ORNSTEIN-UHLENBECK PROCESSES IN HILBERT SPACE WITH NON-GAUSSIAN STOCHASTIC VOLATILITY

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ABSTRACT. We propose a non-Gaussian operator-valued extension of the Barndorff-Nielsen and Shephard stochastic volatility dynamics, defined as the square-root of an operator-valued Ornstein-Uhlenbeck process with Lévy noise and bounded drift. We derive conditions for the positive definiteness of the Ornstein-Uhlenbeck process, where in particular we must restrict to operator-valued Lévy processes with "non-decreasing paths". It turns out that the volatility model allows for an explicit calculation of its characteristic function, showing an affine structure. We introduce another Hilbert space-valued Ornstein-Uhlenbeck process with Wiener noise perturbed by this class of stochastic volatility dynamics. Under a strong commutativity condition between the covariance operator of the Wiener process and the stochastic volatility, we can derive an analytical expression for the characteristic functional of the Ornstein-Uhlenbeck process perturbed by stochastic volatility if the noises are independent. The case of operator-valued compound Poisson processes as driving noise in the volatility is discussed as a particular example of interest. We apply our results to futures prices in commodity markets, where we discuss our proposed stochastic volatility model in light of ambit fields.

1. INTRODUCTION

In this paper we introduce and analyze an Ornstein-Uhlenbeck (OU) process

$$dX(t) = \mathcal{A}X(t)\,dt + \sigma(t)\,dB(t)$$

taking values in a separable Hilbert space H. Here, $\mathcal{A}$ is a densely defined unbounded operator on H, B is an H-valued Wiener process and $\sigma(t)$ is a predictable operator-valued process being integrable with respect to B. We shall be concerned with a particular class of stochastic volatility models $\sigma(t)$ of a non-Gaussian nature.

OU processes with values in Hilbert space provide a natural infinite dimensional formulation for many linear (parabolic) stochastic partial differential equations (see, e.g., Da Prato and Zabczyk [16], Gawarecki and Mandrekar [20] and Peszat and Zabczyk [27]). Our main motivation for studying Hilbert space-valued OU processes comes from the modeling of futures prices in commodity markets, where the dynamics

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follow a class of hyperbolic stochastic partial differential equations (see Benth and Krühner [11, 12]). We refer to Applebaum [2] for a general survey on recent developments of Hilbert-valued OU processes.

Barndorff-Nielsen and Shephard [8] proposed a flexible class of stochastic volatility (SV) models based on real-valued OU processes driven by a subordinator (a pure-jump Lévy process with non-negative drift and positive jumps). This class, which we name the BNS SV model, has been applied to model financial time series like exchange rates and stock prices (see e.g. Barndorff-Nielsen and Shephard [8]). Benth [10] proposed the BNS SV model in an exponential mean-reversion dynamics to model gas prices collected from the UK market. Later, Benth and Vos [13, 14] extended this to a multifactor framework to model prices in energy markets. Their extension of the BNS SV model to a multivariate context is based on the work by Barndorff-Nielsen and Stelzer [9]. There are several papers dealing, both empirically and theoretically, with stochastic volatility in commodity prices (see e.g., Geman [21], Hikspoors and Jaimungal [23] and Schwartz and Trolle [29]).

In the present paper we lift the multivariate BNS SV model by Barndorff-Nielsen and Stelzer [9] to an operator-valued stochastic process, providing a very general stochastic volatility dynamics. In particular, we consider the "stochastic variance process" $\mathcal{Y}(t)$ taking values in the space of Hilbert-Schmidt operators on H,

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t)\,dt + d\mathcal{L}(t)\,,$$

where $\mathcal{L}$ is a square-integrable Lévy process in the space of Hilbert-Schmidt operators on H and $\mathbb{C}$ a bounded operator on the same space. We state conditions on $\mathbb{C}$ and $\mathcal{L}$ to ensure that $\mathcal{Y}$ is a non-negative definite self -adjoint operator, and in this case we define $\sigma(t) := \mathcal{Y}^{1/2}(t)$. In fact, the paths of the process $t \mapsto (\mathcal{L}(t)f, f)_H$ must be increasing for every $f \in H$ to have non-negative definite $\mathcal{Y}$. This property is analogous to the assumption the real-valued BNS SV model is driven by a subordinator process. since $t \mapsto (\mathcal{L}(t)f, f)_H$ is equal to the scalar product of $\mathcal{L}(t)$ with $f \otimes f$ in the space of Hilbert-Schmidt operators on H, and thus a real-valued Lévy process with non-decreasing paths (i.e., a subordinator). We say that $\mathcal{L}$ has "non-decreasing paths" and we show that such Lévy processes have a continuous martingale part with covariance operator having all symmetric Hilbert-Schmidt operators in its kernel.

As a particular example a compound Poisson process is considered, where the jumps are defined to be the tensor product of a Hilbert space valued Gaussian random variable with itself. We demonstrate that such a model leads to Gamma distributed jumps for certain interesting real-valued projections of the Lévy process. Furthermore, from a result of Fraisse and Viguier-Pla [19] the jumps will in general be Wishart distributed in infinite dimensions, and we can compute the characteristic functional of $\mathcal{L}$ for self-adjoint test operators.

Our operator-valued BNS SV model $\mathcal{Y}$ has a convenient affine structure, and we can compute its characteristic function. Moreover, if $\mathcal{L}$ is independent of B, it is possible to derive an analytical expression for the characteristic function of the OU-process $X(t)$ in terms of the semigroups associated with the drift in X and $\mathcal{Y}$ and the characteristic functional of $\mathcal{L}$. To achieve this result, we must impose a rather strong commutativity condition between the covariance operator of the Wiener noise B and the stochastic volatility $\mathcal{Y}^{1/2}$. We find that X is affine in itself and the stochastic volatility. Also, we show that the "mean-reversion adjusted returns" of X are H-valued conditional Gaussian random variables, if these are conditioned on the volatility $\mathcal{Y}^{1/2}$, which can be considered to be an observable in a simplified filtering problem (see Remark 3.5 in Section 3). The "mean-reversion adjusted returns" are defined as the increments of X corrected by the semigroup of $\mathbb{C}$.

We relate our general analysis to commodity futures markets. In this respect, we focus on a process X defined on a specific Hilbert space of functions on $\mathbb{R}_+$, the positive real-line, and with the unbounded operator in the drift being $\mathcal{A} = \partial/\partial x$. Then, $X(t,x)$ can be interpreted as the futures price at time $t \geq 0$ for a contract delivering the commodity at time $x \geq 0$, with a dynamics specified under the Heath-Jarrow-Morton-Musiela (HJMM) modelling paradigm (see Heath, Jarrow and Morton [22] and Musiela [25]). We connect our general SV modeling approach to the analysis in Benth and Krühner [11, 12] and the ambit field approach in Barndorff-Nielsen, Benth and Veraart [5, 6]. We remark that this discussion can be extended to forward rate modeling under the HJM paradigm in fixed-income theory (see Filipovic [18] and Carmona and Theranchi [15] for an analysis of HJM models in infinite dimensions for fixed-income markets.).

Our results are presented as follows: In the next section we introduce the operator-valued BNS SV model and analyze its properties. Section 3 defines the volatility-modulated OU process X along with a discussion of its characteristics. Finally, in Section 4, we discuss our model X in the context of commodity futures price modeling.

2. Operator-valued BNS stochastic volatility model

Throughout the paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a given filtered probability space. Let H be a separable Hilbert space with inner product denoted by $(\cdot,\cdot)_H$ and associated norm $|\cdot|_H$. Introduce $\mathcal{H} := L_{\text{HS}}(H)$, the space of Hilbert-Schmidt operators on H into itself, with the usual inner product denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and associated norm $\|\cdot\|_{\mathcal{H}}$. As H is a separable Hilbert space, $\mathcal{H}$ becomes a separable Hilbert space as well.

Introduce $\mathbb{C} \in L(\mathcal{H})$, that is, a bounded linear operator from $\mathcal{H}$ into itself. In this paper, we shall pay particular attention to two specific cases of $\mathbb{C}$, namely, the operator

$$\mathbb{C}_1 : \mathcal{H} \to \mathcal{H}, \qquad \mathcal{T} \mapsto \mathcal{C}\mathcal{T}\mathcal{C}^* \tag{2.1}$$

or the operator

$$\mathbb{C}_2 : \mathcal{H} \to \mathcal{H}, \qquad \mathcal{T} \mapsto \mathcal{C}\mathcal{T} + \mathcal{T}\mathcal{C}^* \,. \tag{2.2}$$

Here, $\mathcal{C} \in L(H)$, $L(H)$ denoting the space of bounded linear operators in H into itself. We shall exclusively focus on $\mathcal{C} \neq 0$. The following lemma provides us with crucial properties for $\mathbb{C}_i$, $i = 1, 2$:

Lemma 2.1. *It holds that $\mathbb{C}_i \in L(\mathcal{H})$ for $\mathbb{C}_i$ defined in* (2.1) *and* (2.2)*, with $\|\mathbb{C}_1\|_{op} \leq \|\mathcal{C}\|_{op}^2$ and $\|\mathbb{C}_2\|_{op} \leq 2\|\mathcal{C}\|_{op}$ and $\mathcal{C} \in L(H)$. Moreover, $(\mathbb{C}_i\mathcal{T})^* = \mathbb{C}_i\mathcal{T}^*$ for every $\mathcal{T} \in \mathcal{H}$ and $i = 1, 2$.*

Proof. For $\mathcal{S}, \mathcal{T} \in \mathcal{H}$, $\mathbb{C}_i(\mathcal{S} + \mathcal{T}) = \mathbb{C}_i\mathcal{S} + \mathbb{C}_i\mathcal{T}$, where $i = 1, 2$. Hence, linearity holds. Moreover, for an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in H,

$$\begin{aligned}
\|\mathbb{C}_1\mathcal{T}\|_{\mathcal{H}}^2 &= \|\mathcal{C}\mathcal{T}\mathcal{C}^*\|_{\mathcal{H}}^2 \\
&= \sum_{n=1}^{\infty} |\mathcal{C}\mathcal{T}\mathcal{C}^* e_n|_H^2 \\
&\leq \|\mathcal{C}\|_{\text{op}}^2 \sum_{n=1}^{\infty} |\mathcal{T}\mathcal{C}^* e_n|_H^2 \\
&\leq \|\mathcal{C}\|_{\text{op}}^4 \sum_{n=1}^{\infty} |\mathcal{T} e_n|_H^2 \\
&= \|\mathcal{C}\|_{\text{op}}^4 \|\mathcal{T}\|_{\mathcal{H}}^2 \,.
\end{aligned}$$

Here we have used that $\|\mathcal{T}\mathcal{C}^*\|_{\mathcal{H}} = \|\mathcal{C}\mathcal{T}^*\|_{\mathcal{H}}$. For $\mathbb{C}_2$, we have by the triangle inequality,

$$\begin{aligned}
\|\mathbb{C}_2\mathcal{T}\|_{\mathcal{H}} &= \|\mathcal{C}\mathcal{T} + \mathcal{T}\mathcal{C}^*\|_{\mathcal{H}} \\
&\leq \|\mathcal{C}\mathcal{T}\|_{\mathcal{H}} + \|\mathcal{T}\mathcal{C}^*\|_{\mathcal{H}} \\
&\leq \|\mathcal{C}\|_{\text{op}}\|\mathcal{T}\|_{\mathcal{H}} + \|\mathcal{C}\|_{\text{op}}\|\mathcal{T}^*\|_{\mathcal{H}} \\
&= 2\|\mathcal{C}\|_{\text{op}}\|\mathcal{T}\|_{\mathcal{H}} \,.
\end{aligned}$$

Hence, the first claim of the lemma holds.

For $\mathcal{T} \in \mathcal{H}$, it follows that

$$\begin{aligned}
(\mathbb{C}_1\mathcal{T}f, g)_H &= (\mathcal{C}\mathcal{T}\mathcal{C}^* f, g)_H \\
&= (f, \mathcal{C}\mathcal{T}^*\mathcal{C}^* g)_H \\
&= (f, \mathbb{C}_1\mathcal{T}^* g)_H \,.
\end{aligned}$$

An analogous computation shows that also $\mathbb{C}_2\mathcal{T} = \mathbb{C}_2\mathcal{T}^*$, and the second claim of the lemma holds. Hence, the proof is complete. □

Since $\mathbb{C} \in L(\mathcal{H})$, it follows that $\mathbb{C}$ generates a uniformly continuous C_0-semigroup $\mathbb{S}(t)$, $t \geq 0$, with $\mathbb{S}(t) = \exp(t\mathbb{C})$ (see for example Gawarecki and Mandrekar [20, Thm. 1.1]). We note the following for $\mathbb{C}_i$, $i = 1, 2$:

Lemma 2.2. *For the C_0-semigroup $\mathbb{S}_i$ generated by $\mathbb{C}_i$ in* (2.1) *and* (2.2)*, $i = 1, 2$, resp., with $\mathcal{C} \in L(H)$, we have*

$$\mathbb{S}_1(t)\mathcal{T} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{C}^n \mathcal{T} (\mathcal{C}^*)^n \,,$$

and

$$\mathbb{S}_2(t)\mathcal{T} = \exp(t\mathcal{C})\mathcal{T}\exp(t\mathcal{C}^*) \,,$$

for every $\mathcal{T} \in \mathcal{H}$.

Proof. For $\mathcal{T} \in \mathcal{H}$, we find for $n \geq 1$

$$\mathbb{C}_1^n \mathcal{T} = \mathbb{C}_1^{n-1}(\mathcal{C}\mathcal{T}\mathcal{C}^*) \,,$$

and iterating this yields

$$\mathbb{C}_1^n \mathcal{T} = \mathcal{C}^n \mathcal{T} \mathcal{C}^{*n} \,.$$

Hence, the result for $\mathbb{S}_1$ follows.

For the case $\mathbb{C}_2$, note that

$$\exp(t\mathcal{C})\mathcal{T}\exp(t\mathcal{C}^*) = \sum_{n,m=0}^{\infty} \frac{t^{n+m}}{n!m!} \mathcal{C}^n \mathcal{T} \mathcal{C}^{*m} \,.$$

On the other hand,

$$\exp(t\mathbb{C}_2)\mathcal{T} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{C}_2^n \mathcal{T} \,.$$

Spelling out $\mathbb{C}_2^n \mathcal{T}$ and comparing with the terms in the double-sum above, we show the second result. The proof of the lemma is complete. □

We now introduce the operator-valued BNS stochastic volatility model. To this end, assume that $\{\mathcal{Y}(t)\}_{t\geq 0}$ is a $\mathcal{H}$-valued stochastic process satisfying the dynamics

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t)\,dt + d\mathcal{L}(t) \qquad \mathcal{Y}(0) = \mathcal{Y}_0 \,. \tag{2.3}$$

Here, $\mathcal{L}$ is an $\mathcal{H}$-valued Lévy process and $\mathcal{Y}_0 \in \mathcal{H}$. We suppose that $\mathcal{L}$ is square-integrable, with covariance operator $\mathbb{Q}_{\mathcal{L}}$. Recall that $\mathbb{Q}_{\mathcal{L}}$ is a self-adjoint non-negative definite trace class operator on $\mathcal{H}$. We have,

Lemma 2.3. *For every $t \geq 0$, it holds*

$$\int_0^t \|\mathbb{S}(t-s)\mathbb{Q}_{\mathcal{L}}^{1/2}\|_{L_{HS}(\mathcal{H})}^2 \, ds \leq \frac{\mathrm{Tr}(\mathbb{Q}_{\mathcal{L}})}{2\|\mathbb{C}\|_{op}} (\mathrm{e}^{2t\|\mathbb{C}\|_{op}} - 1) < \infty \,.$$

Proof. Note first that for any $\mathcal{T} \in \mathcal{H}$, we have by the representation of $\mathbb{S}$,

$$\begin{aligned}\|\mathbb{S}(u)\mathcal{T}\|_{\mathcal{H}} &\leq \|\mathbb{S}(u)\|_{\text{op}}\|\mathcal{T}\|_{\mathcal{H}} \\ &\leq \|\mathcal{T}\|_{\mathcal{H}} \sum_{k=0}^{\infty} \frac{u^k}{k!}\|\mathbb{C}\|_{\text{op}}^k \\ &= \mathrm{e}^{u\|\mathbb{C}\|_{\text{op}}}\|\mathcal{T}\|_{\mathcal{H}}\,.\end{aligned}$$

But then, for an orthonormal basis $\{\mathcal{T}_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$,

$$\begin{aligned}\|\mathbb{S}(u)\mathbb{Q}_{\mathcal{L}}^{1/2}\|_{L_{\text{HS}}(\mathcal{H})}^2 &= \sum_{n=1}^{\infty} \|\mathbb{S}(u)\mathbb{Q}_{\mathcal{L}}^{1/2}\mathcal{T}_n\|_{\mathcal{H}}^2 \\ &\leq \mathrm{e}^{2u\|\mathbb{C}\|_{\text{op}}} \sum_{n=1}^{\infty} \|\mathbb{Q}_{\mathcal{L}}^{1/2}\mathcal{T}_n\|_{\mathcal{H}}^2 \\ &= \mathrm{e}^{2u\|\mathbb{C}\|_{\text{op}}}\mathrm{Tr}(\mathbb{Q}_{\mathcal{L}})\,.\end{aligned}$$

Here we have used the fact that

$$\mathrm{Tr}(\mathbb{Q}_{\mathcal{L}}) = \sum_{n=1}^{\infty}\langle \mathbb{Q}_{\mathcal{L}}\mathcal{T}_n, \mathcal{T}_n\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty}\|\mathbb{Q}_{\mathcal{L}}^{1/2}\mathcal{T}_n\|_{\mathcal{H}}^2\,.$$

Hence, since $\|\mathbb{C}\|_{\text{op}} < \infty$ and $\mathbb{Q}_{\mathcal{L}}$ is a trace class operator, the result follows. □

Invoking this lemma, it follows from the theory of Hilbert-space valued stochastic differential equations (see *e.g.* Peszat and Zabczyk [27]) that there exists a unique mild solution to (2.3)

$$\mathcal{Y}(t) = \mathbb{S}(t)\mathcal{Y}_0 + \int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\,, \tag{2.4}$$

for $t \geq 0$. In the next lemma we derive a bound for the L^2-norm of $\mathcal{Y}$:

Lemma 2.4. *It holds that*

$$\mathbb{E}\left[\|\mathcal{Y}(t)\|_{\mathcal{H}}^2\right] \leq c\mathrm{e}^{2t\|\mathbb{C}\|_{op}}$$

for a constant $c > 0$ *given by* $c = 2\|\mathcal{Y}_0\|_{\mathcal{H}}^2 + \mathrm{Tr}(\mathbb{Q}_{\mathcal{L}})/\|\mathbb{C}\|_{op}$.

Proof. From the mild solution of $\mathcal{Y}(t)$ in (2.4) and the triangle inequality we find,

$$\mathbb{E}\left[\|\mathcal{Y}(t)\|_{\mathcal{H}}^2\right] \leq 2\|\mathbb{S}(t)\mathcal{Y}_0\|_{\mathcal{H}}^2 + 2\int_0^t \|\mathbb{S}(t-s)\mathbb{Q}_{\mathcal{L}}^{1/2}\|_{L_{\text{HS}}(\mathcal{H})}^2\,ds\,,$$

where we used Cor. 8.17 in Peszat and Zabczyk [27]. But from Lemma 2.3 above the result follows. □

Let us compute the conditional characteristic function of $\mathcal{Y}(t)$: To this end, let $t \geq s$ and note that $\mathcal{Y}(t)$ given $\mathcal{Y}(s)$ has the representation

$$\mathcal{Y}(t) = \mathbb{S}(t-s)\mathcal{Y}(s) + \int_s^t \mathbb{S}(t-u)\, d\mathcal{L}(u)\,. \tag{2.5}$$

Before proceeding, we recall the cumulant of $\mathcal{L}$, that is, the characteristic exponent of the Lévy process $\mathcal{L}$ defined to be $\mathbb{E}[\exp(\mathrm{i}\langle \mathcal{L}(t), \mathcal{T}\rangle_{\mathcal{H}}] = \exp(t\Psi_{\mathcal{L}}(\mathcal{T}))$ for $\mathcal{T} \in \mathcal{H}$ (see Peszat and Zabczyk [27, Thm. 4.27]):

$$\Psi_{\mathcal{L}}(\mathcal{T}) = \mathrm{i}\langle \mathcal{D}, \mathcal{T}\rangle_{\mathcal{H}} - \frac{1}{2}\langle \mathbb{Q}_{\mathcal{L}}^0 \mathcal{T}, \mathcal{T}\rangle_{\mathcal{H}} + \int_{\mathcal{H}} (\mathrm{e}^{\mathrm{i}\langle \mathcal{Z}, \mathcal{T}\rangle_{\mathcal{H}}} - 1 - \mathrm{i}\mathbf{1}_{\|\mathcal{Z}\|_{\mathcal{H}} \leq 1}\langle \mathcal{Z}, \mathcal{T}\rangle_{\mathcal{H}})\, \nu(d\mathcal{Z})\,. \tag{2.6}$$

Here, following Peszat and Zabczyk [27, Thms. 4.44 and 4.47], $\mathbb{Q}_{\mathcal{L}}^0$ is the covariance operator of the continuous martingale part, ν is the Lévy measure on $\mathcal{H}$ satisfying

$$\int_{\mathcal{H}} \|\mathcal{Z}\|_{\mathcal{H}}^2 \nu(d\mathcal{Z}) < \infty\,,$$

and $\mathcal{D} \in \mathcal{H}$ is the drift of the Lévy process, where for $\mathcal{T} \in \mathcal{H}$,

$$\mathbb{E}[\langle \mathcal{L}(1), \mathcal{T}\rangle_{\mathcal{H}}] = \langle \mathcal{D}, \mathcal{T}\rangle_{\mathcal{H}} + \int_{\mathcal{H}\setminus\{\|\mathcal{Z}\|_{\mathcal{H}} < 1\}} \langle \mathcal{Z}, \mathcal{T}\rangle_{\mathcal{H}}\, \nu(d\mathcal{Z})\,.$$

Furthermore, the covariance operator of $\mathcal{L}$ is $\mathbb{Q}_{\mathcal{L}} = \mathbb{Q}_{\mathcal{L}}^0 + \mathbb{Q}_{\mathcal{L}}^1$ with

$$\langle \mathbb{Q}_{\mathcal{L}}^1 \mathcal{T}, \mathcal{U}\rangle_{\mathcal{H}} = \int_{\mathcal{H}} \langle \mathcal{T}, \mathcal{Z}\rangle_{\mathcal{H}} \langle \mathcal{U}, \mathcal{Z}\rangle_{\mathcal{H}}\, \nu(d\mathcal{Z})\,, \qquad \mathcal{T}, \mathcal{U} \in \mathcal{H}\,.$$

We have the following proposition, showing that $\mathcal{Y}$ is an *affine* process in $\mathcal{H}$:

Proposition 2.5. *For any $\mathcal{T} \in \mathcal{H}$ it holds that*

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \mathcal{Y}(t), \mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] = \exp\left(\mathrm{i}\langle \mathcal{Y}(s), \mathbb{S}^*(t-s)\mathcal{T}\rangle_{\mathcal{H}} + \int_0^{t-s} \Psi_{\mathcal{L}}(\mathbb{S}^*(u)\mathcal{T})\, du\right)\,.$$

Proof. From (2.5) we find for $\mathcal{T} \in \mathcal{H}$,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \mathcal{Y}(t), \mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] &= \mathrm{e}^{\mathrm{i}\langle \mathbb{S}(t-s)\mathcal{Y}(s), \mathcal{T}\rangle_{\mathcal{H}}} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\, d\mathcal{L}(u), \mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] \\ &= \mathrm{e}^{\mathrm{i}\langle \mathcal{Y}(s), \mathbb{S}^*(t-s)\mathcal{T}\rangle_{\mathcal{H}}} \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\, d\mathcal{L}(u), \mathcal{T}\rangle_{\mathcal{H}}}\right]\,.\end{aligned}$$

Here, we have appealed to the independent increment property of Lévy processes. Hence, from Peszat and Zabczyk [27, Thm. 4.27] it holds that

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\, d\mathcal{L}(u), \mathcal{T}\rangle_{\mathcal{H}}}\right] = \exp\left(\int_0^{t-s} \Psi_{\mathcal{L}}(\mathbb{S}^*(u)\mathcal{T})\, du\right)\,,$$

with $\Psi_{\mathcal{L}}$ defined in (2.6). The result follows. □

To define a stochastic volatility based on $\mathcal{Y}$ in (2.4) we must impose positivity constraints. This means that we want to restrict our attention to $\mathcal{Y}$'s which are self-adjoint, non-negative definite Hilbert-Schmidt operators on H for each $t \geq 0$. We now analyze additional conditions on $\mathbb{C}$ and $\mathcal{L}$ ensuring non-negative definiteness of $\mathcal{Y}$. First, we show that $\mathcal{Y}(t)$ is self-adjoint whenever $\mathcal{L}(t)$ is under a mild condition on $\mathbb{C}$:

Proposition 2.6. *Suppose that $(\mathbb{C}\mathcal{T})^* = \mathbb{C}\mathcal{T}^*$ for any $\mathcal{T} \in \mathcal{H}$. If $\{\mathcal{L}(t)\}_{t\geq 0}$ is a family of self-adjoint operators on H and $\mathcal{Y}_0$ is self-adjoint, then $\mathcal{Y}(t)$ is a self-adjoint operator on H for every $t \geq 0$.*

Proof. Let $f, g \in H$. Then we compute, using the dynamics of $\mathcal{Y}$ in (2.3), the assumption on $\mathbb{C}$, the self-adjointness of $\mathcal{L}(t)$ and the definition of Bochner integration:

$$\begin{aligned}(\mathcal{Y}(t)f, g)_H &= \int_0^t (\mathbb{C}\mathcal{Y}(s)f, g)_H \, ds + (\mathcal{L}(t)f, g)_H \\ &= \int_0^t (f, \mathbb{C}\mathcal{Y}^*(s)g)_H \, ds + (f, \mathcal{L}(t)g)_H \,.\end{aligned}$$

Thus, as $f, g \in H$ are arbitrary, we find that

$$d\mathcal{Y}^*(t) = \mathbb{C}\mathcal{Y}^*(t)\, dt + d\mathcal{L}(t) \,,$$

with initial condition $\mathcal{Y}^*(0) = \mathcal{Y}_0$. But by uniqueness of solutions of this linear stochastic differential equation, $\mathcal{Y}^*(t) = \mathcal{Y}(t)$. □

Recall from Lemma 2.1 that $(\mathbb{C}_i\mathcal{T})^* = \mathbb{C}_i\mathcal{T}^*$ for $i = 1, 2$.

Example 2.7. *A trivial way to introduce a self-adjoint Lévy process $\mathcal{L}$ in $\mathcal{H}$ is to take any real-valued square-integrable Lévy process L and multiply it with a self-adjoint operator $\mathcal{U} \in \mathcal{H}$, i.e., $\mathcal{L}(t) = L(t)\mathcal{U}$. For $\mathcal{S}, \mathcal{T} \in \mathcal{H}$,*

$$\mathbb{E}\left[\langle \mathcal{L}(t), \mathcal{S}\rangle_{\mathcal{H}} \langle \mathcal{L}(t), \mathcal{T}\rangle_{\mathcal{H}}\right] = \mathbb{E}[L^2(t)]\langle \mathcal{U}, \mathcal{S}\rangle_{\mathcal{H}}\langle \mathcal{U}, \mathcal{T}\rangle_{\mathcal{H}} = \mathbb{E}[L^2(t)]\langle \mathcal{U}^{\otimes 2}\mathcal{S}, \mathcal{T}\rangle_{\mathcal{H}} \,.$$

Thus, the covariance operator for this Lévy process becomes $\mathbb{Q}_{\mathcal{L}} = Var(L(1))\mathcal{U}^{\otimes 2}$, i.e., the tensor product of $\mathcal{U}$ with itself scaled by the variance of $L(1)$. We show that $\mathbb{Q}_{\mathcal{L}}$ is a self-adjoint, non-negative definite trace class operator. Indeed, it is obviously linear and

$$\|\mathbb{Q}_{\mathcal{L}}\mathcal{T}\|_{\mathcal{H}} = Var(L(1))\|\mathcal{U}\langle \mathcal{U}, \mathcal{T}\rangle_{\mathcal{H}}\|_{\mathcal{H}} \leq Var(L(1))\|\mathcal{U}\|_{\mathcal{H}}^2\|\mathcal{T}\|_{\mathcal{H}} \,,$$

which shows $\mathbb{Q}_{\mathcal{L}} \in L(\mathcal{H})$. Moreover,

$$\langle \mathbb{Q}_{\mathcal{L}}\mathcal{S}, \mathcal{T}\rangle_{\mathcal{H}} = Var(L(1))\langle \mathcal{U}, \mathcal{S}\rangle_{\mathcal{H}}\langle \mathcal{U}, \mathcal{T}\rangle_{\mathcal{H}} = Var(L(1))\langle \mathcal{S}, \mathcal{U}^{\otimes 2}\mathcal{T}\rangle_{\mathcal{H}} = \langle \mathcal{S}, \mathbb{Q}_{\mathcal{L}}\mathcal{T}\rangle_{\mathcal{H}}$$

and

$$\langle \mathbb{Q}_{\mathcal{L}}\mathcal{S}, \mathcal{S}\rangle_{\mathcal{H}} = Var(L(1))\langle \mathcal{U}, \mathcal{S}\rangle_{\mathcal{H}}^2 \geq 0 \,,$$

which show that $\mathbb{Q}_{\mathcal{L}}$ is a self-adjoint and non-negative definite operator on $\mathcal{H}$. Finally, for an orthonormal basis $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$,

$$\mathrm{Tr}(\mathbb{Q}_{\mathcal{L}}) = \sum_{n=1}^{\infty}\langle \mathbb{Q}_{\mathcal{L}}\mathcal{T}_n, \mathcal{T}_n\rangle_{\mathcal{H}} = Var(L(1))\sum_{n=1}^{\infty}\langle \mathcal{U}, \mathcal{T}_n\rangle_{\mathcal{H}}^2 = Var(L(1))\|\mathcal{U}\|_{\mathcal{H}}^2$$

where we used Parseval's identity. Hence, $\mathbb{Q}_{\mathcal{L}}$ is trace class. Of course, if we add the assumption that $\mathcal{U}$ is positive definite and $L(t)$ is taking values on $\mathbb{R}_+$,[1] it follows that

$$(\mathcal{L}(t)f, f)_H = L(t)(\mathcal{U}f, f)_H \geq 0\,,$$

for any $f \in H$, and thus $\mathcal{L}(t)$ is non-negative definite.

This simple example of an operator-valued Lévy process $\mathcal{L}$ brings us to the question of non-negative definiteness of $\mathcal{Y}$, which we investigate next. First, let us define what we mean by non-decreasing paths of $\mathcal{L}$:

Definition 2.8. *We say that the $\mathcal{H}$-valued Lévy process $\mathcal{L}$ has non-decreasing paths if $\mathcal{L}(t)$ is self-adjoint and $t \mapsto (\mathcal{L}(t)f, f)_H$ is non-decreasing in $t \geq 0$ for every $f \in H$, $a.s.$*

Note that as $\mathcal{L}(0) = 0$ by definition of the Lévy process, the non-decreasing paths property implies $(\mathcal{L}(t)f, f)_H \geq 0$ for every $t \geq 0$, $a.s.$. But then it follows that $\mathcal{L}(t)$ is a non-negative definite operator. In fact, something slightly stronger holds:

Lemma 2.9. *Assume $\mathcal{L}$ is an $\mathcal{H}$-valued Lévy process with non-decreasing paths. Then $\mathcal{L}(t) - \mathcal{L}(s)$ is $a.s$ non-negative definite for every $t > s \geq 0$.*

Proof. For $t > s \geq 0$, we have for $f \in H$

$$((\mathcal{L}(t) - \mathcal{L}(s))f, f)_H = (\mathcal{L}(t)f, f)_H - (\mathcal{L}(s)f, f)_H$$

which is non-negative $a.s$ by the non-decreasing path property of $t \mapsto (\mathcal{L}(t)f, f)_H$. The assertion follows. □

As we shall see, this monotonicity property of the paths of $\mathcal{L}$ is exactly what we need in order to show that $\mathcal{Y}(t)$ is non-negative definite for every $t \geq 0$. But first, let us do some analysis of Lévy processes in $\mathcal{H}$ with non-decreasing paths.

---

[1] This means that the Lévy process is a so-called subordinator, that is, a process with only positive jumps and non-negative drift.

Define for the moment $L_f(t) := (\mathcal{L}(t)f, f)_H$ for given $f \in H$. We show that this is a Lévy process on the real line. To this end, consider the functional $\mathcal{F}_f : \mathcal{H} \to \mathbb{R}$ defined as $\mathcal{F}_f(\mathcal{T}) = (\mathcal{T}f, f)_H$. This is obviously linear, and since

$$|\mathcal{F}_f(\mathcal{T})| = |(\mathcal{T}f, f)_H| \leq |\mathcal{T}f|_H |f|_H \leq \|\mathcal{T}\|_{\text{op}} |f|_H^2 \,,$$

we have $\mathcal{F}_f \in \mathcal{H}^*$. Hence, there exists a unique element in $\mathcal{H}$, which we also denote by $\mathcal{F}_f$,

$$\mathcal{F}_f(\mathcal{T}) = \langle \mathcal{T}, \mathcal{F}_f \rangle_{\mathcal{H}} \,.$$

In the following we identify Hilbert-Schmidt operators on H with $H \otimes H$. Similarly, the Hilbert-Schmidt operator $h^* \otimes h$ for $h \in H$ is written as $h \otimes h$. Then for any Hilbert-Schmidt operator $\mathcal{V}$ we have the following identity

$$\langle \mathcal{V}, h \otimes h \rangle_{\mathcal{H}} = (\mathcal{V}h, h)_H.$$

Thus, we have $\mathcal{F}_f = f \otimes f$. Indeed, a straightforward calculation shows,

$$\begin{aligned}
\|f \otimes f\|_{\mathcal{H}}^2 &= \sum_{n=1}^{\infty} ((f \otimes f)e_n, (f \otimes f)e_n)_H \\
&= \sum_{n=1}^{\infty} ((f, e_n)_H f, (f, e_n)_H f)_H \\
&= |f|_H^2 \sum_{n=1}^{\infty} (f, e_n)_H^2 \\
&= |f|_H^4 \,.
\end{aligned}$$

Hence, $f \otimes f \in \mathcal{H}$ with norm $\|f \otimes f\|_{\mathcal{H}} = |f|_H^2$. Furthermore,

$$\begin{aligned}
\langle \mathcal{T}, f \otimes f \rangle_{\mathcal{H}} &= \sum_{n=1}^{\infty} (\mathcal{T}e_n, (f \otimes f)e_n)_H \\
&= \sum_{n=1}^{\infty} (\mathcal{T}e_n, (f, e_n)_H f)_H \\
&= \sum_{n=1}^{\infty} (\mathcal{T}(f, e_n)_H e_n, f)_H \\
&= (\mathcal{T}f, f)_H \,.
\end{aligned}$$

By definition of an $\mathcal{H}$-valued Lévy process, $t \mapsto \langle \mathcal{L}(t), \mathcal{T} \rangle_{\mathcal{H}}$ is a real-valued Lévy process for any $\mathcal{T} \in \mathcal{H}$. Therefore, in particular, $L_f(t) = (\mathcal{L}(t)f, f)_H$ is a real-valued Lévy process by choosing $\mathcal{T} = f \otimes f$. If, furthermore, $\mathcal{L}$ has non-decreasing paths, it follows that L_f is a Lévy process with non-decreasing paths, i.e., a subordinator. We have the following property of the continuous martingale part of $\mathcal{L}$:

Proposition 2.10. *Let $\mathcal{L}$ be the Lévy process defined after* (2.3) *with non-decreasing paths, and denote the covariance operator of the continuous martingale part by $\mathbb{Q}_{\mathcal{L}}^0$. Let $\mathcal{T}$ be a symmetric Hilbert-Schmidt operator. Then $\mathbb{Q}_{\mathcal{L}}^0\mathcal{T} = 0$, that is, $\mathcal{T} \in \ker(\mathbb{Q}_{\mathcal{L}}^0)$.*

Proof. Let first $\mathcal{T} = f \otimes f$ with $f \in H$. Then the continuous martingale part of the characteristic function of $L_f(t) = \langle \mathcal{L}(t), \mathcal{F}_f \rangle_{\mathcal{H}}$ is $\langle \mathbb{Q}_{\mathcal{L}}^0 \mathcal{F}_f, \mathcal{F}_f \rangle_{\mathcal{H}}$, which must be zero due to the non-decreasing paths of $L_f(t)$. But then

$$\langle \mathbb{Q}_{\mathcal{L}}^0 \mathcal{F}_f, \mathcal{F}_f \rangle_{\mathcal{H}} = \|(\mathbb{Q}_{\mathcal{L}}^0)^{1/2} \mathcal{F}_f\|_{\mathcal{H}}^2 = 0\,,$$

and thus $\mathcal{F}_f$ is in the kernel of $(\mathbb{Q}_{\mathcal{L}}^0)^{1/2}$. As it holds,

$$\mathbb{Q}_{\mathcal{L}}^0 \mathcal{F}_f = (\mathbb{Q}_{\mathcal{L}}^0)^{1/2} (\mathbb{Q}_{\mathcal{L}}^0)^{1/2} \mathcal{F}_f = (\mathbb{Q}_{\mathcal{L}}^0)^{1/2} 0 = 0\,,$$

we can conclude that $\mathcal{F}_f \in \ker(\mathbb{Q}_{\mathcal{L}}^0)$.

Now let $\mathcal{T}$ be a symmetric Hilbert-Schmidt operator as in the proposition. It can be shown that $\mathcal{T}$ must be of the form

$$\mathcal{T} = \sum_{k,l \in \mathbb{N}} \gamma_{k,l} e_k \otimes e_l,$$

with $\sum_{k,l} \gamma_{k,l}^2 < \infty$ and $\gamma_{k,l} = \gamma_{l,k}$, see Lemma A.1 for a sketch of the arguments. Therefore we can write

$$\begin{aligned}
\mathcal{T} &= \sum_{k \in \mathbb{N}} \gamma_{k,k} e_k \otimes e_k + \sum_{k \in \mathbb{N}} \sum_{l<k} \gamma_{k,l} (e_k \otimes e_l + e_l \otimes e_k) \\
&= \sum_{k \in \mathbb{N}} \gamma_{k,k} e_k \otimes e_k + \sum_{k \in \mathbb{N}} \sum_{l<k} \gamma_{k,l} \big((e_k + e_l) \otimes (e_k + e_l) - e_k \otimes e_k - e_l \otimes e_l \big).
\end{aligned}$$

With this we compute

$$\begin{aligned}
\mathbb{Q}_{\mathcal{L}}^0 \mathcal{T} = &\sum_{k \in \mathbb{N}} \gamma_{k,k} \mathbb{Q}_{\mathcal{L}}^0 (e_k \otimes e_k) \\
&+ \sum_{k \in \mathbb{N}} \sum_{l<k} \gamma_{k,l} \big(\mathbb{Q}_{\mathcal{L}}^0 ((e_k + e_l) \otimes (e_k + e_l)) - \mathbb{Q}_{\mathcal{L}}^0 (e_k \otimes e_k) - \mathbb{Q}_{\mathcal{L}}^0 (e_l \otimes e_l) \big),
\end{aligned}$$

which ends the proof since $\mathbb{Q}_{\mathcal{L}}^0$ applied to $f \otimes f$ for any $f \in H$ is zero by the first part of the proof. □

As the space of symmetric Hilbert-Schmidt operators does not span $\mathcal{H}$, we cannot conclude that $\mathbb{Q}_{\mathcal{L}}^0 = 0$, i.e., that $\mathcal{L}$ does not have a continuous martingale part. Recall that subordinators on $\mathbb{R}$ do not have any continuous martingale part.

Denote now by $\mathcal{H}_+$ the convex cone of non-negative definite operators in $\mathcal{H}$.

Proposition 2.11. *Assume that $\mathbb{C}(\mathcal{H}_+) \subset \mathcal{H}_+$. If $\mathcal{L}(t)$ is an $\mathcal{H}$-valued Lévy process with non-decreasing paths and $\mathcal{Y}_0$ is non-negative definite, then $\mathcal{Y}$ is non-negative definite.*

Proof. Recall that

$$\mathcal{Y}(t) = \mathbb{S}(t)\mathcal{Y}_0 + \int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\,.$$

It holds,

$$\mathbb{S}(t)\mathcal{Y}_0 = \mathrm{e}^{t\mathbb{C}}\mathcal{Y}_0 = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbb{C}^k\mathcal{Y}_0\,,$$

which is then a non-negative definite operator by the assumption on $\mathbb{C}$.

Next, we know that $\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)$ is defined as the strong limit of $\sum_{m=1}^M \mathbb{S}(t-s_m)\,\Delta\mathcal{L}(s_m)$ in $L^2(\Omega;\mathcal{H})$. Here, $\{s_m\}_{m=1}^M$ is a nested partition of $[0,t]$, and $\Delta\mathcal{L}(s_m) := \mathcal{L}(s_{m+1}) - \mathcal{L}(s_m)$ is an increment of $\mathcal{L}$. But $\Delta\mathcal{L}(s_m)$ is non-negative definite $a.s.$ by Lemma 2.9, and therefore each term in the sum above is positive, $a.s.$, since $\mathbb{S}$ preserves non-negative definiteness by assumption on $\mathbb{C}$. Hence it follows that $\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)$ is non-negative definite $a.s.$, and the proof is complete. □

From Proposition 2.6 and Proposition 2.11 it follows that under the assumptions

a) $(\mathbb{C}\mathcal{T})^* = \mathbb{C}\mathcal{T}^*$,

b) $\mathbb{C}(\mathcal{H}_+) \subset \mathcal{H}_+$,

c) $\mathcal{L}(t)$ is a self-adjoint and non-negative definite square-integrable Lévy process with values in $\mathcal{H}$, and

d) $\mathcal{Y}_0$ is self-adjoint and non-negative definite,

then $\mathcal{Y}(t)$ becomes a self-adjoint, non-negative definite square integrable process with values in $\mathcal{H}$. Hence, we have a unique square root, $\mathcal{Y}^{1/2}(t)$ for every $t \geq 0$. We shall use this to model the stochastic volatility.

Lemma 2.12. *It holds that* $\mathbb{C}_1(\mathcal{H}_+) \subset \mathcal{H}_+$.

Proof. We recall the definition of $\mathbb{C}_1$ in (2.1). Let $\mathcal{T} \in \mathcal{H}_+$. Then, for any $f \in H$

$$(\mathbb{C}_1\mathcal{T}f, f)_H = (\mathcal{C}\mathcal{T}\mathcal{C}^*f, f)_H = (\mathcal{T}\mathcal{C}^*f, \mathcal{C}^*f)_H \geq 0\,.$$

Hence, the result follows. □

In fact, for $\mathbb{C}_2$ we cannot prove that it preserves the property of non-negative definiteness. But recalling the proof of Prop. 2.11, it is indeed the associated semigroup of $\mathbb{C}$ that must preserve non-negative definiteness. As we have that $\mathbb{S}_2(t)\mathcal{T} = \exp(t\mathcal{C})\mathcal{T}\exp(t\mathcal{C}^*)$ from Lemma 2.2, it follows that $\mathbb{S}_2(t)(\mathcal{H}_+) \subset \mathcal{H}_+$, and we can conclude that $\mathcal{Y}$ with $\mathbb{C} = \mathbb{C}_2$ is also non-negative definite whenever $\mathcal{L}$ has non-decreasing paths and $\mathcal{Y}_0$ is non-negative definite. Indeed, by inspection of the proof of Prop. 2.11, we can substitute the condition b) $\mathbb{C}(\mathcal{H}_+) \subset \mathcal{H}_+$ on $\mathbb{C}$ with the condition

b') $\quad \mathbb{S}(t)(\mathcal{H}_+) \subset \mathcal{H}_+, \quad t \geq 0\,.$

In conclusion, if we use $\mathbb{C} = \mathbb{C}_i$ for either $i = 1$ or $i = 2$ in the definition of the volatility process $\mathcal{Y}$, we obtain a non-negative definite operator under appropriate conditions on $\mathcal{L}$ and $\mathcal{Y}_0$. We recall that the choice $\mathbb{C} = \mathbb{C}_2$ can be seen as the analogue of the matrix-valued volatility model by Barndorff-Nielsen and Stelzer [9].

Let us discuss the particular case when $\mathcal{L}$ is a compound Poisson process. To this end, we define

$$\mathcal{L}(t) = \sum_{i=1}^{N(t)} \mathcal{X}_i \,, \tag{2.7}$$

where N is a real-valued Poisson process with intensity $\lambda > 0$ and $\{\mathcal{X}_i\}_{i\in\mathbb{N}}$ are $i.i.d.$ square-integrable $\mathcal{H}$-valued random variables. Note that for $f \in H$, we find from the linearity of the inner product

$$\langle \mathcal{L}(t), f \otimes f \rangle_{\mathcal{H}} = \langle \sum_{i=1}^{N(t)} \mathcal{X}_i, f \otimes f \rangle_{\mathcal{H}} = \sum_{i=1}^{N(t)} \langle \mathcal{X}_i, f \otimes f \rangle_{\mathcal{H}} = \sum_{i=1}^{N(t)} (\mathcal{X}_i f, f)_H \,.$$

Hence, $L_f(t) := \langle \mathcal{L}(t), f \otimes f \rangle_{\mathcal{H}}$ is a real-valued compound Poisson process with jumps given by the $i.i.d$ random variables $(\mathcal{X}_i f, f)_H$. The process $L_f(t)$ has non-decreasing paths if and only if $\mathcal{X}$ is self-adjoint and $(\mathcal{X} f, f)_H$ is distributed on $\mathbb{R}_+$, where the latter holds if and only if $\mathcal{X}$ is non-negative definite, i.e., $\mathcal{X} \in \mathcal{H}_+$. Next, introduce the map $\phi_f : \mathcal{H}_+ \to \mathbb{R}_+$ by

$$\phi_f(\mathcal{Z}) = \langle \mathcal{Z}, f \otimes f \rangle_{\mathcal{H}} \,.$$

For any Borel set $A \subset \mathbb{R}_+$, we define $P_{\phi_f}(A) := P_{\mathcal{X}}(\phi_f^{-1}(A))$ where $P_{\mathcal{X}}$ is the law of $\mathcal{X}$. But then

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}uz} P_{\phi_f}(dz) = \int_{\mathcal{H}_+} (\mathrm{e}^{\mathrm{i}u\cdot} \circ \phi_f)(\mathcal{Z}) P_{\mathcal{X}}(d\mathcal{Z}) = \int_{\mathcal{H}_+} \mathrm{e}^{\mathrm{i}u\langle \mathcal{Z}, f\otimes f\rangle_{\mathcal{H}}} P_{\mathcal{X}}(d\mathcal{Z}) \,,$$

and $P_{X_f}(A) = P_{\mathcal{X}}(\phi_f^{-1}(A))$ with P_{X_f} being the law of $X_f := (\mathcal{X} f, f)_H$.

Suppose that Z is an H-valued centered square-integrable Gaussian random variable with covariance operator $\mathcal{Q}_Z$. Let $\mathcal{X}_i = Z_i^{\otimes 2}, i = 1, 2, \dots$, where $\{Z_i\}_{i\in\mathbb{N}}$ are independent copies of Z. First, it is simple to see $Z^{\otimes 2}$ is also a Hilbert-Schmidt operator, that is $Z^{\otimes 2} \in \mathcal{H}$, since

$$\|Z^{\otimes 2}\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |Z^{\otimes 2} e_n|_H^2 = \sum_{n=1}^{\infty} (Z, e_n)_H^2 |Z|_H^2 = |Z|_H^4 < \infty \,.$$

This $\mathcal{H}$-valued random variable has expected ($\mathcal{H}$-valued) value $\mathbb{E}[Z^{\otimes 2}] = \mathcal{Q}_Z$, which can be seen from the following calculation: given $\mathcal{T} \in \mathcal{H}$, then by linearity of the expectation operator

$$\begin{aligned}
\langle \mathbb{E}\left[Z^{\otimes 2}\right], \mathcal{T} \rangle_{\mathcal{H}} &= \sum_{n=1}^{\infty} (\mathbb{E}[Z^{\otimes 2}] e_n, \mathcal{T} e_n)_H \\
&= \sum_{n=1}^{\infty} \mathbb{E}\left[(Z^{\otimes 2} e_n, \mathcal{T} e_n)_H\right] \\
&= \sum_{n=1}^{\infty} \mathbb{E}\left[(Z, e_n)_H (Z, \mathcal{T} e_n)_H\right]
\end{aligned}$$

$$= \sum_{n=1}^{\infty} (\mathcal{Q}_Z e_n, \mathcal{T} e_n)_H$$
$$= \langle \mathcal{Q}_Z, \mathcal{T} \rangle_{\mathcal{H}} .$$

Furthermore, $Z^{\otimes 2}$ is self-adjoint and non-negative definite, since $(Z^{\otimes 2} f, f)_H = (Z, f)^2 \geq 0$. From this we also see that the jumps of $L_f(t)$, the compound Poisson process $\mathcal{L}$ evaluated at $f \otimes f$, is given by $(Z, f)_H^2$, with $(Z, f)_H$ being a real valued centered Gaussian variable with variance $|\mathcal{Q}_Z^{1/2} f|_H^2$. Hence, $(Z, f)_H^2$ becomes Gamma distributed with scale parameter $2|\mathcal{Q}_Z^{1/2} f|_H^2$ and shape parameter $1/2$. In fact, something much more general can be said about the compound Poisson process $\mathcal{L}$ for jumps given by $\mathcal{X} = Z^{\otimes 2}$. Indeed, if $\mathcal{T} \in \mathcal{H}$ is self-adjoint, then it follows from Prop. 3 in Fraisse and Viguier-Pla [19] that the characteristic functional of $\langle Z^{\otimes 2}, \mathcal{T} \rangle_{\mathcal{H}}$ is,

$$\mathbb{E}\left[\exp(\mathrm{i}\langle Z^{\otimes 2}, \mathcal{T} \rangle_{\mathcal{H}})\right] = \left(\det(I - 2\mathrm{i}\mathcal{T}\mathcal{Q}_Z)\right)^{-1/2} . \tag{2.8}$$

Here, I is the identity operator on H and det is the Fredholm determinant. We can interpret $Z^{\otimes 2}$ as being infinite dimensional Wishart distributed. By conditioning of $N(t)$ and appealing to the independence of the jumps $\mathcal{X}_i$, we find the cumulant $\Psi_{\mathcal{L}}$ of $\mathcal{L}$ defined in (2.6) to be

$$\Psi_{\mathcal{L}}(\mathcal{T}) = \ln \mathbb{E}\left[\exp(\mathrm{i}\langle \mathcal{L}(1), \mathcal{T} \rangle_{\mathcal{H}})\right] = \lambda \left(\left(\det(I - 2\mathrm{i}\mathcal{T}\mathcal{Q}_Z)\right)^{-1/2} - 1 \right) , \tag{2.9}$$

for any self-adjoint $\mathcal{T} \in \mathcal{H}$.

Suppose now in more generality that Z is an H-valued centered square-integrable random variable. Then Z has a self-adjoint non-negative definite continuous linear covariance operator $\mathcal{Q}_Z$, too. Let $\mathcal{X}_i = Z_i^{\otimes 2}, i = 1, 2, \ldots,$, where $\{Z_i\}_{i \in \mathbb{N}}$ are independent copies of Z. Then by the same calculations as before $\|Z^{\otimes 2}\|_{\mathcal{H}}^2 = |Z|_H^4 < \infty$ and $\langle \mathbb{E}\left[Z^{\otimes 2}\right], \mathcal{T} \rangle_{\mathcal{H}} = \langle \mathcal{Q}_Z, \mathcal{T} \rangle_{\mathcal{H}}$. Also here $Z^{\otimes 2}$ is self-adjoint and non-negative definite, since $(Z^{\otimes 2} f, f)_H = (Z, f)_H^2 \geq 0$ and the jumps of $L_f(t)$, the compound Poisson process $\mathcal{L}$ evaluated at $f \otimes f$, is given by $(Z, f)_H^2$, with $(Z, f)_H$ being a real valued centered variable with variance $|\mathcal{Q}_Z^{1/2} f|_H^2$. The cumulant has then to be computed for each case separately.

3. A VOLATILITY-MODULATED ORNSTEIN-UHLENBECK PROCESS

Let X be a stochastic process with values in H given by the Ornstein-Uhlenbeck process

$$dX(t) = \mathcal{A}X(t)\,dt + \mathcal{Y}^{1/2}(t)\,dB(t) \qquad X(0) = X_0 . \tag{3.1}$$

Here, B is an H-valued Wiener process with covariance operator $\mathcal{Q}$, which is a self-adjoint, non-negative definite trace class operator on H. Furthermore, $X_0 \in H$ and $\mathcal{Y}$ is given in (2.4), being the solution of the dynamics (2.3) from the previous section, where we assume that $\mathcal{Y}_0$ is self-adjoint, non-negative definite and $\mathcal{L}$ is a $\mathcal{H}$-valued Lévy process with non-decreasing paths. We suppose that $(\mathbb{C}\mathcal{T})^* = \mathbb{C}\mathcal{T}^*$ for every $\mathcal{T} \in \mathcal{H}$ and $\mathbb{C}(\mathcal{H}_+) \subset \mathcal{H}_+$ (or, that the semigroup $\mathbb{S}(t)$ of $\mathbb{C}$ has this property). Then by Props. 2.6 and

2.11, $\mathcal{Y}(t)$ is self-adjoint, non-negative definite, and we can define its square root $\mathcal{Y}^{1/2}(t)$. Finally, $\mathcal{A}$ is a (possibly unbounded) linear operator on H, densely defined, generating a C_0-semigroup $\mathcal{S}$.

Let us first show that the stochastic integral in (3.1) makes sense. The following proposition is crucial:

Proposition 3.1. *For every $t \geq 0$, it holds that*

$$\mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{Y}(t)\mathcal{Q}^{1/2})\right] = \mathrm{Tr}(\mathcal{Q}^{1/2}\mathbb{S}(t)\mathcal{Y}_0\mathcal{Q}^{1/2}) + \mathrm{Tr}(\mathcal{Q}^{1/2}\int_0^t \mathbb{S}(s)\,ds\mathbb{E}[\mathcal{L}(1)]\mathcal{Q}^{1/2})$$

where $\int_0^t \mathbb{S}(s)\,ds$ is the Bochner integral of $s \mapsto \mathbb{S}(s) \in L_{HS}(\mathcal{H})$ and $\mathbb{E}[\mathcal{L}(1)]$ is the operator-valued expected value of $\mathcal{L}(1)$.

Proof. First, note that the trace is linear, to give

$$\mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{Y}(t)\mathcal{Q}^{1/2})\right] = \mathrm{Tr}(\mathcal{Q}^{1/2}\mathbb{S}(t)\mathcal{Y}_0\mathcal{Q}^{1/2}) + \mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\mathcal{Q}^{1/2})\right].$$

Suppose for a moment that $\mathcal{X}$ is a $\mathcal{H}$-valued integrable random variable. Then

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{X}\mathcal{Q}^{1/2})\right] &= \sum_{n=1}^{\infty}\mathbb{E}\left[(\mathcal{Q}^{1/2}\mathcal{X}\mathcal{Q}^{1/2}e_n, e_n)_H\right] \\ &= \sum_{n=1}^{\infty}\mathbb{E}\left[(\mathcal{X}\mathcal{Q}^{1/2}e_n, \mathcal{Q}^{1/2}e_n)_H\right].\end{aligned}$$

But $(\mathcal{X}f, f)_H = \langle \mathcal{X}, f \otimes f\rangle_{\mathcal{H}}$, which holds due to the isomorphy of the Hilbert-Schmidt operators with tensor products of Hilbert spaces, and

$$\mathbb{E}[(\mathcal{X}f, f)_H] = \mathbb{E}[\langle \mathcal{X}, f \otimes f\rangle_{\mathcal{H}}] = \langle \mathcal{M}, f \otimes f\rangle_{\mathcal{H}},$$

for some $\mathcal{M} \in \mathcal{H}$. This operator is called the mean of $\mathcal{X}$, and we write $\mathbb{E}[\mathcal{X}] = \mathcal{M}$, the operator-valued expectation of $\mathcal{X}$. Thus

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{X}\mathcal{Q}^{1/2})\right] &= \sum_{n=1}^{\infty}\mathbb{E}\left[\langle \mathcal{X}, \mathcal{Q}^{1/2}e_n \otimes \mathcal{Q}^{1/2}e_n\rangle_{\mathcal{H}}\right] \\ &= \sum_{n=1}^{\infty}\langle \mathcal{M}, \mathcal{Q}^{1/2}e_n \otimes \mathcal{Q}^{1/2}e_n\rangle_{\mathcal{H}} \\ &= \sum_{n=1}^{\infty}(\mathcal{M}\mathcal{Q}^{1/2}e_n, \mathcal{Q}^{1/2}e_n)_H \\ &= \mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{M}\mathcal{Q}^{1/2}) \\ &= \mathrm{Tr}(\mathcal{Q}^{1/2}\mathbb{E}[\mathcal{X}]\mathcal{Q}^{1/2}).\end{aligned}$$

Letting $\mathcal{X} = \int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)$, we hence obtain

$$\mathbb{E}[\mathrm{Tr}(\mathcal{Q}^{1/2}\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\mathcal{Q}^{1/2})] = \mathrm{Tr}\left(\mathcal{Q}^{1/2}\mathbb{E}\left[\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\right]\mathcal{Q}^{1/2}\right)$$

We derive an expression for the mean of the stochastic integral.

Recalling (the proof of) Prop. 2.5, we find that with $\theta \in \mathbb{R}$

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s),\theta\mathcal{T}\rangle_{\mathcal{H}}}\right] = \exp\left(\int_0^t \Psi_{\mathcal{L}}(\mathbb{S}^*(u)(\theta\mathcal{T}))\,du\right)$$
$$= \exp\left(\int_0^t \Psi_{\mathcal{L}}(\theta\mathbb{S}^*(u)(\mathcal{T}))\,du\right),$$

with $\Psi_{\mathcal{L}}$ defined in (2.6). Since $\Psi_{\mathcal{L}}(0)=0$, we find

$$\frac{d}{d\theta}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s),\theta\mathcal{T}\rangle_{\mathcal{H}}}\right]|_{\theta=0} = \int_0^t \frac{d}{d\theta}\Psi_{\mathcal{L}}(\theta\mathbb{S}^*(u)\mathcal{T})\,du|_{\theta=0}\,.$$

But, for any $\mathcal{S}\in\mathcal{H}$,

$$\frac{d}{d\theta}\Psi_{\mathcal{L}}(\theta\mathcal{S}) = \mathrm{i}\langle\mathcal{D},\mathcal{S}\rangle_{\mathcal{H}} - \theta\langle\mathbb{Q}^0_{\mathcal{L}}\mathcal{S},\mathcal{S}\rangle_{\mathcal{H}} + \mathrm{i}\int_{\mathcal{H}}(\langle\mathcal{Z},\mathcal{S}\rangle_{\mathcal{H}}\mathrm{e}^{\mathrm{i}\theta\langle\mathcal{Z},\mathcal{S}\rangle_{\mathcal{H}}} - \langle\mathcal{Z},\mathcal{S}\rangle_{\mathcal{H}}1(\|\mathcal{Z}\|_{\mathcal{H}}<1))\,\nu(d\mathcal{Z})\,.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[\langle\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s),\mathcal{T}\rangle_{\mathcal{H}}\right] &= (-\mathrm{i})\int_0^t \frac{d}{d\theta}\Psi_{\mathcal{L}}(\theta\mathbb{S}^*(u)\mathcal{T})|_{\theta=0}\,du\\
&= \int_0^t \langle\mathcal{D},\mathbb{S}^*(u)\mathcal{T}\rangle_{\mathcal{H}} + \int_{\|\mathcal{Z}\|_{\mathcal{H}}>1}\langle\mathcal{Z},\mathbb{S}^*(u)\mathcal{T}\rangle_{\mathcal{H}}\,\nu(d\mathcal{Z})\,du\\
&= \int_0^t \langle\mathbb{S}(u)\mathcal{D},\mathcal{T}\rangle_{\mathcal{H}} + \int_{\|\mathcal{Z}\|_{\mathcal{H}}>1}\langle\mathbb{S}(u)\mathcal{Z},\mathcal{T}\rangle_{\mathcal{H}}\,\nu(d\mathcal{Z})\,du\\
&= \langle\int_0^t \mathbb{S}(u)(\mathcal{D}+\int_{\|\mathcal{Z}\|_{\mathcal{H}}>1}\mathcal{Z}\,\nu(d\mathcal{Z})\,du),\mathcal{T}\rangle_{\mathcal{H}}\\
&= \langle\int_0^t \mathbb{S}(u)\,du(\mathcal{D}+\int_{\|\mathcal{Z}\|_{\mathcal{H}}>1}\mathcal{Z}\,\nu(d\mathcal{Z}),\mathcal{T}\rangle_{\mathcal{H}}\,.\end{aligned}$$

Thus, since $\mathbb{E}[\mathcal{L}(1)] = \mathcal{D}+\int_{\|\mathcal{Z}\|_{\mathcal{H}}>1}\mathcal{Z}\,\nu(d\mathcal{Z})$, we get

$$\mathbb{E}\left[\int_0^t \mathbb{S}(t-s)\,d\mathcal{L}(s)\right] = \int_0^t \mathbb{S}(u)\,du\mathbb{E}[\mathcal{L}(1)]\,.$$

This completes the proof. □

To have the stochastic integral $\int_0^t \mathcal{Y}^{1/2}(s)\,dB(s)$ well-defined, the integrand must satisfy the condition

$$\mathbb{E}\left[\int_0^t \|\mathcal{Y}^{1/2}(s)\mathcal{Q}^{1/2}\|^2_{\mathcal{H}}\,ds\right] < \infty\,. \tag{3.2}$$

But $\|\mathcal{Y}^{1/2}(s)\mathcal{Q}^{1/2}\|^2_{\mathcal{H}} = \mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{Y}(s)\mathcal{Q}^{1/2})$. From Prop. 3.1 above we see that the expected value of this trace is integrable in time on any compact set. Thus, $\mathcal{Y}^{1/2}$ can be used as a stochastic volatility operator in the dynamics of X in (3.1).

If the stochastic integral $\int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s)$ exists, then we have the mild solution of (3.1)

$$X(t) = \mathcal{S}(t)X_0 + \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s)\,, \tag{3.3}$$

for a given initial condition $X(0)=X_0\in H$. The stochastic integral is well-defined since

$$\|\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\mathcal{Q}^{1/2}\|_{\mathcal{H}}\leq\|\mathcal{S}(t-s)\|_{\mathrm{op}}\|\mathcal{Y}^{1/2}(s)\mathcal{Q}^{1/2}\|_{\mathcal{H}}\,.$$

By Yosida [31], the operator norm of the semigroup $\mathcal{S}$ is at most exponentially growing with time. Hence, in view of Prop. 3.1, integrability holds.

Here is a result on the characteristic function of the process $X(t)$:

Proposition 3.2. *Suppose that there exists a self-adjoint, positive definite operator $\mathcal{D}\in L(H)$ such that $\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)=\mathcal{D}^{1/2}\mathcal{Y}(s)\mathcal{D}^{1/2}$ for all $s\geq 0$. Then, if $\mathcal{L}$ is independent of B,*

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}(X(t),f)_H}\right]=\exp\left(\mathrm{i}(X_0,\mathcal{S}^*(t)f)_H-\frac{1}{2}\langle\mathcal{Y}_0,\int_0^t\mathbb{S}^*(s)((\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)\otimes(\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f))\,ds\rangle_{\mathcal{H}}\right)$$
$$\times\exp\left(\int_0^t\Psi_{\mathcal{L}}\left(\frac{\mathrm{i}}{2}\int_0^s\mathbb{S}^*(s-u)(\mathcal{D}^{1/2}\mathcal{S}^*(u)f\otimes\mathcal{D}^{1/2}\mathcal{S}^*(u)f)\,du\right)\,ds\right)\,,$$

for any $f\in H$.

Proof. First, from the mild solution of $X(t)$ we find for $f\in H$

$$(X(t),f)_H=(\mathcal{S}(t)X_0,f)_H+(\int_0^t\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s),f)_H\,.$$

We compute the characteristic function of the random variable $(\int_0^t\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s),f)_H$: Since $\mathcal{L}$ and B are independent, we have that $\mathcal{Y}$ and B are independent. From the tower property of conditional expectation, we therefore get after conditioning on the σ-algebra generated by the paths of $\mathcal{Y}$:

$$\mathbb{E}\left[\exp\left(\mathrm{i}(\int_0^t\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s),f)_H\right)\right]$$
$$=\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t(\mathcal{Q}\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)f,\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)f)_H\,ds\right)\right]\,.$$

From the property of the operator $\mathcal{D}$,

$$\begin{aligned}(\mathcal{Q}\mathcal{Y}^{1/2}\mathcal{S}^*(t-s)f,\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)f)_H&=(\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)f,\mathcal{S}^*(t-s)f)_H\\&=(\mathcal{D}^{1/2}\mathcal{Y}(s)\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f,\mathcal{S}^*(t-s)f)_H\\&=(\mathcal{Y}(s)\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f,\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)_H\\&=\langle\mathcal{Y}(s),(\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)\otimes(\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)\rangle_{\mathcal{H}}\,.\end{aligned}$$

For simplicity, introduce for the moment the notation $\mathcal{T}(s)\in\mathcal{H}$ for the family of operators parametrized by time $s\geq 0$, defined by

$$\mathcal{T}(s)=(\mathcal{D}^{1/2}\mathcal{S}^*(s)f)\otimes(\mathcal{D}^{1/2}\mathcal{S}^*(s)f)\,.$$

Thus, from the mild solution of $\mathcal{Y}$,

$$\int_0^t\langle\mathcal{Y}(s),\mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds=\int_0^t\langle\mathbb{S}(s)\mathcal{Y}_0,\mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds+\int_0^t\langle\int_0^s\mathbb{S}(s-u)\,d\mathcal{L}(u),\mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds$$

We have that

$$\int_0^t \langle \mathbb{S}(s)\mathcal{Y}_0, \mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds = \langle \mathcal{Y}_0, \int_0^t \mathbb{S}^*(s)\mathcal{T}(t-s)\,ds\rangle_{\mathcal{H}}$$

where the integral on the right-hand side is interpreted in the Bochner sense. It holds, after appealing to a Fubini theorem for stochastic integrals in Hilbert space (see Peszat and Zabczyk [27, Theorem 8.14])

$$\int_0^t \langle \int_0^s \mathbb{S}(s-u)\,d\mathcal{L}(u), \mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds = \int_0^t \langle \int_u^t \mathbb{S}^*(s-u)\mathcal{T}(t-s)\,ds, d\mathcal{L}(u)\rangle_{\mathcal{H}}\,.$$

The ds-integral inside the inner product is again interpreted as a Bochner integral. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \langle \int_0^s \mathbb{S}(s-u)\,d\mathcal{L}(u), \mathcal{T}(t-s)\rangle_{\mathcal{H}}\,ds\right)\right] \\
&\qquad = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \langle \int_u^t \mathbb{S}^*(s-u)\mathcal{T}(t-s)\,ds, d\mathcal{L}(u)\rangle_{\mathcal{H}}\right)\right] \\
&\qquad = \exp\left(\int_0^t \Psi_{\mathcal{L}}\left(\frac{\mathrm{i}}{2}\int_u^t \mathbb{S}^*(s-u)(\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f\otimes\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)\,ds\right)\,du\right) \\
&\qquad = \exp\left(\int_0^t \Psi_{\mathcal{L}}\left(\frac{\mathrm{i}}{2}\int_0^s \mathbb{S}^*(s-u)(\mathcal{D}^{1/2}\mathcal{S}^*(u)f\otimes\mathcal{D}^{1/2}\mathcal{S}^*(u)f)\,du\right)\,ds\right)\,.
\end{aligned}$$

This proves the Proposition. □

We remark that we could have expressed the characteristic functional of $X(t)$ in terms of the Laplace transform cumulant of $\mathcal{L}$ rather that its characteristic functional $\Psi_{\mathcal{L}}$.

The result in the Proposition above shows that we recover an affine structure of X in terms of X_0 and $\mathcal{Y}_0$. Note that if $\mathcal{Q}$ commutes with $\mathcal{Y}(s)$, then $\mathcal{Q}^{1/2}$ commutes with $\mathcal{Y}^{1/2}(s)$, and we find

$$\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s) = \mathcal{Q}^{1/2}\mathcal{Y}(s)\mathcal{Q}^{1/2}\,.$$

Hence, in this case $\mathcal{D} = \mathcal{Q}$. Indeed, this puts rather strong restrictions on the volatility model $\mathcal{Y}$. A sufficient condition for $\mathcal{Y}$ commuting with $\mathcal{Q}$ is that $\mathcal{Q}$ commutes with $\mathcal{Y}_0$ and $\mathcal{L}(t)$ for all $t \geq 0$, and that $\mathbb{C}(\mathcal{T})\mathcal{Q} = \mathbb{C}(\mathcal{T}\mathcal{Q})$ and $\mathcal{Q}\mathbb{C}(\mathcal{T}) = \mathbb{C}(\mathcal{Q}\mathcal{T})$ for every $\mathcal{T} \in \mathcal{H}$. If this is the case, we have from the dynamics of $\mathcal{Y}$ in (2.3)

$$\mathcal{Q}\mathcal{Y}(t) = \mathcal{Q}\mathcal{Y}_0 + \int_0^t \mathbb{C}(\mathcal{Q}\mathcal{Y}(s))\,ds + \mathcal{Q}\mathcal{L}(t)\,, \tag{3.4}$$

and

$$\mathcal{Y}(t)\mathcal{Q} = \mathcal{Q}\mathcal{Y}_0 + \int_0^t \mathbb{C}(\mathcal{Y}(s)\mathcal{Q})\,ds + \mathcal{Q}\mathcal{L}(t)\,. \tag{3.5}$$

Introduce now the notation

$$\mathcal{L}_{\mathcal{Q}}(t) := \mathcal{Q}\mathcal{L}(t)\,, \tag{3.6}$$

which is an $\mathcal{H}$-valued process. It is in fact a Lévy process with values in $\mathcal{H}$. Indeed, its conditional characteristic function is (here $\mathcal{T} \in \mathcal{H}$ and $t \geq s$)

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\mathcal{L}_{\mathcal{Q}}(t)-\mathcal{L}_{\mathcal{Q}}(s),\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\mathcal{Q}(\mathcal{L}(t)-\mathcal{L}(s)),\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] \\ &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\mathcal{L}(t)-\mathcal{L}(s),\mathcal{Q}\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] \\ &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\mathcal{L}(t)-\mathcal{L}(s),\mathcal{Q}\mathcal{T}\rangle_{\mathcal{H}}}\right] \\ &= \exp\left((t-s)\Psi_{\mathcal{L}}(\mathcal{Q}\mathcal{T})\right)\end{aligned}$$

by the independent increment property and the definition of the cumulant of $\mathcal{L}$. Hence, $\mathcal{L}_{\mathcal{Q}}(t) - \mathcal{L}_{\mathcal{Q}}(s)$ is independent of $\mathcal{F}_s$ with a stationary distribution, which implies that $\mathcal{L}_{\mathcal{Q}}$ is a Lévy process. Its covariance operator is given by $\mathbb{Q}_{\mathcal{L}_{\mathcal{Q}}} = \mathcal{Q}\mathbb{Q}_{\mathcal{L}}\mathcal{Q}$, which is easily seen from

$$\begin{aligned}\mathbb{E}\left[\langle\mathcal{L}_{\mathcal{Q}}(t),\mathcal{T}\rangle_{\mathcal{H}}\langle\mathcal{L}_{\mathcal{Q}}(t),\mathcal{S}\rangle_{\mathcal{H}}\right] &= \mathbb{E}\left[\langle\mathcal{L}(t),\mathcal{Q}\mathcal{T}\rangle_{\mathcal{H}}\langle\mathcal{L}(t),\mathcal{Q}\mathcal{S}\rangle_{\mathcal{H}}\right] \\ &= \langle\mathbb{Q}_{\mathcal{L}}\mathcal{Q}\mathcal{T},\mathcal{Q}\mathcal{S}\rangle_{\mathcal{H}} \\ &= \langle\mathcal{Q}\mathbb{Q}_{\mathcal{L}}\mathcal{Q}\mathcal{T},\mathcal{S}\rangle_{\mathcal{H}}\,,\end{aligned}$$

with $\mathcal{T}, \mathcal{S} \in \mathcal{H}$. Therefore, we have a mild solution of the equation for $\mathcal{Y}_{\mathcal{Q}} := \mathcal{Q}\mathcal{Y}$ in (3.4) given as

$$\mathcal{Y}_{\mathcal{Q}}(t) = \mathbb{S}(t)\mathcal{Q}\mathcal{Y}_0 + \int_0^t \mathbb{S}(t-s)\,d\mathcal{L}_{\mathcal{Q}}(s)\,. \tag{3.7}$$

Moreover, we see that $\mathcal{Y}(t)\mathcal{Q}$ in (3.5) solves the same equation, and thus $\mathcal{Q}\mathcal{Y}(t) = \mathcal{Y}(t)\mathcal{Q}$ by uniqueness of solutions, and the claimed commutativity follows. We remark that if $\mathcal{Q}$ commutes with $\mathcal{C}$, then the assumed property of $\mathbb{C} = \mathbb{C}_i$ holds for $i = 1, 2$. Also, if $\mathcal{L}$ is the simple choice as in Ex. 2.7, it commutes with $\mathcal{Q}$ whenever $\mathcal{U}$ commutes with $\mathcal{Q}$.

Let us expand the result in Prop. 3.2 to a joint conditional characteristic functional for $(X(t), \mathcal{Y}(t)) \in H \times \mathcal{H}$.

Proposition 3.3. *Under the assumptions of Prop. 3.2 it holds for* $t \geq s \geq 0$ *and* $f \in H, \mathcal{T} \in \mathcal{H}$,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}(X(t),f)_H+\mathrm{i}\langle\mathcal{Y}(t),\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] = \exp\left((X(s),a(t-s;f))_H + \langle\mathcal{Y}(s),b(t-s;f,\mathcal{T})\rangle_{\mathcal{H}} + c(t-s;f,\mathcal{T})\right)$$

where

$$\begin{aligned}a(u;f) &= \mathrm{i}\mathcal{S}^*(u)f \\ b(u;f,\mathcal{T}) &= \mathrm{i}\mathbb{S}^*(u)\mathcal{T} + \frac{1}{2}\int_0^u \mathbb{S}^*(u-v)(\mathcal{D}^{1/2}a(v;f))^{\otimes 2}\,dv \\ c(u;f,\mathcal{T}) &= \int_0^u \Psi_{\mathcal{L}}\left(-\mathrm{i}b(v;f,\mathcal{T})\right)\,dv\,,\end{aligned}$$

for $u \geq 0$.

Proof. We follow similar arguments as in the proof of Prop. 3.2.

We observe from the semigroup property of $\mathcal{S}$ and $\mathbb{S}$ that for $t \geq s$

$$X(t) = \mathcal{S}(t-s)X(s) + \int_s^t \mathcal{S}(t-u)\mathcal{Y}^{1/2}(u)\,dB(u)$$
$$\mathcal{Y}(t) = \mathbb{S}(t-s)\mathcal{Y}(s) + \int_s^t \mathbb{S}(t-u)\,d\mathcal{L}(u)\,.$$

Letting $f \in H$ and $\mathcal{T} \in \mathcal{H}$, it follows from $\mathcal{F}_s$-measurability of $\mathcal{Y}(s)$ and $X(s)$,

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\mathrm{i}(X(t),f)_H + \mathrm{i}\langle \mathcal{Y}(t),\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(\mathrm{i}(\mathcal{S}(t-s)X(s),f)_H + \mathrm{i}\langle \mathbb{S}(t-s)\mathcal{Y}(s),\mathcal{T}\rangle_{\mathcal{H}}\right) \\
&\qquad \times \mathbb{E}\left[\exp\left(\mathrm{i}(\int_s^t \mathcal{S}(t-u)\mathcal{Y}^{1/2}(u)\,dB(u),f)_H + \mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\,d\mathcal{L}(u),\mathcal{T}\rangle_{\mathcal{H}}\right) \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(\mathrm{i}(X(s),\mathcal{S}^*(t-s)f)_H + \mathrm{i}\langle \mathcal{Y}(s),\mathbb{S}^*(t-s)\mathcal{T}\rangle_{\mathcal{H}}\right) \\
&\qquad \times \mathbb{E}\left[\exp\left(\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\,d\mathcal{L}(u),\mathcal{T}\rangle_{\mathcal{H}}\right)\mathbb{E}\left[\exp\left(\mathrm{i}(\int_s^t \mathcal{S}(t-u)\mathcal{Y}^{1/2}(u)\,dB(u),f)_H\right) \mid \mathcal{F}_s^{\mathcal{Y}}\right] \mid \mathcal{F}_s\right]\,.
\end{aligned}$$

In the last equality we conditioned on the σ-algebra $\mathcal{F}_s^{\mathcal{Y}} := \sigma(\mathcal{Y}(u), u \leq t) \vee \mathcal{F}_s$, i.e., the σ-algebra generated by the paths of $\mathcal{Y}(u), u \leq t$ and $\mathcal{F}_s$, and appealed to the tower property of conditional expectation. From the Gaussianity of the stochastic integral with respect to B together with independent increment property it follows,

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\mathrm{i}(X(t),f)_H + \mathrm{i}\langle \mathcal{Y}(t),\mathcal{T}\rangle_{\mathcal{H}}} \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(\mathrm{i}(X(s),\mathcal{S}^*(t-s)f)_H + \mathrm{i}\langle \mathcal{Y}(s),\mathbb{S}^*(t-s)\mathcal{T}\rangle_{\mathcal{H}}\right) \\
&\qquad \times \mathbb{E}\left[\exp\left(\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\,d\mathcal{L}(u),\mathcal{T}\rangle_{\mathcal{H}} - \frac{1}{2}\int_s^t \langle \mathcal{Y}(u),(\mathcal{D}^{1/2}\mathcal{S}^*(t-u)f)^{\otimes 2}\rangle_{\mathcal{H}}\,du\right) \mid \mathcal{F}_s\right]\,.
\end{aligned}$$

Just for the moment, introduce the short-hand notation $\mathcal{T}_f(v) = (\mathcal{D}^{1/2}\mathcal{S}^*(v)f) \otimes (\mathcal{D}^{1/2}\mathcal{S}^*(v)f)$ for $v \geq 0$, and we have, by definition of the Bochner integral and the stochastic Fubini Theorem (see Peszat and Zabczyk [27, Thm. 8.14]),

$$\begin{aligned}
&\int_s^t \langle \mathcal{Y}(u),\mathcal{T}_f(t-u)\rangle_{\mathcal{H}}\,du \\
&\quad = \langle \mathcal{Y}(s),\int_s^t \mathbb{S}^*(u-s)\mathcal{T}_f(t-u)\,du\rangle_{\mathcal{H}} + \int_s^t \langle \int_s^u \mathbb{S}(u-v)\,d\mathcal{L}(v),\mathcal{T}_f(t-u)\rangle_{\mathcal{H}}\,du \\
&\quad = \langle \mathcal{Y}(s),\int_s^t \mathbb{S}^*(u-s)\mathcal{T}_f(t-u)\,du\rangle_{\mathcal{H}} + \int_s^t \langle \int_v^t \mathbb{S}^*(u-v)\mathcal{T}_f(t-u)\,du,d\mathcal{L}(v)\rangle_{\mathcal{H}}
\end{aligned}$$

Therefore, by the independent increment property,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\mathrm{i}\langle \int_s^t \mathbb{S}(t-u)\,d\mathcal{L}(u),\mathcal{T}\rangle_{\mathcal{H}} - \frac{1}{2}\int_s^t \langle \mathcal{Y}(u),\mathcal{T}_f(t-u)\rangle_{\mathcal{H}}\,du\right) \mid \mathcal{F}_s\right] \\
&\quad = \exp\left(-\frac{1}{2}\langle \mathcal{Y}(s),\int_s^t \mathbb{S}^*(u-s)\mathcal{T}_f(t-u)\,du\rangle_{\mathcal{H}}\right)
\end{aligned}$$

$$\times \mathbb{E}\left[\exp\left(\int_s^t \langle \mathrm{i}\mathbb{S}^*(t-v)\mathcal{T} - \frac{1}{2}\int_v^t \mathbb{S}^*(u-v)\mathcal{T}_f(t-u)\,du, d\mathcal{L}(v)\rangle_{\mathcal{H}}\right)\right]$$

$$= \exp\left(-\frac{1}{2}\langle \mathcal{Y}(s), \int_s^t \mathbb{S}^*(u-s)\mathcal{T}_f(t-u)\,du\rangle_{\mathcal{H}}\right)$$

$$\times \exp\left(\int_s^t \Psi_{\mathcal{L}}\left(\mathbb{S}^*(t-v)\mathcal{T} + \frac{\mathrm{i}}{2}\int_v^t \mathbb{S}^*(u-v)\mathcal{T}_f(t-u)\,du\right)dv\right).$$

The result follows. □

The above result shows that $(X(t), \mathcal{Y}(t))_{t\geq 0}$ is an affine time-homogeneous Markov process in $H \times \mathcal{H}$. Moreover, the functions $a(\cdot; f) : \mathbb{R}_+ \to H$, $b(\cdot; f, \mathcal{T}) : \mathbb{R}_+ \to \mathcal{H}$ and $c(\cdot; f, \mathcal{T})$ are mild solutions of the system of Riccati equations

$$\begin{aligned} \frac{da(u;f)}{du} &= \mathcal{A}^* a(u;f) \qquad a(0;f) = \mathrm{i}f \\ \frac{db(u;f,\mathcal{T})}{du} &= \mathbb{C}^* b(u;f,\mathcal{T}) + \frac{1}{2}(\mathcal{D}^{1/2}a(u;f))^{\otimes 2}, \qquad b(0;f,\mathcal{T}) = \mathrm{i}\mathcal{T} \\ \frac{dc(u;f,\mathcal{T})}{du} &= \Psi_{\mathcal{L}}\left(-\mathrm{i}b(u;f,\mathcal{T})\right), \qquad c(0;f,\mathcal{T}) = 0\,. \end{aligned} \tag{3.8}$$

Affine processes play an important role in financial applications, as argued in Duffie, Filipovic and Schachermayer [17]. There, general affine Markov processes with Euclidean state space are analysed in detail, and, among other things, a system of generalised Riccati equations (see Eqs. (6.1) and (6.2) in [17]) are presented for the characteristic function. The system of Riccati equations in (3.8) is in an analogous form expressed in a Hilbert space context.

Prop. 3.3 and the system of Riccati equations in (3.8) propose an alternative approach to the model proposed in this paper. Indeed, we could define $(X, \mathcal{Y})$ as an affine Markov process with characteristic functional as in Prop. 3.3 and a, b and c satisfying the system of Riccati equations in (3.8). Such an approach also opens for a state space of $(X, \mathcal{Y})$ beyond the separable Hilbert spaces we use in this paper.

For example, if H, the state space of X, is a Banach space, we consider $\langle X(t), F\rangle$ as a dual pairing of the Banach space with its dual, and $F \in H^*$. We would immediately have the natural extension of $a(\cdot, f)$ above to a function $a(\cdot; F) : \mathbb{R}_+ \to H^*$ defined as $a(u; F) = \mathrm{i}\mathcal{S}^*(u)F$ for $\mathcal{S}^*(u) \in L(H^*)$. Here, $\mathcal{S}(u)$ is the C_0-semigroup on the Banach space H with adjoint $\mathcal{S}^*(u)$. The extension of X would also involve a stochastic integral of the stochastic volatility process $\mathcal{Y}$. Following van Neerven, Veraar and Weis [26], one could choose B as a Hilbert-valued cylindrical Wiener process and consider a stochastic volatility process which belongs to the space of bounded linear operators mapping into H. In their formulation, H must be a so-called UMD Banach space. We have that $\mathcal{Y}$ is also naturally being a Banach space valued process, and we must ensure that we can define a Lévy process $\mathcal{L}$ on this that gives a positive operator process. Applebaum [1] discusses Lévy processes in Banach space and Ornstein-Uhlenbeck process. A Banach valued Ornstein-Uhlenbeck process with Lévy noise can be obtained also by integrating with respect to real valued Lévy process M-type 2 valued integrands. A theory similar to the one of van Neerven, Veraar and Weis [26] can be found for such stochastic integrals in Mandrekar and Rüdiger [24, Chapter 3]. We will not analyse the extension of our Ornstein-Uhlenbeck stochastic volatility process to state spaces beyond separable Hilbert spaces further in this paper.

Let us return to our proposed OU model with stochastic volatility, and investigate the implied ”adjusted returns”. To this end, fix $\Delta t > 0$, and define the ”adjusted return” by

$$R(t,\Delta t) = X(t+\Delta t) - \mathcal{S}(\Delta t)X(t)\,.$$

From (3.3), we find after using the semigroup property of $\mathcal{S}$,

$$R(t,\Delta t) = \int_t^{t+\Delta t} \mathcal{S}(t+\Delta t - s)\mathcal{Y}^{1/2}(s)\,dB(s)\,.$$

We have:

Lemma 3.4. *Let $\mathcal{F}^{\mathcal{Y}}$ be the σ-algebra generated by the paths of $\mathcal{Y}$. Then $R(t,\Delta t)|\mathcal{F}^{\mathcal{Y}}$ is a mean zero H-valued Gaussian random variable, with covariance operator*

$$\mathcal{Q}_{R(t,\Delta t)|\mathcal{Y}} := \int_t^{t+\Delta t} \mathcal{S}(t+\Delta t - s)\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t+\Delta t - s)\,ds\,.$$

Proof. By inspection of the proof of Prop. 3.2, we find for $f \in H$

$$\mathbb{E}\left[\exp(\mathrm{i}(R(t,\Delta t), f)_H)\,|\,\mathcal{F}^{\mathcal{Y}}\right] = \exp\left(-\frac{1}{2}\int_t^{t+\Delta t} |\mathcal{Q}^{1/2}\mathcal{Y}^{1/2}\mathcal{S}^*(t+\Delta t - s)f|_H^2\,ds\right)\,.$$

This is the characteristic function of a Gaussian mean-zero real valued random variable. Hence, $R(t,\Delta t)|\mathcal{F}^{\mathcal{Y}}$ is Gaussian in H with mean equal to zero. The conditional covariance operator follows by a direct computation. □

The stochastic volatility model yields a Gaussian variance-mixture model for the adjusted returns (see Barndorff-Nielsen and Shephard [8] for mean-variance mixture models and stochastic volatility in finance). Remark that if $\mathcal{Q}$ and $\mathcal{Y}$ commute, the conditional covariance operator becomes

$$\mathcal{Q}_{R(t,\Delta t)|\mathcal{Y}} := \int_t^{t+\Delta t} \mathcal{S}(t+\Delta t - s)\mathcal{Y}_{\mathcal{Q}}(s)\mathcal{S}^*(t+\Delta t - s)\,ds\,,$$

with the definition of $\mathcal{Y}_{\mathcal{Q}}$ given above.

Remark 3.5. *In Lemma 3.4 a simplified filtering problem for the adjusted returns of the model is solved. Here the observable is the volatility $\mathcal{Y}^{1/2}$. Compared to more general filtering models, as e.g. those described in Xiong* [30]*, our filtering model is simple, as the observable does not depend on the signal X, for which the adjusted returns are computed.*

4. Application to forward price modelling

Let $H = H_w$, the Filipovic space of all absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|f|_w^2 := f(0)^2 + \int_0^\infty w(x)|f'(x)|^2\,dx < \infty\,, \tag{4.1}$$

where $w : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $w(0) = 1$. We assume that $\int_0^\infty w^{-1}(x)\,dx < \infty$, and denote the (naturally defined) inner product $(\cdot,\cdot)_w$. It turns out that H_w is a separable Hilbert space equipped with the norm $|\cdot|_w$. Moreover, the evaluation functional $\delta_x(f) = f(x)$ is continuous on H_w. As a linear functional, we can express δ_x by $(\cdot, h_x)_w$, with

$$h_x(y) = 1 + \int_0^{x\wedge y} w^{-1}(z)\,dz, y \in \mathbb{R}_+\,. \tag{4.2}$$

See Filipovic [18] for the introduction of this space and its properties (see also Benth and Krühner [11] for a further analysis of this space).

Consider X defined in (3.1) for $H = H_w$ and $A = \partial/\partial x$, the derivative operator. Then X can be considered as the dynamics of the forward curve, that is, $f(t,x) := \delta_x(X(t)) = X(t)(x)$, where $f(t,x) := F(t, t+x)$, and $t \mapsto F(t,T), t \leq T$ is the arbitrage-free forward price dynamics of a contract delivering an asset (commodity or stock) at time T (see Benth and Krühner [11]). We note that the semigroup of A will be the right shift operator $\mathcal{S}(t)f = f(\cdot + t)$, and that

$$\delta_x\mathcal{S}(t)g = g(t+x) = \delta_{x+t}g\,.$$

for any $g \in H_w$. We find from the mild solution of X in (3.3) that

$$f(t,x) = f_0(t+x) + \delta_x \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s) \tag{4.3}$$

where $f_0(t+x) = \delta_x\mathcal{S}(t)X_0$. Note that by Lemma 3.2 in Benth and Krühner [11], it holds

$$\lim_{t\to\infty}(X_0, \mathcal{S}^*(t)h_x)_H = \lim_{t\to\infty}\delta_x\mathcal{S}(t)X_0 = \lim_{t\to\infty} f_0(t+x) = f_0(\infty)\,.$$

Here, $f_0(\infty)$ denotes the limit of $f_0(y)$ as $y \to \infty$, which exists. Hence, from the mild solution in (4.3), the mean of $f(t,x)$ for given $x \in \mathbb{R}_+$ has a limit $f_0(\infty)$ as time tends to infinity.

Now we investigate the stochastic integral in (4.3) in more detail. More specifially, we consider $x = T - t$ for a given $T \geq t$, for which we find the forward price dynamics $F(t,T) := f(t, T-t)$ of a contract delivering the underlying commodity at time T.

By Thm. 2.1. in Benth and Krühner [11], there exists a real-valued Brownian motion b_T on $t \in [0,T]$ such that

$$\delta_{T-t}\int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s) = \int_0^t \sigma(s, T-s)\,db_T(s)\,, \tag{4.4}$$

where, for $x \geq 0$,

$$\begin{aligned}\sigma^2(s,x) &= (\delta_0 \mathcal{S}(x)\mathcal{Y}^{1/2}(s)\mathcal{Q}(\mathcal{S}(x)\mathcal{Y}^{1/2}(s))^*\delta_0^*)(1) \\ &= \delta_0\mathcal{S}(x)(\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s))(\delta_0\mathcal{S}(x))^*(1) \\ &= \delta_x(\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s))\delta_x^*(1)\,.\end{aligned}$$

We remark that the Brownian motion b_T depends on T, since the representation in Thm. 2.1. in Benth and Krühner [11] is for a given linear functional, which in this case δ_0 since we have $\delta_{T-t}\mathcal{S}(t-s) = \delta_0\mathcal{S}(T-s)$. We know that $\delta_x^*(1) = h_x(\cdot)$ (see e.g. Benth and Krühner [11]), and therefore

$$\sigma^2(s,x) = (\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)(h_x(\cdot)))(x)\,. \tag{4.5}$$

Hence, we map the function h_x by the operator $\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)$, and evaluate the resulting function in H_w at x. As $\mathcal{Y}^{1/2}$ is stochastic, we get a stochastic volatility process $s \mapsto \sigma(s,x)$, which is depending on the "spatial" variable $x = T - s$, i.e., "time-to-maturity". In particular, the spot price dynamics $S(t) := f(t,0)$ becomes

$$S(t) = f_0(t) + \int_0^t \sigma(s, t-s)\,db_t(s)\,.$$

I.e., the spot price dynamics follows a Volterra-like process where the integrand $\sigma(s,t-s)$ is stochastic. We refer to Barndorff-Nielsen, Benth and Veraart [4] for an application to Volterra processes (and more specifically, Brownian and Lévy semistationary processes) to model spot prices in energy markets.

Let us carry our discussion further, and *suppose* that $\mathcal{Q}$ commutes with $\mathcal{Y}_0$ and $\mathcal{L}(t)$ for $t \geq 0$, as well as that we have $\mathbb{C}(\mathcal{T})\mathcal{Q} = \mathbb{C}(\mathcal{T}\mathcal{Q})$ and $\mathcal{Q}\mathbb{C}(\mathcal{T}) = \mathbb{C}(\mathcal{Q}\mathcal{T})$ for any $\mathcal{T} \in \mathcal{H}$. Then we recall from the previous Section that $\mathcal{Y}(s)$ will commute with $\mathcal{Q}$ for every $s \geq 0$. The process $\mathcal{Y}^{1/2}(s)$ will also commute with $\mathcal{Q}$, and

$$\sigma^2(s,x) = (\mathcal{Y}(s)\mathcal{Q}h_x)(x)\,.$$

Recalling the definition of $\mathcal{Y}_{\mathcal{Q}}$ in (3.7), we find

$$\begin{aligned}\sigma^2(s,x) &= \delta_x(\mathcal{Y}_{\mathcal{Q}}(s)(h_x)) \\ &= (\mathcal{Y}_{\mathcal{Q}}(s)h_x, h_x)_w \\ &= \langle \mathcal{Y}_{\mathcal{Q}}(s), h_x \otimes h_x\rangle_{\mathcal{H}}\end{aligned}$$

since $\delta_z(f) = (f, h_z)_w$ for any $f \in H_w$. Similar as in Prop. 2.5, we can calculate the cumulant of the process $\mathcal{Y}_{\mathcal{Q}}$ for any $\mathcal{T} \in \mathcal{H}$, and in particular we can calculate the cumulant of the process $s \mapsto \langle \mathcal{Y}_{\mathcal{Q}}(s), h_x \otimes h_x\rangle_{\mathcal{H}}$ for $s \leq t$ by choosing $\mathcal{T} = h_x \otimes h_x$. A simple calculation using the definition of h_x in (4.2) shows that

$$(h_x \otimes h_x)(f) = \left(f(0) + \int_0^x f'(y)\,dy\right) h_x = \mathcal{I}_x(f)h_x\,,$$

where $\mathcal{I}_x \in H_w^*$ is defined as $\mathcal{I}_x(f) = \delta_0(f) + \int_0^x f'(y)\,dy$ for any $f \in H_w$.

In the above considerations we obtain a ”marginal” dynamics, in the sense of a dynamics for a forward contract with fixed time to maturity x. We now represent the forward price dynamics as a space-time random field to emphasize also its spatial dynamics (i.e., its dynamics in time-to-maturity x). First, from Prop. 3.9 in Benth and Krühner [12] we find for any $f \in H_w$,

$$
\begin{aligned}
(\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)f)(x) &= (\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)f, h_x)_w \\
&= (f, (\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s))^*(h_x))_w \\
&= (\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s))^*(h_x)(0)f(0) \\
&\qquad + \int_0^\infty w(y)(\mathcal{S}(t-s)\mathcal{Y}^{1/2}(s))^*(h_x)'(y)f'(y)\,dy \\
&= (\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)h_x)(0)f(0) \\
&\qquad + \int_0^\infty w(y)(\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t-s)h_x)'(y)f'(y)\,dy\,.
\end{aligned}
$$

Again from Prop. 3.9 in Benth and Krühner [12],

$$
\mathcal{S}^*(t)h_x(\cdot) = h_x(0)(\mathcal{S}(t)h_\cdot)(0) + \int_0^\infty w(y)(\mathcal{S}(t)h_\cdot)'(y)h_x'(y)\,dy\,.
$$

But $\mathcal{S}(t)h_x(y) = h_x(y+t)$ and $h_x'(y) = w^{-1}(y)\mathbf{1}(y<x)$. Hence,

$$
\begin{aligned}
\mathcal{S}^*(t)h_x(\cdot) &= h_\cdot(t) + \int_0^x w^{-1}(y+t)\mathbf{1}(y+t<\cdot)\,dy \\
&= h_t(\cdot) + \int_t^{x+t} w^{-1}(y)\mathbf{1}(y<\cdot)\,dy \\
&= h_{t+x}(\cdot)\,.
\end{aligned}
$$

If we use the notation that $B(ds,dy) := \partial_x B(ds,y)\,dy$, we find

$$
\begin{aligned}
\delta_x \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dB(s) &= \int_0^t (\mathcal{Y}^{1/2}(s)h_{x+t-s})(0)\,dB(s,0) \\
&\qquad + \int_0^t\int_0^\infty w(y)(\mathcal{Y}^{1/2}(s)h_{x+t-s})'(y)\,B(ds,dy)\,,
\end{aligned}
$$

or, a representation of $f(t,x) := \delta_x(X(t))$ as a spatio-temporal random field

$$
f(t,x) = f_0(t+x) + \int_0^t (\mathcal{Y}^{1/2}(s)h_{x+t-s})(0)\,dB(s,0) + \int_0^t\int_0^\infty w(y)(\mathcal{Y}^{1/2}(s)h_{x+t-s})'(y)\,B(ds,dy)\,.
$$

Note that $B(t,0) = \delta_0 B(t) = (B(t),h_0)_w = (B(t),1)_w$ is a real-valued Brownian motion with variance $|\mathcal{Q}^{1/2}1|_w^2$. Hence, $b_0(t) := B(t,0)/|\mathcal{Q}^{1/2}1|_w$ is a real-valued standard Brownian motion and we can view the first integral as an Ito integral of a volatility process given by $s \mapsto (\mathcal{Y}^{1/2}(s)h_{x+t-s})(0)$ for $s \le t$ where x is a parameter. It becomes a real-valued Volterra process with parameter x. For the second integral, we integrate with respect to a spatio-temporal random field $(s,y) \mapsto B(s,y)$ over $[0,t]\times\mathbb{R}_+$, thus becoming a stochastic Volterra random field. This part is analogous to an ambit field, a class of spatio-temporal random fields defined in Barndorff-Nielsen and Schmiegel [7]. In a special case, the ambit fields take the form

$$A(t,x) = \int_0^t \int_0^\infty g(t,s,x,y)\eta(s,y)B(ds,dy)$$

for a stochastic random field η and a deterministic kernel function g. Under appropriate integrability conditions, the ambit field $A(t,x)$ is well-defined (see e.g. Barndorff-Nielsen, Benth and Veraart [3]). We observe that we can identify $w(y)$ with the kernel function g, giving a very simple kernel. On the other hand, the volatility field η is more complex in f, as it is also x-dependent and not only s and y dependent. Our stochastic volatility model serves as a motivation for an extension of the ambit field models. We refer to Barndorff-Nielsen, Benth and Veraart [5] and [6] for an application of ambit fields to energy forward price modeling.

We finally remark that in many commodity markets one observes an increasing volatility with decreasing time to delivery, known as the Samuelson effect (see Samuelson [28]). To include this in our dynamics of X, we can add an operator $\Psi(t) \in \mathcal{H}$, possibly time-dependent, such that

$$dX(t) = \mathcal{A}X(t)\,dt + \Psi(t)\mathcal{Y}^{1/2}(t)\,dB(t)\,.$$

Much of the analysis above can, under natural integrability conditions on Ψ, be carried through for this model.

Appendix A. A result on symmetric Hilbert-Schmidt operators

In this section we provide the arguments for a claim in the proof of Proposition 2.10 for the convenience of the reader.

Lemma A.1. *Let H be a separable Hilbert space and let $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of H. Then any symmetric Hilbert-Schmidt operator $\mathcal{V}$ on H can be written as*

$$\mathcal{V} = \sum_{k,l\in\mathbb{N}} \gamma_{k,l} e_k \otimes e_l,$$

with $\sum_{k,l}\gamma_{k,l}^2 < \infty$ and $\gamma_{k,l} = \gamma_{l,k}$.

Proof. Recall that the space of Hilbert-Schmidt operators on H, denoted by $\mathcal{H}$, is isometrically isomorphic to $H^* \otimes H$, which we identify as $H \otimes H$. Therefore, any Hilbert-Schmidt operator $\mathcal{V}$ can be written as

$$\mathcal{V} = \sum_{k,l\in\mathbb{N}} \gamma_{k,l} e_k \otimes e_l,$$

for a sequence of constants $\{\gamma_{k,l}\}_{k,l\in\mathbb{N}}$.

As for the square-summability of the constants, we have necessarily

$$\begin{aligned}\|\mathcal{V}\|_{\mathcal{H}}^2 = \sum_{i\in\mathbb{N}} \Big| \sum_{k,l\in\mathbb{N}} \gamma_{k,l}(e_k\otimes e_l)e_i \Big|_H^2 &= \sum_{i\in\mathbb{N}} \Big| \sum_{l\in\mathbb{N}} \gamma_{i,l}e_l \Big|_H^2 \\ &= \sum_{i\in\mathbb{N}} \Big(\sum_{l\in\mathbb{N}} \gamma_{i,l}e_l, \sum_{m\in\mathbb{N}} \gamma_{i,m}e_m \Big)_H \\ &= \sum_{i,m\in\mathbb{N}} \gamma_{i,m}^2 (e_m, e_m)_H \\ &= \sum_{i,m\in\mathbb{N}} \gamma_{i,m}^2 \,.\end{aligned}$$

So, in order for the norm to be finite, the double sequence $\{\gamma_{k,l}\}_{k,l\in\mathbb{N}}$ has to be square-summable.

As for the property of being symmetric, we need to have for all $f, g \in \mathrm{dom}(\mathcal{V})$ that

$$\big(\mathcal{V}f, g\big)_H = \big(f, \mathcal{V}g\big)_H. \tag{A.1}$$

Since Hilbert-Schmidt operators are bounded (even compact), one has $\mathrm{dom}(\mathcal{V}) = H$. As a side-remark, this furthermore implies that $\mathrm{dom}(\mathcal{V}^*) \supseteq \mathrm{dom}(\mathcal{V}) = H$, which in turn implies that symmetric Hilbert-Schmidt operators are already self-adjoint. The terms in (A.1) can be evaluated to be

$$\big(\mathcal{V}f, g\big)_H = \sum_{k,l} \gamma_{k,l} f_k g_l \quad \text{and} \quad \big(f, \mathcal{V}g\big)_H = \sum_{k,l} \gamma_{k,l} f_l g_k,$$

where $f_k = (f, e_k)_H$, and similarly for g_l. These terms can only be equal for all $f, g \in \mathrm{dom}(\mathcal{V}) = H$ if either $k = l$ or if $\gamma_{k,l} = \gamma_{l,k}$, which implies the assertion. □

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