

CLASSIFICATION OF NON-KAC COMPACT QUANTUM GROUPS OF $SU(n)$ TYPE

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ABSTRACT. We classify up to isomorphism all non-Kac compact quantum groups with the same fusion rules and dimension function as $SU(n)$. For this we first prove, using categorical Poisson boundary, the following general result. Let G be a coamenable compact quantum group and K be its maximal quantum subgroup of Kac type. Then any dimension-preserving unitary fiber functor $\text{Rep } G \rightarrow \text{Hilb}_f$ factors, uniquely up to isomorphism, through $\text{Rep } K$. Equivalently, we have a canonical bijection $H^2(\hat{G}; \mathbb{T}) \cong H^2(\hat{K}; \mathbb{T})$. Next, we classify autoequivalences of the representation categories of twisted q -deformations of compact simple Lie groups.

1. INTRODUCTION

In his fundamental paper [Wor88] on quantization of $SU(n)$ and the Tannaka–Krein duality for compact quantum groups, Woronowicz formulated the following problem: classify quantum groups having the same representation theory, meaning the same fusion rules and dimensions of irreducible representations, as $SU(n)$. Since then, this and similar questions for other representation rings have been studied by a number of authors, see e.g., [WZ94, PM98, Ban96, Ohn99, Hai00, Bic03, Ohn05, Mro14, Mro15] and references therein.

Despite some success, a complete answer has only been obtained in a few low rank cases. At the current stage this is hardly surprising, and a complete explicit answer to Woronowicz’s question is not to be expected. Indeed, such an answer would include understanding of all unitary fiber functors on $\text{Rep } SU(n)$, which is equivalent to classifying full multiplicity ergodic actions of $SU(n)$ on C^* -algebras [Was88b, Lan92]. As it was known already to Wassermann [Was88b] (see also the discussion in [NT11b, pp. 1240–1241]) this, in turn, includes classifying the finite central type factor groups inside $SU(n)$ up to conjugacy. For small n there are not many such subgroups, and for $n = 2, 3$ it is, indeed, possible to classify all full multiplicity ergodic actions [Was88c, Was88a]. However, the problem rapidly becomes unfeasible as n grows larger. Since fiber functors on $\text{Rep } SU(n)$ lead to Kac quantum groups, we therefore should not expect an explicit answer to Woronowicz’s question for the class of compact quantum groups of Kac type. Note that nevertheless for genuine compact groups we have a complete answer: by a result of McMullen [McM84], a compact group with the same fusion rules as for $SU(n)$ is itself isomorphic to $SU(n)$, see also [Han93, KLV14].

We can also forget about compactness and try to describe all cosemisimple Hopf algebras with corepresentation theory of $SU(n)$. For $n = 2$ this was done by Podleś and Müller [PM98], and by Bichon [Bic03] without the restriction on the dimension function, extending a result of Banica in the compact case [Ban96]. For $n = 3$, a complete classification of such Hopf algebras was obtained by Ohn [Ohn99, Ohn05], where he obtained a long list of various multiparametric deformations through a formidable amount of direct computations. Doing anything similar for $n \geq 4$ seems like an overwhelming task, even with computer assistance.

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Our main result is that if we stay away from the Kac and noncompact cases, the question of Woronowicz has a very simple answer: the only quantum groups we have are $SU_q(n)$ for $0 < q < 1$, the categorical twists $SU_q^\tau(n)$ of $SU_q(n)$ studied in our previous paper [NY15], and the deformations of such quantum groups by 2-cocycles on the dual of the maximal torus, see Theorems 5.1 and 5.2 for the precise statement. To put it differently, all such compact quantum groups are obtained by quantization of one of the Poisson–Lie group structures on $SU(n)$ [LS91] and by twisting by a 3-cocycle on the Pontryagin dual of the center.

It is worth remembering that, on the purely algebraic level, compact quantum groups are simply Hopf $*$ -algebras generated by matrix coefficients of their finite dimensional unitary corepresentations. Therefore our result classifies all such Hopf $*$ -algebras with corepresentation theory of $SU(n)$ and noninvolutive antipode. For $n = 3$ this implies that a majority of Hopf algebras in the list of Ohn are of Kac type, do not admit a compatible $*$ -structure, or have nonunitarizable corepresentations, which can also be checked by a careful inspection of his classification.

In view of the Tannaka–Krein duality, the classification problem can be divided into three parts:

- classification of rigid C^* -tensor categories \mathcal{C} with fusion rules of $SU(n)$;
- classification of monoidal autoequivalences of \mathcal{C} ;
- classification of unitary fiber functors $\mathcal{C} \rightarrow \text{Hilb}_f$ inducing the classical dimension function on the representation ring of $SU(n)$.

The first problem was solved in the purely algebraic setting by Kazhdan and Wenzl [KW93], and the modifications needed in the C^* -setting have been carried out in the recent paper by Jordans [Jor14]. Their result states that any such C^* -tensor category is obtained by twisting $\text{Rep } SU_q(n)$ by a 3-cocycle on its chain group (which is naturally isomorphic to the dual of the center of $SU(n)$) for a uniquely determined $q \in (0, 1]$. It should be remarked that there exist only partial results extending this classification to other compact connected simple Lie groups [TW05], and this is essentially the only reason which does not allow us to extend our results to all such groups in place of $SU(n)$.

As for the second problem, the classification of monoidal autoequivalences of $\text{Rep } SU_q(n)$, and more generally of $\text{Rep } G_q$ for any compact connected semisimple Lie group G , was obtained in [NT11a, NT12a]. From this it is easy to deduce a similar result for the 3-cocycle twists of $\text{Rep } SU_q(n)$. The situation for twists of $\text{Rep } G_q$ is slightly more complicated, exactly because we do not have a complete classification of categories with fusion rules of G . Nevertheless, with a bit more effort we can obtain a complete classification of monoidal autoequivalences for simple and simply connected G .

Therefore the main remaining problem is classification of unitary fiber functors on the twists of $\text{Rep } SU_q(n)$ (or more generally of $\text{Rep } G_q$), which induce the classical dimension function on the representation ring. In an earlier paper [NY15] we already constructed some fiber functors and studied the corresponding compact quantum groups G_q^τ . In the present paper we show that for $q \neq 1$, which exactly corresponds to the non-Kac case, all dimension-preserving unitary fiber functors on $\text{Rep } G_q^\tau$ factor through $\text{Rep } T$ in an essentially unique way, where T is the maximal torus. This is deduced from a general result on dual 2-cohomology of coamenable compact quantum groups obtained using a universal property of categorical Poisson boundaries we established in [NY14b]. Finally, let us note that currently this last part is the only one place where it is essential to work within the operator algebraic framework, as the proofs in [NY14b] heavily utilize the techniques from operator theory and subfactor theory.

2. PRELIMINARIES

2.1. Monoidal categories. In this paper we deal with nonstrict C^* -tensor categories, following the conventions of [NT13]. For the convenience of the reader we briefly recall the basic terminology and notation.

A \mathbb{C} -linear category \mathcal{C} is a C^* -category if its morphism sets $\mathcal{C}(X, Y)$ are Banach spaces endowed with a conjugate linear and isometric involution $\mathcal{C}(X, Y) \rightarrow \mathcal{C}(Y, X)$, $T \mapsto T^*$, satisfying

$$(ST)^* = T^*S^*, \quad \|T^*T\| = \|T\|^2.$$

We always assume that such categories are closed under direct sums and subobjects.

A C^* -tensor category is a C^* -category together with a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a distinguished object $\mathbf{1} = \mathbf{1}_{\mathcal{C}} \in \text{Obj } \mathcal{C}$, together with natural unitary isomorphisms

$$\mathbf{1} \otimes X \rightarrow X \leftarrow X \otimes \mathbf{1}, \quad \Phi: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$$

satisfying a number of conditions. We will always assume that the unit object is simple, meaning that $\mathcal{C}(\mathbf{1}, \mathbf{1}) \simeq \mathbb{C}$.

A *unitary tensor functor* between two C^* -tensor categories is given by a triple $F = (F_0, F_1, F_2)$ of the following form: F_1 is a linear functor $\mathcal{C} \rightarrow \mathcal{D}$ satisfying $F_1(T^*) = F_1(T)^*$, F_0 is a unitary isomorphism $\mathbf{1} \rightarrow F_1(\mathbf{1})$, and F_2 is a natural unitary isomorphism $F_1(X) \otimes F_1(Y) \rightarrow F_1(X \otimes Y)$, satisfying the following compatibility condition for the associativity morphisms:

$$\begin{array}{ccc} (F_1(X) \otimes F_1(Y)) \otimes F_1(Z) & \xrightarrow{F_2 \otimes \iota} & F_1(X \otimes Y) \otimes F_1(Z) \xrightarrow{F_2} F_1((X \otimes Y) \otimes Z), \\ \Phi^{\mathcal{D}} \downarrow & & \downarrow F_1(\Phi^{\mathcal{C}}) \\ F_1(X) \otimes (F_1(Y) \otimes F_1(Z)) & \xrightarrow{\iota \otimes F_2} & F_1(X) \otimes F_1(Y \otimes Z) \xrightarrow{F_2} F_1(X \otimes (Y \otimes Z)) \end{array} \quad (2.1)$$

and another set of compatibility conditions for the tensor units $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ and the isomorphism F_0 . If there is no fear of confusion we also write F in place of F_1 . If we are given two unitary tensor functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $F': \mathcal{D} \rightarrow \mathcal{D}'$, their composition $F'F: \mathcal{C} \rightarrow \mathcal{D}'$ is given by the triple $(F'_0F'_1(F_0), F'_1F_1, F'_1(F_2)F'_2)$.

Given two unitary tensor functors F and F' from \mathcal{C} to \mathcal{D} , a natural unitary monoidal transformation $\eta: F \rightarrow F'$ is given by a natural transformation $\eta_X: F_1(X) \rightarrow F'_1(X)$ of functors F_1 and F'_1 by unitary morphisms $(\eta_X)_{X \in \text{Obj } \mathcal{C}}$ which satisfy $F_0 = F'_0\eta_{\mathbf{1}}$ and $\eta_{X \otimes Y}F_2 = F'_2(\eta_X \otimes \eta_Y)$. Such F and F' are said to be *naturally unitarily monoidally isomorphic* if there are natural unitary monoidal transformations $\eta: F \rightarrow F'$ and $\eta': F' \rightarrow F$ such that $\eta'_X\eta_X = \text{id}_{F_1(X)}$ and $\eta_X\eta'_X = \text{id}_{F'_1(X)}$. Moreover, a unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be a unitary monoidal equivalence if there is another unitary tensor functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that GF and FG are naturally unitarily monoidally isomorphic to the identity functors of \mathcal{C} and \mathcal{D} , respectively. We denote the group of unitary monoidal autoequivalences of \mathcal{C} , considered up to natural unitary monoidal isomorphisms, by $\text{Aut}^{\otimes}(\mathcal{C})$.

An important class of C^* -tensor categories is *rigid* C^* -tensor categories, in which any object has a dual. Namely, an object $Y \in \text{Obj } \mathcal{C}$ is a dual of $X \in \text{Obj } \mathcal{C}$, denoted as $Y = \bar{X}$, if there are morphisms $R_X \in \mathcal{C}(\mathbf{1}, \bar{X} \otimes X)$ and $\bar{R}_X \in \mathcal{C}(\mathbf{1}, X \otimes \bar{X})$ which satisfy the *conjugate equations*

$$(R_X^* \otimes \iota_{\bar{X}})\Phi^{-1}(\iota_{\bar{X}} \otimes \bar{R}_X) = \iota_{\bar{X}}, \quad (\bar{R}_X^* \otimes \iota_X)\Phi^{-1}(\iota_X \otimes R_X) = \iota_X,$$

up to the structure morphisms of the unit. The *intrinsic dimension* (also called the quantum dimension) of X is defined as

$$d^{\mathcal{C}}(X) = \inf_{(R_X, \bar{R}_X)} \|R_X\| \|\bar{R}_X\|,$$

where (R_X, \bar{R}_X) runs through the solutions of the conjugate equations for X . We will often omit the superscript \mathcal{C} . A choice of (R_X, \bar{R}_X) such that $d(X)^{1/2} = \|R_X\| = \|\bar{R}_X\|$ is called a *standard solution*.

2.2. Compact quantum groups. Next let us review a few standard facts about compact quantum groups. See again [NT13] for the details.

Let G be a compact quantum group, given by a unital cancellative C^* -bialgebra $(C(G), \Delta)$. A finite dimensional *unitary representation* of G is given by a unitary element $U \in B(H_U) \otimes C(G)$ for some finite dimensional Hilbert space H_U (the underlying Hilbert space of U), which satisfies the

equality $U_{12}U_{13} = (\iota \otimes \Delta)(U)$ in $B(H_U) \otimes B(H_U) \otimes C(G)$. Here, U_{ij} is given by distributing the legs of U to the positions indicated by the matching numbers, so that U_{13} equals $\sum_i x_i \otimes 1 \otimes y_i$ if U is given by $\sum_i x_i \otimes y_i$. We denote by $\dim U$ the dimension of H_U and call it the *classical dimension* of U .

The category of finite dimensional unitary representations of G , denoted by $\text{Rep } G$, has a natural structure of a C^* -tensor category with the tensor product $U \oplus V = U_{13}V_{23}$, so that $H_{U \oplus V} = H_U \otimes H_V$. Any object U of $\text{Rep } G$ has a dual object \bar{U} , realized on the conjugate space \bar{H}_U . Woronowicz's Tannaka–Krein duality theorem says that any compact quantum group G can be recovered from $\text{Rep } G$, which has an irreducible unit and duality of objects, and the unitary tensor functor (the *canonical fiber functor of G*) $\text{Rep } G \rightarrow \text{Hilb}_f, U \mapsto H_U$.

A crucial part of the proof of the Tannaka–Krein duality is the reconstruction of the *regular algebra* $\mathbb{C}[G] \subset C(G)$, defined as the linear span of the elements $(\omega \otimes \iota)(U)$, where $\omega \in B(H_U)_*$ and U runs through the finite dimensional unitary representations of G . It is a Hopf $*$ -algebra, with the coproduct defined by the restriction of Δ . As a linear space it can be identified with

$$\bigoplus_{[U] \in \text{Irr } G} \bar{H}_U \otimes H_U \cong \bigoplus_{[U] \in \text{Irr } G} B(H_U)_*. \quad (2.2)$$

The projection h onto the direct summand $\bar{H}_1 \otimes H_1 \simeq \mathbb{C}$ extends to a unique invariant state on $C(G)$, called the *Haar state*. The C^* -algebra generated by $C(G)$ in the GNS-representation associated with h , called the *reduced form* of G , is denoted by $C^r(G)$. On the other hand, the universal C^* -algebraic envelope of $\mathbb{C}[G]$ is denoted by $C^f(G)$.

There is a unique element $\rho = f_1 \in \mathbb{C}[G]^*$, called the *Woronowicz character*, such that for any finite dimensional unitary representation U of G the operator $\rho_U = (\iota \otimes \rho)(U) \in B(H_U)$ is positive, $\text{Tr}(\rho_U) = \text{Tr}(\rho_U^{-1})$ and

$$(\iota \otimes S^2)(U) = (\rho_U \otimes 1)U(\rho_U^{-1} \otimes 1),$$

where S is the antipode on $\mathbb{C}[G]$. Then the dual unitary representation \bar{U} can be defined by

$$\bar{U} = (j \otimes \iota)((\rho_U^{-1/2} \otimes 1)U^*(\rho_U^{1/2} \otimes 1)) \in B(\bar{H}_U) \otimes \mathbb{C}[G],$$

where $j: B(H_U) \rightarrow B(\bar{H}_U)$ is the canonical $*$ -anti-isomorphism given by $j(T)\bar{\xi} = \overline{T^*\xi}$. Consider the maps $r_U: \mathbb{C} \rightarrow \bar{H}_U \otimes H_U$ and $\bar{r}_U: \mathbb{C} \rightarrow H_U \otimes \bar{H}_U$ defined by $r_U(1) = \sum_i \bar{e}_i \otimes e_i$ and $\bar{r}_U(1) = \sum_i e_i \otimes \bar{e}_i$, where $\{e_i\}_i$ is any choice of an orthonormal basis in H_U (the maps r_U and \bar{r}_U are independent of this choice). Then, with \bar{U} defined as above, as a standard solution of the conjugate equations for U we can take

$$R_U = (\iota \otimes \rho_U^{-1/2})r_U \quad \text{and} \quad \bar{R}_U = (\rho_U^{1/2} \otimes \iota)\bar{r}_U.$$

The element ρ defines also a one-parameter group $(\tau_t)_{t \in \mathbb{R}}$ of Hopf $*$ -algebra automorphisms of $\mathbb{C}[G]$ by

$$(\iota \otimes \tau_t)(U) = (\rho_U^{it} \otimes 1)U(\rho_U^{-it} \otimes 1),$$

which is called the *scaling group*.

A compact quantum group G is said to be *coamenable* if the counit of $\mathbb{C}[G]$ extends to a bounded linear functional on $C^r(G)$. This is equivalent to $C^f(G) = C^r(G)$, but for our purposes, the following characterization is more important. Let Γ_U be the operator on $\ell^2(\text{Irr}(G))$ defined by

$$\text{Irr}(G) \ni [V] \mapsto \sum_{[W] \in \text{Irr}(G)} \dim(\text{Hom}_G(H_W, H_{V \oplus U}))[W].$$

Then $\|\Gamma_U\| \leq \dim U$, and G is coamenable if and only if the equality holds for all U .

Another important class of compact quantum groups is quantum groups of *Kac type*, characterized by any of the following conditions: 1) the antipode S of $\mathbb{C}[G]$ satisfies $S^2 = \text{id}$, 2) $\dim U = d(U)$ for all U in $\text{Rep } G$, 3) h is a tracial state, 4) $(\tau_t)_{t \in \mathbb{R}}$ is trivial.

2.3. Maximal Kac quantum subgroup. When G is a compact quantum group, there exists a unique *maximal Kac quantum subgroup* K of G , which first appeared in work of Sołtan [Soł05]. In other words, K is a compact quantum group of Kac type, we have a surjective homomorphism $\mathbb{C}[G] \rightarrow \mathbb{C}[K]$ of Hopf $*$ -algebras, and if H is any other compact quantum group with the same properties, then $\mathbb{C}[G] \rightarrow \mathbb{C}[H]$ factors through $\mathbb{C}[G] \rightarrow \mathbb{C}[K]$.

Explicitly, the quantum group K can be described as follows. The ideal $I \subset \mathbb{C}[G]$ generated by the elements $a - S^2(a)$ for all $a \in \mathbb{C}[G]$ is easily seen to be a Hopf $*$ -ideal. Hence the quotient $\mathbb{C}[G]/I$ is a Hopf $*$ -algebra with involutive antipode, so it defines a closed quantum subgroup K of G of Kac type. Clearly, K has the required maximality property. One also says that the Hopf $*$ -algebra $\mathbb{C}[K]$ is the *canonical Kac quotient* of $\mathbb{C}[G]$.

Alternatively, K can be described as follows, which is the original definition of Sołtan [Soł05]. Let J be the intersection of the kernels of the GNS-representations of $C^f(G)$ defined by all tracial states. Then we can put $C(K) = C^f(G)/J$. In other words, $I = J \cap \mathbb{C}[G]$, so the ideal I consists of the elements $a \in \mathbb{C}[G]$ such that $\tau(a^*a) = 0$ for all tracial states τ on $\mathbb{C}[G]$.

If G is coamenable, K can be also found from the noncommutative Poisson boundary of \hat{G} [Izu02] as $L^\infty(G/K) \simeq H^\infty(\hat{G}; \mu)$ for any ergodic probability measure μ on $\text{Irr}(G)$ [Tom07, NY14a]. One of our main observations, which will be exploited in Section 3, is that this characterization of maximal Kac quantum subgroups manifests itself already at the categorical level.

2.4. Cohomology of the discrete dual. Deformation problems for compact quantum groups are controlled by cohomology theory of the dual quantum groups, which plays the central role in this paper. We again refer the reader to [NT13] for a more thorough discussion.

Denote by $\mathcal{U}(G)$ the dual space $\mathbb{C}[G]^*$ of $\mathbb{C}[G]$. It is a $*$ -algebra, which is canonically isomorphic to the algebraic direct product $\prod_{[U] \in \text{Irr } G} B(H_U)$. Namely, a linear functional $\omega \in \mathbb{C}[G]^*$ defines operators $\pi_U(\omega) = (\iota \otimes \omega)(U) \in B(H_U)$, and by (2.2) the information about ω is precisely given by the family $(\pi_U(\omega))_{[U] \in \text{Irr } G}$. More generally, we put

$$\mathcal{U}(G^k) = (\mathbb{C}[G]^{\otimes k})^* \cong \prod_{[U_1], \dots, [U_k] \in \text{Irr } G} B(H_{U_1}) \otimes \cdots \otimes B(H_{U_k}).$$

We may interpret $\mathcal{U}(G^k)$ as the space of (possibly unbounded) k -point functions on the “discrete dual” quantum group \hat{G} . If G is a genuine commutative compact group, this agrees with the usual notion of functions on the direct product of the Pontryagin dual group \hat{G} .

A 2-cochain on \hat{G} is an invertible element $\mathcal{E} \in \mathcal{U}(G^2)$. It is said to be *invariant* if it commutes with the image of the “coproduct” $\hat{\Delta}: \mathcal{U}(G) \rightarrow \mathcal{U}(G^2)$ obtained by duality from the product on $\mathbb{C}[G]$. A 2-cochain T is said to be a 2-cocycle if it satisfies

$$(\mathcal{E} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta})(\mathcal{E}).$$

Invariant cocycles are also called *lazy* in the algebraic literature. If c is an invertible element in the center of $\mathcal{U}(G)$ (an *invariant 1-cochain*), then $(c \otimes c)\hat{\Delta}(c^{-1})$ is a 2-cocycle. Such cocycles are called *invariant 2-coboundaries*. The set of invariant 2-cocycles form a group under multiplication, and the invariant 2-coboundaries form a subgroup. The quotient is denoted by $H_G^2(\hat{G}; \mathbb{C}^\times)$, and called the *invariant 2-cohomology group* of \hat{G} . If we restrict to the unitary elements instead of invertible elements throughout, the corresponding group is denoted by $H_G^2(\hat{G}; \mathbb{T})$.

If \mathcal{E} is an invariant unitary 2-cocycle, the multiplication by \mathcal{E}^{-1} on $H_U \otimes H_V$ can be considered as a unitary endomorphism of $U \oplus V$ in $\text{Rep } G$. Such endomorphisms form a natural unitary transformation of the bifunctor \oplus into itself. The cocycle condition corresponds to the fact that this transformation is a monoidal autoequivalence of $\text{Rep } G$. Up to natural unitary monoidal isomorphisms, any autoequivalence of $\text{Rep } G$ fixing the irreducible classes can be obtained in this way. Moreover, the cohomology relation of cocycles corresponds to the natural unitary monoidal isomorphism of autoequivalences. Thus $H_G^2(\hat{G}; \mathbb{T})$ can be considered as the subgroup of $\text{Aut}^\otimes(\text{Rep } G)$

consisting of autoequivalences that preserve the isomorphism classes of objects. Without the unitarity, $H_G^2(\hat{G}; \mathbb{C}^\times)$ corresponds to a subgroup of monoidal autoequivalences of $\text{Rep } G$ as a tensor category over \mathbb{C} .

Let \mathcal{E} be an arbitrary unitary 2-cocycle on \hat{G} , invariant or not, and $F: \text{Rep } G \rightarrow \text{Hilb}_f$ be the canonical fiber functor. Then the triple $F_{\mathcal{E}} = (\text{id}_{\mathbb{C}}, U \mapsto H_U, \mathcal{E}^{-1})$ defines another unitary tensor functor $\text{Rep } G \rightarrow \text{Hilb}_f$. By Woronowicz's Tannaka–Krein duality $F_{\mathcal{E}}$ can be considered as the canonical fiber functor of another compact quantum group $G_{\mathcal{E}}$ satisfying $\text{Rep } G = \text{Rep } G_{\mathcal{E}}$. Concretely, $\mathcal{U}(G_{\mathcal{E}})$ coincides with $\mathcal{U}(G)$ as a $*$ -algebra, but is endowed with the modified coproduct $\hat{\Delta}_{\mathcal{E}}(T) = \mathcal{E} \hat{\Delta}(T) \mathcal{E}^{-1}$. By duality, $\mathbb{C}[G_{\mathcal{E}}]$ is the same coalgebra as $\mathbb{C}[G]$, but has a modified $*$ -algebra structure dual to $\hat{\Delta}_{\mathcal{E}}$. Up to natural unitary monoidal isomorphisms, the functors $F_{\mathcal{E}}$ exhaust all unitary tensor functors $F': \text{Rep } G \rightarrow \text{Hilb}_f$ satisfying $\dim F'(U) = \dim H_U$. Moreover, if T is a unitary element in $\mathcal{U}(G)$ (a *unitary 1-cochain*), then $\mathcal{E}_T = (T \otimes T) \mathcal{E} \hat{\Delta}(T^{-1})$ defines another unitary fiber functor which is naturally unitarily monoidally isomorphic to $F_{\mathcal{E}}$. Thus, the *2-cohomology* $H^2(\hat{G}; \mathbb{T})$, which is the quotient of the set of unitary 2-cocycles by the cohomology relation $\mathcal{E} \sim \mathcal{E}_T$, gives a complete parametrization of such unitary fiber functors up to natural unitary monoidal isomorphisms. We also note that there is an action of the group $H_G^2(\hat{G}; \mathbb{T})$ on the set $H^2(\hat{G}; \mathbb{T})$ given by multiplication on the right. This corresponds to the restriction of the obvious right action of $\text{Aut}^{\otimes}(\text{Rep } G)$ on the natural unitary monoidal isomorphism classes of unitary fiber functors $\text{Rep } G \rightarrow \text{Hilb}_f$.

A *3-cocycle* on \hat{G} is an invertible element $\Phi \in \mathcal{U}(G^3)$ which satisfies

$$(1 \otimes \Phi)(\iota \otimes \hat{\Delta} \otimes \iota)(\Phi)(\Phi \otimes 1) = (\iota \otimes \iota \otimes \hat{\Delta})(\Phi)(\hat{\Delta} \otimes \iota \otimes \iota)(\Phi).$$

Again, such a cocycle is called *invariant* if it commutes with the image of $\hat{\Delta}^{(2)} = (\hat{\Delta} \otimes \iota) \hat{\Delta} = (\iota \otimes \hat{\Delta}) \hat{\Delta}$. Such cocycles are also called *associators*. If Φ is an invariant unitary 3-cocycle and \mathcal{E} is an invariant unitary 2-cochain, the 3-cochain

$$\Phi_{\mathcal{E}} = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta})(\mathcal{E})\Phi(\hat{\Delta} \otimes \iota)(\mathcal{E}^{-1})(\mathcal{E}^{-1} \otimes 1)$$

is again an invariant unitary 3-cocycle. Cocycles of the form $\Phi_{\mathcal{E}}$ are said to be cohomologous to Φ . This defines an equivalence relation on the set of invariant unitary 3-cocycles, and the quotient set is called the *invariant unitary 3-cohomology* $H_G^3(\hat{G}; \mathbb{T})$.

If Φ is an invariant unitary 3-cocycle, its action on $H_U \otimes H_V \otimes H_W$ can be considered as a new associativity morphism on the C^* -category $\text{Rep } G$ with bifunctor \oplus . This gives us a new C^* -tensor category $(\text{Rep } G, \Phi)$, which has the same data as $\text{Rep } G$ except for the new associativity morphisms defined by the action of Φ . It is not clear whether the new category is automatically rigid, but this is at least the case if Φ acts as a scalar on $H_U \otimes H_V \otimes H_W$ for irreducible U, V and W . If \mathcal{E} is an invariant unitary 2-cochain, the categories $(\text{Rep } G, \Phi)$ and $(\text{Rep } G, \Phi_{\mathcal{E}})$ are naturally unitarily monoidally equivalent, by means of the unitary tensor functor $(\text{id}_{\mathbb{1}}, \text{Id}_{\text{Rep } G}, \mathcal{E}^{-1}): (\text{Rep } G, \Phi) \rightarrow (\text{Rep } G, \Phi_{\mathcal{E}})$. This way $H_G^3(\hat{G}; \mathbb{T})$ gives a parametrization of the categories of the form $(\text{Rep } G, \Phi)$ considered up to unitary monoidal equivalences that preserve the isomorphism classes of objects.

2.5. Twisted q -deformations of compact Lie groups. Finally, let us recall from [NY15] how to construct new quantum groups whose representation categories are nontrivial twists of $\text{Rep } G$. Let G be a simply connected semisimple compact Lie group, and T be its maximal torus. We denote the weight lattice and the root lattice by P and Q respectively, so that P/Q is naturally isomorphic to the Pontryagin dual of the center of G .

Suppose that c is a \mathbb{T} -valued 2-cochain on the dual group $\hat{T} = P$, such that its coboundary ∂c happens to be invariant under Q in each variable. Then ∂c can be considered a 3-cocycle Φ^c on $P/Q = \widehat{Z(G)}$. In turn, Φ^c can be considered as an invariant 3-cocycle on \hat{G} , and hence an associator on $\text{Rep } G$. Concretely, for irreducible representations U, V and W , the associator $\Phi^c(U, V, W)$ acts as the scalar $\Phi^c(\omega_U, \omega_V, \omega_W)$, where $\omega_U \in \widehat{Z(G)}$ is the central character of U .

Since Φ^c is the coboundary of c over \hat{T} , we have a unitary fiber functor $F_c: (\text{Rep } G, \Phi^c) \rightarrow \text{Hilb}_f$, which is identical to the canonical fiber functor on $\text{Rep } G$, except that the tensor structure is given by

$$H_U \otimes H_V \rightarrow H_{U \otimes V}, \quad \xi \otimes \eta \mapsto c^*(\xi \otimes \eta).$$

Analogous to the case of twisting by 2-cocycles, this functor defines a new compact quantum group G^c such that $\text{Rep } G^c$ is monoidally equivalent to $(\text{Rep } G, \Phi^c)$. Explicitly, $\mathbb{C}[G^c] = \mathbb{C}[G]$ as coalgebras, while the new $*$ -algebra structure is defined by duality from $(\mathcal{U}(G), c\hat{\Delta}(\cdot)c^*)$.

Notice that, since c is not assumed to be invariant, the category $(\text{Rep } G, \Phi^c)$ need not be monoidally equivalent to $\text{Rep } G$. In fact, for simple and simply connected G , we later show that these categories are monoidally equivalent if and only if the class of ∂c is trivial in $H^3(P/Q; \mathbb{T})$.

More generally, for any $0 < q < \infty$, the finite dimensional admissible unitary representations of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ define the q -deformed compact quantum group G_q (\mathfrak{g} is the complexified Lie algebra of G). This quantum group contains T as a closed subgroup, and the image of $Z(G)$ is central in $\mathcal{U}(G_q)$. Thus, by the same construction we obtain a new quantum group G_q^c such that $\mathcal{U}(G_q^c) = (\mathcal{U}(G_q), c\hat{\Delta}_q(\cdot)c^*)$. Because c is defined on \hat{T} , the coproduct of any element $a \in \mathcal{U}(T)$ computed in $\mathcal{U}(G_q^c)$ is the same as $\hat{\Delta}_q(a) = \hat{\Delta}(a)$. In particular, T is still a closed subgroup of G_q^c .

3. FACTORIZATION OF FIBER FUNCTORS THROUGH KAC QUANTUM SUBGROUPS

Let G be a compact quantum group. We are interested in unitary tensor functors $F: \text{Rep } G \rightarrow \mathcal{A}$ defining the classical dimension function on $\text{Rep } G$, that is, such that $\dim U = d^A(F(U))$ for all U . By [NY14b, Theorem 4.1], if G is coamenable, then there exists a universal functor $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ with this property. More precisely, to apply the results of [NY14b] we in addition have to assume that $\text{Irr } G$ is at most countable. This will be the case in our applications in the later sections. In the general case, we could consider quotient compact quantum groups of G corresponding to countably generated full rigid monoidal subcategories of $\text{Rep } G$, and then pass to the limit. We leave the details of this reduction to the countable case to the interested reader.

The universal functor $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ was constructed as the Poisson boundary of $\text{Rep } G$ with respect to an ergodic measure, but for our current purposes it is more instructive to understand that up to an isomorphism it can be described as follows, see [NY14b, Section 4]. Assume that $\text{Rep } G$ is a C^* -tensor subcategory of a strict C^* -tensor category \mathcal{A} such that $\dim U = d^A(U)$ for all objects U in $\text{Rep } G$. Choose a standard solution (R_U, \bar{R}_U) of the conjugate equations for U in $\text{Rep } G$. There exists a unique positive automorphism a_U of the object U in \mathcal{A} such that

$$(\iota \otimes a_U^{1/2})R_U \quad \text{and} \quad (a_U^{-1/2} \otimes \iota)\bar{R}_U$$

form a standard solution of the conjugate equation for U in \mathcal{A} . Then as \mathcal{P} we can take the C^* -tensor category obtained by the subobject completion (idempotent completion) of the monoidal subcategory of \mathcal{A} generated by $\text{Rep } G$ and the morphisms a_U for all U , and as $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ we can take the embedding functor.

Theorem 3.1. *Let G be a coamenable compact quantum group and K be its maximal Kac quantum subgroup. Then the forgetful functor $\text{Rep } G \rightarrow \text{Rep } K$ is a universal unitary tensor functor defining the classical dimension function on $\text{Rep } G$.*

Proof. We will give two proofs. For the first proof, we consider $\text{Rep } G$ as a subcategory of Hilb_f and use the description of the universal functor $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ given above. Then the above characterization of the morphisms a_U imply that they are exactly equal to ρ_U .

Let \mathcal{P} be the C^* -tensor category obtained by the subobject completion of the monoidal subcategory of Hilb_f generated by $\text{Rep } G$ and the morphisms ρ_U . By Woronowicz's Tannaka–Krein duality, \mathcal{P} defines a closed quantum subgroup $H < G$. Then, on the one hand, since ρ_U is a morphism in the new category $\mathcal{P} = \text{Rep } H$, the eigenspaces of ρ_U are representations of H . On the other hand, the

conjugation by ρ_U^{it} gives the action of the scaling group τ_t of on the direct summand $B(H_U)_* \subset \mathbb{C}[G]$. Since the quotient map $\mathbb{C}[G] \rightarrow \mathbb{C}[H]$ intertwines the scaling groups, we see that the scaling group of H must be trivial. Thus H is of Kac type.

Next, if H' is a closed quantum subgroup of G of Kac type, then using again that the quotient map $\mathbb{C}[G] \rightarrow \mathbb{C}[H']$ intertwines the scaling groups, we see that $\text{Rep } H'$ must contain the morphisms ρ_U , and consequently $\text{Rep } H \subset \text{Rep } H'$. Thus, H is maximal among such quantum subgroups, so $H = K$. This completes the first proof.

For the second proof, we argue more formally. Let $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ be a universal functor. Since the forgetful functor $\text{Rep } G \rightarrow \text{Hilb}_f$ factors through $\Pi: \text{Rep } G \rightarrow \mathcal{P}$, by Woronowicz's Tannaka–Krein duality we can identify $\Pi: \text{Rep } G \rightarrow \mathcal{P}$ with the forgetful functor $\text{Rep } G \rightarrow \text{Rep } H$ for a closed quantum subgroup H of G . Since the quantum dimension function on $\text{Rep } H$ coincides with the classical dimension function, the quantum group H is of Kac type. Hence, for any finite dimensional unitary representation U of G , we have $\text{End}_K(H_U) \subset \text{End}_H(H_U)$. On the other hand, by assumption the forgetful functor $\text{Rep } G \rightarrow \text{Rep } K$ factors through $\text{Rep } G \rightarrow \text{Rep } H$, so that $\text{End}_H(H_U)$ can be embedded into $\text{End}_K(H_U)$. Hence $\text{End}_K(H_U) = \text{End}_H(H_U)$, and since this is true for all U , we conclude that $H = K$. \square

We will need only the following particular case of this result.

Corollary 3.2. *There is a bijective correspondence between the (natural unitary monoidal) isomorphism classes of unitary fiber functors $\text{Rep } K \rightarrow \text{Hilb}_f$ and the isomorphism classes of unitary fiber functors $F: \text{Rep } G \rightarrow \text{Hilb}_f$ such that $\dim U = \dim F(U)$. Namely, the correspondence maps a functor $\text{Rep } K \rightarrow \text{Hilb}_f$ into its composition with the forgetful functor $\text{Rep } G \rightarrow \text{Rep } K$.*

In cohomological terms, this can be rephrased as follows.

Corollary 3.3. *The natural map $H^2(\hat{K}; \mathbb{T}) \rightarrow H^2(\hat{G}; \mathbb{T})$ induced by the inclusion $\mathcal{U}(K) \rightarrow \mathcal{U}(G)$ is a bijection.*

We finish this section with the following simple observation.

Proposition 3.4. *Let G be a compact quantum group, K be its maximal Kac quantum subgroup and \mathcal{E} be a unitary 2-cocycle on \hat{K} . Then $K_{\mathcal{E}}$ is the maximal Kac quantum subgroup of $G_{\mathcal{E}}$.*

Proof. The quantum dimensions of the unitary representations of $K_{\mathcal{E}}$ are the same as those of K , since $\text{Rep } K_{\mathcal{E}} = \text{Rep } K$ and quantum dimension is an intrinsic notion to the representation category. The classical dimensions also remain the same by construction. Thus the quantum and classical dimensions on $\text{Rep } K_{\mathcal{E}}$ coincide, hence $K_{\mathcal{E}}$ is of Kac type.

It follows that $K_{\mathcal{E}}$ is contained in the maximal Kac quantum subgroup of $G_{\mathcal{E}}$. Since \mathcal{E}^* is a unitary 2-cocycle on $\hat{G}_{\mathcal{E}}$ and $(G_{\mathcal{E}})_{\mathcal{E}^*} = G$, by swapping the roles of G and $G_{\mathcal{E}}$ we conclude that $K_{\mathcal{E}}$ must be maximal. \square

4. COMPACT QUANTUM GROUPS OF LIE TYPE

In this section G denotes a simply connected semisimple compact Lie group. Recall that if $q > 0$ and c is a \mathbb{T} -valued 2-cochain on the dual $\hat{T} = P$ of the maximal torus $T \subset G$, such that its coboundary ∂c defines a 3-cocycle Φ^c on $\widehat{Z(G)} = P/Q$, then we can define a compact quantum group G_q^c .

4.1. Dimension-preserving fiber functors. We want to apply the results of Section 3 to G_q^c . For this we have to find the maximal Kac quantum subgroups of these quantum groups.

As was shown by Tomatsu [Tom07, Lemma 4.10], for $q \neq 1$ the maximal Kac quantum subgroup of G_q is T . We have the following generalization of this result.

Theorem 4.1. *For any $q > 0$, $q \neq 1$, and any \mathbb{T} -valued 2-cochain c on P such that ∂c descends to P/Q , the maximal Kac quantum subgroup of G_q^c is the maximal torus T .*

Before going to the proof we need to establish a simple lemma. It is well-known that the Woronowicz character $\rho = f_1$ of G_q lies in $\mathcal{U}(T) \subset \mathcal{U}(G_q)$, see for example [NT13, Proposition 2.4.10]. Namely, it equals $q^{-2\rho^*}$, where $\rho^* \in \mathfrak{h}$ (the complexified Lie algebra of T) is the unique vector satisfying $\alpha(\rho^*) = d_\alpha = (\alpha, \alpha)/2$ for each simple root α . In other words, under the isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$ the element ρ^* corresponds to half the sum of positive roots. Recall also that by construction we have an identification $\mathcal{U}(G_q^c) = \mathcal{U}(G_q)$ as $*$ -algebras.

Lemma 4.2. *Under the identification of $\mathcal{U}(G_q^c)$ with $\mathcal{U}(G_q)$, the Woronowicz character f_1 of G_q^c is again given by $q^{-2\rho^*}$.*

Proof. Since G_q^c does not change when we multiply c by a 2-cochain living on $\widehat{Z(G)}$, without loss of generality we may assume that the cocycle Φ^c is normalized.

For an irreducible unitary representation U of G_q , put $\lambda_U = \Phi^c(\omega_{\bar{U}}, \omega_U, \omega_{\bar{U}})$. Then in the category $(\text{Rep } G_q, \Phi^c)$, the pair $(\lambda_U R_U, \bar{R}_U)$ solves the conjugate equations for U . When we apply the functor $F_c: (\text{Rep } G_q, \Phi) \rightarrow \text{Hilb}_f$, which defines G_q^c , this solution is transformed into $(\lambda_U c R_U, c \bar{R}_U)$.

Since we also have $c \in \mathcal{U}(T \times T)$, it follows that if we choose an orthonormal basis $\{\xi_i\}_i$ in H_U consisting of weight vectors, then the vector $c R_U(1) \in \bar{H}_U \otimes H_U$ has the form $\sum_i \bar{\xi}_i \otimes \beta_i \rho^{-1/2} \xi_i$ for some $\beta_i \in \mathbb{T}$, where ρ denotes the Woronowicz character of G_q . It follows that the Woronowicz character $f_{-1/2}$ of G_q^c has the form $\rho^{-1/2} v$ for a unitary v commuting with $\mathcal{U}(T)$. Hence $f_1 = \rho$ and $v = 1$. \square

Proof of Theorem 4.1. Observe that the ambiguity of the correspondence $c \mapsto [\Phi^c]$ is in that one may perturb c by a 2-cocycle on $\hat{T} = P$ or by a 2-cochain on $\widehat{Z(G)}$. By Proposition 3.4, if Theorem 4.1 is true for a cochain c then it is true for any other cochain that differs from c by a \mathbb{T} -valued 2-cocycle on \hat{T} . Also, G_q^c does not change when we multiply c by a 2-cochain living on $\widehat{Z(G)}$. It follows that for the proof of the theorem it suffices to consider particular cochains c such that $[\Phi^c]$ exhaust the classes in $H^3(\widehat{Z(G)}; \mathbb{T})$ of the form $[\Phi^c]$. Such representatives c_τ were constructed in [NY15, Proposition 2.6], and the corresponding quantum groups were denoted by G_q^τ .

Let us recall the structure of the corresponding C^* -algebras $C(G_q^\tau)$ as studied in [NY15, Section 3]. Given $\tau = (\tau_i)_{i=1}^{\text{rk } G} \in Z(G)^{\text{rk } G}$, we consider the finite subgroup T_τ of T generated by the components τ_i . Then there is a group homomorphism $\psi: \hat{T}_\tau \rightarrow T/Z(G)$ characterized by $\langle \psi(\chi), \alpha_i \rangle = \tau_i$ for any $\chi \in \hat{T}_\tau$ and any simple root α_i . Composing this homomorphism with the conjugation action of $T/Z(G)$ on $C(G_q)$, we obtain an action of \hat{T}_τ on $\mathbb{C}[G_q]$, denoted by $\text{Ad } \psi$. The crossed product $C(G_q) \rtimes_{\text{Ad } \psi} \hat{T}_\tau$ has an action of T_τ which is given by right translations on the copy of $C(G_q)$ and by the dual action on the copy of \hat{T}_τ . The C^* -algebra $(C(G_q) \rtimes_{\text{Ad } \psi} \hat{T}_\tau)^{T_\tau}$ has the structure of a compact quantum group, induced by those of G_q and \hat{T}_τ . This is our quantum group G_q^τ .

Using the crossed product presentation of $C(G_q^\tau)$ and results of Soibelman on the representation theory of $C(G_q)$ for $q \neq 1$, it is not difficult to obtain information on representations of the C^* -algebra $C(G_q^\tau)$, see [NY15, Proposition 3.4]:

$$\text{Prim}(C(G_q^\tau)) = \coprod_{w \in W} (\theta_w(\hat{T}_\tau) \backslash T/T_\tau) \times \widehat{\theta_w^{-1}(T_\tau)}, \quad (4.1)$$

where W is the Weyl group and θ_w is a certain homomorphism of \hat{T}_τ into T expressed in terms of ψ and $w \in W$.

By Lemma 4.2, the scaling groups of G_q^τ and G_q are both given by the conjugation action by $q^{-2it\rho^*} \in T$, $t \in \mathbb{R}$. We denote this common scaling group by $(\tau_t)_t$. We claim that the only irreducible representations of $C(G_q^\tau)$ that are fixed, up to isomorphism, under $(\tau_t)_t$ are the evaluations at the points of T .

Since $\text{Ad } \psi$ commutes with τ_t , we obtain a natural extension of τ_t to $\mathbb{C}[G_q] \rtimes_{\text{Ad } \psi} \hat{T}_\tau$ by letting it act trivially on the copy of \hat{T}_τ . The embedding of $\mathbb{C}[G_q]$ into the crossed product is compatible with this extension of τ_t . The action of T_τ on $\mathbb{C}[G_q] \rtimes_{\text{Ad } \psi} \hat{T}_\tau$, being implemented by the right torus translation action rt and the dual action, also commutes with the above extension of τ_t . Thus, up to a strong Morita equivalence, the action of the scaling group $(\tau_t)_t$ on $C(G_q)$ can be identified with the action of \mathbb{R} on $C(G_q) \rtimes_{\text{rt}} T_\tau \rtimes_{\text{Ad } \psi, \hat{\text{rt}}} \hat{T}_\tau$ such that \mathbb{R} acts by $(\tau_t)_t$ on $C(G_q)$ and trivially on the rest.

The description (4.1) of the primitive spectrum of $C(G_q)$ is obtained from the Morita equivalence $C(G_q^\tau) \sim C(G_q) \rtimes_{\text{rt}} T_\tau \rtimes_{\text{Ad } \psi, \hat{\text{rt}}} \hat{T}_\tau$, the Effros–Hahn machinery for crossed product by finite groups, and the identification of $\text{Prim}(C(G_q))$ with $\coprod_{w \in W} T$. It follows from the above considerations that on the part of the spectrum of $C(G_q^\tau)$ labeled by $w \in W$ in (4.1), the scaling group induces the translation by $q^{-2it(w\rho^* - \rho^*)}$, $t \in \mathbb{R}$, on $\theta_w(\hat{T}_\tau) \backslash T/T_\tau$ (see [NT12b, Lemma 3.4; Yam13, Lemma 8] for the action of T on $\text{Prim}(C(G_q))$ induced by the translations on $C(G_q)$). Since $w\rho^* \neq \rho^*$ unless $w = e$, we obtain the claim.

The rest of the argument is identical to the proof of [Tom07, Lemma 4.10]. Namely, let K be the maximal Kac quantum subgroup of G_q^τ , and π be an irreducible representation of $C(K)$. Composing π with the restriction map $C(G_q^\tau) \rightarrow C(K)$, we obtain an irreducible representation of $C(G_q^\tau)$, again denoted by π . Since the restriction map $C(G_q^\tau) \rightarrow C(K)$ intertwines the scaling groups, and the scaling group of K is trivial, π has to be the evaluation at some point of T . This proves that K is contained in T , hence $K = T$. \square

Remark 4.3. After this paper was submitted for publication we found another proof of the above theorem, which is of a more categorical flavour and is based on considerations of the categorical Poisson boundary. It applies to a more general setting than the above and will appear in a forthcoming joint work with Julien Bichon.

Combining this theorem with Corollary 3.2 we obtain the following result.

Corollary 4.4. *Let G_q^c ($q > 0$, $q \neq 1$) be as in Theorem 4.1. Then any dimension-preserving unitary fiber functor $\text{Rep } G_q^c \rightarrow \text{Hilb}_f$ factors, uniquely up to isomorphism, through $\text{Rep } T$. Equivalently, the inclusion map $\mathcal{U}(T) \rightarrow \mathcal{U}(G_q^c)$ defines a bijection $H^2(\hat{T}; \mathbb{T}) \cong H^2(\hat{G}_q^c; \mathbb{T})$.*

Remark 4.5. The above corollary is not true for $q = 1$. The map $H^2(\hat{T}; \mathbb{T}) \rightarrow H^2(\hat{G}; \mathbb{T})$ is neither surjective, if G is sufficiently large as discussed in the introduction, nor injective, as it factors through the quotient of $H^2(\hat{T}; \mathbb{T})$ by the action of the Weyl group.

4.2. Monoidal equivalences. If c' is a \mathbb{T} -valued 2-cocycle on $\hat{T} = P$ and c is a 2-cochain as in the previous section, the twisting of G_q^c by c' is equal to $G_q^{cc'}$, as seen from the coproduct formula for $\mathcal{U}(G_q^c)$. Thus, we know from Corollary 4.4 that the class of compact quantum groups of the form G_q^c is closed under cocycle twisting when $q \neq 1$. Our goal is to classify such quantum groups up to isomorphism, which will be completed only for the simple case in this paper.

In general, a quantum group isomorphism $G \rightarrow G'$ is the same thing as a unitary monoidal equivalence $F: \text{Rep } G' \rightarrow \text{Rep } G$ and a natural unitary monoidal isomorphism $\eta: F^G F \simeq F^{G'}$, where F^G and $F^{G'}$ are the canonical fiber functors of G and G' . Therefore, in view of injectivity of the map $H^2(\hat{T}; \mathbb{T}) \rightarrow H^2(\hat{G}_q^c; \mathbb{T})$, in order to classify the quantum groups G_q^c up to isomorphism it remains to understand the groups $\text{Aut}^\otimes(\text{Rep } G_q^c)$ and classify the categories $\text{Rep } G_q^c$ up to unitary monoidal equivalence.

We start by computing the invariant 2-cohomology. Let us remark that our arguments in this section will work also for $q = 1$.

Proposition 4.6. *For any $q > 0$ and any \mathbb{T} -valued 2-cochain c on P such that ∂c descends to P/Q , we have group isomorphisms*

$$H^2(P/Q; \mathbb{T}) \cong H_{G_q^c}^2(\hat{G}_q^c; \mathbb{T}) \cong H_{G_q^c}^2(\hat{G}_q^c; \mathbb{C}^\times),$$

induced by the inclusion $\mathcal{U}(Z(G)) \rightarrow \mathcal{U}(G_q^c)$.

Proof. For the trivial c , the assertion is already proved in [NT12a], see also [NT13]. The general case can be reduced to this. Let us again give two arguments for this, one categorical and the other Hopf algebraic.

Let E be a unitary monoidal autoequivalence of $(\text{Rep } G_q, \Phi^c)$ which maps every object to an isomorphic object. Since for any simple objects X, Y and Z in $\text{Rep } G_q$ the associativity morphism $\Phi^c(X, Y, Z)$ is scalar, it is easy to see that E can be also considered as a unitary monoidal autoequivalence \tilde{E} of $\text{Rep } G_q$. Then, by [NT12a, Theorem 1], \tilde{E} is naturally unitarily monoidally isomorphic to an autoequivalence coming from some $[c'] \in H^2(P/Q; \mathbb{T})$. In turn, any such isomorphism defines a natural unitary monoidal isomorphism between E and the autoequivalence induced by c' on $(\text{Rep } G_q, \Phi^c)$. The same works for nonunitary autoequivalences, and we obtain the assertion.

The second proof goes as follows. Recall that the coproduct of $\mathcal{U}(G_q^c)$ is given by $c\hat{\Delta}_q(x)c^*$, where $\hat{\Delta}_q$ is the coproduct on $\mathcal{U}(G_q)$. Thus, an element $\mathcal{E} \in \mathcal{U}(G_q^c \times G_q^c)$ is an invariant 2-cochain on \hat{G}_q^c if and only if $c^*\mathcal{E}c$ is an invariant 2-cochain on \hat{G}_q . Moreover, we claim that \mathcal{E} is an invariant 2-cocycle on \hat{G}_q^c if and only if $c^*\mathcal{E}c$ is an invariant 2-cocycle on \hat{G}_q , and so we can apply to $c^*\mathcal{E}c$ the results of [NT12a].

Let us verify our claim. Suppose that \mathcal{E} is an invariant 2-cocycle on \hat{G}_q^c , so

$$(\mathcal{E}c \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E})(c^* \otimes 1) = (1 \otimes \mathcal{E}c)(\iota \otimes \hat{\Delta}_q)(\mathcal{E})(1 \otimes c^*).$$

We then want to verify the corresponding condition for $c^*\mathcal{E}c$ on \hat{G}_q ,

$$(c^*\mathcal{E}c \otimes 1)(\hat{\Delta}_q \otimes \iota)(c^*\mathcal{E}c) = (1 \otimes c^*\mathcal{E}c)(\iota \otimes \hat{\Delta}_q)(c^*\mathcal{E}c).$$

Using the invariance of \mathcal{E} with respect to the coproduct of \hat{G}_q^c , the left hand side is seen to be equal to $y = (\hat{\Delta} \otimes \iota)(c^*)(c^* \otimes 1)(\mathcal{E}c \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E})(\hat{\Delta} \otimes \iota)(c)$, while the right hand side becomes $z = (\iota \otimes \hat{\Delta})(c^*)(1 \otimes c^*)(1 \otimes \mathcal{E}c)(\iota \otimes \hat{\Delta}_q)(\mathcal{E})(\iota \otimes \hat{\Delta})(c)$. (Note that $c \in \mathcal{U}(T^2)$ implies the equalities of the form $(\hat{\Delta}_q \otimes \iota)(c) = (\hat{\Delta} \otimes \iota)(c)$, and that the terms like $c \otimes 1$ and $(\iota \otimes \hat{\Delta})(c)$ commute with each other.) Then, using that $\partial(c) = (1 \otimes c)(\iota \otimes \hat{\Delta})(c)(\hat{\Delta} \otimes \iota)(c^*)(c^* \otimes 1)$, we obtain

$$z^{-1}y = (\iota \otimes \hat{\Delta})(c^*)(\iota \otimes \hat{\Delta}_q)(\mathcal{E})^{-1}(1 \otimes \mathcal{E}c)^{-1}\partial(c)(\mathcal{E}c \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E})(\hat{\Delta} \otimes \iota)(c).$$

Now, the assumption $\partial(c) \in \mathcal{U}(Z(G)^3)$ implies that we may move $\partial(c)$ in the middle to outside, then the 2-cocycle condition for \mathcal{E} implies that

$$z^{-1}y = \partial(c)(\iota \otimes \hat{\Delta})(c^*)(1 \otimes c^*)(c \otimes 1)(\hat{\Delta} \otimes \iota)(c) = 1.$$

Therefore $y = z$, which is what we wanted to show. The reverse implication can be argued in the same way. \square

Let us denote the based root datum of G by Ψ . There is a natural action of $\sigma \in \text{Aut}(\Psi)$ on the quantized universal enveloping algebra $U_q(\mathfrak{g})$ by Hopf $*$ -algebra automorphisms, which is given on the generators by $E_j \mapsto E_{\sigma j}$, $K_j \mapsto K_{\sigma j}$. This induces an action of $\text{Aut}(\Psi)$ on $(\mathcal{U}(G_q), \hat{\Delta}_q)$. The restriction of this action to T , hence to $Z(G)$, does not depend on q . For a given c , we denote by $\text{Aut}(\Psi)_{c,q}$ the stabilizer subgroup of $\text{Aut}(\Psi)$ for the image of ∂c in $H_{G_q^c}^3(\hat{G}_q^c; \mathbb{T})$.

Theorem 4.7. *For any $q > 0$ and any \mathbb{T} -valued 2-cochain c on P such that ∂c descends to P/Q , we have a short exact sequence*

$$1 \rightarrow H^2(P/Q; \mathbb{T}) \rightarrow \text{Aut}^\otimes(\text{Rep } G_q^c) \rightarrow \text{Aut}(\Psi)_{c,q} \rightarrow 1.$$

Proof. Let $R^+(G)$ be the representation semiring of G . By a result of McMullen [McM84], the natural map $\text{Aut}(\Psi) \rightarrow \text{Aut}(R^+(G))$ is an isomorphism. Thus, if F is an autoequivalence of $\text{Rep } G_q^c$, the induced automorphism of $R^+(G_q^c) = R^+(G)$ can be considered as an element of $\text{Aut}(\Psi)$. This way we obtain a group homomorphism $\text{Aut}^\otimes(\text{Rep } G_q^c) \rightarrow \text{Aut}(\Psi)$. The kernel of the homomorphism $\text{Aut}^\otimes(\text{Rep } G_q^c) \rightarrow \text{Aut}(\Psi)$ consists of the monoidal autoequivalences preserving the isomorphism classes of objects, so it is isomorphic to $H^2(P/Q; \mathbb{T})$ by Proposition 4.6.

Let us next show that the image of $\text{Aut}^\otimes(\text{Rep } G_q^c) \rightarrow \text{Aut}(\Psi)$ is contained in $\text{Aut}(\Psi)_{c,q}$. Consider an autoequivalence of $\text{Rep } G_q^c$ and let σ be its image in $\text{Aut}(\Psi)$. The action of $\text{Aut}(\Psi)$ on $\mathcal{U}(G_q)$ defines an isomorphism $G_q^c \cong G_q^{\sigma(c)}$, hence a unitary monoidal equivalence between $(\text{Rep } G_q, \Phi^c)$ and $(\text{Rep } G_q, \Phi^{\sigma(c)})$. Since σ is defined by an autoequivalence of $(\text{Rep } G_q, \Phi^c)$, it follows that there exists a unitary monoidal equivalence $F = (F_0, F_1, F_2)$ between $(\text{Rep } G_q, \Phi^c)$ and $(\text{Rep } G_q, \Phi^{\sigma(c)})$ which preserves the isomorphism classes of objects of $\text{Rep } G_q$. Replacing F by a naturally unitarily monoidally isomorphic functor, we may assume that F_1 is simply the identity functor on $\text{Rep } G_q$. Then F_2 is given by the action of \mathcal{E}^{-1} for an invariant unitary 2-cochain \mathcal{E} on \hat{G}_q , and (2.1) reads as $\Phi^{\sigma(c)} = \Phi_{\mathcal{E}}^c$. Thus σ preserves the cohomology class of Φ^c in $H_{G_q}^3(\hat{G}_q; \mathbb{T})$.

Reversing the above argument, it also becomes clear that the map $\text{Aut}^\otimes(\text{Rep } G_q^c) \rightarrow \text{Aut}(\Psi)_{c,q}$ is surjective. \square

At this point the precise form of $\text{Aut}(\Psi)_{c,q}$ for general G , c and q is not clear to us. However, for a class of simple G containing $\text{SU}(n)$, we have the following result.

Proposition 4.8. *If G is a simply connected simple compact Lie group not of type D_{2m} , we have $\text{Aut}^\otimes(\text{Rep } G_q^c) \cong \text{Aut}(\Psi)$ for any c and $q > 0$.*

Proof. By assumption P/Q is a cyclic group, which implies that $H^2(P/Q; \mathbb{T})$ is trivial. Moreover, $\text{Aut}(\Psi)$ is either of order 2 (type A_n, D_{2m+1}, E_6) or trivial, and for the latter case we have nothing to do. Notice that the natural map $H^3(P/Q; \mathbb{T}) \rightarrow H_{G_q}^3(\hat{G}_q; \mathbb{T})$ is equivariant with respect to the action of $\text{Aut}(\Psi)$. We claim that even when $\text{Aut}(\Psi)$ contains a nontrivial element σ , it acts trivially already on $H^3(P/Q; \mathbb{T})$.

Let us first consider the case of type A_{n-1} . The element σ induces the automorphism of the Hopf $*$ -algebra $U_q(\mathfrak{su}(n))$ characterized by $E_i \mapsto E_{n-i}$, $1 \leq i \leq n-1$. On the center of $\text{SU}_q(n)$, identified with $\mathbb{Z}/n\mathbb{Z}$, it is given by $a \mapsto -a$. The induced map at the level of cocycles is $\phi \mapsto \psi$, where $\psi(a, b, c) = \phi(-a, -b, -c)$. But these cocycles are cohomologous: if ϕ is given by $\phi(a, b, c) = \omega(\lfloor \frac{a+b}{n} \rfloor - \lfloor \frac{a}{n} \rfloor - \lfloor \frac{b}{n} \rfloor)^c$ for an n th root of unity ω , we see that $\phi\psi^{-1}$ is the coboundary of the 2-cochain $(a, b) \mapsto \omega^{-\lfloor \frac{a}{n} \rfloor + \lfloor -\frac{a}{n} \rfloor}^b$ on $\mathbb{Z}/n\mathbb{Z}$.

It remains to consider the types D_{2m+1} and E_6 . We either have $Z(G) \cong \mathbb{Z}/4\mathbb{Z}$ or $Z(G) \cong \mathbb{Z}/3\mathbb{Z}$. The action of σ must be either $a \mapsto -a$ or trivial, and again everything goes in the same way as in the A_n case. \square

Remark 4.9. Proposition 4.8 fails for the D_{2m} case in two ways. First, $H^2(P/Q; \mathbb{T})$ is no longer trivial. Next, as follows from Proposition 4.10 below, $\text{Aut}(\Psi)_{c,q}$ can be a proper subgroup of $\text{Aut}(\Psi)$, depending on the choice of c (but still independent of q).

If the action of $\text{Aut}(\Psi)$ on $H^3(P/Q; \mathbb{T})$ is nontrivial, then computation of $\text{Aut}(\Psi)_{c,q}$ is part of the more general task of finding when Φ^{c_1} and Φ^{c_2} define the same class in $H_{G_q}^3(\hat{G}_q; \mathbb{T})$.

When G is simple, the cocycles Φ^c exhaust the whole group $H^3(P/Q; \mathbb{T})$, as observed in [NY15, Proposition 2.6]. Therefore in this case what we want is to understand the map $H^3(P/Q; \mathbb{T}) \rightarrow H_{G_q}^3(\hat{G}_q)$. We have the following result.

Proposition 4.10. *If G is a simply connected simple compact Lie group, then the natural map $H^3(P/Q; \mathbb{T}) \rightarrow H_{G_q}^3(\hat{G}_q)$ is injective for any $q > 0$.*

Proof. For $G = SU(n)$, the assertion already follows from the Kazhdan–Wenzl theorem. We follow their scheme also for the other cases, cf. [KW93, Section 4]. The basic idea is to extract a numerical invariant of $(\text{Rep } G_q, \Phi^c)$ using morphisms determined up to scalar multiples by the fusion rules and their compositions, in a way such that the ambiguity of the scalar factors is absorbed by the symmetry of the overall formula.

Let us first consider the cases other than the type D_n . In these cases we know that P/Q can be identified with $\mathbb{Z}/k\mathbb{Z}$ for some k . Thus, $\text{Rep } G$ is graded over $\mathbb{Z}/k\mathbb{Z}$. Moreover, the group $H^3(P/Q; \mathbb{T})$ is also cyclic and we may enumerate its elements by the cocycles $\Phi^j(a, b, c) = \omega_j^{(\lfloor \frac{a+b}{k} \rfloor - \lfloor \frac{a}{k} \rfloor - \lfloor \frac{b}{k} \rfloor)c}$, with $\omega_j = e^{\frac{2\pi\sqrt{-1}j}{k}}$ and $j = 0, 1, \dots, k-1$.

Let us make the convention that for given $U \in \text{Rep } G_q$, c , and $k > 1$, the notation $U^{\otimes k}$ stands for the power of U computed in the category $(\text{Rep } G_q, \Phi^c)$ defined inductively as $U^{\otimes k} = U \oplus U^{\otimes(k-1)}$, so that one has $U^{\otimes k} = U \oplus (U \oplus \dots (U \oplus U) \dots)$. We want to show that there is a way to choose $U \in \text{Rep } G_q$ in the homogeneous component of $[1] \in \mathbb{Z}/k\mathbb{Z}$ such that $U^{\otimes k}$ contains a unique copy of $\mathbb{1}$, and such that for an isometric morphism $f: \mathbb{1} \rightarrow U^{\otimes k}$ in $(\text{Rep } G_q, \Phi^c)$, unique up to a phase factor, the composition

$$U \simeq \mathbb{1} \oplus U \xrightarrow{f \otimes \iota} (U \oplus U^{\otimes(k-1)}) \oplus U \xrightarrow{\Phi_{U, U^{\otimes(k-1)}, U}^j} U \oplus (U^{\otimes(k-1)} \oplus U) \xrightarrow{\iota \otimes f^*} U \oplus \mathbb{1} \simeq U \quad (4.2)$$

in the category $(\text{Rep } G_q, \Phi^j)$ is equal to $\omega_j a_q \iota_U$ for some nonzero number a_q which only depends on q . We note that there is an implicit use of the associator to go from $U \oplus (U^{\otimes(k-1)} \oplus U)$ to $U \oplus U^{\otimes k}$ in order for $\iota \otimes f^*$ to be applicable. However, since we have $\Phi^j(a, b, c) = 1$ for $0 \leq a, b, a+b < k$, that associator acts trivially. By definition of Φ^j we also see that $\Phi_{U, U^{\otimes(k-1)}, U}^j$ simply contributes by the factor ω_j . By rigidity of $\text{Rep } G_q$ we have $(\iota \otimes f^*)(f \otimes \iota) \neq 0$ in $\text{Rep } G_q$, once the multiplicity of $\mathbb{1}$ in $U^{\otimes k}$ equals one. Thus, all we have to show is that there exists $U \in \text{Rep } G_q$ in the homogeneous component of $[1] \in \mathbb{Z}/k\mathbb{Z}$ such that $U^{\otimes k}$ contains a unique copy of $\mathbb{1}$.

In the following argument we refer to standard texts such as [FH91] for the details on the representation theory of simple compact Lie groups.

For $k = 2$, we want to find a self-conjugate irreducible representation U in the homogeneous component of $[1] \in \mathbb{Z}/2\mathbb{Z}$ in $\text{Rep } G$. For the type B_n ($G = \text{Spin}(2n+1)$) we can take the spin representation as U . For the type C_n ($G = \text{Sp}(n)$), we can take the defining representation of $\mathfrak{sp}(2n, \mathbb{C})$ as U . For the type E_7 , the unique irreducible representation of dimension 56 is self-conjugate.

For $k = 3$, we only need to consider the type E_6 group. There are two irreducible representations of dimension 27, U_α and U_β , which are conjugate to each other. Moreover, the representation $U_\alpha \oplus U_\alpha$ contains U_β with multiplicity one (and the complement is spanned by two inequivalent irreducible representations of dimension 351). Hence the multiplicity of $\mathbb{1}$ in $U_\alpha^{\otimes 3}$ equals one.

Now, let us consider the type D_n case for n odd. In this case P/Q is still cyclic of order 4, but there is no irreducible representation U in the homogeneous component of $[1] \in \mathbb{Z}/4\mathbb{Z}$ such that $U^{\otimes 4}$ contains $\mathbb{1}$ with multiplicity one. However, we will see that there is still a way to specify $f: \mathbb{1} \rightarrow U^{\otimes 4}$ only using the fusion rules, such that the composition (4.2) is nonzero.

Let U_\pm denote the half-spin representations. In this case, U_+ is in the homogeneous component of $[1] \in \mathbb{Z}/4\mathbb{Z}$. Since the intertwiners $\mathbb{1} \rightarrow U_+^{\otimes 4}$ are not unique, we make the following choice. In terms of the defining representation $V \simeq \mathbb{C}^{2n}$ (the *vector representation*) of $\mathfrak{so}(2n, \mathbb{C})$, we have a decomposition into irreducible representations

$$U_+ \oplus U_+ \simeq U_{++} \oplus \wedge^{n-2} V \oplus \wedge^{n-4} V \oplus \dots \oplus V \quad (4.3)$$

in $\text{Rep } G_q$, with an additional representation U_{++} . Since the representation ring of G_q is the same as that of G , we continue to denote the corresponding irreducible representations in $\text{Rep } G_q$ by the same symbols U_{++} and $\wedge^k V$.

Since V is self-conjugate, it follows from (4.3) that the multiplicity of U_- in $V \oplus U_+$ equals one. Consider an isometry $f: \mathbb{1} \rightarrow U_+^{\otimes 4}$ obtained as the composition of morphisms $\mathbb{1} \rightarrow U_+ \oplus U_- \rightarrow U_+ \oplus V \oplus U_+$ and $U_+ \oplus V \oplus U_+ \rightarrow U_+^{\otimes 4}$, the latter induced by $V \rightarrow U_+^{\otimes 2}$. This determines f up to a phase factor. Then Theorem A.1 implies that $(\iota \otimes f^*)(f \otimes \iota) \neq 0$ in $\text{Rep } G_q$.

It remains to consider the type D_n case for n even. The group G is again $\text{Spin}(2n)$, but $Z(\text{Spin}(2n))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, consisting of I_{2n} in the Clifford algebra $M_{2n}(\mathbb{C})$, $I_{2n-1} \oplus (-I_{2n-1})$ (the chiral element), $-I_{2n}$ (the nontrivial element in the kernel of $\text{Spin}(2n) \rightarrow \text{SO}(2n)$), and $(-I_{2n-1}) \oplus I_{2n-1}$ (their product). In this case, we also know that the representations U_{\pm} are mutually inequivalent, irreducible, and self-conjugate, and that the decomposition of $U_+ \oplus U_-$ in $\text{Rep } G$ into irreducible representations is given by

$$\wedge^{n-1}V \oplus \wedge^{n-3}V \oplus \dots \oplus V.$$

As in the odd case, we continue to use the same symbols for representations of G_q . In terms of the grading of $\text{Rep } G_q$ over $P/Q \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, U_+ has degree $(1, 0)$ while U_- has degree $(0, 1)$, and V has $(1, 1)$. Moreover, the generators of $H^3(P/Q; \mathbb{T}) \simeq (\mathbb{Z}/2)^3$ can be represented by the following cocycles:

$$\begin{aligned} \phi_1((a, a'), (b, b'), (c, c')) &= (-1)^{abc}, & \phi_2((a, a'), (b, b'), (c, c')) &= (-1)^{a'b'c'}, \\ \phi_3((a, a'), (b, b'), (c, c')) &= (-1)^{abc'}. \end{aligned}$$

Note that the first two are induced by the embeddings $\mathbb{Z}/2\mathbb{Z} \rightarrow P/Q$. In terms of the central characters of U_{\pm} and V , denoting them by ω_{\pm} and ω_V , we have

$$\begin{aligned} \phi_1(\omega_+, \omega_+, \omega_+) &= \phi_1(\omega_V, \omega_V, \omega_V) = -1, & \phi_1(\omega_-, \omega_-, \omega_-) &= 1, \\ \phi_2(\omega_-, \omega_-, \omega_-) &= \phi_2(\omega_V, \omega_V, \omega_V) = -1, & \phi_2(\omega_+, \omega_+, \omega_+) &= 1, \\ \phi_3(\omega_+, \omega_+, \omega_+) &= \phi_3(\omega_-, \omega_-, \omega_-) = 1, & \phi_3(\omega_V, \omega_V, \omega_V) &= -1. \end{aligned}$$

Now, take a category of the form $(\text{Rep } G_q, \Phi^c)$. First, using each of U_{\pm} as U as in the above argument for with $k = 2$, one may test whether ∂c contains the classes of ϕ_1 and ϕ_2 . Next, using V in that argument, one may test if ∂c contains the class of ϕ_3 . This way it is possible to recover the cohomology class of ∂c from the self-duality morphisms of U_{\pm} and V . \square

Remark 4.11. It follows that $[\partial c]^2 \in H^3(P/Q; \mathbb{T})$ is a cohomological obstruction to the existence of braiding on $(\text{Rep } G_q; \Phi^c)$ for simple G , see [NY15, Remark 4.4].

Finally, let us make the following simple observation.

Proposition 4.12. *If $\text{Rep } G_{q_1}^{c_1}$ and $\text{Rep } G_{q_2}^{c_2}$ are unitarily monoidally equivalent for some $q_1, q_2 > 0$ and \mathbb{T} -valued 2-cochains c_1, c_2 on P such that $\partial c_1, \partial c_2$ descend to P/Q , then either $q_1 = q_2^{-1}$ or $q_1 = q_2$.*

Proof. It is well-known that there is an isomorphism between G_q and $G_{q^{-1}}$ which maps the maximal torus into the maximal torus. It follows that $G_q^c \cong G_{q^{-1}}^{c'}$ for some c' . Therefore without loss of generality we may assume that $q_1, q_2 \leq 1$. Next, a monoidal equivalence between $\text{Rep } G_{q_1}^{c_1}$ and $\text{Rep } G_{q_2}^{c_2}$ defines an automorphism of the representation semiring $R^+(G)$, so by the same argument as in the proof of Theorem 4.7 we conclude that for some $\sigma \in \text{Aut}(\Psi)$, $\text{Rep } G_{q_1}^{\sigma(c_1)}$ and $\text{Rep } G_{q_2}^{c_2}$ are monoidally equivalent via an equivalence which defines the identity map on $R^+(G)$.

Now, the proposition follows by observing that the quantum dimension on $(\text{Rep } G_q, \Phi^c)$ is independent of c , and if U_1 is any nontrivial irreducible representation of G_{q_1} and U_2 is the corresponding representation of G_{q_2} (via the identification of the representation rings), then $\dim_{q_1} U_1 > \dim_{q_2} U_2$ if $q_1 < q_2 \leq 1$. Indeed, as we already reminded before Lemma 4.2, the quantum dimension on $\text{Rep } G_q$ is defined by the trace of $q^{-2\rho^*}$, where $\rho^* \in \mathfrak{h}$ is the element corresponding to half the sum of positive roots. The element ρ^* acts in every representation of G by an operator with symmetric spectrum,

since the longest element in the Weyl group maps ρ^* into $-\rho^*$. Therefore our claim follows from the fact that the function $q \mapsto q + q^{-1}$ is strictly decreasing on the interval $(0, 1]$. \square

4.3. Isomorphisms. Let us summarize what kind of isomorphisms between the quantum groups G_q^c we have for $q \leq 1$. First, if b is a 2-cochain defined on P/Q , the quantum group G_q^{cb} is *by definition* the same as G_q^c . Second, if we replace c by $cd e$ for some 1-cochain e on P , we have the isomorphism $G_q^c \simeq G_q^{cd e}$ implemented on the level of $\mathcal{U}(G_q)$ by conjugation by e , which can be regarded as a unitary in $\mathcal{U}(T)$. Finally, any $\sigma \in \text{Aut}(\Psi)$ induces an automorphism of $\mathcal{U}(G_q)$, which in turn induces an isomorphism $G_q^c \cong G_q^{\sigma(c)}$.

The following theorem states that for $q < 1$ and simply connected simple groups this exhausts all the possibilities.

Theorem 4.13. *Let G be a simply connected simple compact Lie group, $q_1, q_2 \in (0, 1)$, and c_1, c_2 be \mathbb{T} -valued 2-cochains on P such that $\partial c_1, \partial c_2$ descend to P/Q . Then the quantum groups $G_{q_1}^{c_1}$ and $G_{q_2}^{c_2}$ are isomorphic if and only if $q_1 = q_2$ and there exist an element $\sigma \in \text{Aut}(\Psi)$ and a \mathbb{T} -valued 2-cochain b on P/Q such that $c_1 \sigma(c_2)^{-1} b^{-1}$ is a coboundary on P .*

Proof. The “if” part is clear from the discussion preceding the theorem. In order to prove the “only if” part let us assume that $G_{q_1}^{c_1} \cong G_{q_2}^{c_2}$. By Proposition 4.12 we have $q_1 = q_2 = q$. Next, the isomorphism defines an automorphism of the representation semiring of G , and consequently an element $\sigma^{-1} \in \text{Aut}(\Psi)$. Since $G_q^{c_2} \cong G_q^{\sigma(c_2)}$, replacing c_2 by $\sigma(c_2)$ we may assume that σ is trivial. Then the isomorphism $G_q^{c_1} \cong G_q^{c_2}$ gives us a unitary monoidal equivalence between $(\text{Rep } G_q, \Phi^{c_1})$ and $(\text{Rep } G_q, \Phi^{c_2})$ which preserves the isomorphism classes of objects of $\text{Rep } G_q$. Hence Φ^{c_1} and Φ^{c_2} define the same class in $H_{G_q}^3(\hat{G}_q; \mathbb{T})$. By Proposition 4.10 we conclude that the 3-cocycles ∂c_1 and ∂c_2 on P/Q are cohomologous. Let b be a 2-cochain on P/Q satisfying $(\partial c_1)(\partial c_2)^{-1} = \partial b$. Replacing c_2 by $c_2 b$ (which does not change $G_q^{c_2}$), we may further assume that $\partial c_1 = \partial c_2$.

Let $F: \text{Rep } G_q^{c_1} \rightarrow \text{Hilb}_f$ be the unitary fiber functor corresponding to the dual cocycle $c_2 c_1^{-1}$ on the maximal torus $T \subset G_q$. Then F defines the quantum group $G_q^{c_2}$, and our isomorphism $G_q^{c_1} \cong G_q^{c_2}$ defines an autoequivalence E of $\text{Rep } G_q^{c_1}$, which maps an object to an isomorphic object, such that FE is naturally unitarily monoidally isomorphic to the canonical fiber functor $\text{Rep } G_q^{c_1} \rightarrow \text{Hilb}_f$. Then, Proposition 4.6 implies that E is naturally unitarily monoidally isomorphic to the functor defined by a 2-cocycle b' on P/Q (trivial in all except the type D_{2n} case). Replacing once again c_2 by $c_2 b'$ we may assume that $b' = 1$. Then F is naturally unitarily monoidally isomorphic to the canonical fiber functor $\text{Rep } G_q^{c_1} \rightarrow \text{Hilb}_f$. It follows that $c_2 c_1^{-1}$, considered as a dual cocycle on $G_q^{c_1}$, is a coboundary. Finally, by Corollary 4.4 we conclude that the cocycle $c_2 c_1^{-1}$ on $P = \hat{T}$ is a coboundary. \square

Remark 4.14. The theorem is not true for $q = 1$, since as we already observed in Remark 4.5, the map $H^2(\hat{T}; \mathbb{T}) \rightarrow H^2(\hat{G}; \mathbb{T})$ is not injective.

5. CLASSIFICATION OF NON-KAC COMPACT QUANTUM GROUPS OF $SU(n)$ -TYPE UP TO ISOMORPHISM

For $G = SU(n)$ the results of the previous section can be further strengthened thanks to a classification theorem of Kazhdan and Wenzl [KW93]. It states that any semisimple rigid monoidal category with fusion rules of $SL(n)$ must be equivalent to one of the categories $(\text{Rep } SL_q(n), \Phi^c)$, where q is not a nontrivial root of unity and c is a 2-cochain as in the previous section. The corresponding result for C^* -tensor categories states that any rigid C^* -tensor category with fusion rules of $SU(n)$ is unitarily monoidally equivalent to $(\text{Rep } SU_q(n), \Phi^c)$ for some $q \in (0, 1]$ and c , see [Jor14] for details, as well as [Pin07; PR11, Section 7] for related slightly weaker results.

Let us say that a compact quantum group G is of $SU(n)$ -type if there is a dimension-preserving isomorphism of the representation semirings $R^+(G) \cong R^+(SU(n))$. Then the fact that any rigid C^* -tensor category with fusion rules of $SU(n)$ is unitarily monoidally equivalent to $\text{Rep } SU_q^c(n)$, together with Corollary 4.4 implying that the class of quantum groups $SU_q^c(n)$ is closed under cocycle twisting for $q \neq 1$, show that any non-Kac compact quantum group of $SU(n)$ -type is isomorphic to $SU_q^c(n)$. Combined with Theorem 4.13 this gives a complete classification of such compact quantum groups up to isomorphism, answering, in the non-Kac case, the question of Woronowicz raised at the end of [Wor88].

To formulate the result more precisely, recall that for $SU(n)$ the group $\text{Aut}(\Psi)$ contains one nontrivial automorphism θ . Its action on $\mathcal{U}(SU_q(n))$, which already appeared in the proof of Proposition 4.8, is characterized by $E_i \mapsto E_{n-i}$, $1 \leq i \leq n-1$. Restricted to the maximal torus T , this action can be expressed as

$$\text{diag}(t_1, \dots, t_n) \mapsto \text{diag}(t_n^{-1}, \dots, t_1^{-1}). \quad (5.1)$$

Our main classification result can now be formulated as follows.

Theorem 5.1. *Let G be a non-Kac compact quantum group of $SU(n)$ -type for some $n \geq 2$. Then $G \cong SU_q^c(n)$ for some $q \in (0, 1)$ and a \mathbb{T} -valued 2-cochain c on $\hat{T} \cong \mathbb{Z}^{n-1}$ such that ∂c descends to a 3-cocycle on $Z(\widehat{SU(n)}) \cong \mathbb{Z}/n\mathbb{Z}$. Furthermore, two such quantum groups $SU_{q_1}^{c_1}(n)$ and $SU_{q_2}^{c_2}(n)$ are isomorphic if and only if $q_1 = q_2$ and there exists a \mathbb{T} -valued 2-cochain b on $Z(\widehat{SU(n)})$ such that either $c_1 c_2^{-1} b^{-1}$ or $c_1 \theta(c_2)^{-1} b^{-1}$ is a coboundary on \hat{T} .*

Let us now find an explicit parameter set for the quantum groups G of $SU(n)$ -type. Such a set can be obtained by the following procedure. Choose cochains c_1, \dots, c_n such that ∂c_i exhaust the group $H^3(Z(\widehat{SU(n)}); \mathbb{T}) \cong \mathbb{Z}/n\mathbb{Z}$. Then $G \cong SU_q^{c_k \omega}(n)$ for some $q \in (0, 1)$, $1 \leq k \leq n$ and a skew-symmetric bicharacter ω on \hat{T} . The numbers q and k are uniquely determined, and the above theorem tells us what the ambiguity in the choice of ω is.

In order to present c_k and ω concretely, it is convenient to enlarge the maximal torus $T \simeq \mathbb{T}^{n-1}$ of $SU(n)$ to that of $U(n)$, that is, to the group $\tilde{T} \cong \mathbb{T}^n$ of diagonal unitary matrices. Lifting 2-cocycles on \hat{T} to the group $\hat{\tilde{T}} \cong \mathbb{Z}^n$ with the base $(L_i)_{i=1}^n$ dual to the basis $(E_{ii})_{i=1}^n$ of the Lie algebra of diagonal matrices, we conclude that up to coboundaries such cocycles are represented by skew-symmetric bicharacters $\omega: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{T}$ which satisfy $\omega(L_1 + \dots + L_n, x) = 1$ for any $x \in \mathbb{Z}^n$. Putting $\omega_{ij} = \omega(L_i, L_j) \in \mathbb{T}$, the matrix $(\omega_{ij})_{i,j=1}^n$ satisfies $\omega_{ii} = 1$, $\omega_{ji} = \bar{\omega}_{ij}$ and $\prod_i \omega_{ij} = 1$ for any j . Two such matrices $\omega = (\omega_{ij})_{i,j}$ and $\tilde{\omega} = (\tilde{\omega}_{ij})_{i,j}$ represent the same element of $H^2(\hat{\tilde{T}}; \mathbb{T})$ if and only if $\omega_{ij}^2 = \tilde{\omega}_{ij}^2$ for all i, j .

As we already mentioned in the proof of Theorem 4.1, explicit examples of cochains c were constructed in [NY15]. Namely, every $(n-1)$ -tuple $\tau = (\tau_1, \dots, \tau_{n-1})$ of roots of unity of order n defines a cochain c_τ such that

$$c_\tau(\lambda, \mu + \alpha_i) = c_\tau(\lambda, \mu) \quad \text{and} \quad c_\tau(\lambda + \alpha_i, \mu) = \tau_i^{-|\mu|} c_\tau(\lambda, \mu),$$

where $\alpha_i = L_i - L_{i+1}$ are the simple roots and $|\cdot|: P \rightarrow \mathbb{Z}$ is defined by $|L_1| = n-1$, $|L_i| = -1$ for $2 \leq i \leq n$. Such a cochain is not uniquely defined, but the ambiguity only contributes to a cochain on P/Q and the corresponding quantum group $SU_q^\tau(n) = SU_q^{c_\tau}(n)$ is independent of any choices. Moreover, by [NY15, Proposition 4.1] we have $[\partial c_\tau] = [\partial c_\nu]$ in $H^3(P/Q; \mathbb{T})$ if and only if

$$\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \nu_i^i.$$

Therefore the required representatives c_k can be obtained by taking, for example,

$$\tau^{(k)} = (e^{2\pi(k-1)i/n}, 1, \dots, 1).$$

We next have to deform $SU_q^{c_\tau}(n)$ by a skew-symmetric bicharacter ω on \hat{T} to get a quantum group $SU_q^{c_\tau, \omega}(n)$, which we will also denote by $SU_q^{\tau, \omega}(n)$. In terms of the quantum groups $SU_q^{\tau, \omega}(n)$ Theorem 5.1 can be reformulated as follows.

Theorem 5.2. *Let G be a non-Kac compact quantum group of $SU(n)$ -type for some $n \geq 2$. Then $G \cong SU_q^{\tau, \omega}(n)$ for some $q \in (0, 1)$, τ and ω as above. Furthermore, two such quantum groups $SU_q^{\tau, \omega}(n)$ and $SU_{q'}^{\tau', \omega'}(n)$ are isomorphic if and only if $q = q'$, $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tau'_i{}^i$ and one of the following holds for $\omega_{ij} = \omega(L_i, L_j)$ and $\omega'_{ij} = \omega'(L_i, L_j)$:*

- (i) $\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \omega'_{ij}{}^2 \prod_{k=i}^{j-1} \tau'_k$ for all $1 \leq i < j \leq n-1$;
- (ii) $\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \omega'_{n-i+1, n-j+1}{}^2 \prod_{k=i}^{j-1} \tau'_{n-k}$ for all $1 \leq i < j \leq n-1$.

Proof. We only have to show that the condition that there exists a cochain b on P/Q such that $c_\tau \omega (c_{\tau'} \omega')^{-1} b^{-1}$ (resp., $c_\tau \omega \theta (c_{\tau'} \omega')^{-1} b^{-1}$) is a coboundary, is equivalent to $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tau'_i{}^i$ and condition (i) (resp., condition (ii)).

Consider the first case. Let us first of all observe that the condition on the existence of b is equivalent to saying that the 3-cocycles $\partial(c_\tau \omega)$ and $\partial(c_{\tau'} \omega')$ on P/Q are cohomologous and if

$$\partial(c_\tau \omega) \partial(c_{\tau'} \omega')^{-1} = \partial b \quad (5.2)$$

holds for a 2-cochain b on P/Q , then $c_\tau \omega (c_{\tau'} \omega')^{-1} b^{-1}$ is a coboundary on P . This is an immediate consequence of the triviality of the group $H^2(P/Q; \mathbb{T})$, as it implies that the cochain b satisfying (5.2) is uniquely determined up to a coboundary on P/Q .

Now, as we already stated before the theorem, the condition that $\partial(c_\tau \omega)$ and $\partial(c_{\tau'} \omega')$ are cohomologous is equivalent to $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tau'_i{}^i$, since ω and ω' are cocycles. Once this condition is satisfied, we can define a bicharacter f on P by

$$f(L_i, L_j) = \prod_{k=i}^{n-1} \tau_k \tau'_k.$$

It has the property that

$$f(\lambda, \mu + \alpha_i) = f(\lambda, \mu), \quad f(\lambda + \alpha_i, \mu) = \tau_i^{-|\mu|} \tau'_i{}^{|\mu|} f(\lambda, \mu).$$

It follows that $c_\tau c_{\tau'}^{-1} = f b$ for a 2-cochain b on P/Q . Since f, ω and ω' are 2-cocycles, b satisfies (5.2). Therefore the condition that $c_\tau \omega (c_{\tau'} \omega')^{-1} b^{-1}$ is a coboundary means that the cocycle $f \omega \omega'^{-1}$ is a coboundary. This is equivalent to saying that this cocycle is symmetric, and since it is a bicharacter and the bicharacters ω and ω' are skew-symmetric, this is equivalent to

$$f(L_i, L_j) \omega(L_i, L_j)^2 = f(L_j, L_i) \omega'(L_i, L_j)^2 \quad \text{for } 1 \leq i < j \leq n-1.$$

This is exactly condition (i) in the formulation of the theorem.

Turning to the second case, we see from (5.1) that $\theta(c_{\tau'} \omega')$ is equal to $c_{\tau'} \tilde{\omega}'$, given by $\tilde{\tau}'_i = \tau'_{n-i}$ and $\tilde{\omega}'(L_i, L_j) = \omega'(L_{n-i+1}, L_{n-j+1})$. We then proceed as in the first case. \square

Finally, let us present explicit generators and relations of $\mathbb{C}[SU_q^{\tau, \omega}(n)]$. The algebra $\mathbb{C}[SU_q^\tau(n)]$ is generated by the matrix coefficients v_{ij} , $1 \leq i, j \leq n$, of the canonical n -dimensional representation. Then $\mathbb{C}[SU_q^{\tau, \omega}(n)] = \mathbb{C}[SU_q^\tau(n)]$ as coalgebras, while the new product \cdot_ω is determined by the following rule: if $x, y \in \mathbb{C}[SU_q^\tau(n)]$ are such that

$$(\pi \otimes \iota \otimes \pi) \Delta^{(2)}(x) = z_i \otimes x \otimes z_j \quad \text{and} \quad (\pi \otimes \iota \otimes \pi) \Delta^{(2)}(y) = z_k \otimes x \otimes z_l,$$

where $\pi: \mathbb{C}[\mathrm{SU}_q^T(n)] \rightarrow \mathbb{C}[T]$ is the restriction map and we write $t = \mathrm{diag}(z_1(t), \dots, z_n(t))$ for elements $t \in T$, so that $\pi(v_{ij}) = \delta_{ij}z_i$, then

$$x \cdot_{\omega} y = \omega_{ik}\bar{\omega}_{jl}xy.$$

From the relations in $\mathbb{C}[\mathrm{SU}_q^T(n)]$ given in [NY15, Section 4.3] we conclude that $\mathbb{C}[\mathrm{SU}_q^{\tau, \omega}]$ can be described as the universal algebra with generators v_{ij} , $1 \leq i, j \leq n$, and relations

$$\begin{aligned} v_{ij}v_{il} &= \left(\prod_{j \leq p < l} \tau_p^{-1} \right) q \bar{\omega}_{jl}^2 v_{il}v_{ij} \quad (j < l), \quad v_{ij}v_{kj} = \left(\prod_{i \leq p < k} \tau_p \right) q \omega_{ik}^2 v_{kj}v_{ij} \quad (i < k), \\ v_{ij}v_{kl} &= \left(\prod_{k \leq p < i} \tau_p^{-1} \right) \left(\prod_{j \leq p < l} \tau_p^{-1} \right) \omega_{ik}^2 \bar{\omega}_{jl}^2 v_{kl}v_{ij} \quad (i > k, j < l), \\ \left(\prod_{j \leq p < l} \tau_p \right) \omega_{jl}^2 v_{ij}v_{kl} - \left(\prod_{i \leq p < k} \tau_p \right) \bar{\omega}_{ki}^2 v_{kl}v_{ij} &= (q - q^{-1})v_{il}v_{kj} \quad (i < k, j < l), \end{aligned}$$

and

$$\sum_{\sigma \in S_n} \tau^{m(\sigma)} (-q)^{|\sigma|} \bar{\omega}(1, \dots, n) \omega(\sigma(1), \dots, \sigma(n)) v_{1\sigma(1)} \cdots v_{n\sigma(n)} = 1,$$

where $m(\sigma) = (m(\sigma)_1, \dots, m(\sigma)_{n-1})$ is the multi-index given by $m(\sigma)_i = \sum_{k=2}^n (k-1)m_i^{(k, \sigma(k))}$, with

$$m_i^{(k, j)} = \begin{cases} 1, & \text{if } k \leq i < j, \\ -1, & \text{if } j \leq i < k, \\ 0, & \text{otherwise,} \end{cases}$$

and the function $\omega(i_1, \dots, i_n)$ is defined by $\prod_{k < l} \omega_{ik, il}$. The $*$ -structure is uniquely determined by requiring the invertible matrix $(v_{ij})_{i, j}$ to be unitary.

APPENDIX A. CLEBSCH–GORDAN COEFFICIENTS OF $\mathrm{Spin}(2n)$

Throughout this appendix $n \geq 3$ is an odd integer. We review an explicit decomposition of tensor powers of the half-spin representations of $\mathrm{Spin}(2n)$ (cf. [FH91]), and its extension to $\mathrm{Spin}_q(2n)$. Our goal is to prove the following theorem.

Theorem A.1. *Let $q > 0$, and $f: \mathbf{1} \rightarrow U_+^{\otimes 4}$ be the unique up to a phase factor isometric embedding of $\mathbf{1}$ into $U_+^{\otimes 4}$ in $\mathrm{Rep} \mathrm{Spin}_q(2n)$ which factors through $U_+ \otimes V \otimes U_+$. Then one has $(\iota \otimes f^*)(f \otimes \iota) \neq 0$.*

A.1. Classical case. Let us first consider the classical case $q = 1$. Let V be a complex vector space of dimension $2n$ endowed with a nondegenerate quadratic form $Q(\xi)$. The associated symmetric bilinear form is denoted by $(\xi, \eta)_Q = \frac{1}{2}(Q(\xi + \eta) - Q(\xi) - Q(\eta))$, so that one has $Q(\xi) = (\xi, \xi)_Q$. The vector space $\wedge^2 V$ becomes a Lie algebra with Lie bracket

$$[\xi \wedge \eta, \xi' \wedge \eta'] = (\eta, \eta')_Q \xi \wedge \xi' - (\eta, \xi')_Q \xi \wedge \eta' + (\xi, \xi')_Q \eta \wedge \eta' - (\xi, \eta')_Q \eta \wedge \xi',$$

which is isomorphic to $\mathfrak{so}(2n, \mathbb{C})$. It acts on V by $(\xi \wedge \eta) \cdot \zeta = -(\eta, \zeta)_Q \xi + (\xi, \zeta)_Q \eta$. We denote the coproduct on the universal enveloping algebra by $\hat{\Delta}$.

Let W be a maximal isotropic subspace of V . We identify its dual W^* with a subspace of V by means of Q . Let e_1, \dots, e_n be a basis of W , and (e_{n+1}, \dots, e_{2n}) be the dual basis in $W^* \subset V$. We represent the linear operators on V using the matrix units $E_{i, j}$ with respect to (e_1, \dots, e_{2n}) . For example, $e_{n+2} \wedge e_1$ is represented by $E_{1, 2} - E_{n+2, n+1}$.

For a Cartan subalgebra \mathfrak{h} of $\wedge^2 V$, we take the linear span of the elements $e_{n+i} \wedge e_i = E_{i, i} - E_{n+i, n+i}$ for $i = 1, \dots, n$, and let $(L_i)_{i=1}^n \subset \mathfrak{h}^*$ be the dual weights, so $(L_i, e_{n+j} \wedge e_j) = \delta_{i, j}$. A standard choice of simple positive roots is $\alpha_i = L_i - L_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha_n = L_{n-1} + L_n$. The corresponding generators of the Lie algebra are $X_i = e_{n+i+1} \wedge e_i = E_{i, i+1} - E_{n+i+1, n+i}$ for $i = 1, \dots, n-1$ and $X_n = e_n \wedge e_{n-1} = E_{n-1, 2n} - E_{n, 2n-1}$, $Y_i = e_{n+i} \wedge e_{i+1} = E_{i+1, i} - E_{n+i, n+i+1}$ for $i = 1, \dots, n-1$ and

$Y_n = e_{2n} \wedge e_{2n-1} = E_{2n-1,n} - E_{2n,n-1}$. We note that V has the highest weight L_1 , with the highest weight vector e_1 .

The complex Clifford algebra $\text{Cl}(Q)$ associated with Q is generated by $c(\xi)$ for $\xi \in V$, subject to the relations

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2(\xi, \eta)_Q = Q(\xi) + Q(\eta) - Q(\xi + \eta).$$

Considering $\text{Cl}(Q)$ as a Lie algebra with the commutator bracket, we have a Lie algebra embedding $\wedge^2 V \rightarrow \text{Cl}(Q)$ given by $\xi \wedge \eta \mapsto \frac{1}{4}[c(\xi), c(\eta)] = \frac{1}{2}c(\xi)c(\eta) + \frac{1}{2}(\xi, \eta)_Q$. Then the action of $\xi \wedge \eta \in \wedge^2 V$ on V can be identified with the adjoint action of $\frac{1}{4}[c(\xi), c(\eta)]$ on $c(\zeta)$ for $\zeta \in V$. In the following we put $c(e_i) = c_i$ for $i = 1, \dots, 2n$.

Put $S = \wedge^* W$, $U_+ = \wedge^{\text{odd}} W$, and $U_- = \wedge^{\text{even}} W$. We have a representation of $\text{Cl}(Q)$ on S given by $c_i w = \sqrt{2}e_i \wedge w$ and $c_{n+i} w = -\sqrt{2}e_i \lrcorner w$ for $i = 1, \dots, n$. Then, the induced action of $\wedge^2 V$ preserves both U_+ and U_- . If $X = \{x_1 < x_2 < \dots < x_k\}$ is a subset of $\{1, \dots, n\}$, we let e_X denote the vector $e_{x_1} \wedge \dots \wedge e_{x_k}$. Then $e_{n+i} \wedge e_i$ acts as the multiplication by $-\frac{1}{2}$ on e_X if i does not belong to X , and by $\frac{1}{2}$ if it does. In other words,

$$\text{the vector } e_X \text{ has weight } \frac{1}{2} \sum_{i \in X} L_i - \frac{1}{2} \sum_{i \notin X} L_i. \quad (\text{A.1})$$

The representation U_+ has the highest weight $\frac{1}{2}(L_1 + \dots + L_n)$, while U_- has the lowest weight $-\frac{1}{2}(L_1 + \dots + L_n)$ (they are conjugate to each other). One sees that the corresponding highest/lowest weight vectors are given by $e_{\{1, \dots, n\}} = e_1 \wedge \dots \wedge e_n \in \wedge^n W \subset U_+$ and $e_\emptyset = 1 \in \wedge^0 W \subset U_-$.

Now, put $v_i = \frac{\sqrt{-1}}{\sqrt{2}}(e_i - e_{n+i})$ and $v_{n+i} = \frac{1}{\sqrt{2}}(e_i + e_{n+i})$ for $i = 1, \dots, n$, and denote their span over \mathbb{R} by V_0 . Then Q restricts to a positive definite bilinear form on V_0 , with the orthonormal basis v_1, \dots, v_{2n} . Thus, $\wedge^2 V_0$ is a real Lie subalgebra of $\wedge^2 V$ which is isomorphic to $\mathfrak{so}(2n, \mathbb{R})$. Moreover, if we define a Hermitian inner product on S so that the vectors e_X ($X \subset \{1, \dots, n\}$) form an orthonormal basis, the elements $c(v_i)$ ($i = 1, \dots, 2n$) act as skew-adjoint operators. Consequently, our model of $\text{Spin}(2n)$ is the closed connected subgroup of $\text{GL}_1(\text{Cl}(Q))$ with Lie algebra $\wedge^2 V_0$, and the half-spin representations are U_\pm with the orthonormal bases consisting of vectors e_X with odd/even $|X|$. Note also that that we can define a $\wedge^2 V_0$ -invariant Hermitian inner product on V by letting $(e_i)_{i=1}^{2n}$ to be an orthonormal basis.

In order to find a copy of V in $U_+ \otimes U_+$ (see the decomposition (4.3)), we need to find a highest weight vector of weight L_1 . For this purpose, given a subset $X \subset \{1, \dots, n\}$, let us define

$$I_1(X) = \sum_{k \in X} k - |X|n \quad \text{and} \quad \sigma_1(X) = (-1)^{I_1(X)}.$$

Let us also consider the set Ω_1 of pairs of sets (X', X'') such that $X' \cup X'' = \{1, \dots, n\}$, $X' \cap X'' = \{1\}$ and $|X'|$ is odd. Then, we claim that

$$\tilde{e}_1 = \sum_{(X', X'') \in \Omega_1} \sigma_1(X') e_{X'} \otimes e_{X''}$$

is a highest weight vector of weight L_1 in $U_+ \otimes U_+$. That the weight is correct follows immediately from (A.1). To see that \tilde{e}_1 belongs to the kernel of the operators X_i , we compute $\hat{\Delta}(X_i)\tilde{e}_1$ as follows. If $i = 1$, then we clearly get zero, since $e_1 \wedge e_X = 0$ for any X containing 1. Consider now $2 \leq i \leq n-1$. The action of $\hat{\Delta}(e_{n+i+1} \wedge e_i)$ is given by the sum of the actions by $-e_{i+1} \lrcorner e_i \wedge$ on the tensor components, so the image of \tilde{e}_1 can be expressed as

$$\sum_{\substack{(X', X'') \in \Omega_1 \\ i \notin X', i+1 \in X'}} \sigma_1(X') e_{X' \cup \{i\} \setminus \{i+1\}} \otimes e_{X''} + \sum_{\substack{(Y', Y'') \in \Omega_1 \\ i \in Y', i+1 \notin Y'}} \sigma_1(Y') e_{Y'} \otimes e_{Y'' \cup \{i\} \setminus \{i+1\}}.$$

We see that X' in the first sum and $Y' = X' \cup \{i\} \setminus \{i+1\}$ in the second contribute with the same terms up to a sign. Since $I_1(X') = I_1(Y') + 1$, these terms in fact cancel. Thus, $\hat{\Delta}(e_{n+i} \wedge e_i)\tilde{e}_1 = 0$.

As for the remaining element $X_n = e_n \wedge e_{n-1}$, we compute $\hat{\Delta}(X_n)\tilde{e}_1$ as

$$- \sum_{\substack{(X', X'') \in \Omega_1 \\ n-1, n \notin X'}} \sigma_1(X')e_{X' \cup \{n-1, n\}} \otimes e_{X''} - \sum_{\substack{(Y', Y'') \in \Omega_1 \\ n-1, n \in Y'}} \sigma_1(Y')e_{Y'} \otimes e_{Y'' \cup \{n-1, n\}}.$$

This time, the term in the first sum corresponding to some X' cancels the term in the second sum corresponding to $Y' = X' \cup \{n-1, n\}$, since $I_1(X') = I_1(Y') + 1$.

Now, let us find the images \tilde{e}_i of e_i under the intertwiner $V \rightarrow U_+ \otimes U_+$ mapping e_1 to \tilde{e}_1 . This can be done by observing that $e_{i+1} = Y_i e_i$ and $e_{n+i} = -Y_i e_{n+i+1}$ for $1 \leq i \leq n-1$, and $e_{2n} = -Y_n e_{n-1}$. In order to formulate the result, we need to introduce more notation. For $X \subset \{1, \dots, n\}$ and $1 \leq i \leq n$ put

$$\begin{aligned} I_i(X) &= I_1(X) + |\{k \in X \mid k < i\}| - (i-1), & \sigma_i(X) &= (-1)^{I_i(X)}, \\ I_{n+i}(X) &= I_1(X) + |\{k \in X \mid k < i\}| - (n-1), & \sigma_{n+i}(X) &= (-1)^{I_{n+i}(X)}, \end{aligned}$$

and

$$\begin{aligned} \Omega_i &= \{(X', X'') \mid X' \cup X'' = \{1, \dots, n\}, X' \cap X'' = \{i\}, |X'| \text{ is odd}\}, \\ \Omega_{n+i} &= \{(X', X'') \mid X' \cup X'' = \{1, \dots, n\} \setminus \{i\}, X' \cap X'' = \emptyset, |X'| \text{ is odd}\}. \end{aligned}$$

Then

$$\tilde{e}_i = \sum_{(X', X'') \in \Omega_i} \sigma_i(X')e_{X'} \otimes e_{X''}, \quad \tilde{e}_{n+i} = (-1)^i \sum_{(X', X'') \in \Omega_{n+i}} \sigma_{n+i}(X')e_{X'} \otimes e_{X''}. \quad (\text{A.2})$$

Let us check, for example, that $\tilde{e}_{2n} = -\hat{\Delta}(Y_n)\tilde{e}_{n-1}$. As $Y_n = e_{2n} \wedge e_{2n-1}$, the vector $-\hat{\Delta}(Y_n)\tilde{e}_{n-1}$ equals

$$- \sum_{\substack{(X', X'') \in \Omega_{n-1} \\ n \in X'}} \sigma_{n-1}(X')e_{X' \setminus \{n-1, n\}} \otimes e_{X''} - \sum_{\substack{(Y', Y'') \in \Omega_{n-1} \\ n \notin Y'}} \sigma_{n-1}(Y')e_{Y'} \otimes e_{Y'' \setminus \{n-1, n\}}.$$

This expression equals \tilde{e}_{2n} , since for any X' in the first sum we have $I_{n-1}(X') = I_{2n}(X' \setminus \{n-1, n\})$ and for any Y' in the second sum we have $I_{n-1}(Y') = I_{2n}(Y')$.

Next, since V is irreducible and self-conjugate, $U_+ \otimes V$ contains U_- with multiplicity one. In order to find this inclusion we need to find a lowest weight vector of weight $-\frac{1}{2}(L_1 + \dots + L_n)$. It is given by

$$e_1 \otimes e_{n+1} + \dots + e_n \otimes e_{2n}. \quad (\text{A.3})$$

Again, the weight is correct by (A.1), since the weight of $e_{n+i} \in V$ is $-L_i$. To see that we indeed get a lowest weight vector, we can compute the action of $Y_i = e_{n+i} \wedge e_{i+1}$ as

$$-e_i \lrcorner e_{i+1} \wedge e_i \otimes e_{n+i} + e_{i+1} \otimes (E_{i+1, i} - E_{n+i, n+i+1})e_{n+i+1} = e_{i+1} \otimes e_{n+i} - e_{i+1} \otimes e_{n+i} = 0,$$

while the action of $Y_n = e_{2n} \wedge e_{2n-1}$ clearly gives zero, since $e_i \in U_+$ and $e_{n+i} \in V$ are both annihilated by Y_n .

It follows that the unique up to a scalar factor inclusion $g: U_- \rightarrow U_+^{\otimes 3}$ which factors through $U_+ \otimes V$ is characterized by $1 \mapsto \sum_{i=1}^n e_i \otimes \tilde{e}_{n+i}$. Similarly, we can define another inclusion $h: U_- \rightarrow U_+^{\otimes 3}$ which factors through $V \otimes U_+$, by $1 \mapsto \sum_{i=1}^n \tilde{e}_{n+i} \otimes e_i$.

Proposition A.2. *We have $g^*h = (-1)^{\frac{n+1}{2}}n(n-1)$.*

Proof. We just need to compute the inner product of the corresponding lowest weight vectors $\sum_{i=1}^n e_i \otimes \tilde{e}_{n+i}$ and $\sum_{i=1}^n \tilde{e}_{n+i} \otimes e_i$ inside $U_+^{\otimes 3}$. Putting $X_{i,j} = \{1, \dots, n\} \setminus \{i, j\}$, we see that the only contributing terms are

$$\sum_{i \neq j} (-1)^i \sigma_{n+i}(X_{i,j})e_i \otimes e_{X_{i,j}} \otimes e_j$$

from $\sum_{i=1}^n e_i \otimes \tilde{e}_{n+i}$, and

$$\sum_{i \neq j} (-1)^j \sigma_{n+j}(\{i\}) e_i \otimes e_{X_{i,j}} \otimes e_j$$

from $\sum_{i=1}^n \tilde{e}_{n+j} \otimes e_j$. If $i > j$, we have $I_{n+i}(\{j\}) = -2n + j + 2$ and

$$I_{n+i}(X_{i,j}) = -\frac{n(n-3)}{2} - j - 1,$$

so that $\sigma_{n+i}(\{j\}) = (-1)^j$ and $\sigma_{n+i}(X_{i,j}) = (-1)^{\frac{n+1}{2}+j+1}$. On the other hand, if $i < j$, then $I_{n+i}(\{j\}) = -2n + j + 1$ and $I_{n+i}(X_{i,j}) = -\frac{n(n-3)}{2} - j$, so that $\sigma_{n+i}(\{j\}) = (-1)^{j+1}$ and $\sigma_{n+i}(X_{i,j}) = (-1)^{\frac{n+1}{2}+j}$. Thus in any case we have

$$(-1)^{i+j} (\sigma_{n+i}(X_{i,j}) e_i \otimes e_{X_{i,j}} \otimes e_j, \sigma_{n+j}(\{i\}) e_i \otimes e_{X_{i,j}} \otimes e_j) = (-1)^{\frac{n+1}{2}}, \quad (\text{A.4})$$

and the assertion follows. \square

Proof of Theorem A.1 for $q = 1$. By the uniqueness of the embedding of $\mathbb{1}$ inside $U_+ \otimes V \otimes U_+$, we may also regard f either as the composition of $\bar{R}_+ : \mathbb{1} \rightarrow U_+ \otimes U_-$ and $\lambda \otimes h$, or as the composition of $R_+ : \mathbb{1} \rightarrow U_- \otimes U_+$ and $\mu g \otimes \iota$, for some $\lambda, \mu \in \mathbb{C}^\times$. Then the conjugate equations and the previous proposition imply the assertion. \square

A.2. Quantum case. Next let us move on to the quantum case with $q > 0$, $q \neq 1$. We follow the conventions of [NT13] for the quantized universal enveloping algebra $U_q(\mathfrak{so}_{2n})$: it is generated by elements $E_1, \dots, E_n, F_1, \dots, F_n$, and $K_1^{\pm 1}, \dots, K_n^{\pm 1}$ subject to a standard set of relations based on the root datum. For our purpose, it is convenient to consider the elements $X_i = K_i^{-\frac{1}{2}} E_i$ and $Y_i = X_i^* = F_i K_i^{\frac{1}{2}}$, so that one has

$$\hat{\Delta}_q(X_i) = X_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes X_i, \quad \hat{\Delta}_q(Y_i) = Y_i \otimes K_i^{-\frac{1}{2}} + K_i^{\frac{1}{2}} \otimes Y_i. \quad (\text{A.5})$$

Keeping the Hermitian forms on V and U_\pm , we have unitary representations of $U_q(\mathfrak{so}_{2n})$ on these spaces given by the same formulas for the elements X_i and Y_i as in the case $q = 1$, see [Ros90]. To see this, consider $H_i = [X_i, Y_i]$ computed in \mathfrak{so}_{2n} , so that we have $H_i = e_{n+i} \wedge e_i - e_{n+i+1} \wedge e_{i+1}$ for $i < n$ and $H_n = e_n \wedge e_{2n} + e_{n-1} \wedge e_{2n-1}$. We then observe that since the weights appearing in V are of the form $\pm L_i$, the elements H_i act with eigenvalues $0, \pm 1$, and therefore

$$\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} = H_i \quad \text{on } V.$$

We let K_i act on V as q^{H_i} . The quantum Serre relations are trivially satisfied, e.g., because $X_i^2 = 0$ on V and therefore the quantum Serre relations for the elements X_i are equivalent to the classical ones. Similar arguments apply to U_+ and U_- , since by (A.1) the elements H_i again act with eigenvalues $0, \pm 1$.

We thus need to find intertwiners $V \rightarrow U_+ \otimes U_+$ and $U_- \rightarrow U_+ \otimes V$ with respect to the new coproduct (A.5) for the same representations of X_i and Y_i on U_\pm and V as for $q = 1$. The formulas are similar to (A.2) and (A.3) but involve extra factors of q . More precisely, put $\sigma_i^q(X) = (-q)^{I_i(X)}$ for $1 \leq i \leq 2n$. Then replacing $\sigma_i(X)$ by $\sigma_i^q(X)$ in (A.2) we get vectors $\tilde{e}_i^q \in U_+ \otimes U_+$ defining an embedding $V \rightarrow U_+ \otimes U_+$ which maps $e_i \in V$ into \tilde{e}_i^q . This is proved similarly to the case $q = 1$. For example, we compute $\hat{\Delta}_q(X_i) \tilde{e}_1^q$ for $2 \leq i \leq n-1$ as

$$\sum_{\substack{(X', X'') \in \Omega_1 \\ i \notin X', i+1 \in X'}} q^{-\frac{1}{2}} \sigma_1^q(X') e_{X' \cup \{i\} \setminus \{i+1\}} \otimes e_{X''} + \sum_{\substack{(Y', Y'') \in \Omega_1 \\ i \in Y', i+1 \notin Y'}} q^{\frac{1}{2}} \sigma_1^q(Y') e_{Y'} \otimes e_{Y'' \cup \{i\} \setminus \{i+1\}}.$$

As in the case $q = 1$, if we take some X' in the first sum and then consider $Y' = X' \cup \{i\} \setminus \{i+1\}$ in the second, then the corresponding terms cancel, since $I_1(X') = I_1(Y') + 1$. Thus $\hat{\Delta}_q(X_i) \tilde{e}_1^q = 0$.

Concerning (A.3), a correct formula for the lowest weight vector defining an embedding $U_- \rightarrow U_+ \otimes V$ is

$$\sum_{i=1}^n q^i e_i \otimes e_{n+i},$$

while an embedding $U_- \rightarrow V \otimes U_+$ can be defined using the lowest weight vector

$$\sum_{j=1}^n q^{-j} e_{n+j} \otimes e_j.$$

Again, this is easy to check similarly to the case $q = 1$.

Proof of Theorem A.1 for $q \neq 1$. We need to establish an analogue of Proposition A.2. The only difference from the case $q = 1$ is that in the analogue of equation (A.4) we get an extra factor $q^{i-j+I_{n+i}(X_{i,j})+I_{n+j}(\{i\})}$. Since this factor is positive, we still have $g^*h \neq 0$, and the rest of the argument is the same as for $q = 1$. \square

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