ON A SELECTION PROBLEM FOR SMALL NOISE PERTURBATION IN THE MULTIDIMENSIONAL CASE

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ABSTRACT. The problem on identification of a limit of an ordinary differential equation with discontinuous drift that perturbed by a zero-noise is considered in multidimensional case. This problem is a classical subject of stochastic analysis, see, for example, [6, 29, 11, 20]. However the multidimensional case was poorly investigated. We assume that the drift coefficient has a jump discontinuity along a hyperplane and is Lipschitz continuous in the upper and lower half-spaces. It appears that the behavior of the limit process depends on signs of the normal component of the drift at the upper and lower half-spaces in a neighborhood of the hyperplane, all cases are considered.

Zero-noise limit; Peano phenomenon.
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1. INTRODUCTION

Consider the Cauchy problem

$$X_t = x + \int_0^t b(X_s)ds, t \geq 0 \tag{1.1}$$

for $x \in \mathbb{R}^d$, where $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable vector field.

If b satisfies a Lipschitz and linear growth condition, it is well-known that there exists a global unique solution $X_\cdot \in C([0,\infty);\mathbb{R}^d)$ to (1.1).

However, if b is not Lipschitzian, the situation may change dramatically and well-posedness of (1.1) in the sense of uniqueness or even existence of solutions may fail.

An example of such a function is given by

$$b(x) = 2sgn(x)\sqrt{|x|} \tag{1.2}$$

for $X_0 = 0$ and dimension $d = 1$, where the extremal trajectories $X_t = +t^2, -t^2$ and the zero curve $X_t = 0, t \geq 0$ are solutions to (1.1) among infinitely other ones.

Another example in the case of a discontinuous vector field is

$$b(x) = sgn(x) \tag{1.3}$$

for $X_0 = 0$ with infinitely many solutions, where $X_t = +t, -t, t \geq 0$ are extremal solutions.

If we merely require that the vector field b is continuous, satisfying a growth condition of the form $\langle b(x), x\rangle \leq K(|x|^2 + 1)$ then it follows from Peano's theorem and the theorem of Arzelà-Ascoli that the set $C(x)$ of solutions $X_\cdot \in C([0,\infty);\mathbb{R}^d)$ of (1.1) is non-empty and compact in $C([0,\infty);\mathbb{R}^d)$. Moreover, $C(x)$ is connected. See [28].

Here the problem of uniqueness of solutions of (1.1) for an initial distribution μ_0 on $\mathbb{R}^d$, which corresponds to the case, when $C(x)$ is a singleton for x μ_0-a.e., can be characterized by unique solutions of narrowly mesurable families of probability measures $(\mu_t)_{t\geq 0}$ on $C([0,\infty);\mathbb{R}^d)$ satisfying the continuity equation

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(b \cdot \nabla f)ds, t \geq 0 \tag{1.4}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ (space of smooth functions with compact support). It turns out that such solutions have the representation $\mu_t = \pi_t \mu, t \geq 0$ for projections π_t and a probability measure μ on $C([0,\infty);\mathbb{R}^d)$ called a superposition of solutions of the ODE (1.1). See [13], [1] and [15] for more information on the concept of superposition of solutions.

Using the concept of renormalized solutions, we mention that for $b \in W^{1,1}(\mathbb{R}^d)$ with $div(b) = 0$ and initial distributions μ_0 with $\frac{\partial \mu_0}{\partial x} \in L^\infty(\mathbb{R}^d)$ it can be shown that the continuity equation has a unique solution $(\mu_t)_{t\geq 0}$ in the subclass of solutions for which $\frac{\partial \mu_t}{\partial x} \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$. See [13], [1] and [2].

On the other hand, the case, when $C(x)$ is not a singleton, gives rise to the natural question of how an "appropriate" or "meaningful" solution to (1.1) can be selected.

One important method in connection with this selection problem is due to Krylov [21], who constructed Markov selections, that is families of superposition solutions $(\mu^x)_{x\in\mathbb{R}^d}$ such that

$$\mu_{t+s}^x = \int_{\mathbb{R}^d} \mu_s^y \mu_t^x(dy), t, s \geq 0, x \in \mathbb{R}^d$$

holds for $\mu_t^x = \pi_t \mu^x$.

Another crucial approach which we want to employ in this paper is that of zero-noise selection, that is the selection of a solution $X_\cdot$ to (1.1) as a limiting value of solutions $X_\cdot^\varepsilon$ of the ODE (1.1) perturbed by a small noise $\varepsilon w(\cdot)$ given by the stochastic differential equation (SDE)

$$X_t^\varepsilon = x + \int_0^t b(X_s)ds + \varepsilon w(t) \tag{1.5}$$

for $\varepsilon \searrow 0$ in the sense of convergence in law, where $w(\cdot)$ is a d-dimensional Wiener process. The motivation for this selection principle comes from the desire to construct solutions to (1.1), which are stable under random perturbations.

The first results in this direction, that is when $C(x)$ is not a singleton ("Peano phenomenon"), was obtained in the foundational papers of Bafico [5] and Bafico, Baldi [6] in the case of one-dimensional time-homogeneous vector fields b, where the authors prove under certain conditions the existence of a unique limiting law on a small time interval which is concentrated on at most two trajectories. The proof of the latter results relies on estimates of mean exit times of $X_\cdot^\varepsilon$ with respect to (small) neighbourhoods of isolated singular points of b by means of solutions of an associated boundary value problem. We also mention the papers [17], [18], where the authors use large deviation techniques to study the convergence rate of the laws of $X_\cdot^\varepsilon$ for a concrete class of one-dimensional time-homogeneous functions b related to (1.2). In this context it is also worth mentioning the work of [9], which among other things deals with the study of the small noise problem (1.5) based on viscosity solutions of (perturbed) backward Kolmogorov equations in the scalar case. See also the Malliavin calculus approach in [25] and the article [29] based on local time techniques.

We also remark that extensions of the paper [6] to the case of zero-noise limits of linear transport equations associated with the one-dimensional ODE

$$dX_t = 2sgn(X_t)\,|X_t|^\gamma\,, \gamma \in (0,1)$$

were analyzed in [3], [4]. See also [14], [24] in the case of zero-noise limits of non-linear PDE's.

Let us now have a look at the small noise problem (1.5) in the multidimensional case. In fact the muldimensional problem is scarcely treated in the literature. Here we shall distinguish between the continuous and discontinuous vector fields:

In the case of bounded and continuous functions b Zhang [31] gives a characterization of the limiting values $X_\cdot^\varepsilon$ of (1.5) by using viscosity solutions of Hamilton-Jacobi-Bellman equations in connection with so-called exit time functions, which partially extends results in [6] to the multidimensional setting.

To the best of our knowledge, the case of discontinuous multidimensional vector fields b has been only examined in the papers of Delarue, Flandoli, Vincenzi [12] and [8]. In the remarkable work [12] the authors study small noise perturbations of the Vlasov-Poisson equation by means of estimates of probabilities for exit times in connection with a zero-noise limit for ODE's in four dimensions. The paper [8] deals with ODE's (1.1) for merely measurable b. However, the concept of solutions to (1.1) in the latter work is in the sense of Filippov, which we don't want to consider in this article.

The objective of our paper is the analysis of zero-noise limits in the case of discontinuous time-inhomogeneous vector fields b in $\mathbb{R}^d$. More precisely, we aim at considering vector fields b, whose discontinuity points are located in a hyperplane. Our method for the construction of zero-noise limits, which is different from the techniques of the above mentioned authors, is based on estimates of probabilities for exit times of $X_\cdot^\varepsilon$ at discontinuity points in a hyperplane. We comment that our approach extends the one in [6] to the multidimensional case. In contrast to [6] our technique does not require knowledge of the explicit distribution of $X_\cdot^\varepsilon$. We in fact show that the behavior of the limiting process depends on the normal component of the drift at the upper and lower half-spaces in a neighbourhood of the hyperplane.

2. Framework and formulation of the problem

Consider an ODE

$$\frac{dX(t)}{dt} = b(t, X(t)),\ t \geq 0, \tag{2.1}$$

or its integral version

$$X(t) = x^0 + \int_0^t b(s, X(s))ds,\ t \geq 0, \tag{2.2}$$

where $x^0 \in \mathbb{R}^d$.

Assume that

$$b(t,x) = \begin{cases} b^+(t,x),\ x_d \geq 0 \\ b^-(t,x),\ x_d < 0 \end{cases} = b^+(t,x)1\!\!1_{x_d\geq 0} + b^-(t,x)1\!\!1_{x_d<0} = b^+(t,x)1\!\!1_{x\in\mathbb{R}^d_+} + b^-(t,x)1\!\!1_{x\in\mathbb{R}^d_-},$$

where $x = (x_1, \ldots, x_d) = (\bar{x}, x_d)$, $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times [0, \infty)$, $\mathbb{R}^d_- = \mathbb{R}^{d-1} \times (-\infty, 0]$, $b^\pm$ are continuous functions that satisfy a Lipschitz condition in x on $\mathbb{R}^d$.

Denote by H the hyper-plane $\mathbb{R}^{d-1} \times \{0\}$.

Since b is Lipschitz continuous in x outside of H, equation (2.2) has a unique solution up to the moment τ_H of hitting H,

$$\tau_H := \inf\{t \geq 0 : \; X(t) \in H\}.$$

However, uniqueness and existence of a solution may fail after the moment τ_H. It is not difficult to find examples with no solution to (2.2) or examples with multiple solutions for $t \geq \tau_H$.

Consider an SDE in $\mathbb{R}^d$ with a small noise parameter

$$X^\varepsilon(t) = x^\varepsilon + \int_0^t b(s, X^\varepsilon(s))ds + \varepsilon w(t), \; t \geq 0, \tag{2.3}$$

where $\{w(t), t \geq 0, \}$ is a Wiener process, $\lim_{\varepsilon \to 0} x^\varepsilon = x^0$.

In contrast to the case of ODE (2.2), by Veretennikov's theorem [30] there exists a unique strong solution to SDE (2.3) even if the function b is just measurable and satisfies the linear growth condition in x:

$$\forall T > 0 \; \exists C = C(T) \; \forall t \in [0, T] \; \forall x \in \mathbb{R}^d \quad |b(t, x)| \leq C(1 + |x|).$$

Moreover, the solution is a strong Markov process. Therefore, if the limit $\lim_{\varepsilon \to 0} X^\varepsilon$ exists (in distributions or in any other sense), then it is natural to call this limit a natural solution to (2.2).

Note that X^ε spends zero time at the hyper-plane $H := \{x \in \mathbb{R}^d \; : \; x_d = 0\}$, so it does not matter how to define the drift coefficient if $x_d = 0$.

The following observation is useful for further considerations. For any initial condition $x^0 \in \mathbb{R}^d$ the sequence of distributions of $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ is weakly relatively compact in $C([0, \infty); \mathbb{R}^d)$.

For any limit point X of $\{X^\varepsilon\}$ as $\varepsilon \to 0+$ and for any $t_0 \geq 0$ the following equality holds a.s.

$$X(t) = X(t_0) + \int_{t_0}^t b(s, X(s))ds, t \in [t_0, \tau_{t_0, H}],$$

where $\tau_{t_0,H} = \inf\{t \geq t_0 : X(t) \in H\}$. Proof see in Appendix. To prove that the sequence $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ converges in distribution to a process X^0 as $\varepsilon \to 0+$, it is sufficient to show that for any sequence $\{\varepsilon_k\}$, $\lim_{k \to 0} \varepsilon_k = 0$ there exists a subsequence $\{\varepsilon_{k_l}\}$ such that $X^{\varepsilon_{k_l}} \Rightarrow X^0, \; l \to \infty$. Since the family $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ is weakly relatively compact, without loss of generality we may initially assume that $\{X^{\varepsilon_k}\}$ is already convergent. The problem is only to prove convergence to X^0 described in the corresponding theorems.

Equations (2.3) and (2.2) use values of $b^-(t, x)$ and $b^+(t, x)$, where x lies in the lower half-space and the upper half-space, respectively. It will be convenient to assume sometimes that $b^\pm$ are defined on the whole $\mathbb{R}^d$.

Denote by $\{X^\pm(t), t \in [0, T]\}$ and $\{X^{\pm,\varepsilon}(t), t \in [0, T]\}$ solutions of the integral equations

$$X^\pm(t) = x^0 + \int_0^t b^\pm(s, X(s))ds, t \geq 0, \tag{2.4}$$

$$X^{\pm,\varepsilon}(t) = x^\varepsilon + \int_0^t b^+(s, X^{\pm,\varepsilon}(s))ds + \varepsilon w(t), t \geq 0.$$

Obviously we have the equality $X(\cdot \wedge \tau_H) = X^+(\cdot \wedge \tau_H)$ if $x^0 \in \mathbb{R}^d_+$ and $X(\cdot \wedge \tau_H) = X^-(\cdot \wedge \tau_H)$ if $x^0 \in \mathbb{R}^d_-$.

Set

$$\bar{\tau}_H := \begin{cases} \inf\{t \geq 0 : \ X_d^+(t) < 0\}, & \text{if } x^0 \in \mathbb{R}^d_+ \setminus H, \\ \inf\{t \geq 0 : \ X_d^-(t) > 0\}, & \text{if } x^0 \in \mathbb{R}^d_- \setminus H, \end{cases}$$

$$\tau_H^{(\varepsilon)} := \inf\{t \geq 0 : \ X^\varepsilon(t) \in H\}.$$

The following lemma yields that if $x^0 \notin H$, then we have even the locally uniform convergence $X^\varepsilon \to X$ until the process X hits the hyperplane. Let $x^0 \notin H$. Then we have the following convergence with probability 1

$$X^\varepsilon(\cdot \wedge \tau_H) \to X(\cdot \wedge \tau_H), \ \varepsilon \to 0,$$

where X^ε, X are considered as random elements with values in the space of continuous functions $C([0,\infty), \mathbb{R}^d)$ with the topology of uniform convergence on compact sets.

Moreover, if

$$\tau_H = \bar{\tau}_H, \tag{2.5}$$

then $\lim_{\varepsilon\to 0} \tau_H^{(\varepsilon)} = \tau_H, \ \lim_{\varepsilon \to 0} X^\varepsilon(\tau_H^{(\varepsilon)}) = X(\tau_H)$ and we have convergence

$$X^\varepsilon(\cdot \wedge \tau_H^{(\varepsilon)}) \to X(\cdot \wedge \tau_H), \ \varepsilon \to 0, \quad \text{a.s.}$$

The proof is standard, we postpone it to the appendix and make some comments here.

If $\tau_H^\varepsilon < \infty$, then the process $\tilde{X}^\varepsilon(t) := X^\varepsilon(\tau_H^\varepsilon + t)$ satisfies an analogue of equation (2.3)

$$\tilde{X}^\varepsilon(t) = \tilde{x}^\varepsilon + \int_0^t b(s, \tilde{X}^\varepsilon(s))ds + \varepsilon \tilde{w}(t),$$

with a new initial data $\tilde{x}^\varepsilon := X^\varepsilon(\tau_H^\varepsilon)$ and a new Wiener process $\tilde{w}(t) := w(\tau_H^\varepsilon + t), t \geq 0$. So, in case of (2.5), without loss of generality we may assume from the very beginning that x^0 and x^ε belong to H. Note that equality (2.5) holds, for example, if

- $x^0 \in \mathbb{R}^d_+, \tau_H < \infty$, and $b_d^+(\tau_H, X(\tau_H)) < 0$
  or
- $x^0 \in \mathbb{R}^d_-, \tau_H < \infty$, and $b_d^-(\tau_H, X(\tau_H)) > 0$,

If $x^0 \in \mathbb{R}^d_+ \setminus H$ and

$$\forall t \geq 0 \, \forall x \in H : \ b_d^+(t,x) \geq 0 \tag{2.6}$$

then $\tau_H = \bar{\tau}_H = +\infty$ and therefore $X^\varepsilon \to X, \varepsilon \to 0$ a.s. in $C([0,\infty), \mathbb{R}^d)$. Indeed, by the Lipschitz condition we have the inequality

$$\forall t \geq 0 \, \forall x \in \mathbb{R}^d : \ |b_d(t,x)| \geq -L|x_d|.$$

So, $|X_d(t)| \geq |X_d(0)|e^{-Lt} = |x_d^0|e^{-Lt} > 0, \ t \geq 0$, and $\tau_H = +\infty$. Actually, similarly to the reasoning above it can be proved that if $x^0 \in \mathbb{R}^d \setminus H$, then $X(\tau_H) \notin D \times \{0\}$ for any open set $D \subset \mathbb{R}^{d-1}$ such that for some $\delta > 0$

$$\forall t \in [\tau_H - \delta, \tau_H] \, \forall \bar{x} \in D : \ b_d^+(t, (\bar{x}, 0)) \geq 0$$

The problem whether $\tau_H = \lim_{\varepsilon \to 0} \tau_H^\varepsilon$ and what is the limit of $\{X^\varepsilon\}$ if $\tau_H \neq \bar{\tau}_H$ is not trivial. It follows from the above reasoning that $\tau_H \neq \bar{\tau}_H$ for $x^0 \in \mathbb{R}^d_+$ may happen only if

$b_d^+(\tau_H, X(\tau_H)) = 0$ and in any neighborhood of the point $(\tau_H, X(\tau_H)) \in [0,\infty)\times H$ there is a point $(t,x) \in [0,\infty)\times H$ such that $b_d^+(t,x) < 0$. In some sense the function b_d^+ should be degenerate in $(\tau_H, X(\tau_H))$. Theoretically, there may be a case when $x^0 \in \mathbb{R}_+^d \setminus H$, the function X_+ only touches H at the instant τ_H, then goes up and never hits H again. If ω is such that $w_d(\tau_H) > 0$ then the process X^ε will be most likely above H for a long may be infinite time, $\lim_{\varepsilon\to 0}\tau_H^\varepsilon = \infty$, and $X^\varepsilon \to X^+$ in $C([0,\infty),\mathbb{R}^d)$ for this ω. (This is only non-rigorous suggestion!) However, if ω is such that $w_d(\tau_H) < 0$ and the function b_d^- is negative in $\mathbb{R}_-^d$, then $X^\varepsilon(\tau_H)$ may lie in the lower half space and may be pulled down by the drift b^- to the negative half-space. So, the limit for this ω should be $1\!\!1_{t\le\tau_H} X^+(t) + 1\!\!1_{t>\tau_H} X^-_{\tau_H, X^+(\tau_H)}(t)$, where $X^-_{\tau_H, X^+(\tau_H)}(t)$ is a solution to the equation

$$X^-_{\tau_H, X^+(\tau_H)}(t) = X^+(\tau_H) + \int_{\tau_H}^t b^-(s, X^-_{\tau_H, X^+(\tau_H)}(s))ds, t \ge \tau_H.$$

A particular example of such situation was considered in [12]. The general theory in degenerate cases is unstudied yet and is not in the scope of this paper.

Let us briefly describe cases considered in this paper and the corresponding limit processes. We assume that starting points x^0 and x^ε lie in H. As it was mentioned above (see Lemma 2), this does not reduce the generality in the most of situations.

We consider the following four types of behavior of vertical components $b_d^\pm$ of the drift at the hyperplane in a neighborhood of the starting point x^0.

Case 1 (repulsion from the hyperplane). Assume that

$$b_d^+(0,x^0) > 0 \text{ and } b_d^-(0,x^0) < 0. \tag{2.7}$$

Since,the drift is continuous, condition (2.7) means that the drift b points outwards H in a neighborhood of $(0,x^0)$; so solutions X^+ and X^-, defined in (2.4), immediately leave H to the upper and the lower half-space, respectively.

In this case, see Theorem 3 in §3, the limit process with certain probabilities moves as $X^\pm$ until the moment when the corresponding process (X^+ or X^-) reaches H again (this moment is strictly positive). This situation is similar to the one-dimensional case.

Case 2 (attraction to the hyperplane). Assume that

$$b_d^+(0,x^0) < 0 \text{ and } b_d^-(0,x^0) > 0. \tag{2.8}$$

In this case, the drift b points towards H in a neighborhood of $(0,x^0)$. So, if the ODE (2.1) starts close to the hyperplane H, then it reaches H fast. This observation hints us that the limit process cannot leave H. Theorem 5 in §5 states that the limit process slides along H (in a neighborhood of $(0,x^0)$) according to a certain ODE. This case differs from the one-dimensional one. In one-dimensional case, the only possibility for the limit is the process that do not move at all.

We also can treat weaker assumption than (2.8)

$$b_d^+(0,x^0) < 0 \text{ and } \exists T>0\ \exists\delta>0\ \forall t\in[0,T],\ x\in H, |x-x^0|<\delta:\ b_d^-(t,x)\ge 0 \tag{2.9}$$

or

$$b_d^-(0,x^0) > 0 \text{ and } \exists T>0\ \exists\delta>0\ \forall t\in[0,T],\ x\in H, |x-x^0|<\delta:\ b_d^+(t,x)\le 0, \tag{2.10}$$

i.e., we allow that one strict inequality in (2.8) at the point $(0, x^0)$ is replaced by the non-strict inequality in a neighborhood of $(0, x^0)$.

Case 3 (attraction/repulsion). Assume that either

$$b_d^+(0, x^0) > 0 \text{ and } b_d^-(0, x^0) \geq 0 \tag{2.11}$$

or

$$b_d^+(0, x^0) \leq 0 \text{ and } b_d^-(0, x^0) < 0. \tag{2.12}$$

In this case, see Theorem 4 in §4, the limit process behaves as X^+ or X^-, respectively, until the moment when the corresponding process reaches H again.

Note, that in (2.11) the function b_d^- may be even negative everywhere except of $(0, x^0)$.

Case 4 (critical). Assume that vertical coordinates of the drift equal zero in H in some neighborhood of $(0, x^0)$, i.e., there is $T > 0$ and $\delta > 0$ such that

$$b_d^+(t, x) = b_d^-(t, x) = 0 \tag{2.13}$$

for any $t \in [0, T],\ x \in H, |x - x^0| < \delta$.

This case is the most interesting one. The limit process may be random and not Markov. It satisfies certain SDE (in the neighborhood of $(0, x^0)$), see §6, that may depend on the vertical component $\{w_d(t), t \geq 0\}$ of the Wiener process. That is, the limit may depend on a nontrivial noise, despite the random perturbation $\varepsilon w(t)$ from (2.3) converges to 0 as $\varepsilon \to 0$.

All cases 1-4 consider assumptions on the drift at the point $(0, x^0)$ or at some neighborhood of $(0, x^0)$. That's why, all results will be formulated as results on convergence of processes stopped at the instant of exit from this neighborhood. The following statement on localization states that the behavior of the drift outsides a neighborhood of x^0 does not have impact on the limit before the exit time from this neighborhood. Let $\tilde{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function that satisfies the linear growth condition in x:

$$\forall T > 0\ \exists C = C(T)\ \forall x \in \mathbb{R}^d \quad |b(t, x)| \leq C(1 + |x|).$$

Assume that $b(t, x) = \tilde{b}(t, x)$ for all $t \in [0, T]$ and $x \in D$, where D is an open set. Let $\{\tilde{X}(t), t \geq 0\}$ be a solution to the SDE

$$\tilde{X}^\varepsilon(t) = x^\varepsilon + \int_0^t \tilde{b}(s, \tilde{X}^\varepsilon(s))ds + \varepsilon w(t),\ t \geq 0,$$

where $x^\varepsilon \in D$. Define the exit moments

$$\sigma_D^\varepsilon := \inf\{t \geq 0 \ :\ X^\varepsilon(t) \notin D\},\ \tilde{\sigma}_D^\varepsilon := \inf\{t \geq 0 \ :\ \tilde{X}^\varepsilon(t) \notin D\}.$$

Then the limits (in any sense) $\lim_{\varepsilon \to 0} X^\varepsilon(\cdot \wedge \sigma_D^\varepsilon \wedge T)$ and $\lim_{\varepsilon \to 0} \tilde{X}^\varepsilon(\cdot \wedge \tilde{\sigma}_D^\varepsilon \wedge T)$ are equal if at least one of them exists (in the corresponding sense). The proof follows from uniqueness of the solution and the fact that if coefficients of equations coincide in a domain, then solutions coincide until the exit from this domain (if coefficients are locally Lipschitz, then this is classical result on localization; if coefficients are measurable, then the result follows from reasoning of Veretennikov's paper [30]).

When we study the global analogues of cases 1-4, then usually we will have convergence in distribution in $C([0, \infty), \mathbb{R}^d)$. The following standard result states that under some natural assumptions, convergence in distribution in $C([0, \infty), \mathbb{R}^d)$ of processes yields convergence of processes stopped at exit times from open sets. Let $\{Y_n, n \geq 0\}$ be a sequence of continuous $\mathbb{R}^d$-valued stochastic processes, $Y^n \Rightarrow Y^0, n \to \infty$ in distribution (considered as random elements in $C([0,\infty), \mathbb{R}^d)$).

Define exit moments

$$\sigma_D^n := \inf\{t \geq 0 \ : \ Y^n(t) \notin D\}.$$

If $D \subset \mathbb{R}^d$ is an open set such that $P(\sigma_D^0 = \sigma_{\overline{D}}^0) = 1$, where $\overline{D}$ is the closure of D, then for any $T \in [0, \infty]$ we have convergence in distribution of triples

$$(Y^n(\sigma_D^n \wedge T), \sigma_D^n \wedge T, Y^n(\cdot \wedge \sigma_D^n \wedge T)) \Rightarrow (Y^0(\sigma_D^0 \wedge T), \sigma_D^0 \wedge T, Y^0(\cdot \wedge \sigma_D^0 \wedge T)), n \to \infty.$$

The proof see in Appendix.

Next statement shows that assumption $P(\sigma_D^0 = \sigma_{\overline{D}}^0) = 1$ is not restrictive, in general. Let Y be a continuous $\mathbb{R}^d$-valued stochastic process, $Y(0) = x^0$. Then for almost all $r > 0$ w.r.t. the Lebesgue measure we have $P(\sigma_{B_r(x^0)} = \sigma_{\bar{B}_r(x^0)}) = 1$, where

$$B_r(x^0) = \{x \in \mathbb{R}^d \ : \ |x - x^0| < r\}, \ \bar{B}_r(x^0) = \{x \in \mathbb{R}^d \ : \ |x - x^0| \leq r\}.$$

The proof see in Appendix.

We will apply localization results of Lemmas 2, 2, and reduce all considerations to the case when the global analogues of cases 1-4 are satisfied.

Indeed, consider, for example, case 1. It follows from (2.7) and continuity of $b^{\pm}$ that there is $c > 0, \delta > 0$, and $T > 0$ such that

$$b_d^+(t, x) \geq c \ \text{ and } \ b_d^-(t, x) < -c \tag{2.14}$$

for any $t \in [0, T], |x - x^0| < \delta$.

Let $\tilde{b}^{\pm}$ be any extensions of $b^{\pm}$ outside of the set $D := [0, T] \times \{x \in \mathbb{R}^d \ : \ |x - x^0| \geq \delta\}$ such that $\tilde{b}^{\pm}$ are continuous, satisfy Lipschitz condition in x, and

$$\tilde{b}_d^+(t, x) \geq c \ \text{ and } \ \tilde{b}_d^-(t, x) < -c \ \text{ for all } t \geq 0 \ \text{ and } \ x \in H.$$

If we identify the limit $\lim_{\varepsilon \to 0} \tilde{X}^\varepsilon(\cdot \wedge \tilde{\sigma}_D^\varepsilon \wedge T)$, then the limit $\lim_{\varepsilon \to 0} X^\varepsilon(\cdot \wedge \sigma_D^\varepsilon \wedge T)$ will be the same (we do this in Theorem 3). Moreover, if $\tilde{X}^\varepsilon \Rightarrow X^0$ and $P(\sigma_D^0 = \sigma_{\overline{D}}^0)) = 1$, then

$$X^\varepsilon(\cdot \wedge \sigma_D^\varepsilon \wedge T) = \tilde{X}^\varepsilon(\cdot \wedge \tilde{\sigma}_D^\varepsilon \wedge T) \Rightarrow X^0(\cdot \wedge \sigma_D^0 \wedge T).$$

So, in order to prove convergence $X^\varepsilon(\cdot \wedge \sigma_D^\varepsilon \wedge T) \Rightarrow X^0(\cdot \wedge \sigma_D^0 \wedge T)$, without loss of generality, we may initially assume that assumption (2.14) is satisfied for all $t \in [0, T], x \in \mathbb{R}^d$, instead of (2.7).

Moreover, it can be shown in case 1 that the limit process X will be not in H at the exit moment from D with probability 1 if δ is small. The prelimit equation for the solution after the exit moment has the form similar to (2.3), see the equation in Remark 2. Further we apply Lemma 2 and obtain the behavior of the limit process until it hits H again, etc.

We don't know how to identify the limit of $\{X^\varepsilon\}$ if the starting point $x^0 \in H$ is "critical-repulsive-attractive", i.e., if, for example, $b_d^+(0, x^0) \leq 0, \ b_d^-(0, x^0) = 0$ and any neighborhood of x^0 contains $x, y \in H$ such that $b_d^-(0, x) > 0$ and $b_d^-(0, y) < 0$. However, using Lemmas 2, 2, 2, 2 and Theorems 3, 4, 5, and 6 below, the limit of $\{X^\varepsilon\}$ can be identified until the instant when the limit process reaches such "critical-repulsive-attractive" point.

3. Repulsive Case

Consider the repulsive case (2.7). Let functions $X^{\pm}$ be defined in (2.4). They leave H to the positive half-space or negative half-space, respectively, and

$$\tau_H^{\pm} = \inf\{t > 0 \ : \ X^{\pm}(t) \in H\} \tag{3.1}$$

are strictly greater that 0.

Assume that (2.7) is satisfied. The distribution of $X^{\varepsilon}(\cdot \wedge \tau_H^+ \wedge \tau_H^-)$ converges weakly as $\varepsilon \to 0$ to the measure

$$p_- \delta_{X^-(\cdot \wedge \tau_H^+ \wedge \tau_H^-)} + p_+ \delta_{X^+(\cdot \wedge \tau_H^+ \wedge \tau_H^-)},$$

where

$$p_{\pm} = \frac{\pm b_d^{\pm}(0, x^0)}{b_d^+(0, x^0) - b_d^-(0, x^0)}.$$

Convergence in Theorem 3 could not be a.s. or be convergence in probability. Indeed, assume that a sequence $\{X^{\varepsilon}\}$ converges a.s. to a process $X^-(t)1\!\!1_{\Omega_-} + X^+(t)1\!\!1_{\Omega_+}$, where $\Omega_{\pm}$ are disjoint measurable sets, $P(\Omega_{\pm}) = p_{\pm}$. It can be shown that $\Omega_{\pm} \in \mathcal{F}_{0+}$, so their probabilities are either 0 or 1.

Proof. Without loss of generality, see Remark 2, we will assume that

$$\exists c > 0 \, \forall t \geq 0 \, \forall x \in \mathbb{R}^d \setminus H : \ \ \mathrm{sgn}(x_d) b_d(t, x) \geq c. \tag{3.2}$$

Let us estimate the time spent by X^{ε} in the neighborhood of H. By Ito-Tanaka formula, see [27], we have

$$|X_d^{\varepsilon}(t)| = \int_0^t \mathrm{sgn}(X_d^{\varepsilon}(s)) b_d(s, X^{\varepsilon}(s)) ds + \varepsilon \int_0^t \mathrm{sgn}(X_d^{\varepsilon}(s)) dw_d(s) + L_d^{\varepsilon}(t) =$$

$$= \int_0^t \mathrm{sgn}(X_d^{\varepsilon}(s)) b_d(s, X^{\varepsilon}(s)) ds + \varepsilon B^{\varepsilon}(t) + L_d^{\varepsilon}(t),$$

where B^{ε} is a new one-dimensional Brownian motion, L_d^{ε} is a non-decreasing, continuous process, $L_d^{\varepsilon}(0) = 0$.

Therefore, the pair $(|X_d^{\varepsilon}|, L_d^{\varepsilon})$ is a solution of Skorokhod's reflecting problem for the driving process $\xi_{\varepsilon}(t) = \int_0^t \mathrm{sgn}(X_d^{\varepsilon}(s)) b_d(s, X^{\varepsilon}(s)) ds + \varepsilon B^{\varepsilon}(t)$. Hence, see for example [26],

$$|X_d^{\varepsilon}(t)| = \xi_{\varepsilon}(t) - \min_{s \in [0,t]} \xi_{\varepsilon}(s).$$

Let $f, g \in C([0, \infty))$. Assume that the function $t \to g(t) - f(t)$ is non-negative and non-decreasing. Then

$$g(t) - \min_{s \in [0,t]} g(s) \geq f(t) - \min_{s \in [0,t]} f(s), \ \ t \geq 0. \tag{3.3}$$

Proof. Let $t > 0$ be fixed. Assume that the minimum $\min_{s \in [0,t]} f(s)$ is attained at a point $s^* \in [0, t]$. Then

$$\min_{s \in [0,t]} g(s) - \min_{s \in [0,t]} f(s) = \min_{s \in [0,t]} g(s) - f(s^*) \leq g(s^*) - f(s^*) \leq g(t) - f(t).$$

This implies (3.3). □

It follows from Lemma 3 and (3.2) that

$$|X_d^\varepsilon(t)| \geq (ct + \varepsilon B^\varepsilon(t)) - \min_{s\in[0,t]}(cs + \varepsilon B^\varepsilon(s)).$$

Denote $\sigma_{H_\delta}^\varepsilon := \inf\{t \geq 0 : \ |X_d^\varepsilon(t)| \geq \delta\}$.

Therefore

$$P(|X_d^\varepsilon(t)| \geq \delta, \ t \geq 2\delta/c) \to 1, \ \varepsilon \to 0+, \tag{3.4}$$

$$P(\sigma_{H_\delta}^\varepsilon > 2\delta/c) \to 0, \ \varepsilon \to 0+ . \tag{3.5}$$

Moreover if $\delta < 1 \wedge c/2M$, where $M = \max_{t\in[0,1], \ |x-x^0|\leq 1} |\bar{b}(t,x)|$, then

$$P(|\bar{X}(\sigma_{H_\delta}^\varepsilon) - \bar{x}^0| > 2\delta M/c) \to 0, \ \varepsilon \to 0+ . \tag{3.6}$$

It follows from (3.4), Lemma 2, and assumption (3.2) that for any limit point X we have with probability 1:

$$X(t) = x^0 + 1\!\!1_{\Omega_+}\int_0^t b^+(s, X(s))ds + 1\!\!1_{\Omega_-}\int_0^t b^-(s, X(s))ds,$$

where $\Omega_+ = \{\omega : \ X(t) > 0 \text{ for all } t > 0\}, \ \Omega_- = \{\omega : \ X(t) < 0 \text{ for all } t > 0\}$.

Moreover

$$P(\Omega_+ \cup \Omega_-) = 1, \ P(\Omega_- \Delta \{X_d(\sigma_{H_\delta}) = -\delta\}) = 0, \text{ and } P(\Omega_+ \Delta \{X_d(\sigma_{H_\delta}) = \delta\}) = 0 \tag{3.7}$$

for any $\delta > 0$.

Let $\delta > 0, \ \beta > 0$ be sufficiently small fixed numbers and α be such that for all $t \in [0, \delta]$ and $x \in \mathbb{R}^d$, $|x - x^0| < \beta$:

$$0 < b_d^+(t,x) - \alpha < b_d^+(0,x^0) < b_d^+(t,x) + \alpha,$$

$$0 < -b_d^-(t,x) - \alpha < -b_d^-(0,x^0) < -b_d^-(t,x) + \alpha.$$

Define the processes

$$X^{\varepsilon,\pm\alpha}(t) = \int_0^t \left(b_d^+(0,x^0)1\!\!1_{X^{\varepsilon,+\alpha}(s)\geq 0} + b_d^-(0,x^0)1\!\!1_{X^{\varepsilon,+\alpha}(s)<0} \pm \alpha\right) ds + \varepsilon w(t), t \geq 0.$$

By comparison theorem, see [19], we have a.s.

$$X^{\varepsilon,-\alpha}(t) \leq X^d(t) \leq X^{\varepsilon,+\alpha}(t)$$

for all $t \in [0, \sigma_{H_\delta}^\varepsilon \wedge \inf\{s : |\bar{X}(s) - \bar{x}^0| > \beta\}]$.

The processes $X^{\varepsilon,\pm\alpha}$ are one-dimensional homogeneous diffusions. So, see [16, 19],

$$P(X_d^{\varepsilon,\pm\alpha}(\sigma_{H_\delta}^{\pm\alpha}) = \delta) =$$

$$\frac{(\pm\alpha - b_d^-(0,x^0))^{-1}(1 - e^{2\delta\varepsilon^{-2}(b_d^-(0,x^0)\pm\alpha)})}{(\pm\alpha - b_d^-(0,x^0))^{-1}(1 - e^{2\delta\varepsilon^{-2}(b_d^+(0,x^0)\pm\alpha)}) + (\pm\alpha + b_d^+(0,x^0))^{-1}(1 - e^{2\delta\varepsilon^{-2}(b_d^-(0,x^0)\pm\alpha)})},$$

where $\sigma_{H_\delta}^{\pm\alpha} := \inf\{t \geq 0 : \ |X^{\varepsilon,\pm\alpha}(t)| \geq \delta\}$.

This and (3.7) conclude the proof. □

4. Repulsive/Attractive Case

Let $x^0 \in H$ and (2.11) or (2.12) be satisfied. Then

$$X^\varepsilon(\cdot \wedge \tau_H^\pm) \to X^\pm(\cdot \wedge \tau_H^\pm),\ \varepsilon \to 0,\ \text{a.s.},$$

where "$+$" is selected for the case (2.11) and "$-$" is selected for (2.12), correspondingly; $\tau_H^\pm$ is defined in (3.1). If condition (2.12) or (2.11) is satisfied at the point $X^\pm(\tau_H^\pm)$, then we may define a moment $\tau_\pm^2$ similarly to $\tau_\pm^1 := \tau_H^\pm$ and obtain the similar convergence of processes $\{X^\varepsilon\}$ on $[\tau_\pm^1, \tau_\mp^2]$, and so on. Note that if (2.11) is satisfied in x^0, then (2.11) cannot be true in $X^+(\tau_+^1)$.

Proof. Assume that (2.11) is satisfied.

Then for any $\bar\alpha > 0$ there are $T_1 > 0, \delta_1 > 0$ such that

$$\forall t \in [0, T_1]\ \forall x,\ |x - x^0| \le \delta_1 :\ \ b_d^-(t, x) \ge -\bar\alpha$$

and there are $c > 0, T_2 > 0, \delta_2 > 0$ such that

$$\exists c > 0\ \forall t \in [0, T_2]\ \forall x,\ |x - x^0| \le \delta_2 :\ \ b_d^+(t, x) \ge c.$$

It follows from Lemmas 2, 2, 2, and 2 (see also Remarks 2 and 2) that without loss of generality we may assume that

$$\forall t \ge 0\ \forall x \in \mathbb{R}^d :\ \ b_d^+(t, x) \ge c$$

and

$$\forall t \ge 0\ \forall x \in \mathbb{R}^d :\ \ b_d^-(t, x) \ge -\bar\alpha.$$

Lemma 2 and Remark 2 yield that to prove the Theorem it suffices to verify that any limit point X of $\{X^\varepsilon\}$ as $\varepsilon \to 0$ is such that

$$\int_0^T 1\!\!1_{\{X_d(s) \le 0\}} ds = 0\ \text{ a.s.} \tag{4.1}$$

For any $\alpha > 0$ set

$$\widetilde{X}_d^{\varepsilon,\alpha}(t) = \int_0^t \left(-\alpha 1\!\!1_{\{\widetilde{X}_d^{\varepsilon,\alpha}(s) < 0\}} + c 1\!\!1_{\{\widetilde{X}_d^{\varepsilon,\alpha}(s) \ge 0\}}\right) ds + \varepsilon w_d(t).$$

By comparison theorem, $X_d^\varepsilon(t) \ge \widetilde{X}_d^{\varepsilon,\alpha}(t), t \ge 0$ a.s. if $\alpha > \bar\alpha$.

Let $\widetilde{X}_d^\alpha$ be a limit of $\{\widetilde{X}_d^{\varepsilon,\alpha}\}$ as $\varepsilon \to 0$. Then by Theorem 3

$$P(\widetilde{X}_d^\alpha(t) > 0,\ t > 0) = \frac{c}{c + \alpha}.$$

Therefore, any limit point X of $\{X^\varepsilon\}$ is such that for any $\alpha > \bar\alpha$ and any $\delta > 0$

$$P(X_d(t) \ge \delta,\ t \ge 2c/\delta) \ge \limsup_{\varepsilon \to 0} P(X_d^\varepsilon(t) \ge \delta,\ t \ge 2c/\delta)$$

$$\ge \limsup_{\varepsilon \to 0} P(\widetilde{X}_d^{\varepsilon,\alpha}(t) \ge \delta,\ t \ge 2c/\delta) \ge$$

$$P(\widetilde{X}_d^\alpha(t) \ge \delta,\ t \ge 3c/2\delta) = P(\widetilde{X}_d^\alpha(t) > 0,\ t > 0) = \frac{c}{c + \alpha}.$$

Since $\bar\alpha > 0$ and $\alpha > \bar\alpha$ were arbitrary, we have (4.1). □

5. ATTRACTIVE CASE

Assume that $x^0 \in H$, and one of conditions (2.8), (2.9), (2.10), or (2.13) is satisfied. Let X^0 be any (weak) limit point for $\{X^\varepsilon\}$.

Denote

$$\sigma_\delta = \sigma_\delta(X^0) := \inf\{t \geq 0 \ : \ |X^0(t) - x^0| \geq \delta\}. \tag{5.1}$$

Let $\delta > 0$ and $T > 0$ be such that

$$b_d^+(t, x) \leq 0, b_d^-(t, x) \geq 0$$

for any $t \in [0, T], \ x \in H, |x - x^0| < \delta.$

Then with probability 1 the process X^0 cannot leave the hyperplane H before the moment $\sigma_\delta \wedge T$:

$$P(X^0(t) \in H, t \in [0, \sigma_\delta \wedge T]) = 1.$$

Proof. We prove the lemma if the corresponding global condition (2.8), (2.9), (2.10), or (2.13) is satisfied. For example, in this case (2.13) is of the form

$$\forall t \geq 0 \, \forall x \in H : \ \ b_d^\pm(t, x) = 0.$$

Assume that $X^{\varepsilon_k} \Rightarrow X^0, k \to \infty$. By Skorokhod's theorem on a single probability space, we may assume that the convergence is a.s.:

$$\forall T > 0 : \ \sup_{t \in [0,T]} |X^{\varepsilon_k}(t) - X^0(t)| \to 0, \ k \to \infty. \tag{5.2}$$

Assume that for some ω and t_1 we have $X^0_d(t_1) > 0$. Denote by $t_0 \in [0, t_1]$ the instant of the last visit of H by X^0, i.e.,

$$t_0 = \sup\{s \in [0, t_1] : \ X^0_d(s) = 0\}.$$

Due to (5.2) and the Lebesgue dominated convergence theorem we have

$$X^0_d(t) = \int_{t_0}^t b_d^+(s, X^0(s))ds, \ t \in [t_0, t_1].$$

Since $X^0_d(t_0) = 0, X^0_d(t) \geq 0, \ t \in [t_0, t_1]$, the Lipschitz condition implies

$$b_d^+(s, X^0(s)) \leq b_d^+(s, (\bar{X}^0(s), 0)) + LX^0_d(s) \leq LX^0_d(s).$$

Thus

$$X^0_d(t) \leq L \int_{t_0}^t X^0_d(s)ds, \ t \in [t_0, t_1].$$

The application of Gronwall's lemma completes the proof. □

Assume that $x^0 = (\bar{x}^0, 0) \in H$ and (2.8) (or (2.9), or (2.10)) is satisfied, functions $b^\pm$ are continuous in (t, x). Denote

$$p^\pm(s, \bar{x}) = \frac{\pm b_d^\mp(s, (\bar{x}, 0))}{b_d^-(s, (\bar{x}, 0)) - b_d^+(s, (\bar{x}, 0))}.$$

Let $\bar{X}(t)$ be a solution of the ordinary differential equation in $\mathbb{R}^{d-1}$

$$\bar{X}^0(t) = \bar{x}^0 + \int_0^t \big(\bar{b}^+(s,(\bar{X}^0(s),0))p^+(s,\bar{X}^0(s)) +$$
$$+\bar{b}^-(s,(\bar{X}^0(s),0))p^-(s,\bar{X}^0(s))\big)\,ds,\ t\in[0,\sigma_\delta(X^0)\wedge T]. \tag{5.3}$$

Then

$$X^\varepsilon(\cdot\wedge\sigma_\delta(X^0)\wedge T)\xrightarrow{P}(\bar{X}^0(\cdot\wedge\sigma_\delta(X^0)\wedge T),0),\varepsilon\to 0,$$

Coefficients of the equation (5.3) are locally bounded and Lipschitz continuous in $\bar{x}$ in any neighborhood of $(0,x^0)$. Thus there exists a unique solution to (5.3).

Proof. We will only consider the case

$$\exists c>0\ \forall t\geq 0\ \forall x\in\mathbb{R}^d\setminus H:\ \ \mathrm{sgn}(x_d)b_d(t,x)\leq -c.$$

Since X^0 is non-random, the convergence in distribution is equivalent to the convergence in probability. Therefore we may prove the convergence in distribution only. Observe that equation (2.3) can be represented in the form

$$X^\varepsilon(t) = x^0 + \int_0^t b^+(s,X^\varepsilon(s))dl^\varepsilon_+(s) + \int_0^t b^-(s,X^\varepsilon(s))dl^\varepsilon_-(s) + \varepsilon w(t), \tag{5.4}$$

where $l^\varepsilon_\pm(t) = \int_0^t 1\!\!1_{\pm X^\varepsilon_d(s)>0}ds, t\geq 0$, is the time spent by X^ε in upper and lower half-spaces, respectively. Observe that a sequence of processes $\{l^\varepsilon_\pm\}_{\varepsilon>0}$ is weakly relatively compact. Let $\{\varepsilon_k\}$ be any subsequence such that $\lim_{k\to\infty}\varepsilon_k = 0$ and $\{(X^{\varepsilon_k},l^{\varepsilon_k}_\pm)\}$ is weakly convergent (see Lemma 2). By Skorokhod's theorem on a single probability space, we may assume that for almost all ω convergence (5.2) holds and also for any $T>0$ (see Lemma 5)

$$\sup_{t\in[0,T]}|\int_0^t b_d^+(s,X^{\varepsilon_k}(s))dl^{\varepsilon_k}_+(s) + \int_0^t b_d^-(s,X^{\varepsilon_k}(s))dl^{\varepsilon_k}_-(s)| = \tag{5.5}$$

$$\sup_{t\in[0,T]}|\int_0^t b_d(s,X^{\varepsilon_k}(s))ds| = \sup_{t\in[0,T]}|X^{\varepsilon_k}_d(t)| = \sup_{t\in[0,T]}|X^{\varepsilon_k}_d(t)-X^0_d(t)|\to 0, k\to\infty.$$

$$\sup_{t\in[0,T]}|l^{\varepsilon_k}_\pm(t)-l^0_\pm(t)|\to 0, k\to\infty, \tag{5.6}$$

Since $l^\varepsilon_+(t)+l^\varepsilon_-(t)=t$, convergence (5.5) implies that

$$\lim_{k\to\infty}\sup_{t\in[0,T]}|\int_0^t\big(b_d^+(s,X^{\varepsilon_k}(s))-b_d^-(s,X^{\varepsilon_k}(s))\big)\,dl^{\varepsilon_k}_+(s) + \int_0^t b_d^-(s,X^{\varepsilon_k}(s))ds| = 0. \tag{5.7}$$

The following result is well known. Assume that $\{f_n\},\ \{g_n\}\subset C([0,T])$ are uniformly convergent sequences of continuous functions

$$f_n\to f_0 \text{ and } g_n\to g_0,\ \ n\to\infty,$$

and each function g_n is non-decreasing.

Then we have the uniform convergence of the integrals

$$\sup_{t\in[0,T]}\left|\int_0^t f_n(s)dg_n(s) - \int_0^t f_0(s)dg_0(s)\right|\to 0,\ n\to\infty.$$

It follows from this Lemma, (5.6) and (5.7) that

$$\int_0^t \left(b_d^-(s,X^0(s)) - b_d^+(s,X^0(s))\right) dl_+^0(s) = \int_0^t b_d^-(s,X^0(s))ds$$

Hence

$$l_\pm^0(t) = \int_0^t \frac{\pm b_d^\mp(s,(\bar X^0(s),0))}{b_d^-(s,(\bar X^0(s),0)) - b_d^+(s,(\bar X^0(s),0))} ds,\ t\ge 0.$$

The application of Lemma 5 to (5.4) completes the proof. □

6. Critical Case

Consider the critical case (2.13).

Assume that the limits exist

$$c^\pm(t,\bar x) := \lim_{x_d\to 0\pm} \frac{b_d^\pm(t,x)}{x_d}, t\in[0,T], |\bar x - \bar x^0| < \delta.$$

They equal partial derivatives $\frac{\partial b_d^\pm(t,(\bar x,0))}{\partial x_d}$ if they exist. Since $b_d^\pm(t,x)=0, t\in[0,T], |x-x^0|<\delta$, and Lipschitz in x, functions $c^\pm$ are bounded for $t\in[0,T], |\bar x-\bar x^0|<\delta$. We assume that functions $c^\pm$ are extended for all t,x, and $c^\pm$ are measurable and are globally bounded.

Consider the system

$$\begin{cases} \bar X(t) = \bar x^0 + \int_0^t \left(\bar b^+(s,(\bar X(s),0))\mathbb{1}_{Y(s)\ge 0} + \bar b^-(s,(\bar X(s),0))\mathbb{1}_{Y(s)<0}\right) ds, \\ Y(t) = \int_0^t \Big(c^+(s,\bar X(s))\mathbb{1}_{Y(s)\ge 0} + c^-(s,\bar X(s))\mathbb{1}_{Y(s)<0}\Big)Y(s)ds + w_d(t). \end{cases} \tag{6.1}$$

1) There exists a unique weak solution to (6.1) defined up to the moment $\sigma_\delta(\bar X)\wedge T$, where $\sigma_\delta(\bar X) := \inf\{t\ge 0\ :\ |\bar X(t)-\bar x^0|\ge\delta\}$.

2) (a) There exists a unique strong solution to (6.1) defined up to the moment $\sigma_\delta(\overline{X})\wedge T$, if the functions $c^\pm(s,\overline{x}) = c^\pm(s)$ are independent of $\overline{x}$.

b) The system (6.1) has a unique maximal solution, if e.g. the functions $\overline{b}^\pm(s,(\overline{x},0)), c^\pm(s,\overline{x})$ belong to $C^{3,3}([0,T]\times\mathbb{R}^{d-1})$.

Proof. Observe that the function $C([0,T]) \ni F : y \to \bar x \in C([0,T])$ that is defined by equation

$$\bar x(t) = \bar x^0 + \int_0^t \left(\bar b^+(s,(\bar x(s),0))\mathbb{1}_{y(s)\ge 0} + \bar b^-(s,(\bar x(s),0))\mathbb{1}_{y(s)<0}\right) ds,\ t\in[0,T],$$

is a measurable non-anticipative mapping.

Hence, the second equation in (6.1) can be written as

$$Y(t) = \int_0^t a(Y,s)ds + w_d(t),$$

where

$$a(y,s) = \left(c^+(s,F(y)(s))\mathbb{1}_{y(s)\ge 0} + c^-(s,F(y)(s))\mathbb{1}_{y(s)<0}\right) y(s),\ s\in[0,T].$$

This equation has a unique weak solution by the Girsanov theorem, see Chapter VII, §2 in [23]. The process $\bar X$ is also defined uniquely by the formula $\bar X(t) = F(Y)(t), t\in[0,T]$.

Proof in the case 2a) is obvious because all coefficients are Lipschitz continuous in the spatial variable.

As for the proof of case 2b), see Theorem 3.2 in [22]. □

Assume that $x^0 \in H$, functions $c^{\pm}$ are continuous in (t,x), and assumption (2.13) is satisfied.

Then for almost all $\delta \in (0,\delta)$ we have the weak convergence
$\big(\bar X^\varepsilon(\cdot \wedge \sigma_{\delta_1}(\bar X^\varepsilon)\wedge T), \varepsilon^{-1}X^\varepsilon_d(\cdot \wedge \sigma_{\delta_1}(\bar X^\varepsilon)\wedge T)\big) \Rightarrow$

$$\Rightarrow \big(\bar X(\cdot \wedge \sigma_{\delta_1}(\bar X)\wedge T), Y(\cdot \wedge \sigma_{\delta_1}(\bar X)\wedge T)\big),\ \ \varepsilon\to 0,$$

where $(\bar X, Y)$ is a solution of (6.1).

In particular,

$$X^\varepsilon(\cdot\wedge\sigma_{\delta_1}(\bar X^\varepsilon)\wedge T) = \big(\bar X^\varepsilon(\cdot \wedge \sigma_{\delta_1}(\bar X^\varepsilon)\wedge T), X^\varepsilon_d(\cdot \wedge \sigma_{\delta}(\bar X^\varepsilon)\wedge T)\big) \Rightarrow \big(\bar X(\cdot \wedge \sigma_{\delta_1}(\bar X)\wedge T), 0\big).$$

Moreover, if there exists a strong solution to (6.1), then not only weak convergence holds but also convergence in probability. Weak uniqueness and strong existence yield uniqueness of the strong solution, see reasoning of [10].

Proof. Without loss of generality (see Lemmas 2 and 2) we may assume that

$$\forall t\ge 0\ \forall x\in H:\ \ b^{\pm}_d(t,x)=0. \tag{6.2}$$

Set $Y^\varepsilon(t) := X^\varepsilon_d(t)/\varepsilon$. Then

$$\begin{cases} \bar X^\varepsilon(t) = \bar x^0 + \int_0^t \big(\bar b^+(s,(\bar X^\varepsilon(s),\varepsilon Y^\varepsilon(s)))1\!\!1_{Y^\varepsilon(s)\ge 0} + \bar b^-(s,(\bar X^\varepsilon(s),\varepsilon Y^\varepsilon(s)))1\!\!1_{Y^\varepsilon(s)<0}\big)\,ds + \varepsilon\bar w(t),\\ Y^\varepsilon(t) = \int_0^t \Big(\frac{b_d^+(s,(\bar X^\varepsilon(s),\varepsilon Y^\varepsilon(s)))}{\varepsilon Y^\varepsilon(s)}1\!\!1_{Y^\varepsilon(s)\ge 0} + \frac{b_d^-(s,(\bar X^\varepsilon(s),\varepsilon Y^\varepsilon(s)))}{\varepsilon Y^\varepsilon(s)}1\!\!1_{Y^\varepsilon(s)<0}\Big)Y^\varepsilon(s)ds + w_d(t). \end{cases}$$

$\int_0^\infty 1\!\!1_{Y^\varepsilon(s)=0}ds = 0$ a.s. It can be readily shown that a family $\{(\bar X^\varepsilon, Y^\varepsilon), \varepsilon\in(0,1)\}$ is weakly relatively compact in $C([0,\infty);\mathbb{R}^d)$. Let $\{(\bar X^{\varepsilon_n}, Y^{\varepsilon_n})\}$ be a convergent subsequence, where $\lim_{n\to\infty}\varepsilon_n = 0$. By Skorokhod's theorem on a single probability space, there is a sequence of copies $(\tilde{\bar X}^{\varepsilon_n}, \tilde Y^{\varepsilon_n}, \tilde w^{\varepsilon_n}) \overset{d}{=} (\bar X^{\varepsilon_n}, Y^{\varepsilon_n}, w)$ such that

$$\begin{cases} \tilde{\bar X}^{\varepsilon_n}(t) = \bar x^0 + \int_0^t \Big(\bar b^+(s,(\tilde{\bar X}\varepsilon_n(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))1\!\!1_{\tilde Y^{\varepsilon_n}(s)\ge 0} + \\ \qquad\qquad \bar b^-(s,(\tilde{\bar X}^{\varepsilon_n}(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))1\!\!1_{\tilde Y^{\varepsilon_n}(s)<0}\Big)\,ds + \varepsilon_n\tilde{\bar w}^{\varepsilon_n}(t),\\ \tilde Y^{\varepsilon_n}(t) = \int_0^t \Big(\frac{b^+(s,(\tilde{\bar X}^{\varepsilon_n}(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))}{\varepsilon_n\tilde Y^{\varepsilon_n}(s)}1\!\!1_{\tilde Y^{\varepsilon_n}(s)\ge 0} + \\ \qquad\qquad \frac{b^-(s,(\tilde{\bar X}^{\varepsilon_n}(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))}{\varepsilon_n\tilde Y^{\varepsilon_n}(s)}1\!\!1_{\tilde Y^{\varepsilon_n}(s)<0}\Big)\tilde Y^{\varepsilon_n}(s)ds + \tilde w_d^{\varepsilon_n}(t). \end{cases}$$

and

$$(\tilde{\bar X}^{\varepsilon_n}, \tilde Y^{\varepsilon_n}, \tilde w^{\varepsilon_n}) \to (\tilde{\bar X}, \tilde Y, \tilde w),\ n\to\infty$$

almost surely.

It follows from (6.2), the Lipschitz condition and the uniform convergence of $\{\tilde Y^{\varepsilon_n}\}$ that

$$\lim_{n\to\infty}\int_0^t \Big(\frac{b_d^+(s,(\tilde{\bar X}^{\varepsilon_n}(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))}{\varepsilon_n\tilde Y^{\varepsilon_n}(s)}1\!\!1_{\tilde Y^{\varepsilon_n}(s)\ge 0} + \frac{b_d^-(s,(\tilde{\bar X}^{\varepsilon_n}(s),\varepsilon_n\tilde Y^{\varepsilon_n}(s)))}{\varepsilon_n\tilde Y^{\varepsilon_n}(s)}1\!\!1_{\tilde Y^{\varepsilon_n}(s)<0}\Big)\tilde Y^{\varepsilon_n}(s)ds$$

is locally Lipschitz function. Therefore $\tilde{Y}$ is of the form

$$\tilde{Y}(t) = \int_0^t \xi(s)ds + \tilde{w}_d(t),$$

where ξ is locally bounded and $\xi(t)$ is independent of σ-algebra generated by $\{(\tilde{w}_d(s) - \tilde{w}_d(t)), s \geq t\}$. So, $P(\tilde{Y}(s) = 0) = 0$, for any $s > 0$, and

$$1\!\!1_{\tilde{Y}^{\varepsilon_n}(s)\geq 0} \to 1\!\!1_{\tilde{Y}(s)\geq 0} \text{ and } 1\!\!1_{\tilde{Y}^{\varepsilon_n}(s)<0} \to 1\!\!1_{\tilde{Y}(s)<0} \text{ as } n \to \infty,$$

almost surely.

It follows from the Lebesgue dominated convergence theorem that the limit process $(\tilde{\bar{X}}, \tilde{Y})$ is a solution of (6.1) with $\tilde{w}_d$ in place of w_d. Since $\{(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n})\}$ was arbitrary convergent subsequence, the proof of the Theorem follows from the weak uniqueness of the solution to (6.1).

If there exists a unique strong solution to (6.1), consider then a.s. convergent sequence of copies of $(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n}, w, \bar{X}, Y)$:

$$(\tilde{\bar{X}}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}, \tilde{w}^{\varepsilon_n}, \hat{\bar{X}}^{\varepsilon_n}, \hat{Y}^{\varepsilon_n}) \to (\tilde{\bar{X}}, \tilde{Y}, \tilde{w}, \hat{\bar{X}}, \hat{Y}).$$

It can be seen that the limit processes $(\tilde{\bar{X}}, \tilde{Y})$ and $(\hat{\bar{X}}, \hat{Y})$ satisfy the same equation with the same Wiener process $\tilde{w}$. It follows from uniqueness of the strong solution that $(\tilde{\bar{X}}, \tilde{Y}) = (\hat{\bar{X}}, \hat{Y})$ a.s. So $(\tilde{\bar{X}}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}) - (\hat{\bar{X}}^{\varepsilon_n}, \hat{Y}^{\varepsilon_n}) \to 0$ a.s. Therefore $(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n}) - (\bar{X}, Y) \to 0$ in probability.

□

A limit of X^ε may be a non-Markov process in case (2.13). Indeed, assume that $\bar{b}^\pm(t,x) = \bar{b}^\pm = \text{const}, \bar{b}^+ \neq \bar{b}^-, b_d^\pm(t,x) = 0$, and $x^0 \in H$.

It follows from Theorem 6 that

$$X^\varepsilon(\cdot) \to (\bar{X}^0(\cdot), 0), \varepsilon \to 0,$$

where

$$\bar{X}^0(t) := x^0 + \bar{b}^+ l^+(t) + \bar{b}^- l^-(t),$$

$l^\pm(t) = \int_0^t 1\!\!1_{\pm w_d(s)>0} ds$ is the time spent by $w_d(s), s \in [0,t]$, in the positive half-line or negative half-line, respectively.

The process $\bar{X}^0(t), t \geq 0$, is not a Markov process.

Appendix A.

Proof of Lemma 2. The proof of weak relative compactness follows easily from Theorem 7.3 in [7] because our assumptions yield that the drift b satisfies the linear growth condition in x:

$$\forall T > 0\, \exists C > 0\, \forall t \in [0,T]\, \forall x \in \mathbb{R}^d \ : \quad |b(t,x)| \leq C(1 + |x|).$$

Let $\{\varepsilon_k\}$ be such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $\{X^{\varepsilon_k}\}$ converges in distribution to a process X.

Then by Skorokhod's theorem on a single probability space, there are copies

$$\tilde{X}^{\varepsilon_k} \overset{d}{=} X^{\varepsilon_k}, \quad \tilde{X} \overset{d}{=} X,$$

and Wiener processes $\{\tilde{w}^k\}$ such that

$$\tilde{X}^{\varepsilon_k}(t) = \tilde{X}^{\varepsilon_k}(0) + \int_0^t b^+(s, \tilde{X}^{\varepsilon_k}(s))ds + \varepsilon_k \tilde{w}^{\varepsilon_k}(t), t \geq 0,$$

$$\forall T > 0 \quad \sup_{t\in[0,T]} |\tilde{X}^{\varepsilon_k}(t) - \tilde{X}(t)| \to 0, \ \ k \to \infty, \text{ a.s.,} \tag{A.1}$$

$$\forall T > 0 \quad \sup_{t\in[0,T]} \varepsilon_k |\tilde{w}^{\varepsilon_k}(t)| \to 0, \ \ k \to \infty, \text{ a.s.} \tag{A.2}$$

Let ω be such that (A.1) holds and $t_0 < t_1$ be any fixed numbers such that $\tilde{X}(z) \notin H, z \in [t_0, t_1]$. Let for clarity $\tilde{X}(z) \in \mathbb{R}^d_+ \setminus H, z \in [t_0, t_1]$. Then $\tilde{X}^{\varepsilon_k}(z) \in \mathbb{R}^d_+ \setminus H, z \in [t_0, t_1]$ for sufficiently large k, and

$$\tilde{X}^{\varepsilon_k}(t) = \tilde{X}^{\varepsilon_k}(t_0) + \int_{t_0}^t b^+(s, \tilde{X}^{\varepsilon_k}(s))ds + \varepsilon_k \tilde{w}^{\varepsilon_k}(t), \ t \in [t_0, t_1].$$

It follows from (A.1), (A.2) that

$$\tilde{X}(t) = \tilde{X}(t_0) + \int_{t_0}^t b^+(s, \tilde{X}(s))ds, \ t \in [t_0, t_1].$$

This yields

$$\tilde{X}(t) = \tilde{X}(t_0) + \int_{t_0}^t b(s, \tilde{X}(s))ds,$$

for $t \in [t_0, \tau_{t_0,H})$ and therefore the equality for all $t \in [t_0, \tau_{t_0,H}]$ because all processes in the equality are continuous.

Lemma 2 is proved. □

Proof of Lemma 2. Assume for clarity that x^0 belongs to the upper half-space.

Since b_+ is locally bounded and Lipschitz in x, we have the following uniform convergence for any ω and $T > 0$:

$$\lim_{\varepsilon\to 0} \max_{t\in[0,T]} |X_+(t) - X_+^\varepsilon(t)| = 0.$$

Observe that $X(t) = X_+(t), t \leq \tau_H$ and $X^\varepsilon(t) = X_+^\varepsilon(t), t \leq \tau_H^{(\varepsilon)}$.

For any $T < \tau_H$ the process $X(t) = X_+(t), t \in [0, T]$ is separated from the hyperplane. Thus for sufficiently small $\varepsilon > 0$ the process $X_+^\varepsilon(t), t \in [0, T]$ is also separated from the hyperplane. So $X^\varepsilon(t) = X_+^\varepsilon(t), t \in [0, T]$ and $\tau_H^{(\varepsilon)} > T$ for small ε. Hence we have

$$\liminf_{\varepsilon\to 0} \tau_H^{(\varepsilon)} \geq \tau_H, \ \ \lim_{\varepsilon\to 0} (\tau_H^{(\varepsilon)} - \tau_H)_+ = 0,$$

and the uniform convergence

$$\lim_{\varepsilon\to 0} \max_{t\in[0,T]} |X(t) - X^\varepsilon(t)| = 0.$$

for any $T < \tau_H$.

Let now t_0 be arbitrary. Then

$$\limsup_{\varepsilon\to 0} \max_{t\in[0,t_0]} |X(t\wedge\tau_H) - X^\varepsilon(t\wedge\tau_H)| \leq$$

$$\limsup_{\varepsilon\to 0} 1\!\!1_{\tau_H \leq \tau_H^{(\varepsilon)}} \max_{t\in[0,t_0\wedge\tau_H]} |X(t) - X^\varepsilon(t)| + \limsup_{\varepsilon\to 0} 1\!\!1_{\tau_H > \tau_H^{(\varepsilon)}} \max_{t\in[0,t_0\wedge\tau_H^{(\varepsilon)}]} |X(t) - X^\varepsilon(t)| +$$

$$+\limsup_{\varepsilon\to 0}1\!\!1_{\tau_H>\tau_H^{(\varepsilon)}}\int_{\tau_H^{(\varepsilon)}}^{\tau_H}(|b(s,X(s))|+|b(s,X^\varepsilon(s))|)ds+\limsup_{\varepsilon\to 0}\varepsilon\sup_{s\in[0,t_0]}|w(s)|\leq$$

$$2\limsup_{\varepsilon\to 0}\max_{t\in[0,t_0]}|X_+(t)-X_+^\varepsilon(t)|+\limsup_{\varepsilon\to 0}1\!\!1_{\tau_H>\tau_H^{(\varepsilon)}}\int_{\tau_H^{(\varepsilon)}}^{\tau_H}(|b(s,X(s))|+|b(s,X^\varepsilon(s))|)ds=$$

$$\limsup_{\varepsilon\to 0}1\!\!1_{\tau_H>\tau_H^{(\varepsilon)}}\int_{\tau_H^{(\varepsilon)}}^{\tau_H}(|b(s,X(s))|+|b(s,X^\varepsilon(s))|)ds\leq$$

$$\sup_{s\in[0,\tau_H]}\left(|b(s,X(s))|+\sup_{\varepsilon\in(0,1]}|b(s,X^\varepsilon(s))|\right)\limsup_{\varepsilon\to 0}(\tau_H^{(\varepsilon)}-\tau_H)_+. \tag{A.3}$$

Since b satisfies the linear growth condition, we have

$$\sup_{s\in[0,\tau_H]}\left(|b(s,X(s))|+\sup_{\varepsilon\in(0,1]}|b(s,X^\varepsilon(s))|\right)<\infty.$$

So the right hand side of (A.3) equals 0. □

Proof of Lemma 2. By Skorokhod's theorem on a single probability space, there are copies

$$\tilde{Y}^n\stackrel{d}{=}Y^n, n\geq 0,$$

defined on the joint probability space and such that

$$\forall T>0\quad \sup_{t\in[0,T]}|\tilde{Y}^n(t)-\tilde{Y}^0(t)|\to 0,\ \ n\to\infty,\ \text{a.s.} \tag{A.4}$$

Define $\tilde{\sigma}_D^0,\tilde{\sigma}_{\overline{D}}^0$ analogously to $\sigma_D^0,\sigma_{\overline{D}}^0$ but using the process $\tilde{Y}^0$.

It is easy to see that if ω is such that (A.4) holds and $\tilde{\sigma}_D^0=\tilde{\sigma}_{\overline{D}}^0$, then $\tilde{\sigma}_D^n\to\tilde{\sigma}_D^0$. Thus for any $T>0$ and that ω

$$\lim_{n\to\infty}(\tilde{Y}^n(\tilde{\sigma}_D^n\wedge T),\tilde{\sigma}_D^n\wedge T,\tilde{Y}^n(\cdot\wedge\tilde{\sigma}_D^n\wedge T))=(\tilde{Y}^0(\tilde{\sigma}_D^0\wedge T),\tilde{\sigma}_D^0\wedge T,\tilde{Y}^0(\cdot\wedge\tilde{\sigma}_D^0\wedge T)).$$

This yields the proof because

$$(\tilde{Y}^n(\tilde{\sigma}_D^n\wedge T),\tilde{\sigma}_D^n\wedge T,\tilde{Y}^n(\cdot\wedge\tilde{\sigma}_D^n\wedge T))\stackrel{d}{=}(Y^n(\sigma_D^n\wedge T),\sigma_D^n\wedge T,Y^n(\cdot\wedge\sigma_D^n\wedge T)).$$

□

Proof of Lemma 2. The inequality $\sigma_{B_r(x^0)}\neq\sigma_{\bar{B}_r(x^0)}$ may be true only if $|Y(\sigma_{B_r(x^0)})-x^0|=r$ and there exists $\delta>0$ such that $|Y(t)-x^0|\leq r, t\in[\sigma_{B_r(x^0)},\sigma_{B_r(x^0)}+\delta]$.

For each ω the set of such r is at most countable set. So, by Fubini's theorem for almost all $r>0$ w.r.t. the Lebesgue measure we have $P(\sigma_{B_r(x^0)}=\sigma_{\bar{B}_r(x^0)})=1$. □

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