C^*-ALGEBRAS OF ALGEBRAIC DYNAMICAL SYSTEMS AND RIGHT LCM SEMIGROUPS

NATHAN BROWNLOWE, NADIA S. LARSEN, AND NICOLAI STAMMEIER

ABSTRACT. We introduce algebraic dynamical systems, which consist of an action of a right LCM semigroup by injective endomorphisms of a group. To each algebraic dynamical system we associate a C^*-algebra and describe it as a semigroup C^*-algebra. As part of our analysis of these C^*-algebras we prove results for right LCM semigroups. More precisely we discuss functoriality of the full semigroup C^*-algebra and compute its K-theory for a large class of semigroups. We introduce the notion of a Nica-Toeplitz algebra of a product system over a right LCM semigroup, and show that it provides a useful alternative to study algebraic dynamical systems.

1. INTRODUCTION

Dynamical systems arising from injective endomorphisms of countable, discrete groups have provided an important source of algebraic data from which to build C^*-algebras. The case of group automorphisms can by now be considered classical, and is modelled C^*-algebraically by the much-studied crossed product. More recent work in the area has focussed on injective endomorphisms which are not surjective. The study of C^*-algebras associated to a single injective endomorphism of a group with finite cokernel includes Hirshberg's work [14] on endomorphisms of amenable groups, and Cuntz and Vershik's work [10] on endomorphisms of abelian groups, while the situation with an injective endomorphism with infinite cokernel was considered by Vieira [30].

Through recent work in [2, 28] the authors have sought to build a unifying framework for the above constructions, while at the same time significantly broadening the scope of the theory. In [28] the third-named author generalised the single-endomorphism setting to actions of semigroups of injective endomorphisms via the notion of an *irreversible algebraic dynamical system*, which consists of an action of a countably generated, free abelian monoid P (with identity) by injective group endomorphisms subject to an independence condition which originated from [10]. A construction of very similar flavour was proposed by the first two authors, see [16, Example 5.10] for details. The C^*-algebra from [28] associated to an irreversible algebraic dynamical system has a presentation by generators and relations that encodes the action, and can be viewed alternatively as a Cuntz-Nica-Pimsner algebra. In [2] the authors looked at similar actions $\theta : P \curvearrowright G$, and studied them via the semidirect product semigroup $G \rtimes_\theta P$. The focus of [2] is on the semigroup C^*-algebra, in the sense of Li [21], associated to $G \rtimes_\theta P$.

In many cases, the semidirect products $G \rtimes_\theta P$ from [2] are examples of *right LCM semigroups*. These are left cancellative semigroups with the property that principal right ideals intersect to be either empty or another principal right ideal. Their C^*-algebras,

---

both the semigroup C^*-algebra and a boundary-quotient type C^*-algebra, have garnered recent attention [25, 4, 29]. Right LCM semigroups lead to a tractable and yet rich class of semigroup C^*-algebras that includes the C^*-algebras associated to quasi-lattice ordered pairs as introduced by Nica [24], and the Toeplitz type C^*-algebras associated to a self-similar actions in [18] as a counterpart to the Cuntz-Pimsner algebras from [23]. In [2], the authors pursued the study of the semigroup C^*-algebras of right LCM semigroups by establishing uniqueness results.

This paper serves two major purposes. The first, and our original motivation for this work, is to significantly extend the work of [28] by introducing the notion of an algebraic dynamical system and studying C^*-algebras that encode the action. An *algebraic dynamical system* consists of an action $\theta : P \curvearrowright G$ of a right LCM semigroup (with identity) P by injective endomorphisms of a group G which harmonise with the structure of the principal right ideals in P. This setting is more general than the one considered in [28] and gives a large class of new examples. To every algebraic dynamical system (G, P, θ) we associate a C^*-algebra $\mathcal{A}[G, P, \theta]$ that is universal for a unitary representation of G and a representation of P by isometries satisfying relations which encode the dynamics. These relations lead to a Wick ordering on $\mathcal{A}[G, P, \theta]$. Our first main result says that $\mathcal{A}[G, P, \theta]$ is canonically isomorphic to the full semigroup C^*-algebra of $G \rtimes_\theta P$. This explains and motivates our renewed interest in the study of the full semigroup C^*-algebra of right LCM semigroups. Therefore, a continuation of the work in [2] in order to build a general theory of semigroup C^*-algebras associated to right LCM semigroups constitutes the second major purpose of this paper.

Conceptionally, $\mathcal{A}[G, P, \theta]$ resembles a Toeplitz type crossed product for a generalised Exel-Larsen system naturally arising from (G, P, θ). To make this idea precise we appeal to Fowler's product systems of right-Hilbert bimodules over semigroups [12]. Explicitly, we show that his theory, that was originally developed for quasi-lattice ordered pairs, can be taken a considerable step further to the setting of right LCM semigroups.

Motivated by our analysis of the C^*-algebra associated to algebraic dynamical systems, we prove a number of results for right LCM semigroups:

(1) We investigate the functoriality of $S \mapsto C^*(S)$ and determine the right notion of a morphism for the category with objects right LCM semigroups so that the construction of the full semigroup C^*-algebra becomes functorial. We also characterise surjectivity of the induced morphism, and injectivity in the case that both left regular representations are faithful.

(2) We use the machinery of Cuntz, Echterhoff and Li [7] to show that, for a large class of right LCM semigroups S, the K-theory of $C^*(S)$ is given by the K-theory of $C_r^*(S^*)$, where S^* denotes the group of units in S.

(3) We extend the notion of a compactly aligned product system of right-Hilbert bimodules, and of a Nica covariant representation of such a product system, from the quasi-lattice ordered setting first introduced by Fowler [12] to the setting of right LCM semigroups. This leads to the definition of a Nica-Toeplitz algebra for product systems over right LCM semigroups.

Our description of $\mathcal{A}[G, P, \theta]$ as the semigroup C^*-algebra $C^*(G \rtimes_\theta P)$ of the right LCM semigroup $G \rtimes_\theta P$ allows us to apply the general theory from (1) and (2). We thus characterise the morphisms between algebraic dynamical systems which induce

morphisms of their C^*-algebras, and we compute the K-theory of $\mathcal{A}[G, P, \theta]$ for a large class of algebraic dynamical systems (G, P, θ). We also apply the general theory built in (3) to the setting of algebraic dynamical systems, where we construct a compactly aligned product system M from an algebraic dynamical system (G, P, θ) whose Nica-Toeplitz algebra $\mathcal{NT}(M)$ is canonically isomorphic to $\mathcal{A}[G, P, \theta]$. This allows us to conclude that, for right LCM semigroups S, $C^*(S)$ can be modelled as the Nica-Toeplitz algebra of the product system over S with fibres $\mathbb{C}$. We would like to stress the point that there is an abundance of compactly aligned product systems M, built from algebraic dynamical systems, for which the left action on some (or all) of the fibres of M is not by generalised compacts, as has frequently been the case in the context of quasi-lattice ordered pairs.

We finish with a section on examples, the most basic of which is the system $(\mathbb{Z}, P, \cdot)$ in which $P \subset \mathbb{Z}^\times$ is generated by a family of relatively prime nonzero integers acting on $\mathbb{Z}$ by multiplication. Despite the straightforward nature of these systems, we show that they provide an interesting class of C^*-algebras that warrant further study. Other examples of algebraic dynamical systems include the more general $(R, P, \cdot)$ in which R is a ring of integers in a number field and P a right LCM subsemigroup of $R^\times$ acting by multiplication, and shift systems $(\bigoplus_P G_0, P, \theta)$ in which G_0 is countable, P right LCM, and θ the natural shift action on $\bigoplus_P G_0$.

The structure of the paper is as follows: in Section 2 we introduce algebraic dynamical systems (G, P, θ) and their C^*-algebras $\mathcal{A}[G, P, \theta]$. In Section 3 we address the question of functoriality of $S \mapsto C^*(S)$, and in Section 4 we prove that $\mathcal{A}[G, P, \theta] \cong C^*(G \rtimes_\theta P)$, and apply the results of Section 3 to the context of $\mathcal{A}[G, P, \theta]$. In Section 5 we prove that for a large class of right LCM S we have $K_*(C^*(S)) \cong K_*(C_r^*(S^*))$, where $C_r^*(S^*)$ is the reduced C^*-algebra of the group of units S^* in S. In Section 6 we introduce the Nica-Toeplitz algebra for a product system over a right LCM semigroup, and then in Section 7 we describe $\mathcal{A}[G, P, \theta]$ as such a Nica-Toeplitz algebra. We finish in Section 8 with a discussion of examples of algebraic dynamical systems and their C^*-algebras. The flow of the paper is a steady back and forth between algebraic dynamical systems and general right LCM semigroups. We feel that this arrangement reflects in a proper way the fruitful interplay between the two topics in the context of C^*-algebraic constructions.

We would like to point out that algebraic dynamical systems are not the same as *algebraic dynamics*, a terminology going back to [26]. The objects of interest in this area are actions of countable, discrete groups on compact (abelian) groups, see [22] for a recent survey of highly interesting connections to other fields. We remark that algebraic dynamical systems (G, P, θ) can be thought of as algebraic dynamics provided that G is abelian and P is a group. We hope that our choice does not cause confusion, but rather stimulates an examination of the connections between these two concepts.

2. Algebraic dynamical systems and their C^*-algebras

In [28, Definition 1.5], the third author introduced the notion of *irreversible algebraic dynamical systems* and studied associated C^*-algebras, which under mild assumptions turn out to be unital UCT Kirchberg algebras. Recall that an irreversible algebraic dynamical system (G, P, θ) is given by a countable, discrete group G, a countably generated, free abelian semigroup P with identity, and a P-action θ on G by injective group endomorphisms such that

$$\theta_p(G) \cap \theta_q(G) = \theta_{pq}(G) \text{ if and only if } pP \cap qP = pqP.$$

Note that due to the above displayed condition and injectivity of θ_p, the group G needs to be infinite whenever P is nontrivial. Moreover, the only group automorphism of G present in such a dynamical system is id_G.

We are now going to provide a vast generalisation of the concept of an irreversible algebraic dynamical system (G, P, θ), where P is allowed to be any countable right LCM semigroup with identity, for example a group, and relations of the images of θ_p and θ_q no longer imply relations for pP and qP. Recall that a semigroup S is right LCM if it is left cancellative and for any p, q in S, the intersection of principal right ideals $pS \cap qS$ is either empty or of the form rS for some $r \in S$, see [4].

All identities for semigroups will be denoted by 1. If S is a semigroup with identity, we let S^* denote the group of invertible elements in S.

Definition 2.1. An *algebraic dynamical system* is a triple (G, P, θ) consisting of a countable, discrete group G, a countable, right LCM semigroup P with identity, and an action θ of P by injective group endomorphisms of G such that for all $p, q \in P$:

$$(2.1) \qquad \theta_p(G) \cap \theta_q(G) = \theta_r(G) \text{ whenever } r \in P \text{ satisfies } pP \cap qP = rP.$$

Following [2] we say that θ *respects the order on* P whenever (2.1) is satisfied. Note that if $r' \in P$ satisfies $rP = r'P$, then there is $x \in P^*$ such that $r' = rx$. Hence one has $\theta_{r'}(G) = \theta_r(\theta_x(G)) = \theta_r(G)$ as θ_x is a group automorphism of G.

The class of algebraic dynamical systems contains various types of examples, see Section 8. In particular, it includes the irreversible algebraic dynamical systems from [28]. We showed in [2, Proposition 8.2] that $G \rtimes_\theta P$ is a right LCM semigroup whenever (G, P, θ) is a triple as in Definition 2.1. In later sections we shall exploit this connection, but for the moment we focus our attention on the triple (G, P, θ).

Definition 2.2. For an algebraic dynamical system (G, P, θ) let $\mathcal{A}[G, P, \theta]$ be the universal C^*-algebra generated by a unitary representation u of G and an isometric representation s of P satisfying

(A1) $s_p u_g = u_{\theta_p(g)} s_p$ for all $p \in P, g \in G$, and

$$(A2) \quad s_p^* u_g s_q = \begin{cases} u_k s_{p'} s_{q'}^* u_\ell^* & \text{if } pP \cap qP = rP, \ pp' = qq' = r \\ & \text{and } g = \theta_p(k)\theta_q(\ell^{-1}) \text{ for some } k, \ell \in G, \\ 0 & \text{otherwise.} \end{cases}$$

We need to check that (A2) does not depend on the choice of r or (k, ℓ). If $r' \in P$ satisfies $r'P = rP$, then there exists $x \in P^*$ such that $r' = rx$. If $p'', q'' \in P$ satisfy $pp'' = qq'' = r'$, then $p'' = p'x, q'' = q'x$, and hence $s_{p''}s_{q''}^* = s_{p'}s_x s_x^* s_{q'}^* = s_{p'}s_{q'}^*$.

Now suppose $k_1, \ell_1 \in G$ satisfy $\theta_p(k_1)\theta_q(\ell_1^{-1}) = g = \theta_p(k)\theta_q(\ell^{-1})$ and $pP \cap qP = rP$ for some $r \in P$. But G is a group, so we have $\theta_p(k^{-1}k_1) = \theta_q(\ell^{-1}\ell_1)$. As θ respects the order on P, this element belongs to $\theta_r(G)$. Since θ_p and θ_q are injective, we get $k^{-1}k_1 = \theta_{p'}(k_2)$ and $\ell^{-1}\ell_1 = \theta_{q'}(\ell_2)$ for some $k_2, \ell_2 \in G$. Note that $\theta_p(k^{-1}k_1) = \theta_q(\ell^{-1}\ell_1)$ is the same as $\theta_r(k_2) = \theta_r(\ell_2)$, and injectivity of θ_r then implies that $k_2 = \ell_2$. Using (A1) we conclude

$$u_{k_1}s_{p'}s_{q'}^*u_{\ell_1}^* = u_k u_{k^{-1}k_1}s_{p'}s_{q'}^* u_{\ell^{-1}\ell_1}^* u_\ell^* = u_k s_{p'} u_{k_2} u_{\ell_2}^* s_{q'}^* u_\ell^* = u_k s_{p'}s_{q'}^* u_\ell^*.$$

Remark 2.3. We chose (A2) as one of our defining relations of $\mathcal{A}[G, P, \theta]$ because the expressions of the form $s_p^* u_g s_q$ are precisely the ones which need to be expressible in terms of $u_{g_1}s_{p_1}s_{p_2}^*u_{g_2}^*, g_i \in G, p_i \in P$ in order to get a *Wick ordering* on $\mathcal{A}[G, P, \theta]$, i.e. monomials in the generators where all starred generators appear on the right of all non-starred generators, see [17].

We like to think of $\mathcal{A}[G, P, \theta]$ as a Toeplitz type crossed product of $C^*(G)$ by the P-action θ. If one looks into the literature for semigroup crossed products, say by abelian semigroups, one encounters a covariance condition that deals with the analogue of $s_p^* u_g s_p$ by means of a transfer operator. This was introduced by Ruy Exel in [11] for $\mathbb{N}$ and was extended to abelian semigroups by the second author in [19]. This development gave rise to the notion of Exel-Larsen systems. In fact, for each algebraic dynamical system, one can associate a transfer operator L to the action $\alpha : C^*(G) \to C^*(G)$ induced by θ. Explicitly, L is the semigroup homomorphism from the opposite semigroup P^{op} of P to the semigroup of unital, positive, linear maps $C^*(G) \to C^*(G)$ given by $L_p(\delta_g) = \chi_{\theta_p(G)}(g)\delta_{\theta_p^{-1}(g)}$, where δ_g denote the generating unitaries in $C^*(G)$. In this way, we can regard $\mathcal{A}[G, P, \theta]$ as a Toeplitz type crossed product for the generalised Exel-Larsen system (G, P, α, L), see Section 7 for details.

The step of moving from abelian to more general semigroups P requires taking care of expressions of the form $s_p^* s_q$ as well. If P happens to be right LCM, Nica covariance as it appeared first in [24] provides a natural means to replace $s_p^* s_q$ by $s_{q'}s_{p'}^*$ if $pP \cap qP = rP$ with $r = pp' = qq'$ and declare it to vanish otherwise.

Altogether we see that (A2) combines the idea of a covariance condition coming from a transfer operator for θ and Nica covariance for the semigroup P. Note, however, that (A2) cannot be replaced by these two conditions in general, e.g. they say nothing about terms of the form $s_p^* u_g s_q$ with $g \neq 1$ and $p, q \in P \setminus P^*$ such that $rP \supset pP \cup qP$ implies $r \in P^*$ for all $r \in P$.

For each $g \in G, p \in P$ we denote the projection $u_g s_p s_p^* u_g^*$ by $e_{(g,p)}$. In the following lemma we show that relation (A2) is equivalent to a more familiar looking relation involving the product of these projections $e_{(g,p)}$.

Lemma 2.4. *Let (G, P, θ) be an algebraic dynamical system, u a unitary representation of G and s an isometric representation of P satisfying (A1). Then u and s satisfy (A2)*

if and only if

$$(2.2) \qquad e_{(g,p)}e_{(h,q)} = \begin{cases} e_{(g\theta_p(k),r)} & \text{if } pP \cap qP = rP \text{ for some } r \in P \text{ and} \\ & g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose (A2) holds. Since $e_{(g,p)}e_{(h,q)} = u_g s_p(s_p^* u_{g^{-1}h} s_q)s_q^* u_h^*$, we know from (A2) that $e_{(g,p)}e_{(h,q)}$ is zero unless $pP \cap qP = rP$ and $g^{-1}h = \theta_p(k)\theta_q(\ell^{-1})$ for some $k, \ell \in G$. If these conditions hold, and $p', q' \in P$ with $pp' = qq' = r$, then (A1) and (A2) give

$$e_{(g,p)}e_{(h,q)} = u_g s_p(s_p^* u_{g^{-1}h} s_q)s_q^* u_h^* = u_g s_p u_k s_{p'} s_{q'}^* u_\ell^* s_q^* u_h^* = u_{g\theta_p(k)} s_r s_r^* u_{h\theta_q(\ell)}^* = e_{(g\theta_p(k),r)}.$$

Now (2.2) holds because $g^{-1}h = \theta_p(k)\theta_q(\ell^{-1})$ for some $k, \ell \in G$ if and only if $g\theta_p(k) \in h\theta_q(G)$ for some $k \in G$.

Suppose (2.2) holds. Let $g \in G$ and $p, q \in P$. Then $s_p^* u_g s_q = s_p^* e_{(1,p)} e_{(g,q)} u_g s_q$ vanishes according to (2.2) unless $pP \cap qP = rP$ for some $r \in P$ and $1\theta_p(k) = g\theta_q(\ell)$ for some $k, \ell \in G$. Note that this is the same condition as in (A2). If we let $p', q' \in P$ satisfy $pp' = qq' = r$, we arrive at

$$s_p^* u_g s_q = s_p^* e_{(1,p)} e_{(g,q)} u_g s_q = s_p^* u_{\theta_p(k)} s_{pp'} s_{qq'}^* u_{g\theta_q(\ell)}^* u_g s_q = u_k s_{p'} s_{q'}^* u_\ell^*,$$

which establishes (A2). $\qquad \square$

Notation 2.5. Given $p \in P$, we will refer to a complete set of representatives for $G/\theta_p(G)$ as a *transversal* of $G/\theta_p(G)$ and denote it by T_p.

Remark 2.6. Note that (2.2) implies that for each $p \in P$ and each transversal T_p of $G/\theta_p(G)$, the projections $\{e_{(g,p)} \mid g \in T_p\}$ form a collection of mutually orthogonal projections.

We would like to discuss the question of functoriality for the mapping $(G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]$. To this end, we introduce a natural notion of a morphism between algebraic dynamical systems.

Definition 2.7. Suppose (G_1, P_1, θ_1) and (G_2, P_2, θ_2) are algebraic dynamical systems. A morphism from (G_1, P_1, θ_1) to (G_2, P_2, θ_2) is a pair (ϕ_G, ϕ_P), where

(i) $\phi_G : G_1 \to G_2$ is a group homomorphism,
(ii) $\phi_P : P_1 \to P_2$ is an identity preserving semigroup homomorphism, and
(iii) $\phi_G \circ \theta_{1,p} = \theta_{2,\phi_P(p)} \circ \phi_G$ holds for all $p \in P_1$.

The proof of the following lemma is straightforward and therefore omitted.

Lemma 2.8. *Suppose* (G_1, P_1, θ_1) *and* (G_2, P_2, θ_2) *are algebraic dynamical systems. If* (ϕ_G, ϕ_P) *is a morphism from* (G_1, P_1, θ_1) *to* (G_2, P_2, θ_2), *then* $(g, p) \mapsto (\phi_G(g), \phi_P(p))$ *defines an identity preserving semigroup homomorphism* $\phi_G \rtimes \phi_P : G_1 \rtimes_{\theta_1} P_1 \to G_2 \rtimes_{\theta_2} P_2$.

Remark 2.9. By Lemma 2.8, every morphism between algebraic dynamical systems gives rise to a homomorphism between the corresponding semigroups. The converse is false in general. More precisely, for given algebraic dynamical systems (G_1, P_1, θ_1) and (G_2, P_2, θ_2), a homomorphism $\varphi : G_1 \rtimes_{\theta_1} P_1 \to G_2 \rtimes_{\theta_2} P_2$ is induced by a morphism of the algebraic dynamical systems if and only if $\varphi(G_1 \times \{1\}) \subset G_2 \times \{1\}$ and $\varphi(\{1\} \times P_1) \subset \{1\} \times P_2$.

It is a natural question whether every morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ induces a $*$-homomorphism $\varphi_G \rtimes \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \to \mathcal{A}[G_2, P_2, \theta_2]$. We shall postpone an answer to this question until the end of Section 4.

3. C^*-ALGEBRAS OF RIGHT LCM SEMIGROUPS AND FUNCTORIALITY

In [21], Li investigated the functoriality of the assignment $S \mapsto C^*(S)$ in the context of $ax + b$-semigroups over integral domains, and applied his findings to the Toeplitz type C^*-algebras associated to number fields in [6]. He remarked that functoriality is not likely to be easily described for arbitrary S. Here we show that it can be successfully approached in case that S is a right LCM semigroup with identity.

We first recall Li's construction of the full and the reduced C^*-algebra associated to a discrete left cancellative semigroup S. A set $X \subseteq S$ is a right ideal if it is closed under right multiplication with any element of S. For each right ideal X, the sets

$$pX = \{px \mid x \in X\} \quad \text{and} \quad p^{-1}X = \{y \in S \mid py \in X\}.$$

are also right ideals. Li [21, p.4] defines $\mathcal{J}(S)$ to be the smallest family of right ideals of S satisfying

(a) $S, \emptyset \in \mathcal{J}(S)$; and
(b) $X \in \mathcal{J}(S)$ and $p \in S$ implies pX and $p^{-1}X \in \mathcal{J}(S)$.

The elements of $\mathcal{J}(S)$ are called *constructible right ideals*. The general form of a constructible right ideal is given in [21, Equation (5)]. We note that $\mathcal{J}(S)$ is also closed under finite intersections, a fact that can be derived from (a) and (b) using $pS \cap qS = p(p^{-1}(qS))$.

Definition 3.1. The set of constructible right ideals $\mathcal{J}(S)$ is called *independent* if for every $X, X_1, \ldots, X_n \in \mathcal{J}(S)$ we have

$$X_j \subsetneq X \text{ for all } 1 \leq j \leq n \Longrightarrow \bigcup_{j=1}^{n} X_j \subsetneq X.$$

Alternatively, $\mathcal{J}(S)$ is independent if $\bigcup_{j=1}^{n} X_j = X \Longrightarrow X_j = X$ for some j.

Let us recall Li's definition of the full semigroup C^*-algebra for S.

Definition 3.2. Let S be a discrete left cancellative semigroup. The *full semigroup C^*-algebra $C^*(S)$* is the universal C^*-algebra generated by isometries $(v_p)_{p \in S}$ and projections $(e_X)_{X \in \mathcal{J}(S)}$ satisfying

(L1) $v_p v_q = v_{pq}$;
(L2) $v_p e_X v_p^* = e_{pX}$;
(L3) $e_\emptyset = 0$ and $e_S = 1$; and
(L4) $e_X e_Y = e_{X \cap Y}$,

for all $p, q \in S$, $X, Y \in \mathcal{J}(S)$. Note that (L2) and (L3) give $v_p v_p^* = e_{pS}$ for all $p \in S$.

The reduced semigroup C^*-algebra $C_r^*(S)$ is defined by the left regular representation of S on $l^2(S)$: if $\{\varepsilon_p\}_{p \in S}$ is the canonical orthonormal basis of $\ell^2(S)$, then $C_r^*(S)$ is generated by the isometries V_p in $\mathcal{L}(\ell^2(S))$ given by $V_p(\varepsilon_q) = \varepsilon_{pq}$. The universal property

of the full semigroup C^*-algebra gives a $*$-homomorphism $\lambda : C^*(S) \to C^*_r(S)$ which sends v_p to V_p for each $p \in S$; this is the *left regular representation*.

A particularly nice class of semigroups in terms of the structure $\mathcal{J}(S)$ is the class of right LCM semigroups S: Recall from [2, Lemma 3.3] that $\mathcal{J}(S) = \{\emptyset\} \cup \{pS \mid p \in S\}$ provided that S is a right LCM semigroup with identity. Further, $\mathcal{J}(S)$ is always independent, see [2, Corollary 3.6]. Moreover, $C^*(S)$ is the closed span of products of the form $v_p v_q^*$ for $p, q \in S$, see [2, Lemma 3.11].

Theorem 3.3. *Suppose S_1 and S_2 are right LCM semigroups with identity. Let $(\tilde{v}_p)_{p \in S_1}$ and $(v_p)_{p \in S_2}$ be families of generating isometries in $C^*(S_1)$ and $C^*(S_2)$, respectively. An identity preserving semigroup homomorphism $\phi : S_1 \to S_2$ induces a $*$-homomorphism $\varphi : C^*(S_1) \to C^*(S_2)$ such that $\tilde{v}_p \mapsto v_{\phi(p)}$ if and only if ϕ satisfies*

$$(3.1) \qquad \phi(p)S_2 \cap \phi(q)S_2 = \phi(pS_1 \cap qS_1)S_2 \text{ for all } p, q \in S_1.$$

Proof. Let $\{\tilde{e}_{pS_1}\}_{p \in S_1}$ and $\{e_{pS_2}\}_{p \in S_2}$ be the standard projections in $C^*(S_1)$ and $C^*(S_2)$, so $\tilde{e}_{pS_1} = \tilde{v}_p \tilde{v}_p^*$ for $p \in S_1$ and similarly for S_2.

We show first that (3.1) is a necessary condition. Assume therefore that ϕ induces a $*$-homomorphism $\varphi : C^*(S_1) \to C^*(S_2)$. Given $p, q \in S_1$, we observe that $pS_1 \cap qS_1 \neq \emptyset$ forces $\phi(p)S_2 \cap \phi(q)S_2 \supset \phi(pS_1 \cap qS_1)S_2 \neq \emptyset$. Conversely, $\phi(p)S_2 \cap \phi(q)S_2 \neq \emptyset$ together with the equation

$$e_{\phi(p)S_2 \cap \phi(q)S_2} = e_{\phi(p)S_2} e_{\phi(q)S_2} = \varphi(\tilde{e}_{pS_1})\varphi(\tilde{e}_{qS_1}) = \varphi(\tilde{e}_{pS_1}\tilde{e}_{qS_1}) = \varphi(\tilde{e}_{pS_1 \cap qS_1})$$

implies that $pS_1 \cap qS_1 \neq \emptyset$. Hence we have $\phi(p)S_2 \cap \phi(q)S_2 \neq \emptyset$ if and only if $pS_1 \cap qS_1 \neq \emptyset$. Assuming both intersections to be non-empty, the fact that S_1 and S_2 are right LCM implies that there exist $r_1 \in S_1$ and $r_2 \in S_2$ such that $\phi(p)S_2 \cap \phi(q)S_2 = r_2 S_2$ and $pS_1 \cap qS_2 = r_1 S_1$. It follows that $r_2 S_2 \supseteq \phi(r_1)S_2$. If $\phi(r_1)S_2$ was a proper subset of $r_2 S_2$, then the image of $e_{r_2 S_2} - e_{\phi(r_1)S_2}$ under the left regular representation would be non-zero. Since the assumption on ϕ implies that $e_{r_2 S_2} = \varphi(\tilde{e}_{r_1 S_1}) = e_{\phi(r_1)S_2}$, we would obtain a contradiction. Hence, $r_2 \in \phi(r_1)S_2^*$ and thus (3.1) is satisfied.

Conversely, let us assume (3.1) holds. We claim that $(v_{\phi(p)})_{p \in S_1}$ and $(e_{\phi(p)S_2})_{p \in S_1}$ satisfy (L1)-(L4). Since S_1 and S_2 are right LCM, it suffices to prove (L4) because (L1)-(L3) are immediate. Let $p, q \in S_1$. If $pS_1 \cap qS_1 = \emptyset$, which by (3.1) is equivalent to $\phi(p)S_2 \cap \phi(q)S_2 = \emptyset$, then

$$e_{\phi(p)S_2} e_{\phi(q)S_2} = e_{\phi(p)S_2 \cap \phi(q)S_2} = 0,$$

as required for (L4). If $pS_1 \cap qS_1 \neq \emptyset$, then $pS_1 \cap qS_1 = rS_1$ for some $r \in S_1$. Now (3.1) ensures that

$$e_{\phi(p)S_2} e_{\phi(q)S_2} = e_{\phi(p)S_2 \cap \phi(q)S_2} = e_{\phi(pS_1 \cap qS_1)S_2} = e_{\phi(r)S_2}.$$
$\square$

Remark 3.4. Note that the set on the right hand side in (3.1) is always contained in the set on the left. As we can see from the proof, (3.1) is equivalent to the requirement that the map $\phi_{\mathcal{J}} : \mathcal{J}(S_1) \to \mathcal{J}(S_2)$ given by $\emptyset \mapsto \emptyset$ and $pS_1 \mapsto \phi(p)S_2$ is compatible with finite intersections. In fact, as S_1 and S_2 are right LCM semigroups with identity, the image $\mathcal{J}(\phi) := \phi_{\mathcal{J}}(\mathcal{J}(S_1))$ is the smallest family of right ideals in S_2 which satisfies

(a) $\emptyset, S_2 \in \mathcal{J}(\phi)$; and
(b) $X \in \mathcal{J}(\phi)$ and $p \in S_1$ implies $\phi(p)X$ and $\phi(p)^{-1}X \in \mathcal{J}(\phi)$.

Therefore we interpret (3.1) as a condition which characterises when $\mathcal{J}(\phi)$ is a *subfamily of constructible right ideals* of S_2 corresponding to $\phi(S_1)$.

Remark 3.5. The condition (3.1) is essentially the same as (b) from [21, Lemma 2.18]. The apparent difference is that the similar identity in [21, Lemma 2.18] has images under ϕ of right ideals in the left-hand side as well as in the right-hand side. Since we deal with right LCM semigroups, we can phrase the condition only using images of right ideals in the right-hand side of (3.1). It is striking that for right LCM semigroups there is no need to require [21, Lemma 2.18 (a)], and (b) turns out to be the precise condition which is needed to ensure that ϕ induces a $*$-homomorphism on the level of the full semigroup C^*-algebra.

Proposition 3.6. *Let S_1 and S_2 be right LCM semigroups with identity and suppose $\phi : S_1 \to S_2$ is a semigroup homomorphism satisfying (3.1). Then the induced $*$-homomorphism $\varphi : C^*(S_1) \to C^*(S_2)$ is surjective if and only if ϕ is surjective. If φ is injective, then ϕ is injective. The converse holds when the left regular representation implements an isomorphism $C^*(S_i) \cong C_r^*(S_i)$ for each $i = 1, 2$.*

Proof. Since $\phi(S_1)$ is a subsemigroup of S_2 and $\varphi(C^*(S_1)) \cap \{v_q \mid q \in S_2\} = \{v_{\phi(p)} \mid p \in S_1\}$, the statement about surjectivity of φ is immediate. Assume that ϕ is not injective. If $p, q \in S_1, p \neq q$ are such that $\phi(p) = \phi(q)$, then $\tilde{v}_p \neq \tilde{v}_q$ but $\varphi(\tilde{v}_p) = v_{\phi(p)} = v_{\phi(q)} = \varphi(\tilde{v}_q)$, showing that φ is not injective either.

Finally, suppose that ϕ is injective and that the left regular representation implements canonical isomorphisms $C^*(S_i) \cong C_r^*(S_i)$ for $i = 1, 2$. The left regular representation of S_2 on $\ell^2(S_2)$ restricts to a representation of $\phi(S_1)$. Compressing this by the projection onto $\phi(S_1)$ gives rise to a representation on $\ell^2(\phi(S_1)) \subset \ell^2(S_2)$. As ϕ is injective, this is simply a representation of S_1 which is unitarily equivalent to the left regular representation on $\ell^2(S_1)$. Since $C^*(S_1) \cong C_r^*(S_1)$, we deduce that this compression of φ is injective. Hence φ itself is injective. $\square$

We shall examine (3.1) more closely for right LCM semigroups built from algebraic dynamical systems in the next section.

4. A SEMIGROUP C^*-ALGEBRA DESCRIPTION OF $\mathcal{A}[G, P, \theta]$

In this section we show that, for an algebraic dynamical system (G, P, θ), the C^*-algebra $\mathcal{A}[G, P, \theta]$ is isomorphic to the full semigroup C^*-algebra of the semidirect product $G \rtimes_\theta P$, which is a right LCM semigroup by [2, Proposition 8.2]. As an application of this identification we study functoriality of the assignment $(G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]$ based on the findings of Section 3. Throughout this section we let (G, P, θ) be an algebraic dynamical system. We start with a description of the structure of the constructible right ideals $\mathcal{J}(G \rtimes_\theta P)$ of $G \rtimes_\theta P$.

Notation 4.1. We denote the principal right ideals $(g, p)\big(G \rtimes_\theta P\big)$ by $X_{(g,p)}$.

Proposition 4.2. *Let $X_{(g,p)}$ and $X_{(h,q)}$ be principal right ideals of $G \rtimes_\theta P$, for $g, h \in G$ and $p, q \in P$. Then*

$$X_{(g,p)} \cap X_{(h,q)} = \begin{cases} X_{(g\theta_p(k),r)} & \text{if } pP \cap qP = rP \text{ for some } r \in P \text{ and} \\ & g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence the family of constructible right ideals $\mathcal{J}(G \rtimes_\theta P)$ is independent. If $(T_p)_{p \in P}$ is a family of transversals, then $\mathcal{J}(G \rtimes_\theta P) = \{\emptyset\} \cup \{X_{(g,p)} \mid p \in P, g \in T_p\}$.

Proof. The formula for the intersection of principal ideals can be found in the proof of [2, Proposition 8.2]. The remaining claims follow from [2, Lemma 3.3 and Corollary 3.6]. $\qquad\square$

Corollary 4.3. $C^*(G \rtimes_\theta P)$ *is the closed linear span of $\{v_{(g,p)}v_{(h,q)}^* \mid g, h \in G, p, q \in P\}$.*

Proof. Since $G \rtimes_\theta P$ is right LCM, this follows from [2, Lemma 3.11]. $\qquad\square$

We now state the main result of this section.

Theorem 4.4. *There is an isomorphism $\varphi : \mathcal{A}[G, P, \theta] \to C^*(G \rtimes_\theta P)$ satisfying*

$$\varphi(u_g) = v_{(g,1)} \quad and \quad \varphi(s_p) = v_{(1,p)},$$

for all $g \in G$, $p \in P$.

Proof. Straightforward calculations show that $g \mapsto v_{(g,1)}$ is a unitary representation of G and $p \mapsto v_{(1,p)}$ is an isometric representation of P. We show that these representations satisfy (A1) and (A2). Recall that $e_{X_{(g,p)}} = v_{(g,p)}v_{(g,p)}^*$. Fix $g, h \in G$ and $p, q \in P$. Then

$$v_{(1,p)}v_{(g,1)} \overset{\text{(L1)}}{=} v_{(1,p)(g,1)} = v_{(\theta_p(g),p)} \overset{\text{(L1)}}{=} v_{(\theta_p(g),1)}v_{(1,p)},$$

so (A1) holds. To get (A2), we just check that (2.2) holds. We have

$$v_{(g,1)}v_{(1,p)}v_{(1,p)}^*v_{(g,1)}^* = v_{(g,p)}v_{(g,p)}^* = e_{X_{(g,p)}}.$$

Proposition 4.2 gives that

$$e_{X_{(g,p)}}e_{X_{(h,q)}} = \begin{cases} e_{X_{(g\theta_p(k),r)}} & \text{if } pP \cap qP = rP \text{ and } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ \emptyset & \text{otherwise,} \end{cases}$$

which is (2.2). The universal property of $\mathcal{A}[G, P, \theta]$ gives therefore a homomorphism $\varphi : \mathcal{A}[G, P, \theta] \to C^*(G \rtimes_\theta P)$ satisfying $\varphi(u_g) = v_{(g,1)}$ and $\varphi(s_p) = v_{(1,p)}$, for all $g \in G$, $p \in P$. To prove the result we use the universal property of $C^*(G \rtimes_\theta P)$ to find an inverse φ' for φ.

Consider the elements $u_g s_p, e_{(g,p)} \in \mathcal{A}[G, P, \theta]$. Each $u_g s_p$ is an isometry with range projection equal to $e_{(g,p)}$. The elements $\{u_g s_p, e_{(h,q)} \mid g, h \in G, p, q \in P\} \cup \{0\}$ are easily seen to satisfy (L1)–(L4): condition (L1) follows from an application of (A1), (L2) and (L3) are immediate, and (L4) follows from (2.2) and the formula for intersection of ideals from Proposition 4.2. The universal property of $C^*(G \rtimes_\theta P)$ now gives a homomorphism $\varphi' : C^*(G \rtimes_\theta P) \to \mathcal{A}[G, P, \theta]$ which satisfies $v_{(g,1)} \mapsto u_g$ and $v_{(1,p)} \mapsto s_p$, for all $g \in G$, $p \in P$. Hence φ and φ' are mutually inverse, so φ is an isomorphism. $\qquad\square$

Let us return to the question of functoriality for $(G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]$ raised at the end of Section 2. Now that we have realised $\mathcal{A}[G, P, \theta]$ as the semigroup C^*-algebra of the right LCM semigroup $G \rtimes_\theta P$, we can appeal to (3.1) which gives us a precise condition under which a morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems induces a $*$-homomorphism $\varphi_G \rtimes \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \to \mathcal{A}[G_2, P_2, \theta_2]$.

Definition 4.5. A morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems is called *admissible*, if, in addition to (i)–(iii) of Definition 2.7, the following two conditions hold for all $p, q \in P_1$:

(iv) $\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2$.
(v) $\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2)\big) = \theta_{1,p}(G_1)\theta_{1,q}(G_1)$ if $pP_1 \cap qP_1 \neq \emptyset$.

Remark 4.6. Let us comment briefly on these extra requirements:

(a) For both (iv) and (v), the inclusion $\supset$ holds for every morphism (ϕ_G, ϕ_P), so the statement is about the reverse inclusion.
(b) Condition (iv) is nothing but (3.1) for $\phi_P : P_1 \to P_2$. As long as we do not have more knowledge on P_1 and P_2, we cannot expect to get anything better than this, compare Theorem 3.3. But the existence of a semigroup homomorphism $\phi_P : P_1 \to P_2$ satisfying (iv) has structural consequences: If P_2 is such that the family of principal right ideals ordered by reverse inclusion is directed, then the same is true of P_1. In particular, this is the case if P_2 is a group.
(c) Condition (v) holds if ϕ_G is injective or the action θ_1 is by automorphisms of G_1.
(d) Note that (v) forces $\theta_{1,p} \in \mathrm{Aut}(G_1)$ whenever $\theta_{2,\phi_P(p)} \in \mathrm{Aut}(G_2)$. Hence a morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ with $P_1^* \subsetneq \phi_P^{-1}(P_2^*)$ is not admissible.

Before we proceed to the general considerations on functoriality we look at admissible morphisms in two particular situations.

Proposition 4.7. *Let P_1 be a group. Then every morphism $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ of algebraic dynamical systems is admissible.*

Proof. If P_1 is a group, then $P_1^* = P_1$ and $\phi_P(P_1) \subseteq P_2^*$. Hence $\phi_P(p)P_2 \cap \phi_P(q)P_2 = P_2 = \phi_P(pP_1 \cap qP_1)P_2$ for all $p, q \in P_1$, which establishes (iv). Moreover, θ_1 and $\theta_{2,\phi_P(\cdot)}$ are P_1-actions by group automorphisms of G_1 and G_2, respectively, so

$$\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2)\big) = G_1 = \theta_{1,p}(G_1)\theta_{1,q}(G_1)$$

for all $p, q \in P_1$. This shows (v). $\qquad\square$

Proposition 4.8. *Let $P_1 = \mathbb{F}_n^+$ for some $1 \leq n \leq \infty$. A morphism of algebraic dynamical systems $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ is admissible if and only if it satisfies $\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\big) = \theta_{1,p}(G_1)$ for all $p \in P_1$.*

Proof. Fix $p, q \in P_1$. Since pP_1 and qP_1 are disjoint unless $pP_1 \subset qP_1$ or $qP_1 \subset pP_1$, we can assume $q \in pP_1$, i.e., $q = pr$ for some $r \in P_1$. In this case,

$$\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(p)P_2 \cap \phi_P(p)\phi_P(r)P_2 = \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2$$

shows (iv). In addition, we get $\theta_{1,p}(G_1)\theta_{1,q}(G_1) = \theta_{1,p}(G_1)$ and

$$\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2)\big) = \phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\big),$$

so (v) is valid if and only if $\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\big) = \theta_{1,p}(G_1)$ for all $p \in P_1$. $\quad\square$

The next two results show that admissible morphisms yield the right class of morphisms with respect to functoriality for $(G, P, \theta) \mapsto \mathcal{A}[G, P, \theta]$.

Lemma 4.9. *Suppose* $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ *is a morphism of algebraic dynamical systems. Then* $\phi_G \rtimes \phi_P : G_1 \rtimes_{\theta_1} P_1 \to G_2 \rtimes_{\theta_2} P_2$ *satisfies* (3.1) *if and only if* (ϕ_G, ϕ_P) *is admissible.*

Proof. For convenience, let us denote $S_i := G_i \rtimes_{\theta_i} P_i$ for $i = 1, 2$ and $\phi := \phi_G \rtimes \phi_P$. Pick $p, q \in P_1$. We need to show that the equality

(4.1) $\phi(g, p)S_2 \cap \phi(h, q)S_2 = \phi\big((g, p)S_1 \cap (h, q)S_1\big)S_2$

holds for all $g, h \in G_1$ if and only if we have

(iv) $\phi_P(p)P_2 \cap \phi_P(q)P_2 \;=\; \phi_P(pP_1 \cap qP_1)P_2,$ and

(v) $\phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2)\big) \;=\; \theta_{1,p}(G_1)\theta_{1,q}(G_1)$ if $pP_1 \cap qP_1 \neq \emptyset$.

It was noticed in Remark 3.4 and Remark 4.6 (a) that, for all of these equations, the set on the right is contained in the set on the left. Moreover, Proposition 4.2 implies that $(g, p)S_1 \cap (h, q)S_1$ is non-empty if and only if

$$pP_1 \cap qP_1 \neq \emptyset \text{ and } g^{-1}h \in \theta_{1,p}(G_1)\theta_{1,q}(G_1).$$

Likewise, $\phi(g, p)S_2 \cap \phi(h, q)S_2$ is non-empty if and only if

(4.2) $\phi_P(p)P_2 \cap \phi_P(q)P_2 \neq \emptyset \text{ and } \phi_G(g^{-1}h) \in \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2).$

Now suppose (4.1) holds. For $g = h = 1$ and $p, q \in P_1$, (4.1) implies $\phi_P(p)P_2 \cap \phi_P(q)P_2 = \phi_P(pP_1 \cap qP_1)P_2$, which is (iv). Next, assume that $pP_1 \cap qP_1 \neq \emptyset$ and take $g = 1, h \in \phi_G^{-1}\big(\phi_G(G_1) \cap \theta_{2,\phi_P(p)}(G_2)\theta_{2,\phi_P(q)}(G_2)\big)$. We then have $\phi(1, p)S_2 \cap \phi(h, q)S_2 \neq \emptyset$. By (4.1), this implies $(1, p)S_1 \cap (h, q)S_1 \neq \emptyset$. Hence $h \in \theta_{1,p}(G_1)\theta_{1,q}(G_1)$ and (v) holds.

Conversely, suppose (iv) and (v) are satisfied. If $\phi(g, p)S_2 \cap \phi(h, q)S_2$ is empty, then there is nothing to show, so assume it is non-empty. As noted before, this means that we have (4.2). In this case, (v) implies $g^{-1}h \in \theta_{1,p}(G_1)\theta_{1,q}(G_1)$. Together with (iv) this yields (4.1) for (g, p) and (h, q).   □

Corollary 4.10. *A morphism* $(\phi_G, \phi_P) : (G_1, P_1, \theta_1) \to (G_2, P_2, \theta_2)$ *of algebraic dynamical systems induces a* $*$-*homomorphism* $\varphi_G \rtimes \varphi_P : \mathcal{A}[G_1, P_1, \theta_1] \to \mathcal{A}[G_2, P_2, \theta_2]$ *if and only if* (ϕ_G, ϕ_P) *is admissible.*

Proof. This follows from applying Lemma 4.9, Theorem 3.3 and Theorem 4.4.   □

5. THE K-THEORY FOR RIGHT LCM SEMIGROUPS THAT ARE LEFT ORE

The goal of this section is to compute the K-theory of $C_r^*(S)$ for a large class of right LCM semigroups. In view of Theorem 4.4, we will thus obtain the K-theory of $\mathcal{A}[G, P, \theta]$ whenever $G \rtimes_\theta P$ is left Ore and the left regular representation from $C^*(G \rtimes_\theta P)$ onto $C_r^*(G \rtimes_\theta P)$ is an isomorphism. As we shall see, right LCM left Ore semigroups form a natural class to which the results in [7, §7] apply nicely: thus, as an outcome we will obtain a stronger statement than just a computation of the K-groups.

Recall that a semigroup S is *right reversible* if every pair of non-empty left ideals has non-empty intersection. If S is cancellative and right reversible, then it is called a *left Ore semigroup*. It is well-known and an essential reason for the importance of the class

of left Ore semigroups that a semigroup S is left Ore if and only if it embeds into a group $\mathcal{G}$ such that $\mathcal{G} = S^{-1}S$. The standing assumption in [7, §7] is that S is a left Ore semigroup such that the family $\mathcal{J}(S)$ of constructible right ideals is independent. By [2, Corollary 3.6], every right LCM semigroup S with identity satisfies that $\mathcal{J}(S)$ is independent. For the remainder of the section we assume that S is a right LCM left Ore semigroup with identity, and we let $\mathcal{G}(S)$ denote its enveloping group. Recall also that S^* is the group of invertible elements in S.

We recall now notation and facts from [7]. For $q \in S$ and $X \in \mathcal{J}(S) \setminus \{\emptyset\}$, we let $q^{-1} \cdot X$ denote the subset $\{q^{-1}r \mid r \in X\}$ of $\mathcal{G}(S)$. This should not be confused with the right ideal $q^{-1}X$ defined in Section 3. However, if X is a subset of qS, then $q^{-1}X \subset S$ can be identified with $q^{-1} \cdot X \subset \mathcal{G}(S) = S^{-1}S$ using the canonical embedding $S \subset \mathcal{G}(S) = S^{-1}S$ because $q^{-1} \cdot X \subset 1^{-1}S$. Let Y denote the set

$$S^{-1} \cdot (\mathcal{J}(S) \setminus \{\emptyset\}) := \{q^{-1} \cdot X \mid q \in S, X \in \mathcal{J}(S) \setminus \{\emptyset\}\}.$$

The group $\mathcal{G}(S)$ acts on Y by left multiplication. This action can be described as follows: for $s_1^{-1}s_2 \in \mathcal{G}(S)$ and $t_1^{-1} \cdot t_2 S$ with $s_i, t_i \in S$, the left Ore condition for S implies that there are $r_1, r_2 \in S$ such that $s_1^{-1}s_2 t_1^{-1}t_2 = r_1^{-1}r_2$ and therefore

$$s_1^{-1}s_2 \cdot (t_1^{-1} \cdot t_2 S) = r_1^{-1} \cdot r_2 S.$$

For $X \in \mathcal{J}(S) \setminus \{\emptyset\}$, the stabiliser subgroup is

$$\mathcal{G}(S)^X = \{g \in \mathcal{G}(S) \mid g \cdot (1^{-1} \cdot X) = 1^{-1} \cdot X\}.$$

We need to choose a complete set of representatives $\mathcal{X}$ for the quotient $\mathcal{G}(S) \backslash Y$ with the property that $\mathcal{X} \subseteq \mathcal{J}(S) \setminus \{\emptyset\}$.

Lemma 5.1. *The action of $\mathcal{G}(S)$ on Y is transitive. In particular, $\{S\}$ is a complete set of representatives for the quotient $\mathcal{G}(S) \backslash Y$.*

Proof. Let $X_1 = t_1 S$ and $X_2 = t_2 S$ for $t_1, t_2 \in S$ be two elements of $\mathcal{J}(S) \setminus \{\emptyset\}$, see [2, Lemma 3.3]. Let $s_1, s_2 \in S$. We need to find $g \in \mathcal{G}(S)$ such that $g \cdot (s_1^{-1} \cdot X_1) = s_2^{-1} \cdot X_2$. Note that $r \cdot (1^{-1} \cdot S) = 1^{-1} \cdot rS$ for all $r \in S$. Hence, the choice $g = s_2^{-1} t_2 t_1^{-1} s_1$ works. $\square$

Lemma 5.2. *The stabiliser group $\mathcal{G}(S)^S$ of $S \in Y$ is S^*.*

Proof. It suffices to note that for all s, t in S we have

$$
\begin{aligned}
s^{-1}t \in \mathcal{G}(S)^S &\iff s^{-1}t \cdot (1^{-1} \cdot S) = (1^{-1} \cdot S) \\
&\iff 1^{-1} \cdot tS = 1^{-1} \cdot sS \\
&\iff tS = sS \\
&\iff s^{-1}t \in S^*.
\end{aligned}
$$

$\square$

It is proved in [7, Lemma 4.2] that the reduced semigroup C^*-algebra $C_r^*(S)$ is Morita equivalent to a certain reduced crossed product $D_r^{(\infty)}(S) \rtimes_r \mathcal{G}(S)$. Let $\{V_s\}_{s \in S}$ denote the family of isometries on $l^2(S)$ that generates $C_r^*(S)$ and consider the homomorphism

$$\Psi_S : C_r^*(S^*) \to C_r^*(S), \lambda_{s^{-1}t} \mapsto V_s^* V_t \text{ for } s, t \in S$$

from [7, Lemma 7.2].

We are now ready to state the main result of this subsection. With the preparation above, the proof is a direct application of [7, Corollary 7.4]. As in [7], K_* stands for the direct sum of K_0 and K_1 viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded abelian group and K^* is the K-homology.

Theorem 5.3. *If the enveloping group $\mathcal{G}(S)$ of S satisfies the Baum-Connes conjecture with coefficients in the $\mathcal{G}(S)$-algebras $c_0(Y)$ and $D_r^{(\infty)}(S)$, then $K_*(\Psi_S) : K_*(C_r^*(S^*)) \to K_*(C_r^*(S))$ is an isomorphism. If, moreover, $c_0(Y) \rtimes_r \mathcal{G}(S)$ and $D_r^{(\infty)}(S) \rtimes_r \mathcal{G}(S)$ satisfy the UCT or $\mathcal{G}(S)$ satisfies the strong Baum-Connes conjecture with coefficients in $c_0(Y)$ and $D_r^{(\infty)}(S)$, then $K^*(\Psi_S) : K^*(C_r^*(S^*)) \to K^*(C_r^*(S))$ is an isomorphism.*

We now apply Theorem 5.3 to the semigroups arising from algebraic dynamical systems (G, P, θ). Our main result is a considerable generalisation of [8, Theorem 6.3.4].

Proposition 5.4. *If (G, P, θ) is an algebraic dynamical system, then the group of invertible elements $(G \rtimes_\theta P)^*$ is isomorphic to $G \rtimes_\theta P^*$ and $G \rtimes_\theta P$ is left Ore precisely when P is left Ore.*

Proof. The claim about the group of units follows, for example, from [2, Lemma 2.4]. It is clear that P needs to be right reversible and right cancellative whenever $G \rtimes_\theta P$ has these two properties. In other words, P itself has to be a left Ore semigroup as left cancellation is assumed throughout. So let us suppose that this P is left Ore. If we have $(g_1\theta_{p_1}(h), p_1 q) = (g_2\theta_{p_2}(h), p_2 q)$, then right cancellation for P and then for G forces $p_1 = p_2$ and $g_1 = g_2$. To see that $G \rtimes_\theta P$ is right reversible, consider the left ideals $(G \rtimes_\theta P)(g, p)$ and $(G \rtimes_\theta P)(h, q)$. By assumption, there exists $r \in P$ with $r = p'p = q'q$ and hence

$$(1, q')(h, q) = (\theta_{q'}(h), r) = (\theta_{q'}(h)\theta_{p'}(g)^{-1}, p')(g, p) \in (G \rtimes_\theta P)(g, p) \cap (G \rtimes_\theta P)(h, q).$$

So $G \rtimes_\theta P$ is right reversible, and hence is a left Ore semigroup. $\qquad\square$

Recall that $\lambda : C^*(G \rtimes_\theta P) \to C_r^*(G \rtimes_\theta P)$ denotes the left regular representation and $\varphi : \mathcal{A}[G, P, \theta] \to C^*(G \rtimes_\theta P)$ is the isomorphism from Theorem 4.4. We have the following consequence of Theorem 5.3.

Corollary 5.5. *Let (G, P, θ) be an algebraic dynamical system such that P is a left Ore semigroup and let $\mathcal{A}[G, P, \theta]$ be its associated C^*-algebra. Assume that λ is an isomorphism. If $\mathcal{G}(G \rtimes_\theta P)$ satisfies the Baum-Connes conjecture with coefficients in the $\mathcal{G}(G \rtimes_\theta P)$-algebras $c_0(Y)$ and $D_r^{(\infty)}(G \rtimes_\theta P)$, then*

$$K_*\big((\lambda \circ \varphi)^{-1} \circ \Psi_{G \rtimes_\theta P}\big) : K_*(C_r^*(G \rtimes_\theta P^*)) \to K_*(\mathcal{A}[G, P, \theta])$$

is an isomorphism.

If, moreover, $c_0(Y) \rtimes_r \mathcal{G}(G \rtimes_\theta P)$ and $D_r^{(\infty)}(G \rtimes_\theta P) \rtimes_r \mathcal{G}(G \rtimes_\theta P)$ satisfy the UCT or $\mathcal{G}(G \rtimes_\theta P)$ satisfies the strong Baum-Connes conjecture with coefficients in $c_0(Y)$ and $D_r^{(\infty)}(G \rtimes_\theta P)$, then

$$K^*((\lambda \circ \varphi)^{-1} \circ \Psi_{G \rtimes_\theta P}) : K^*(C_r^*(G \rtimes_\theta P^*)) \to K^*(\mathcal{A}[G, P, \theta])$$

is an isomorphism.

Corollary 5.5 applies whenever $\mathcal{G}(G \rtimes_\theta P)$ satisfies the strong Baum-Connes conjecture with commutative coefficients, in particular it applies when $\mathcal{G}(G \rtimes_\theta P)$ is amenable. To check amenability, one possible strategy is as follows: by [8, Proposition 6.1.3], there is an embedding of $G \rtimes_\theta P$ into a semidirect product $G^{(\infty)} \rtimes_{\theta(\infty)} P^{-1}P$, where $G^{(\infty)}$ is the inductive limit of (G, θ_p) over P. The universal property of $\mathcal{G}(G \rtimes_\theta P)$ gives an injective group homomorphism $\mathcal{G}(G \rtimes_\theta P) \hookrightarrow G^{(\infty)} \rtimes_{\theta(\infty)} P^{-1}P$. Thus amenability of $\mathcal{G}(G \rtimes_\theta P)$ will follow from that of $G^{(\infty)} \rtimes_{\theta(\infty)} P^{-1}P$. For instance, the latter is amenable whenever G and $P^{-1}P$ are amenable.

6. Nica-Toeplitz algebras of product systems over right LCM semigroups

Our next aim is to describe $\mathcal{A}[G, P, \theta]$ as a C^*-algebra associated to a product system of right-Hilbert $C^*(G)$-bimodules. As the natural product system to consider is fibred over the right LCM semigroup P, we need to extend the existing theory to build Nica-Toeplitz algebras for such product systems. In this section we define the notions of a compactly aligned product system over a right LCM semigroup, and of a Nica covariant representation of such a product system. We use Parseval frames to find a condition under which a product system over a right LCM semigroup is compactly aligned, and we show that the Fock representation is Nica covariant.

Let A be a C^*-algebra and S a (countable, discrete) left cancellative semigroup with identity 1. Recall from [12] that a *product system* of right-Hilbert A-bimodules over S is a semigroup $M = \bigsqcup_{p \in S} M_p$, where each M_p is a right-Hilbert A-bimodule; $x \otimes_A y \mapsto xy$ determines an isomorphism of $M_p \otimes_A M_q$ onto M_{pq} for all $p, q \in S$ such that $p \neq 1$; the bimodule M_1 is ${}_A A_A$; and the products from $M_1 \times M_p \to M_p$ and $M_p \times M_1 \to M_p$ are given by the module actions of A on M_p.

Recall that for each $p \in S$ and each $\xi, \eta \in M_p$ the operator $\Theta_{\xi,\eta} : M_p \to M_p$ defined by $\Theta_{\xi,\eta}(\mu) := \xi \cdot \langle \eta, \mu \rangle_p$ is adjointable with $\Theta_{\xi,\eta}^* = \Theta_{\eta,\xi}$. The space $\mathcal{K}(M_p) := \overline{\text{span}}\{\Theta_{\xi,\eta} \mid \xi, \eta \in M_p\}$ is a closed two-sided ideal in $\mathcal{L}(M_p)$ called the *algebra of generalised compact operators* on M_p. For $p, q \in S$, there is a homomorphism $\iota_p^{pq} : \mathcal{L}(M_p) \to \mathcal{L}(M_{pq})$ characterised by

(6.1) $$\iota_p^{pq}(T)(\xi\eta) = (T\xi)\eta \quad \text{for all } \xi \in M_p, \eta \in M_q.$$

In other words, $\iota_p^{pq}(T) = T \otimes \text{id}_{M_q}$. For $q \notin pS$ we set $\iota_p^q = 0$.

A *representation* ψ of M in a C^*-algebra B is a map $M \to B$ such that

(1) each $\psi_p := \psi|_{M_p} : M_p \to B$ is linear, $\psi_1 : A \to B$ is a homomorphism,
(2) $\psi_p(\xi)\psi_q(\eta) = \psi_{pq}(\xi\eta)$ for all $p, q \in S$, $\xi \in M_p$, and $\eta \in M_q$; and
(3) $\psi_p(\xi)^*\psi_p(\eta) = \psi_1(\langle \xi, \eta \rangle_A^p)$ for all $p \in S$, and $\xi, \eta \in M_p$.

For each $p \in S$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(M_p) \to B$ characterised by

$$\psi^{(p)}(\Theta_{\xi,\eta}) = \psi_p(\xi)\psi_p(\eta)^*.$$

In [12, Definition 5.7] Fowler introduced the notion of a compactly aligned product system for quasi-lattice ordered pairs. We now extend this notion to product systems over right LCM semigroups.

Definition 6.1. A product system M over a right LCM semigroup S is called *compactly aligned* if for all $p, q, r \in S$ such that $pS \cap qS = rS$ and all $T_p \in \mathcal{K}(M_p), T_q \in \mathcal{K}(M_q)$ we have $\iota_p^r(T_p)\iota_q^r(T_q) \in \mathcal{K}(M_r)$.

Remark 6.2. Recall that if r is a right least common multiple of p and q, then all right least common multiples of p and q are of the form rx for $x \in S^*$. Note that M_x is a Morita equivalence from A to A for every $x \in S^*$ because $M_x \otimes_A M_{x^{-1}} \cong {}_A A_A \cong M_{x^{-1}} \otimes_A M_x$. Hence $T \mapsto T \otimes \text{id}$ defines an adjunction between M_x and $M_{x^{-1}}$ in the sense of [5, Definition 2.17]. Combining this with [5, Lemma 3.7], we deduce that the natural isomorphism $\iota_r^{rx} : \mathcal{L}(M_r) \to \mathcal{L}(M_{rx})$ restricts to an isomorphism from $\mathcal{K}(M_r)$ to $\mathcal{K}(M_{rx})$. Thus, $\iota_p^r(T_p)\iota_q^r(T_q)$ belongs to $\mathcal{K}(M_r)$ if and only if $\iota_p^{rx}(T_p)\iota_q^{rx}(T_q) \in \mathcal{K}(M_{rx})$ for all $x \in S^*$.

We will shortly present a class of compactly aligned product systems. Before doing so we recall that a *Parseval frame* for a right-Hilbert module Y over a unital C^*-algebra A is a countable family $(\xi_i)_{i \in I} \subset Y$ that satisfies the *reconstruction formula* $\eta = \sum_{i \in I} \xi_i \cdot \langle \xi_i, \eta \rangle$ for all $\eta \in Y$, where the convergence is in norm in Y. Parseval frames are also referred to as standard normalised tight frames, which are the main objects of interest in [13]. For our purposes the reconstruction formula is more relevant than the equivalent Parseval-type identity in A that defines a standard normalised tight frame.

Lemma 6.3. *Let A be a unital C^*-algebra, S a right LCM semigroup with identity, and $M = \bigsqcup_p M_p$ a product system of right-Hilbert A-bimodules such that the following condition is satisfied:*

(CA) *Whenever $p, q, r \in S$ satisfy $pS \cap qS = rS$, there are Parseval frames $(\xi_i^{(p)})_{i \in I_p}$ for M_p and $(\xi_i^{(q)})_{i \in I_q}$ for M_q, such that $\iota_p^r(\Theta_{\xi_i^{(p)}, \xi_i^{(p)}})\iota_q^r(\Theta_{\xi_j^{(q)}, \xi_j^{(q)}})$ is in $\mathcal{K}(M_r)$ for all $i \in I_p, j \in I_q$.*

Then M is compactly aligned.

Proof. Let $p, q, r \in S$ satisfy $pS \cap qS = rS$. By linearity and continuity it suffices to show $\iota_p^r(\Theta_{\eta_1, \eta_2})\iota_q^r(\Theta_{\zeta_1, \zeta_2}) \in \mathcal{K}(M_r)$ for all $\eta_1, \eta_2 \in M_p, \zeta_1, \zeta_2 \in M_q$. We claim that, due to the reconstruction formula, we have

$$\sum_{i \in F} \Theta_{\eta_1, \eta_2} \Theta_{\xi_i^{(p)}, \xi_i^{(p)}} \xrightarrow{F \subset I_p \text{ finite}} \Theta_{\eta_1, \eta_2}, \text{ and}$$
$$\sum_{j \in F'} \Theta_{\xi_j^{(q)}, \xi_j^{(q)}} \Theta_{\zeta_1, \zeta_2} \xrightarrow{F' \subset I_q \text{ finite}} \Theta_{\zeta_1, \zeta_2},$$

with convergence in norm in $\mathcal{K}(M_r)$. Firstly, note that it suffices to consider the second case, where the rank one operators corresponding to the elements from the Parseval frame are to the left, because the involution on $\mathcal{L}(M_p)$ is continuous. We start by observing that

$$\sum_{j \in F'} \Theta_{\xi_j^{(q)}, \xi_j^{(q)}} \Theta_{\zeta_1, \zeta_2} - \Theta_{\zeta_1, \zeta_2} = \Theta_{(\sum_{j \in F'} \xi_j^{(q)} \cdot \langle \xi_j^{(q)}, \zeta_1 \rangle) - \zeta_1, \zeta_2}.$$

It is routine to see that $\|\Theta_{\xi_1,\xi_2}\| \leq \|\xi_1\| \cdot \|\xi_2\|$ holds for all $\xi_1, \xi_2 \in M$, where M is a right-Hilbert module over A. Using this observation, we obtain

$$\|\sum_{j \in F'} \Theta_{\xi_j^{(q)},\xi_j^{(q)}}\Theta_{\zeta_1,\zeta_2} - \Theta_{\zeta_1,\zeta_2}\| = \|\Theta_{(\sum_{j \in F'} \xi_j^{(q)} \cdot \langle \xi_j^{(q)},\zeta_1 \rangle) - \zeta_1, \zeta_2}\|$$
$$\leq \|\sum_{j \in F'} \xi_j^{(q)} \cdot \langle \xi_j^{(q)}, \zeta_1 \rangle - \zeta_1\|\|\zeta_2\|.$$

Therefore $\sum_{j \in F'} \Theta_{\xi_j^{(q)},\xi_j^{(q)}}\Theta_{\zeta_1,\zeta_2}$ converges in norm to Θ_{ζ_1,ζ_2} as $F' \nearrow I_q$ and, likewise, $\sum_{i \in F} \Theta_{\eta_1,\eta_2}\Theta_{\xi_i^{(p)},\xi_i^{(p)}}$ converges in norm to Θ_{η_1,η_2} as $F \nearrow I_p$. From this we deduce

$$\sum_{(i,j) \in F \times F'} \iota_p^r(\Theta_{\eta_1,\eta_2})\iota_p^r(\Theta_{\xi_i^{(p)},\xi_i^{(p)}})\iota_q^r(\Theta_{\xi_j^{(q)},\xi_j^{(q)}})\iota_q^r(\Theta_{\zeta_1,\zeta_2}) \xrightarrow{F \times F' \subset I_p \times I_q \text{ finite}} \iota_p^r(\Theta_{\eta_1,\eta_2})\iota_q^r(\Theta_{\zeta_1,\zeta_2}),$$

where, again, we have convergence with respect to the operator norm on $\mathcal{L}(M_r)$. By condition (CA), the operator

$$\sum_{(i,j) \in F \times F'} \iota_p^r(\Theta_{\xi_i^{(p)},\xi_i^{(p)}})\iota_q^r(\Theta_{\xi_j^{(q)},\xi_j^{(q)}})$$

is in $\mathcal{K}(M_r)$ for each finite set $F \times F' \subset I_p \times I_q$. Since $\mathcal{K}(M_r)$ is a norm closed ideal in $\mathcal{L}(M_r)$, the result follows. $\square$

To associate a C^*-algebra to product systems over right LCM semigroups we now introduce the notion of Nica covariant representations of such product systems, which is an extension of Fowler's Nica covariance [12, Definition 5.7] for product systems of quasi-lattice ordered pairs.

Definition 6.4. A representation $\psi : M \to B$ of a compactly aligned product system M over a right LCM semigroup S in a C^*-algebra B is called *Nica covariant* if for all $p, q \in S$ and $T_p \in \mathcal{K}(M_p), T_q \in \mathcal{K}(M_q)$ we have

$$(6.2) \qquad \psi^{(p)}(T_p)\psi^{(q)}(T_q) = \begin{cases} \psi^{(r)}\big(\iota_p^r(T_p)\iota_q^r(T_q)\big) & \text{if } pS \cap qS = rS \text{ for some } r \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.5. An explanation is in order about this definition. Suppose we have $p, q \in S$ such that $pS \cap qS = rS$ for some $r \in S$. Let $T_p \in \mathcal{K}(M_p)$ and $T_q \in \mathcal{K}(M_q)$. Let $x \in S^*$, so that rx is another right LCM for p and q. Then a necessary condition for a representation $\psi : M \to B$ to be Nica covariant is that

$$(6.3) \qquad\qquad \psi^{(r)}\big(\iota_p^r(T_p)\iota_q^r(T_q)\big) = \psi^{(rx)}\big(\iota_p^{rx}(T_p)\iota_q^{rx}(T_q)\big).$$

Since $\iota_p^{rx} = \iota_r^{rx} \circ \iota_p^r$ and similarly for ι_q^{rx}, and since ι_r^{rx} is a homomorphism on $\mathcal{L}(M_r)$, we have

$$\iota_p^{rx}(T_p)\iota_q^{rx}(T_q) = \iota_r^{rx}(\iota_p^r(T_p)\iota_q^r(T_q)).$$

Using this, (6.3) will follow if $\iota_r^{rx}(\iota_p^r(T_p)\iota_q^r(T_q)) \in \mathcal{K}(M_{rx})$ and $\psi^{(rx)} \circ \iota_r^{rx} = \psi^{(r)}$. The first requirement is always satisfied because by Remark 6.2, the homomorphism ι_r^{rx} restricts to an isomorphism from $\mathcal{K}(M_r)$ onto $\mathcal{K}(M_{rx})$ and $\iota_p^r(T_p)\iota_q^r(T_q) \in \mathcal{K}(M_r)$ by compact alignment. In the next result we present a class of product systems for which the requirement $\psi^{(rx)} \circ \iota_r^{rx} = \psi^{(r)}$ holds for every representation ψ of M.

Lemma 6.6. *Suppose A is a unital C^*-algebra, S is a right LCM semigroup, and M is a product system of right-Hilbert A-bimodules over S. If for each $x \in S^*$ there is an element $1_x \in M_x$ satisfying $\Theta_{1_x,1_x} = \mathrm{id}_{M_x}$, then $\psi^{(rx)} \circ \iota_r^{rx} = \psi^{(r)}$ holds for every representation ψ of M and all $r \in S$.*

Proof. We claim that for each $r \in S$, $x \in S^*$ and $\mu, \nu \in M_r$ we have $\iota_r^{rx}(\Theta_{\mu,\nu}) = \Theta_{\mu 1_x, \nu 1_x} \in \mathcal{K}(M_{rx})$. To see this, let $\xi\eta \in M_{rx}$ with $\xi \in M_r$ and $\eta \in M_x$. We have

$$\Theta_{\mu 1_x, \nu 1_x}(\xi\eta) = (\mu 1_x) \cdot \langle \nu 1_x, \xi\eta \rangle_{rx} = (\mu 1_x) \cdot \langle 1_x, \varphi_x(\langle \nu, \xi \rangle_r)\eta \rangle_x$$
$$= \mu\big(\Theta_{1_x,1_x}(\varphi_x(\langle \nu, \xi \rangle_r)\eta)\big)$$
$$= (\mu \cdot \langle \nu, \xi \rangle_r)\eta = (\Theta_{\mu,\nu}(\xi))\eta$$
$$= \iota_r^{rx}(\Theta_{\mu,\nu})(\xi\eta).$$

Then the desired conclusion follows by linearity and continuity from

$$\psi^{(rx)} \circ \iota_r^{rx}(\Theta_{\mu,\nu}) = \psi^{(rx)}(\Theta_{\mu 1_x, \nu 1_x}) = \psi_r(\mu)\psi_x(1_x)\psi_x(1_x)^*\psi_r(\nu)^*$$
$$= \psi_r(\mu)\psi^{(x)}(\Theta_{1_x,1_x})\psi_r(\nu)^*$$
$$= \psi_r(\mu)\psi^{(x)}(\mathrm{id}_{M_x})\psi_r(\nu)^*$$
$$= \psi^{(r)}(\Theta_{\mu,\nu}).$$

$\square$

Example 6.7. Suppose that M is a product system of right-Hilbert A-bimodules over a right LCM semigroup S. Suppose that we have a collection of automorphisms $\{\alpha_x : x \in S^*\}$ such that each M_x is given by ${}_A A_{\alpha_x}$ with inner product $\langle a, b \rangle_x = \alpha_x^{-1}(a^*b)$. Then the elements $1_x := 1 \in A$ satisfy the hypothesis of Lemma 6.6. The product systems we form in the next section fall in this class of examples.

It was observed [12] that a natural representation for product systems M is the *Fock representation*. The details of the construction are as follows: let $\varphi_p : A \to \mathcal{L}(M_p)$ be the $*$-homomorphism that implements the left action in M_p. The vector space

$$F(M) = \big\{(\xi_p)_{p \in S} \mid \xi_p \in M_p, \sum_{p \in S} \|\xi_p\|_p^2 < \infty\big\}$$

becomes a right-Hilbert A-bimodule when equipped with the inner product

$$\langle (\xi_p)_{p \in S}, (\eta_p)_{p \in S} \rangle = \sum_{p \in S} \langle \xi_p, \eta_p \rangle_p,$$

together with left and right actions

$$\varphi_{F(M)}(a)(\xi_p)_{p \in S} = (\varphi_p(a)\xi_p)_{p \in S} \text{ and } (\xi_p)_{p \in S} \cdot a = (\xi_p \cdot a)_{p \in S}.$$

It is routine to check that the map $\psi_F : M \to \mathcal{L}(F(M))$ given by

$$\psi_{F,p}(\xi)(\eta_q)_{q \in S} = (\chi_{pS}(q) \cdot \xi\eta_r)_{q \in S}$$

defines a representation of M, where χ_{pS} denotes the characteristic function on pS and $q = pr$. Note that $M_p \otimes_A M_q \cong M_{pq}$ allows us to write $\eta_{pq} = \sum_{i \in I_{pq}} \eta'_{pq,i} \otimes \eta''_{pq,i}$ for suitable $\eta'_{pq,i} \in M_p, \eta''_{pq,i} \in M_q$ and we have

$$\psi_{F,p}(\xi)^*(\eta_q)_{q \in S} = \Big(\sum_{i \in I_{pq}} \varphi_{p^{-1}q}(\langle \xi, \eta'_{pq,i} \rangle_p)\eta''_{pq,i}\Big)_{q \in S}.$$

Proposition 6.8. *If M is a compactly aligned product system over a right LCM semigroup, then the Fock representation ψ_F is Nica covariant.*

Proof. The following is a straightforward adaptation of the proof for the quasi-lattice ordered case from [16, Subsection 2.3]. Let $p, q \in S$ and $\xi, \eta \in M_p, \xi', \eta' \in M_q$. If $pS \cap qS = \emptyset$, then $\psi_F^{(p)}(\Theta_{\xi,\eta})\psi_F^{(q)}(\Theta_{\xi',\eta'})(\zeta_r)_{r \in S} = 0$ for all $(\zeta_r)_{r \in S} \in F(M)$, so $\psi_F^{(p)}(\Theta_{\xi,\eta})\psi_F^{(q)}(\Theta_{\xi',\eta'}) = 0$. So let $r \in S$ satisfy $pS \cap qS = rS$. Since M is compactly aligned we know that $\iota_p^r(\Theta_{\xi,\eta})\iota_q^r(\Theta_{\xi',\eta'}) \in \mathcal{K}(M_r)$. If $\iota_p^r(\Theta_{\xi,\eta})\iota_q^r(\Theta_{\xi',\eta'}) = \Theta_{\tilde{\xi},\tilde{\eta}}$ for some $\tilde{\xi}, \tilde{\eta} \in M_r$, then it is routine to check

$$\psi_F^{(p)}(\Theta_{\xi,\eta})\psi_F^{(q)}(\Theta_{\xi',\eta'}) = \psi_F^{(r)}(\Theta_{\tilde{\xi},\tilde{\eta}}).$$

By linearity and continuity, the required calculations extend to the general case of $\iota_p^r(\Theta_{\xi,\eta})\iota_q^r(\Theta_{\xi',\eta'}) \in \mathcal{K}(M_r)$. In other words, we have

$$\psi_F^{(p)}(\Theta_{\xi,\eta})\psi_F^{(q)}(\Theta_{\xi',\eta'}) = \psi_F^{(r)}(\iota_p^r(\Theta_{\xi,\eta})\iota_q^r(\Theta_{\xi',\eta'})),$$

so ψ_F is Nica covariant. $\square$

Definition 6.9. For a compactly aligned product system M over a right LCM semigroup S, the Nica-Toeplitz algebra $\mathcal{NT}(M)$ is the universal C^*-algebra generated by a Nica covariant representation of M. We denote the Nica covariant representation generating $\mathcal{NT}(M)$ by j_M.

Note that $\mathcal{NT}(M)$ is always nonzero due to Proposition 6.8.

7. Nica-Toeplitz algebras for algebraic dynamical systems

In this section we describe the C^*-algebra $\mathcal{A}[G, P, \theta]$ as the Nica-Toeplitz algebra of a product system of right-Hilbert $C^*(G)$-bimodules over the right LCM semigroup P. Throughout this section, let (G, P, θ) be an algebraic dynamical system. We denote the canonical generating unitaries in $C^*(G)$ by δ_g, $g \in G$. The action θ induces an action $\alpha : P \to \text{End}\, C^*(G)$, i.e. $\alpha_p(\delta_g) = \delta_{\theta_p(g)}$. Note that injectivity passes from θ to α, so α is an action by unital injective $*$-homomorphisms. We also have an action of the opposite semigroup P^{op} by unital, positive, linear maps $L_p : C^*(G) \to C^*(G)$ given by $L_p(\delta_g) = \chi_{\theta_p(G)}(g)\delta_{\theta_p^{-1}(g)}$. Each L_p is a transfer operator for $(C^*(G), \alpha_p)$, in the sense that $L_p(\alpha_p(a)b) = aL_p(b)$ for all $a, b \in C^*(G)$, and so $(C^*(G), P, \alpha, L)$ is a dynamical system in the style of the Exel-Larsen systems studied in [3, 19].

Remark 7.1. We use the by now standard construction for associating a right-Hilbert $C^*(G)$-bimodule to each $p \in P$: we take $C^*(G)$ as a vector space and give it a right action $a \cdot b = a\alpha_p(b)$, and a pre-inner product $\langle a, b\rangle_p = L_p(a^*b)$. We mod out by $N_p := \{a \in C^*(G) \mid L_p(a^*a) = 0\}$ and complete to get a right-Hilbert $C^*(G)$-module M_p. Left multiplication by elements of $C^*(G)$ extends to a left action φ_p on M_p by adjointable operators. We denote the quotient map by $\pi_p : C^*(G) \to C^*(G)/N_p$ and note that the image of π_p is dense in M_p.

Notation 7.2. For the remainder of this section we will write $\mathcal{E}_{g,p}$ for the generalised rank one projection $\Theta_{\pi_p(\delta_g),\pi_p(\delta_g)} \in \mathcal{K}(M_p)$, where $p \in P$ and $g \in G$.

Proposition 7.3. *Suppose $\{M_p \mid p \in P\}$ is the collection of right-Hilbert $C^*(G)$-bimodules from Remark 7.1. For each $p \in P$ and each transversal T_p of $G/\theta_p(G)$ the family $\{\mathcal{E}_{(g,p)} \mid g \in T_p\}$ consists of pairwise orthogonal projections in $\mathcal{K}(M_p)$. In particular, the set $\{\pi_p(\delta_g) \mid g \in T_p\}$ is an orthonormal basis for M_p. Furthermore, $\varphi_p(C^*(G)) \subset \mathcal{K}(M_p)$ holds if and only if $G/\theta_p(G)$ is finite.*

Proof. For each $g, h \in T_p$ we have

$$\langle \pi_p(\delta_g), \pi_p(\delta_h) \rangle_p = L_p(\delta_{g^{-1}h}) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{E}_{g,p}\mathcal{E}_{h,p} = 0$ whenever $g, h \in T_p$ are distinct. For each $h \in G$ there is a unique $g' \in T_p$ with $h \in g'\theta_p(G)$, and hence

$$\sum_{g \in T_p} \mathcal{E}_{g,p}(\pi_p(\delta_h)) = \sum_{g \in T_p} \chi_{g\theta_p(G)}(h)\pi_p(\delta_h) = \chi_{g'\theta_p(G)}(h)\pi_p(\delta_h) = \pi_p(\delta_h).$$

By linearity and continuity, this means $\sum_{g \in T_p} \mathcal{E}_{g,p}(m) = m$ for all $m \in M_p$, and hence $\{\pi_p(\delta_g) \mid g \in T\}$ is an orthonormal basis for M_p.

For the third assertion note that $\varphi_p(1) = \sum_{g \in T_p} \mathcal{E}_{g,p}$, which, in case T_p is finite, means that $\varphi_p(1) \in \mathcal{K}(M_p)$, and hence $\varphi_p(C^*(G)) \subset \mathcal{K}(M_p)$. If T_p is infinite, then $\varphi_p(1)$ is an infinite sum (in the strict topology) of mutually orthogonal, equivalent, non-zero projections $\mathcal{E}_{g,p}$. Thus, $\varphi_p(1)$ is not compact. $\square$

We can use the bimodules M_p to construct a product system of right-Hilbert $C^*(G)$-bimodules over P.

Proposition 7.4. *Suppose $\{M_p \mid p \in P\}$ is the collection of right-Hilbert $C^*(G)$-bimodules from Remark 7.1. Then the semigroup $M := \bigsqcup_{p \in P} M_p$ with multiplication characterised by*

$$\pi_p(\delta_g)\pi_q(\delta_h) = \pi_{pq}(\delta_{g\theta_p(h)}) \quad \text{for } p, q \in P, \ g, h \in G,$$

is a product system of right-Hilbert $C^*(G)$-bimodules over P.

Proof. The map $(\pi_p(\delta_g), \pi_q(\delta_h)) \mapsto \pi_{pq}(\delta_{g\theta_p(h)})$ is bilinear, and extends to a surjective map from the algebraic balanced tensor product $M_p \odot_{C^*(G)} M_q$ to M_{pq}. This map is equivariant with respect to the left and right actions of $C^*(G)$. It also follows from the calculation

$$\begin{aligned}
\langle \pi_p(\delta_{g_1}) \otimes \pi_q(\delta_{h_1}), \pi_p(\delta_{g_2}) \otimes \pi_q(\delta_{h_2}) \rangle &= \langle \pi_q(\delta_{h_1}), \langle \pi_p(\delta_{g_1}), \pi_p(\delta_{g_2}) \rangle_p \cdot \pi_q(\delta_{h_2}) \rangle_q \\
&= L_q(\delta_{h_1^{-1}} L_p(\delta_{g_1^{-1}}\delta_{g_2})\delta_{h_2}) \\
&= L_q(L_p(\alpha_p(\delta_{h_1^{-1}})\delta_{g_1^{-1}}\delta_{g_2}\alpha_p(\delta_{h_2}))) \\
&= L_{pq}(\delta_{\theta_p(h_1^{-1})}\delta_{g_1^{-1}}\delta_{g_2}\delta_{\theta_p(h_2)}) \\
&= \langle \pi_{pq}(\delta_{g_1\theta_p(h_1)}), \pi_{pq}(\delta_{g_2\theta_p(h_2)}) \rangle
\end{aligned}$$

that $M_p \odot_{C^*(G)} M_q \to M_{pq}$ preserves the inner product, and hence extends to an isomorphism $M_p \otimes_{C^*(G)} M_q \to M_{pq}$. The remaining conditions for $M := \bigsqcup_{p \in P} M_p$ to be a product system are straightforward to check. $\square$

Proposition 7.5. *Given $p, q \in P$ and transversals T_p, T_q of $G/\theta_p(G)$, $G/\theta_q(G)$, respectively, the map $m_{p,q} : T_p \times T_q \to G$ given by $(g, h) \mapsto g\theta_p(h)$ is injective and its image is a transversal for $G/\theta_{pq}(G)$. In particular, $\{\pi_{pq}(\delta_{g\theta_p(h)}) \mid g \in T_p, h \in T_q\}$ is an orthonormal basis for M_{pq}.*

Proof. Suppose we have $m_{p,q}(g_1, h_1) = m_{p,q}(g_2, h_2)$ for some $g_i \in T_p, h_i \in T_q$. This amounts to $g_1 = g_2\theta_p(h_2 h_1^{-1})$, so $g_1 = g_2$ as T_p is a transversal. As θ_p is injective, we then immediately get $h_1 = h_2$. Thus $m_{p,q}$ is injective. Now given $g \in G$, let $g_1 \in T_p$ be the element satisfying $g \in g_1\theta_p(G)$. Next, choose $h_1 \in T_q$ such that $\theta_p^{-1}(g_1^{-1}g) \in h_1\theta_q(G)$. Then

$$(g_1\theta_p(h_1))^{-1}g = \theta_p(h_1^{-1}\theta_p^{-1}(g_1^{-1}g)) \in \theta_p(\theta_q(G)) = \theta_{pq}(G),$$

so $g \in m_{p,q}(g_1, h_1)\theta_{pq}(G)$. Thus $m_{p,q}(T_p \times T_q)$ is a transversal for $G/\theta_{pq}(G)$.   □

Proposition 7.5 can also be proven using the fact that orthonormal bases for M_p and M_q yield an orthonormal basis for $M_p \otimes_{C^*(G)} M_q \cong M_{pq}$, see [20, Lemma 4.3].

In order to prove that the product system M from Proposition 7.4 is compactly aligned, we need the following lemma. Within its proof, we will make multiple applications of the identity

$$\Theta_{\pi_p(\delta_{g_1}), \pi_p(\delta_{g_2})}(\pi_p(\delta_h)) = \chi_{g_2\theta_p(G)}(h)\pi_p(\delta_{g_1 g_2^{-1} h}) \quad \text{for all } p \in P, \ g_1, g_2, h \in G.$$

Lemma 7.6. *For each $p, q, r \in P$ with $pP \cap qP = rP$ and $g_1, g_2, h_1, h_2 \in G$ we have*

$$\iota_p^r(\Theta_{\pi_p(\delta_{g_1}), \pi_p(\delta_{g_2})})\iota_q^r(\Theta_{\pi_q(\delta_{h_1}), \pi_q(\delta_{h_2})})$$

$$= \begin{cases} \Theta_{\pi_r(\delta_{g_1\theta_p(k)}), \pi_r(\delta_{h_2\theta_q(\ell)})} & \text{if } g_2\theta_p(k) = h_1\theta_q(\ell) \text{ for some } k, \ell \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Choose $p', q' \in P$ with $pp' = qq' = r$. Let $T_{p'}$ and $T_{q'}$ be transversals for $G/\theta_{p'}(G)$ and $G/\theta_{q'}(G)$, respectively. We know from Proposition 7.3 that $\{\pi_{p'}(\delta_i) \mid i \in T_{p'}\}$ and $\{\pi_{q'}(\delta_j) \mid j \in T_{q'}\}$ are orthonormal bases for $M_{p'}$ and $M_{q'}$, respectively. We claim that

$$(7.1) \qquad \iota_p^r(\Theta_{\pi_p(\delta_{g_1}), \pi_p(\delta_{g_2})}) = \sum_{i \in T_{p'}} \Theta_{\pi_r(\delta_{g_1\theta_p(i)}), \pi_r(\delta_{g_2\theta_p(i)})}$$

and

$$(7.2) \qquad \iota_q^r(\Theta_{\pi_q(\delta_{h_1}), \pi_q(\delta_{h_2})}) = \sum_{j \in T_{q'}} \Theta_{\pi_r(\delta_{h_1\theta_q(j)}), \pi_r(\delta_{h_2\theta_q(j)})}.$$

To see that (7.1) holds, fix $s \in G$. Using (6.1) we get

$$\iota_p^r(\Theta_{\pi_p(\delta_{g_1}), \pi_p(\delta_{g_2})})(\pi_r(\delta_s)) = \Theta_{\pi_p(\delta_{g_1}), \pi_p(\delta_{g_2})}(\pi_p(\delta_s))\pi_{p'}(\delta_1)$$
$$= \chi_{g_2\theta_p(G)}(s)\pi_p(\delta_{g_1 g_2^{-1} s})\pi_{p'}(\delta_1)$$
$$= \chi_{g_2\theta_p(G)}(s)\pi_r(\delta_{g_1 g_2^{-1} s}).$$

For $i \in T_{p'}$ we have

$$\Theta_{\pi_r(\delta_{g_1\theta_p(i)}), \pi_r(\delta_{g_2\theta_p(i)})}(\pi_r(\delta_s)) = \chi_{g_2\theta_p(i)\theta_r(G)}(s)\pi_r(\delta_{g_1 g_2^{-1} s}).$$

Since $g_2\theta_p(G) = \bigsqcup_{i \in T_{p'}} g_2\theta_p(i)\theta_r(G)$, we know that s belongs to $g_2\theta_p(G)$ if and only if there is an $i \in T_{p'}$ with $s \in g_2\theta_p(i)\theta_r(G)$. Note that i is uniquely determined in this case.

It follows that $\iota_p^r(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})$ and $\sum_{i \in T_{p'}} \Theta_{\pi_r(\delta_{g_1\theta_p(i)}),\pi_r(\delta_{g_2\theta_p(i)})}$ agree on the image of π_r, and hence on M_r. So (7.1) holds. A similar argument gives (7.2).

Now observe that, for all $i \in T_{p'}, j \in T_{q'}$, we have

$$\Theta_{\pi_r(\delta_{g_1\theta_p(i)}),\pi_r(\delta_{g_2\theta_p(i)})}\Theta_{\pi_r(\delta_{h_1\theta_q(j)}),\pi_r(\delta_{h_2\theta_q(j)})}$$
$$= \chi_{g_2\theta_p(i)\theta_r(G)}(h_1\theta_q(j))\Theta_{\pi_r(\delta_{g_1g_2^{-1}h_1\theta_q(j)}),\pi_r(\delta_{h_2\theta_q(j)})}.$$

It then follows from (7.1) and (7.2) that $\iota_p^r(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\iota_q^r(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})}) \neq 0$ if and only if

$$h_1\theta_q(j) \in g_2\theta_p(i)\theta_r(G) \text{ for some } i \in T_{p'},\, j \in T_{q'}.$$

As $r \in pP$, we have $\theta_p(G)\theta_r(G) = \theta_p(G)$, so the equation from above is equivalent to $g_2^{-1}h_1 \in \theta_p(G)\theta_q(G)$. This in turn is equivalent to the existence of $k, \ell \in G$ satisfying $g_2\theta_p(k) = h_1\theta_q(\ell)$. In this case, the summand of $\iota_p^r(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\iota_q^r(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})$ corresponding to the unique pair (i,j) with $k \in i\theta_{p'}(G)$ and $\ell \in j\theta_{q'}(G)$ is non-zero.

Now suppose $k_1, k_2, \ell_1, \ell_2 \in G$ satisfy $g_2\theta_p(k_n) = h_1\theta_q(\ell_n), n = 1, 2$. We need to show that $k_1^{-1}k_2 \in \theta_{p'}(G)$ and $\ell_1^{-1}\ell_2 \in \theta_{q'}(G)$. Rewriting the two equations as

$$\theta_p(k_1)\theta_q(\ell_1^{-1}) = g_2^{-1}h_1 = \theta_p(k_2)\theta_q(\ell_2^{-1})$$

gives $\theta_p(k_1^{-1}k_2) = \theta_q(\ell_1^{-1}\ell_2)$. Since θ respects the order, we conclude that $\theta_p(k_1^{-1}k_2) \in \theta_p(G) \cap \theta_q(G) = \theta_r(G)$ and, similarly, $\theta_q(\ell_1^{-1}\ell_2) \in \theta_r(G)$. Using that θ_p and θ_q are injective and $pp' = qq' = r$, we conclude that $k_1^{-1}k_2 \in \theta_{p'}(G)$ and $\ell_1^{-1}\ell_2 \in \theta_{q'}(G)$. $\square$

Remark 7.7. Note that if we apply Lemma 7.6 to $p, q, r \in P$ with $pP \cap qP = rP$, $g = g_1 = g_2 \in G$ and $h = h_1 = h_2 \in G$, we get the identity

$$(7.3) \qquad \iota_p^r(\mathcal{E}_{g,p})\iota_q^r(\mathcal{E}_{h,q}) = \begin{cases} \mathcal{E}_{g\theta_p(k),r} & \text{if } g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.8. *The product system M from Proposition 7.4 is compactly aligned.*

Proof. This follows immediately from Lemma 6.3 and (7.3). $\square$

We can now state the main result of this section. Recall from Section 6 that j_M denotes to the universal Nica covariant representation of a compactly aligned product system M; for each $p \in P$ we denote by j_p the restriction $j_M|_{M_p}$.

Theorem 7.9. *Let M be the product system given in Proposition 7.4. There is an isomorphism $\varphi : \mathcal{A}[G, P, \theta] \to \mathcal{NT}(M)$ satisfying*

$$\varphi(u_g) = j_1(\delta_g) \quad \text{and} \quad \varphi(s_p) = j_p(\pi_p(1)),$$

for all $g \in G, p \in P$.

We will use the following result which provides a characterisation of Nica covariance in terms of the generalised rank one projections $\mathcal{E}_{g,p}$.

Proposition 7.10. *Let M be the product system given in Proposition 7.4. A representation ψ of M is Nica covariant if and only if for all $g, h \in G$ and $p, q \in P$ we*

have

$$(7.4) \quad \psi^{(p)}(\mathcal{E}_{g,p})\psi^{(q)}(\mathcal{E}_{h,q}) = \begin{cases} \psi^{(r)}(\mathcal{E}_{g\theta_p(k),r}) & \text{if } pP \cap qP = rP \text{ for some } r \in P \text{ and} \\ & g\theta_p(k) \in h\theta_q(G) \text{ for some } k \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If ψ is Nica covariant, then (7.4) follows immediately from (7.3). For the converse direction, we note that, by linearity and continuity, it suffices to check (6.2) for $\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})}$ and $\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})}$ for all $p, q \in P$ and $g_i, h_i \in G$. We have

$$\psi^{(p)}(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\psi^{(q)}(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})$$
$$= \psi^{(p)}(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})}\mathcal{E}_{g_2,p})\psi^{(q)}(\mathcal{E}_{h_1,q}\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})$$
$$= \psi^{(p)}(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\psi^{(p)}(\mathcal{E}_{g_2,p})\psi^{(q)}(\mathcal{E}_{h_1,q})\psi^{(q)}(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})}),$$

which we know by (7.4) is equal to

$$(7.5) \qquad \psi^{(p)}(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\psi^{(r)}(\mathcal{E}_{g_2\theta_p(k),r})\psi^{(q)}(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})$$

if $pP \cap qP = rP$ for some $r \in P$ and $g_2\theta_p(k) \in h\theta_q(G)$ for some $k \in G$, and is zero, otherwise. We immediately see from (7.4) and Lemma 7.6 that (6.2) is satisfied when $pP \cap qP = \emptyset$ or $g_2^{-1}h_1 \notin \theta_p(G)\theta_q(G)$.

So suppose $pP \cap qP = rP$ for some $r \in P$ and $g_2\theta_p(k) \in h_1\theta_q(G)$ for some $k \in G$. Let $p', q' \in P$ with $pp' = qq' = r$, and $\ell \in G$ with $g_2\theta_p(k) = h_1\theta_q(\ell)$. The expression in (7.5) can be written as

$$\psi_p(\pi_p(\delta_{g_1}))\psi_p(\pi_p(\delta_{g_2}))^*\psi^{(r)}(\mathcal{E}_{g_2\theta_p(k),r})\psi_q(\pi_q(\delta_{h_1}))\psi_q(\pi_q(\delta_{h_2}))^*.$$

Rewriting the middle term gives

$$\psi^{(r)}(\mathcal{E}_{g_2\theta_p(k),r}) = \psi_r(\pi_r(\delta_{g_2\theta_p(k)}))\psi_r(\pi_r(\delta_{g_2\theta_p(k)}))^*$$
$$= \psi_r(\pi_r(\delta_{g_2\theta_p(k)}))\psi_r(\pi_r(\delta_{h_1\theta_q(\ell)}))^*$$
$$= \psi_p(\pi_p(\delta_{g_2}))\psi_{p'}(\pi_{p'}(\delta_k))\psi_{q'}(\pi_{q'}(\delta_\ell))^*\psi_q(\pi_q(\delta_{h_1}))^*.$$

Using $\psi_1(\langle\pi_p(\delta_{g_2}),\pi_p(\delta_{g_2})\rangle_p) = 1 = \psi_1(\langle\pi_q(\delta_{h_1}),\pi_q(\delta_{h_1})\rangle_q)$, we turn the expression in (7.5) into

$$\psi_p(\pi_p(\delta_{g_1}))\psi_{p'}(\pi_{p'}(\delta_k))\psi_{q'}(\pi_{q'}(\delta_\ell))^*\psi_q(\pi_q(\delta_{h_2}))^* = \psi_r(\pi_r(\delta_{g_1\theta_p(k)}))\psi_r(\pi_r(\delta_{h_2\theta_q(\ell)}))$$
$$= \psi^{(r)}(\Theta_{\pi_r(\delta_{g_1\theta_p(k)}),\pi_r(\delta_{h_2\theta_q(\ell)})}).$$

But we know from Lemma 7.6 that

$$\psi^{(r)}\big(\iota_p^r(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\iota_q^r(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})\big) = \psi^{(r)}(\Theta_{\pi_r(\delta_{g_1\theta_p(k)}),\pi_r(\delta_{h_2\theta_q(\ell)})}).$$

So we get

$$\psi^{(p)}(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\psi^{(q)}(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})}) = \psi^{(r)}\big(\iota_p^r(\Theta_{\pi_p(\delta_{g_1}),\pi_p(\delta_{g_2})})\iota_q^r(\Theta_{\pi_q(\delta_{h_1}),\pi_q(\delta_{h_2})})\big),$$

and hence (6.2) is satisfied. $\qquad\square$

The conclusion we draw from Proposition 7.10 is that, for the product system M, it suffices to check Nica covariance on rank one projections coming from suitable orthonormal bases.

Proof of Theorem 7.9. The proof follows closely that of Theorem 4.4. The map $g \mapsto j_1(\delta_g)$ is a unitary representation of G in $\mathcal{NT}(M)$, and $p \mapsto j_p(\pi_p(1))$ is a representation of P by isometries in $\mathcal{NT}(M)$. We have

$$j_p(\pi_p(1))j_1(\delta_g) = j_p(\pi_p(\delta_{\theta_p(g)})) = j_1(\delta_{\theta_p(g)})j_p(\pi_p(1)),$$

which is (A1). Instead of (A2), we show (2.2), see Lemma 2.4. First note that $j_1(\delta_g)j_p(\pi_p(1))j_p(\pi_p(1))^*j_1(\delta_g)^* = j^{(p)}(\mathcal{E}_{g,p})$. Now (2.2) follows from applying (7.4) to the representation j. So the universal property of $\mathcal{A}[G, P, \theta]$ gives a homomorphism $\varphi : \mathcal{A}[G, P, \theta] \to \mathcal{NT}(M)$ satisfying $\varphi(u_g) = j_1(\delta_g)$ and $\varphi(s_p) = j_p(\pi_p(1))$, for all $g \in G$, $p \in P$. This homomorphism is surjective because each $j_p(\pi_p(\delta_g)) = \varphi(u_g s_p)$ is in the range of φ.

It is routine to show that $\pi_p(\delta_g) \mapsto u_g s_p$ defines a representation of the product system M, and (2.2) gives (7.4), which is Nica covariance. The induced homomorphism $\psi : \mathcal{NT}(M) \to \mathcal{A}[G, P, \theta]$ is inverse to φ, and hence φ is an isomorphism. $\qquad\square$

We note the following immediate consequence:

Corollary 7.11. *If S is a right LCM semigroup, then $C^*(S)$ is isomorphic to the Nica-Toeplitz algebra $\mathcal{NT}(M)$ of the product system of right-Hilbert bimodules M over S with $M_p = \mathbb{C}$ for all $p \in S$.*

Proof. The triple $(\{1\}, S, \mathrm{id})$ defines an algebraic dynamical system and $\{1\} \rtimes_{\mathrm{id}} S \cong S$, so Theorem 4.4 and Theorem 7.9 imply

$$C^*(S) \cong C^*(\{1\} \rtimes_{\mathrm{id}} S) \cong \mathcal{A}[\{1\}, S, \mathrm{id}] \cong \mathcal{NT}(M)$$

for the product system M from Proposition 7.4 associated to $(\{1\}, S, \mathrm{id})$. $\qquad\square$

8. Examples

Observing that algebraic dynamical systems are a natural generalisation of irreversible algebraic dynamical systems as introduced in [28, Definition 1.5], we already have various examples at our disposal, namely [28, Examples 1.8–1.11,1.14].

Remark 8.1. For each irreversible algebraic dynamical system (G, P, θ), the C^*-algebra $\mathcal{A}[G, P, \theta]$ can be viewed as the natural Toeplitz extension of the C^*-algebra $\mathcal{O}[G, P, \theta]$ from [28]. In fact, $\mathcal{A}[G, P, \theta]$ is the Nica-Toeplitz algebra of the product system M associated to (G, P, θ) according to Theorem 7.9 and $\mathcal{O}[G, P, \theta]$ is the Cuntz-Nica-Pimsner algebra of M if (G, P, θ) is of finite type, see [28, Theorem 5.9].

But the class of algebraic dynamical systems is much larger:

Example 8.2. Suppose P is a discrete group acting on another discrete group G by automorphisms $\theta_p, p \in P$. Then (G, P, θ) is an algebraic dynamical system and $\mathcal{A}[G, P, \theta] = C^*(G \rtimes_\theta P) = C^*(G) \rtimes P$. In particular, we can take $P = \{1\}$ and obtain the full group C^*-algebra for G.

Example 8.3. Similar to the last part in Example 8.2, if we let $G = \{1\}$, then any right LCM semigroup P acts on G by the trivial action and $\mathcal{A}[G, P, \theta]$ is nothing but the full semigroup C^*-algebra of P.

Remark 8.4. Since we know that, conversely, $G \rtimes_\theta P$ is a right LCM semigroup for every algebraic dynamical system (G, P, θ), we see that studying algebraic dynamical systems is a priori more sensible than studying right LCM semigroups. The difference is that an algebraic dynamical system comes equipped with a decomposition of the right LCM semigroup as a semidirect product of a group by a right LCM semigroup. A way of extracting a specific semidirect product description for certain cancellative semigroups out of their semigroup structure is presented in [2, Proposition 2.11].

Example 8.5. The semigroup $\mathbb{N}^\times$ is a right LCM semigroup which acts upon $\mathbb{Z}$ by multiplication in an order preserving way. Hence we get an algebraic dynamical system $(\mathbb{Z}, \mathbb{N}^\times, \cdot)$. By Theorem 4.4, we have $\mathcal{A}[\mathbb{Z}, \mathbb{N}^\times, \cdot] \cong C^*(\mathbb{Z} \rtimes \mathbb{N}^\times)$, and by Theorem 7.9, they both are isomorphic to a Nica-Toeplitz algebra for a product system of right-Hilbert $C^*(\mathbb{Z})$-modules over $\mathbb{N}^\times$. It was observed in [1, Example 3.9] and in [15] that the latter C^*-algebra is isomorphic to the additive boundary quotient $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, the Toeplitz algebra of the affine semigroup over the natural numbers. We note that $(\mathbb{Z}, \mathbb{N}^\times, \cdot)$ is in fact an irreversible algebraic dynamical system and $\mathcal{O}[\mathbb{Z}, \mathbb{N}^\times, \cdot] \cong \mathcal{Q}_\mathbb{N}$.

Example 8.5 yields an alternative perspective on the boundary quotient diagram of [1] which seems well-suited for generalisations to other cases that are similar to $\mathbb{N} \subset \mathbb{Z}$. The next example provides a reasonable framework for such a task.

Example 8.6. Suppose R is the ring of integers in a number field K and let (r) denote the principal ideal generated by $r \in R$. If we take $G = R$ (equipped with addition) and $P \subset R^\times$ is a subsemigroup satisfying $(p) \cap (q) \in \{(r) \mid r \in P\}$ for all $p, q \in P$, then $(G, P, \cdot)$ is an algebraic dynamical system. Note that the intersection condition is automatically satisfied if R is a principal ideal domain, i.e. the class number of K is one. In this case, we recover the natural extension $C^*(R \rtimes R^\times)$ of the ring C^*-algebra $\mathfrak{A}[R]$ from [9].

Remark 8.7. Even though there are examples of semigroups P as in Example 8.6 for number fields K with class number bigger than one, it would be convenient to have a systematic treatment which is available for arbitrary subsemigroups of $R^\times$ for arbitrary number fields K. This asks for a relaxation of the condition

$$\theta_p(G) \cap \theta_q(G) = \theta_r(G) \text{ whenever } pP \cap qP = rP$$

to the constraint

$$[\theta_p(G) \cap \theta_q(G) : \theta_r(G)] < \infty \text{ whenever } pP \cap qP \supset rP \text{ and } r \text{ is minimal}$$

in the sense that $pP \cap qP \supset r'P$ and $r'P \supset rP$ imply $r'P = rP$. We would like to point out that this condition is somewhat reminiscent of Spielberg's notion of finite alignment for categories of paths, see [27, Section 3].

Let us end with two basic constructions for algebraic dynamical systems, which have been described in [28, Examples 1.12 and 1.13] for the irreversible case. The first one is built by shifting some suitable group G_0 along a right LCM semigroup P.

Proposition 8.8. *Let P be a countable, right LCM semigroup with identity and G_0 a countable group. Then P admits a shift action θ on $G := \bigoplus_P G_0$ and (G, P, θ) is an algebraic dynamical system.*

Proof. The action θ is given by $(\theta_p((g_q)_{q\in P}))_r = \chi_{pP}(r)\ g_{p^{-1}r}$ for all $p, r \in P$. It is apparent that θ_p is an injective group endomorphism for all $p \in P$. Observing that $\theta_p(G) = 0 \oplus \left(\bigoplus_{q\in pP} G_0\right)$, we see that θ also respects the order on P. Hence we get an algebraic dynamical system in the sense of Definition 2.1. $\square$

Example 8.9. If we apply the construction of Proposition 8.8 to $G_0 = \mathbb{Z}/n\mathbb{Z}$ and $P = \mathbb{N}$, the resulting algebra $\mathcal{A}[G, P, \theta]$ is canonically isomorphic to the Toeplitz extension $\mathcal{T}_n$ of the Cuntz algebra $\mathcal{O}_n$, compare [28, Example 3.30].

Example 8.10. Suppose we are given a family $(G^{(i)}, P, \theta^{(i)})_{i\in I}$ of algebraic dynamical systems. Then we can consider $G := \bigoplus_{i\in I} G^{(i)}$ and let P act on G component-wise, i.e. $\theta_p(g_i)_{i\in I} := (\theta_p^{(i)}(g_i))_{i\in I}$. In this way we get a new algebraic dynamical system (G, P, θ).

Remark 8.11. It is also possible to decompose an algebraic dynamical system (G, P, θ). Let $G = \bigoplus_{i\in I} G_i$ be a decomposition of G into θ-invariant subgroups, i.e. $\theta_p(G_i) \subset G_i$ for all $i \in I$. Then $(G_i, P, \theta_{|G_i})$ defines an algebraic dynamical system for each $i \in I$. Indeed, it is easy to see that the subsystem $(G_i, P, \theta_{|G_i})$ inherits the required properties from (G, P, θ). Besides, we note that the procedure from Example 8.10 is inverse to this method of decomposing.

Example 8.12. For $G = \mathbb{Z}$, the injective group endomorphisms are given by multiplication with elements from $\mathbb{Z}^\times$. For $\mathcal{P} \subset \mathbb{Z}^\times$, let $|\mathcal{P}\rangle$ denote the multiplicative subsemigroup of $\mathbb{Z}^\times$ with identity generated by $\mathcal{P}$. Let us consider subsemigroups $P_1 = |2, 3\rangle$, $P_2 = |-2, 3\rangle$, $P_3 = |2, -3\rangle$ and $P_4 = |-2, -3\rangle$. Each of them gives rise to an irreversible algebraic dynamical system $(\mathbb{Z}, P_i, \cdot)$. It is an instructive exercise to show that the corresponding semidirect products $\mathbb{Z} \rtimes P_i$ are mutually non-isomorphic. However, if we add -1 as a generator to each of P_i, we always arrive at $P_5 = |-1, 2, 3\rangle$. Moreover, the corresponding embedding $\mathbb{Z} \rtimes P_i \subset \mathbb{Z} \rtimes P_5$ yields injective, admissible morphisms from $(\mathbb{Z}, P_i, \theta)$ into $(\mathbb{Z}, P_5, \cdot)$. Hence, by Corollary 4.10 there are canonical $*$-homomorphisms from $\mathcal{A}[\mathbb{Z}, P_i, \theta]$ to $\mathcal{A}[\mathbb{Z}, P_5, \theta]$ for $i = 1, \ldots, 4$.

In this example the left regular representation implements an isomorphism between the full and the reduced semigroup C^*-algebras for $\mathbb{Z} \rtimes P_i$ for each $i = 1, \ldots, 5$, see [2, Example 6.3] or [21]. Thus by Proposition 3.6, the C^*-algebras $\mathcal{A}[\mathbb{Z}, P_i, \cdot], i = 1, \ldots, 4$ are proper subalgebras of $\mathcal{A}[\mathbb{Z}, P_5, \cdot]$ in a natural way. Note that the families of constructible right ideals $\mathcal{J}(\mathbb{Z} \rtimes P_i)$ are isomorphic for all five semidirect products. Thus, their diagonal subalgebras are canonically isomorphic, see [21]. Since this identification is compatible with translation by $\mathbb{Z}$, we also have an isomorphism on the level of the core subalgebras $\mathcal{F} \cong \mathcal{D} \rtimes \mathbb{Z}$, a fact that follows from [2, Definition 3.12] and [28, Theorem A.5].

It is therefore natural to ask if, and in how far the C^*-algebras $\mathcal{A}[\mathbb{Z}, |2, 3\rangle, \cdot], \mathcal{A}[\mathbb{Z}, |-2, 3\rangle, \cdot], \mathcal{A}[\mathbb{Z}, |2, -3\rangle, \cdot], \mathcal{A}[\mathbb{Z}, |-2, -3\rangle, \cdot]$ and $\mathcal{A}[\mathbb{Z}, |-1, 2, 3\rangle, \cdot]$ are different. Corollary 5.5 shows that $K_*(\mathcal{A}[\mathbb{Z}, |P_i\rangle, \cdot])$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ for $i = 1, \ldots, 4$, and that

$$K_*\big(\mathcal{A}[\mathbb{Z}, |P_5\rangle, \cdot]\big) = K_*\big(C^*(\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z})\big).$$

Thus, K-theory distinguishes $\mathcal{A}[\mathbb{Z}, |P_5\rangle, \cdot]$ from any of the C^*-algebras $\mathcal{A}[\mathbb{Z}, |P_i\rangle, \cdot]$ for $1, \ldots, 4$. However, finer invariants are needed in order to determine the exact relationship between those four C^*-algebras.

References

[1] N. Brownlowe, A. an Huef, M. Laca and I. Raeburn, *Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers*, Ergodic Theory Dynam. Systems, **32** (2012), 35–62, DOI:10.1017/S0143385710000830.

[2] N. Brownlowe, N.S. Larsen and N. Stammeier, *On C^*-algebras associated to right LCM semigroups*, Trans. Amer. Math. Soc. **369** (2017), no. 1, 31–68.

[3] N. Brownlowe and I. Raeburn, *Two families of Exel-Larsen crossed products*, J. Math. Anal. Appl., **398** (2013), 68–79, DOI:10.1016/j.jmaa.2012.08.026.

[4] N. Brownlowe, J. Ramagge, D. Robertson and M.F. Whittaker, *Zappa-Szép products of semigroups and their C^*-algebras*, J. Funct. Anal. **266** (2014), 3937–3967, DOI:10.1016/j.jfa.2013.12.025.

[5] P. Clare, T. Crisp, N. Higson, *Adjoint functors between categories of Hilbert modules*, preprint arxiv:1409.8656.

[6] J. Cuntz, C. Deninger and M. Laca, *C^*-algebras of Toeplitz type associated with algebraic number fields*, Math. Ann. **355** (2013), 1383–1423, DOI:10.1007/s00208-012-0826-9.

[7] J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of the C^*-algebra generated by the left regular representation of an Ore semigroup*, to appear in JEMS, preprint arXiv:1201.4680v2.

[8] J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of crossed products by automorphic semigroup actions*, Q. J. Math. **64** (2013), no. 3, 747-784, DOI:10.1093/qmath/hat021.

[9] J. Cuntz, X. Li, *The regular C^*-algebra of an integral domain*, Quanta of maths, 149–170, Clay Math. Proc. **11** (2010), Amer. Math. Soc., Providence, RI.

[10] J. Cuntz and A. Vershik, *C^*-algebras associated with endomorphisms and polymorphisms of compact abelian groups*, Comm. Math. Phys. **321** (2013), 157–179, DOI:10.1007/s00220-012-1647-0.

[11] R. Exel, *A new look at the crossed-product of a C^*-algebra by an endomorphism*, Ergodic Theory Dynam. Systems **23** (2003), 1733–1750, DOI:10.1017/S0143385707000302.

[12] N.J. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. **204** (2002), 335–375, DOI:10.2140/pjm.2002.204.335.

[13] M. Frank and D.R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras*, J. Operator Theory **48** (2002), 273–314.

[14] I. Hirshberg, *On C^*-algebras associated to certain endomorphisms of discrete groups*, New York J. Math. **8** (2002), 99–109, nyjm.albany.edu/j/2002/8-6.html.

[15] J.H. Hong, N.S. Larsen and W. Szymański, *The Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$ and C^*-algebras of product systems*, Progress in operator algebras, noncommutative geometry, and their applications, Theta Ser. Adv. Math. **15** (2012), Theta, Bucharest, 97–109.

[16] J.H. Hong, N.S. Larsen and W. Szymański, *KMS states on Nica-Toeplitz algebras of product systems*, Internat. J. Math. **23** (2012), 38pp, DOI:10.1142/S0129167X12501236.

[17] P.E.T. Jørgensen, L.M. Schmitt and R.F. Werner, *Positive representations of general commutation relations allowing Wick ordering*, J. Funct. Anal. **134** (1995), 33–99.

[18] M. Laca, I. Raeburn, J. Ramagge and M.F. Whittaker, *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, J. Funct. Anal. **266** (2014), 6619–6661, DOI:10.1016/j.jfa.2014.03.003.

[19] N.S. Larsen, *Crossed products by abelian semigroups via transfer operators*, Ergodic Theory Dynam. Systems **30** (2010), 1147–1164, DOI:10.1017/S0143385709000509.

[20] N.S. Larsen and I. Raeburn, *Projective multi-resolution analyses arising from direct limits of Hilbert modules*, Math. Scand. **100** (2007), 317–360, mscand/15026.

[21] X. Li, *Semigroup C^*-algebras and amenability of semigroups*, J. Funct. Anal. **262** (2012), 4302–4340, DOI:10.1016/j.jfa.2012.02.020.

[22] D. Lind and K. Schmidt, *A survey of algebraic actions of the discrete Heisenberg group*, preprint arXiv:1502.06243.

[23] V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory **52** (2004), 223–249.

[24] A. Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory **27** (1992), 17–52.

[25] M.D. Norling, *Inverse semigroup C*-algebras associated with left cancellative semigroups*, Proc. Edinb. Math. Soc. **57** (2014), 533–564, DOI:10.1017/S0013091513000540.

[26] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.

[27] J. Spielberg, *Groupoids and C*-algebras for categories of paths*, Trans. Amer. Math. Soc. **366** (2014), no. 11, 5771–5819, DOI:10.1090/S0002-9947-2014-06008-X.

[28] N. Stammeier, *On C*-algebras of irreversible algebraic dynamical systems*, J. Funct. Anal. **269** (2015), no. 4, 1136–1179, DOI:10.1016/j.jfa.2015.02.005.

[29] C. Starling, *Boundary quotients of C*-algebras of right LCM semigroups*, J. Funct. Anal. **268** (2015), no. 11, 3326–3356, DOI:10.1016/j.jfa.2015.01.001.

[30] F. Vieira, *C*-algebras associated with endomorphisms of groups*, preprint arXiv:1308.1386v2.

SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF SYDNEY, NSW 2006, AUSTRALIA

E-mail address: nathan.brownlowe@sydney.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, 0316 OSLO, NORWAY

E-mail address: nadiasl@math.uio.no

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, 0316 OSLO, NORWAY

E-mail address: nicolsta@math.uio.no