1,j])*dt + theta*sqrt(dt)*rnorm(1,0,1)
    r[i,j] <- r[i-1,j] + dr
  }
}

# Simulation of the stock price with GBM considering Vasicek model
```

```
for(j in 1:n){
  for(i in 2:(m+1)){
    dS <- r[i-1,j]*S[i-1,j]*dt + sigma*sqrt(dt)*S[i-1,j]*rnorm(1,0,1)
    S[i,j] <- S[i-1,j] + dS
  }
}

F = matrix(0,m+1,n)  # the matrix holding futures price consisting of ZeroCoupon
CFD = matrix(0,m+1,n)
# prices and the stock prices
F = S / ZeroCouponPrices
CFD = F*ZeroCouponPrices*(1-ZeroCouponPrices)
CFD_payoff = (S-S0)*(1-ZeroCouponPrices)
TimeAxis <- seq(0, T, dt)

matplot(TimeAxis, S, type="l",lty=1,
        main="Thirty Stocks' Prices", ylim=c(0,600),
        ylab="Stock Price", xlab="time", col=1)
matplot(TimeAxis, F, type="l",lty=1,
        main="Thirty Futures' Prices", ylim=c(0,600),
        ylab="Futures Prices", xlab="time", col=1)
matplot(TimeAxis, CFD, type="l",lty=1,
        main="Thirty CFDs' prices",
        ylab="Payoff", xlab="time", col=1)
matplot(TimeAxis, 1-ZeroCouponPrices, type="l",lty=1,
        main="1-Z(t,T)",
        ylab="1-Z(0,t)", xlab="time", col=1)

matplot(TimeAxis, S, type="l",lty=1,
        main="Thirty Stocks' Prices (logarithmic coordinates)",
        ylab="Stock Price", xlab="time", col=1, log="y")
matplot(TimeAxis, F, type="l",lty=1,
        main="Thirty Futures' Prices (logarithmic coordinates)",
        ylab="Futures Prices", xlab="time", col=1, log="y")
matplot(TimeAxis, CFD, type="l",lty=1,
        main="Thirty CFDs' prices (logarithmic coordinates)",
        ylab="Payoff", xlab="time", col=1, log="y")
matplot(TimeAxis, 1-ZeroCouponPrices, type="l",lty=1,
        main="1-Z(t,T) (logarithmic coordinates)",
        ylab="1-Z(0,t)", xlab="time", col=1, log="y")
```

A.5   Code Generates Figure 2.1

```
 # First, we should have a path of Drifted Brownian motion

# Pamameters
mu = 0.005
eta = 0.02 # scaling the variance of Brownian motion for X

# we can change the values as we want
```

```
rho_1 = 0.95  # correlation bewteen V(.) for X, and V_1(.) for Y_1 (area)
rho_2 = 0.05  # correlation bewteen V(.) for X, and V_2(.) for Y_2 (system)
# Remark the correlation between V_1(.) and V_2(.) is
# rho_12 = rho_1 * rho_2 + sqrt(1-rho_1) * sqrt(1-rho_2)
rho_12 = rho_1 * rho_2 + sqrt(1-rho_1) * sqrt(1-rho_2)


n <- 30     # amounts of simulation trials??
T <- 10     # total time-interval
m <- 1000   # amount of sub-time-intervals
dt <- T/m   # difference in time for each subinterval


# And we need 3 pairwise correlated Brownian motions: V, V_1 and V_2
# V_1 = rho_1*V + sqrt(1-rho_1)*V_ind
# V_2 = rho_2*V + sqrt(1-rho_2)*V_ind
# Because we have n trials, so we should have 3 matrices


# the m paths of Brownian motion for X
set.seed(1006)
V = matrix(rnorm(m*n,0,1), ncol = m, byrow = T)
# the m paths of the independent Brownian motion
V_ind = matrix(rnorm(m*n,0,1), ncol = m, byrow = T)
# the m paths of Brownian motion for Y_1
V_1 = rho_1*V + sqrt(1 - rho_1) * V_ind
V_2 = rho_2*V + sqrt(1 - rho_2) * V_ind


X0 = 1      # Initial value
X <- matrix(0,m+1,n)  # matrix to hold Drifted Brownian motion's path
X[1,] <- X0  # put initial values into the first line


# Generate the paths
for(j in 1:n){
  for(i in 2:(m+1)){
    dX <- mu*dt + eta*sqrt(dt)*V[j,i-1]
    X[i,j] <- X[i-1,j] + dX
  }
}


# X has n paths of Drifted Brownian motions
# ----------------------------------


# Then, we should have two Ornstein Uhlenbeck processes, one for area-price
# one for system-price. We assume the area-price has bigger variance.
mu_1 = 0.005
eta_1 = 0.06 # scaling the variance of Gaussian, for area price


Y_area_init = 0.5
Y_area <- matrix(0,m+1,n)  # matrix to hold OU-process
Y_area[1,] <- Y_area_init  # put initial values into the first line


for(j in 1:n){
  for(i in 2:(m+1)){
```

```r
    dY <- -mu_1 * Y_area[i-1,j] * dt + eta_1 * sqrt(dt) * V_1[j,i-1]
    Y_area[i,j] <- Y_area[i-1,j] + dY
  }
}


mu_2 = 0.005
eta_2 = 0.02 # scaling the variance of Gaussian, for system price

Y_sys_init = 0.5
Y_sys <- matrix(0,m+1,n)  # matrix to hold OU-process
Y_sys[1,] <- Y_sys_init  # put initial values into the first line

for(j in 1:n){
  for(i in 2:(m+1)){
    dY <- -mu_2 * Y_sys[i-1,j] * dt + eta_2 * sqrt(dt) * V_2[j,i-1]
    Y_sys[i,j] <- Y_sys[i-1,j] + dY
  }
}


# Both Y_area and Y_sys have n paths of OU-process

# Generate paths of area-price and system-price with X, Y_area and Y_sys
S_area = exp(X+Y_area)
S_sys = exp(X+Y_sys)



# Copy from the ZeroCouponPrice.R
## function to find ZCB price using Vasicek model

r0=0 # intinal interest rate
ZeroCoupon <-   # run the Z(t,T), t=0 in our simulation
  function(r0, a, b, theta, T){
    A = (1 - exp(-a*T))/a
    B = (A-T) * ( b - theta^2/(4*a^2) ) - theta^2/(8*a) * A^2
    return(exp(-A*r0 + B))
  }

## define model parameters for plotting yield curves

r0 <- 0.005  # the initial interest rate  (Norges Bank's current interest rate)
b <- 0.005   # objective of mean-reverting (Norges Bank's current interest rate)
a <- 0.5     # controll the tendency of reverting
theta <- 0.01 # adjust the variance of Gaussian

ZeroCouponPrices <- matrix(0,m+1,n)
ZeroCouponPrices[1,] <- ZeroCoupon(r0=r0, a, b, theta, 0)

for(j in 1:n){
  for(i in (2:m+1)){
    ZeroCouponPrices[i,j] <- ZeroCoupon(r0, a, b, theta, i*T/m)
  }
```

```
}

# The matrix ZeroCouponPrices, is what we need

# Generate n paths of EPAD
#EPAD = (S_area - S_sys) * ZeroCouponPrices

TimeAxis <- seq(0, T, dt)

# expectation of the EPAD
TimeLines = matrix(rep(TimeAxis,n), nrow=m+1) # we need n time lines for n trials
C_1 = exp(X + mu*TimeLines + Y_area*exp(-mu_1*TimeLines))
C_2 = exp(X + mu*TimeLines + Y_sys*exp(-mu_2*TimeLines))
a1 = 1
a2 = rho_1
a3 = sqrt(1-rho_1^2)
a4 = rho_2
a5 = sqrt((rho_12 - rho_1*rho_2)/(1-rho_1^2))
a6 = sqrt(1 - rho_2^2 - (rho_12 - rho_1*rho_2)^2 / (1 - rho_1^2))


omega1 = sqrt(eta^2*(TimeLines)
              + 2*a2 * eta * eta_1 * (1-exp(-mu_1*TimeLines))/mu_1
              + a2^2 * eta_1^2 * (1 - exp(-2*mu_1*TimeLines))/(2*mu_1) )
omega2 = sqrt(a3^2 * eta_1^2 * (1 - exp(-2*mu_1*TimeLines))/(2*mu_1) )
omega3 = sqrt(eta^2*(TimeLines)
              + 2*a4 * eta * eta_2 * (1-exp(-mu_2*TimeLines))/mu_2
              + a4^2 * eta_2^2 * (1 - exp(-2*mu_2*TimeLines))/(2*mu_2) )
omega4 = sqrt(a5^2 * eta_2^2 * (1 - exp(-2*mu_2*TimeLines))/(2*mu_2) )
omega5 = sqrt(a6^2 * eta_2^2 * (1 - exp(-2*mu_2*TimeLines))/(2*mu_2) )

EPAD.RiskNeutral = ZeroCouponPrices * (C_1*exp(0.5*(omega1^2 + omega2^2))
                                   - C_2*exp(0.5*(omega3^2 + omega4^2 + omega5^2)))

# Generate price of futures on the system-price
Future = C_2*exp(0.5*(omega3^2 + omega4^2 + omega5^2))

matplot(TimeAxis ,EPAD.RiskNeutral[,1:n], type="l", lty=1, col=1,
        main="Thirty Paths of risk-neural price of EPAD",
        ylab="EPAD.RiskNeutral(t)", xlab="TimeAxis", ylim=c(-1.5,2.5))
```

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List of Figures