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American Options in Financial Quotes

Managing the Risk of Being Picked Off

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This master's thesis is submitted under the master's programme *Modelling and Data Analysis*, with programme option *Finance, Insurance and Risk*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

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> *I could paint for a hundred years, a thousand years without stopping and I would still feel as though I knew nothing.*

Paul Cézanne

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Introduction

In this thesis we build a mathematical model for the risk involved when a person makes a binding fixed-price offer to buy or sell something that fluctuates in value. This situation often arises in financial markets, where such an offer is called a *quote*. Quotes involve a risk for the person giving them, and an opportunity for the person receiving them. This is best seen through an example: say trader A offers trader B to buy 100 shares in company X for 10 kroner per share, and the offer is binding for the next 5 seconds. If the market price of the share drops to 6 kroner after 3 seconds, trader B can profit from accepting the offer. If she does, she is said to be *picking off* the quote.

We investigate two different versions of quotes: offers that cannot be canceled before a certain time has passed ("minimum resting times"), and quotes that are automatically canceled if the price moves past a specified barrier ("last look").

Our model of financial quotes is in many respects similar to option pricing models. Quotes with minimum resting times are in a certain sense similar to American options, and quotes with last look are similar to American barrier options. There are also differences however, for example are the time scales of financial quotes orders of magnitude shorter than the time scales of traditional options. These differences lead us to use different modeling approaches than what is used for traditional option pricing models, in particular we investigate the class of integer-valued Levy processes.

This thesis is applied work. Therefore, the emphasis of the work has been to formulate the situation into a tractable mathematical model, and then solving the resulting problems using the appropriate mathematical tools. It is my belief that applications are important for mathematics; they can motivate new questions and give us interesting representations of problems that could otherwise seems dry and uninteresting. We therefore weave the application and the mathematics tightly together in this thesis.

The work use various concepts and results from stochastic analysis and stochastic optimization. It is assumed throughout that the reader is familiar with basic probability theory. Many basic concepts are used without definition. Some key concepts that we use repeatedly are defined in Appendix B for the reader's convenience. Here we also state many standard results without proof.

Structure

In chapter 1 we first describe the problem in words, and then proceed to write down a formal mathematical model.

In chapter 2 we give several results of a general character. We do not pursue proves of these results for the most general settings possible, but rather aim to find a level of abstraction that covers the specific stochastic models we use in chapters 3-5.

In chapter 3 we use a Brownian motion to model randomness. We derive results that are similar to well-known results from the theory of option pricing. The similarities can be gauged by comparing chapter 3 with the option pricing literature review in Appendix A.1.

In chapter 4 we extend our results to a Skellam process; perhaps the simplest possible point process applicable to our situation. The main contribution of this chapter is the derivation of a reflection principle for the Skellam process, which we then use to provide analytical results. In particular the density of the stopped Skellam process (Proposition 18) is to the best of our knowledge a new result.

In chapter 5 we go a step further and model randomness with a general integer-valued Levy process (chapter 1 explains why this situation is of relevance to our application). The relevant literature available on this topic is scarce, and available sources are cited the text. Useful background material for this chapter is Appendix A.2, where we clarify the connection between integer-valued Levy processes and the compound Poisson process. We show how our problem can be reduced to a discrete-time problem in the special case of an infinite time-horizon, and then apply theory from the study of Markov decision processes to attain an analytical solution. The most challenging version of our problem is in section 5.3.3. Here we turn to discrete approximation and use standard results from the theory of dynamic programming. We develop an algorithm documented in Appendix C in order to apply the dynamic programming-approach.

Starting point and contribution
The major work and contribution of this thesis is the development of a *mathematical model* suited for analyzing a particular decision problem, and then *solving the problems* posed by the model using the appropriate mathematical tools.

I have not taken any courses on the mathematics of American options nor optimal stopping-problems before or during my work with this thesis. The exposition of the material related to these subjects is therefore a product of my own literature review. Relevant academic papers are cited where appropriate. In addition, I have used several books as a general reference to the subject matter. The most important of these books are Jeanblanc et al. (2009); Øksendal and Sulem (2005); Oksendal (2013); Björk (2004); Lamberton and Lapeyre (2011).

Some of the results and proofs in this thesis are original, while others have been adapted from books and papers. The symbol ☆ is used to mark sections and proofs that is original, while the symbol ✤ is used to mark sections and proofs where significant independent work has been done. Many standard results are stated without proof in Appendix B.

Chapter 1

Financial quotes

1.1 What is a financial quote?

A financial quote is an offer one market participant provides to another, to buy or sell a specified number of a given security for a specified price. For example, trader A makes an offer to trader B to buy 10 shares in company Z for a price for 100 kroner per share. In many modern electronic marketplaces participants can solicit quotes from those willing to provide them through a mechanism called "request for quote".

Quotes can be divided into two general categories: "indicative" and "firm". An indicative quote is an offer that is not binding to the trader who provided it, she is merely indicating to potential trading partners which price she is interested in trading at. Someone interested in trading at the indicated price can contact the supplier of the quote, for instance by phone or electronic chat, and enter into bilateral negotiation over terms.

We will not concern ourselves with indicative quotes in this thesis. Rather we are interested only in firm quotes. These are offers that are binding to the one who makes it, for a specified period of time. The receiver of this quote has the right, but not the obligation, to enter into a trade at the specified price, at any time up to the quote expires. He who supplies a quote is exposed to the risk that the fair value of whatever is being quoted changes before the quote expires. This risk is sometimes referred to as the **risk of being picked off**.

Consider our earlier example, where trader A made an offer to buy from trader B. Trader A now has the obligation to buy at the specified price at any time until the quote expires, even if the fair value of shares moves to A's disfavor. If now the fair value of shares in company Z falls to 90 kroner before the quote has expired, trader B can earn 10 kroner from selling to A for the quoted price of 100 kroner.

The position of someone who has supplied a quote is in fact very much like that of someone who is short an American option, an observation first made by Copeland and Galai (1983). The quoted price takes the role as the strike price of the option, and the validity time of the quote is the expiry time of the option. The receiver of the quote is in a position as if he were long an American option. If the quote was an offer to buy, the receiver is long an American put; if the quote was an offer to sell, the receiver is long an American call.

In the classical framework of option pricing introduced by Black and Scholes (1973), the price of a traded option is found using hedging and the principle of no arbitrage. In this model the price is unique, since the options (and all financial claims) are perfectly replicable. The fundamental building block of this theory is the replicating portfolio: a portfolio of assets that exactly replicates the payoff from the option. The market model is called complete. The no arbitrage pricing principle provides an interval for prices also when the market model is incomplete, that is when financial claims are not always perfectly replicable. It is however not clear how we should apply the idea of no arbitrage pricing and replicating portfolios in the current setting: this portfolio would normally involve a position in the underlying asset, but the process of acquiring such a position is exactly what we are modeling in the first place.

Non arbitrage prices can also be found using a so-called risk-neutral evaluation. That is, the fair price is given by the expected discounted payoff under a risk-neutral probability measure. If the market is complete, the measure is unique and it correspondingly provides the unique non-arbitrage price. If the market model is not complete, the interval of arbitrage prices corresponds to all the risk-neutral measures of the incomplete model. Therefore, one method of finding arbitrage free option prices is via risk neutral valuation (see for example Karatzas and Shreve (1998). In short, the idea is that if the market is complete and there exists a (unique) risk neutral probability measure, the arbitrage free price of *any* traded payoff is the (unique) expectation of the discounted payoff under the risk-neutral probability measure (Jeanblanc et al., 2009).

We will apply the *idea* of risk neutral pricing to evaluate the risk of posting firm quotes, although it will not lead us to a no arbitrage-price of the option embedded in the quote: We will ask for the expectation of the discounted payoff of the quote, under a given martingale measure. This expectation represents a monetary measure of the risk involved in supplying a firm quote.

To see why the expectation under a risk neutral measure can be a useful benchmark, imagine a hypothetical complete market where one is able to continuously trade in the quoted asset. In this market the quote can be replicated by an American option, and the initial value of the hedging portfolio of the American option can be found via its risk-neutral expectation.

1.1.1 Minimum resting times

A **minimum resting time** refers to an arrangement wherein a quote has to be active for a minimum period of time before it can be canceled.

The attention surrounding minimum resting times has increased, as regulators and market participants have become increasingly concerned about the rise of algorithmic trading. Algorithmic trading can cause a phenomenon known as "phantom liquidity", in which quotes are submitted and subsequently canceled within a very short time frame (Blocher et al., 2016). There is an ongoing debate on whether this and related phenomena is detrimental to the quality of markets, see for example Hendershott et al. (2011); Budish et al. (2015); Foucault et al. (2016). Several regulatory responses have been proposed, among them a rule requiring all quotes to have a minimum resting time (Jorgensen et al., 2016). It is therefore of interest to regulators how the introduction of a minimum resting time will affect trading costs (Furse et al., 2011). It seems intuitive that a minimum resting time entails a larger risk of being picked off, because the supplier of the quote can not adjust the quoted prices in reaction to new information until the minimum resting time has expired.

We aim to develop a model that can help regulators evaluate the effect of minimum resting times, as well as to provide precise theoretical predictions to empirical researchers. We will model the minimum resting time as the expiry time T of the firm quote. We will see that the picking-off risk faced by the supplier of a quote increases if regulators impose a longer resting time T. In competitive markets it is likely that this increased risk will somehow be transfered to the trading costs of the market participants. We will therefore study how the picking off-risk varies with the expiry time T, and how the relationship is affected by other parameters in the model.

1.1.2 Last look

A **last look-quote** is one where the supplier of the quote retains the right to *not* enter into the trade if there has been a sufficiently adverse movement in the spot price.

There are various ways in which this last look feature can be implemented in practice (Oomen, 2016). In this thesis we model last look as a constant threshold $B \in \mathbb{R}$ such that the quote becomes invalid if and when the fair value hits this threshold. We shall see that the mathematical structure of evaluating the picking off-risk in a last look-quote is very similar to the pricing of American barrier options.

There is a current controversy concerning the last look-feature in foreign exchange markets, which makes a mathematical model of last look-quotes of particular practical and regulatory interest [1].

1.2 Stylized facts of high-frequency financial prices

The randomness in our model will stem from the fair price of an asset being quoted. In order to construct a suitable model we shall start by considering some stylized facts about financial prices relevant for time frames of seconds or less.

Prices take values on a discrete set
Financial prices take values on a discrete set of points called *ticks*, see Angel (1997); Werner et al. (2015) for more details on the tick grid.

In mathematical finance it is common practice to model prices as random processes with values on the real line, in contrast to this fact. This incoherence is not very important for models whose domain are time spans of days, months and weeks, but matters when the model is to be applied for time spans of seconds or milliseconds.

Prices change value in continuous time
Most trading systems treat time as continuous, in the sense that orders are processed continuously on a first come-first served basis and recorded in chronological order (Budish et al., 2015). An accurate model for financial prices should therefore have a time index which takes values on the real line.

Note that trading systems give recorded trades a digital time-stamp. This means that although trading takes place in continuous time[2], transaction *data* have discrete-valued time indices. For the applications in this thesis it is the actual events that are important, not how they are stored in digital systems.

[1] See for example "Currency Trading's Last Look' Rules Are Changing, BOE Says", Bloomberg news, 28th July 2016.

[2] To the extent that physical time is continuous

Table 1.1: Comparison of random processes used in thesis

Stylized fact	Brownian motion	Skellam	ΔNB Levy
Continuous time	✓	✓	✓
Discrete state-space	✗	✓	✓
More than one tick	✗	✗	✓
Time-clustering	✗	✗	✗

Prices sometimes change by more than one tick
Price changes of several ticks in one increment are commonly seen in practice. The frequency distribution of the increment size is however rapidly decreasing - Barndorff-Nielsen et al. (2012) analyze a particular data set and show that price increments of more than five ticks are very rare.

Figure 1.1 illustrates how the granularity of the price grid becomes apparent at very short time intervals.

Price changes are clustered in time
Price changes are clustered in time in two different manners.

First, prices change more frequently at certain times of the day. The frequency of price changes is lower during night time, and higher around the release of important information such as company reports and macroeconomic announcements. If one is modeling prices over the length of a day this form of clustering is important. It is not important when we are modeling time spans of seconds or less however; we are either dealing with a daytime-second or a nighttime-second, and we are either dealing with a second when important information is released or we are not. We can therefore disregard the time-of-day clustering effect during the time intervals we are modeling.

There is however a second clustering effect, present also in very short time intervals: we a more likely to see price change in the next few seconds if another price change has just occurred. A variety of modeling approaches has been suggested for dealing with this effect, see Bauwens and Hautsch (2009) for more details. The economic mechanisms behind this clustering effect is still being debated, possible explanations include heterogeneous information arrivals (Andersen and Bollerslev, 1997), investor learning (Banerjee and Green, 2015) and behavioral models (Cont, 2007).

In this thesis we consider three different random processes for the price. The three processes give three different trade-offs between mathematical tractability and realism. In table 1.1 we summarize the stylized facts of high-frequency prices, and how the four different random processes match up against these facts. A review of some key facts concerning Levy processes can be found in the Appendix, section A.2.

None of the price processes considered in this thesis matches the stylized fact that price changes are clustered in time - this extension is left for future work.

1.3 Discounting

We have argued that financial quotes are in some respects very much like American options. Unlike common option pricing models however, we will abstract from the concept of discounting. There are three reasons for this choice.

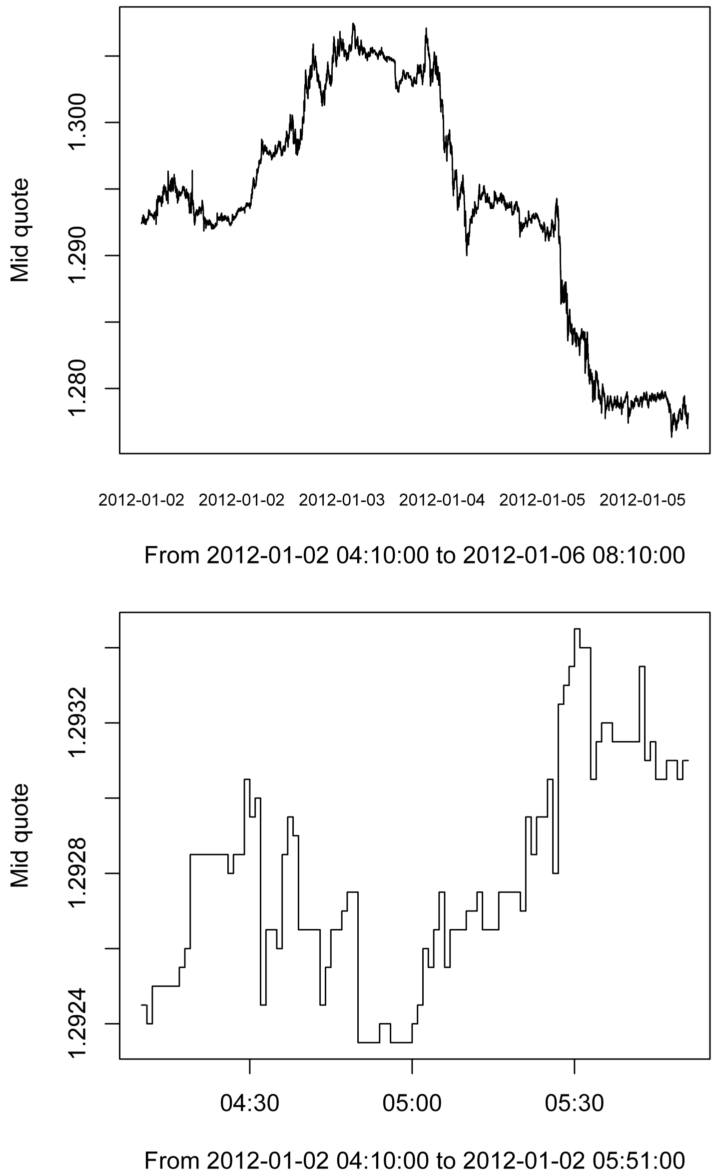


Figure 1.1: EURUSD exchange rate over different time intervals.
See Appendix C for data description.

First, the time scales involved in modeling financial quotes are very different from those of traditional option pricing models. The expiration time of a quote in modern markets will typically be measured in seconds or milliseconds, while American stock options have expiry dates measured in months or years. Cash flows occurring months or years into the future are significantly affected by discounting, and we therefore cannot abstract it away without fundamentally changing the structure of decision problems. For cash flows that are seconds or milliseconds into the future things are different. The empirical literature on intraday interest rates finds evidence of rates in the range of 0.1 to 0.9 basis points per hour (Furfine, 2001; Kraenzlin and Nellen, 2010; Jurgilas and Žikeš, 2014). These numbers implies per-second interest rates in the range of 2.8×10^{-9} to 2.5×10^{-7} per cent. On the other hand, a volatility coefficient of 20% per year, commonly found for stocks indices, translates into a per-second volatility of $\frac{.2}{(252*8*60*60)^{1/2}} \approx 7 \times 10^{-5}$, several orders of magnitude larger than any realistic discount rate. We see that discounting is simply not of any real significance over very short time horizons.

Secondly, it is not clear that per-second or per-millisecond discount rates are even meaningful economic concepts. The hourly discount rates mentioned earlier are thought to be due to structural properties of payment systems. It is not clear that discounting over much shorter time intervals makes any economic sense.

Thirdly, modeling financial quotes using discounting leads to the same problem formulations as when pricing American options. These are problems that are already thoroughly studied in the literature, and usually one has to resort to numerics for their solution. A review can be found in Appendix A.1. We will on the other hand be able to attain several analytical results because we exclude discounting.

1.4 Mathematical formulation

In this section we formulate the discussion of sections 1.1 and 1.3 into a mathematical model.

We assume a probability space $(\Omega, P, \mathcal{F})$, and an adapted random process (S_t) called the **price process**, taking values in the space E. The price process represents the "fair value" of the asset being quoted [3]. We will look at different cases for E, both $E = \mathbb{R}$ and $E = \mathbb{Z}$

1.4.1 Quotes with a minimum resting time

Definition 1. *A* ***quote*** *given at time* 0 *is a pair* $(K, T) \in E \times \mathbb{R}_+$

The quote is a ***sell quote*** *if* $S_0 < K$.

The quote is a ***buy quote*** *if* $K < S_0$.

Note that the lack of discounting in this model shows up in the fact that we are comparing S_0 with K directly, rather than $e^{-rt}K$.

[3] Exactly what should be understood by the word "fair" depends on the particular context. In the stock market-example discussed earlier, one could take "fair value" to mean the net present value of expected future dividend payments discounted at the relevant risk-adjusted discount rate. In general, (S_t) represents the current value of the asset, against which all quotes are compared. We assume that (S_t) is common knowledge to all participants in the market.

The value K should be interpreted as the price at which the supplier of the quote commits to buying or selling one unit of a given asset to the receiver of the quote, and T is the time at which the offers expires.

We shall not consider quotes where $K = S_0$.

We shall in many places write $\bar{K}$ when discussing the sell quote and $\underline{K}$ when discussing the buy quote.

We say that a quote is *executed* when and if the receiver of a quote decides to trade. A sell quote executed at time $\tau \in [0, T]$ gives payoff $(S_\tau - \bar{K})$ to receiver of the quote. Similarly, the supplier loses $-(S_\tau - \bar{K})$. We will assume that the receiver only executes the quote if he does not lose relative to the fair value, and so we write the payoff of the receiver as $(S_\tau - \bar{K})_+$, and of the supplier as $-(S_\tau - \bar{K})_+$.

Definition 2. *The* ***payoff*** *to the receiver from a quote executed at time τ is denoted $\psi(S_\tau)$.*

If the quote is a sell quote, we have $\psi_{sell} = (S_\tau - \bar{K})_+$

If the quote is a buy quote, we have $\psi_{buy} = (\underline{K} - S_\tau)_+$

For both buys and sells, ψ is a non-negative convex function. This fact will be used repeatedly. Many of the arguments that follows will in fact only require these properties, and not the functional form itself.

The expected payoff to the receiver and the supplier of the financial quote depends on the value of the underlying price process at the time of execution. In order to evaluate the risk associated with financial quotes, we will ask for the largest possible expected payoff that can be attained.

Definition 3. *Let $\mathcal{T}$ be the collection of stopping times taking values in $[0, T]$.*

We refer to Appendix B for definition and basic properties of stopping times.

Definition 4. *Take as given a quote (K, T). We denote the* ***picking off-risk*** *from supplying the quote by $V(K, T)$:*

$$
\begin{aligned}
V_{sell}(\bar{K}, T) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\psi_{sell}(S_\tau)\right] \\
V_{buy}(\underline{K}, T) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\psi_{buy}(S_\tau)\right]
\end{aligned} \tag{1.1}
$$

Definition 5. *Any stopping time that attains the supremum in (1.1) is called an* ***optimal stopping time****.*

An optimal stopping time may not be unique. In such cases we will be particularly interested in the first optimal stopping time:

Definition 6. *Let $\tau^* \in \mathcal{T}$ be an optimal stopping time. We say τ^* is the* ***first optimal stopping time*** *if, for any optimal stopping time $\sigma^* \in \mathcal{T}$, we have*

$$P(\tau^* \leq \sigma^*) = 1$$

1.4.2 Quotes with last look

The last look quote is associated with a boundary $B \in E$ such that the quote becomes invalid ("knocked out") if the fair price ever crosses B.

Definition 7. *A* ***quote with last look*** *is a triplet* $(K, T, B) \in E \times \mathbb{R}_+ \times E$

The quote is a ***sell quote*** *if* $S_0 < K < B$.

The quote is a ***buy quote*** *if* $B < K < S_0$.

We shall use the notation $(\bar{K}, T, \bar{B})$ for a sell quote and $(\underline{K}, T, \underline{B})$ for a buy quote.

Definition 8. *Take as given a stochastic process* (S_t)*. For a constant* $B \in E$ *we define the* ***first hitting time*** *as*

$$T_B := \inf\{t \geq 0 \mid S_t = B\}$$

Note that our definition of hitting time is the time when the process (S_t) *takes the value* B. Many books on stochastic calculus use a similar notation to denote the first time the process is greater than or equal to B, a time we will here refer to as T_B^+:

$$T_B^+ = \inf\{t \geq 0 \mid S_t \geq B\}$$

The two random times T_B and T_B^+ are equal almost surely if the process (S_t) has continuous paths. If the process has jumps, however, these two times can differ. The distinction between T_B and T_B^+ is in fact crucial for many of the problems and arguments in this thesis.

Definition 9. *Take as given a stochastic process* (S_t)*. For a constant* $B \in E$ *we define the* ***knockout time*** $T_\dagger$ *as*

$$T_\dagger(B) := \begin{cases} \inf\{t \geq 0 \mid S_t > B\} & \text{if } B > S_0 \\ \inf\{t \geq 0 \mid S_t < B\} & \text{if } B < S_0 \end{cases}$$

Definition 10. *We define the* ***running maximum*** (M_t) *as*

$$M_t := \sup_{u \in [0,t]} S_u$$

We define the ***running minimum*** (m_t) *as*

$$m_t := \inf_{u \in [0,t]} S_u$$

Definition 11. *The payoff to the receiver from a* ***last look sell quote*** *executed at time* τ *is*

$$\mathbf{1}_{\{M_\tau \leq B\}} \psi_{sell}(S_\tau)$$

The payoff to the receiver from a ***last look buy quote*** *executed at time* τ *is*

$$\mathbf{1}_{\{m_\tau \geq B\}} \psi_{buy}(S_\tau)$$

One could make a definition equivalent to Definition 11 using the random times T_B or $T_\dagger$ rather than the running maximum and minimum. The reason for the definition made here is that we shall in later chapters attain the joint law of a process and its running maximum/minimum.

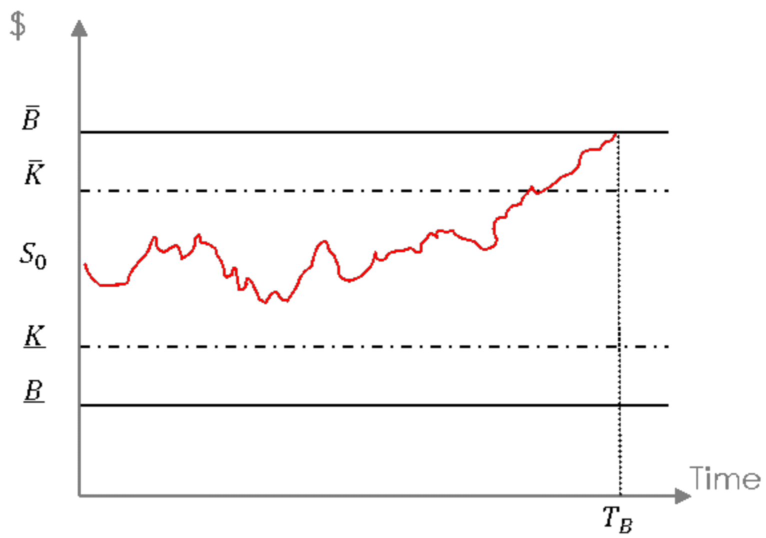
We note that the payoff functions from last look-quotes are the same as American barrier options; up-and-out calls in the case of sell quotes, down-and-out puts in the case of buy quotes (see Appendix A.1).

Definition 12. *We define the risk from supplying a last look-quote by*

$$\begin{aligned}\hat{V}_{sell}(\bar{K}, T, \bar{B}) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{M_\tau \leq \bar{B}\}} \psi_{sell}(S_\tau)\right] \\ \hat{V}_{buy}(\underline{K}, T, \underline{B}) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{m_\tau \geq \underline{B}\}} \psi_{buy}(S_\tau)\right]\end{aligned} \tag{1.2}$$

Figure 1.2 illustrates the problem.

Figure 1.2: Illustration of the problem



Chapter 2

General results

In this section we give several results that does not depend on the choice of price process. In later chapters we apply these results to various special cases.

We take as given the probability space $(\Omega, \mathcal{F}, P)$ and a random process (S_t) taking values in the measurable space $(E, \mathcal{E})$. When nothing else is said we shall always work with the natural filtration of (S_t), denoted by $\mathbb{F}$. We make the following important assumptions:

i) The probability space satisfies the usual conditions (see Appendix B).

ii) The process (S_t) is a Levy process (see Appendix A.2)

iii) The process (S_t) is a martingale.

We also suppose that the integrability condition $\mathbb{E}[|\psi(S_t)|] < \infty$ holds for all $t \in [0, T]$.

Proposition 1 (Optimal stopping with a minimum resting time)**.** ☆
The stopping time $\tau = T$ is optimal for (1.1)*.*

The idea behind Proposition 1 is that if we regard the payoff from the quote as a stochastic process, $(\psi(S_t), t \geq 0)$, it is a submartingale since it is a convex function of a martingale. And since a submartingale is increasing in expectations, it is clearly optimal to stop it at the latest possible time. Here is a formal proof:

Proof. We must show that $\mathbb{E}[\psi(S_T)] = \sup_{\tau \in \mathcal{T}} \mathbb{E}[\psi(S_\tau)]$. Take any $\tau \in \mathcal{T}$. Note that $\tau \leq T$ by the definition of $\mathcal{T}$, and therefore $\mathcal{F}_\tau \subseteq \mathcal{F}_T$. We can therefore apply Doob's optional sampling theorem (Appendix Theorem 57) to the martingale (S_t) and the bounded stopping times τ and T. Moreover we can apply Jensen's inequality for conditional expectations to the convex function ψ. We therefore have

$$\begin{aligned}\mathbb{E}[\psi(S_\tau)] &= \mathbb{E}[\psi(\mathbb{E}[S_T \mid \mathcal{F}_\tau])] \\ &\leq \mathbb{E}[\mathbb{E}[\psi(S_T) \mid \mathcal{F}_\tau]] \\ &= \mathbb{E}[\psi(S_T)]\end{aligned}$$

The proposition now follows from observing that $T \in \mathcal{T}$. □

Proposition 2 (Optimal stopping of last look)**.** ☆
Define the stopping time

$$\tau^* := T \wedge T_B$$

Assume that

$$P(M_{\tau^*} > B) = 0 \quad \textit{if } S_0 < B$$
$$P(m_{\tau^*} < B) = 0 \quad \textit{if } S_0 > B$$

Then the stopping time $\tau^* = T \wedge T_B$ *is the first optimal stopping time for* (1.2).

Proposition 2 says that the receiver of a last look quote can expect to do no better than to wait until either the quote expires, or the price process hits the boundary B. The reason is the following: either the sample space realization is such that the quote is going to be killed by the last look-feature, in which case one cannot expect to do better than wait until the process (S_t) hits the boundary B. Or, the quote is not going to be killed, in which case one cannot expect to do better than wait until the quote expires at time T. In either case, stopping before $T_B \wedge T$ gives a lower expected payoff than continuing. The key assumption we have to make for this argument to hold essentially amounts that the underlying process never "jumps past" the barrier ($P(M_{\tau^*} > B) = 0$), meaning that it is safe to wait until the exact moment when $S_t = B$. We now give the formal proof:

Proof. We prove the case $S_0 < B$ (the sell quote). Take any stopping time $\tau \in \mathcal{T}$. We shall first show that

$$\mathbb{E}[\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}}] \le \mathbb{E}[\psi(S_{\tau^*})\mathbf{1}_{\{M_{\tau^*} \le B\}}]$$

Let $A := \{\omega \in \Omega \mid M_{T \wedge T_B} \le B\}$.

We have that

$$\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}} = \underbrace{\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{\tau < T\wedge T_B\}}\mathbf{1}_{\{A\}}}_{\text{I}} + \underbrace{\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{\tau \ge T\wedge T_B\}}\mathbf{1}_{\{A\}}}_{\text{II}} + \underbrace{\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{A^c\}}}_{\text{III}}$$

We consider the terms I and II separately. We shall see that term III vanish in expectations, since A^c is a null set.

First consider term I. On the set $\{\tau < T \wedge T_B\}$ we have $\mathcal{F}_\tau \subseteq \mathcal{F}_{T\wedge T_B}$, and hence we can use Doob's optional sampling theorem on the martingale (S_t). Furthermore we apply Jensen's inequality for conditional expectations, and write

$$\begin{aligned}
\psi(S_\tau)\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{\tau < T\wedge T_B\}}\mathbf{1}_{\{A\}} =& \psi(\mathbb{E}[S_{T\wedge T_B} \mid \mathcal{F}_\tau])\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{\tau < T\wedge T_B\}}\mathbf{1}_{\{A\}} \\
&\le \mathbb{E}[\psi(S_{T\wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_\tau \le B\}}\mathbf{1}_{\{\tau < T\wedge T_B\}}\mathbf{1}_{\{A\}} \\
&= \mathbb{E}[\psi(S_{T\wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_{T\wedge T_B} \le B\}}\mathbf{1}_{\{\tau < T\wedge T_B\}}\mathbf{1}_{\{A\}}
\end{aligned}$$

The last equality use that on the set $\{\tau < T \wedge T_B\} \cap A$ we have $\mathbf{1}_{\{M_\tau \le B\}} = \mathbf{1}_{\{M_{T\wedge T_B} \le B\}} = 1$.

The above inequality also show that it is never optimal to stop before $T \wedge T_B$. Hence, if $T \wedge T_B$ is indeed optimal, it must also be the first optimal stopping time.

Now consider term II. By the definition of T_B and since ψ is monotonically increasing we have $\psi(S_{T_B}) = \psi(B) > \psi(x)$ for all $x < B$. The assumption $\tau \in \mathcal{T}$ means that $\tau \le T$. Therefore on the set $\{\tau \ge T \wedge T_B\} \cap A$ it must be the case that $T_B \le T$. On the set A we also have

$\mathbf{1}_{\{M_{T_B} \leq B\}} = 1 \geq \mathbf{1}_{\{M_\tau \leq B\}}$. Therefore,

$$\begin{aligned}
\mathbf{1}_{\{\tau \geq T_B\}}\mathbf{1}_{\{M_\tau \leq B\}}\psi(S_\tau)\mathbf{1}_{\{A\}} &\leq \mathbf{1}_{\{\tau \geq T_B\}}\mathbf{1}_{\{M_\tau \leq B\}}\psi(S_{T_B})\mathbf{1}_{\{A\}} \\
&\leq \mathbf{1}_{\{\tau \geq T_B\}}\mathbf{1}_{\{M_{T_B} \leq B\}}\psi(S_{T_B})\mathbf{1}_{\{A\}} \\
&= \mathbf{1}_{\{\tau \geq T_B \wedge T\}}\mathbf{1}_{\{M_{T_B \wedge T} \leq B\}}\psi(S_{T_B \wedge T})\mathbf{1}_{\{A\}} \\
&= \mathbf{1}_{\{\tau \geq T_B \wedge T\}}\mathbf{1}_{\{M_{T_B \wedge T} \leq B\}}\psi(\mathbb{E}[S_{T_B \wedge T} \mid \mathcal{F}_\tau])\mathbf{1}_{\{A\}}
\end{aligned}$$

The last line use that $\mathcal{F}_{T \wedge T_B} \subseteq \mathcal{F}_\tau$ for $\{\omega \in \Omega \mid \tau \geq T \wedge T_B\}$, and thus $S_{T_B \wedge T}$ is $\mathcal{F}_\tau$-measurable.

Combining our considerations for term I and term II, we get

$$\begin{aligned}
\psi(S_\tau)\mathbf{1}_{\{M_\tau \leq B\}} \leq& \\
&\mathbb{E}[\psi(S_{T \wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_{T \wedge T_B} \leq B\}}\mathbf{1}_{\{\tau < T \wedge T_B\}}\mathbf{1}_{\{A\}} + \\
&\mathbb{E}[\psi(S_{T \wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_{T \wedge T_B} \leq B\}}\mathbf{1}_{\{\tau \geq T \wedge T_B\}}\mathbf{1}_{\{A\}} + \\
&\psi(S_\tau)\mathbf{1}_{\{M_\tau \leq B\}}\mathbf{1}_{\{A^c\}} \\
&= \mathbb{E}[\psi(S_{T \wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_{T \wedge T_B} \leq B\}}\mathbf{1}_{\{A\}} + \psi(S_\tau)\mathbf{1}_{\{M_\tau \leq B\}}\mathbf{1}_{\{A^c\}}
\end{aligned}$$

We shall apply the expectation operator on the preceding inequality. Note that on $\{A\}$ we have $\mathbf{1}_{\{M_{T \wedge T_B}\}} = 1$. Moreover since $P(A) = 1$ by assumption we have $P(A^c) = 0$ and hence $\mathbb{E}[Y\mathbf{1}_{\{A^c\}}] = 0$ for any random variable Y by the properties of the Lebesgue integral. Therefore, by the Tower property of conditional expectations,

$$\begin{aligned}
\mathbb{E}\left[\psi(S_\tau)\mathbf{1}_{\{M_\tau \leq B\}}\right] &\leq \mathbb{E}\left[\mathbb{E}[\psi(S_{T \wedge T_B}) \mid \mathcal{F}_\tau]\mathbf{1}_{\{M_{T \wedge T_B} \leq B\}}\mathbf{1}_{\{A\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}[\psi(S_{T \wedge T_B}) \mid \mathcal{F}_\tau]\right] \\
&= \mathbb{E}\left[\psi(S_{T \wedge T_B})\right] \\
&= \mathbb{E}\left[\psi(S_{T \wedge T_B})\mathbf{1}_{\{M_{T \wedge T_B} \leq B\}}\right]
\end{aligned}$$

Since $T \wedge T_B \in \mathcal{T}$ and τ was arbitrary, we have proved that $T \wedge T_B$ is the first optimal stopping time. The case $S_0 > B$ (the buy quote) follows the same steps. □

Proposition 3 (Symmetry of buy and sell quotes)**.** ☆
Suppose we have the sell and buy quotes $(\bar{K}, T, \bar{B})$ *and* $(\underline{K}, T, \underline{B})$ *satisfying*

$$\bar{K} = -\underline{K}$$
$$\bar{B} = -\underline{B}$$

and the price process satisfies

$$S_0 = 0$$

If the price process is symmetric, meaning that we have the equality of law

$$S_t \overset{law}{=} -S_t \quad \textit{all } t \geq 0$$

Then,

i)

$$V_{sell}(\bar{K}, T) = V_{buy}(\underline{K}, T)$$

ii)

$$\hat{V}_{sell}(\bar{K}, T, \bar{B}) = \hat{V}_{buy}(\underline{K}, T, \underline{B})$$

Proof. **We first prove i)**. Note that $\bar{K} - S_t \overset{law}{=} S_t - \underline{K}$ for any $t \geq 0$ and constants $\bar{K}, \underline{K}$. Since the law of S_t is symmetric, we have that for any continuous and bounded function f, $\mathbb{E}[f(S_t)] = \mathbb{E}[f(-S_t)]$ (see Appendix B). Therefore, for any $t \in [0, T]$ we have that

$$\begin{aligned}
\mathbb{E}[\psi_{\text{sell}}(S_t)] &= \mathbb{E}[\max(0, S_t - \bar{K})] \\
&= \mathbb{E}[\max(0, \underline{K} - S_t)] \\
&= \mathbb{E}[\psi_{\text{buy}}(S_t)]
\end{aligned}$$

Using that the stopping time $T \in \mathcal{T}$ is optimal (Proposition 1), we have

$$\begin{aligned}
V_{\text{sell}}(\bar{K}, T) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}[\psi_{\text{sell}}(S_\tau)] \\
&= \sup_{\tau \in \mathcal{T}} \mathbb{E}[\psi_{\text{buy}}(S_\tau)] \\
&= V_{\text{buy}}(\underline{K}, T)
\end{aligned}$$

We now prove ii), for the last look-quotes $(\bar{K}, T, \bar{B})$ and $(\underline{K}, T, \underline{B})$. First note that, for any $t \in [0, T]$, we have

$$\begin{aligned}
P(S_t > \bar{B}) &= P(S_t > -\underline{B}) \\
&= P(-S_t < \underline{B}) \\
&= P(S_t < \underline{B})
\end{aligned}$$

Which implies that

$$\begin{aligned}
T_\dagger(\bar{B}) &:= \inf\{t \geq 0 \mid S_t > \bar{B}\} \\
&\overset{law}{=} \inf\{t \geq 0 \mid S_t < \underline{B}\} \\
&=: T_\dagger(\underline{B})
\end{aligned}$$

Also note that for any $\tau \in \mathcal{T}$,

$$\{M_\tau \leq \bar{B}\} = \{\tau < T_\dagger(\bar{B})\}$$

and

$$\{m_\tau \geq \underline{B}\} = \{\tau < T_\dagger(\underline{B})\}$$

Therefore we have that

$$\begin{aligned}
P(M_\tau \leq \bar{B}) &= P(\tau < T_\dagger(\bar{B})) \\
&= P(\tau < T_\dagger(\underline{B})) \\
&= P(m_\tau \geq \underline{B})
\end{aligned}$$

Using the above we get that

$$\begin{aligned}
\mathbb{E}\left[\psi_{\text{sell}}(S_\tau)\mathbf{1}_{\{M_\tau \leq \bar{B}\}}\right] &= \mathbb{E}[\max(0, S_\tau - \bar{K}) \mid \tau < T_\dagger(\bar{B})]\, P(\tau < T_\dagger(\bar{B})) \\
&= \mathbb{E}[\max(0, (\underline{K} - S_\tau)) \mid \tau < T_\dagger(\underline{B})]\, P(\tau < T_\dagger(\underline{B})) \\
&= \mathbb{E}[\psi_{\text{buy}}(S_\tau)\mathbf{1}_{\{m_\tau \geq \underline{B}\}}]
\end{aligned}$$

By assumption we have that

$$S_{T_{\bar{B}}} = \bar{B} = -\underline{B} = -S_{T_{\bar{B}}}$$

and since S_t is symmetric we have $T_{\bar{B}} \overset{law}{=} T_{\underline{B}}$. This means that

$$\begin{aligned} P(S_{T \wedge T_{\bar{B}}} \leq x) &= P(S_T \leq x \mid T < T_{\bar{B}}) + P(S_{T_{\bar{B}}} \leq x \mid T \geq T_{\bar{B}}) \\ &= P(-S_T \leq x \mid T < T_{\underline{B}}) + P(S_{T_{\underline{B}}} \leq x \mid T \geq T_{\underline{B}}) \\ &= P(-S_{T \wedge T_{\underline{B}}} \leq x) \end{aligned}$$

Showing that the we have symmetry of distribution also for the random variables $(S_{T \wedge T_{\bar{B}}}, S_{T \wedge T_{\underline{B}}})$. Using that $T \wedge T_B$ is optimal (Proposition 2) , we have that

$$\begin{aligned} \hat{V}_{\text{sell}}(\bar{K}, T, \bar{B}) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\psi_{\text{sell}}(S_\tau) \mathbf{1}_{\{M_\tau \leq \bar{B}\}}\right] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}[\psi_{\text{buy}}(S_\tau) \mathbf{1}_{\{m_\tau \geq \underline{B}\}}] \\ &= \hat{V}_{\text{buy}}(\underline{K}, T, \underline{B}) \end{aligned}$$

And the proof is complete. □

Because of Proposition 3 we shall mostly describe sell quotes in the remainder of this thesis. Chapter 3 is an exception, as we there give several results explicitly also for the buy quote to illustrate the symmetry between buys and sells. Moreover, we shall omit the bar in $\bar{K}$ and $\bar{B}$, and simply refer to sell quotes as (K, T) and (K, T, B), keeping in mind that these quotes satisfy

$$S_0 < K < B$$

The *put-call parity* is a well-known result from option pricing theory. In the current context we have a similar result, stated in Proposition 4.

Proposition 4 (Quote "put-call" parity)**.** *Let $(K, T) \in E \times \mathbb{R}_+$ and $S_0 \in E$ be given. The following relation ("put-call" parity) holds:*

$$V_{sell} - V_{buy} = S_0 - K$$

Proof. Observe that

$$(x - K) = (x - K)_+ - (K - x)_+$$

Therefore, for any stopping time τ,

$$(S_\tau - K) = (S_\tau - K)_+ - (K - S_\tau)_+$$

Taking expectations, and exploiting the fact that by our martingality assumption and Doob's optional sampling theorem we have $\mathbb{E}[S_\tau] = S_0$ for any stopping time τ, we get

$$S_0 - K = \mathbb{E}[(S_\tau - K)_+] - \mathbb{E}[(K - S_\tau)_+]$$

Now take the supremum over stopping times in $\mathcal{T}$,

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[(S_\tau - K)_+] - \sup_{\tau \in \mathcal{T}} \mathbb{E}[(K - S_\tau)_+] = S_0 - K$$

Which completes the proof. □

Chapter 3

Picking-off risk under the Brownian motion

In this chapter we retain the assumptions of Chapter 2, apart from the following restrictions:

Let (W_t) be the standard Brownian motion with continuous paths, and let $\mathbb{F}$ be the P-augmented natural filtration associated with (W_t). Furthermore let x and σ be two given positive real numbers. The price process is given by

$$S_t = \sigma W_t + x_0 \tag{3.1}$$

We shall in several cases provide results for both the buy and the sell quote.

3.1 Minimum resting time

Recall that our modeling considerations around minimum resting times and picking off-risk led us to equation (1.1):

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\psi(S_\tau)\right]$$

where ψ is $(S_\tau - K)_+$ and $(K - S_\tau)$ for the sell and buy quote respectively.

Proposition 5. *Under the assumptions of this chapter, the optimal stopping time of* (1.1) *is* $\tau^* = T$.

Proof. We will show that (S_t) is a $\mathbb{F}$-martingale. The claim then follows from Proposition 1.

The Brownian motion (W_t) is clearly measurable with regards to its own natural filtration $\mathbb{F}$. Since the function $f(x) = ax + b$ is continuous and hence Borel measurable, the process (S_t) is also $\mathbb{F}$-measurable.

The process is (S_t) is in L^1:

$$\mathbb{E}[|S_t|] = \sigma\sqrt{\frac{2T}{\pi}} + x_0 < \infty$$

The process (S_t) has the martingale property:

$$\begin{aligned}\mathbb{E}[S_t \mid \mathcal{F}_u] &= \mathbb{E}[S_t - S_u \mid \mathcal{F}_u] + S_u \\ &= S_u\end{aligned}$$

Hence the process (S_t) is an $\mathbb{F}$-martingale. The claim therefore follows from Proposition 1. □

We can manipulate the normal distribution to compute the expected value from following the optimal strategy explicitly.

Proposition 6. ☆
Let $\Phi(\cdot)$ be the standard normal CDF, and n be $+1$ for the sell quote and -1 for the buy quote:

$$n = \begin{cases} +1 & \text{if } \psi(y) = (y-K)_+ \\ -1 & \text{if } \psi(y) = (K-y)_+ \end{cases}$$

Under the assumptions of this chapter, the value of (1.1) is given by:

$$\sup_{\tau \in \mathcal{T}} \{\mathbb{E}[\psi(S_\tau)]\} = n \left[\sigma\sqrt{\frac{T}{2\pi}} \exp\left(-\frac{(K-x_0)^2}{2\sigma^2 T}\right) + (x_0 - K)\Phi\left(\frac{(x_0-K)}{\sigma\sqrt{T}}\right)\right] \tag{3.2}$$

Proof. We know from Proposition 5 that the largest expected value is attained by stopping at the final time T. Hence, our task is to evaluate $\mathbb{E}[\psi(S_T)]$. We start with the sell quote:

Note that $S_T \overset{law}{=} x_0 + \sigma\sqrt{T}Z$, where Z is standard normal. Let f denote the standard normal pdf. We have

$$\begin{aligned}
\mathbb{E}[(S_T - K)_+] &= \mathbb{E}[(x_0 + \sigma\sqrt{T}Z - K)_+] \\
&= \int_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} (x_0 + \sigma\sqrt{T}z - K) f(z) dz \\
&= \sigma\sqrt{T}\int_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} z f(z) dz - (x_0 - K) P\left(Z \geq \frac{K-x_0}{\sigma\sqrt{T}}\right) \\
&= \sigma\sqrt{T}\int_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} z f(z) dz - (x_0 - K)\Phi\left(\frac{x_0-K}{\sigma\sqrt{T}}\right)
\end{aligned}$$

We continue working on the first integral, using the substitution $y = \frac{-z^2}{2}$:

$$\begin{aligned}
\int_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} z f(z) dz &= \frac{1}{\sqrt{2\pi}} \int_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} z e^{-z^2/2} dz \\
&= -\frac{1}{\sqrt{2\pi}} \int_b^{\infty} e^y dy \\
&= -\frac{1}{\sqrt{2\pi}} \left[e^{-\frac{z^2}{2}}\right]_{\frac{K-x_0}{\sigma\sqrt{T}}}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(K-x_0)^2}{2\sigma^2 T}}
\end{aligned}$$

Thereby we get

$$\mathbb{E}[(S_T - K)_+] = \sigma\sqrt{\frac{T}{2\pi}} e^{-\frac{(K-x_0)^2}{2\sigma^2 T}} + (x_0 - K)\Phi\left(\frac{x_0-K}{\sigma\sqrt{T}}\right)$$

For the buy quote we follow the same steps:

$$\mathbb{E}\left[(K - S_T)_+\right] = \mathbb{E}\left[K - S_T \mid S_T \le K\right] P(S_T \le K) + 0$$

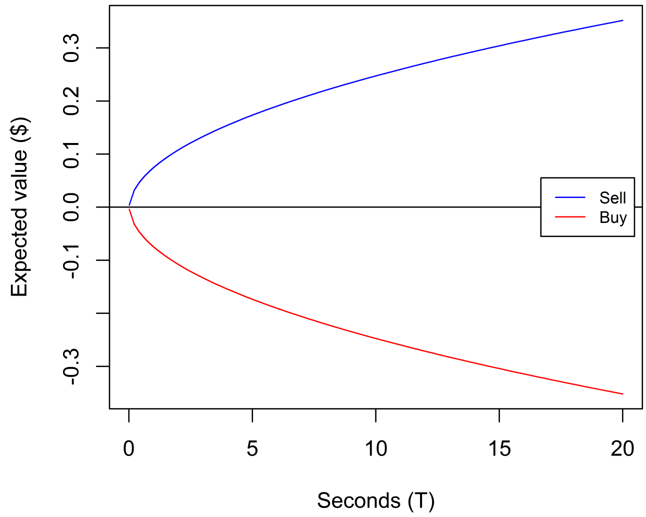
$$= -\mathbb{E}[Y \mid Y \le K - x_0] P(Y \le K - x_0) + (K - x_0) P(S_T \le K)$$

$$= -\sigma\sqrt{\frac{T}{2\pi}} e^{-\frac{(K-x_0)}{2\sigma^2 T}} - (x_0 - K)\Phi\left(-\frac{x_0 - K}{\sigma\sqrt{T}}\right)$$

□

The formula in Proposition 6 is of course closely related to the famous Black-Scholes formula from option pricing theory (see Appendix A.1).

Figure 3.1: Picking-off risk as function of minimum resting time (*The buy quote has been reflected on the x-axis for illustrative purpose*).



3.2 Last look

Our considerations around the last look quote led us to equations (1.2):

$$\hat{V}_{\text{sell}} = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{M_\tau \le B\}}(S_\tau - K)_+\right]$$

$$\hat{V}_{\text{buy}} = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{m_\tau \ge B\}}(K - S_\tau)_+\right]$$

Proposition 7. *Under the assumptions of this chapter, the optimal stopping time for* (1.2) *is*

$$\tau^* = T_B \wedge T$$

Proof. Since the Brownian motion has continuous sample paths, we have that

$$P(M_{T \wedge T_B} > B) = 0$$

The claim now follows from Proposition 2. □

We want to find an explicit formula for the expected value $\mathbb{E}\left[\psi(S_{T_B\wedge T})\right]$ attained by following the optimal rule. For this we will need the density of the absorbed process $S_{T\wedge T_B}$.

Lemma 8. *Let (S_t) and T_B be as defined above, and let $\Phi(\cdot)$ be the standard normal density. The density f of the stopped process $S_{T\wedge T_B}$ is given by*

$$f(u) = \phi\left(\frac{u-x_0}{\sigma\sqrt{T}}\right) - \phi\left(\frac{u+x_0-2B}{\sigma\sqrt{T}}\right)$$

Proof. The proof is a consequence of the reflection principle for the Brownian motion (see Appendix B. Let (W_t) be the standard Brownian motion, and let here (Y_t) be the running supremum

$$(Y_t) := (\sup_{u\leq t} W_t \,, t \geq 0)$$

Take first the joint probability law of the terminal value W_T of the Brownian motion and the running supremum over $[0,T]$:

$$\begin{aligned}
P(W_T \leq x, Y_T \leq y) &= P(W_T \leq x) - P(W_T \leq x, Y_T \geq y) \\
&= \Phi\left(\frac{x}{\sqrt{T}}\right) - P(W_T \leq x, Y_T \geq y) \\
&= \Phi\left(\frac{x}{\sqrt{T}}\right) - P(W_T \leq x-2y) \\
&= \Phi\left(\frac{x}{\sqrt{T}}\right) - \Phi\left(\frac{x-2y}{\sqrt{T}}\right)
\end{aligned}$$

Where the first line use the law of total probability and the second-to-last line use the reflection principle. Set $\hat{x} = \frac{x-x_0}{\sigma}$ and $\hat{y} = \frac{y-x_0}{\sigma}$, and note that by (3.1) and the properties for the Normal distribution we have

$$P\left(S_T \leq x\right) = P\left(W_T \leq \hat{x}\right) \quad \text{and} \quad P\left(\sup_{t\leq T} S_t > y\right) = P\left(\sup_{t\leq T} W_t > \hat{y}\right)$$

Hence, if we let $F(x,y)$ denote the joint distribution of $(S_T, \sup_{t\leq T} S_T)$, we get

$$\begin{aligned}
F(x,y) &:= P(S_T \leq x, \sup S_T \leq y) \\
&= P(W_T \leq \hat{x}, Y_T \leq \hat{y}) \\
&= \Phi\left(\frac{x-x_0}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{x+x_0-2y}{\sigma\sqrt{T}}\right)
\end{aligned}$$

Or, equivalently in terms of the joint density function,

$$f(u,y) = \phi\left(\frac{u-x_0}{\sigma\sqrt{T}}\right) - \phi\left(\frac{u+x_0-2y}{\sigma\sqrt{T}}\right)$$

The lemma now follows from setting $y = B$. □

We are now ready to compute the expected value from following the optimal stopping rule. It turns out that this value can be expressed as a linear combination of quotes without the last look feature. These we know how to compute from Proposition 5. This result has an analogy in the pricing of European barrier options, see Appendix A.1 for more details on this.

Proposition 9. ✤
Let (S_T) *and* T_B *be as defined above, and let* $\mathbb{E}_r[f(S_t)]$ *denote the expectation of* $f(S_t)$ *when one changes the starting point of* (S_t) *from* x_0 *to* r*. We have*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_{x_0}\left[\mathbf{1}_{\{M_\tau\leq B\}}\psi(S_\tau)\right] = \mathbb{E}_{x_0}\left[\psi(S_T)\right] - \mathbb{E}_{2B-x_0}\left[\psi(S_T)\right]$$

Proof. From Proposition 7 we know the optimal stopping time to be $\tau^* = T_B \wedge T$, so we must evaluate $\mathbb{E}_{x_0}\left[\psi(S_{T_B\wedge T})\right]$. This is straightforward when we use the density of the stopped process from Lemma 8 :

$$\mathbb{E}_{x_0}\left[\psi(S_{T_B\wedge T})\right] = \int_{\mathbb{R}} \psi(u)\phi\left(\frac{u-x_0}{\sigma\sqrt{T}}\right)du - \int_{\mathbb{R}} \psi(u)\phi\left(\frac{u+x_0-2B}{\sigma\sqrt{T}}\right)du$$

The first integral is the expectation of $\psi(S_T)$ when the process (S_t) starts in x_0. The second integral can also been seen as the expectation of $\psi(S_T)$, but now the starting point of the process (S_t) has been shifted to $2B-x$. □

The result of Proposition 9 has an analogy in the theory of pricing Barrier options, shown in Appendix A.1.

We can also compute the probability that the last look-feature is activated:

Proposition 10. ✤
Let (S_t) *and* T_B *be as defined above. Then,*

i) The hitting time T_B *has the scaled inverse chi-square distribution,*

$$T_B \overset{law}{=} \frac{(B-x_0)^2}{\sigma^2 Z^2}, \quad Z\sim\mathcal{N}(0,1)$$

ii) The probability of the last look-feature coming into effect is given by

$$P(T_B\leq T) = 2-2\Phi\left(\frac{B-x_0}{\sigma\sqrt{T}}\right)$$

Proof. We first prove $i)$ for the sell quote, meaning that $B > S_0$. From (3.1) we have

$$\begin{aligned} M_T &:= \sup_{t\in[0,T]} S_t \\ &= \sup_{t\in[0,T]} \frac{W_T - x_0}{\sigma} =: Y_T \end{aligned}$$

Define

$$y := \frac{B-x_0}{\sigma}$$

We apply the following corollary to the reflection principle for the Brownian motion (see Appendix B):

$$P(M_T\leq B) = P(|W_T|\leq B)$$

Hence, we have

$$\begin{aligned} P(T_B\leq T) &= P(B\leq M_T) \\ &= P(y\leq Y_T) \\ &= P(y\leq |W_T|) \\ &= P(y\leq |Z|\sqrt{T}) \\ &= P\left(\frac{y}{Z^2}\leq T\right) \end{aligned}$$

Which implies that the hitting time T_B has the scaled inverse chi-square law,

$$T_B \stackrel{law}{=} \frac{(B - x_0)^2}{\sigma^2 Z^2}$$

To prove ii), we could use the CDF of the scaled inverse chi-square distribution, properties of the incomplete gamma function and it's relation to the normal CDF. However, the claim can also be derived using only the reflection principle and the symmetry of the normal distribution:

$$\begin{aligned}
P(T_B \leq T) &= P(M_T > B) \\
&= P(Y_T > y) \\
&= 1 - P(Y_T \leq y) \\
&= 1 - (P(W_T \leq y) - P(W_T \leq -y)) \\
&= 1 - P(W_T \leq y) + 1 - P(W_T \geq -y) \\
&= 2 - P(W_T \leq y) - P(-W_T \leq y) \\
&= 2 - P(W_T \leq y) - P(W_T \leq y) \\
&= 2 - 2\Phi\left(\frac{y}{\sqrt{T}}\right) \\
&= 2 - 2\Phi\left(\frac{B - x_0}{\sigma\sqrt{T}}\right)
\end{aligned}$$

This completes the proof. □

Chapter 4

Picking-off risk under the Skellam process

4.1 Construction and general properties

The Skellam process is in a certain precise way a discrete counterpart to the Brownian motion. It is therefore a natural starting point when we want to go from the Brownian motion to a process that takes values in a countable set.

Let $(\Omega, \mathcal{F}^N, \mathbb{F}, P)$ be a complete filtered probability space, let (N_t^+) and (N_t^-) be two $\mathbb{F}$-adapted independent Poisson processes with rates λ^+ and λ^-. Definition and basic properties of the Poisson process are collected in Appendix A.2.

Let $x_0 \in \mathbb{N}$ be given. The Skellam process evolves as the the difference between the two Poisson processes:

$$S_t := N_t^+ - N_t^- + x_0 \tag{4.1}$$

We shall for simplicity set $x_0 = 0$ in the remainder of this chapter.

Proposition 11. *The Skellam process is an $\mathbb{F}$-martingale if and only if $\lambda^+ = \lambda^-$.*

Proof. Integrability follows from observing that for any given t, we have

$$\begin{aligned} \mathbb{E}\left[|S_t|\right] &\leq \mathbb{E}\left[N_t^+\right] + \mathbb{E}\left[N_t^-\right] \\ &= (\lambda^+ + \lambda^-)t < \infty \end{aligned}$$

The process is adapted since both (N_t^+) and (N_t^-) are $\mathbb{F}$-adapted.

The martingale property follows from

$$\begin{aligned} \mathbb{E}[S_t \mid \mathcal{F}_u] &= \mathbb{E}[N_t^+ \mid \mathcal{F}_u] - \mathbb{E}[N_t^- \mid \mathcal{F}_u] \\ &= (\lambda^+ - \lambda^-)t \end{aligned}$$

We see that the martingale property holds if and only if $\lambda^+ = \lambda^-$. □

The Skellam process is particularly tractable in our application because a double-jump is a zero probability event, a property inherited from the constituent Poisson processes. This is a nice property because the process can only reach a given $m \in \mathbb{N}$ if it has already taken every integer value $1, 2, \ldots, m-1$ for a positive amount of time, which implies that the stopped maximum process (almost surely) never exceeds B, $P(M_{T\wedge T_B} > B) = 0$. If, on the other hand, there was a positive probability of a double-jump occurring, we could not be sure that the process didn't jump from say $B-1$ directly to $B+1$. We shall come back to the issue of double-jumps in the next chapter.

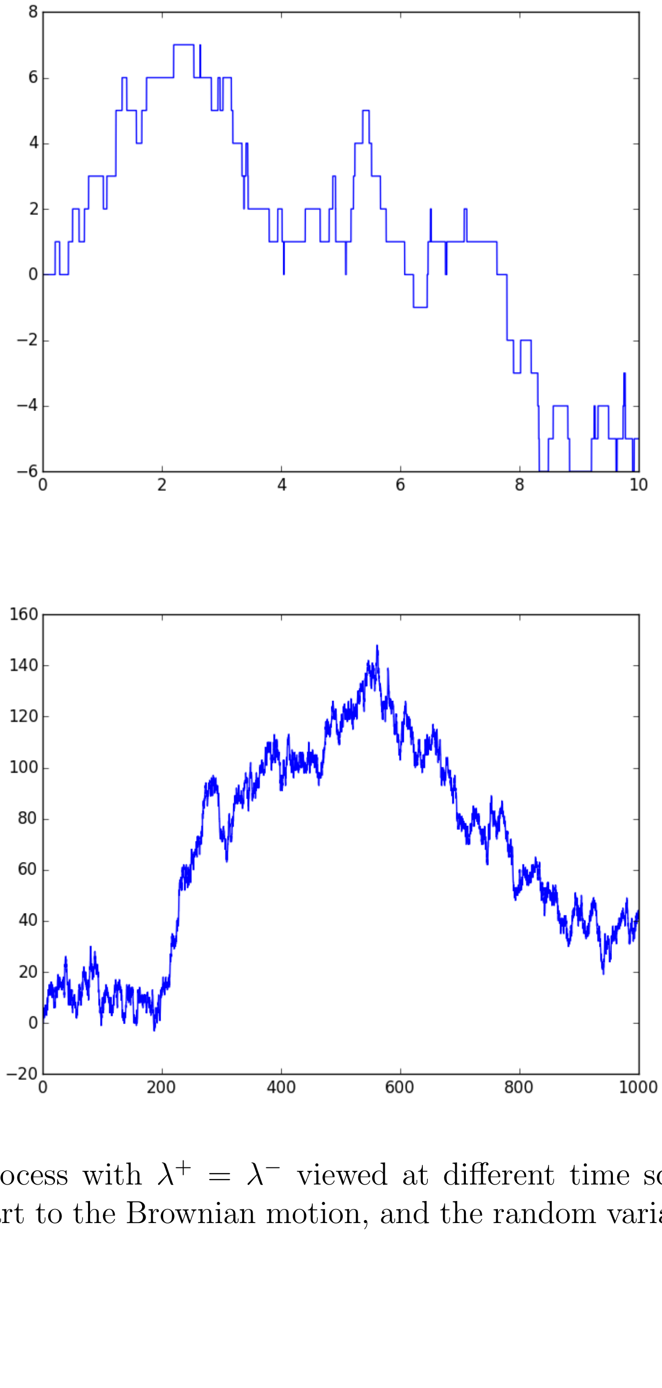


Figure 4.1: Skellam process with $\lambda^+ = \lambda^-$ viewed at different time scales. The process is a discrete-valued countpart to the Brownian motion, and the random variable $Y_t/\sqrt{t}$ converges to a $\mathcal{N}(0,1)$.

Lemma 12. *Under the assumptions of this chapter, the following equality holds:*

$$P(M_{T\wedge T_B} > B) = 0$$

Proof. We prove only the case where $0 < B$. The proof of the case $B < 0$ follows the same arguments.

Define the set N and note from the definition of the hitting time T_B and knockout time $T_\dagger$ that we have

$$\begin{aligned} N &:= \{M_{T\wedge T_B} > B\} \\ &= \{T_\dagger < T_B\} \\ &= \{S_k \neq B, \text{ all } k \leq T\} \cap \{S_k > B, \text{ some } k \leq T\} \end{aligned}$$

Hence there must be a "double jump" for at least some $k \leq T$,

$$N \subseteq \{U_k - U_{k^-} \geq 2\}$$

But by the properties of the Poisson process $\{U_k - U_{k^-} \geq 2, \, t \geq 0\}$ is a null set, hence $P(N) = 0$. □

Lemma 13. *The Skellam process is a Levy process.*

Proof. We can write the Skellam process as the sum of two independent compound Poisson processes:

$$S_t = \sum_{n=1}^{N_t^+} 1 + \sum_{n=1}^{N_t^-} (-1)$$

The compound Poisson process is a Levy process, and the sum of two independent Levy processes is again a Levy process (Cont and Tankov, 2004, Theorem 4.1). □

Lemma 14. *Fix $t > 0$ and suppose $S_0 = 0$. The probability mass function for the random variable S_t is given by*

$$p(k) = e^{-t(\lambda^+ + \lambda^-)} \left(\frac{\lambda^+}{\lambda^-}\right)^{k/2} I_{|k|}\left(2t\sqrt{\lambda^+\lambda^-}\right)$$

Where $I_k(x)$ is the modified Bessel function of the first kind

$$I_k(x) = \left(\frac{1}{2}x\right) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^n}{n!\Gamma(k+n+1)}$$

Proof. See Barndorff-Nielsen et al. (2012). □

We shall for the remainder for this chapter assume that our Skellam process is a martingale, meaning that $\lambda^+ = \lambda^-$ (cf. Proposition 11). Associated with our Skellam martingale we define the "joint intensity" λ by

$$\lambda := \frac{1}{2}\lambda^+ = \frac{1}{2}\lambda^-$$

We note that the Skellam process is also known as a **simple birth and death-process** (Grimmett and Stirzaker, 2001, pg. 270)

4.2 Minimum resting time

Take as given a quote $(K, T) \in \mathbb{N} \times \mathbb{R}_+$, and let (S_t) be a Skellam martingale (meaning that $\lambda^+ = \lambda^-$).

Proposition 15. *The optimal stopping time for the quote with a minimum resting time (problem 1.1) is $\tau^* = T$.*

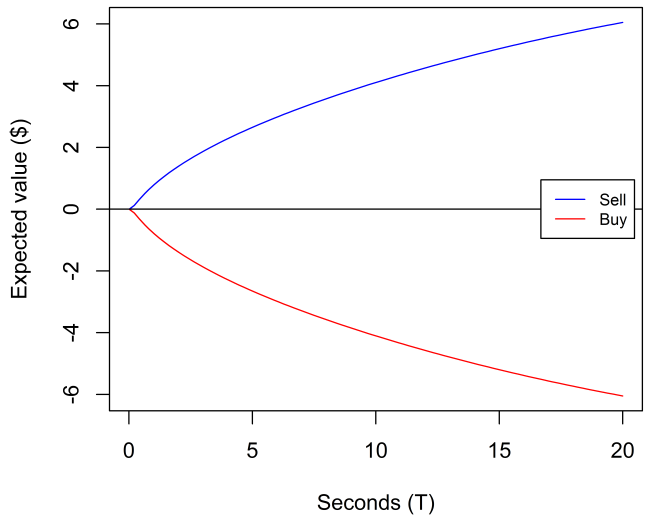
Proof. The claim follows from Propositions 1 and 11 and Lemma 12. □

Proposition 16. *The value of the quote, problem (1.1), is given by:*

$$\sup_{\tau \in \mathcal{T}} \{\mathbb{E}[\psi(S_\tau)]\} = e^{-\lambda T} \sum_{k \in \mathbb{Z}} \left[\psi(k) I_{|k|}(\lambda T)\right] \tag{4.2}$$

Proof. The claim follows by inserting $\lambda^+ = \lambda^-$ and $\lambda := \frac{1}{2}\lambda^+$ in Lemma 14. □

Figure 4.2: Picking-off risk as function of minimum resting time (Skellam process). (*The buy quote has been reflected on the x-axis for illustration purpose)*



4.3 Last look

Take as given a last-look sell quote $(K, T, B) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{N}$, and let (S_t) be a Skellam martingale with $S_0 = 0$.

Proposition 17. *The optimal stopping time for the quote with a last look-feature (problem 1.2) is $\tau^* = T \wedge T_B$.*

Proof. The claim follows from Propositions 2 and 11 and Lemma 12. □

Proposition 18. ☆
The probability mass function of the stopped process $S_{T \wedge T_B}$ is given by

$$P(S_{T \wedge T_B} = x) = \begin{cases} 0 & \text{if } x > B \\ P(N_T = B) & \text{if } x = B \\ P(S_T = x) - P(S_T = 2B - x) & \text{if } x < B \end{cases}$$

Proof. The case $x > B$ follows from noting that $P(M_{T_B} > B) = 0$ (Lemma 12). Therefore $P(S_{T \wedge T_B} > B) = 0$.

For the case $x < B$ we first note that

$$\begin{aligned} P(S_{T \wedge T_B} = x) &= P(S_T = x, M_T < B) \\ &= P(S_T = x) - P(S_T = x, M_T \geq B) \end{aligned}$$

Where the second equality follows from the law of total probability.

We shall derive an adapation of the reflection principle to the Skellam process in order to turn the expression $P(S_T = x, M_T \geq B)$ into one not involving M_T.

Since $P(M_{T_B} > B) = 0$ we can conclude that $S_{T_B} = B$ almost surely. Recall that the Skellam process is a Levy process. Moreover, the assumption $\lambda^+ = \lambda^-$ implies that the process is symmetric, meaning that for any t,

$$S_t \stackrel{\text{law}}{=} -S_t$$

Using these properties we deduce that

$$\begin{aligned} P(S_T = x, M_T \geq B) &= P(S_T = x, T_B \leq T) \\ &= P(S_T = x \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_T - S_{T_B} = x - B \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_{T-T_B} = x - B \mid T_B \leq T)P(T_B \leq T) \\ &= P(-S_{T-T_B} = x - B \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_{T-T_B} = B - x \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_T - S_{T_B} = B - x \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_T = 2B - x \mid T_B \leq T)P(T_B \leq T) \\ &= P(S_T = 2B - x, M_T \geq B) \\ &= P(S_T = 2B - x) \end{aligned}$$

The last equality use that since $x < B$ we have $2B - x > B$, and thus on the set $\{S_t > 2B - x\}$ we have $M_t > B$ almost surely. Hence, we have

$$P(S_{T \wedge T_B} = x) = P(S_T = x) - P(S_T = 2B - x)$$

For the case $x = B$ we first note that

$$P(S_{T \wedge T_B} = x) = P(M_T = B)$$

We shall now use another adaptation of the reflection principle to prove that

$$M_t \stackrel{\text{law}}{=} N_t, \text{ any } t \geq 0$$

Following the same steps as in the previous case, we have that

$$
\begin{aligned}
P(M_T \le B) &= P(S_T \le B, M_T \le B) \\
&= P(S_T \le B) - P(S_T \le B, M_T > B) \\
&= P(S_T \le B) - P(S_T \le B \mid T_B < T)P(T_B < T) \\
&= \dots \\
&= P(S_T \le B) - P(S_T \ge 2B - x, M_T > B) \\
&= P(S_T \le B) - P(S_T \ge 2B - x) \\
&= P(S_T \le B) - P(S_T \ge 2B - B) \\
&= P(S_T \le B) - P(S_T \ge B) \\
&= P(S_T \le B) - P(-S_T \ge B) \\
&= P(S_T \le B) - P(S_T \le -B) \\
&= P(|S_T| \le B) \\
&= P(|U_T + D_T| \le B) \\
&= P(N_T \le B)
\end{aligned}
$$

Equality of laws implies that $P(M_T = B) = P(N_T = B)$.

□

Proposition 19. *Let $I_k(x)$ be the modified Bessel function of the first kind. The value of (1.2) is given by*

$$
\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{M_\tau \le B\}} \psi(S_\tau)\right] = e^{-\lambda T} \left(\sum_{x<B} \psi(x) \left[I_{|x|}(\lambda T) - I_{|2B-x|}(\lambda T) \right] + \psi(B) \frac{(\lambda T)^B}{B!} \right)
$$

Proof. We know from Proposition 17 that the optimal stopping time is $T \wedge T_B$. Hence, we get

$$
\begin{aligned}
\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\mathbf{1}_{\{M_\tau \le B\}} \psi(S_\tau)\right] &= \mathbb{E}\left[\psi(S_{T \wedge T_B})\right] \\
&= \sum_{x \in \mathbb{Z}} \psi(x) P(S_{T \wedge T_B} = x)
\end{aligned}
$$

The claim now follows from applying the density of the stopped process (Lemma 18). □

We can also compute the probability that the last look comes into effect, as for the case when the price follows a Brownian motion.

Proposition 20. *The probability of the last look coming into effect, $P(T_B \le T)$, is given by*

$$
\begin{aligned}
P(T_B \le T) &= 1 - P\left(N_T \le B - 1\right) \\
&= 1 - e^{-\lambda T} \sum_{n=0}^{B-1} \frac{(\lambda T)^n}{n!}
\end{aligned}
$$

Proof. We have

$$\begin{aligned} P(T_B \leq T) &= P(M_T \geq B) \\ &= 1 - P(M_T \leq B - 1) \\ &= 1 - P(N_T \leq B - 1) \\ &= 1 - e^{-\lambda T} \sum_{n=0}^{B-1} \frac{(\lambda T)^n}{n!} \end{aligned}$$

And the proof is complete. □

Chapter 5

Picking-off risk under an integer-valued Levy process

5.1 Construction and general properties

The Skellam process is unrealistic in that it can only change by one tick in each increment. We remedy this shortcoming here, as we construct a class of integer-valued Levy processes better suited to our purpose. We show that the Skellam process is a special case of this class. As an example of an integer-valued Levy process we shall use the ΔNB Levy process proposed by Barndorff-Nielsen et al. (2012).

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, and let (N_t^+) and (N_t^-) be two independent $\mathbb{F}$-adapted Poisson processes with intensities λ^+ and λ^-. Let $(u_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}})$ be two sequences of iid random variables taking values in $\mathbb{N}$, with $(u_n), (d_n), (N_t^+)$ and (N_t^-) all independent from each other. We also require that u_n and d_n are $\mathcal{F}_{T_n}$-measurable for all n, and assume that $\mathbb{E}[u_1] < \infty$ and $\mathbb{E}[d_1] < \infty$.

Define the **up** and **down** processes as

$$U_t := \sum_{n=1}^{N_t^+} u_n \tag{5.1}$$

$$D_t := \sum_{n=1}^{N_t^-} d_n \tag{5.2}$$

We define the **integer-valued Levy price process** (S_t) as

$$S_t := U_t - D_t \tag{5.3}$$

The Skellam process is clearly the particular instance of the general process in (5.3) attained with $u_n = d_n = 1$ for all n. The Skellam process was shown to be a a martingale when the intensities of the up- and down-tick processes were the same (Proposition 11). There was an overlap between the notion of being a martingale and symmetry of distribution. The general class of integer-valued Levy process allows for other situations as well; one can for example have an asymmetric martingale price process which exhibits frequent small downward jumps and rare large upward jumps, in which the relative frequency of the two jumps adjust to ensure martingality (see Figure 5.1 for an example).

Proposition 21. ☆
The process (S_t) *is an* $\mathbb{F}$*-martingale if and only if*

$$\frac{\lambda^+}{\lambda^-} = \frac{\mathbb{E}[d_1]}{\mathbb{E}[u_1]}$$

Proof. Integrability follows from applying Wald's equation (see Proposition 48 in the Appendix):

$$\begin{aligned}\mathbb{E}[|S_t|] &= \mathbb{E}[U_t] + \mathbb{E}[D_t] \\ &= (\lambda^+\mathbb{E}[u_1] + \lambda^-\mathbb{E}[d_1])t < \infty\end{aligned}$$

The process (S_t) is $\mathbb{F}$-adapted because (N_t^+), (N_t^-), (u_n) and (d_n) are all assumed $\mathbb{F}$-adapted.

For the martingale property, observe that

$$\begin{aligned}\mathbb{E}[S_t \mid \mathcal{F}_j] &= \mathbb{E}[U_t - D_t \mid \mathcal{F}_j] \\ &= U_j + (t-j)\lambda^+\mathbb{E}[u_1] - D_j - (t-j)\lambda^-\mathbb{E}[d_1] \\ &= S_j + (t-j)(\lambda^+\mathbb{E}[u_1] - \lambda^-\mathbb{E}[d_1])\end{aligned}$$

We see that the martingale property holds if and only if

$$\lambda^+\mathbb{E}[u_1] - \lambda^-\mathbb{E}[d_1] = 0$$

Which is equivalent to the stated claim.

□

Example. Barndorff-Nielsen et al. (2012) propose a model that they call the Δ NB Levy process. This process is a special case of our general class of integer-valued Levy processes.

The process Δ NB is constructed by setting u_1 and d_1 to be logarithmically distributed random variables. Recall that a random variable X has the logarithmic distribution with parameter $q \in (0,1)$ if X has a discrete distribution on $\mathbb{N}_0$ with probability density function f given by

$$f(n) = \frac{1}{-\ln(1-q)} \frac{q^n}{n}$$

$$n = 0, 1, 2, \ldots$$

We shall later use this process for our numerical applications.

5.2 Minimum resting time

Proposition 1 applies in the case when (S_t) is an integer-valued Levy martingale.

If we specify a *particular* process, we may or may not be able to write down an expression for the picking off-risk $\mathbb{E}[\psi(S_T)]$, depending on whether the distribution of the random variable S_T is known. The ΔNB Levy process in our earlier example is one such case.

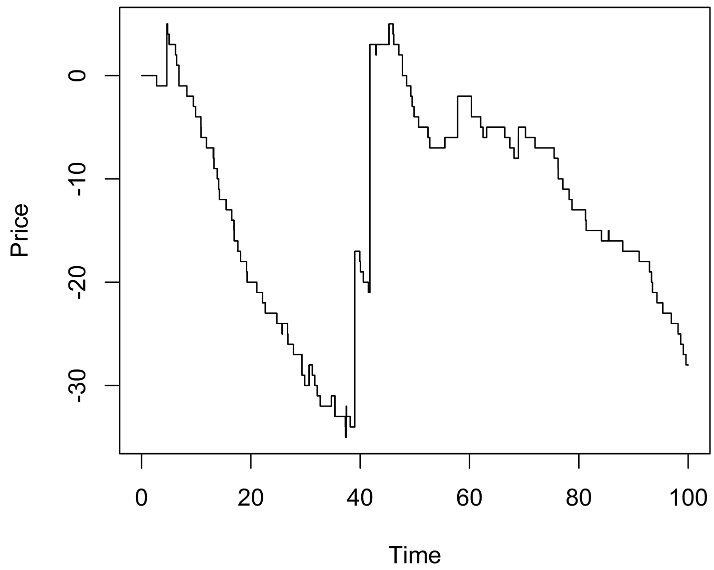


Figure 5.1: ΔNB, an integer valued Levy process with multi-tick increments. Parameters $p \approx 0.2125, q^+ = .9, q^- = .1$. These parameter values result in an asymmetric martingale.

5.3 Last look

Throughout this section we take as given a last-look sell quote, meaning a triplet $(K, T, B) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{N}$ satisfying

$$S_0 < K < B$$

.

The optimal stopping of a last look-quote becomes a much more subtle affair under a general integer-valued Levy jump process than in the other cases we have studied in this thesis. The reason is that now we have a positive probability of the price jumping past the barrier, in other words we have

$$P(M_{T \wedge T_B} > B) > 0$$

The problem of finding the optimal stopping time is in general a finite horizon, continuous time stopping problem in a countable state space. When the underlying stochastic process is a Levy type, it is possible to show that the optimal stopping rule takes the form of a stopping boundary, meaning that one should stop at the first time the process cross the boundary and enters into the stopping set (Oksendal, 2013). In general the stopping set will depend on time, meaning whether a point x in the sample space is in the stopping set depends on the amount of time remaining $(T - t)$. If, however, the quote never expires $(T = \infty)$, it turns out that the stopping region is time-invariant, greatly simplifying the problem.[1] For this reason we shall investigate the infinite horizon-case to get better acquainted with our problem. We shall thereafter use the ideas from the infinite-horizon solution to develop an approximation for the finite-horizon case.

5.3.1 Extending the state space

To attain computational results we shall use techniques that exploits the markovianity of Levy processes. General properties and definitions regarding Markov processes are collected in Appendix A.2.6. We note that the techniques used in this section could equally well be used for

[1]The situation is analogous to the optimal execution boundary of American put options. The explicit form of the boundary is not known in a continuous-time finite-horizon framework, while it is constant and relatively easy to obtain in an infinite horizon framework.

the Brownian motion and the Skellam process

Recall that we are working with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and an integer-valued Levy price process (S_t) adapted to the filtration $\mathbb{F}$.

We now face the issue that the payoff from the last look-quote (Definition 11) is path-dependent, which means that the *process* $(\mathbf{1}_{\{M_t \leq B\}}\psi(S_t)\,,\, t \geq 0)$ is non-markovian. We solve this issue by introducing the **graveyard state** $\{\dagger\}$.

First we define the measurable space $(E, \mathcal{E})$ by

$$E := \mathbb{Z} \cup \{\dagger\}, \quad \mathcal{E} := 2^E$$

We modify the payoff function to reflect the enlarged state space,

$$\psi(\{\dagger\}) = 0$$

We define the process $(\tilde{S}_t)$ taking values in E by

$$\tilde{S}_t := \begin{cases} S_t & \text{if } t < T_\dagger \\ \{\dagger\} & \text{if } t \geq T_\dagger \end{cases}$$

We see that $\{\dagger\}$ is an absorbing state for $(\tilde{S}_t)$.

Lemma 22. *The process $(\tilde{S}_t)$ is $\mathbb{F}$-adapted.*

Proof. (☆) We need to show that the random variable $\tilde{S}_t : \Omega \to E$ is a $(\mathcal{F}_t, \mathcal{E})$-measurable function for all $t \geq 0$. Take any $t \geq 0$. The random variable $\tilde{S}_t$ is said to be $(\mathcal{F}_t, \mathcal{E})$-measurable if for all $A \in \mathcal{E}$, we have $\tilde{S}_t^{-1}(A) \in \mathcal{F}_t$. Since (S_t) is adapted to $\mathbb{F}$ we know that S_t is $\mathcal{F}_t$-measurable, and so we only have to check the isolated point $\{\dagger\}$. But clearly

$$\tilde{S}_t^{-1}(\{\dagger\}) = \{\omega \in \Omega \mid M_t > B\} \subset \mathcal{F}_t$$

We conclude that $\tilde{S}_t$ is $\mathcal{F}_t$-measurable for any arbitrary $t \geq 0$, and hence adapted to the filtration $\mathbb{F}$. □

Proposition 23. *The process $(\tilde{S}_t)$ has the Markov property.*

Proof. (☆) We shall prove that for all $0 \leq s \leq t$ and any bounded Borel-measurable function $f : E \to \mathbb{R}$, we have

$$\mathbb{E}[f(\tilde{S}_t) \mid \mathcal{F}_u] = \mathbb{E}[f(\tilde{S}_t) \mid \tilde{S}_u]$$

Where the notation $\mathbb{E}[\cdot \mid X]$ is understood to mean $\mathbb{E}[\cdot \mid \sigma(X)]$. Note that the expectations involved are well-defined by our standing assumptions from chapter 2. We have that

$$\mathbb{E}[f(\tilde{S}_t) \mid \mathcal{F}_u] = \underbrace{\mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u \neq \dagger\}} \mid \mathcal{F}_u]}_{(*)} + \underbrace{\mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u = \dagger\}} \mid \mathcal{F}_u]}_{(**)}$$

For the term $(*)$ we have

$$\begin{aligned}
\mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u \neq \dagger\}} \mid \mathcal{F}_u] &= \Big(\mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_t = \dagger\}} \mid \mathcal{F}_u] + \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_t \neq \dagger\}} \mid \mathcal{F}_u]\Big)\,\mathbf{1}_{\{M_u \leq B\}} \\
&= \Big(\mathbb{E}[\mathbf{1}_{\{\sup_{j\in[u,t]} S_j > B\}} \mid \mathcal{F}_u] f(\dagger) + \mathbb{E}[f(S_t)\mathbf{1}_{\{\sup_{j\in[u,t]} S_j \leq B\}} \mid \mathcal{F}_u]\Big)\,\mathbf{1}_{\{M_u \leq B\}} \\
&= \Big(\mathbb{E}[\mathbf{1}_{\{\sup_{j\in[u,t]} S_j > B\}} \mid S_u] f(\dagger) + \mathbb{E}[f(S_t)\mathbf{1}_{\{\sup_{j\in[u,t]} S_j \leq B\}} \mid S_u]\Big)\,\mathbf{1}_{\{M_u \leq B\}} \\
&= \Big(\mathbb{E}[\mathbf{1}_{\{\sup_{j\in[u,t]} S_j > B\}} \mid \tilde{S}_u] f(\dagger) + \mathbb{E}[f(S_t)\mathbf{1}_{\{\sup_{j\in[u,t]} S_j \leq B\}} \mid \tilde{S}_u]\Big)\,\mathbf{1}_{\{M_u \leq B\}} \\
&= \Big(\mathbb{E}[\mathbf{1}_{\{\sup_{j\in[u,t]} S_j > B\}} \mid \tilde{S}_u] f(\dagger) + \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\sup_{j\in[u,t]} S_j \leq B\}} \mid \tilde{S}_u]\Big)\,\mathbf{1}_{\{M_u \leq B\}} \\
&= \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u \neq \dagger\}} \mid \tilde{S}_u]
\end{aligned}$$

For the term $(**)$ we have

$$\begin{aligned}\mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u=\dagger\}} \mid \mathcal{F}_u] &= f(\dagger)\mathbb{E}[\mathbf{1}_{\{\tilde{S}_u=\dagger\}} \mid \mathcal{F}_u] \\ &= f(\dagger)\mathbb{E}[\mathbf{1}_{\{\tilde{S}_u=\dagger\}} \mid \tilde{S}_u] \\ &= \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u=\dagger\}} \mid \tilde{S}_u]\end{aligned}$$

Combining $(*)$ and $(**)$ gives

$$\begin{aligned}\mathbb{E}[f(\tilde{S}_t) \mid \mathcal{F}_u] &= \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u\neq\dagger\}} \mid \tilde{S}_u] + \mathbb{E}[f(\tilde{S}_t)\mathbf{1}_{\{\tilde{S}_u=\dagger\}} \mid \tilde{S}_u] \\ &= \mathbb{E}[f(\tilde{S}_t) \mid \tilde{S}_u]\end{aligned}$$

And the proof is complete. □

5.3.2 Infinite horizon

In this section we will see that the stopping boundary is constant when $T = \infty$. The case of infinite horizon is not the most meaningful in light of our application, but we shall use the ideas from this section to construct an approximation procedure to the finite-horizon case.

The assumption that $T = \infty$ will allow us to concentrate only on the jump times of the process, essentially transforming our continuous-time problem into one in discrete-time. Discrete-time stopping problems are in some regards much easier to handle, and we derive a recursive solution that allows an iterative computational algorithm.

We denote by $\mathcal{T}_t^s$ the collection of stopping times taking values in $[t, s)$, and by $\mathcal{T}_t$ the collection of stopping times larger than t but finite almost surely. For consistency with our earlier notation we write $\mathcal{T}$ for the collection $\mathcal{T}_0$.

Suppressing the fixed quote (K, T, B) from our notation, we define the **conditional value function** $v : E \to \mathbb{R}$ as

$$v(x) := \sup_{\tau\in\mathcal{T}} \left[\psi(\tilde{S}_\tau) \mid \tilde{S}_0 = x\right]$$

We regain the (unconditional) value function $\hat{V}$ defined in (1.2) from $v(0) = \hat{V}$.

Since $(\tilde{S}_t)$ is a (strong) Markov process we can express the expected continuation value at time t through the conditional value function:

$$\begin{aligned}\sup_{\tau\in\mathcal{T}_t} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \mathcal{F}_t] &= \sup_{\tau\in\mathcal{T}_t} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_t = x] \\ &= \sup_{\tau\in\mathcal{T}} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_0 = x] \\ &= v(x)\end{aligned}$$

If τ^* is an optimal stopping time, in the sense that it attains the supremum of (1.2), it must at time τ^* be better to stop than to continue,

$$\begin{aligned}\psi(\tilde{S}_{\tau^*}) &\geq \sup_{\tau\in\mathcal{T}_{\tau^*}} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \mathcal{F}_{\tau^*}] \\ &= v(\tilde{S}_{\tau^*})\end{aligned}$$

Hence we define the infinite horizon stopping region as

$$D_t := \{(x, t) \in E \times \mathbb{R}_+ \mid \psi(\tilde{S}_t) \geq v(\tilde{S}_t)\}$$

An important feature of the infinite-horizon problem is that the stopping region is time-invariant; if it is optimal to stop at the time-space pair (t, x) then it is also optimal to stop at the time-space pair (s, x):

Lemma 24. ☆
The stopping region is constant:

$$D_t = D_u$$

for any $t, u \geq 0$.

Proof. Because the increments of a Levy process are stationary, we have

$$\sup_{\tau \in \mathcal{T}_t} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_t = x] = \sup_{\tau \in \mathcal{T}_u} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_u = x]$$

From the definition of the stopping region,

$$\begin{aligned} D_t &= \{x \in \mathbb{Z} \mid \psi(x) \geq \sup_{\tau \in \mathcal{T}_t} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_t = x]\} \\ &= \{x \in \mathbb{Z} \mid \psi(x) \geq \sup_{\tau \in \mathcal{T}_u} \mathbb{E}[\psi(\tilde{S}_\tau) \mid \tilde{S}_u = x]\} \\ &= D_u \end{aligned}$$

□

Definition 13. *We say that* $n \geq 0$ *is a* ***jump time*** *of the process* (S_t) *if*

$$|S_n - S_{n^-}| > 0$$

We denote the set of jump times by $\mathcal{J}$,

$$\mathcal{J} := \{n \geq 0 \; : \; |S_n - S_{n^-}| > 0\}$$

Since (S_t) is a Levy process, the set of jump times is a countable set. For notational convenience we relabel the jump times of (S_t) as $1, 2, 3, \ldots$. We shall in this section only use the subscript n when referring to a jump time.

Define the process $X_n : \mathbb{N} \times \Omega \to E$ by

$$X_n = \tilde{S}_n, \; n \in \mathcal{J}$$

The process $(S_n)_{n \in \mathcal{J}}$ is called the discrete-time Markov chain embedded in (S_t). Similarly, (X_n) is the discrete-time Markov chain embedded in $(\tilde{S}_t)$. We shall therefore use a notation standard in the study of such chains,

$$p_(x, y) := P(X_{n+1} = y \mid X_n = x)$$

The next proposition tells us that we only have to consider the jump times when looking for the first optimal stopping time.

Proposition 25. ☆
Under the assumptions of this section, assume there exists a stopping time for (1.2) *that is finite with probability one. Then, the* ***first*** *optimal stopping time is a jump time of* (S_t).

Proof. Assume τ is optimal for (1.2) and finite with probability one. For any given $\omega \in \Omega$ there exists a jump time $n \leq \tau$ such that $\tilde{S}_\tau = \tilde{S}_n$. Since the value of $\tilde{S}_\tau$ is in the stopping region, so is the value of $\tilde{S}_n$ by Lemma 24. Hence n is also an optimal stopping time. Since $n \leq \tau$ and τ was an arbitrary optimal stopping time, the first optimal stopping time is a jump time. ☐

Proposition 26. *For any measurable function $f : \mathbb{Z} \to \mathbb{R}_0^+$ define the functional $f \mapsto Uf$ by*

$$
\begin{aligned}
(Uf)(x) &:= \sum_{y \in \mathbb{Z}} p(x, y) f(y) \\
&=: \mathbb{E}_x[f(X_1)]
\end{aligned}
$$

Under the assumptions of this section the following holds:

i) The conditional value function $v(x)$ is the least superharmonic majorant of the payoff function ψ (cf. Definition (34) in the appendix)

ii) $v = \max\{\psi, Uv\}$

iii) $v = \lim_{k \to \infty} v_k$*, where*

$$
\begin{aligned}
v_0 &:= \psi \\
v_{k+1} &:= \max(v_k, Uv_k), \quad k \geq 0
\end{aligned}
$$

Proof. We adapt the proof of Geiger (n.d.).
i) We clearly have $v \geq \psi$. To show that v is superharmonic let τ_j be a sequence of finite stopping times such that for every $x \in \mathbb{Z}$

$$
\mathbb{E}_x[\psi(X_{\tau_j})] \uparrow v(x) \quad \text{as } j \to \infty
$$

We condition on the first transition, use the strong Markov property and the monotone convergence theorem to get

$$
\begin{aligned}
v(x) &\geq \mathbb{E}_x[\psi(X_{1+\tau_j})] \\
&= \sum_{y \in \mathbb{Z}} p(x, y) \mathbb{E}_y[\psi(X_{1+\tau_j})] \\
&= \sum_{y \in \mathbb{Z}} p(x, y) \mathbb{E}_y[\psi(X_{\tau_j})] \\
&\longrightarrow \sum_{y \in \mathbb{Z}} p(x, y) v(y) \quad \text{as } j \to \infty \\
&= (Uv)(x)
\end{aligned}
$$

To show that v is the least superharmonic majorant, suppose that another function h satisfy $h \geq \psi$ and $h \geq Uh$. Then, for every $x \in \mathbb{Z}$ and every finite stopping time τ,

$$
h(x) = \mathbb{E}_x[h(X_0)] \geq \mathbb{E}_x[h(X_\tau)] \geq \mathbb{E}_x[\psi(X_\tau)]
$$

Where the first inequality follows from the stopping theorem for supermartingales, cf. Lemma 58 in the appendix. We take the supremum over all stopping times that are almost surely finite to get $h \geq v$.

iii) To show that the sequence v_k converge to v we first define

$$\tilde{v} := \lim_{k\to\infty} v_k$$

We will show that $\tilde{v}$ is the least superharmonic majorant of v, and so *iii*) will follow from *i*).

By the monotone convergence theorem and the definition of v we have

$$U\tilde{v} = \lim_{k\to\infty} Uv_k \leq \lim_{n\to\infty} v_{k+1} = \tilde{v}$$

Now suppose that another function h has the properties that $h \geq \psi$ and $h \geq Uh$. We will use induction to show that

$$h \geq v_k, \quad k \geq 0$$

For $k = 0$ the claim follows directly from the definition of v_k. Now suppose the claim holds for k. Then,

$$Uv_k \leq Uh \leq h$$

which implies

$$v_{k+1} := \max\{v_k, Uv_k\} \leq h$$

iii) We will show by induction that

$$v_{k+1} = \max(\psi, Uv_k)$$

From *ii)* we then know that the claim holds when we let k go to infinity.

By definition we have that

$$\begin{aligned} v_1 &:= \max\{v_0, Uv_0\} \\ &= \max\{\psi, Uv_0\} \end{aligned}$$

Suppose the claim holds for k. Then,

$$\begin{aligned} v_{k+1} &= \max\{v_k, Uv_k\} \\ &= \max\{\psi, Uv_{k-1}, Uv_k\} \\ &= \max\{\psi, Uv_k\} \end{aligned}$$

Which completes the proof. □

Proposition 26 has the intuitive interpretation that we stop whenever the payoff from stopping equals the conditional expected value from continuing. Under a suitable regularity condition, that stopping time turns out to be the smallest optimal stopping time, justifying our earlier definition of the stopping region.

Corollary 27. *Under the assumptions of this section, the stopping time*

$$\tau^* := \inf\{t \geq 0 \mid v = \psi\}$$

is the smallest optimal stopping time for (1.2).

We note that for the class of integer-valued Levy processes considered in this chapter we can readily compute $\mathbb{E}_x[\psi(X_1)]$ by conditioning on the direction of the first jump:

$$\mathbb{E}_x[\psi(X_1)] = p\mathbb{E}[\psi(x + u_1)] + (1 - p)\mathbb{E}[\psi(x - d_1)], \quad x \leq B$$

This expression is useful for numerical implementation of our solution algorithm.

5.3.3 Finite horizon

We solved the optimal stopping with an infinite time horizon by restricting our attention only to the jump times of the price process. We used arguments where the discrete-time Markov chain were allowed to take an unlimited number of transitions before reaching the stopping region. With a finite time horizon these arguments no longer work.

Instead we will settle for an approximation; we discretize the time span $[0, T]$ into a finite number of intervals, and solve for the optimal stopping time provided that the decision maker only considers stopping at the endpoint of each interval. This discrete-time decision problem can be solved via backward induction.

Let G be a set of N equally spaced points that partitions $[0, T]$,

$$G = \left\{0, 1\frac{T}{N}, 2\frac{T}{N}, \dots, N\frac{T}{N}\right\}$$

We define the discrete-time process $X_n : G \times \Omega \to E$ as

$$X_n = \tilde{S}_{n\frac{T}{N}}, \quad n = 0, 1, 2, \dots, N$$

Note that the subscript n now refers to the *deterministic* points in our discretization grid G, not the *random* jump times it did in the previous section.

Intuitively, as long as the price process (S_t) has not crossed the boundary B, the process (X_n) takes the same value as (S_t) at the points $n \in G$. When (S_t) has crossed the boundary for the first time, the process (X_n) enters the absorbing graveyard state.

Lemma 28. *The process (X_n) is a discrete-time Markov process*

Proof. Since $(\tilde{S}_t)$ is a Markov process, and $\sigma(S_{1\frac{T}{N}}, S_{2\frac{T}{N}}, \dots, S_T) \subset \mathcal{F}_t$, we have

$$\begin{aligned} P(X_n \in A \mid X_{n-1}) &= P(\tilde{S}_{n\frac{T}{N}} \in A \mid \tilde{S}_{(n-1)\frac{T}{N}}) \\ &= P(\tilde{S}_{n\frac{T}{N}} \in A \mid \tilde{S}_{(n-1)\frac{T}{N}}, \dots, \tilde{S}_0) \\ &= P(X_n \in A \mid X_{n-1}, \dots, X_0) \end{aligned}$$

Which concludes the proof. □

Let $\mathcal{M}$ be the collection of stopping times almost surely finite taking values in G.

Define the **discrete time finite-horizon optimal stopping problem** as

$$\sup_{\sigma \in \mathcal{M}} \mathbb{E}[\psi(X_\sigma)] \tag{5.4}$$

We note that in (5.4) the indicator function $\mathbf{1}_{\{M_\sigma \leq B\}}$ is absent, reason of course being that the knockout boundary is now baked into the definition of (X_n).

Define the **Bellman equations** as

$$v_N(x) := \psi(x) \tag{5.5}$$

$$v_n(x) := \max\{\psi(x), \sum_{y \in E} p(x, y) v_{n+1}(y)\}, \quad n = 0, 1, \dots, N-1 \tag{5.6}$$

Lemma 29. *The process*

$$Z_n := v_n(X_n)\,, n = 0, 1, \dots, N$$

is a supermartingale.

Proof. We have

$$\begin{aligned}\mathbb{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[Z_{n+1} \mid X_n] \\ &= \mathbb{E}[v_{n+1}(X_{n+1}) \mid X_n] \\ &\leq v_n(X_n) \\ &= Z_n\end{aligned}$$

Which proves the claim. □

Proposition 30. *Under the assumptions of this section the stopping time*

$$\sigma^* := T \wedge \inf\{n \in G \mid \psi(X_n) = v_n\}$$

is optimal for 5.4.

Proof. Let the process (Z_n) be as defined in Lemma 24. From that result and from the optional stopping-theorem for supermartingales, we have that, for any stopping time $\sigma \in \mathcal{M}$,

$$\mathbb{E}[Z_\sigma] \leq \mathbb{E}[Z_0] = v_0(x)$$

We use this and the definition of v_n to conclude that for any n and any stopping time $\sigma \in \mathcal{M}$,

$$v_0(x) \geq \mathbb{E}[Z_\sigma] \geq \mathbb{E}[\psi(X_\sigma)]$$

We shall show that inequality holds with equality for σ^*, which then proves our claim.

For this purpose we consider, for $n = 0, 1, \dots, N-1$, the stopping time $\sigma^* \wedge (n+1)$ and the stopped process $Z_{\sigma^* \wedge (n+1)}$. Note that the random variables $\mathbf{1}_{\{\sigma^* \leq n\}}$ and $\mathbf{1}_{\{\sigma^* > n\}}$ are both $\mathcal{F}_n$-measurable. We can therefore write

$$\begin{aligned}\mathbb{E}[Z_{\sigma^* \wedge (n+1)} \mid X_n] &= \mathbb{E}[\mathbf{1}_{\{\sigma^* \leq n\}} Z_{\sigma^*} + \mathbf{1}_{\{\sigma^* > n\}} Z_{\sigma^*} \mid X_n] \\ &= \mathbf{1}_{\{\sigma^* \leq n\}} Z_{\sigma^*} + \mathbb{E}[\mathbf{1}_{\{\sigma^* > n\}} Z_{\sigma^*} \mid X_n] \\ &= \mathbf{1}_{\{\sigma^* \leq n\}} Z_{\sigma^*} + \mathbb{E}[\mathbf{1}_{\{\sigma^* > n\}} V_n \mid X_n] \\ &= \mathbf{1}_{\{\sigma^* \leq n\}} Z_{\sigma^*} + \mathbf{1}_{\{\sigma^* > n\}} V_n \\ &= Z_{\sigma^* \wedge n}\end{aligned}$$

Therefore,

$$\begin{aligned}v_0(X_0) = \mathbb{E}[Z_0] = \mathbb{E}[Z_{\sigma^* \wedge 0}]\mathbb{E}[Z_{\sigma^* \wedge 1}] = \dots = \mathbb{E}[Z_{\sigma^* \wedge N}] = \mathbb{E}[Z_{\sigma^*}] \\ = \mathbb{E}[v_{\sigma^*}(X_{\sigma^*})] \\ = \mathbb{E}[\psi(X_\sigma^*)]\end{aligned}$$

Which concludes the proof. □

Example. (☆) We illustrate the application of the Bellman equations with a numerical algorithm and a numerical solution.

To use the Bellman equations we have to specify transition probabilities between discrete time points. Recall that the probability that the Poisson process jumps twice in any time interval vanish as the length of the interval goes to zero (see Appendix A.2). Moreover, the probability of a Levy process making a double jump is also zero (Cont and Tankov, 2004, Proposition 5.3), which means that the constituent Poisson processes of (S_t) almost surely does not jump simultaneously. Therefore we approximate the transition probabilities for (X_n) by:

$$P(X_{n+1} = y \mid X_n = x) = \begin{cases} P(u_1 = y - x)\frac{T}{N}\lambda^+ & \text{if } y > x \\ P(d_1 = x - y)\frac{T}{N}\lambda^- & \text{if } y < x \\ 1 - (\lambda^+ + \lambda^-)\frac{T}{N} & \text{if } y = x \end{cases}$$

Where as before λ^+ and λ^- are the intensities of the respective Poisson processes for up- and downticks, u_1 and d_1 are random variables signifying the size of upward and downward jumps respectively, and $\frac{T}{N}$ is the length of the time intervals in our discretization grid.

To illustrate an actual execution boundary we take u_1 and d_1 to be logarithmically distributed random variables, implying that (S_t) is the ΔNB Levy process described in our earlier example. Moreover we set the parameter values such that (S_t) is a martingale, as prescribed by Proposition 21. The details of the computational algorithm and the numerical parameter values are in Appendix C. Figure 5.2 shows the resulting execution boundary,

$$D_n := \{(x, n) \in E \times G \mid \psi(x) = v_n(x)]\}$$

It is interesting to note that the boundary is decreasing as we near the expiry time T.

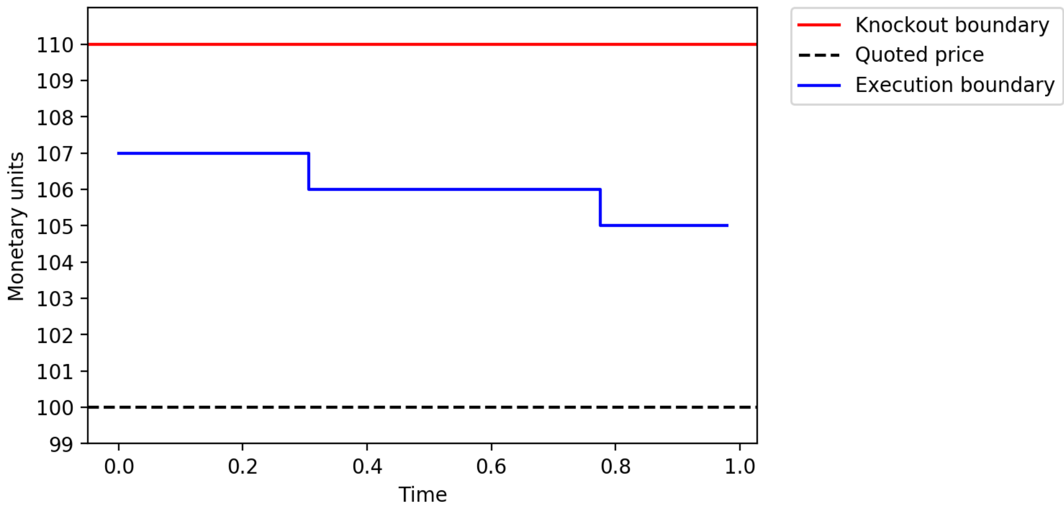


Figure 5.2: Example of an optimal execution boundary.

Chapter 6

Conclusions and future work

In this thesis we build a mathematical model suited for analysis of an important practical decision problem. We study our model and its solution under increasingly complex and realistic random dynamics.

We start out by looking at the case where randomness comes from a Brownian motion. The Brownian motion is a continuous process and therefore does not fit very well with the stylized fact we're trying to model, but it is nonetheless an important benchmark case. The Brownian case approximates the results we get when we use a very simple "tick-valued" random process, the Skellam process. This is not surprising, since the Skellam process in a sense is a discrete counterpart to the Brownian motion. In both cases we solve our two optimal stopping problems using elementary arguments, with martingality, convexity and the properties of conditional expectations as our fundamental building blocks. We consider the explicit, intuitive and constructive solutions to be a great advantage of these simpler models.

When we have a general integer-valued Levy process in our model, the possibility of large jumps fundamentally change the nature of our problem and its solution. We resort the method of dynamic programming to tackle the problem. This is a general method that could equally well be applied to the Brownian and Skellam cases, but the method does require an approximation procedure and does not yield explicit results.

There is an alternative approach to solving continuous-time optimal stopping problems, namely by the so-called *variational approach* and the associated *free boundary-problem*. I include a heuristic account of this approach and how it will look in the context of an integer-valued Levy process as an afterword.

6.0.1 Afterword: The variational inequalities

In this thesis we have used iterative procedures based on the general theory of *dynamic programming* to solve the last look-problem under a general integer-valued Levy process. An alternative approach is to use the *variational inequalities* for optimal stopping problems, which we describe here. This section is largely heuristic, and the work is included in anticipation of future research. The exposition is by no means meant to be complete or rigorous.

We first state the free-boundary problem in its general form, based on Peskir and Shiryaev (2006). Assume (X_t) is a strong Markov process taking values in the measurable space E. Given a measurable function $G : E \to \mathbb{R}$ satisfying certain regularity conditions, called the *reward function*, the general optimal stopping problem has the form

$$V(x) = \sup_{\tau} \mathbb{E}_x[G(X_\tau)]]$$

Where the supremum is taken over all stopping times (belonging to a set of admissable stopping times), and $X_0 = x$ with $x \in E$. It can be shown that the optimal stopping problem is equivalent to finding the smallest superharmonic function $\hat{V} : E \to \mathbb{R}$ which dominates the reward function on E (see Appendix A.1 for more details). Moreover, when we utilize the optimal stopping time τ^*, the stopped value process $(V(X_{t\wedge\tau^*}), t \geq 0)$ is a martingale (Peskir and Shiryaev, 2006, Theorem 2.4). The first entry time into the stopping set $D = \{\hat{V} = G\}$ is optimal, and $C = \{\hat{V} > G\}$ is the continuation set. From this one can deduce that $\hat{V}$ and C solve the *free-boundary problem*,

$$A\hat{V} \leq 0$$
$$\hat{V} \geq 0$$

Where A is the infinitesimal generator of (X_t) (cf. Appendix B Definition B.1). After invoking certain sufficient conditions (see Peskir and Shiryaev (2006, chapter 1) for details) and identifying $\hat{V} = V$ one can deduce that

$$AV = 0 \quad \text{in } C$$
$$V|_D = G|_D \tag{6.1}$$

It should be noted that a solution to this system consists of **both** a function V **and** an unknown stopping region D.

A heuristic argument for the variational formulation above goes as follows. If we are in the continuation region $(V > G)$, then the stopped value process $(V(X_{t\wedge\tau^*}), t \geq 0)$ is a martingale. Combining this fact with Dynkin's formula (Appendix B Theorem B.3) we get that $AV = 0$. A very loose interpretation of this condition is that since (X_t) is a Markov process, it has a transition operator satisfying the semigroup property (Appendix A.2.6). The infinitesimal generator can be understood loosely as a "derivative" of this transition operator, and hence the condition $AV = 0$ is similar to a first-order optimality condition. The second condition, $V|_D = G|_D$, comes from observing that in the stopping region the optimal value is just the payoff from stopping.

Our class of integer-valued Levy processes are based on the compound Poisson process. For any smooth function f with compact support, the infinitesimal generator of a compound Poisson process is given by (Applebaum, 2009, Example 3.3.7)

$$(Af)(x) = \int_E [f(x+y) - f(x)]\nu(dy)$$

Where ν is the Levy measure (Appendix A.2) of the process, and by convention $\nu(\{0\}) = 0$. In the case where E is a countable set, the generator reduce to

$$(Af)(x) = \lambda \sum_{n\in E} [a_n f(x+n) - f(x)]$$

In the finite-horizon case the free-boundary problem (6.1) would contain a time derivative term $\partial/\partial t$. In the infinite horizon case this term is not present however, simplifying matters further

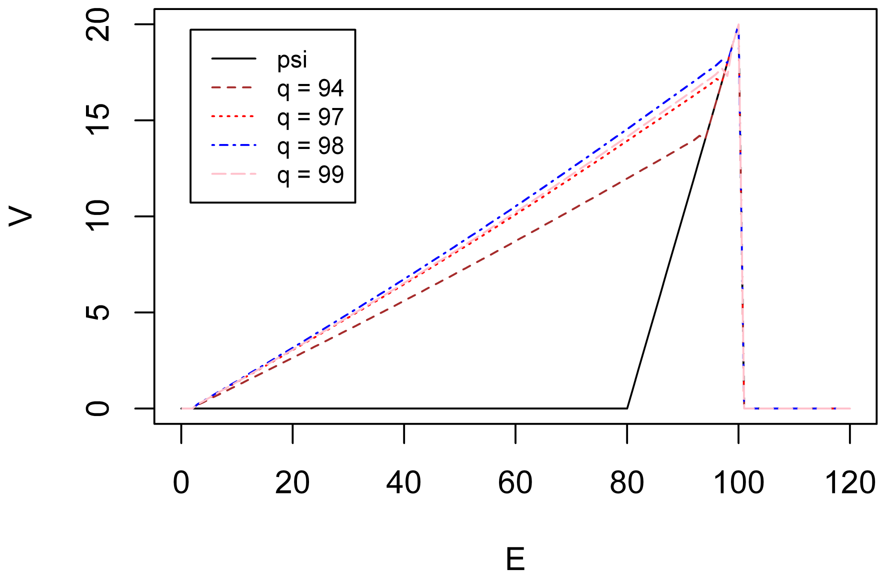


Figure 6.1: Candidate value functions V for different possible boundaries q. The quote has $B = 100$, $K = 80$, and Levy measure ν with support on $\{-3, -2, -1, +1, +2, +3\}$. More details in Appendix C.

(mirroring the fact that the stopping set is time-invariant, cf. Lemma 24). If we set $q := \inf\{D\}$, the free-boundary problem (6.1) in the infinite-horizon case becomes a linear homogeneous difference equation with undetermined boundary conditions $\psi(n)$:

$$\begin{aligned} V_n &= \psi(n) && n \geq q \\ V_n &= \sum_{k \in E} a_k V_{n+k} && n < q \end{aligned} \tag{6.2}$$

Where $a_0 = 0$. Based on our (heuristic) treatment we conjecture that a solution to the optimal stopping problem (1.2) under an integer-levy price process and an infinite time-horizon is a boundary $q \in E$ and a sequence $(V_n)_{n \in E}$ solving (6.2). Figure 6.1 shows an example where candidate value functions $(V)^{(i)}$ has been computed for different possible values of the boundary q. The computational algorithm use a fixed-point iteration procedure to compute $(V)^{(i)}$, details of which are documented in Appendix C.

The variational approach taken here differs from iterative approach of subsection 5.3.2 in that for the latter we iterated in (discrete) time using the transition probabilities of the process. In the variational approach, on the other hand, we sum infinitesimal transition probabilities over points in the (discrete) state space.

Appendices

Appendix A

Literature Review

A.1 American and European Barrier Options

In this section we review several important results on American options, and in particular the optimal execution boundary. The interested reader can thereby see how our results on financial quotes relates to earlier literature on American options. We shall also review results on Barrier options, which relates to last look-quotes. We only review European Barrier options here.

We take as given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ satisfying the usual conditions and a random process (S_t). The filtration $\mathbb{F}$ is the natural filtration associated with (S_t). When we later write $\mathbb{E}(X)$ for some random variable X, it should be understood that the expectation operator works under the probability measure $\mathbb{Q}$. We assume a complete financial market in the sense of Black and Scholes (1973). For brevity we omit the details of the market model, see Karatzas and Shreve (1998) for more on this.

Let the risk-free rate be constant and equal to r. Let the dynamics of the fair price S_t under the measure $\mathbb{Q}$ be given by

$$dS_t = S_t(rdt + \sigma dW_t) \tag{A.1}$$

$$S_0 = x \in \mathbb{R}_+$$

Let $\mathcal{T}$ be the collection of stopping times taking values in $[0, T]$, and let ψ be a non-negative convex deterministic function. Define the function $V(T, K)$ by

$$V(T, K; S_0, r) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}\psi(\tau, S_\tau)] \tag{A.2}$$

The function V is referred to as the value of an American option with payoff ψ and expiry T. It can be shown that (A.2) is the no-arbitrage price of the American option in a complete market, hence the use of the word *value* (see Pascucci (2011) for details).

A stopping time τ^* that attains the supremum in (A.2) is called an optimal execution strategy. What can be said about the function V and the optimal strategy τ^* when S_t has the dynamics described by (A.1)? To answer this question we will need to introduce some general results on optimal stopping.

A.1.1 Elements of optimal stopping

Two approaches to solving optimal stopping problems have been developed. The first is Snell's envelope (Snell, 1952), and the second is Dynkin's superharmonic characterization of the value function (Dynkin, 1963). We will base our discussion on Dynkin's characterization, as laid out in Oksendal (2013), but will also clarify the connection to Snell's envelope.

The general problem (A.2) is called inhomogeneous in time, because time enters as an argument in the reward function ψ. We will however first examine the time homogeneous problem, and later see that the inhomogeneous one can be reduced to this simpler case.
A basic concept in the solution of (A.2) is superharmonic functions:

Definition 14. *A lower semicontinuous measurable function $f : \mathbb{R}^n \to [0, \infty]$ is said to be* ***superharmonic*** *wrt. X_t if*

$$f(x) \geq \mathbb{E}_x [f(X_\tau)]$$

Let $\mathcal{A}$ be the characteristic operator of f (Appendix B). It follows from Dynkin's formula (Appendix B) that if $f \in C_0^2(\mathbb{R}^n)$ then f is superharmonic wrt. X_t if and only if

$$\mathcal{A}f \leq 0$$

There is a relation between superharmonic functions and supermartingales (indeed, this relation is where the name *supermartingale* comes from):

Lemma 31. *If X_t is a Markov process and f is a superharmonic function, the process $f(X_t)$ is a supermartingale wrt. the σ-algebras generated by X_t.*

Proof. For $t > s$ we have

$$\mathbb{E}_x[f(X_t \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[f(X_{t-s})] \quad \text{(the Markov property)}$$

$$\leq f(X_S)$$

Since f is a measurable function the process $\xi_t := f(X_t)$ is measurable wrt. to the σ-algebra generated by X_t. $\square$

Definition 15. *Let h be a real measurable function on $\mathbb{R}^n$, and f be a superharmonic function. If $f \geq h$ we say that f is a* ***superharmonic majorant*** *of h.*

Let F_h be the collection of all superharmonic majorants of h. The function

$$\bar{h}(x) = \inf_{f \in F_h} f(x)$$

is called the ***least superharmonic majorant*** *of h.*

The function $\bar{h}$ is in fact superharmonic, see Oksendal (2013, lemma 10.1.3 c)).

We are now ready to state Dynkin's superharmonic characterization of the value function $G(x)$:

Theorem 32. *Let*

$$G(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{(x)} [\psi(S_\tau)]$$

where (S_t) is a strong Markov process.

Define the ***continuation region*** C

$$C = \{x \mid G(x) > \psi(x)\}$$

the ***stopping region*** D

$$D = C^c$$

and the stopping time τ_D *as the first passage time into the stopping region*

$$\tau_D = \inf\{t \geq 0 \mid S_t \in D\}$$

Suppose that the stopping time τ^* *is optimal. Then,*

i) *The value function* G *is the least superharmonic majorant for the reward function* ψ.

ii) $\tau_D \leq \tau^*$ Q^x*-almost surely*

iii) *The stopping time* τ_D *is optimal.*

iv) *The process* $V(S_{t\wedge\tau_D})$ *is a martingale.*

The time-inhomogeneous case

When the time-horizon is infinite and there is no discounting, the optimal stopping-problem is homogeneous in time. When there is a fixed time horizon T this is no longer the case. One can however reduce this time-inhomogeneous case to the homogeneous one by increasing the dimensionality of the problem.

Let (X_t) be the random process to be stopped, and let $[0, T]$ be the time available. One can introduce the two-dimensional process $(X_t, T - t)$ and proceed with the same arguments as for the time-homogeneous case. More details and a worked example can be found in Oksendal (2013) chapter 10.

A.1.2 Application of optimal stopping to American Options

Proposition 33. *Let* $f(x)$ *denote the payoff of the American option, meaning* $f(x) = (x - K)_+$ *and* $f(x) = (K - x)_+$ *for the call and put respectively, and let* $\mathcal{T}$ *be the set of all stopping times taking values in* $[0, T]$. *The optimal value function* V_t *is given by*

$$V_t = \Phi(t, S_t) \tag{A.3}$$

$$\tag{A.4}$$

$$= \sup_{\tau\in\mathcal{T}} \mathbb{E}\left(e^{-r(\tau-t)} f(xe^{(r-\sigma^2/2)(\tau-t)+\sigma(W\tau-W_t)}\right) \tag{A.5}$$

Proof. (Lamberton and Lapeyre, 2011, chapter 4.4) □

Proposition 34. *The process*

$$e^{-\int_0^t r(s,X_s)ds}\Phi(t, Xt)$$

is the smallest martingale (Snell envelope) that dominates the process $f(X_t)$ *at all times.*

Proof. Lamberton and Lapeyre (2011, chapter 5.3) □

There is a connection between optimal stopping and a set of variational inequalities. For the American option-problem this connection is the following.

Theorem 35. *Assume that u is a regular solution of the following system of partial differential inequalities:*

$$\frac{\partial u}{\partial t} + A_t u - ru \leq 0 \quad \text{for } u \geq f$$

$$\left(\frac{\partial u}{\partial t} + A_t u - ru\right)(f - u) = 0$$

$$u(T, x) = f(x)$$

Then

$$u(t, x) = \Phi(t, x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left(e^{-\int_t^T r(s, X_s^{t,x})ds} f(X_\tau^{t,x})\right)$$

Proof. See for example Lamberton and Lapeyre (2011, chapter 5.4), Pascucci (2011) and Oksendal (2013). □

A.1.3 Explicit formulae for American options

We have seen that the general optimal stopping problem can be rewritten in terms of a set of variational inequalities, for which there exists numerical solution methods.

In this section, however, we will use other arguments to find explicit formulae for put and call options.

The call option

Recall that the payoff to the holder of an American call option executed at time τ with strike $\bar{K}$ is $(S_\tau - \bar{K})_+$. We are interested in the largest expected discounted payoff,

$$V_{\text{call}}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\left(e^{-r\tau}(S_\tau - \bar{K})_+\right)$$

Let $C_E(S_0, T; K, \sigma, r)$ be the Black-Scholes price of a European call option written at time zero with maturity T, strike K, volatility σ and risk-free rate r:

$$C_E(S_0, T, K, \sigma, r) = S_0 \mathcal{N}\left[d_1\left(\frac{S_0}{Ke^{-rT}}, T\right)\right] - Ke^{-rT}\mathcal{N}\left[d_2\left(\frac{S_0}{Ke^{-rT}}, T\right)\right]$$

where

$$d_1(y, u) = \frac{1}{\sqrt{\sigma^2 u}} \ln(y) + \frac{1}{2}\sqrt{\sigma^2 u}$$

$$d_2(y, u) = d_1(y, u) - \sqrt{\sigma^2 u}$$

and $\mathcal{N}$ denotes the standard Gaussian cumulative distribution:

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

In what follows we hold the arguments (T, σ, K, r) fixed, and so we suppress them in the notation and write simply $C_E(S_0)$.

The following theorem tells us that the Black-Scholes price of a European call option in fact also describes value of the American call.

Theorem 36. *The value V_{call} of the American call option is given by*

$$V_{call}(x) = C_E(x)$$

Proof. (✤)
We first observe that from risk-neutral pricing in the Black-Scholes market, we have

$$C_E(x) = \mathbb{E}_x\left(e^{-rT}(S_T - \bar{K})_+\right)$$

Since $\mathcal{T}$ is the set of stopping times taking values in $[0, T]$, the following inequality is clear:

$$\begin{aligned} V_{\text{sell}}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\left(e^{-r\tau}(S_\tau - \bar{K})_+\right) \\ &\geq \mathbb{E}_x\left(e^{-rT}(S_T - \bar{K})_+\right) \\ &= C_E(x) \end{aligned}$$

The idea is now to show the reverse inequality using the martingale property of discounted prices, the convexity of the max function and Jensen's inequality. Consider the payoff from any stopping time τ bounded by T:

$$\begin{aligned} e^{-r\tau}(S_\tau - \bar{K})_+ &\leq (e^{-r\tau}S_\tau - e^{-rT}\bar{K})_+ \\ &= \left(\mathbb{E}\left(e^{-rT}S_T \mid \mathcal{F}_\tau\right) - e^{-rT}\bar{K}\right)_+ \\ &= \left(\mathbb{E}\left(e^{-rT}S_T - e^{-rT}\bar{K} \mid \mathcal{F}_\tau\right)\right)_+ \\ &\leq \mathbb{E}\left(\left(e^{-rT}S_T - e^{-rT}\bar{K}\right)_+ \mid \mathcal{F}_\tau\right) \\ &= \mathbb{E}\left(e^{-rT}\left(S_T - \bar{K}\right)_+ \mid \mathcal{F}_\tau\right) \end{aligned}$$

We take expectations in the preceding inequality, to get

$$\mathbb{E}\left(e^{-r\tau}(S_\tau - \bar{K})_+\right) \leq \mathbb{E}\left(e^{-rT}\left(S_T - \bar{K}\right)_+\right)$$

and note that since the relation holds for any stopping time τ, it holds in particular for the stopping time attaining the supremum:

$$\begin{aligned} V_{\text{sell}}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}_x\left(e^{-r\tau}(S_\tau - \bar{K})_+\right) \\ &\leq C_E(x) \end{aligned}$$

□

From the proof we also note that late stopping ($\tau = T$) is always optimal for the call option under the assumptions of this section [1].

[1] If we introduce stock dividends or some other payoff from holding the option, late stopping would no longer be optimal in general

The put option

We recall that the payoff to the holder of an American put executed at time τ is $(\underline{K} - S_\tau)_+$. We have seen that the American call option in the Black-Scholes model is worth the same as the European call option. For the American put things are not this simple. However, in the special case when the interest rate is zero, we will see that there is an equivalence between the American and European put option.
Recall that we want to evaluate the function

$$V_{\text{put}}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left(e^{-r\tau}(\bar{K} - S_\tau)_+\right)$$

Recall also that the Put-Call parity implicitly gives us the no arbitrage price $P_E(x)$ of the European put option in terms of the European call option, the spot price and the strike:

$$P_E(x) = x - C_E(x) - e^{-rT}K$$

Theorem 37. *Assume the risk-free rate is zero ($r = 0$). Then, the value of the American put is given by the no arbitrage price of an European put option,*

$$V_{put}(x) = P_E(x)$$

Proof. (✤)
The proof follows the same structure as for the call option. We consider the payoff from stopping at some time $\tau \le T$:

$$\begin{aligned}(K - S_\tau)_+ &= (K - \mathbb{E}(S_T \mid \mathcal{F}_\tau))_+ \\ &= (\mathbb{E}(K - S_T \mid \mathcal{F}_\tau))_+ \\ &\le \mathbb{E}\left((K - S_T)_+ \mid \mathcal{F}_\tau\right)\end{aligned}$$

Taking expectations, we get

$$\mathbb{E}\left((K - S_\tau)_+\right) \le \mathbb{E}\left((K - S_T)_+\right)$$

Since the relation holds for any stopping time τ bounded by T, it must hold for the supremum, and hence

$$V_{\text{buy}}(x) \le P_E(x)$$

The inequality $V_{\text{buy}}(x) \ge P_E(x)$ follows from the following observation:

$$\begin{aligned}V_{\text{buy}}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left((K - S_\tau)_+\right) \\ &\ge \mathbb{E}\left((K - S_T)_+\right) \\ &= P_E(x)\end{aligned}$$

□

Treating the risk-free as zero will be a very good approximation when the time to maturity is very short. For example, if the continuously compounded yearly rate is 5%, the interest rate for a minute is $\frac{0.05}{60*24*365} = 1.9 * 10^{-8}$.

But can we still say something about the value of an American put if the risk-free rate is not zero? The next theorem shows that we get an additional term stemming from the risk-free rate.

Theorem 38. *Write $P_A(x,t)$ for the no arbitrage price of an American put option at time t on a stock with initial price x and time to maturity T, and $P_E(x,0)$ for the corresponding European option. Define the* ***exercise boundary*** *D_t as the critical stock price below which the American put should be exercised:*

$$\text{if } S_t \leq D_t, \text{ then } P_A(x,t) = \max[0, K - S_t]$$

$$\text{if } S_t > D_t, \text{ then } P_A(x,t) > \max[0, K - S_t]$$

We can decompose the American put price as follows:

$$P_A(x,0) = P_E(x,0) + rK \int_0^T e^{-rt} \mathcal{N}\left(\frac{\ln(D_t/x) - rt(1-r/2)}{\sigma\sqrt{t}}\right) dt$$

Proof. See Carr et al. (1992) □

The equation in Theorem 38 is intuitive: the price of the American put is the price of the European put, plus a premium for being able to exercise early. The early exercise premium is the present value of the strike price times the probability that the put has been exercised at a particular time, summed of all times (Chung et al., 2011).

Unfortunately, Theorem 38 does not provide us with an explicit formula, since the unknown price P_A enters into the exercise boundary B_t. The issue remains to determine this boundary.

If the time to maturity T is very short, Barles et al. (1995) provides an approximation of the exercise boundary.

Proposition 39. *If the time maturity is very short, a good approximation to the default boundary is*

$$B_{T-t} \approx K(1 - \sigma\sqrt{(T-t)|\ln(T-t)|})$$

Substituting the approximation of B_{T-t} into Theorem 38 yields an explicit formula for the price of the American put.

A.1.4 European barrier options

The no-arbitrage price of an European barrier option in a complete market is characterized by the expected payoff (under the martingale measure)

$$U_{\text{sell}}(x) = \mathbb{E}_x\left[e^{-rs}\mathbf{1}_{\{T_B \geq T\}}(S_T - \bar{K})_+\right] \tag{A.6}$$

and

$$U_{\text{buy}}(x) = \mathbb{E}_x\left[e^{-rs}\mathbf{1}_{\{T_B \geq T\}}(\underline{K} - S_T)_+\right] \tag{A.7}$$

Where T_B denotes the first time the process (S_t) hits the barrier B. The continuity of the Brownian motion means that we know the value of the price process when it hits the barrier B:

$$T_B = \inf\{t \geq 0 \mid S_t = B\}$$

This nice property allows many explicit results on hitting times, which we shall see plays an important role in this section.

We will approach the problem by abstracting away from the specific form of the payoff function, and look for ways to evaluate functionals of the general form

$$\hat{F}(x, B, \psi) = e^{-rT}\mathbb{E}_x\left[\mathbf{1}_{\{T_B > T\}}\psi(S_T)\right] \tag{A.8}$$

Hence, in our application $\psi(x)$ equals $(x-\bar{K})_+$ for the call option and $(\underline{K}-x)_+$ for the put option.

When the initial point S_0 is above the barrier B, that is $S_0 > B$, the problem is called a "down-and-out problem", because the payoff is "knocked out" if the process goes down to B. We will first consider down-and-out problem, and thereafter see that the same principles applies to "up-and-out problems" where $S_0 < B$.

The main idea is to transform the problem of evaluating $\hat{F}$ into the evaluation of simpler functionals F:

$$F(x, \psi) = e^{-rT}\mathbb{E}_x\left[\underline{\psi}(S_T)\right] \tag{A.9}$$

Where $\underline{\psi}$ defines a "chopped off" payoff function,

$$\underline{\psi}(y) := \begin{cases} \psi(y) & y \geq B \\ 0 & y < B \end{cases}$$

Such a transformation is useful because the functional F does not feature the hitting time T_B, and therefore it can be evaluated with simpler methods: in fact we will find explicit formulae. The mathematics that follows has been adapted from the chapters on barrier options in Björk (2004) and Jeanblanc et al. (2009).

The main theorem is the following:

Theorem 40 (Evaluation of down-and-out claims)**.** *Let F, $\hat{F}$, ψ and $\underline{\psi}$ be as defined above. Then, the following relation holds:*

$$\hat{F}(x, B, \psi) = F\left(x, \underline{\psi}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2 x^{-1}, \underline{\psi}\right)$$

Proof.
First we define the process X_t as

$$X_t := \ln S_t$$

From Itô's formula we find that the process X_t is a Brownian motion with drift:

$$dX_t = \underbrace{(r - \frac{1}{2}\sigma^2)}_{\equiv\lambda}\, dt + \sigma dW_t$$
$$X_0 = \ln x$$

At the hitting time T_B we have already noted that $S_{T_B} = B$ by the continuity of Brownian motion and the exponential function. This implies that at time T_B we also have

$$X_{T_B} = \ln B$$

Therefore, the following two stopping times are equal:

$$\inf\{t \geq 0 \mid S_t = B\} = \inf\{t \geq 0 \mid X_t = \ln B\}$$

We use the notation $X_T^{\ln B}$ and S_T^B for the stopped processes:

$$X_T^{\ln B} := X_{T\wedge T_{\ln B}}$$
$$S_T^B := S_{T\wedge T_B}$$

The density f of the random variable $X_T^{\ln B}$ is attained from the reflection principle (see Appendix B and chapter 3), and equals

$$\begin{aligned} f(u) &= \phi\left(u, \lambda T + \ln x, \sigma\sqrt{T}\right) - e^{-\frac{\lambda(\ln x - \ln B)}{\sigma^2}} \phi\left(u, \lambda T - \ln x + 2\ln B, \sigma\sqrt{T}\right) \\ &= \phi\left(u, \lambda T + \ln x, \sigma\sqrt{T}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \phi\left(u, \lambda T + \frac{B^2}{x}, \sigma\sqrt{T}\right) \end{aligned}$$

Where $\lambda = \left(r - \frac{1}{2}\sigma^2\right)$ and $\phi(u, \mu, \sigma)$ denotes the density of the normal distribution with mean μ and variance σ.

Observe that we can write

$$\begin{aligned} e^{rT}\hat{F}(x, B, \psi) &= \mathbb{E}_x\left[\mathbf{1}_{\{T_B > T\}}\psi(S_T)\right] \\ &= \mathbb{E}_x\left[\mathbf{1}_{\{T_B > T\}}\psi(S_T^B)\right] \\ &= \mathbb{E}_x\left[\mathbf{1}_{\{T_B > T\}}\underline{\psi}(S_T^B)\right] \\ &= \mathbb{E}_x\left[\underline{\psi}(S_T^B)\right] \\ &= \mathbb{E}_x\left[\underline{\psi}(e^{X_T^{\ln B}})\right] \\ &= \mathbb{E}_x\left[(\underline{\psi} \circ \exp)(X_T^{\ln B})\right] \\ &= \int_{-\infty}^{\infty} \underline{\psi}(e^u) f(u) du \\ &= \int_{\ln B}^{\infty} \underline{\psi}(e^u) f(u) du \\ &= \underbrace{\int_{\ln B}^{\infty} \underline{\psi}(e^u) N\left(u, \lambda T + \ln x, \sigma\sqrt{T}\right) du}_{I_1} \\ &\quad - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \underbrace{\int_{\ln B}^{\infty} \underline{\psi}(e^u) N\left(u, \lambda T + \ln\left(\frac{B^2}{x}\right), \sigma\sqrt{T}\right) du}_{I_2} \end{aligned}$$

We may freely replace the lower integration limit in I_1 and I_2 by $-\infty$, because of the chopped-off payoff function $\underline{\psi}$ is anyway zero in that region. When doing this we note that the density in I_1 is that of the random variable X_T, with the usual starting point $S_0 = x$. The density in the integral I_2 would be that of the random variable X_T if the starting point was $S_0 = \frac{B^2}{x}$. Therefore we have

$$\begin{aligned} \hat{F}(x, B, \psi) &= e^{-rT}\mathbb{E}_x\left[\mathbf{1}_{\{T_B > T\}}\psi(S_T)\right] \\ &= e^{-rT} I_1 - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} I_2 \\ &= e^{-rT}\mathbb{E}_x\left[\underline{\psi}(S_T)\right] - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} e^{-rT}\mathbb{E}_{B^2x^{-1}}\left[\underline{\psi}(S_T)\right] \\ &= F\left(x, \underline{\psi}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2x^{-1}, \underline{\psi}\right) \end{aligned}$$

□

Theorem 40 described down-and-out claims, that is claims where the barrier is *below* the starting value of the process (S_t). An identical results holds for up-and-out claims, with an almost identical proof, except that the chopped-off payoff function ψ now takes the form

$$\bar{\psi}(y) := \begin{cases} \psi(y) & y \leq B \\ 0 & y > B \end{cases}$$

We write down the result as a corollary to Theorem 40, more details can be found in Björk (2004, theorem 18.12).

Corollary 41 (Evaluation of an up-and-out claim)**.** *Let F, $\hat{F}$, ψ and $\bar{\psi}$ be as defined above. Then, the following relation holds:*

$$\hat{F}(x, B, \psi) = F\left(x, \bar{\psi}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2 x^{-1}, \bar{\psi}\right)$$

We will have use of the two following lemmas.

Lemma 42 (Linearity of claim functional)**.** *The functional $F(x, \underline{\psi})$ is linear in its function argument $\underline{\psi}$, meaning that for functions f and g and scalars α and β, we have*

$$F(x, \alpha f + \beta g) = \alpha F(x, f) + \beta F(x, g)$$

Proof. Observing that the functional F is simply a scaled expectation, the lemma follows directly from the linearity of the integral. □

Lemma 43. *Assume that x, K, B are three fixed numbers, and that $K < B$. Then we have*

$$(x - K)_+ \mathbf{1}_{x \leq B} = (x - K)_+ - (x - B)_+ - (B - K)\mathbf{1}_{x > B}$$

Proof. The lemma is seen clearly by simply drawing a picture. For a formal proof, consider first the case when $x \leq B$. Then, the equation in the lemma becomes

$$(x - K)_+ = (x - K)_+ - 0 - 0$$

If on the other hand $x > B$, the assumption that $B > K$ implies that we also have $x > K$. Now the equation in the lemma becomes

$$0 = x - K - x + B - B + K$$

□

Let us consider the call option. The functional $\hat{F}$ is given by

$$\begin{aligned} \hat{F}(x, \psi) &= e^{-rT} \mathbb{E}_x \left[(S_T - \bar{K})_+ \mathbf{1}_{T_B > T} \right] \\ S_0 &= x < \bar{K} < B \end{aligned}$$

The claim is of an up-and-out type. The chopped off payoff function $\bar{\psi}$ becomes

$$\bar{\psi}(y) := \begin{cases} (y - \bar{K})_+ & y \leq B \\ 0 & y > B \end{cases}$$

Applying Corollary 41 and the two previous lemmas, we get

$$\begin{aligned}
\hat{F}(x,(S_T-K)_+) &= F\left(x,\bar{\psi}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2x^{-1},\bar{\psi}\right) \\
&= F(x,(S_T-K)_+\mathbf{1}_{S_T<B}) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2x^{-1},(S_T-K)_+\mathbf{1}_{S_T<B})\right) \\
&= F\left(x,(S_T-K)_+ - (S_T-B)_+ - (B-K)\mathbf{1}_{S_T<B}\right) \\
&- \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(B^2x^{-1},(S_T-K)_+ - (S_T-B)_+ - (B-K)\mathbf{1}_{S_T<B})\right) \\
&= F\left(x,(S_T-K)_+\right) - F\left(x,(S_T-B)_+\right) - (B-K)F\left(x,\mathbf{1}_{S_T<B}\right) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \\
&\cdot \left[F\left(B^2x^{-1},(S_T-K)_+\right) - F\left(B^2x^{-1},(S_T-B)_+\right) - (B-K)F\left(B^2x^{-1},\mathbf{1}_{S_T<B}\right)\right]
\end{aligned}$$

The functional $F(x,(S_T-K)_+)$ is well-known from the Black-Scholes formula for call options, a quantity for which we have introduced the short-hand notation $C_E(x,K)$:

$$\begin{aligned}
F\left(x,(S_T-K)_+\right) &= e^{-rT}\mathbb{E}\left[(S_T-K)_+\right] \\
&=: C_E(x,K)
\end{aligned}$$

The quantity $\mathbb{E}_x\left(\mathbf{1}_{S_T<B}\right)$ is straightforward to compute using the log-normal cumulative distribution, and we adopt the short-hand notation $\mathcal{H}(x,B)$:

$$\begin{aligned}
F(x,\mathbf{1}_{S_T<B}) &= e^{-rT}\mathbb{E}_x\left(\mathbf{1}_{S_T<B}\right) \\
&= e^{-rT}\mathcal{N}\left[\frac{\ln(\frac{x}{B})-\lambda T}{\sigma\sqrt{T}}\right] \\
&=: \mathcal{H}(x,B)
\end{aligned}$$

Note that $\mathcal{H}(x,B)$ is of the same form as the no-arbitrage price of a binary option.

Going back to the evaluation of $\hat{F}(x,(S_T-K)_+)$, we have arrived at the following result:

$$\begin{aligned}
&\hat{F}(x,(S_T-K)_+) = \\
&C_E(x,K) - C_E(x,B) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \left[C_E(B^2x^{-1},K) - C_E(B^2x^{-1},B)\right] \\
&- (B-K)\left[\mathcal{H}(x,B) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \mathcal{H}(B^2x^{-1},B)\right]
\end{aligned}$$

We see that the problem of evaluating the risk-neutral expectation of an European up-and-out barrier call option has been reduced to the evaluation of a portfolio of calls and binary claims. The latter are not path-dependent claims; in the formula the hitting time T_B is absent.

To evaluate the down-and-out European put option, we will use the following lemma, which is the equivalent of the put-call parity for Barrier options:

Lemma 44 (Barrier option put-call parity)**.** *Let x,K,B be given real numbers with $B<K$. Then,*

$$(K-x)_+\mathbf{1}_{x>B} = K\mathbf{1}_{x>B} - x\mathbf{1}_{x>B} + (x-K)_+\mathbf{1}_{x>B}$$

Proof. If $x \le B$, all terms in the lemma are zero, and there is nothing to prove.

If on the other hand $x > B$, there are three possibilities for the relationship between x and K. First, say that $K > x$. Then the equation in the lemma becomes

$$K - x = K - x + 0$$

Second, say that $K < x$. The equation then becomes

$$0 = K - x + x - K$$

Third, say that $K = x$. The equation becomes

$$0 = K - x + 0 = 0$$

$\square$

We can now apply Theorem 40 together with lemmas 42 and 44 to evaluate the buy quote. The relevant payoff functions are

$$\psi(y) = (\underline{K} - y)_+$$

and

$$\underline{\psi}(y) := \begin{cases} \psi(y) & y \ge B \\ 0 & y < B \end{cases}$$

From Theorem 40 we get

$$\begin{aligned} e^{-rT}\mathbb{E}_x\left[\mathbf{1}_{T_B>T}(\underline{K} - S_T)_+\right] &= \hat{F}(x, \psi\mathbf{1}_{T_B>T}) \\ &= F(x, \underline{\psi}) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(\frac{B^2}{x}, \underline{\psi}\right) \\ &= F(x, (K - S_T)_+\mathbf{1}_{S_T>B}) - \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} F\left(\frac{B^2}{x}, (K - S_T)_+\mathbf{1}_{S_T>B}\right) \\ &= K \cdot F(x, \mathbf{1}_{S_T>B}) - F(x, S_T\mathbf{1}_{S_T>B}) + F(x, (S_T - K)_+\mathbf{1}_{S_T>B}) \\ &- \left(\frac{B}{x}\right)^{\frac{2\lambda}{\sigma^2}} \left[K \cdot F\left(\frac{B^2}{x}, \mathbf{1}_{S_T>B}\right) + F\left(\frac{B^2}{x}, (S_T - K)_+\mathbf{1}_{S_T>B}\right)\right] \end{aligned}$$

The functionals F have interpretations in terms of contingent claims: $F(\cdot, \mathbf{1}_{S_T>B})$ is the pricing functional of a binary option with barrier B, which can be evaluated using the CDF of the lognormal distribution. The term $F(\cdot, S_T\mathbf{1}_{S_T>B})$ prices a down-and-out contract on the underlying asset (no option involved), which in turn can be decomposed into a binary option and a call option. The term $F(\cdot, (S_T - K)_+)$ is the pricing functional of a down-and-out call option, which can be evaluated by another application of Theorem 40, as for evaluation of the put option.

A.2 Integer-valued random processes

In this thesis we are interested in continuous-time random processes that take values in a countable set. This countable set is called the *tick grid* in our application. Recalling that a set E is countable if there exists an injective function from E to the natural numbers $\mathbb{N}$, we might as well simplify our notation and treat the tick grid as a subset of $\mathbb{Z}$, hence our interest in integer-valued processes.

A.2.1 Poisson process

A fundamental building block of Levy jump processes are the Poisson process. We recall that the Poisson process is a counting process that

i) Starts at zero

ii) Has independent increments

iii) The number of increments in any interval of length t is a Poisson random variable with parameter λt

There is an alternative definition of the Poisson process that is often used.This definition states that a counting process N_t is a Poisson process with rate λ if

i) It starts at zero

ii) Has stationary independent increments

iii) $P(N_h = 1) = \lambda h + o(h)$

iv) $P(N_h = 2) = o(h)$

Where the small-o notation $f(h) = o(h)$ means $\lim\limits_{h \to 0} \frac{f(h)}{h} = 0$.

The property *iv)* says that the probability of two events counted by the Poisson process happens at the same time is zero. Or, in other words, a double-jump is a zero-probability event. This property is used repeatedly in this thesis.

Although not immediately clear, the two definitions given are in fact equivalent. More details can be found in Ross (2014).

Proposition 45 (Thinning property). *Let (N_t) be a Poisson process with intensity λ, and let $(T_1, T_2, \dots)$ be the sequence of arrival times. Let (Y_n) be an iid sequence of Bernoulli trials with parameter p, independent from (N_t). Then, the processes*

$$U_t := \sum_{n=0}^{\infty} \mathbf{1}_{\{t \leq T_n\}} \mathbf{1}_{\{Y_n = 1\}}$$

and

$$D_t := \sum_{n=0}^{\infty} \mathbf{1}_{\{t \leq T_n\}} \mathbf{1}_{\{Y_n = 0\}}$$

are two independent Poisson processes.

Proof. See Ross (2014).

□

Proposition 46 (Superposition property). *Let (U_t) and (D_t) be two independent Poisson processes with intensities λ_U and λ_D respectively. Then, the process*

$$N_t := U_t + D_t$$

is a Poisson process with intensity $\lambda_U + \lambda_D$.

Proof. See Ross (2014). ∎

Definition 16. *The **arrival times** of the Poisson process are the times that the counting process N_t jumps, meaning the times $\{t \geq 0 \mid N_t > N_{t^-}\}$.*

The next proposition is useful for simulating the Poisson process and related processes, and it is applied in the algorithms of Appendix C.

Proposition 47. *Given that $N_t = k$, the k arrival times are uniformly distributed on the interval $[0, t]$.*

A proof can be found in Ross (2014).

A.2.2 Compound Poisson

Let (N_t) be a a Poisson process with rate λ, and (U_j) be an iid sequence of random variables independent from (N_t). The compound Poisson process has the form

$$Y_t = \sum_{j=1}^{N_t} U_j, \quad U_j \sim \text{i.i.d.}, \quad N \perp\!\!\!\perp U$$

We use a result known as *Wald's equation* in order to compute the expectation of the compound Poisson.

Proposition 48 (Wald's equation)**.** *Let (X_n) be a sequence of real-valued, iid random variables, and let N be a nonnegative integer-valued random variable that is independent of the sequence (X_n). Suppose that N and X_n have finite expectations. Then,*

$$\mathbb{E}[X_1 + X_2 + \ldots + X_N] = \mathbb{E}[N]\mathbb{E}[X_1]$$

Corollary 49. *The expectation of the compound Poisson process is given by*

$$\mathbb{E}[Y_t] = \lambda t \mathbb{E}[U_1]$$

Proof.

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[\mathbb{E}[Y_t \mid N_t]] \\
&= \mathbb{E}[N_t \mathbb{E}[U_1]] \\
&= \mathbb{E}[N_t]\mathbb{E}[U_1] \\
&= \lambda t \mathbb{E}[U_1]
\end{aligned}$$

∎

A.2.3 Levy processes

Levy processes are frequently applied as models of financial prices (Cont and Tankov, 2004), and are therefore a natural starting point for this review. Recall that a random process (L_t) is a Levy process if

i) It starts at zero (a.s.)

ii) It has independent and stationary increments

iii) It is stochastically continuous , meaning that for all $a > 0$ and all $s \geq 0$,

$$\lim_{t \to s} P\left(|L_t - L_s| > a\right) = 0$$

The stochastic continuity of Levy processes allows for discontinuous sample paths; informally what we require is that the probability of encountering a discontinuity ("jump") vanishes as the time increments goes to zero, see Applebaum (2009) for more details.

It is clear from the definition that Levy processes are homogeneous Markov processes. In fact the stronger statement that they are strong Markov processes also holds.

Another crucial fact about Levy processes is that every Levy process has a cadlag modification that is itself a Levy process (Applebaum, 2009, Theorem 2.1.8).

If we replace the condition iii) in the definition of a Levy process with the much stronger requirement of continuous sample paths, we get the definition of the Brownian motion. Since continuity of paths implies stochastic continuity, the Brownian motion is an example of a Levy process.

A.2.4 Poisson random measures

For any Levy process L_t we can define the *jump process* ΔL_t by

$$\Delta L_t = L_t - L_{t^-}$$

Note that the limit L_{t^-} is well-defined because the Levy process has cadlag paths.

The Poisson process is a very simple example of a Levy process, because it only jumps in increments of 1. We can make this observation precise by actually *defining* the Poisson process $X(t)$ to be an integer-valued Levy process that is increasing (a.s.) and is such that $(\Delta X(t))$ takes values in $\{0, 1\}$ (the third definition of the Poisson process we have seen. There are several more). Moreover, the converse is also true: the Poisson process is the only Levy process with those properties (Applebaum, 2009, Theorem 2.2.13).

We can express the Poisson process by counting the number of jumps:

$$X_t = \#\{0 \leq s \leq t, \, \Delta X_s = 1\}$$

The intensity of the process can be expressed as $\lambda = \mathbb{E}[X_1]$.

To study more general Levy processes we generalize the above idea by counting jumps of a particular size. Specifically, for a Levy process L_t and any $A \in \mathcal{B}(\mathbb{R} - \{0\})$, define the *random Poisson measure* as

$$N(t, A) := \#\{0 \leq s \ \leq t \mid \Delta L_s \in A\}$$

We define the *intensity measure* μ as

$$\mu(A) := \mathbb{E}[N(1, A)]$$

The reason for the name "Poisson measure" is the following. Let $A \in \mathbb{R}$ be bounded from below, meaning that $\{0\} \notin \bar{A}$. Then, when we vary t, we find that $N(t, A)$ is a Poisson process with intensity $\mu(A)$. Note that it now follows that $\mu(A) < \infty$ whenever A is bounded below.

We can integrate functions against the Poisson measure as follows. Let A be bounded from below, and let f be a Borel measurable real-valued deterministic function. For each $t > 0$ and $\omega \in \Omega$ we define the *Poisson integral* of f as

$$\int_A f(x)N(t,dx)(\omega) := \sum_{x\in A} f(x)N(t,\{x\})(\omega)$$

Note that the sum $\sum_{x\in A}$ is well-defined because $L(t)$ can only have finitely many jumps in A. The assumption that A is bounded from below is essential here, see Applebaum (2009) for more details. As we vary t the Poisson integral becomes a cadlag random process, in fact it turns out that it is a compound Poisson process.

For a set A bounded from below and $f \in L^1$ we also define the *compensated Poisson integral* with respect to the *compensated Poisson random measure* $\tilde{N}$:

$$\int_A f(x)\tilde{N}(t,dx) = \int_A f(x)N(t,dx) - t\int_A f(x)\mu(dx)$$

A.2.5 Levy-Itô decomposition and integer-valued processes

The Levy-Itôdecomposition allows us to express any Levy process L_t as a combination of a deterministic drift, a Brownian part, a compensated Poisson integral handling the small jumps, and a regular Poisson integral handling the large jumps. More formally, for real-valued numbers b and σ and the Brownian motion W_t, we have

$$L_t = bt + \sigma W(t) + \int_{|x|<1} x\tilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx)$$

Note that certain technicalities are necessary to define the integral over small jumps, see Applebaum (2009, pg. 122). We do not have to use 1 as the threshold between small and large jumps: we could accommodate any constant R in the Poisson integral $\int_{|x|<R} x\tilde{N}(t,dx)$ by changing the constant b accordingly.

(☆)
In this thesis we are interested in Levy processes that take values on the integers. For such a process the Levy-Ito decomposition makes it clear that there can be no drift, no Brownian part, no small jumps and in fact no jumps of non-integer size:

$$\begin{aligned} b &= 0 \\ \sigma &= 0 \\ \mu(A) &= 0 \text{ for all } A \text{ where } A\cap\mathbb{Z} = \emptyset \end{aligned}$$

Thus, our integer-valued process Y_t is a Poisson integral (or, equivalently a compound Poisson process):

$$\begin{aligned} Y_t &= \int_{|y|\geq 1} yN(t,dy) \\ &= \sum_{n\in\mathbb{Z}} xN(t,\{n\}) \end{aligned}$$

Now, for any A bounded from below we know that $N(t, A)$ is a Poisson process with intensity $\mu(A)$ as we vary t. Hence $N(t, \{n_j\})$ is a Poisson process with intensity $\lambda_j \in \mathbb{R}_+$.

Let $\{n_1, n_2, \dots\}$ be the integers contained in A, meaning $\{n_1, n_2, \dots\} = A \cap \mathbb{Z}$. Since $\mu(A)$ only has mass at these integers, we have $N(t, A) = N(t, \{n_1, n_2, \dots\})$. Furthermore, $\mu(A) = \mathbb{E}[N(1, \{n_1, n_2, \dots\})]$. Since $N(t, \{n_j\})$ and $N(t, \{n_k\})$ are independent for $n \neq k$ (this is a general property of Levy processes), we have $\mu(\{n_j, n_k\}) = \lambda_k + \lambda_j$. Now, by the thinning property of the Poisson process, we conclude that any integer-valued Levy process must have a Poisson random measure of the form

$$N(t, A) = \sum_{j \in A \cap \mathbb{Z}} N_t^{(j)}$$

Where $(N_t^{(j)})$ are independent Poisson processes with intensities $\{\lambda^j\}$.

In other words, any integer-valued Levy process can be expressed as a superposition of independent Poisson processes, where each process corresponds to a jump of a certain integer increment:

$$L_t = \sum_{k \in \mathbb{Z}} k N_t^{(k)}$$

We have seen that every integer-valued Levy process can be written as a Poisson integral,

$$L_t = \int_{|x| \geq 1} x N(t, dx)$$

It is also a fact that $(\int_A f(x) N(t, dx), t \geq 0)$ is a compound Poisson process, see Applebaum (2009, Theorem 2.3.9), which means that *any* integer-valued Levy process can be written as a compound Poisson process.

Skellam process

An integer-valued Levy process where the intensity measure only has mass at the points $\{-1, 1\}$ is known as a *Skellam process*, and according to the preceding discussion it can be written as the difference between two Poisson processes:

$$Y_t = X_t^+ - X_t^-$$

The process is thus named because it is linked to the Skellam distribution, introduced by Irwin (1937). The Skellam distribution comes about as the difference between two independent Poisson distributions, with parameters λ^- and λ^+. For a fixed t, the distribution of L_t is given by

$$Y_t \sim \text{Sk}(t\lambda^+, t\lambda^-)$$

and

$$Y_t - Y_s \sim \text{Sk}((t-s)\lambda^+, (t-s)\lambda^-), \quad t > s$$

If we set $\lambda^+ = \lambda^- = 1/2$, Y_t is a martingale with unit variance per unit of time, and as $t \to \infty$ we have $Y_t/\sqrt{t} \to \mathcal{N}(0, 1)$, see Barndorff-Nielsen et al. (2012). Therefore we can think of the Skellam process as a discrete-value analogy of the Brownian motion.

We can express the "upticks" and the "downticks" of the Skellam process as compound Poissons by setting the sequence of random variables (U_j) equal to plus/minus one:

$$X_t^+ - X_t^- = \sum_{i=1}^{X_t^+} (+1) + \sum_{i=1}^{X_t^-} (-1)$$

A.2.6 Markov processes

In section A.2 we were concerned with processes that have independent and stationary increments with paths satisfying a certain stochastic continuity criterion. In this section we explore another concept that will be crucial to the arguments in this thesis, namely *markovianity*.

We take as given a probability space $(\Omega, \mathcal{F}, P)$ and a compact separable metric space E called the state space. We denote by $\mathcal{E}$ the Borel σ-algebra on E. Let (X_t) be a random process.

Let $\mathbb{F}_t^X$ be the filtration generated by (X_t), and assume that $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \geq 0$, in other words we have that (X_t) is adapted to $\mathcal{F}_t$. Moreover, define the sigma-algebra of future events as

$$\mathcal{F}_t' := \sigma\{X_u, u \geq t\}$$

Definition 17. *We say that (X_t) is a **Markov process** (with respect to $\mathcal{F}_t$) if for any $t \geq 0$ and $B \in \mathcal{F}_t'$,*

$$P(B \mid \mathcal{F}_s) = P(B \mid X_s)$$

Definition 18. *The collection $\{P_{s,t}(\cdot, \cdot) \mid 0 \leq s < t < \infty\}$ is a **Markov transition function** on $(E, \mathcal{E})$ if, for all $s < t < u$ we have*

i) for every $x \in E$, the map $B \mapsto P(x, B)$ is a probability measure on $(E, \mathcal{E})$,

ii) for every $B \in \mathcal{E}$, the map $x \mapsto P(x, B)$ is $\mathcal{E}$-measurable.

iii) for every $x \in E$ and every $A \in \mathcal{E}$ we have

$$P_{s,u}(x, A) = \int_E P_{s,t}(x, dy) P_{t,u}(y, A)$$

The condition in *iii*) is called the Chapman-Kolmogorov equation, and is a manifestation of the Markov property. One should interpret the transition function as

$$P_{s,t}(x, A) = P(X_t \in A \mid X_s = x)$$

Note that we have allowed for the situation where

$$P_{s,t}(x, E) \leq 1$$

Such a transition function is called **submarkovian**. When the inequality holds with equality, the transition function is called strictly Markovian. One can convert the submarkovian case into the strictly markovian one by extending the state space. For this, we introduce a new point $\dagger \neq E$, and set

$$E_\dagger = E \cup \{\dagger\}, \quad \mathcal{E}_\dagger = \sigma(\mathcal{E}, \{\dagger\})$$

We now define a new transition function $P_{s,t}'$ as follows, for $A \in \mathcal{E}$:

$$\begin{aligned} P_{s,t}'(x, A) &= P_{s,t}(x, A) \\ P_{s,t}'(x, \{\dagger\}) &= 1 - P_{s,t}(x, E),\ x \neq \dagger \\ P_{s,t}'(\dagger, E) &= 0 \\ P_{s,t}'(\dagger, \{\dagger\}) &= 1 \end{aligned}$$

One can verify that $P_{s,t}'$ is a Markov transition function. Moreover it is strictly Markovian. The state $\{\dagger\}$ is an absorbing state.

Definition 19. *A Markov process is said to be* ***time homogeneous*** *if for any $x \in E_\dagger$, $A \in \mathcal{E}_\dagger$ and $s \leq t$,*

$$P_{s,t}(x, A) = P_{t-s}(x, A)$$

We can introduce a class of operators related to the transition function.

Definition 20. *Let $P_t(x, B)$ be a transition function on the measurable space $(E, \mathcal{E})$, and let f be a nonnegative measurable function on E. Define the* ***transition operator $\boldsymbol{P}$*** *by*

$$\boldsymbol{P}f := \int f(y)P(x, dy)$$

The operator $\mathbf{P}$ is a bounded linear operator on the space of bounded, measurable functions on E. Using this notation, we can restate the Chapman-Kolmogorov equation as

$$P_{t+s} = P_t P_s$$

Which means that the family $(\mathbf{P}_t)$ forms a semigroup[2] of operators on the space of bounded, measurable functions on E.

The strong Markov property is essentially about Markov at random times. We now state a theorem about the markovian properties of Levy processes that is essential for many arguments used in this thesis.

Theorem 50. *If (X_t) is a Levy process and τ is a stopping time, then on $\{\tau < \infty\}$,*

i) The process $(X_{\tau+t} - X_\tau, t \geq 0)$ is a Levy process that is independent of $(\mathcal{F}_\tau)$

ii) For each $t \geq 0$, $(X_{\tau+t} - X_\tau, t \geq 0)$ has the same law as X_t

iii) The process $(X_{\tau+t} - X_\tau, t \geq 0)$ has cadlag paths and is $\mathcal{F}_{\tau+t}$-adapted.

Proof. See Applebaum (2009, Theorem 2.2.11) □

[2]A **semigroup** is a set together with an associate binary operation.

Appendix B

Referenced theorems and definitions

Definition 21. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(E, \mathcal{E})$ be a measurable space. A* ***random variable*** *on Ω is a measurable function $X : \Omega \to E$.*

Definition 22. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable.*

a) For every Borel subset of the real line ($B \in \mathcal{B}(\mathbb{R})$), define $P_X(B) = P(X \in B)$.

b) The resulting function $P_X : \mathcal{B} \to [0, 1]$ is called ***the probability law*** *of X*

Lemma 51. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. Then, the law P_X of X is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

Definition 23. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X and Y be two random variables with laws P_X and P_Y. If*

$$P_X(B) = P_Y(B) \quad \text{for all } B \in \mathcal{B}(\mathbb{R})$$

Then we say that X and Y are ***equal in law*** *and write*

$$X \overset{law}{=} Y$$

Proposition 52. *Let E be a measure space, and X, Y be two random variables taking values in E. If $X \overset{law}{=} Y$, then*

$$\int_E f dP_X = \int_E f dP_Y$$

for all continuous and bounded functions f on E.

Definition 24. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration on $(\Omega, \mathcal{F}, P)$ is a family of σ-algebras $(\mathcal{F}_t, t \geq 0)$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$.*

Definition 25. *A filtration $\mathbb{F}$ is called right-continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$, where*

$$\mathcal{F}_{t^+} := \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$$

Definition 26. *A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to satisfy* ***the usual conditions*** *if the following three conditions hold:*

i) $\mathcal{F}$ is P-complete, meaning that if $B \subset A \in \mathcal{F}$ and $P(A) = 0$ then $B \in \mathcal{F}$.

ii) $\mathcal{F}_0$ contains all zero-probability sets.

iii) The filtration $\mathbb{F}$ is right-continuous

Definition 27. ***The natural filtration*** *generated by the process (S_t) is denoted $\mathbb{F}^S$ and defined by*

$$\mathbb{F}^S = \{\mathcal{F}_t^S := \sigma(S_u, u \leq t\}_{t \geq 0}$$

Definition 28. *A random process (S_t) is $\mathbb{F}$-**adapted** if for all $t \geq 0$ the random variable S_t is $\mathcal{F}_t$-measurable.*

In particular, a random process is always adapted to its own filtration.

Definition 29. *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space. A random process (M_t) is said to be a **martingale** with respect to $\mathbb{F}$ (and P) if*

i) *M_t is $\mathcal{F}_t$ for all t*

ii) *$\mathbb{E}[|M_t|] < \infty$ for all t*

iii) *$\mathbb{E}[M_t \mid \mathcal{F}_u] = M_u$ for all $t \geq u$.*

Lemma 53. *Let (S_t) be a cadlag Levy process. The natural filtration generated by (S_t) is right-continuous.*

Definition 30. *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, a random variable τ is said to be a $\mathbb{F}$-**stopping time** if*

$$\{\tau \leq t\} \in \mathcal{F}_t, \quad \text{for all } t \geq 0$$

Definition 31. *Let B be a set and (S_t) a random process. We define the **first entry time** of (S_t) into B as*

$$\tau_B := \inf\{t \geq 0 \mid S_t \in B\}$$

Lemma 54. *Let B be a closed set and (S_t) a random process. The first entry time of (S_t) into B, τ_B , is a stopping time (with respect to the natural filtration of (S_t).*

Lemma 55. *Let B be an open set and (S_t) a random process. Assume that the filtration $\mathbb{F}$ is right-continuous. Then, the first entry time of (S_t) into B, τ_B is an $\mathbb{F}$-stopping time.*

Definition 32. *Let τ be an $\mathbb{F}$-stopping time. We define the **stopped sigma-algebra** as*

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \mid \text{ for each } t > 0,\ A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$$

The stopped sigma-algebra represents the sigma-algebra of events occurring up to the random time τ.

Lemma 56. *Assume that (S_t) is an adapted process. Let (S_t) be a random process and τ a stopping time. We refer to $S_{t\wedge\tau}$ as the stopped process. The stopped process is a random variable with respect to the sigma-algebra $\mathcal{F}_t$.*

Theorem 57 (Doob' optional sampling)**.** *Let $\sigma \leq \tau$ be bounded stopping times. For any cadlag martingale (X_t),*

i) *The random variables X_σ, X_τ are integrable.*

ii) *We have*

$$X_\sigma = \mathbb{E}\left[X_\tau | \mathcal{F}_\sigma\right]$$

Definition 33. *A lower semicontinuous measurable function $f : \mathbb{R}^n \to [0, \infty]$ is said to be **superharmonic** wrt. X_t if*

$$f(x) \geq \mathbb{E}_x\left[f(X_\tau)\right]$$

There is a relation between superharmonic functions and supermartingales (indeed, this relation is where the name *supermartingale* comes from):

Lemma 58. *If X_t is a Markov process and f is a superharmonic function, the process $f(X_t)$ is a supermartingale wrt. the σ-algebras generated by X_t.*

Proof. For $t > s$ we have

$$\mathbb{E}_x[f(X_t \mid \mathcal{F}_s] = \mathbb{E}_{X_s}[f(X_{t-s})] \quad \text{(the Markov property)}$$

$$\leq f(X_S)$$

Since f is a measurable function the process $\xi_t := f(X_t)$ is measurable wrt. to the σ-algebra generated by X_t. □

Definition 34. *Let h be a real measurable function on $\mathbb{R}^n$, and f be a superharmonic function. If $f \geq h$ we say that f is a **superharmonic majorant** of h.*

Let F_h be the collection of all superharmonic majorants of h. The function

$$\bar{h}(x) = \inf_{f \in F_h} f(x)$$

*is called the **least superharmonic majorant** of h.*

The function $\bar{h}$ is in fact superharmonic, see Oksendal (2013, lemma 10.1.3 c)).

Definition 35.
*Let (X_t) be a time-homogeneous Markov process taking values in $\mathbb{R}^n$. The **infinitesimal generator** A of X_t is defined by*

$$Af(x) = \lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} \tag{B.1}$$

Definition 36.
*Let (X_t) be an Itô diffusion. The **characteristic operator** $\mathcal{A}$ of X_t is defined by*

$$\mathcal{A}f(x) = \lim_{U \downarrow x} \frac{E^x[f(X_{\tau_U})] - f(x)}{\mathbb{E}^x[\tau_U]} \tag{B.2}$$

Where the U's are open sets decreasing to the point $\{x\}$, and $\tau_U = \inf\{t > 0 \ : \ X_t \neq U\}$ is the first exit time of X_t from U.

The characteristic operator and the generator are closely related: it turns out that $\mathcal{A}f = Af$ for all f where the limit in B.1 exists, see Oksendal (2013, page 129) and Dynkin (1965).

Theorem 59 (Dynkin's formula)**.**
Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time and that $\mathbb{E}^x[\tau] < \infty$. Then,

$$\mathbb{E}^x\left[f(X_\tau)\right] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right] \tag{B.3}$$

Proposition 60 (Reflection principle for Brownian motion)**.** *Let W_t be a standard Brownian motion. Let x and y be given real numbers satisfying $y \geq 0$ and $x \leq y$. Let (M_t) be the running supremum of the (W_t), meaning that $M_t = \sup_{u \in [0,t]} W_u$. The following holds:*

$$P(W_t \leq x, M_t \geq y) = P(W_t \geq 2y - x)$$

Corollary 61. *For a fixed t, we have the equality of law*

$$M_t \overset{law}{=} |W_t|$$

Proof. See Jeanblanc et al. (2009, Chapter 3) □

Appendix C

Computer code and algorithms

C.1 Data description for Figure 1.1

The data shown in the figure is from Thomson Reuters Matching, a leading trading platform for foreign exchange. The plot shows the exchange rate for Euro versus US Dollars, over the time periods stated in the caption of the figures.

C.2 R code for Figure 3.1

```
#Parameter values:
sigma = .2
x = 100
k = 100.01

# Main function:
risk = function(mty,buy) {
    if (buy ==T) {n=1}
    else {n=-1}
    v=n*sigma * sqrt(mty/(2*pi))*exp(-((k-x)^2)/(2*sigma^2*mty)) +
        n*(x-k)*pnorm(n*(x-k)/(sigma*sqrt(mty)))
    return(v)
}

# Call main function:
x.values = seq(from=.01,to=20,length.out = 100)
z = sapply(x.values,risk,T)

#Make plots and save output
```

C.3 R code for Figure 4.2

```
#Parameter values
delta = 1
K = 2

# Probability mass function of the Skellam r.v.
point.prob = function(k, delta, mty) {
```

```
    res = exp(−mty*2*delta)*besselI(2*mty*delta, abs(k))
    return(res)
}

# Payoff functions:
call.option = function(x,K) {
    return(max(0,x−K))
}

put.option = function(x,K) {
    return(max(0,K−x))
}

# Main function
risk = function(mty,delta,K, FUN=call.option) {
    k = seq(from = −50, to = 50, by = 1)
    res = sapply(k, function(x) call.option(x,K)*point.prob(x, delta, mty))
    return(sum(res))
}

# Call main function
x.values = seq(from=.01,to=20,length.out = 100)
y = sapply(x.values, risk, delta, K, call.option)
z = sapply(x.values, risk, delta, K, put.option)

# Make plots and save output
```

C.4 R code for Figure 4.1

```
#This code simulates a Skellam process

import numpy as np
import numpy.random as rnd

#Paramaeter values:
T=10 #Points in timegrid
lam = 4
n = 10**4

# Draw number of up and down jumps
N1 = rnd.poisson(T*lam)
N2 = rnd.poisson(T*lam)

X = np.zeros((n))
Y = np.zeros((n))

# Draw position of jumps
for i in range(N1):
    pos = rnd.randint(0,n)
    X[pos] += 1
```

```
for j in range(N2):
     pos = rnd.randint(0,n)
     Y[pos] += 1

# Make cumulative process (S0 = 100)
Z = np.cumsum(X-Y)+100

# Make plots and save output
```

C.5 R code for Figure 5.1

```
#2017-06-60
#Simulate DeltaNB Levy process

library(extraDistr)

#parameters:
#up
r.u =0.5
p.u=.5
#down
r.d = 0.5
p.d = .5

T=100

set.seed(1)

#draw random variables:
#number of jumps:
n.u = rpois(1,T*r.u*abs(log(1-p.u)))
n.d = rpois(1,T*r.d*abs(log(1-p.d)))

#timing of up jumps
t.u = runif(n.u,0,T)
t.u = sort(t.u)
t.d = runif(n.d,0,T)
t.d = sort(t.d)

#jump sizes
u=rlgser(n.u,p.u)
d=rlgser(n.d,p.d)

#combine jumps sizes and timings in single table
jumps.u = cbind(t.u,u)
jumps.d = cbind(t.d,-d)
jumps = rbind(jumps.u,jumps.d)

#for plotting, need many points on x-axis (no-jump points)
grid = seq(from=0,to=T,by=.01/T)
```

```
x.axis = cbind(grid,rep(0,length(grid)))

#combine jumps and no-jump time points
tab = rbind(jumps,x.axis)
tab = as.data.frame(tab)
colnames(tab) = c("time","jump")

tab.order = tab[with(tab,order(time)), ] #chronological ordering along time d
tab.order$value = cumsum(tab.order$jump) #value of process is cumsum of jumps

# Make plot and save output
```

C.6 Python code for Figure 5.2

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Tue Sep 26 11:52:47 2017

@author: jo
"""

import numpy as np

T = 1
n = 50


#Truncation point decides width of window used for transition probabilities:
trunc_point = 20

K = 100 #Quoted price ("Strike price")
B = 110 #Knockout boundary

pricegrid = np.arange(0,2*B,1)
timegrid = np.linspace(0,T,n)
timestep = timegrid[1]-timegrid[0]

def payoff(x,K=100,B=120):
    pi = np.maximum(x-K,0)
    pi[x>B] = 0
    return pi

def transprob(n=20,q_u = .4, q_d = 0.4, p=.5, lam=6.0, h=.1):
    ### Compute transition probabilities for the integer Levy process,
    ### for jump sizes of -n to +n.
    y = np.arange(1,n+1,1)
    p_up = lam*h*p*(-1/np.math.log(1-q_u))*(q_u**y.astype(float))
                /y.astype(float)
    p_down = lam*h*(1-p)*(-1/np.math.log(1-q_d))*(q_d**y.astype(float))
```

```
                /y.astype(float)
    p_unch = 1 - lam*h
    z = np.arange(-n,+n+1,1)
    p = 0.0*z
    p[z>0] = p_up
    p[z<0] = np.flip(p_down, axis=0)
    p[z==0] = p_unch
    return p

#value matrix:
V = np.zeros((len(pricegrid),len(timegrid)))
V[:,-1] = payoff(pricegrid,K,B)

#execution boundary:
boundary = 0.0*timegrid

#Loop from time T-1 and backwards toward 0:
for i in range(-2,-len(timegrid)-1,-1):
    #Loop over each price in the grid:
    for price,pos in enumerate(pricegrid):
        #skip lower and upper part of pricegrid (due to truncation of
        #transition probabilities. This step implies that the pricegrid must
        #be set wide enough that its length doesnt matter for the end result)
        #Also skip prices where the quote is knocked out
        valid = 1
        if pos <= trunc_point:
            valid = 0
        if price > B:
            valid = 0
        if pos >= len(pricegrid)-trunc_point:
            valid = 0
        if valid == 1:
            probs = transprob(n=trunc_point,h=timestep)
            cont_value = np.inner(probs,V[(pos-trunc_point):(pos+
                                  trunc_point+1),i+1])
            stop_value = V[pos,-1]
           V[pos,i] = np.maximum(cont_value,stop_value)
        if cont_value > stop_value:
            boundary[i] = price + 1

boundary[-1] = np.nan

#Make plot and save output
```

C.7 R code for Figure 6.1

```
# Levy measure parameters
# The process is a the sum of three Skellam processes, with jump size 1, 2 an
lam = c(10,25,50,50,25,10)/100
```

```
L =sum(lam)

K = 80 # "Strike"
B = 100 #Barrier

# Truncated the state space for computational purposes:
upper = 120
lower = 0
E = seq(from=upper, to = lower, by =- 1)
f = rep(0, length(E))
l = length(E)

# Payoff function
psi = function(x) {
  if (x <= B) { return(max(0,x-K))
  } else {
      return(0)
    }

}

V = function(q) {
  nrounds = 1:1000
  # Iterate over the sequence (fixed-point iteration)
  for (n in nrounds) {
    for (e in E[1:(l-3)]) {
      idx = which(E == e)
      if (e >= q) {
        f[idx] = psi(e)
      } else {
        f[idx] = (f[idx - 3]*lam[1] + f[idx-2]*lam[2] + f[idx-1]*lam[3] +
                      f[idx+1]*lam[4] + f[idx+2]*lam[5] + f[idx+3]*lam[6])/L
      }
    }
    f[l-2] = (f[l - 3]*lam[1] + f[l-2]*lam[2] + f[l-1]*lam[3] +
                  f[l+1]*lam[4] + f[l+2]*lam[5])/L
    f[l-1] = (f[l - 3]*lam[1] + f[l-2]*lam[2] + f[l-1]*lam[3] +
                  f[l]*lam[4])/L
    f[l] =  (f[l - 3]*lam[1] + f[l-2]*lam[2] + f[l-1]*lam[3])/L
    f[is.na(f)] = 0
  }
  return(f)
}

# Compute candidate value functions V for different boundaries q
g = lapply(90:100,V)
stop.payoff = sapply(E, psi)

 # Make plots
```

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