J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2016-0044

Journal für die reine und angewandte Mathematik

EPW cubes

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Dedicated to Piotr Pragacz on the occasion of his 60th birthday

Abstract. We construct a new 20-dimensional family of projective six-dimensional irreducible holomorphic symplectic manifolds. The elements of this family are deformation equivalent with the Hilbert scheme of three points on a K3 surface and are constructed as natural double covers of special codimension-three subvarieties of the Grassmannian $G(3,6)$. These codimension-three subvarieties are defined as Lagrangian degeneracy loci and their construction is parallel to that of EPW sextics, we call them the EPW cubes. As a consequence we prove that the moduli space of polarized IHS sixfolds of $K3$-type, Beauville–Bogomolov degree 4 and divisibility 2 is unirational.

1. Introduction

By an irreducible holomorphic symplectic (IHS) $2n$-fold we mean a $2n$-dimensional simply connected compact Kähler manifold with trivial canonical bundle that admits a unique (up to a constant) closed non-degenerate holomorphic two-form and is not a product of two manifolds (see [3]). The IHS manifolds are also known as hyperkähler and irreducible symplectic manifolds, in dimension 2 they are called $K3$ surface.

Moduli spaces of polarized $K3$ surfaces are a historically old subject, studied by the classical Italian geometers. Mukai extended the classical constructions and proved unirationality results for the moduli spaces $\mathcal{M}_{2d}$ parametrising polarized $K3$ surfaces of degree $2d$ for many cases with $d \leq 19$ (see [19, 21, 23]). On the other hand it was proven in [8] that $\mathcal{M}_{2d}$ is of general type for $d > 61$ and some smaller values. Note that when the Kodaria dimension of such moduli space is positive, the generic element of such moduli space is believed to be non-constructible.

There are only five known descriptions of the moduli space of higher dimensional IHS manifolds (all these examples are deformations equivalent to $K3^{[n]}$). In dimension four we have the following unirational moduli spaces:

A. Iliev was supported by SNU grant 0450-20130016, G. Kapustka by NCN grant 2013/08/A/ST1/00312, M. Kapustka by NCN grant 2013/10/E/ST1/00688, and K. Ranestad by RCN grant 239015.

- double EPW sextics with Beauville–Bogomolov degree $q = 2$ (see [24]),
- Fano scheme of lines on four-dimensional cubic hypersurfaces with $q = 6$ (see [4]),
- $\mathrm{VSP}(F, 10)$ where F define a cubic hypersurface of dimension 4 with $q = 38$ (see [13]),
- zero locus of a section of a vector bundle on $G(6, 10)$ with $q = 22$ described in [6].

Moreover, there is only one more known family in dimension 8 with $q = 2$ studied in [17]. Analogously to the case of $K3$ surfaces there are results in [9] about the Kodaira dimension of the moduli spaces of polarized IHS fourfolds of $K3^{[2]}$-type: In particular, it is proven that such moduli spaces with split polarization of Beauville–Bogomolov degree $q \geq 24$ are of general type (and for $q = 18$ or 22 are of positive Kodaira dimension). We expect that the number of constructible families in higher dimension becomes small.

According to O'Grady [24], the 20-dimensional family of natural double covers of special sextic hypersurfaces in $\mathbb{P}^5$ (called EPW sextics) gives a maximal dimensional family of polarized IHS fourfold deformation equivalent to the Hilbert scheme of two points on a $K3$ surface (this is a maximal dimensional family since $b_2(S^{[2]}) = 23$ for S a $K3$ surface). Our aim is to perform a construction parallel to that of O'Grady to obtain a unirational 20-dimensional family (also of maximal dimension) of polarized IHS sixfolds deformation equivalent to the Hilbert scheme of three points on a $K3$ surface (i.e. of $K3^{[3]}$ type). The elements of this family are natural double covers of special codimension-three subvarieties of the Grassmannian $G(3, 6)$ that we call EPW cubes.

Let us be more precise. Let W be a complex six-dimensional vector space. We fix an isomorphism $j : \wedge^6 W \to \mathbb{C}$ and the skew symmetric form

$$\eta : \wedge^3 W \times \wedge^3 W \to \mathbb{C}, \quad (u, v) \mapsto j(u \wedge v). \tag{1.1}$$

We denote by $LG_\eta(10, \wedge^3 W)$ the variety of ten-dimensional Lagrangian subspaces of $\wedge^3 W$ with respect to η. For any three-dimensional subspace $U \subset W$, the ten-dimensional subspace

$$T_U := \wedge^2 U \wedge W \subset \wedge^3 W$$

belongs to $LG_\eta(10, \wedge^3 W)$, and $\mathbb{P}(T_U)$ is the projective tangent space to $G(3, W) \subset \mathbb{P}(\wedge^3 W)$ at $[U]$.

For any $[A] \in LG_\eta(10, \wedge^3 W)$ and $k \in \mathbb{N}$, we consider the following Lagrangian degeneracy locus, with natural scheme structure (see [28]):

$$D_k^A = \{[U] \in G(3, W) \mid \dim A \cap T_U \geq k\} \subset G(3, W).$$

For the fixed $[A] \in LG_\eta(10, \wedge^3 W)$ we call the scheme D_2^A an *EPW cube*. We prove that if A is generic then D_2^A is a sixfold singular only along the threefold D_3^A and that D_4^A is empty. Moreover, D_3^A is smooth such that the singularities of D_2^A are transversal $\frac{1}{2}(1, 1, 1)$ singularities along D_3^A.

Before we state our main theorem we shall need some more notation. The projectivized representation $\wedge^3$ of $\mathrm{PGL}(W)$ on $\wedge^3 W$ splits $\mathbb{P}^{19} = \mathbb{P}(\wedge^3 W)$ into a disjoint union of four orbits:

$$\mathbb{P}^{19} = (\mathbb{P}^{19} \setminus W) \cup (F \setminus \Omega) \cup (\Omega \setminus G(3, W)) \cup G(3, W),$$

where $G(3, W) \subset \Omega \subset F \subset \mathbb{P}^{19}$, $\dim(\Omega) = 14$, $\mathrm{Sing}(\Omega) = G(3, W)$, $\dim(F) = 18$, $\mathrm{Sing}(F) = \Omega$, see [7]. We call the invariant sets G, Ω, F and $\mathbb{P}^{19}$ the (projective) orbits of

$\wedge^3$ for PGL(6). See [16, Appendix] for some results about the geometry of Ω and its relations with EPW sextics. For any nonzero vector $w \in W$, denote by

$$F_{[w]} = \langle w \rangle \wedge (\wedge^2 W)$$

the ten-dimensional subspace of $\wedge^3 W$, such that

$$\bigcup_{[w] \in \mathbb{P}(W)} \mathbb{P}(F_{[w]}) = \Omega \subset \mathbb{P}(\wedge^3 W).$$

We follow the notation of O'Grady [26]:

$$\Sigma = \{[A] \in LG_\eta(10, \wedge^3 W) \mid \mathbb{P}(A) \cap G(3, W) \neq \emptyset\},$$
$$\Delta = \{[A] \in LG_\eta(10, \wedge^3 W) \mid \exists\, w \in W\colon \dim A \cap F_{[w]} \geq 3\}.$$

We also consider a third subset

$$\Gamma = \{A \in LG_\eta(10, \wedge^3 W) \mid \exists\, [U] \in G(3, W)\colon \dim A \cap T_U \geq 4\}.$$

All three subsets Σ, Δ, Γ are divisors (see [26] and Lemma 3.6). Hence,

$$LG_\eta^1(10, \wedge^3 W) := LG_\eta(10, \wedge^3 W) \setminus (\Sigma \cup \Gamma)$$

is a dense open subset of $LG_\eta(10, \wedge^3 W)$. Our main result is the following.

Theorem 1.1. *If $[A] \in LG_\eta^1(10, \wedge^3 W)$, then there exists a natural double cover Y_A of the EPW cube D_2^A branched along its singular locus D_3^A such that Y_A is an IHS sixfold of $K3^{[3]}$-type with polarization of Beauville–Bogomolov degree $q = 4$ and divisibility 2. In particular, the moduli space of polarized IHS sixfolds of $K3^{[3]}$-type, Beauville–Bogomolov degree 4 and divisibility 2 is unirational.*

We prove the theorem in Section 5 at the very end of the paper. The plan of the proof is the following: In Proposition 3.1 we prove that for $[A] \in LG_\eta^1(10, \wedge^3 W)$, the variety D_2^A is singular only along the locus D_3^A, and that it admits a smooth double cover $Y_A \to D_2^A$ branched along D_3^A with a trivial canonical class. The proof of the Proposition is based on a general study of Lagrangian degeneracy loci contained in Section 2. By globalizing the construction of the double cover to the whole affine variety $LG_\eta^1(10, \wedge^3 W)$, we obtain a smooth family

$$\mathcal{Y} \to LG_\eta^1(10, \wedge^3 W)$$

with fibers $\mathcal{Y}_{[A]} = Y_A$. Note that the family $\mathcal{Y}$ is naturally a family of polarized varieties with the polarization given by the divisors defining the double cover.

In Lemma 3.7 we prove that $\Delta \setminus (\Gamma \cup \Sigma)$ is nonempty. Following [26, Section 4.1], we associate to a general $[A_0] \in \Delta \setminus (\Gamma \cup \Sigma)$ a $K3$ surface S_{A_0}. Then, in Proposition 4.1, we prove that there exists a rational two-to-one map from the Hilbert scheme $S_{A_0}^{[3]}$ of length-3 subschemes on S_{A_0} to the EPW cube $D_2^{A_0}$. We infer in Section 5 that in this case the sixfold Y_{A_0} is birational to $S_{A_0}^{[3]}$. Together with the fact that Y_{A_0} is smooth, irreducible and has trivial canonical class, this proves that Y_{A_0} is IHS.

Since flat deformations of IHS manifolds are still IHS, the family $\mathcal{Y}$ is a family of smooth IHS sixfolds. The fact that the obtained IHS manifolds are of $K3^{[3]}$-type is a straightforward consequence of Huybrechts theorem [12, Thm. 4.6].

During the proof of Theorem 1.1 we retrieve also some information on the constructed varieties. We prove in Section 2.3 that the polarization ξ giving the double cover $Y_A \to D_2^A$ has Beauville–Bogomolov degree $q(\xi) = 4$ and is primitive. Moreover, the degree of an EPW cube $D_2^A \subset G(3,6) \subset \mathbb{P}^{19}$ is 480.

Let us recall that the coarse moduli space $\mathcal{M}$ of polarized IHS sixfolds of $K3^{[3]}$-type and Beauville–Bogomolov degree 4 has two components distinguished by divisibility. We conclude the paper by proving that the image of the moduli map $LG_\eta^1(10, \wedge^3 W) \to \mathcal{M}$ defined by $\mathcal{Y}$ is a 20-dimensional open and dense subset of the component of $\mathcal{M}$ corresponding to divisibility 2 (see Proposition 5.3).

Acknowledgement. The authors wish to thank Olivier Debarre, Alexander Kuznetsov and Kieran O'Grady for useful comments, O'Grady in particular for pointing out a proof of Proposition 5.3.

2. Lagrangian degeneracy loci

In this section we study resolutions of Lagrangian degeneracy loci. Let us start with fixing some notation and definitions. We fix a vector space W_{2n} of dimension $2n$ and a symplectic form $\omega \in \wedge^2 W_{2n}^*$. Let X be a smooth manifold and let $\mathcal{W} = W_{2n} \times \mathcal{O}_X$ be the trivial bundle with fiber W_{2n} on X equipped with a non-degenerate symplectic form $\tilde{\omega}$ induced on each fiber by ω. Consider a Lagrangian vector subbundle $J \subset \mathcal{W}$, i.e. a subbundle of rank n whose fibers are isotropic with respect to $\tilde{\omega}$. Let $A \subset W_{2n}$ be a Lagrangian vector subspace inducing a trivial subbundle $\mathcal{A} \subset \mathcal{W}$. For each $k \in \mathbb{N}$ we define the set

$$D_k^A = \{x \in X \mid \dim(J_x \cap \mathcal{A}_x) \geq k\} \subset X,$$

where J_x and $\mathcal{A}_x$ denote the fibers of the bundles J and $\mathcal{A}$ as subspaces in the fiber $\mathcal{W}_x$. Let us now define $LG_\omega(n, W_{2n})$ to be the Lagrangian Grassmannian parameterizing all subspaces of W_{2n} which are Lagrangian with respect to ω. Then J defines a map $\iota : X \to LG_\omega(n, W_{2n})$ in such a way that $J = \iota^* \mathcal{L}$, where $\mathcal{L}$ denotes the tautological bundle on the Lagrangian Grassmannian $LG_\omega(n, W_{2n})$. Moreover, similarly as on X, we can define

$$\mathbb{D}_k^A = \{[L] \in LG_\omega(n, W_{2n}) \mid \dim(L \cap A) \geq k\} \subset LG_\omega(n, W_{2n}),$$

which admits a natural scheme structure as a degeneracy locus. We then have

$$D_k^A = \iota^{-1} \mathbb{D}_k^A,$$

i.e. the scheme structure on D_k^A is defined by the inverse image of the ideal sheaf of $\mathbb{D}_k^A$; see [11, p.163].

2.1. Resolution of $\mathbb{D}_k^A$. For each $k \in \mathbb{N}$, let $G(k, A)$ be the Grassmannian of k-dimensional subspaces of A and let

$$\tilde{\mathbb{D}}_k^A = \{([L],[U]) \in LG_\omega(n, W_{2n}) \times G(k, A) \mid L \supset U\}.$$

By [28], $\tilde{\mathbb{D}}_k^A$ is a resolution of $\mathbb{D}_k^A$. We shall describe the above variety more precisely. First of all we have the following incidence described more generally in [28]:

$$\begin{array}{ccc} & \tilde{\mathbb{D}}_k^A & \\ {}^{\phi}\swarrow & & \searrow^{\pi} \\ \mathbb{D}_k^A & & G(k,A). \end{array}$$

The projection ϕ is clearly birational, whereas π is a fibration with fibers isomorphic to a Lagrangian Grassmannian $LG(n-k, 2n-2k)$. In particular, $\tilde{\mathbb{D}}_k^A$ is a smooth manifold of Picard number two with Picard group generated by H, the pullback of the hyperplane section of $LG(n, W_{2n})$ in its Plücker embedding, and R, the pullback of the hyperplane section of $G(k, A)$ in its Plücker embedding. Denote by $\mathcal{Q}$ the tautological bundle on $G(k, A)$ seen as a subbundle of the trivial symplectic bundle $W_{2n} \otimes \mathcal{O}_{G(k,A)}$. Consider the subbundle $\mathcal{Q}^{\perp} \subset W_{2n} \otimes \mathcal{O}_{G(k,A)}$ perpendicular to $\mathcal{Q}$ with respect the symplectic form. The following was observed in [28].

Lemma 2.1. *The variety $\tilde{\mathbb{D}}_k^A$ is isomorphic to the Lagrangian bundle*

$$\mathcal{F} := LG(n-k, \mathcal{Q}^{\perp}/\mathcal{Q}).$$

Of course the tautological Lagrangian subbundle on $LG(n-k, \mathcal{Q}^{\perp}/\mathcal{Q})$ can be identified with the bundle $\phi^*\mathcal{L}/\pi^*\mathcal{Q} =: \mathcal{W}$. In particular, we have

$$c_1(\mathcal{W}) = \phi^* c_1(\mathcal{L}) - \pi^* c_1(\mathcal{Q}) = R - H.$$

Lemma 2.2. *The relative tangent bundle T_π of $\pi: \mathcal{F} \to G(k, A)$ is the bundle $S^2(\mathcal{W}^\vee)$.*

Proof. This can be seen by globalizing the construction of the tangent space of the Lagrangian Grassmannian described for example in [22]. □

Lemma 2.3. *The canonical class of $\tilde{\mathbb{D}}_k^A$ is $-(n+1-k)H - (k-1)R$.*

Proof. We use the exact sequence

$$0 \to T_\pi \to T_{\mathcal{F}} \to \pi^* T_{G(k,A)} \to 0.$$

Now $\mathcal{W}^\vee$ has rank $n-k$, so

$$c_1(T_\pi) = c_1(S^2(\mathcal{W}^\vee)) = (n+1-k)c_1(\mathcal{W}^\vee) = (n+1-k)(H-R)$$

while $\pi^* c_1(T_{G(k,A)}) = nR$. Hence

$$K_{\mathcal{F}} = -c_1(T_{\mathcal{F}}) = -(n+1-k)H - (k-1)R.$$ □

Lemma 2.4. *The variety $\mathbb{D}_1^A$ is a hyperplane section of $LG_\omega(n, W_{2n})$.*

Proof. Indeed $\mathbb{D}_1^A$ is the intersection of the codimension-one Schubert cycle on the Grassmannian $G(n, 2n)$ with the Lagrangian Grassmannian, hence a hyperplane section of the Lagrangian Grassmannian. □

Let us denote by $\mathbb{E}$ the exceptional divisor of ϕ.

Lemma 2.5. *For $k = 2$ we have $[\mathbb{E}] = [H] - 2[R]$.*

Proof. It is clear that $[\mathbb{E}] = a[H] + b[R]$ for some $a, b \in \mathbb{Z}$. Let us now consider the restriction of $\mathbb{E}$ to a fiber of π, i.e. we fix a vector space $V_2 \subset A$ of dimension 2 and consider $LG(n-2, V_2^{\perp}/V_2)$. Since $\mathbb{E} = \phi^{-1} D_3^A$, we have

$$\mathbb{E} \cap \pi^{-1}[V_2] = \big\{[L] \in LG(n-2, V_2^{\perp}/V_2) \mid \dim(L/V_2 \cap A/V_2) \geq 1\big\}.$$

It is hence a divisor of type $\mathbb{D}_1^{A/V_2}$ which is a hyperplane section of the fiber by Lemma 2.4. It follows that $a = 1$.

To compute the coefficient at $[R]$ we fix a subspace V_{n-2} of dimension $n-2$ in A and consider the Schubert cycle

$$\sigma_{V_{n-2}} = \big\{[U] \in G(2, A) \mid \dim(U \cap V_{n-2}) \geq 1\big\}.$$

The class $[\sigma_{V_{n-2}}]$ in the Chow group of $G(2, A)$ is then the class of a hyperplane section. We now describe $\phi_* \pi^* (\sigma_{V_{n-2}})$ as the class of the Schubert cycle $\sigma_{n-2,n}$ on $LG(n, 2n)$ defined by

$$\sigma_{n-2,n} = \big\{[L] \in LG(n, 2n) \mid \dim(L \cap V_{n-2}) \geq 1,\ \dim(L \cap A) \geq 2\big\}.$$

By [28, Theorem 2.1] we have

$$[\sigma_{n-2,n}] = c_1(\mathcal{L}^{\vee}) c_3(\mathcal{L}^{\vee}) - 2c_4(\mathcal{L}^{\vee})$$

and

$$[\mathbb{D}_2^A] = c_1(\mathcal{L}^{\vee}) c_2(\mathcal{L}^{\vee}) - 2c_3(\mathcal{L}^{\vee}).$$

In terms of intersection on $\tilde{\mathbb{D}}_2^A$ the two equations give

$$H^{\frac{n(n+1)}{2}-3} \cap [\tilde{\mathbb{D}}_2^A] = c_1(\mathcal{L}^{\vee})^{\frac{n(n+1)}{2}-2} c_2(\mathcal{L}^{\vee}) - 2c_1(\mathcal{L}^{\vee})^{\frac{n(n+1)}{2}-3} c_3(\mathcal{L}^{\vee})$$

and

$$H^{\frac{n(n+1)}{2}-4} \cdot R \cap [\tilde{\mathbb{D}}_2^A] = c_1(\mathcal{L}^{\vee})^{\frac{n(n+1)}{2}-3} c_3(\mathcal{L}^{\vee}) - 2c_1(\mathcal{L}^{\vee})^{\frac{n(n+1)}{2}-4} c_4(\mathcal{L}^{\vee}).$$

Since we know that $\mathbb{E}$ is contracted by the resolution to $\mathbb{D}_3^A$, we also have

$$\mathbb{E} \cdot H^{\frac{n(n+1)}{2}-4} = 0.$$

We can now compute b:

$$\begin{aligned}
(2.1)\quad 0 &= \mathbb{E} \cdot H^{\frac{n(n+1)}{2}-4} \\
&= (H + bR) \cdot H^{\frac{n(n+1)}{2}-4} \\
&= H^{\frac{n(n+1)}{2}-3} + bH^{\frac{n(n+1)}{2}-4} \cdot R \\
&= c_1(\mathcal{L}^{\vee})^{\frac{n(n+1)}{2}-4} \big(c_1(\mathcal{L}^{\vee})^2 c_2(\mathcal{L}^{\vee}) + (b-2) c_1(\mathcal{L}^{\vee}) c_3(\mathcal{L}^{\vee}) - 2b c_4(\mathcal{L}^{\vee})\big).
\end{aligned}$$

Using the theorem of Hiller and Boe on relations in the Chow ring of the Lagrangian Grassmannian (see [27, Theorem 6.4]), we get

$$c_1(\mathcal{L}^\vee)^2 = 2c_2(\mathcal{L}^\vee) \quad \text{and} \quad c_2(\mathcal{L}^\vee)^2 = 2\big(c_3(\mathcal{L}^\vee)c_1(\mathcal{L}^\vee) - c_4(\mathcal{L}^\vee)\big).$$

Substituting in (2.1), we get

$$0 = (b+2)\deg\big(c_1(\mathcal{L}^\vee)c_3(\mathcal{L}^\vee) - 2c_4(\mathcal{L}^\vee)\big) = (b+2)\deg \sigma_{n-2,n}.$$

It follows that $b = -2$. □

2.2. The embedding of $G(3,W)$ into $LG_\eta(10, \wedge^3 W)$. Let W be a six-dimensional vector space. Let $G = G(3,W) \subset \mathbb{P}(\wedge^3 W)$ be the Grassmannian of three-dimensional subspaces in W in its Plücker embedding. Now, recall for each $[U] \in G$,

$$T_U = \wedge^2 U \wedge W \subset \wedge^3 W.$$

The projective space $\mathbb{P}(T_U)$ is tangent to $G(3,W)$ at $[U]$. Let $\mathcal{T}$ be the corresponding vector subbundle of $\wedge^3 W \otimes \mathcal{O}_G$. Let A be a ten-dimensional subspace of $\wedge^3 W$ isotropic with respect to the symplectic form η defined by (1.1) and such that $\mathbb{P}(A) \cap G(3,W) = \emptyset$. Recall that for $k = 1,2,3,4$ we defined

$$D_k^A = \big\{[U] \in G \mid \dim(T_U \cap A) \geq k\big\} \subset G.$$

Observe that $\mathcal{T}$ is a Lagrangian subbundle of $\wedge^3 W \otimes \mathcal{O}_G$ with respect to the two-form η. It follows that we are in the general situation described at the beginning of Section 2, with $n = 10$, $W_{20} = \wedge^3 W$, $X = G$, $J = \mathcal{T}$ and $A = A$. Then $\mathcal{T}$ defines a map

$$\iota : G(3,W) \to LG_\eta(10, \wedge^3 W), \quad [U] \mapsto [T_U].$$

We denote by $\mathcal{C}_U := \mathbb{P}(T_U) \cap G(3,W)$ the intersection of $G(3,W)$ with its projective tangent space $[U]$. Then $\mathcal{C}_U$ is linearly isomorphic to a cone over $\mathbb{P}^2 \times \mathbb{P}^2$ with vertex $[U]$. The quadrics containing the cone $\mathcal{C}_U$ plays in this situation a similar role in the local analysis of the singularities of D_k^A as the Plücker quadrics containing the Grassmannian $\mathbb{P}(F_{[w]}) \cap G(3,W)$ in [26]; this will be made more precise in Lemma 2.7.

We aim at proving the following result.

Proposition 2.6. *Let $A \in LG_\eta(10, \wedge^3 W)$ such that $\mathbb{P}(A) \cap G(3,W) = \emptyset$. The map ι is an embedding and $\iota(G(3,W))$ meets transversely all loci $\mathbb{D}_k^A \setminus \mathbb{D}_{k+1}^A$ for $k = 1,2,3$. In particular, each D_k^A is of expected dimension.*

For the proof we shall adapt the idea of [26] to our context. Let us first describe ι more precisely locally around a chosen point $[U_0] \in G(3,W)$. For this, we choose a basis $v_1, \dots, v_6$ for W such that $U_0 = \langle v_1, v_2, v_3\rangle$ and define $U_\infty = \langle v_4, v_5, v_6\rangle$. For any $[U] \in G(3,W)$ we have $T_U = \wedge^2 U \wedge W$, so T_{U_0}, T_{U_∞} are two Lagrangian spaces that intersect only at 0: $T_{U_0} \cap T_{U_\infty} = 0$. By appropriate choice of v_4, v_5, v_6 we can also assume that $T_{U_\infty} \cap A = 0$.

Let

$$\mathcal{V} = \big\{[L] \in LG_\eta(10, \wedge^3 W) \mid L \cap T_{U_\infty} = 0\big\}.$$

The decomposition $\wedge^3 W = T_{U_0} \oplus T_{U_\infty}$ into Lagrangian subspaces and the isomorphism $T_{U_\infty} \to T_{U_0}^\vee$ induced by η allow us to view a Lagrangian space L in $\mathcal{V}$ as the graph of a symmetric linear map $Q_L : T_{U_0} \to T_{U_\infty} = T_{U_0}^\vee$. Let $q_L \in \mathrm{Sym}^2 T_{U_0}^\vee$ be the quadratic form corresponding to Q_L. The map $[L] \mapsto q_L$ defines an isomorphism $\mathcal{V} \to \mathrm{Sym}^2 T_{U_0}^\vee$.

Consider the open neighborhood

$$\mathfrak{U} = \big\{[U] \in G(3, W) \mid T_U \cap T_{U_\infty} = 0\big\}$$

of $[U_0]$ in $G(3, W)$. For $[U] \in \mathfrak{U}$ we denote by $Q_U := Q_{T_U}$ and $q_U := q_{T_U}$ the symmetric linear map and the quadratic form corresponding to the Lagrangian space T_U.

We shall describe q_U in local coordinates. Observe that for any $[U] \in G(3, W)$,

$$T_U \cap T_{U_\infty} = 0 \;\leftrightarrow\; U \cap U_\infty = 0$$

and that any such subspace U is the graph of a linear map $\beta_U : U_0 \to U_\infty$. In particular, there is an isomorphism

$$\rho : \mathfrak{U} \to \mathrm{Hom}(U_0, U_\infty), \quad [U] \mapsto \beta_U$$

whose inverse is the map

$$\alpha \mapsto [U_\alpha] := \big[(v_1 + \alpha(v_1)) \wedge (v_2 + \alpha(v_2)) \wedge (v_3 + \alpha(v_3))\big].$$

In the given basis $(v_1, v_2, v_3), (v_4, v_5, v_6)$ for U_0 and U_∞ we let $B_U = (b_{i,j})_{i,j \in \{1,2,3\}}$ be the matrix of the linear map β_U. In the dual basis we let (m_0, M), with $M = (m_{i,j})_{i,j\in\{1,2,3\}}$, be the coordinates in

$$T_{U_0}^\vee = (\wedge^3 U_0 \oplus \wedge^2 U_0 \otimes U_\infty)^\vee = (\wedge^3 U_0 \oplus \mathrm{Hom}(U_0, U_\infty))^\vee.$$

Note, that under our identification the map $\iota : G(3, W) \to LG(10, \wedge^3 W)$ restricted to $\mathfrak{U}$ is the map $[U] \mapsto q_U$, which justifies our slight abuse of notation in the following lemma.

Lemma 2.7. *In the above coordinates, the map*

$$\iota : \mathfrak{U} \ni [U] \mapsto q_U := q_{T_U} \in \mathrm{Sym}^2 T_{U_0}^\vee$$

is defined by

$$q_U(m_0, M) = \sum_{i,j\in\{1,2,3\}} b_{i,j} M^{i,j} + m_0 \sum_{i,j\in\{1,2,3\}} B_U^{i,j} m_{i,j} + m_0^2 \det B_U,$$

where $M^{i,j}$, $B_U^{i,j}$ are the entries of the matrices adjoint to M and B_U.

Proof. We write in coordinates the map

$$\wedge^3 U_0 \oplus \wedge^2 U_0 \otimes U_\infty \to \wedge^3 U_\infty \oplus \wedge^2 U_\infty \otimes U_0$$

whose graph is $\wedge^3 U \oplus \wedge^2 U \otimes U_\infty$ where U is the graph of the map $U_0 \to U_\infty$ given by the matrix B_U. □

Let now Q_A be the symmetric map $T_{U_0} \to T_{U_\infty} = T_{U_0}^\vee$ whose graph is A and let q_A be the corresponding quadratic form. In this way

$$D_l^A \cap \mathfrak{U} = \{[U] \in \mathfrak{U} \mid \dim T_U \cap A) \geq l\} = \{[U] \in \mathfrak{U} \mid \mathrm{rk}(Q_U - Q_A) \leq 10 - l\},$$

hence D_l^A is locally defined by the vanishing of the $(11-l) \times (11-l)$ minors of the 10×10 matrix with entries being polynomials in $b_{i,j}$.

First we show that the space of quadrics that define $\mathcal{C}_U$ surjects onto the space of quadrics on linear subspaces in $\mathbb{P}(T_U)$.

Lemma 2.8. *If $P \subset \mathbb{P}(T_U) \setminus G(3,6)$ is a linear subspace of dimension at most 2, then the restriction map $\mathbf{r}_P : H^0(\mathbb{P}(T_U), \mathcal{I}_{\mathcal{C}_U}(2)) \to H^0(P, \mathcal{O}_P(2))$ is surjective.*

Proof. We may restrict to the case when P is a plane. Since $\mathcal{C}_U \subset \mathbb{P}(T_U) \cap G(3,6)$ is projectively equivalent to the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ in its Segre embedding, it suffices to show that if $P \subset \mathbb{P}^8$ is a plane that does not intersect $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then the Cremona transformation Cr on $\mathbb{P}^8$ defined by the quadrics containing $\mathbb{P}^2 \times \mathbb{P}^2$ maps P to a linearly normal Veronese surface. Note that the ideal of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is defined by 2×2 minors of a 3×3 matrix with linear forms in $\mathbb{P}^8$, whereas the secant of $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by the determinant of this matrix. Since the first syzygies between the generators of this ideal are generated by linear ones we infer from [1, Proposition 3.1] that they define a birational map. Moreover, this Cremona transformation contracts the secant determinantal cubic hypersurface V_3 to a to a variety linearly isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$, so the inverse Cremona is of the same kind. Furthermore, the fibers of the map $V_3 \to \mathbb{P}^2 \times \mathbb{P}^2$ are three-dimensional linear spaces spanned by quadric surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$. Now, by assumption, P does not intersect $\mathbb{P}^2 \times \mathbb{P}^2$, so the restriction $\mathrm{Cr}|_P$ is a regular, hence finite, morphism. Since the fibers of the Cremona transformation are linear, P intersects each fiber in at most a single point, so the restriction $\mathrm{Cr}|_P$ is an isomorphism. Thus, if $\mathrm{Cr}(P)$ is not linearly normal, the linear span $\langle \mathrm{Cr}(P) \rangle$ is a $\mathbb{P}^4$, being a smooth projected Veronese surface. Assume this is the case. Then $\mathrm{Cr}(P)$ is not contained in any quadric. Since the quadrics that define the inverse Cremona map $\mathrm{Cr}(P)$ to the plane P, these quadrics form only a net when restricted to the four-dimensional space $\langle \mathrm{Cr}(P) \rangle$. In fact, the complement of $\mathbb{P}^2 \times \mathbb{P}^2 \cap \langle \mathrm{Cr}(P) \rangle$ in $\langle \mathrm{Cr}(P) \rangle$ is mapped to P by the inverse Cremona transformation. Therefore $\langle \mathrm{Cr}(P) \rangle$ must be contained in the cubic hypersurface that is contracted by this inverse Cremona. Since this hypersurface is contracted to the original $\mathbb{P}^2 \times \mathbb{P}^2$, we infer that P is contained in $\mathbb{P}^2 \times \mathbb{P}^2$. This contradicts our assumption and concludes our proof. □

Lemma 2.9. *Let $K = A \cap T_{U_0} = \ker Q_A \subset T_{U_0}$ and assume that $k = \dim K \leq 3$. Then for any $l \leq k$ the tangent cone $\mathfrak{C}^l_{A,U_0}$ of $D_l^A \cap \mathfrak{U}$ at U_0 is linearly isomorphic to a cone over the corank-l locus of quadrics in $\mathbb{P}(H^0(\mathbb{P}(K), \mathcal{O}_{\mathbb{P}(K)}(2)))$.*

Proof. We follow the idea of [25, Proposition 1.9]. If we choose a basis Λ of $T_{U_0}^\vee$, the symmetric linear map Q_U is defined by a symmetric matrix $M^\Lambda(B_U)$ with entries being polynomials in $(b_{i,j})_{i,j \in \{1,2,3\}}$.

The linear summands of each entry in $M^\Lambda(B_U)$ form a matrix that we denote by $N^\Lambda(B_U)$. Since $Q_0 = 0$, the entries of $M^\Lambda(B_U)$ have no nonzero constant terms. Moreover, by using Lemma 2.7 and $\Lambda_0 = (m_0, M)$, we see that the map $\mathfrak{U} \ni U \mapsto q'_U \in \mathrm{Sym}^2 T_{U_0}^\vee$, where q'_U is the quadratic form corresponding to the symmetric map defined by the matrix $N^{\Lambda_0}(B_U)$, maps $\mathfrak{U}$ linearly onto the linear system of quadrics containing the cone $\mathcal{C}_{U_0}$. Of course, this surjection is independent of the choice of the basis.

We now choose a basis Λ in T_{U_0} in which Q_A is represented by a diagonal matrix $R_k = \operatorname{diag}\{0, \dots, 0, 1, \dots, 1\}$ with k zeros in the diagonal. Then

$$\begin{aligned} D_l^A \cap \mathfrak{U} &= \{[U] \in \mathfrak{U} \mid \dim(T_U \cap A) \geq l\} \\ &= \{[U] \in \mathfrak{U} \mid \dim \ker(Q_U - Q_A) \geq l\} \\ &= \{[U] \in \mathfrak{U} \mid \operatorname{rk}(M^\Lambda(B_U) - R_k) \leq 10 - l\}. \end{aligned}$$

Hence D_l^A is defined in coordinates $(b_{i,j})_{i,j\in\{1,2,3\}}$ on $\mathfrak{U}$ by $(11-l)\times(11-l)$ minors of the matrix $M^\Lambda(B_U) - R_k$. Furthermore, since $[U_0]$ is the point 0 in our coordinates $(b_{i,j})_{i,j\in\{1,2,3\}}$, the tangent cone to $D_l^A \cap \mathfrak{U}$ at $[U_0]$ is defined by the initial terms of the $(11-l)\times(11-l)$ minors of $M^\Lambda(B_U) - R_k$. Note that we can write

$$M^\Lambda(B_U) - R_k = -R_k + N^\Lambda(B_U) + Z(B_U),$$

where the entries of the matrix $Z(B_U)$ are polynomials with no linear or constant terms. We illustrate this decomposition as follows:

$$\left(\begin{array}{ccc|ccc} & & & N^\Lambda_{1,k+1} + Z^\Lambda_{1,k+1} & \cdots & N^\Lambda_{1,10} + Z^\Lambda_{1,10} \\ & N^\Lambda_{\mathbf{k}} + Z_{\mathbf{k}} & & \vdots & \ddots & \vdots \\ & & & N^\Lambda_{k,k+1} + Z^\Lambda_{k,k+1} & \cdots & N^\Lambda_{k,10} + Z^\Lambda_{k,10} \\ \hline N^\Lambda_{k+1,1} + Z^\Lambda_{k+1,1} & \cdots & N^\Lambda_{k+1,k} + Z^\Lambda_{k+1,k} & -1 + N^\Lambda_{k+1,k+1} + Z^\Lambda_{k+1,k+1} & \cdots & N^\Lambda_{k+1,10} + Z^\Lambda_{k+1,10} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N^\Lambda_{10,1} + Z^\Lambda_{10,1} & \cdots & N^\Lambda_{10,k} + Z^\Lambda_{10,k} & N^\Lambda_{10,k+1} + Z^\Lambda_{10,k+1} & \cdots & -1 + N^\Lambda_{10,10} + Z^\Lambda_{10,10} \end{array}\right).$$

Let Φ be an $(11-l)\times(11-l)$ minor of $M^\Lambda(B_U) - R_k$ and consider its decomposition $\Phi = \Phi_0 + \cdots + \Phi_r$ into homogeneous parts Φ_d of degree d. Observe that $\Phi_d = 0$ for $d \leq k - l$, moreover Φ_{k-l+1} can be nonzero only if the submatrix associated to the minor Φ contains all nonzero entries of R_k. In the latter case Φ_{k-l+1} is a $(k+1-l)\times(k+1-l)$ minor of the $k \times k$ upper left corner submatrix $N^\Lambda_{\mathbf{k}}(B_U)$ of the matrix $N^\Lambda(B_U)$. Let us now denote by q'_U the quadric corresponding to the matrix $N^\Lambda(B_U)$ and by ι^N the map $U \mapsto q'_U$. Then, by changing Φ, we get that the tangent cone of $D_l^A \cap \mathfrak{U}$ is contained in

$$\hat{\mathfrak{C}}^l_{A,U_0} := \{[U] \in \mathfrak{U} \mid \operatorname{rk}(N^\Lambda_{\mathbf{k}}(B_U)) \leq k - l\} = \{[U] \in \mathfrak{U} \mid \operatorname{rk}(q'_U|_K) \leq k - l\}.$$

The latter is the preimage by $\mathbf{r}_K \circ \iota^N$ of the corank-l locus in the projective space of quadrics $\mathbb{P}(H^0(\mathbb{P}(K), \mathcal{O}_{\mathbb{P}(K)}(2)))$.

By Lemma 2.8, we have seen that $\mathbf{r}_K \circ \iota^N$ is a linear surjection. So we conclude that $\hat{\mathfrak{C}}^l_{A,U_0}$ is a cone over the corank-l locus of quadrics in $\mathbb{P}(H^0(\mathbb{P}(K), \mathcal{O}_{\mathbb{P}(K)}(2)))$ with vertex a linear space of dimension $10 - k(k+1)/2$. It follows that $\hat{\mathfrak{C}}^l_{A,U_0}$ is an irreducible variety of codimension $l(l+1)/2$ equal to the codimension of D_l^A. Thus we have equality $\mathfrak{C}^l_{A,U_0} = \hat{\mathfrak{C}}^l_{A,U_0}$ which ends the proof. □

Corollary 2.10. *If A is a Lagrangian space in $\wedge^3 W$ such that $\mathbb{P}(A)$ does not meet $G(3,W)$, then the variety D_l^A is smooth of the expected codimension $l(l+1)/2$ outside D_{l+1}^A. Moreover, if $l = 2$ and* $\dim A \cap T_{U_0} = 3$, *i.e.* $[U_0]$ *is a point in $D_3^A \setminus D_4^A$, then the tangent cone $\mathfrak{C}^2_{A,U_0}$ is a cone over the Veronese surface in $\mathbb{P}^5$ centered in the tangent space of D_3^A.*

Proof of Proposition 2.6. It is clear from Lemma 2.7 that ι is a local isomorphism into its image, and by Corollary 2.10, the subscheme $D_A^k = \iota^{-1}(\iota(G(3,W)) \cap \mathbb{D}_A^k)$ is smooth outside D_A^{k+1}, so $\iota(G(3,W))$ meets the degeneracy loci transversally. □

2.3. Invariants. We shall compute the classes of the Lagrangian degeneracy loci $D_k^A \subset G(3,W)$ in the Chow ring of $G(3,W)$. We consider the embedding

$$\iota : G(3,W) \to LG_\eta(10, \wedge^3 W)$$

defined by the bundle of Lagrangian subspaces $\mathcal{T}$ on $G(3,W)$. According to [28, Theorem 2.1] the fundamental classes of the Lagrangian degeneracy loci D_k^A are

$$[D_1^A] = [c_1(\mathcal{T}^\vee) \cap G(3,W)], \quad [D_2^A] = [(c_2c_1 - 2c_3)(\mathcal{T}^\vee) \cap G(3,W)]$$

and

$$[D_3^A] = [(c_1c_2c_3 - 2c_1^2c_4 + 2c_2c_4 + 2c_1c_5 - 2c_3^2)(\mathcal{T}^\vee) \cap G(3,W)].$$

The $\mathbb{P}^9$-bundle $\mathbb{P}(\mathcal{T})$ is the projective tangent bundle on $G(3,W)$. So $\mathcal{T}^\vee$ fits into an exact sequence

$$0 \to \Omega_{G(3,W)}(1) \to \mathcal{T}^\vee \to \mathcal{O}_{G(3,W)}(1) \to 0$$

and we get

$$\deg D_1^A = 168, \quad \deg D_2^A = 480, \quad \deg D_3^A = 720.$$

Remark 2.11. This may be compared with the degree of the line bundle $2H - 3E$ on $S^{[3]}$, where S is a $K3$ surface of degree 10, H is the pullback of the line bundle of degree 10 on S, and E is the unique divisor class such that the divisor of non-reduced subschemes in $S^{[3]}$ is equivalent to $2E$. The degree, i.e. the value of the Beauville–Bogomolov form, is $q(2H - 3E) = 4$, and the degree and the Euler–Poincaré characteristic of the line bundle are

$$(2H - 3E)^6 = 15q(2H - 3E)^3 = 960 \quad \text{and} \quad \chi(2H - 3E) = 10.$$

So if the map defined by $|2H - 3E|$ is a morphism of degree 2, the image would have degree 480, like D_2^A.

In Section 4, we show that $S^{[3]}$ for a general $K3$ surface S of degree 10 admits a rational double cover of a degeneracy locus D_2^A. However that double cover is not a morphism.

3. The double cover of an EPW cube

Proposition 3.1. *Let* $[A] \in LG_\eta(10, \wedge^3 W)$*. If* $\mathbb{P}(A) \cap G(3,W) = \emptyset$ *and* $D_4^A = \emptyset$*, then* D_2^A *admits a double cover* $f : Y_A \to D_2^A$ *branched over* D_3^A *with* Y_A *a smooth irreducible manifold having trivial canonical class.*

Before we pass to the construction of the double cover let us observe the following.

Lemma 3.2. *Under the assumptions of Proposition* 3.1 *the variety* D_2^A *is integral.*

Proof. We know that D_2^A is of expected dimension. Observe now that by Corollary 2.10 the variety D_2^A is irreducible if and only if it is connected. To prove connectedness we perform a computation in the Chow ring of the Grassmannian $G(3,W)$ showing that the class $[D_2^A]$ does not decompose into a sum of nontrivial effective classes in the Chow group $A^3(G(3,W))$ whose intersection is the zero class in $A^6(G(3,W))$. More precisely we compute

$$[D_2^A] = 16h^3 - 12hs_2 + 12s_3,$$

where h is the hyperplane class on $G(3,W)$, s_2 and s_3 are the Chern classes of the tautological bundle on $G(3,W)$. We then solve in integer coordinates $a, b, c \in \mathbb{Z}$ the equation

$$(ah^3 - bs_2 + cs_3)\big((16-a)h^3 - (12-b)s_2 + (12-c)s_3\big) = 0$$

in the Chow group $A^6(G(3,W))$ which is generated by s_2^3, $h^3s_1s_2$, s_3^2. Multiplying out the equation in the Chow ring and extracting coefficients at the generators, we get a system of three quadratic diophantine equations in a, b, c:

$$\begin{cases} -5a^2 + 4ab - b^2 + 56a - 20b = 0, \\ -6a^2 + 8ab - 2b^2 - 4ac + 2bc + 72a - 52b + 20c = 0, \\ 6a^2 - 6ab + b^2 + 2ac - c^2 - 72a + 36b - 4c = 0. \end{cases} \tag{3.1}$$

The only integer solutions are $(0,0,0)$ and $(16,12,12)$. This ends the proof. □

The plan of the construction of the double cover in Proposition 3.1 is the following. We consider the resolution $\tilde{D}_2^A \to D_2^A$ with exceptional divisor E. We prove that E is a smooth even divisor, and hence that there is a smooth double cover $\tilde{Y} \to \tilde{D}_2^A$ branched over E. Finally, we contract the branch divisor of the double cover using a suitable multiple of the pullback of a hyperplane class on D_2^A by the resolution and the double cover.

Thus, we start by defining the incidences

$$\tilde{D}_2^A = \big\{([U],[U']) \in G(3,W) \times G(2,A) \mid T_U \supset U'\big\}$$

and

$$\tilde{\mathbb{D}}_2^A = \big\{([L],[U']) \in LG_\eta(10,\wedge^3 W) \times G(2,A) \mid L \supset U'\big\}.$$

They fit in the following diagram:

$$\begin{array}{ccc} G(3,W) & \xrightarrow{\ \iota\ } & LG_\omega(10,\wedge^3 W) \\ \cup\!\! | & & \cup\!\! | \\ D_2^A & \xrightarrow{\ \iota|_{D_2^A}\ } & \mathbb{D}_2^A \\ \uparrow \alpha & & \uparrow \phi \\ \tilde{D}_2^A & \xrightarrow{\ \tilde{\iota}\ } & \tilde{\mathbb{D}}_2^A. \end{array}$$

Lemma 3.3. *Under the assumptions of Proposition* 3.1 *the variety $\tilde{D}_2^A$ as well as the exceptional locus E of the map α are smooth. In particular, α is a resolution of singularities of D_2^A.*

Proof. Since we know that $D_4^A = \emptyset$, the resolution $\alpha : \tilde{D}_2^A \to D_2^A$ is just the blow-up of D_2^A along D_3^A. Now, $\tilde{D}_2^A \setminus E$ is isomorphic to $D_2^A \setminus D_3^A$, so, by Corollary 2.10, we deduce that $\tilde{D}_2^A$ is smooth outside E. Let $p \in E \subset \tilde{D}_2^A$. Then $\alpha(p) \in D_3^A$. Take $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ to be three general hyperplanes passing through $\alpha(p)$. Consider $Z_{\mathbf{P}} = D_2^A \cap \mathbf{P}_1 \cap \mathbf{P}_2 \cap \mathbf{P}_3$ and its strict transform $\tilde{Z}_{\mathbf{P}} \subset \tilde{D}_2^A$. We have the following diagram:

$$\begin{array}{ccc} \tilde{Z}_{\mathbf{P}} & \longrightarrow & \tilde{D}_2^A \\ \downarrow{\scriptstyle \alpha_{\mathbf{P}}} & & \downarrow{\scriptstyle \alpha} \\ Z_{\mathbf{P}} & \longrightarrow & D_2^A. \end{array}$$

The map $\alpha_{\mathbf{P}} : \tilde{Z}_{\mathbf{P}} \to Z_{\mathbf{P}}$ is the blow-up of $Z_{\mathbf{P}}$ in $D_3^A \cap \mathbf{P}_1 \cap \mathbf{P}_2 \cap \mathbf{P}_3$, which by Corollary 2.10 is a finite set of isolated points. By the assumption on $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ the strict transform $\tilde{Z}_{\mathbf{P}}$ contains the whole fiber $\alpha^{-1}(p)$ and hence also $p \in \tilde{Z}_{\mathbf{P}}$. Let $\tilde{\mathbf{P}}_i$ be the strict transform of $\mathbf{P}_i$ for $i = 1, 2, 3$. Then $\tilde{\mathbf{P}}_i$ is a Cartier divisor on $\tilde{D}_2^A$ and $\tilde{Z}_{\mathbf{P}} = \tilde{\mathbf{P}}_1 \cap \tilde{\mathbf{P}}_2 \cap \tilde{\mathbf{P}}_3$ is a complete intersection of Cartier divisors on $\tilde{D}_2^A$. Now, from Corollary 2.10, the exceptional divisor $E_{\mathbf{P}} = E \cap \tilde{Z}_{\mathbf{P}}$ of $\alpha_{\mathbf{P}}$ is isomorphic to a finite union of disjoint ($\mathbb{P}^2$), one for each point in $D_3^A \cap \mathbf{P}_1 \cap \mathbf{P}_2 \cap \mathbf{P}_3$. But $E_{\mathbf{P}}$ is itself a Cartier divisor on $\tilde{Z}_{\mathbf{P}}$ by general properties of the blow-up. Therefore $\tilde{Z}_{\mathbf{P}}$ is smooth. We conclude that $\tilde{D}_2^A$ is smooth at p and similarly, that E is smooth at p. □

We compute the first Chern class of the normal bundle of the embedding $\tilde{\iota} : \tilde{D}_2^A \to \tilde{\mathbb{D}}_2^A$.

Lemma 3.4. *One has*

$$c_1(\tilde{\iota}^* N_{\tilde{\iota}(\tilde{D}_2^A)|\tilde{\mathbb{D}}_2^A}) = c_1(\alpha^* \iota^* N_{\iota(G(3,W))|LG_\eta(10,\wedge^3 W)}) = 38h,$$

where h is the pullback via the resolution α of the restriction of the hyperplane class on $G(3,W)$ to D_2^A.

Proof. From the transversality (Proposition 2.6) we have

$$\tilde{\iota}^* N_{\tilde{\iota}(\tilde{D}_2^A)|\tilde{\mathbb{D}}_2^A} = \alpha^* \iota^* N_{\iota(G(3,W))|LG_\eta(10,\wedge^3 W)},$$

which gives the first equality.

To get the second, consider the exact sequence

$$0 \to T_{G(3,W)} \to \iota^*(T_{LG_\eta(10,\wedge^3 W)}) \to \iota^*(N_{\iota(G(3,W))|LG_\eta(10,\wedge^3 W)}) \to 0,$$

and observe that

$$\iota^*(T_{LG_\eta(10,\wedge^3 W)}) = \iota^*(S^2 \mathcal{L}^\vee) = S^2(\iota^* \mathcal{L}^\vee) = S^2 \mathcal{T}^\vee,$$

where $\mathcal{L}$ denotes, as before, the tautological bundle on the Lagrangian Grassmannian $LG_\eta(10, \wedge^3 W)$. We obtain

$$c_1(\alpha^* \iota^* N_{\iota(G(3,W))|LG_\eta(10,\wedge^3 W)}) = -11\alpha^* c_1(\mathcal{T}) - 6h.$$

Now, from

$$0 \to \mathcal{O}_{G(3,W)}(-1) \to \mathcal{T} \to T_{G(3,W)}(-1) \to 0$$

we obtain $\alpha^* c_1(\mathcal{T}) = -4h$, which proves the lemma. □

Note that in our notation we have

$$\tilde{\iota}^* H = \tilde{\iota}^* \phi^* c_1(\mathcal{L}^\vee) = \alpha^* \iota^* c_1(\mathcal{L}^\vee) = \alpha^* c_1(\mathcal{T}^\vee) = 4h.$$

We aim now at constructing a double covering of $\tilde{D}_2^A$ branched along E. It is enough to prove that E is an even divisor. This follows from the exact sequence

$$0 \to T_{\tilde{D}_2^A} \to \tilde{\iota}^* T_{\tilde{\mathbb{D}}_2^A} \to \tilde{\iota}^* N_{\tilde{\iota}(\tilde{D}_2^A)|\tilde{\mathbb{D}}_2^A} \to 0$$

and Lemma 2.3. Indeed, from them we infer

$$c_1(T_{\tilde{D}_2^A}) = \tilde{\iota}^*(9H + R) - 38h = \tilde{\iota}^*(R) - 2h,$$

which, by Lemma 2.5, means

$$E = \mathbb{E} \cap \tilde{D}_2^A = \tilde{\iota}^*(H - 2R) = 2K_{\tilde{D}_2^A}.$$

By Lemma 3.3 there hence exists a smooth double cover $\tilde{f} : \tilde{Y} \to \tilde{D}_2^A$ branched along the exceptional locus E of the resolution α. Moreover, from the adjunction formula for double covers we get $K_{\tilde{Y}} = \tilde{f}^{-1}(E) =: \tilde{E}$.

We now need to contract $\tilde{E} = \tilde{f}^{-1}(E)$ on $\tilde{Y}$. For that, with slight abuse of notation, we denote by h the class of the hyperplane section on $D_2^A \subset G(3, W)$. Then $|\tilde{f}^* \alpha^* h|$ is a globally generated linear system whose associated morphism defines $\alpha \circ \tilde{f}$ and hence contracts E to a threefold and is two-to-one on $\tilde{Y} \setminus \tilde{f}^{-1}(E)$. It follows by standard arguments (for example applying Stein factorization and [10, Proposition 4.4]) that there exists a number n such that the system $|n \tilde{f}^* \alpha^* h|$ defines a morphism $\tilde{\alpha} : \tilde{Y} \to Y$ which is a birational morphism contracting exactly $\tilde{E}$ to a threefold $\mathcal{Z}$ and such that its image Y is normal. We then have the following diagram:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{D}_2^A \\ \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{f} & D_2^A, \end{array}$$

in which Y admits a two-to-one map $f : Y \to D_2^A$ branched along D_3^A.

Proof of Proposition 3.1. We have constructed Y, a normal variety admitting a two-to-one map $f : Y \to D_2^A$ branched along D_3^A. Clearly $K_{\tilde{Y}} = \tilde{E}$ implies $K_Y = 0$. It hence remains to prove that Y is smooth. Since $\tilde{\alpha}$ is a contraction that contracts only $\tilde{E}$, it is clear that Y is smooth outside of $\mathcal{Z} = \tilde{\alpha}(\tilde{E})$. Let now $p \in \mathcal{Z}$ and let $p' = f(p)$. We then choose three general hypersurfaces $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ of degree n in $\mathbb{P}(\wedge^3 W)$ passing through p'. Consider

$$Z_{\mathbf{P}} = D_2^A \cap \mathbf{P}_1 \cap \mathbf{P}_2 \cap \mathbf{P}_3 \quad \text{and} \quad Z'_{\mathbf{P}} = D_3^A \cap \mathbf{P}_1 \cap \mathbf{P}_2 \cap \mathbf{P}_3.$$

Then $Z'_{\mathbf{P}}$ is a finite set of points that includes p'. Consider the following natural restriction of the above diagram:

$$\begin{array}{ccc} \tilde{Y}_{\mathbf{P}} & \xrightarrow{\tilde{f}_{\mathbf{P}}} & \tilde{Z}_{\mathbf{P}} \\ \downarrow{\scriptstyle \tilde{\alpha}_{\mathbf{P}}} & & \downarrow{\scriptstyle \alpha_{\mathbf{P}}} \\ Y_{\mathbf{P}} & \xrightarrow{f_{\mathbf{P}}} & Z_{\mathbf{P}}. \end{array}$$

Here $\alpha_{\mathbf{P}} = \alpha|_{\alpha^{-1}(Z_{\mathbf{P}})} : \tilde{Z}_{\mathbf{P}} \to Z_{\mathbf{P}}$ is just the blow-up of $Z_{\mathbf{P}}$ along $Z'_{\mathbf{P}}$. The exceptional divisor $E_{\mathbf{P}}$ is then, by Corollary 2.10, isomorphic to a finite set of disjoint planes that each have normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ in $\tilde{Z}_{\mathbf{P}}$. Taking the double cover of $\tilde{Z}_{\mathbf{P}}$ branched along the exceptional divisor $E_{\mathbf{P}}$, the preimages of these $(\mathbb{P}^2)$ are the components of $\tilde{E}_{\mathbf{P}} \subset \tilde{Y}_{\mathbf{P}}$, each component a $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$. The contraction $\tilde{\alpha}_{\mathbf{P}}$ contracts the divisor $\tilde{E}_{\mathbf{P}}$ to a finite set of points in $Y_{\mathbf{P}}$. It contracts one of its $(\mathbb{P}^2)$, denote it by $\tilde{E}^p_{\mathbf{P}}$, to the point p. Note also that from the construction, $Y_{\mathbf{P}}$ is the intersection of three Cartier divisors on Y which is smooth outside the finite set of points $Z'_{\mathbf{P}}$. Thus, since we constructed Y to be normal, we deduce that $Y_{\mathbf{P}}$ is also normal. We claim that p must be a smooth point of $Y_{\mathbf{P}}$. Indeed, we know that $\tilde{\alpha}_{\mathbf{P}}$ is a birational morphism onto the normal variety $Y_{\mathbf{P}}$. Moreover, all lines $\mathbf{l} \subset \tilde{E}^p_{\mathbf{P}} = \mathbb{P}^2$ are numerically equivalent on $\tilde{Y}_{\mathbf{P}}$ and satisfy

$$\mathbf{l} \cdot K_{\tilde{Y}_{\mathbf{P}}} = -1 < 0.$$

It follows from [18, Corollary 3.6] that there exists an extremal ray r for $\tilde{Y}_{\mathbf{P}}$ whose associated contraction $\mathrm{cont}_r : \tilde{Y}_{\mathbf{P}} \to \hat{Y}_{\mathbf{P}}$ contracts $\tilde{E}^p_{\mathbf{P}}$ to a point $\hat{p}$ and that $\tilde{\alpha}_{\mathbf{P}}$ factorizes through cont_r. By [18, Theorem 3.3] we have that cont_r is the blow-down of $\tilde{E}^p_{\mathbf{P}}$ and $\hat{p}$ is a smooth point of $\hat{Y}_{\mathbf{P}}$. Let us now denote by $\sigma : \hat{Y}_{\mathbf{P}} \to Y_{\mathbf{P}}$ the morphism satisfying $\tilde{\alpha}_{\mathbf{P}} = \sigma \circ \mathrm{cont}_r$. Consider the restriction σ_o of σ to small open neighborhoods of $\hat{p}$ and p. Then σ_o is a birational proper morphism which is bijective to an open subset of the normal variety $Y_{\mathbf{P}}$. It follows by Zariski's main theorem that σ_o is an isomorphism and in consequence, p is a smooth point on $Y_{\mathbf{P}}$.

The latter implies that Y must also be smooth at p as it admits a smooth complete intersection subvariety which is smooth at p. □

Corollary 3.5. *Let $[A] \in LG_\eta(10, \wedge^3 W)$ be a general Lagrangian subspace with a three-dimensional intersection with some space $F_{[w]} = \{w \wedge \alpha \mid \alpha \in \wedge^2 W\}$. Then there exists a double cover $f_A : Y_A \to D^A_2$ branched over D^A_3, where Y_A is a smooth irreducible sixfold with trivial canonical class.*

Proof. It is enough to make a dimension count to prove that the general Lagrangian space A satisfying the assumptions of the corollary also satisfies the assumptions of Proposition 3.1. Indeed, as in the introduction, let

$$\Delta = \{[A] \in LG_\eta(10, \wedge^3 W) \mid \exists\, w \in W\colon \dim(A \cap F_{[w]}) \geq 3\},$$
$$\Gamma = \{[A] \in LG_\eta(10, \wedge^3 W) \mid \exists\, U \in G(3, W)\colon \dim(A \cap T_U) \geq 4\}.$$

Lemma 3.6. *The set $\Gamma \subset LG_\eta(10, \wedge^3 W)$ is a divisor.*

Proof. Let us consider the incidence

$$\Xi = \{([U],[A]) \in G(3, W) \times LG_\eta(10, \wedge^3 W) \mid \dim(T_U \cap A) \geq 4\}.$$

The dimension of Ξ can be computed by looking at the projection $\Xi \to G(3, 6)$. For a fixed tangent plane we choose first a $\mathbb{P}^3$ inside: this choice has 24 parameters. Then for a fixed $\mathbb{P}^3$ we have $\dim(LG(6, 12)) = 21$ parameters for the choice of A. Thus the dimension of Ξ is $9 + 24 + 21 = 54$. It remains to observe that the projection $\Xi \to LG_\eta(10, \wedge^3 W)$ is finite, and that $\dim(LG_\eta(10, \wedge^3 W)) = 55$. □

Note that in [26, Proposition 2.2] it is proven that Δ is irreducible and not contained in $\Sigma = \{[A] \in LG(10, 20) \mid \mathbb{P}(A) \cap G(3, W) \neq \emptyset\}$. Our corollary is now a consequence of Proposition 3.1 and the following lemma.

Lemma 3.7. *The divisors $\Delta, \Gamma \subset LG_\eta(10, \wedge^3 W)$ have no common components.*

Proof. We need to prove $\dim(\Delta \cap \Gamma) < 54$ which, by the fact that Δ is irreducible and not contained in Σ, is equivalent to $\dim((\Delta \cap \Gamma) \setminus \Sigma) < 54$. For this, observe that if $[A] \in (\Delta \cap \Gamma) \setminus \Sigma$ then there exist $[U] \in G(3, W)$ and $[w] \in \mathbb{P}(W)$ with $\dim(A \cap T_U) = 4$ and $\dim(A \cap F_{[w]}) = 3$. We can hence consider the incidence

$$\Theta = \{([A], [W_3], [W_4], [w], [U]) \mid W_3 = A \cap F_{[w]},\ W_4 = A \cap T_U\}$$
$$\subset LG_\eta(10, \wedge^3 W) \times G(3, \wedge^3 W) \times G(4, \wedge^3 W) \times \mathbb{P}(W) \times G(3, W)$$

such that its projection to $LG_\eta(10, \wedge^3 W)$ contains $(\Delta \cap \Gamma) \setminus \Sigma$. Note also that if we take $([A], [W_3], [W_4], [w], [U]) \in \Theta$ then $W_4 \cap W_3 = W_4 \cap F_{[w]} = W_3 \cap T_U$.

We shall now compute the dimension of Θ by considering fibers under subsequent projections:

$$LG_\eta(10, \wedge^3 W) \times G(3, \wedge^3 W) \times G(4, \wedge^3 W) \times \mathbb{P}(W) \times G(3, W)$$
$$\xrightarrow{\pi_1} G(3, \wedge^3 W) \times G(4, \wedge^3 W) \times \mathbb{P}(W) \times G(3, W)$$
$$\xrightarrow{\pi_2} G(4, \wedge^3 W) \times \mathbb{P}(W) \times G(3, W)$$
$$\xrightarrow{\pi_3} \mathbb{P}(W) \times G(3, W).$$

We have two possibilities for pairs $([w], [U])$ which give us two types of points to consider:

(i) If $w \notin U$, then $\dim T_U \cap F_{[w]} = 3$.

(ii) If $w \in U$, then $\dim T_U \cap F_{[w]} = 7$.

We then have different types of elements in the intersection $\pi_3^{-1}([w], [U]) \cap \pi_2(\pi_1(\Theta))$, depending on the number

$$d_1 := \dim(W_4 \cap F_{[w]}) = \dim(W_4 \cap W_3) \leq 3.$$

If $W_4^\perp$ denotes the orthogonal to W_4 with respect to η in $\wedge^3 W$, then $\dim W_4^\perp \cap F_{[w]} = 6 + d_1$. Now, in order for $[W_3]$ to be an element of $\pi_2^{-1}([W_4], [w], [U]) \cap \pi_1(\Theta)$ we must have

$$W_3 \subset W_4^\perp \cap F_{[w]}.$$

The fiber $\pi_1^{-1}([W_3], [W_4], [w], [U]) \cap \Theta$ is of dimension $(3 + d_1)(4 + d_1)/2$. Hence to compute the dimension of each component of Θ it is enough to compute the dimensions of the spaces F_{i,d_1} of elements $([W_3], [W_4], [w], [U])$ of types (i, d_1), where $i = 1$ if $w \notin U$ and $i = 2$ if $w \in U$.

(i) For $i = 1$ we start with a choice of $[U] \in G(3, W)$. Then $[w]$ belongs to an open subset of $\mathbb{P}^5$. We have $d_1 \leq 3$ and $[W_4]$ belongs to the Schubert cycle consisting of four-spaces in the ten-dimensional space T_U that meet the fixed three-space $T_U \cap F_{[w]}$ in dimension d_1. And $[W_3]$ belongs to the Schubert cycle of three-spaces in the $(6 + d_1)$-dimensional space $W_4^\perp \cap F_{[w]}$ that contains the space $W_4 \cap F_{[w]}$ of dimension d_1

(ii) For $i = 2$ we again start with a choice of $[U] \in G(3, W)$. In this case $[w]$ belongs to $\mathbb{P}(U)$. We have $d_1 \leq 3$ and $[W_4]$ belongs to the Schubert cycle of four-spaces in the ten-dimensional space T_U that meet the fixed seven-space $T_U \cap F_{[w]}$ in dimension d_1. Then $[W_3]$ belongs to the Schubert cycle of three-spaces in the $(6 + d_1)$-dimensional space $W_4^{\perp} \cap F_{[w]}$ that contains the space $W_4 \cap F_{[w]}$ of dimension d_1.

We have

$$\dim F_{i,d_1} = \begin{cases} 9 + 5 + d_1(3 - d_1) + (4 - d_1)6 + (d_1 + 3)(3 - d_1) & \\ \quad = 47 - 3d_1 - 2d_1^2 & \text{for } i = 1, \\ 9 + 2 + d_1(7 - d_1) + (4 - d_1)6 + (d_1 + 3)(3 - d_1) & \\ \quad = 44 + d_1 - 2d_1^2 & \text{for } i = 2. \end{cases}$$

In each case we have

$$\dim F_{i,d_1} + \frac{(3 + d_1)(4 + d_1)}{2} \leq 53.$$

It follows that $\dim \Theta \leq 53$ which implies $\dim(\Delta \cap \Gamma) \leq 53$. Hence Δ and Γ have no common components. □

This also concludes the proof of Corollary 3.5. □

4. Special A

Let us recall from [26] the following construction. Let V and V_0 be two vector spaces of dimensions 5 and 1 respectively. Let $W = V \oplus V_0$. Consider the space $\wedge^3 W$ equipped with the symplectic form η given by the wedge product as above. Let $v_0 \in V_0$, choose a general Lagrangian subspace A of $\wedge^3 W$ such that $A \cap F_{[v_0]}$ is a vector space of dimension 3, i.e. $[A]$ is a general element of the divisor $\Delta \subset LG_\eta(10, \wedge^3 W)$. In particular, we assume $[A] \in \Delta \setminus \Sigma$. Note that, by [26, Proposition 2.2 (2)], for a general $[A] \in \Delta$ there is a unique $[v_0]$ such that $F_{[v_0]} \cap A$ is of dimension 3.

Let $\tilde{K} = A \cap F_{[v_0]}$ and denote by $K \subset \wedge^2 V$ the three-dimensional subspace such that $\tilde{K} = v_0 \wedge K$. Observe that there is a natural isomorphism $\wedge^2 V \to F_{[v_0]}$ given by wedge product with v_0. The latter induces an isomorphism $\wedge^3 V \to F_{[v_0]}^{\vee}$.

Let $[B] \in LG_\eta(10, \wedge^3 W)$ be a Lagrangian space such that

$$B \cap F_{[v_0]} = \{0\} \quad \text{and} \quad B \cap A = \{0\}.$$

Then the symplectic form η defines a canonical isomorphism $B \to F_{[v_0]}^{\vee}$ by which A appears as the graph of a symmetric map $\tilde{Q}_A : F_{[v_0]} \to B = F_{[v_0]}^{\vee}$. Composed with the isomorphisms $\wedge^2 V \to F_{[v_0]}$ and $\wedge^3 V \to F_{[v_0]}^{\vee}$ we get a symmetric map

$$Q_A : \wedge^2 V \to \wedge^3 V \cong (\wedge^2 V)^{\vee}.$$

Clearly $\ker Q_A = K$. Let q_A be the quadric on $\wedge^2 V$ given by Q_A, then q_A is a quadric of rank 7; it is a cone over K. The map Q_A defines an isomorphism $\wedge^2 V/K \to K^{\perp}$ and hence the quadric q_A defines a quadric $K^{\perp} \subset \wedge^3 V$:

$$q_A^* : \beta \mapsto \mathrm{vol}(\alpha \wedge \beta), \quad \text{where} \quad Q_A(\alpha) = \beta.$$

Moreover, to each $v^* \in V^\vee$ we associate the quadric

$$q_{v^*} : \wedge^3 V \ni \omega \mapsto \operatorname{vol}(\omega(v^*) \wedge \omega) \in \mathbb{C}.$$

The quadrics q_{v^*} are the Plücker quadrics defining the Grassmannian $G(3,V) \subset \mathbb{P}(\wedge^3 V)$. We denote by S_A the smooth $K3$ surface (see [26, Corollary 4.9]) of genus 6 defined on $\mathbb{P}(K^\perp)$ by the restrictions of the quadrics q_{v^*} and the quadric q_A^*. Let $S_A^{[2]}$ and $S_A^{[3]}$ denote the appropriate Hilbert schemes of points on S_A. Observe that we have a natural isomorphism

$$W^\vee = V^\vee \oplus V_0^\vee \in v^* + cv_0^* \mapsto q_{v^*} + cq_A^* \in H^0(\mathcal{I}_{S_A}(2)).$$

We then have a rational two-to-one map:

$$\varphi : S_A^{[2]} \dashrightarrow \mathbb{P}(W),$$

well defined on the open subset consisting of reduced subschemes whose span is not contained in $G(3,V)$, by associating to $\{\beta_1,\beta_2\} \subset S_A$ the hyperplane in $W^\vee = H^0(\mathcal{I}_S(2))$ consisting of quadrics containing the line $\langle\beta_1,\beta_2\rangle$. Let us describe this map more precisely. Since $\{\beta_1,\beta_2\} \subset K^\perp \subset \wedge^3 V \subset \wedge^3 W$, we have $\beta_i \wedge \kappa = 0$ for $i = 1,2$ and $\kappa \in K$, and hence also for $\kappa \in \tilde{K}$. Thus $\beta_i \in \wedge^3 W$ is contained in the space spanned by A and $F_{[v_0]}$. It follows that there exists $\alpha_i \in \wedge^2 V$ such that $\beta_i + v_0 \wedge \alpha_i \in A$. Let us fix such α_i (determined up to elements in K). Then $Q_A(\alpha_i) = \beta_i$ and

$$\begin{aligned} q_A^*(\lambda_1\beta_1 + \lambda_2\beta_2) &= \operatorname{vol}\big((\lambda_1\alpha_1 + \lambda_2\alpha_2) \wedge (\lambda_1\beta_1 + \lambda_2\beta_2)\big) \\ &= \lambda_1\lambda_2 \operatorname{vol}(\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1) \end{aligned}$$

since $q_A^*(\beta_1) = q_A^*(\beta_2) = 0$. But A is Lagrangian, so we have

$$\alpha_i \wedge \beta_i = 0 \quad \text{for } i = 1,2,$$

and

$$\operatorname{vol}(\alpha_1 \wedge \beta_2) = \operatorname{vol}(\alpha_2 \wedge \beta_1) := c_{12}.$$

Now, β_1 and β_2 are decomposable, i.e. $q_{v^*}(\beta_i) = 0$, and their linear span is not contained in $G(3,V)$. We may therefore choose a basis $\{v_1,\dots,v_5\}$ for V such that $\beta_1 = v_1 \wedge v_2 \wedge v_3$ and $\beta_2 = v_1 \wedge v_4 \wedge v_5$. A direct computation now shows

$$\Big(t_0 q_A^* + \sum_{i=1}^{5} t_i q_{v_i^*}\Big)(\lambda_1\beta_1 + \lambda_2\beta_2) = 2t_0c_{12}\lambda_1\lambda_2 + 2t_1\lambda_1\lambda_2$$

so

$$\varphi(\{\beta_1,\beta_2\}) = [c_{12}v_0 + v_1] \in \mathbb{P}(W). \tag{4.1}$$

It is proven in [26] that $\varphi(\{\beta_1,\beta_2\})$ lies on the EPW sextic associated to A. Let us present the proof in a way that we will be able to further generalize. It suffices to show that there are nonzero scalars x_1, x_2 and an element $\kappa \in K$ such that

$$\big(x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + v_0 \wedge \kappa\big) \wedge (c_{12}v_0 + v_1) = 0.$$

Indeed, this implies

$$\big[x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + v_0 \wedge \kappa\big] \in \mathbb{P}(F_{[c_{12}v_0+v_1]}) \cap \mathbb{P}(A).$$

Let us now denote by $\kappa_1, \kappa_2, \kappa_3$ a basis of K, then we consider the equation

$$\Big(x_1(\beta_1 + v_0 \wedge \alpha_1) + x_2(\beta_2 + v_0 \wedge \alpha_2) + \sum_{j=1}^{3} y_j v_0 \wedge \kappa_j\Big) \wedge (c_{12} v_0 + v_1) = 0,$$

i.e.

$$\Big(-x_1 c_{12} v_0 \wedge \beta_1 - x_2 c_{12} v_0 \wedge \beta_2 + x_1 v_0 \wedge \alpha_1 \wedge v_1 + x_2 v_0 \wedge \alpha_2 \wedge v_1 + \sum_{j=1}^{3} y_j v_0 \wedge \kappa_j \wedge v_1\Big) = 0.$$

To make this equation into a system of linear equations we multiply with the elements of basis in $\wedge^2 V$ and compose with the volume map vol : $\wedge^6 W \to \mathbb{C}$.

We obtain trivial equations when multiplying by $v_1 \wedge v_i$, $i = 2, 3, 4, 5$. Multiplying with $v_2 \wedge v_3$, we get

$$\begin{aligned}
&\kappa_i \wedge v_1 \wedge v_2 \wedge v_3 = \kappa_i \wedge \beta_1 = 0, \quad i = 1, 2, 3,\\
&\beta_1 \wedge v_2 \wedge v_3 = 0,\\
&\alpha_1 \wedge v_1 \wedge v_2 \wedge v_3 = \alpha_1 \wedge \beta_1 = 0,\\
&\alpha_2 \wedge v_1 \wedge v_2 \wedge v_3 = \alpha_2 \wedge \beta_1 = c_{12} = c_{12}\, \mathrm{vol}(v_0 \wedge \beta_2 \wedge v_2 \wedge v_3).
\end{aligned}$$

So the equation multiplied with $v_2 \wedge v_3$ is also trivial. Similarly, the equation multiplied with $v_4 \wedge v_5$ is trivial. So the only nontrivial linear equations are obtained by multiplying by forms in $\langle v_2 \wedge v_4, v_2 \wedge v_5, v_3 \wedge v_4, v_3 \wedge v_5\rangle$. Each of these 2-vectors annihilates β_1 and β_2, so we get the following four independent equations in five variables, with a unique solution up to scalars:

$$\begin{aligned}
&\Big(x_1\alpha_1 + x_2\alpha_2 + \sum_{j=1}^{3} y_j\kappa_j\Big) \wedge v_0 \wedge v_1 \wedge v_2 \wedge v_4 = 0,\\
&\Big(x_1\alpha_1 + x_2\alpha_2 + \sum_{j=1}^{3} y_j\kappa_j\Big) \wedge v_0 \wedge v_1 \wedge v_2 \wedge v_5 = 0,\\
&\Big(x_1\alpha_1 + x_2\alpha_2 + \sum_{j=1}^{3} y_j\kappa_j\Big) \wedge v_0 \wedge v_1 \wedge v_3 \wedge v_4 = 0,\\
&\Big(x_1\alpha_1 + x_2\alpha_2 + \sum_{j=1}^{3} y_j\kappa_j\Big) \wedge v_0 \wedge v_1 \wedge v_3 \wedge v_5 = 0.
\end{aligned}$$

Let us now consider the rational map $\psi : S_A^{[3]} \to G(3, W)$ defined on general subschemes $\mathfrak{s} \subset S_A$ of length 3 as the codimensional-three space in $W^\vee = H^0(\mathcal{I}_S(2))$ consisting of those quadrics which contain the plane spanned by $\mathfrak{s}$. It is clear that for a subscheme corresponding to a general triple of points $\{\beta_1, \beta_2, \beta_3\}$ we have

$$\psi(\{\beta_1, \beta_2, \beta_3\}) = \big[\langle \varphi(\{\beta_1, \beta_2\}) \wedge \varphi(\{\beta_1, \beta_3\}) \wedge \varphi(\{\beta_2, \beta_3\})\rangle\big]. \tag{4.2}$$

Proposition 4.1. *The map ψ is a generically two-to-one rational map onto D_A^2.*

Proof. Let β_1, β_2, β_3 be three general points on S_A. The proof then amounts to two lemmas:

Lemma 4.2. *The fiber of* ψ*,*

$$\psi^{-1}(\psi(\{\beta_1, \beta_2, \beta_3\})) = \big\{\{\beta_1, \beta_2, \beta_3\}, \{\gamma_1, \gamma_2, \gamma_3\}\big\}$$

is two triples of points on S_A *whose union is a set of six distinct points on a twisted cubic contained in* $G(3, V)$*.*

Proof. Let $U_{\beta_1}, U_{\beta_2}, U_{\beta_3} \subset V$ be the subspaces corresponding to β_1, β_2, β_3. Then there exists a unique three-dimensional subspace $U_{\beta_1,\beta_2,\beta_3}$ meeting each U_{β_i} in a two-dimensional space. It follows that $U_{\beta_1}, U_{\beta_2}, U_{\beta_3}$ are contained in the intersection $C_{\beta_1,\beta_2,\beta_3}$ of $\mathbb{P}^6$ with the Schubert cycle $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ in $G(3, V)$ of three-spaces meeting $U_{\beta_1,\beta_2,\beta_3}$ in a two-dimensional space. Since $\mathcal{S}_{\beta_1,\beta_2,\beta_3}$ is a cone over $\mathbb{P}^1 \times \mathbb{P}^2$, the considered intersection $C_{\beta_1,\beta_2,\beta_3}$ is, in general, a twisted cubic. Moreover, under the generality assumption,

$$C_{\beta_1,\beta_2,\beta_3} \cap S_A = C_{\beta_1,\beta_2,\beta_3} \cap q_A^*$$

consists of six points. Three of them are $\beta_1, \beta_2, \beta_3$ and the residual three will be denoted by $\gamma_1, \gamma_2, \gamma_3$. The linear span of $C_{\beta_1,\beta_2,\beta_3}$ is a $\mathbb{P}^3$, we denote it by $\mathbf{P}$, and its intersection with $G(3, V)$ is $\mathbf{P} \cap G(3, V) = C_{\beta_1,\beta_2,\beta_3}$. We denote by Π the plane $\langle \beta_1, \beta_2, \beta_3 \rangle$. Now, every quadric containing S_A and Π, when restricted to $\mathbf{P}$, decomposes into Π and another plane Π'. Since, in general, Π does not pass through γ_i for $i = 1, 2, 3$, the plane Π' must pass through the points γ_i for $i = 1, 2, 3$. This means that $\Pi' = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$. It is then clear that $\psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1, \gamma_2, \gamma_3\})$.

Assume on the other hand that $\psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1', \gamma_2', \gamma_3'\})$. Then, by (4.2) and (4.1), we deduce that $U_{\beta_1,\beta_2,\beta_3} = U_{\gamma_1',\gamma_2',\gamma_3'}$ hence $C_{\beta_1,\beta_2,\beta_3} = C_{\gamma_1',\gamma_2',\gamma_3'}$. It follows that $\langle \gamma_1', \gamma_2', \gamma_3' \rangle \subset \mathbf{P}$. But the net of quadrics corresponding to $\psi(\{\beta_1, \beta_2, \beta_3\}) = \psi(\{\gamma_1', \gamma_2', \gamma_3'\})$ defines on $\mathbf{P}$ two planes $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ and $\langle \beta_1, \beta_2, \beta_3 \rangle$. It follows that

$$\{\gamma_1', \gamma_2', \gamma_3'\} = \{\beta_1, \beta_2, \beta_3\} \quad \text{or} \quad \{\gamma_1', \gamma_2', \gamma_3'\} = \{\gamma_1, \gamma_2, \gamma_3\},$$

which ends the proof. □

Lemma 4.3. $\dim(T_{\psi(\{\beta_1,\beta_2,\beta_3\})} \cap A) = 2.$

Proof. By appropriate choice of the basis of V we can assume, without loss of generality, that $\beta_1 = v_1 \wedge v_2 \wedge v_3$, $\beta_2 = v_1 \wedge v_4 \wedge v_5$, and $\beta_3 = v_2 \wedge v_4 \wedge (v_3 + v_5)$. Observe as above that $\beta_i \wedge \kappa = 0$ for $i = 1, 2$ and $\kappa \in K$, hence β_i is contained in the space spanned by A and $F_{[v_0]}$. It follows that there exist $\alpha_i \in \wedge^2 V$ such that $\beta_i + v_0 \wedge \alpha_i \in A$. We fix such α_i (determined modulo K). Since A is Lagrangian, we have

$$\begin{aligned} \alpha_i \wedge \beta_i &= 0, \quad i = 1, 2, 3,\\ \alpha_1 \wedge \beta_2 &= \alpha_2 \wedge \beta_1 := c_{12},\\ \alpha_1 \wedge \beta_3 &= \alpha_3 \wedge \beta_1 := c_{13},\\ \alpha_2 \wedge \beta_3 &= \alpha_3 \wedge \beta_2 := c_{23}. \end{aligned}$$

As above, a direct computation gives

$$\begin{aligned}\varphi(\{\beta_1,\beta_2\}) &= c_{12}v_0 + v_1,\\ \varphi(\{\beta_1,\beta_3\}) &= c_{13}v_0 + v_2,\\ \varphi(\{\beta_2,\beta_3\}) &= -c_{23}v_0 + v_4.\end{aligned}$$

It follows that

$$\begin{aligned}T_{\psi(\{\beta_1,\beta_2,\beta_3\})} = \big\{\omega \in \wedge^3 W \mid \omega &\wedge (c_{12}v_0+v_1)\wedge(c_{13}v_0+v_2)\\ &= \omega\wedge(c_{12}v_0+v_1)\wedge(-c_{23}v_0+v_4)\\ &= \omega\wedge(c_{13}v_0+v_2)\wedge(-c_{23}v_0+v_4) = 0\big\}.\end{aligned}$$

Again we denote by $\kappa_1, \kappa_2, \kappa_3$ a basis of K. Now, $\beta_i + v_0\wedge\alpha_i \in A$ and $K\wedge v_0 \subset A$, so to prove the lemma it is enough to prove that the system of equations

$$\begin{cases}\Big(\sum_{i=1}^3 x_i(\beta_i + v_0\wedge\alpha_i) + \sum_{j=1}^3 y_j v_0\wedge\kappa_j\Big)\wedge(c_{12}v_0+v_1)\wedge(c_{13}v_0+v_2) = 0,\\ \Big(\sum_{i=1}^3 x_i(\beta_i + v_0\wedge\alpha_i) + \sum_{j=1}^3 y_j v_0\wedge\kappa_j\Big)\wedge(c_{12}v_0+v_1)\wedge(-c_{23}v_0+v_4) = 0,\\ \Big(\sum_{i=1}^3 x_i(\beta_i + v_0\wedge\alpha_i) + \sum_{j=1}^3 y_j v_0\wedge\kappa_j\Big)\wedge(c_{13}v_0+v_2)\wedge(-c_{23}v_0+v_4) = 0\end{cases}$$

in variables $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ has a two-dimensional set of solutions satisfying $x = (x_1, x_2, x_3) \neq 0$. By reductions as above and rearranging, we get the system

$$\begin{cases} v_0\wedge\Big(-c_{12}x_2\beta_2\wedge v_2 + c_{13}x_3\beta_3\wedge v_1 + \Big(\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i\Big)\wedge v_1\wedge v_2\Big) = 0,\\ v_0\wedge\Big(-c_{12}x_1\beta_1\wedge v_4 - c_{23}x_3\beta_3\wedge v_1 + \Big(\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i\Big)\wedge v_1\wedge v_4\Big) = 0,\\ v_0\wedge\Big(-c_{13}x_1\beta_1\wedge v_4 - c_{23}x_2\beta_2\wedge v_2 + \Big(\sum_{i=1}^3 x_i\alpha_i + y_i\kappa_i\Big)\wedge v_2\wedge v_4\Big) = 0.\end{cases} \tag{4.3}$$

To make the system of equations (4.3) into a system of linear equations we multiply each of the equations by the coordinate vectors and obtain a system of 18 linear equations in six coordinates. If we now denote the three left-hand side expressions dependent on (x, y) in the equations from (4.3) by $u_1(x,y), u_2(x,y), u_3(x,y) \in \wedge^5 W$, a straightforward computation, as above, shows that the following equations are trivial:

$$\begin{aligned}&u_1(x,y)\wedge v_0 = u_1(x,y)\wedge v_1 = u_1(x,y)\wedge v_2 = u_1(x,y)\wedge v_3 = 0,\\ &u_2(x,y)\wedge v_0 = u_2(x,y)\wedge v_1 = u_2(x,y)\wedge v_4 = u_2(x,y)\wedge v_5 = 0,\\ &u_3(x,y)\wedge v_0 = u_3(x,y)\wedge v_2 = u_3(x,y)\wedge v_4 = u_3(x,y)\wedge (v_3+v_5) = 0.\end{aligned}$$

The following products are equal:

$$\begin{aligned} u_1(x,y)\wedge v_4 = -u_2(x,y)\wedge v_2 &= u_3(x,y)\wedge v_1 \\ &= \Big(\sum_{i=1}^{3} x_i\alpha_i + y_i\kappa_i\Big)\wedge v_0\wedge v_1\wedge v_2\wedge v_4, \end{aligned}$$

while

$$u_1(x,y)\wedge v_5 = c_{13}x_3v_0\wedge\ldots\wedge v_5 + \Big(\sum_{i=1}^{3} x_i\alpha_i + y_i\kappa_i\Big)\wedge v_0\wedge v_1\wedge v_2\wedge v_5,$$

$$u_2(x,y)\wedge v_3 = c_{23}x_3v_0\wedge\ldots\wedge v_5 - \Big(\sum_{i=1}^{3} x_i\alpha_i + y_i\kappa_i\Big)\wedge v_0\wedge v_1\wedge v_3\wedge v_4,$$

$$\begin{aligned} u_3(x,y)\wedge(v_3-v_5) &= (c_{13}x_1 - c_{23}x_2)v_0\wedge\ldots\wedge v_5 \\ &\quad - \Big(\sum_{i=1}^{3} x_i\alpha_i + y_i\kappa_i\Big)\wedge v_0\wedge v_2\wedge v_4\wedge(v_3-v_5). \end{aligned}$$

So the 18 linear equations are reduced to the following four independent ones:

$$\begin{aligned} u_1(x,y)\wedge v_4 &= 0, & u_1(x,y)\wedge v_5 &= 0,\\ u_2(x,y)\wedge v_3 &= 0, & u_3(x,y)\wedge(v_3-v_5) &= 0. \end{aligned}$$

It follows that the system of linear equations admits a two-dimensional system of solutions. To prove that nonzero solutions satisfy $x\neq 0$ it is enough to observe that a solution with $x=0$ is a 3-vector $v_0\wedge\kappa$ with $\kappa\in K$ such that

$$\kappa\wedge v_1\wedge v_2 = \kappa\wedge v_1\wedge v_4 = \kappa\wedge v_2\wedge v_4 = 0.$$

But any such κ lies in the space $\langle v_1\wedge v_2, v_1\wedge v_4, v_2\wedge v_4\rangle = \wedge^2\langle v_1,v_2,v_4\rangle$. By assumption, $\mathbb{P}(v_0\wedge K)\subset\mathbb{P}(A)$ does not intersect $G(3,W)$, so this is impossible. Therefore the only solution of system (4.3) satisfying $x=0$ is $(x,y)=(0,0)$. □

Proposition 4.1 follows immediately from Lemmas 4.2 and 4.3. □

Remark 4.4. There is an alternative approach to Proposition 4.1. We consider the intersection F of the quadrics containing S_A together with a generic plane $B=\langle\beta_1,\beta_2,\beta_3\rangle$. This is a complete intersection of degree 8 with six ordinary double points that span a three-space. Three of them are the points of intersection of $B\cap S_A$ and the residual three points span another plane B' contained in our Fano threefold F. Since S_A does not contain any plane curve, if a plane passes through three points of S_A, these points are isolated in the intersection. If the plane is contained in F, the three points must therefore be three of the six ordinary double points. Since the three-space spanned by B and B' cuts F along the sum of $B\cup B'$, it follows that the degree of ψ is 2 at the point corresponding to F. On the other hand the generic complete intersection of three quadrics containing S_A has also six ordinary double points. The six ordinary double points then span a $\mathbb{P}^3$. In this context, a complete intersection that contains S_A corresponds to a point of the EPW cube exactly when the intersection of this $\mathbb{P}^3$ with F is a reducible quadric.

Next, we compute the codimension of the indeterminacy locus and the ramification locus of ψ.

Proposition 4.5. *The rational map ψ is well defined outside a set of codimension 2. Moreover, the ramification locus of ψ is of codimension ≥ 2.*

Before we pass to the proof of the proposition we introduce some more notation. Recall first that, by the assumption on generality of A, we know that S_A does not contain any line, conic or twisted cubic. Let F_A be the Fano threefold obtained as the intersection $G(3, V) \cap \langle S_A \rangle$. By the generality of A, it follows that F_A is smooth. Let $[U] \in G(3, V)$. Consider the Schubert cycle $\mathcal{S}_U = \{U' \in G(3, V) \mid \dim(U \cap U') \geq 2\}$. It is clear that in the Plücker embedding of $G(3, V) \subset \mathbb{P}(\wedge^3 V)$ the variety $\mathcal{S}_U$ is the tangent cone of $G(3, V)$ in $[U]$. It spans the projective tangent space and is a cone over $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex $[U]$. We are interested in intersections $\mathcal{S}_U \cap F_A$. Note that F_A is of degree 5 and has Picard group of rank 1 generated by the hyperplane class. Hence F_A does not contain any surface of degree ≤ 4. It follows that $\mathbb{P}^6_A \cap \mathcal{S}_U = F_A \cap \mathcal{S}_U$ is a cubic curve, a possibly reducible or non-reduced degeneration of a twisted cubic curve. We denote the corresponding subscheme of the Hilbert scheme of twisted cubics in F_A by $\mathcal{H}_A$.

Let $\mathcal{B}_1$ be the subset of $S_A^{[3]}$ consisting of those subschemes that are contained in a conic in $F_A \subset G(3, V)$. Since F_A is a linear section of $G(3, V)$ and contains no planes, the Hilbert scheme of conics in F_A admits a birational map to $\mathbb{P}(V)$ associating to a conic $\mathfrak{c}$ the intersection of three-spaces parametrized by points on $\mathfrak{c}$. It is hence of dimension 4 and we get that $\mathcal{B}_1$ is of dimension 4. Let $\mathcal{B}_2$ be the subset of $S_A^{[3]}$ consisting of those subschemes that meet some line contained in $G(3, V)$ in a scheme of length 2. Then $\mathcal{B}_2$ is also of dimension 4, since the Hilbert scheme of lines in F_A is isomorphic to $\mathbb{P}^2$ (cf. [26, Proposition 5.2] and [14]).

Lemma 4.6. *Let $\mathfrak{s}$ be a subscheme of length 3 in S_A corresponding to a point from $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$. Then there is a unique, possibly degenerate, twisted cubic from $\mathcal{H}_A$ that contains $\mathfrak{s}$. Furthermore, the induced map $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2) \to \mathcal{H}_A$ is dominant.*

Proof. Since $S_A \subset G(3, V) \cong G(2, V^\vee)$, we may characterize the elements of $\sigma \in S_A^{[3]}$ via the incidence of curves C_σ of degree 3 in $\mathbb{P}(V^\vee)$ supported on lines. For a general σ, the curve C_σ is the union of three lines and has a unique transversal line, a line that meets all three lines. If $\sigma \in S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$, then curve C_σ spans $\mathbb{P}(V^\vee)$ and contains no conic. It follows that C_σ admits a unique transversal line hence $\mathfrak{s}_\sigma$ is contained in $\mathcal{S}_U$ for a unique U. We conclude by the definition of $\mathcal{H}_A$. For dominancy of the map we observe that if $\mathfrak{c} \in \mathcal{H}_A$, then $\mathfrak{c} \cap q_A^*$ is contained in S_A and clearly contains a subscheme in $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$. □

We can now pass to the proof of Proposition 4.5

Proof of Proposition 4.5. Any subscheme $\mathfrak{s}$ of length 3 in S_A spans a plane $\Pi_{\mathfrak{s}}$. The map ψ associates to $\mathfrak{s}$ the space $V^q_{\mathfrak{s}}$ of quadrics containing $S_A \cup \Pi_{\mathfrak{s}}$. For general $\mathfrak{s}$ the latter is a space of dimension 3. Now, ψ is well defined exactly on those $\mathfrak{s}$ for which $\dim V^q_{\mathfrak{s}} = 3$. But $V^q_{\mathfrak{s}}$ is the kernel of the restriction map $H^0(S_A, \mathcal{I}_{S_A}(2)) \to H^0(\Pi_{\mathfrak{s}}, \mathcal{I}_{S_A \cap \Pi_{\mathfrak{s}}}(2))$. The latter kernel is three-dimensional unless $\dim H^0(\Pi_{\mathfrak{s}}, \mathcal{I}_{S_A \cap \Pi_{\mathfrak{s}}}(2)) \leq 2$. Hence ψ is not defined only if $S_A \cap \Pi_{\mathfrak{s}}$ has length at least 4. Then the intersection $\Pi_{\mathfrak{s}} \cap G(3, V)$ contains a scheme of length 4. As S_A contains no conics, $\Pi_{\mathfrak{s}}$ cannot be contained in $G(3,V)$. We infer by the proof of [20, Lemma 2.2] that $\Pi_{\mathfrak{s}} \cap G(3,V)$ contains a line or a unique conic. If $\Pi_{\mathfrak{s}} \cap G(3,V)$ contains a line, then it is either a reducible conic or the union of this line with a point. In the latter case, since S_A contains no lines, the intersection $\Pi_{\mathfrak{s}} \cap S_A$ does not contain any subscheme of length 4. It follows that there is a map with finite fibers from the indeterminacy locus of ψ to the Hilbert scheme of conics in $G(3,V) \cap \mathbb{P}^6$ which is of dimension 4. We conclude that the indeterminacy locus is of dimension at most 4. In fact, it is equal to 4 since a general $V_4 \subset V$ defines a conic in $G(3,V) \cap \mathbb{P}^6$ which meets S_A in four points.

Finally, to bound the dimension of the ramification locus, we again let $\mathfrak{s}$ be a subscheme of length 3 in S_A corresponding to a point from $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$. Then by Lemma 4.6 there is a possibly degenerate twisted cubic from $\mathcal{H}_A$ spanning a $\mathbb{P}^3$ and containing $\mathfrak{s}$. Now, from the proof of Proposition 4.1 we know that a point from $S_A^{[3]} \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$ can be in the ramification locus of ψ only if the quadric Q_A is totally tangent to the twisted cubic. The latter is a codimension-three condition on twisted cubics in $G(3,V) \cap \mathbb{P}^6$, hence by Lemma 4.6 a codimension-three condition for the ramification locus. To be more precise we have an incidence:

$$\mathcal{X} = \{(C,Q) \in \mathcal{H}_A \times H^0(O_{\mathbb{P}^6}(2)) \mid Q|_C \text{ is totally non-reduced}\}.$$

We compute its dimension from the projection onto $\mathcal{H}_A$. Indeed, fixing C, we get a codimension-three space of quadrics totally tangent to it. The dimension of the general fiber of the second projection follows giving codimension 3 in $\mathcal{H}_A$. □

5. Proof of Theorem 1.1

Let us choose a generic Lagrangian space A_0 satisfying

$$[A_0] \in \Delta \setminus (\Gamma \cup \Sigma) \subset LG_\eta(10, \wedge^3 W).$$

Note that from Lemma 3.7, we can choose A_0 such that K is generic in $F_{[v_0]}$. From Proposition 4.1 there is a rational two-to-one map

$$\psi : S_{A_0}^{[3]} \to D_2^{A_0}.$$

On the other hand from Proposition 3.1 there exists a double cover $Y_{A_0} \to D_2^{A_0}$ such that Y_{A_0} is a smooth sixfold with trivial canonical bundle. Our aim is to construct a birational map

$$S_{A_0}^{[3]} \dashrightarrow X_{A_0}.$$

We consider the subset $\mathcal{B}$ in $S_{A_0}^{[3]}$, the union of the indeterminacy locus and the ramification locus of the rational two-to-one map $\psi\colon S_{A_0}^{[3]} \to D_2^{A_0}$. Clearly the restriction of the map ψ to $S_{A_0}^{[3]} \setminus \mathcal{B}$ is an étale covering of degree 2 onto a smooth open subset $\mathcal{D} \subset D_2^{A_0}$. In particular, $\mathcal{D} \cap D_3^{A_0} = \emptyset$. Note that $S_{A_0}^{[3]}$ is simply connected and that by Proposition 4.5 the subset $\mathcal{B}$ is of codimension 2. This implies that $S_{A_0}^{[3]} \setminus \mathcal{B}$ is also simply connected. Thus $\pi_1(\mathcal{D}) = \mathbb{Z}_2$ and the restriction of ψ to $S_{A_0}^{[3]} \setminus \mathcal{B}$ is a universal covering.

Since $\mathcal{D}$ is disjoint from $D_3^{A_0}$, the restriction of the double cover $f_{A_0}\colon Y_{A_0} \to D_2^{A_0}$ to $f_{A_0}^{-1}(\mathcal{D})$ is also an étale covering.

By Proposition 3.1 the variety Y_{A_0} is smooth and irreducible. It follows that the restriction of f_{A_0} to $f_{A_0}^{-1}(\mathcal{D})$ is not trivial. We infer that it is also the universal covering. We deduce that Y_{A_0} is birational to $S_{A_0}^{[3]}$ and that $f_{A_0}^{-1}(\mathcal{D})$ is simply connected. It follows that Y_{A_0} is also simply connected because $f_{A_0}^{-1}(\mathcal{D})$ is obtained from the smooth variety Y_{A_0} by removing a subset of real codimension 2. Moreover, since both Y_{A_0} and $S^{[3]}$ have trivial canonical bundle, by [15, Theorem 1.1] they have equal Hodge numbers. Thus

$$h^2(\mathcal{O}_{Y_{A_0}}) = h^2(\mathcal{O}_{S_{A_0}^{[3]}}) = 1.$$

From the Beauville classification theorem [3, Theorem 2] we infer that Y_{A_0} is IHS.

Recall the notation

$$LG_\eta^1(10, \wedge^3 W) := \{[A] \in LG_\eta(10, \wedge^3 W) \mid \mathbb{P}(A) \cap G(3, W) = \emptyset \text{ and } \dim(A \cap T_U) \leq 3 \text{ for all } [U] \in G(3, W)\}.$$

Consider now the varieties

$$\mathcal{D}_k = \{([A], [U]) \in LG_\eta^1(10, \wedge^3 W) \times G(3, W) \mid [U] \in D_k^A\} \quad \text{for } k = 2, 3.$$

By globalizing the construction in Proposition 3.1 to the affine variety $LG_\eta^1(10, \wedge^3 W)$ we construct a variety $\mathcal{Y}$ which is a double cover of $\mathcal{D}_2$ branched in $\mathcal{D}_3$. We get a smooth family

$$\mathcal{Y} \to LG_\eta^1(10, \wedge^3 W)$$

with fibers $\mathcal{Y}_{[A]} = Y_A$ polarized by the divisor defining the double cover. In particular, a special fiber $\mathcal{Y}_{[A_0]} = Y_{A_0}$ is an IHS manifold. Since a smooth deformation of an IHS manifold is still IHS, we obtain that Y_A is IHS for every $A \in LG_\eta^1(10, \wedge^3 W)$. So $\mathcal{Y} \to LG_\eta^1(10, \wedge^3 W)$ is a family of IHS manifolds.

In order to show that the IHS sixfolds in the family $\mathcal{Y}$ are of $K3^{[3]}$-type we use the fact proved above that $S_{A_0}^{[3]}$ and Y_{A_0} are birational. Indeed, two birational IHS manifolds are deformation equivalent from [12, Theorem 4.6]. The Beauville–Bogomolov degree $q = 4$ of our polarization follows from our computation of degree in Section 2.3.

We end the proof of Theorem 1.1 by performing a study of the moduli map defined by the family $\mathcal{Y}$.

Proposition 5.1. *Let $\mathcal{M}$ be the coarse moduli space of polarized IHS sixfolds of $K3^{[3]}$-type and Beauville–Bogomolov degree* 4*. Let*

$$\mathfrak{M}_{\mathcal{Y}} : LG_\eta^1(10, \wedge^3 W) \to \mathcal{M}, \quad [A] \mapsto [Y_A]$$

be the map given by $\mathcal{Y}$. The image of $\mathfrak{M}_{\mathcal{Y}}$ is a dense open subset of a component of dimension 20 *in $\mathcal{M}$.*

For the proof we will need the following lemma.

Lemma 5.2. *Let $A \in LG_\eta^1(10, \wedge^3 W)$. If the linear automorphism $\mathfrak{g} \in \mathrm{PGL}(\wedge^3 W)$ is such that $D_2^A \subset G(3, W) \cap \mathfrak{g}(G(3, W))$, then $G(3, W) = \mathfrak{g}(G(3, W))$.*

Proof. Let us denote by G_1, G_2 the varieties $G(3, W)$ and $\mathfrak{g}(G(3, W))$ respectively.

Let $X \subset G_1 \cap G_2$ be an irreducible component of the intersection that contains D_A^2. Then X has codimension at most 3 in both G_1 and G_2 and spans $\mathbb{P}^{19}$. Furthermore it is contained in a complete intersection of quadric hypersurfaces on each G_i. If X has codimension 3, then $X = D_2^A$ and it lies in a complete intersection of three quadrics. But the complete intersection has degree $8 \cdot 42 = 336$, while D_2^A has degree 480, so this is impossible.

For lower codimension of X we first note that $D_2^A \subset D_1^A$. Since

$$[D_1^A] = [c_1(\mathcal{T}^\vee) \cap G(3, W)] \quad \text{and} \quad c_1(\mathcal{T}^\vee) = 4h,$$

the divisor D_1^A is a quartic hypersurface section of G_1 and G_2. So we may assume that D_2^A is contained in a quartic hypersurface section of X.

Consider the following subvariety in G_1: Let $V_5 \subset W$ be a general five-dimensional subspace, and let V_1 be a general one-dimensional subspace of V_5. Let

$$F(1,5) = \big\{[U] \in G_1 \mid V_1 \subset U \subset V_5\big\} \subset G_1$$

and denote by $P(1,5)$ the span of $F(1,5)$. Then $F(1,5)$ is a four-dimensional smooth quadric and the span $P(1,5)$ is a $\mathbb{P}^5$.

If X has codimension 2, then $X_{(1,5)} := X \cap F(1,5)$ is an irreducible surface. Furthermore, $X_{(1,5)}$ is contained in at least two quadric sections of $F(1,5)$. So $X_{(1,5)}$ has degree at most 8. On the other hand

$$D_{(1,5)} := D_2^A \cap F(1,5) \subset X_{(1,5)}$$

is a curve of degree 56, contained in a quartic hypersurface section of $X_{(1,5)}$, which has degree at most 32. Since this is absurd, we may assume that X has codimension 1, i.e. is a divisor in the varieties G_i.

Since D_2^A spans $\mathbb{P}^9$, the divisor X must be a quadric hypersurface section of each G_i. Then $P(1,5) \cap X$ is a complete intersection of two quadrics, and through every point of $P(1,5)$ there are infinitely many secant lines to X. The union of the spaces $P(1,5)$ as V_5 and V_1 varies is a variety $\Omega_1 \subset \mathbb{P}^{19}$, characterized in [7, Lemma 3.3] as the locus of points in $\mathbb{P}^{19}$ that lie on more than one secant line to G_1. Furthermore G_1 is the singular locus of Ω_1. Similarly, Ω_2 is defined with respect to G_2. By the above argument each $P(1,5)$ in Ω_1 is also contained in Ω_2. Thus $\Omega_1 \subset \Omega_2$. But then they coincide, and since $G_i = \mathrm{Sing}(\Omega_i)$, the two Grassmannians G_1 and G_2 coincide. □

Proof. We claim that $\mathfrak{M}_{\mathcal{Y}}([A_1]) = \mathfrak{M}_{\mathcal{Y}}([A_2])$ if and only if there exists a linear automorphism $\mathfrak{g} \in \mathrm{Aut}(G(3,W)) \simeq \mathbb{Z}/2 \times \mathrm{PGL}(W)$ such that $\mathfrak{g}(A_1) = A_2$. Indeed, assume that $\mathfrak{M}_{\mathcal{Y}}([A_1]) = \mathfrak{M}_{\mathcal{Y}}([A_2])$. Then Y_{A_1} and Y_{A_2}, polarized by ample classes defining double covers to $D_{A_1}^2$ and $D_{A_2}^2$ respectively, are isomorphic. It follows that there is a linear automorphism $\mathfrak{g} \in \mathrm{PGL}(\wedge^3 W)$ such that $\mathfrak{g}(D_{A_1}^2) = D_{A_2}^2$. It follows that

$$D_{A_2}^2 \subset G(3,W) \cap \mathfrak{g}(G(3,W)).$$

By Lemma 5.2, we deduce that $G(3,W) = \mathfrak{g}(G(3,W))$. It follows that $\mathfrak{g} \in \mathrm{Aut}(G(3,W))$.

By [24] the locus $LG_\eta^1(10, \wedge^3 W)$ is contained in the stable locus of the natural linearized $\mathrm{PGL}(W)$ action on $LG_\eta(10, \wedge^3 W)$. From our claim we hence infer that

$$\begin{aligned}\dim\big(\mathfrak{M}_{\mathcal{Y}}(LG_\eta^1(10, \wedge^3 W))\big) &\geq \dim LG_\eta^1(10, \wedge^3 W) - \dim(\mathrm{PGL}(W)) \\ &= 55 - 35 = 20.\end{aligned}$$

But 20 is the dimension of $\mathcal{M}$, so our map is surjective onto an (also by stability) open subset of a component of $\mathcal{M}$ of dimension 20. □

We conclude by determining the component of the moduli space that is filled by our family.

Recall that for $v \in H^2((K3)^{[3]}, \mathbb{Z})$ the divisibility of v is defined as the generator of the subgroup $(v, H^2((K3)^{[3]}, \mathbb{Z})) \subset \mathbb{Z}$ where $(\cdot, \cdot)$ is the scalar product induced by the Beauville–Bogomolov form. Note that for Beauville–Bogomolov degree 4 there are two possible divisibilities for H, either $l = 1$ or 2 (see [9, Proposition 3.6]). It follows from [2, Proposition 2.1 (3) and Corollary 2.4] that there are exactly two components, distinguished by the divisibility, of the coarse moduli space of polarized IHS sixfolds of $K3^{[3]}$-type and Beauville–Bogomolov degree 4. The following proposition, whose proof was pointed out to us by Kieran O'Grady, determines which one of those two components is filled by $\mathfrak{M}_{\mathcal{Y}}$.

Proposition 5.3. *The image of $\mathfrak{M}_{\mathcal{Y}}$ is open and dense in the connected component of the coarse moduli space of IHS sixfolds of $K3^{[3]}$-type, Beauville–Bogomolov degree* 4 *and divisibility* 2*.*

Proof. By the above, it remains to compute the divisibility of our polarization. For this, fix A general and denote the polarization by P. Observe that the involution of the double cover $Y_A \to D_A^2$ defined by the polarization is anti-symplectic. Indeed as an involution on an IHS manifold it is either symplectic or anti-symplectic, but the fixed point locus of a symplectic involution is a symplectic manifold (see [5, Proposition 3]) whereas the fixed locus of our involution is of dimension 3. This means that the involution must be anti-symplectic. Moreover, since we proved that our family is of maximal dimension, we may assume that Y_A has Picard group spanned by the polarization P. It follows that the action of the involution on $H^2(Y_A)$ has an invariant subspace spanned by the class $[P]$. Furthermore, the involution respects the Beauville–Bogomolov bilinear form $(\cdot, \cdot)$. Thus, since $([P], [P]) = 4$, the involution on $H^2(Y_A)$ is of the form

$$v \mapsto -v + \frac{1}{2}(v, [P])[P].$$

Since the involution must map integral cohomology to integral cohomology, it follows that $(v, [P])$ is even for all integral classes v. This implies that the divisibility of $[P]$ is not equal to 1. We infer that it is equal to 2. □

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Eingegangen 23. November 2015, in revidierter Fassung 24. Juni 2016