UiO : Department of Mathematics
University of Oslo

Sextactic Points on Plane Algebraic Curves

Paul Aleksander Maugesten

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

In this thesis we consider sextactic points on plane algebraic curves and a 2-Hessian curve that identifies these points. This curve was first established by Cayley, and we prove that Cayley's 2-Hessian is wrong. Moreover, we correct his mistakes and give the correct defining polynomial of the 2-Hessian curve. In addition, we present a formula for the number of sextactic points on a cuspidal curve.

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CHAPTER 1

Introduction

As long as mathematicians have studied curves, they have been concerned with different types of special points on them and the properties of such points. A problem in classical algebraic geometry could be to find the number and type of singular points on some curve. This thesis joins this classical line of thought and its aim is to investigate aspects of *sextactic points* on plane algebraic curves.

Sextactic points are quite similar to inflection points. An inflection point on a curve C is a smooth point where the tangent intersects C with intersection multiplicity 3 or higher. Similarly, a sextactic point is a smooth point on a curve C where the osculating conic intersects the curve C with intersection multiplicity 6 or higher. Hence, it is clear that the sextactic points are a very natural generalisation of inflection points. In this thesis, we will try to keep this relationship in mind and use it to our advantage while developing new results for sextactic points.

The starting point of the theory of sextactic points is the work done by Cayley in his 1859 article *On the Conic of Five-Pointic Contact at Any Point of a Plane Curve* [Cay59] and the subsequent article *On the Sextactic Points of a Plane Curve* [Cay65] from 1865. In [Cay59], Cayley establishes the defining polynomial of the osculating conic at a smooth point, that is not an inflection point, on a plane curve of any degree. In the same article, Cayley coins the term *sextactic* for a point where the osculating conic intersects the curve with multiplicity higher than five. In the later article [Cay65], Cayley takes the work on sextactic points further and relative to a curve produces an associated curve, which we have called Cayley's 2-Hessian. According to Cayley, this curve intersects the original curve in its sextactic points.

Since Cayley's seminal papers, other mathematicians have made contributions to the theory of sextactic points, but the contributions throughout history are, as far as we have found, sparse.

One of the more classical contributions that we came across was done by Coolidge. In [Coo31] from 1931, Coolidge presents a formula for the number of sextactic points on a curve with some restrictions on the singularities of the curve and its dual. Coolidge's formula is in fact parallel to the classical Plücker formula for the number of inflection points on a plane curve.

A similar, but much more recent result is that of Thorbergsson and Umehara in [TU02], where they give a formula for the number of sextactic points on a smooth algebraic curve. To obtain this result, they use the theory of linear systems and Weierstrass points on smooth curves, see [Mir95].

In the article [Cuk97], Cukierman aims to extend the fact that the Hessian curve of a plane curve intersects the original curve in its inflection points. He considers osculating curves of degree $n \geq 1$ to a smooth curve C to obtain an n-Hessian curve that intersects C in the n-inflections. These points are sometimes referred to as hyperosculating points, since they are the points on C where the osculating curve intersects C with intersection multiplicity strictly greater than $n(n+3)/2$.

The articles by Cayley and the natural connection to inflection points form the foundation of our investigation about sextactic points on plane curves. In the 1990s, the theory of Weierstrass points with respect to linear systems was extended to singular curves, and in [Not99], Notari developed a technique to compute the Weierstrass weight of a singular point with respect to any linear system. Using this theory, we could therefore employ the same approach as Thorbergsson and Umehara did for smooth curves and consider sextactic points on a singular curve as Weierstrass points for a complete linear system. This was the vital aspect that enabled us to prove new results for cuspidal curves.

The last 20 years, the class of plane rational cuspidal curves has been intensively studied. There are many such curves, and defining polynomial and parametrisations can be found in the literature, see [Moe13]. As they are accessible, and can be described by simple invariants, we have chosen these as our main source of examples.

This thesis has led to some quite interesting results regarding sextactic points. The most surprising result to come out of this thesis is perhaps the error of Cayley in his construction of the 2-Hessian. Although not severe, this mistake leads to the wrong defining polynomial of the curve and hence does not give the desired points when intersected with the original curve. We were able to correct the mistake in Cayley's proof and present the right defining polynomial of the 2-Hessian curve.

We give a new proof for a known formula for the number of inflection points on a cuspidal curve in a way that is very natural and straightforward. Using the same approach, we were able to find and prove a new formula for the number of sextactic points on a cuspidal curve. A small investigation into higher order extatic points produced a minor result for binomial curves that is interesting in its own right.

In the process of studying sextactic points, we have encountered several problems and other fascinating sides of the theory that we were not able to investigate further at that time. The first and foremost concerns a general sextactic point formula. Using Notari's technique to compute the Weierstrass weight for any singular point, it would be interesting to pursue a general formula for the number of sextactic points on a plane curve, without restrictions on its type of singularities.

Investigating different series of rational cuspidal curves and examine sextactic points, and possibly higher order extatic points, on such curves could also be an interesting topic to consider further.

Another possible subject for additional research would be go beyond plane curves and consider aspects of sextactic points to curves in $\mathbb{P}^1 \times \mathbb{P}^1$, or rational curves in higher dimensional projective spaces.

In Chapter 2 we give the necessary preliminaries about plane curves to set up the remaining parts of the thesis. This includes standard concepts such as intersection multiplicity, tangents and singular points, as well as notions that are especially vital for our examination of sextactic points, such as the Puiseux parametrisation and Weierstrass points.

In Chapter 3 we cover some of the well-known theory of inflection points, including the Hessian curve. Furthermore, we state several inflection point formulas and give a new proof of one them.

In Chapter 4 the osculating conic, the sextactic points and the 2-Hessian curve found by Cayley are introduced. We see that Cayley's 2-Hessian is wrong, and correct the mistake in his proof. Moreover, we investigate different sextactic point formulas as well as giving a formula of our own for cuspidal curves, in particular for rational curves.

In Chapter 5 we investigate a special class of curves and show that they do not exhibit any sextactic points. Additionally, we consider Weierstrass points of complete linear systems of higher degree and make an attempt to show that there are no hyperosculating points on these curves, under some assumptions.

To consider examples and doing explicit calculations we have used the general purpose computer algebra system [Maple]. Examples of computations and programming can be found in Appendix A and Appendix B.

Appendix A is an extensive collection of tables that show the results of computations done in [Maple] for curves of degree 3, 4 and 5. The tables give easy access to information concerning singular, inflection and sextactic points, as well as intersections with tangents and osculating conics, and the Hessian and 2-Hessian curve. This appendix therefore works as a reference list for examples throughout the thesis.

Appendix B consists mainly of [Maple] code, together with a small program in [Macaulay2]. The [Maple] programs are the implementations of our formula for the 2-Hessian curve and Cayley's osculating conic, as well as the implementation of Cayley's 2-Hessian curve. The [Macaulay2] program is a minor block of code used to find the defining polynomial from a parametrisation of a rational curve.

All figures are made using [GeoGebra]. Note that the figures only visualise the real part of the curves and that some information therefore will be lost.

CHAPTER 2

Background

In this chapter, we recall important definitions and notions regarding plane algebraic curves. In particular, we present the essential theory needed to work with the main objects of the thesis; inflection points and sextactic points.

2.1 Plane algebraic curves

Let $\mathbb{P}^2$ denote the projective plane over $\mathbb{C}$, where a point $p \in \mathbb{P}^2$ is represented by homogeneous coordinates $(x : y : z)$. If $\mathbb{C}[x, y, z]$ is the polynomial ring in three variables x, y, z, and $F(x, y, z) \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial, we define the *algebraic curve*, or simply the *curve* C, as

$$V(F) = \{(x, y, z) \in \mathbb{P}^2 \mid F(x, y, z) = 0\}.$$

Generally, if F is a reducible polynomial $F = F_1^{r_1} \cdot \ldots \cdot F_n^{r_n}$, where each F_i is irreducible and homogeneous, we call $F_1 \cdot \ldots \cdot F_n$ the *minimal polynomial* of the curve $C = V(F)$. Here $V(F) \coloneqq V(F_1) \cup \ldots \cup V(F_n)$ and each $V(F_i)$ is called a *component* of C. The *degree* of C, denoted by $\deg C$, is defined as the degree of the minimal polynomial of the curve and likewise, we will call a curve $C = V(F)$ *irreducible* if F is irreducible. An irreducible curve C that is birationally equivalent to $\mathbb{P}^1$ will be called a *rational* curve, and we note that any such curve admits a parametrisation. Two curves $C = V(F)$ and $D = V(G)$ are said to be *projectively equivalent* if there is some $M \in PGL_3(\mathbb{C})$ such that $G(Mp) = F(p)$ for all $p \in \mathbb{P}^2$.

To any point p on an algebraic curve, we can assign an integer m_p, called the *multiplicity of p on C*, as follows. If $p \notin C$, we let $m_p = 0$. For $p \in C$ we may, after a linear change of coordinates moving p to $(0 : 0 : 1)$, write

$$F(x, y, 1) = f_m(x, y) + f_{m+1}(x, y) + \ldots + f_d(x, y). \tag{2.1}$$

Here each f_i is homogeneous in x and y of degree i, and m is the degree of the smallest non-zero polynomial in the expansion. With this notation, the multiplicity of the point p is given by $m_p \coloneqq m$.

For $F \in \mathbb{C}[x, y, z]$, we denote its partial derivative with respect to x, y and z by F_x, F_y and F_z respectively. A point p on a curve $C = V(F)$ is called *singular* if

$$F_x(p) = F_y(p) = F_z(p) = 0,$$

while a non-singular point is called *smooth*.

Given a curve $C = V(F)$ and a point $p \in C$, we denote the *tangent to C at p* by T_p. For a smooth point p, there is a unique tangent at p given in [Fis01, Proposition 3.6, p. 45–46] as

$$T_p = V(xF_x(p) + yF_y(p) + zF_z(p)).$$

For a singular point p, we can obtain tangent lines as follows. First move p to $(0 : 0 : 1)$ as before. Then, by the fundamental theorem of algebra, the term of lowest degree in equation (2.1) can be written

$$f_m(x, y) = \prod_{i=1}^{k} L_i(x, y)^{r_i},$$

where the L_i are linear factors and $1 \leq k \leq m$. Hence

$$V(f_m) = \bigcup_{i=1}^{k} V(L_i),$$

and so the k *tangent lines of C at p* are defined as the lines $T_i = V(L_i)$, where $i = 1, \ldots, k$. When there is only one tangent to a singular point p and also only a single branch of the curve going through p, the singular point is called a *cusp*. A singular point of multiplicity two, with two distinct tangents is called a *node*. In particular, p is called a *simple* node if $m_p = 2$ and $(C.T_i)_p = 3,\ i = 1, 2$ for both of its tangents. A cusp is similarly called simple if $(C.T)_p = 3$ for the tangent T to C at p. Curves where all singular points are cusps are called *cuspidal*.

Remark 2.1.1. For a smooth point p, this way of obtaining the tangent coincides with the previous definition. This is because for a polynomial, the series (2.1) can be viewed as its Taylor series, so that a linear term here will give the tangent as defined for smooth points before. Hence a point is smooth if and only if $m_p = 1$.

Let ${\mathbb{P}^2}^\vee$ denote the dual projective plane. Any line in $\mathbb{P}^2$ gives a unique point in ${\mathbb{P}^2}^\vee$, so since every smooth point on an algebraic curve C has a unique tangent T_p, these tangents define unique points $T_p^\vee \in {\mathbb{P}^2}^\vee$. The set of tangents to smooth points on an algebraic curve gives a curve in ${\mathbb{P}^2}^\vee$, called the *dual curve*. This is a classical construction, and can be found in e.g. [Fis01, Section 5.1, p. 74], where it is defined as

$$C^\vee = \overline{\{L \in {\mathbb{P}^2}^\vee \mid L = T_p, \text{for } p \in C \text{ smooth}\}}.$$

Suppose that $C = V(F)$ and $D = V(G)$ are two algebraic curves without common components. We call a point $p \in C \cap D$ an *intersection point of C and D*, and we denote the *intersection multiplicity between C and D at p* by $(C.D)_p$. Moving p to $(0 : 0 : 1)$, the intersection multiplicity can be calculated as in [Ful69, Theorem 3, p. 75–76], that is

$$(C.D)_p = \dim_{\mathbb{C}}(\mathbb{C}[x, y]_{(x,y)}/(f, g)),$$

where $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$. If the curves C and D have intersection multiplicity 1 at a point p, we say that C and D intersect *transversally* at p. The total intersection between two curves C and D will be denoted by $C.D$, without specifying an intersection point. That is,

$$C.D = \sum_{p \in C \cap D} (C.D)_p.$$

The global intersection between two plane curves is described by Bézout's theorem [Har77, Corollary 7.8, p. 54].

Theorem 2.1.2 (Bézout's theorem)**.** *Let C and C' be two curves in $\mathbb{P}^2$ without common components of degrees d and d' respectively. Let $C \cap C' = \{p_1, \ldots, p_n\}$. Then*

$$\sum_{i=1}^{n} (C.C')_{p_i} = d \cdot d'.$$

In other words, two curves intersect in precisely the product of their degrees number of points, counting multiplicity.

Let $p_1 = (x_1 : y_1 : z_1)$ and $p_2 = (x_2 : y_2 : z_2)$ be two distinct point in $\mathbb{P}^2$. Recall that they determine a unique line given by $L = ax + by + cz$ because the matrix

$$M = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

is of rank 2 and if L goes through p_1 and p_2, L is determined by

$$M \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The polynomial L can be found explicitly by considering the determinant

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

which is a linear polynomial that vanishes on p_1 and p_2.

In the same way, five points in $\mathbb{P}^2$, where no three are on a line, determine a unique conic through the points. To see this, let p_1, p_2, p_3, p_4, p_5 be five points in general position, i.e. such that no three are on a line. Then, after a change of coordinates, we can assume that $p_1 = (0 : 0 : 1), p_2 = (0 : 1 : 0), p_3 = (1 : 0 : 0)$ and $p_4 = (1 : 1 : 1)$, while $p_5 = (x_0 : y_0 : z_0)$ and not equal to any of the other points. A conic is given by the zero set of some homogeneous polynomial F on the form

$$F = ax^2 + by^2 + cz^2 + dyz + exz + fxy.$$

Considering $F(p_i) = 0$ for $i = 1, 2, 3, 4, 5$ gives that $a = b = c = 0$, $d + e + f = 0$ and $dy_0z_0 + ex_0z_0 + fx_0y_0 = 0$. Hence, there is a unique conic given by F if and only if

$$\begin{pmatrix} 1 & 1 & 1 \\ y_0z_0 & x_0z_0 & x_0y_0 \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a unique solution. This does not happen if and only if the rows are linearly dependent, which in this case means if and only if $y_0z_0 = x_0z_0 = x_0y_0$. But that is impossible because then $p_1, \ldots, p_5$ would not be distinct, and so there is a unique conic trough the five points.

Given five points in general position $p_i = (x_i : y_i : z_i), i = 1, \ldots, 5$, we can find the defining polynomial of the conic through them by considering the determinant

$$\begin{vmatrix} x^2 & y^2 & z^2 & yz & xz & xy \\ x_1^2 & y_1^2 & z_1^2 & y_1z_1 & x_1z_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & x_2z_2 & x_2y_2 \\ x_3^2 & y_3^2 & z_3^2 & y_3z_3 & x_3z_3 & x_3y_3 \\ x_4^2 & y_4^2 & z_4^2 & y_4z_4 & x_4z_4 & x_4y_4 \\ x_5^2 & y_5^2 & z_5^2 & y_5z_5 & x_5z_5 & x_5y_5 \end{vmatrix}.$$

2.2 Multiplicity sequences

To be able to investigate and classify cusps on a plane curve, we will use the so-called *multiplicity sequence*. Before giving the definition, we recall some theory regarding blow-ups of unibranched points.

Let C be a cuspidal plane curve and consider the curve germ of C at p, $(C^{(0)}, p_0) \subset (\mathbb{C}^2, 0)$. Blowing up $X_0 = \mathbb{C}^2$ with centre p_0 gives a smooth surface X_1 and a transform $\pi_0 \colon X_1 \to X_0$. Inside the total transform of $C^{(0)}$, $\pi_0^{-1}(C^{(0)})$, there is an exceptional line E_0 and a strict transform $C^{(1)}$, which meet E_0 at a unique point $p_1 \in X_1$. Repeating this process, gives the following diagram.

$$\begin{array}{ccccccc} \mathbb{C}^2 = X_0 & \xleftarrow{\pi_0} & X_1 & \xleftarrow{\pi_1} & \ldots & \xleftarrow{\pi_n} & X_n \\ \cup & & \cup & & & & \cup \\ C^{(0)} & \longleftarrow & C^{(1)} & \longleftarrow & \ldots & \longleftarrow & C^{(n)} \end{array}$$

If $p = p_0$ is a singularity and we obtain a smooth curve $C^{(n)}$ after n successive blow-ups, we say that $\pi \colon X_n \to X_0$ is a *resolution* of C. We can always obtain a resolution of a singularity by performing a finite number of blow-ups:

Theorem 2.2.1 ([Wal04, Theorem 3.3.1, p. 43])**.** *In the situation described above, there exists an integer N such that $C^{(N)}$ is smooth (and hence $C^{(n)}$ is smooth for $n > N$).*

By [Wal04, Theorem 3.4.4, p. 48], we can in fact obtain a resolution of a plane curve singularity such that all curves in $\pi^{-1}(C)$ are smooth, no three meet in a single point, and all curves intersection transversally. Such a resolution is called *minimal*. An *infinitely near point* of $p \in C$ is a point $p_i \in X_i$ that corresponds to p under a minimal resolution π. By abuse of notation we let $C^{(n)}$ denote the second last strict transform of C in the minimal resolution. We get the following definition.

Definition 2.2.2. The *multiplicity sequence* of a point p on a cuspidal curve C is the sequence of the multiplicities of the infinitely near points of p. If $p_i \in C^{(i)}$

is an infinitely near point of p and $m_{i,p}$ denotes its multiplicity on $C^{(i)}$, then the multiplicity sequence is represented by

$$\overline{m}_p = [m_{0,p}, m_{1,p}, \ldots, m_{n,p}].$$

When the point p is understood it is customary to omit the point from the notation and simply write the sequence as $\overline{m}$ and the individual multiplicities as m_i.

For a cusp p, with multiplicity sequence $\overline{m}_p$, $m_0 \geq m_1 \geq \ldots \geq m_n = 1$ ([FZ96, p. 440]). Furthermore, we have the following characterisation of the multiplicity sequences for a cusp from [FZ96, Proposition 1.2, p. 440].

Proposition 2.2.3. *The multiplicity sequence $\overline{m}_p = [m_0, m_1, \ldots, m_n]$ has the following two properties:*

(i) For each $i = 1, \ldots, n$ there exists $k \geq 0$ such that

$$m_{i-1} = m_i + \cdots + m_{i+k},$$

where

$$m_i = m_{i+1} = \ldots = m_{i+k-1}.$$

(ii) If

$$m_{n-r} > m_{n-r+1} = \ldots = m_n = 1,$$

then $m_{n-r} = r - 1$.

In order to more easily represent a multiplicity sequence for a cusp, we are going to introduce some notation. According to Proposition 2.2.3(ii), a multiplicity sequence always ends with a sequence of ones, the number of which is equal to the index of the first element not equal to one in the multiplicity sequence. Hence, we are going to omit the ones from the sequence. Moreover, whenever a number repeats in the sequence, we will represent it by the number itself with a subindex that is equal to the number of times it repeats. For example, the cuspidal curve given by $F = y^5 - x^3z^2$ has multiplicity sequence $\overline{m}_{p_1} = [3, 2, 1, 1]$ for $p_1 = (0 : 0 : 1)$ and $\overline{m}_{p_2} = [2, 2, 1, 1]$ for $p_2 = (1 : 0 : 0)$. These would then be represented as $\overline{m}_{p_1} = [3, 2]$ and $\overline{m}_{p_2} = [2_2]$.

With the infinitely near points and the multiplicity sequence, we can give the definitions of some important invariants of a given point. The first is from [Har77, Example 3.9.3, p. 393].

Definition 2.2.4. Let p be a point on an algebraic curve C with multiplicity sequence $\overline{m}_p$. Then the *delta-invariant* of p is given by

$$\delta_p = \sum \frac{m_q(m_q - 1)}{2},$$

where the sum is taken over all infinitely near points of p.

For a cusp with multiplicity sequence $\overline{m}_p$, we get that

$$\delta_p = \sum_{i=0}^{n} \frac{m_i(m_i - 1)}{2}.$$

With the delta-invariant established, we can introduce the *geometric genus* of an irreducible algebraic curve C. This is given by the so-called *genus degree formula* ([Fis01, p. 180]):

Proposition 2.2.5. *An irreducible algebraic curve C of degree d has geometric genus*

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{p \in C} \delta_p.$$

When the curve C is understood, we will usually denote this by g alone. The geometric genus is an important invariant of an algebraic curve, and is a nonnegative integer ([Ful69, p. 196]). Moreover, as shown in [Ful69, p. 198] a curve is rational if and only if $g = 0$.

Closely related to the geometric genus is the *arithmetic genus* of an irreducible algebraic curve C of degree d. For a plane curve this is given by [Har77, Exercise 7.2(b), p. 54] as

$$g_a(C) = \frac{(d-1)(d-2)}{2}.$$

As in the case of the geometric genus, we will omit C, when it is clear from the context. The connection between the two numbers is clear, as

$$g = g_a - \sum_{p \in C} \delta_p.$$

Remark 2.2.6. Note that when we refer to the genus of a curve, without specifying if it is the geometric or arithmetic, we mean its geometric genus. To further emphasise this we always use the subscript a when discussing the arithmetic genus.

The multiplicity sequence gives information about intersection multiplicities in the form of the following lemma. This lemma is stated without using the abbreviated form of the multiplicity sequence defined above. Instead, we suppose that all multiplicity sequences are infinite with $m_j = 1$ for all $j > n$.

Lemma 2.2.7. *Let (C, p) be an irreducible plane curve germ with multiplicity sequence $\overline{m}_p = [m_0, m_1, \ldots, m_n, \ldots]$. Then there exists a germ of a smooth curve (Γ, p) through p with $(\Gamma . C)_p = k$ if and only if k satisfies the condition:*

$$k = m_0 + m_1 + \ldots + m_s \text{ for some } s > 0 \text{ with } m_0 = m_1 = \ldots = m_{s-1}.$$

2.3 Puiseux parametrisation

To investigate points on curves, we will frequently use the fact that a unibranched point can be given a local parametrisation. From [Fis01, Cor. 7.7, p. 135] we have that a branch of the germ (C, p), for $p = (0 : 0 : 1)$, can be parametrised by

$$(t^m : ct^l + \cdots : 1),$$

where $l \geq 1$, $c \neq 0$ and "$\cdots$" denote higher order terms in t. Furthermore, by choosing $y = 0$ as the tangent at p, [Fis01, Thm. 8.1, p. 148] says that $m = m_p$ and thus $l = (T_p.C)_p > m$. A parametrisation on this form is called a *Puiseux parametrisation*.

The Puiseux parametrisation can be used to calculate the intersection multiplicity between two curves at a point. The result comes from [Fis01, Theorem 8.7, p. 159].

Theorem 2.3.1. *Let C and C' be algebraic curves given by F and G respectively. Suppose that $p \in C \cap C'$ and that the germ of C' at p has Puiseux parametrisation $(\phi_1(t) : \phi_2(t) : 1)$. Then the intersection multiplicity $(C.C')_p$ is given by*

$$(C.C')_p = \operatorname{ord}_t F(\phi_1(t), \phi_2(t), 1).$$

In other words, the intersection multiplicity between two curves is the order of the power series obtained by substituting the Puiseux parametrisation of one curve into the defining polynomial of the other.

2.4 Weierstrass points

Let C be a curve of degree d and suppose that Q is the collection of all curves in $\mathbb{P}^2$ of degree n. A curve $D \in Q$ gives a divisor on C by associating to a point $p \in C$ the intersection multiplicity between C and D at p. By Bézout's theorem, any such divisor is of degree nd. A curve of degree n is determined by the coefficients of some homogeneous polynomial $F = \sum_{i+j+k=n} a_{ijk} x^i y^j z^k$ up to multiplication by a non-zero scalar. We can therefore identify a curve $D \in Q$ with a point in some projective space. Since the number of monomials in x, y, z of degree n is given by

$$\binom{n+3-1}{3-1} = \frac{(n+1)(n+2)}{2},$$

we can think of Q as $\mathbb{P}^N$, where

$$\begin{aligned} N &= \frac{(n+1)(n+2)}{2} - 1 \\ &= \frac{n(n+3)}{2}. \end{aligned}$$

Hence, Q is a *complete linear system* of dimension $n(n+3)/2$ and degree nd, a so-called $g_{nd}^{n(n+3)/2}$. Traditionally, given a smooth curve C and a complete linear system Q of dimension r, the Weierstrass weight of a point $p \in C$ with respect to Q is calculated by finding the numbers l such that the dimension of the spaces

$$Q \supset Q(-p) \supset \cdots \supset Q(-lp) \cdots$$

changes. Here the space $Q(-lp)$ consists of the divisors in Q that intersect C at p with multiplicity l or higher. The numbers where the dimension changes are known as *gap numbers for Q at p*. Denoting the gap numbers by n_i,

$i = 1, \ldots, r+1$, the *Weierstrass weight of p with respect to Q*, or *Q-Weierstrass weight of p*, is the sum

$$w_p(Q) = \sum_{i=1}^{r+1} (n_i - i).$$

A point p on a plane curve with non-zero Q-Weierstrass weight is called a *Weierstrass point with respect to Q*, or *Q-Weierstrass point* for short.

To find the Weierstrass points on a plane curve C with respect to a complete linear system Q, we are going to use a direct approach involving the calculations of certain intersection multiplicities. The general method works for any irreducible plane curve and thus this enables us to calculate the Q-Weierstrass weights of singular points. The method is due to Notari and is found in [Not99].

Remark 2.4.1. In [Not99] and [BG97] there are two notions of weight, the traditional Q-Weierstrass weight of p, $w_p(Q)$, and what is called the *extraweight* of p with respect to Q, denoted $E_p(Q)$. In general, these notions of weight need not agree, but for a cuspidal curve, $E_p(Q) = w_p(Q)$ by [BG97, p. 153]. Hence, all results in [Not99] and [BG97] involving $E_p(Q)$ are valid when using the Q-Weierstrass weight, when we are dealing with cuspidal curves.

Since we are mainly interested in cuspidal curves, this remark means that our computations are simplified. In addition, Notari's technique depends on calculations done for each branch of C through p, and thus the algorithm is also more manageable when considering cuspidal curves since every point is unibranched. In fact, Notari remarks on [Not99, p. 26] that in the unibranched case, we can consider intersection multiplicities between C and selected curves from Q to compute the Q-Weierstrass weight of a point. For completeness, we present Notari's technique and show why this simplifies in the cuspidal case.

Assuming that C is cuspidal, Notari's technique [Not99, p. 26] for computing the Weierstrass weight of a point $p \in C$ with respect to a complete linear system Q of dimension r is as follows:

Proposition 2.4.2. *1. Let $Q_1 = Q$ and $n_1 = 1 + (C.C_1)_p$, where C_1 is a general curve in Q_1.*

2. For $i \geq 2$, find a condition such that the general curve of Q_i intersect C at p with multiplicity at least n_i.

3. Impose the condition on Q_i to obtain a linear sub-system Q_{i+1}. Set $n_{i+1} = 1 + (C.C_{i+1})_p$, where C_{i+1} is a general curve in Q_{i+1}.

4. Continue step 2 and 3 until Q_{i+1} is empty.

Then the gap numbers $(a_1, \ldots, a_{r+1})$ are in fact the sequence of integers $(n_1, \ldots, n_{r+1})$ computed by the above algorithm. Hence, by [Not99, formula (4), p. 25] and Remark 2.4.1, the Q-Weierstrass weight of a point p is given by

$$w_p(Q) = \sum_{i=1}^{r+1} (n_i - i). \tag{2.2}$$

Remark 2.4.3. Note that the gap numbers for a cuspidal curve computed in Proposition 2.4.2 above are given as $n_i = 1 + (C.C_i)_p$, so the algorithm is reduced to finding curves $C_1, \ldots, C_{r+1} \in Q$ such that the intersections are distinct. If we let $h_{i-1} = (C.C_i)_p$, we can express the Q-Weierstrass weight of a point without having to add 1 to each intersection by a simple change of summation index. Because $n_i = 1 + h_{i-1}$, (2.2) gives the following:

$$\begin{aligned} w_p(Q) &= \sum_{i=1}^{r+1} \big((1 + h_{i-1}) - i\big) \\ &= \sum_{i=1}^{r+1} \big(h_{i-1} - (i-1)\big) \\ &= \sum_{i=0}^{r} (h_i - i). \end{aligned} \tag{2.3}$$

This means that whenever we are going to find the Q-Weierstrass weight of a point p on a cuspidal curve using Proposition 2.4.2, we are actually going to determine distinct intersection multiplicities $h_0, \ldots, h_r$ as described above and use formula (2.3) to calculate the weight.

To obtain results regarding the number of different types of Q-Weierstrass points on a cuspidal curve, we are going to exploit that there is a so-called generalised Plücker formula that gives a connection between the total sum of Q-Weierstrass weights of the Q-Weierstrass points on a curve and the local computations of Q-Weierstrass weights of the points. The desired result is [BG97, Proposition 3.4, p. 153] by Ballico & Gatto, which we state for a cuspidal curve, using our notation.

Proposition 2.4.4. *Let C be a projective, irreducible, cuspidal curve of arithmetic genus g_a, and let Q be a complete linear system of degree d and dimension r. If we let δ_i denote the delta invariant of a singular point and $\delta = \sum_{i=1}^{s} \delta_i$, the following expression holds:*

$$\sum_{p \in C} w_p(Q) = (r+1)d + (r+1)r(g_a - \delta - 1).$$

Remark 2.4.5. Because $g = g_a - \delta$ for a plane curve, we can also express this formula using the geometric genus. This gives

$$\sum_{p \in C} w_p(Q) = (r+1)d + (r+1)r(g-1). \tag{2.4}$$

2.5 Examples of curves

To better understand our objects of interest, we will look at some examples of different curves, starting with the cubics.

2.5.1 Cubic curves

By the genus formula of Proposition 2.2.5, the genus of a cubic curve C is either 1 or 0. If $g = 1$, then C is smooth and is a so-called *elliptic curve*. These are classified according to their j-invariants. If $g = 0$, then C has a unique singular point, and by [Nam84, Proposition 2.2.1, p. 128], there are only two types up to projective equivalence; the nodal cubic and the cuspidal cubic. We have the following examples.

Example 2.5.1. Let $C = V(F)$ be the curve given by

$$F = x^3 + x^2 z - y^2 z.$$

Investigating the partial derivatives of F, we find that $(0:0:1)$ is the only singularity. Furthermore, from the inhomogeneous decomposition of F into homogeneous parts,

$$F(x, y, 1) = x^3 + x^2 - y^2 = x^3 + (x + y)(x - y),$$

$(0 : 0 : 1)$ is a node. If we now intersect the curve and its tangents at the singular point, see Table A.1, we get that the intersection multiplicity is 3 for both tangents, meaning that the node is simple. Figure 2.1 is an illustration of the part of the curve contained in the affine cover $z = 1$, where the singularity is evident.

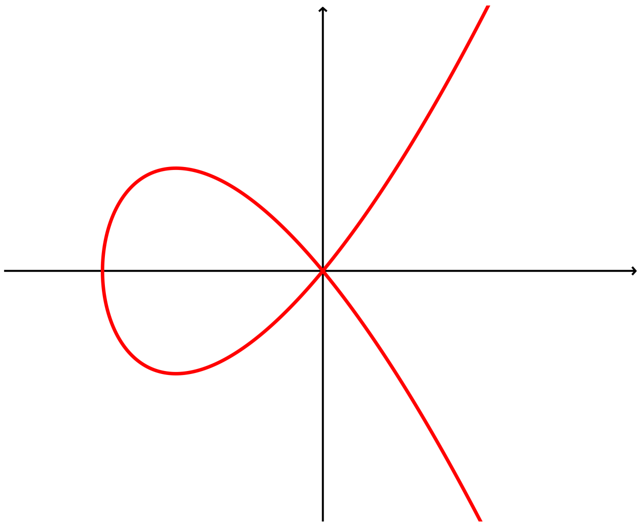

Figure 2.1: Nodal cubic.

Example 2.5.2. Consider now the curve with defining polynomial $F = zy^2 - x^3$. Calculating the derivatives show that the only singular point is $p = (0 : 0 : 1)$. The inhomogeneous decomposition of the polynomial into homogeneous parts is

$$F(x, y, 1) = y^2 - x^3,$$

and so we see that the singular point has multiplicity 2 and a single tangent. Because there is a singe branch going through p, this means that it is a cusp. By Table A.2, we see that the intersection multiplicity between the curve and the tangent at $(0 : 0 : 1)$ is 3, making the singularity simple. Figure 2.2 illustrates the real, affine part of the curve in the cover $z = 1$.

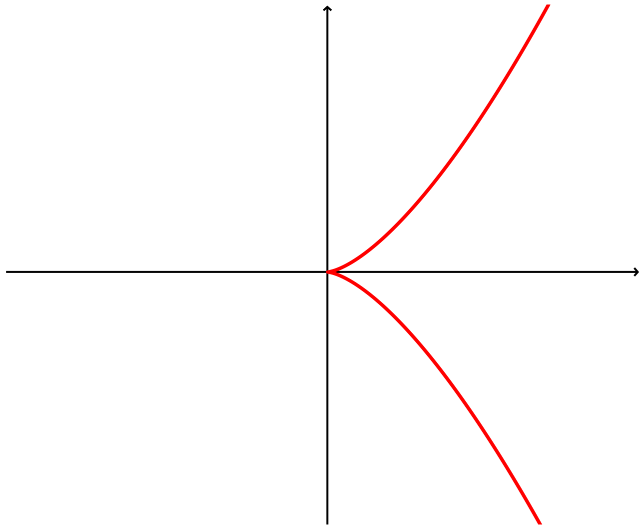

Figure 2.2: Cuspidal cubic.

Example 2.5.3. Lastly, let us look at an example of an elliptic curve, specifically the curve given by

$$F = y^2z - xz^2 - z^3 - x^3.$$

Investigating its partial derivatives shows that there are no singular points. The figure below depicts the curve in the affine cover given by $z = 1$.

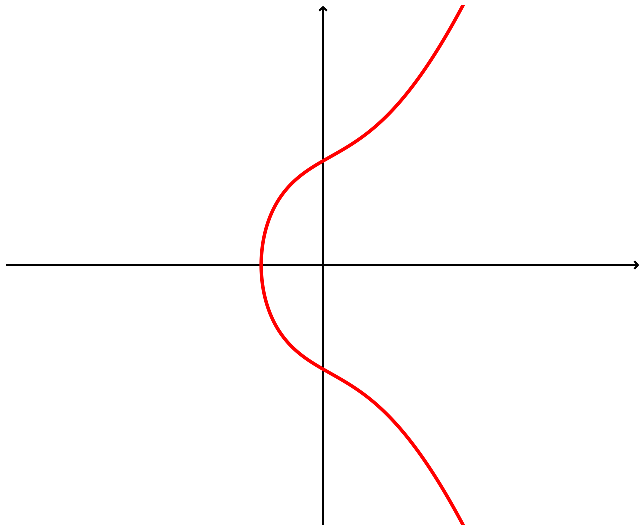

Figure 2.3: Elliptic curve.

2.5.2 Rational cuspidal curves

The next step is to consider curves of higher degree. For each degree there are many curves and so it is natural that we limit our attention to some type of curves that we have a reasonable good understanding of. Rational cuspidal curves have been thoroughly studied during the last 20 years. Many works in the literature, see e.g. [Moe13], give parametrisations and defining polynomials, so the curves are easily accessible. Furthermore, there are many such curves and they can be described by simple invariants, hence we are going to use this class of curves as our primary source of examples.

In [Nam84], Namba has classified the rational cuspidal curves up to degree 5, up to projective equivalence. Table 2.1 gives the five types of rational cuspidal curves of degree four and comes from [Nam84, p. 135, Theorem 2.2.5, p. 146]. Table 2.2 does the same for the rational cuspidal quintics and is a summary of [Nam84, Theorem 2.3.10, pp. 179–182]. Both of these are modified versions of tables given in [Moe13, pp. 42–43].

Curve	Multiplicity sequence	Parametrisation
C_{1A}	[3]	$(t^3s : t^4 : s^4)$
C_{1B}	[3]	$(ts^3 : s^4 : st^3 - t^4)$
C_2	$[2_3]$	$(t^2s^2 : t^4 : s^4 - t^3s)$
C_3	$[2_2]$,[2]	$(s^4 + ts^3 : t^2s^2 : t^4)$
C_4	[2],[2],[2]	$(ts^3 - \frac{1}{2}s^4 : t^2s^2 : t^4 - 2t^3s)$

Table 2.1: Rational cuspidal quartics.

Curve	Multiplicity sequence	Parametrisation
C_{1A}	[4]	$(s^5 : st^4 : t^5)$
C_{1B}	[4]	$(s^5 - s^4t : st^4 : t^5)$
C_{1C}	[4]	$(s^5 + as^4t - (1+a)s^2t^3 : st^4 : t^5),\ a \neq -1$
C_2	$[2_6]$	$(s^4t : s^2t^3 - s^5 : t^5 - 2s^3t^2)$
C_{3A}	[3,2],$[2_2]$	$(s^5 : s^3t^2 : t^5)$
C_{3B}	[3,2],$[2_2]$	$(s^5 : s^3t^2 : st^4 + t^5)$
C_4	[3],$[2_3]$	$(s^4t - \frac{1}{2}s^5 : s^3t^2 : \frac{1}{2}st^4 + t^5)$
C_5	$[2_4]$,$[2_2]$	$(s^4t - s^5 : s^2t^3 - \frac{5}{32}s^5 : -\frac{47}{128}s^5 + \frac{11}{16}s^3t^2 + st^4 + t^5)$
C_6	[3],$[2_2]$, [2]	$(s^4t - \frac{1}{2}s^5 : s^3t^2 : -\frac{3}{2}st^4 + t^5)$
C_7	$[2_2]$,$[2_2]$,$[2_2]$	$(s^4t - s^5 : s^2t^3 - \frac{5}{32}s^5 : -\frac{125}{128}s^5 - \frac{25}{16}s^3t^2 - 5st^4 + t^5)$
C_8	$[2_3]$, [2],[2],[2]	$(s^4t : s^2t^3 - s^5 : t^5 + 2s^3t^2)$

Table 2.2: Rational cuspidal quintics.

CHAPTER 3

Inflection points

There are several types of special points that can occur on plane algebraic curves. We have seen that the singular points inherently are one type, and now we want to consider special types of smooth points. In this chapter we consider inflection points and some of the theory surrounding them.

3.1 Inflection points

Consider a line L that intersects a plane algebraic curve $C = V(F)$ in the point p. If we move p to $(0 : 0 : 1)$, we can find the intersection multiplicity, by considering an affine parametrisation of the line given by

$$\phi(t) = (\alpha t, \beta t).$$

Let $F(x, y, 1) = f(x, y) = \sum_{k=1}^{d} f_k(x, y)$, then by Theorem 2.3.1 the intersection multiplicity between L and C at p is given by the order of t in

$$f(\phi(t)) = \sum_{k=1}^{d} f_k(\alpha, \beta)t^k.$$

If we let $m = m_p$ be the multiplicity of p on C, then this means that $(C.L)_p > m$ if and only if $f_m(\alpha, \beta) = 0$, that is if and only if L is a tangent line to C at p. In particular, if p is a smooth point, $(C.L)_p \geq 2$ if and only if L is the tangent line of C at p.

The smooth points where the intersection multiplicity between the tangent line and the curve is strictly higher than 2 are the subject of this chapter. We have the following definition:

Definition 3.1.1. Let $C = V(F)$ be a plane algebraic curve. For a smooth point $p \in C$, we call p an *inflection point* or *flex* if the tangent T_p to C at p satisfies $(T_p.C)_p \geq 3$. An inflection point is said to be of type v, or equivalently denoted as a v-flex, when $(T_p.C)_p - 2 = v$.

In particular, an inflection point of type 1 will be called a *simple inflection point*, while non-simple inflection points will be denoted as *higher order inflection points*. Figure 3.1 depicts a simple inflection point on the cuspidal cubic, while Figure 3.2 shows a higher order inflection point on the quintic curve C_{1A} from Table 2.2.

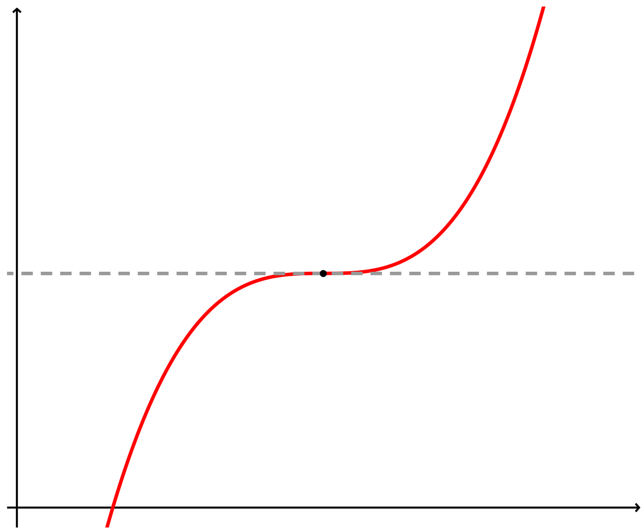

Figure 3.1: A simple inflection point on the cuspidal cubic with tangent.

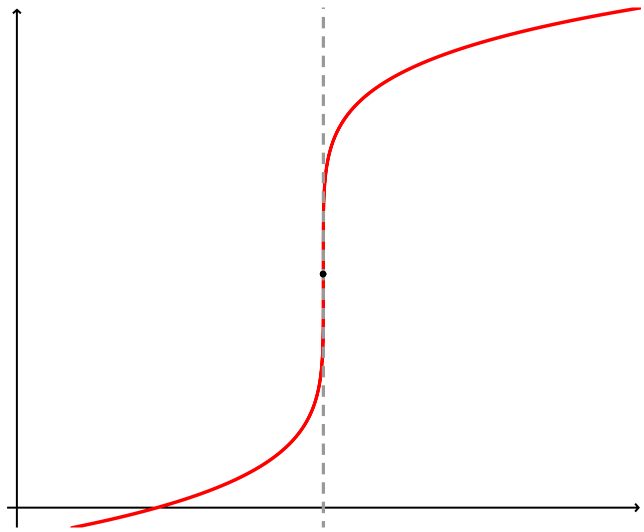

Figure 3.2: A higher order inflection point on quintic $C_{1\mathrm{A}}$ with tangent.

3.2 The Hessian curve

The Hessian curve is the essential tool to identify inflection points on a given curve. The following definition is from [Fis01, p. 65].

Definition 3.2.1. If $F \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree $d \geq 2$, then the symmetric 3×3 matrix

$$H_F := \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix}$$

is called the *Hessian matrix of F*. If F is a minimal polynomial of the curve $C = V(F)$ and $\deg(\det H_F) \geq 1$, then $H_1^C := V(\det H_F)$ is called the *Hessian curve* of C.

When there is no ambiguity regarding the underlying curve we will simply denote the Hessian curve by H_1 or just H. Note that if $\deg C = d$, then $\deg H = 3(d-2)$ as each entry in H_F is a homogeneous polynomial of degree $(d-2)$. Moreover, by [Fis01, p. 67] the Hessian curve is independent of the coordinates, and $\operatorname{Sing} C \subset H$.

The connection between inflection points and the Hessian curve is given in the following result.

Theorem 3.2.2 ([Fis01, 4.5, p. 67])**.** *Let $C = V(F)$ be a curve that contains no lines, and let H be its Hessian curve. Then*

a) $\det H_F \neq 0$*;*

b) a smooth point $p \in C$ is an inflection point if and only if $p \in H$;

c) C and H have no common components;

d) if $p \in C$ is a simple inflection point, then

$$(C.H)_p = 1.$$

Remark 3.2.3. This theorem gives us a direct way of finding the inflection points of a given curve. By intersecting a curve C with its Hessian curve, all intersection points are either singular or inflection points, so by eliminating the singular ones, we are left with the inflection points of C.

3.3 Inflection point formulas

Sometimes we would like information about the number of inflection points on a curve without having to find them all explicitly. To do so, there are different kinds of so-called *inflection point formulas* of various generality. The natural starting point is to see that there is a formula for smooth curves.

Theorem 3.3.1 ([Mir95, Corollary 4.16, p. 241])**.** *Let C be a smooth algebraic curve of degree d. Then C has exactly $3d(d-2)$ inflection points, where an inflection point p whose tangent line meets C at p with multiplicity v is counted $v-2$ times.*

This result can be proved in a number of ways, but we present here a straightforward proof using Bézout's theorem.

Proof of Theorem 3.3.1. Since C is smooth, C and H intersect only in the inflection points by Theorem 3.2.2. Bézout's theorem gives that

$$\sum_{p \in C \cap H} (C.H)_p = d \cdot 3(d-2),$$

so that there are indeed $3d(d-2)$ inflection points on C, counted with multiplicity. ∎

From this result and Bézout's theorem it is clear that singularities on a curve will reduce the number of inflection points the curve can have. Not surprisingly, different singularities account for different reductions in this number, and so there are various formulas for different cases. The perhaps most famous of which is one of the classical Plücker formulas. In our notation, it is as follows:

Theorem 3.3.2 ([Fis01, Inflection point formula, 5.7, p. 89])**.** *Let $C = V(F)$ be an irreducible curve of degree $d \geq 2$ such that the singularities of the curve C and its dual $C^\vee$ are simple nodes and simple cusps. Then the number of inflection points v is given by*

$$v = 3d(d-2) - 6n - 8k,$$

where n and k are the number of nodes and cusps on the curve respectively.

This theorem has a classical proof that we recount for completeness. It is essentially the same proof as Mork presents in [Mor04, pp. 22–23].

Proof of Theorem 3.3.2. As C is irreducible it contains no lines and thus by Theorem 3.2.2 c) C and the Hessian curve of C, H, are distinct. Moreover, by [Fis01, p. 88] C has only simple inflection points because all cusps on $C^\vee$ are simple. Hence $(C.H)_p = 1$ for all inflection points by Theorem 3.2.2 d). Bézout's theorem gives that $C.H = 3d(d-2)$, so if we let q denote a node on C and p a cusp on C, we have that

$$\begin{aligned} 3d(d-2) &= v + \sum_{p \in \operatorname{Sing} C} (C.H)_p \\ &= v + n \cdot (C.H)_q + k \cdot (C.H)_p. \end{aligned} \tag{3.1}$$

because all nodes and cusps on C are simple. Hence, we have reduced the problem to having to find $(C.H)_q$ and $(C.H)_p$. Assume first that q is a node. By [Fis01, 5.6 a), p. 86] we can performing a linear change of coordinates to ensure that $q = (0:0:1)$ and the affine equation $f(x, y) = F(x, y, 1)$ is on the form

$$f = xy - (x^3 + y^3) + \cdots .$$

Calculating the defining polynomial of H, we then obtain

$$\det H_F = (d-1)(d-2)xyz^{3(d-2)-2} + 2d(d-1)(x^3 + y^3)z^{3(d-2)-3} + \cdots .$$

Looking at two branches of C at q separately, they can be given Puiseux parametrisations

$$\begin{aligned} x &= t & \qquad & & x &= t^2 + \cdots \\ y &= t^2 + \cdots & \quad \text{and} & & y &= t \\ z &= 1 & & & z &= 1 \end{aligned}$$

Using Theorem 2.3.1 and substituting these parametrisations into $\det H_F$ will give the intersection for each branch. By symmetry we need only consider one of the parametrisations, and using $x = t, y = t^2 + \cdots, z = 1$ gives

$$\begin{aligned} \varphi(t) &= (d-1)(d-2)(t^3 + \cdots) + d(d-1)(t^3 + t^6 + \cdots) + \cdots \\ &= 2(d-1)^2 t^3 + \cdots . \end{aligned}$$

Hence $\operatorname{ord}(\varphi) = 3$ for each branch and thus $(C \cdot H)_q = 2 \cdot 3 = 6$.

Assuming now that p is a cusp, [Fis01, 5.6 b), p. 87] ensures that we can assume $p = (0 : 0 : 1)$ and that $f(x, y) = F(x, y, 1)$ now is on the form

$$f(x, y) = y^2 - x^3 + ax^2y + \cdots .$$

The defining polynomial of the Hessian curve then becomes

$$4(d-1)(d-2)(3x - ay)y^2z^{3(d-2)-2} - 6(d-1)(d-3)x^4z^{3(d-2)-4} + \cdots .$$

The Puiseux parametrisation of C at p is given by $x = t^2, y = t^3 + \cdots, z = 1$, and substituting this into the equation above gives

$$\varphi(t) = 6(d-1)^2t^8 + \cdots .$$

Hence, $(C \cdot H)_p = 8$. Combining this with the result for nodes means that (3.1) becomes

$$v = 3d(d-2) - 8n + 6k,$$

which is exactly what we wanted to prove. ■

The Plücker inflection point formula can be applied to many curves, but is restricted by the fact that all singularities must be simple. Several formulas for curves with more general singularities exist, for example the one for cuspidal curves given in [Moe13]. We provide a new proof of this formula, using the theory of Weierstrass points with respect to a complete linear system and Ballico & Gatto's Plücker formula to produce a quite straightforward proof.

Theorem 3.3.3 ([Moe13, Theorem 2.1.8, p. 32])**.** *Assume that C is a plane cuspidal curve with n cusps p_j, each having tangent T_j. Let m_j denote the multiplicity, δ_j the delta invariant, and $l_j \coloneqq (T_j.C)_{p_j}$ the tangential intersection multiplicity of p_j. Then the number of inflection points v on C, counted such that an inflection point q_i of type v_i accounts for v_i inflection points, is given by*

$$v = 3d(d-2) - 6\sum_{j=1}^{n} \delta_j - \sum_{j=1}^{n}(m_j + l_j - 3). \tag{3.2}$$

Proof. Let C be a cuspidal curve as described in the theorem. By Section 2.3 we can find a local Puiseux parametrisation of each point on the form

$$(t^m : t^l + \cdots : 1),$$

where m is the multiplicity of the point and l the tangential intersection of the curve at the point in question. To produce the wanted formula, we want to use Proposition 2.4.4. Let therefore Q be the complete linear system on C consisting of lines. Then, it is clear from Bézout's theorem that Q is a g^2_d, i.e. a complete linear system of dimension 2 and degree d. Taking

$$x, y, z$$

as the basis for Q, substituting the local parametrisation of the point into this gives

$$t^m, t^l + \cdots, 1. \tag{3.3}$$

In other words, we have three curves from Q that intersect C at p with distinct intersection multiplicities

$$h_0 = 0,\ h_1 = m,\ h_2 = l.$$

This means that the Q-Weierstrass weight of p is

$$\begin{aligned} w_p(Q) &= \sum_{i=0}^{2}(h_i - i) \\ &= m + l - 3. \end{aligned} \tag{3.4}$$

Note now that if p is a smooth point, which is not an inflection point, then $m = 1$ and $l = 2$, so $w_p(Q) = 0$. For an inflection point, we have that

$$w_p(Q) = l - 2,$$

so the weight of an inflection point is simply its type as a flex.

Now, we use the Ballico & Gatto Plücker formula of Proposition 2.4.4 with $r = 2$ and degree d. We get

$$\sum_{p \in C} w_p(Q) = 3d + 6(g_a - \delta - 1). \tag{3.5}$$

From what we observed above, the sum on the left can be split in two.

$$\begin{aligned} \sum_{p \in C} w_p(Q) &= \sum_{p \in \operatorname{Infl} C} w_p(Q) \ + \sum_{p \in \operatorname{Sing} C} w_p(Q) \\ &= \sum_{p \in \operatorname{Infl} C} (l_p - 2) + \sum_{p \in \operatorname{Sing} C} (m_p + l_p - 3) \end{aligned}$$

Denoting the sum over the inflection points of C by v, it is clear that v is the total number of inflection points counted with multiplicity. Moreover, we have assumed that there are n singular points p_j on C such that the sum over all singularities can be taken from 1 to n. Putting this into equation (3.5) gives

$$v + \sum_{j=1}^{n}(m_j + l_j - 3) = 3(d + 2(g_a - \delta - 1)),$$

which is equivalent to

$$v = 3(d + 2(g_a - \delta - 1)) - \sum_{j=1}^{n}(m_j + l_j - 3).$$

Expressing the arithmetic genus and the sum of the delta invariants explicitly now gives the formula we want.

$$\begin{aligned} v &= 3(d + 2\Big(\tfrac{(d-1)(d-2)}{2} - \sum_{j=1}^{n}\delta_j\Big) - 2) - \sum_{j=1}^{n}(m_j + l_j - 3) \\ &= 3d + 3(d-1)(d-2) - 6\sum_{j=1}^{n}\delta_j - 6 - \sum_{j=1}^{n}(m_j + l_j - 3) \\ &= 3d(d-2) - 6\sum_{j=1}^{n}\delta_j - \sum_{j=1}^{n}(m_j + l_j - 3). \end{aligned}$$

■

Let us consider an example.

Example 3.3.4. Let C be the quintic C_{1C} in Table 2.2. From Table A.11, C has a single cusp $p = (0 : 0 : 1)$ with multiplicity sequence $\overline{m}_p = [4]$ and delta invariant $\delta_p = 6$. By formula (3.2), we then get that the number of inflection points on C is

$$\begin{aligned} v &= 3 \cdot 5 \cdot (5-2) - 6 \cdot 6 - (4+5-3) \\ &= 3, \end{aligned}$$

which is consistent with the number of inflection points we got in Table A.11 when intersecting H and C in [Maple].

Theorem 3.3.3 is again a special case of a result in the classical work [BK86] by Brieskorn and Knörrer. The result from [BK86] is the most general inflection point formula there is, and holds for any irreducible plane curve, without restrictions on the genus or on the singularities.

Theorem 3.3.5 ([BK86, Theorem 2, p. 586])**.** *Suppose C is a plane algebraic curve without multiple components. Let d be the degree of C, let p_i, with $i = 1, \dots, n$, be the singular points of C and let v be the number of inflection points of C. Note that here the inflection points of C are counted with multiplicity, i.e., such that an inflection point of type v_i accounts for v_i inflection points. Furthermore, let m_p be the multiplicity of the point p on C, m_{ij} the multiplicities of the infinitely near singular points of C at p_i and finally, let $m_p^\vee$ be the multiplicity of the singular points of $C^\vee$. Then*

$$v = 3d(d-2) - 3\sum_{i=1}^{n}\sum_{j=1}^{t_i-1} m_{ij}(m_{ij}-1) - \sum_p (2m_p + m_p^\vee - 3)$$

Remark 3.3.6. In the case of a cuspidal curve, this formula becomes the same as in Theorem 3.3.3, because $m_p + m_p^\vee = l_p$ [Moe08, p. 16].

3.4 Local intersections

Recall that by Remark 3.2.3 and Bézout's theorem a curve C of degree d and its Hessian curve H intersect in $3d(d-2)$ points, where these points are either singular or inflections. Hence, formula (3.2) can be rewritten to

$$v = C.H - \sum_{j=1}^{n}(6\delta_j + m_j + l_j - 3), \tag{3.6}$$

so that the number of inflection points is given by the intersection between C and H, and subtracting the contributions made by singular points.

Note now that from the proof of Theorem 3.3.3 we have that the Q-Weierstrass weight of an inflection point p is on the same form as the Q-Weierstrass weight for a singular point, i.e. $w_p(Q) = m_p + l_p - 3$. Furthermore, $\delta_p = 0$ so we can include the inflection points in the sum occurring in the formula above. Moving the sum over to the left and expressing $C.H$ as the sum of local intersection gives:

$$\sum_{p \in C \cap H} (C.H)_p = \sum_{p \in C \cap H} (6\delta_p + m_p + l_p - 3).$$

As shown in [Moe13, Theorem 2.1.9, p. 32], this is a local result.

Theorem 3.4.1. *The intersection multiplicity $(C.H)_p$ of a cuspidal curve C and its Hessian curve H at any point $p \in C$ is*

$$(C.H)_p = 6\delta_p + m_p + l_p - 3.$$

This means that the contributions from singular points subtracted in (3.6) above are indeed the local intersection multiplicities between C and H at the singular points. Let us look at an example.

Example 3.4.2. Let C_3 be the quartic curve from Table 2.1. By Table A.7, we see that it has 2 singular points and a single inflection point. Moreover, we have calculated the multiplicities, delta invariants and tangential intersections at the points. Let $p_1 = (0 : 0 : 1)$, $p_2 = (1 : 0 : 0)$ and p_3 denote the inflection point. Then Theorem 3.4.1 above gives

$$\begin{aligned}
(C.H)_{p_1} &= 6 \cdot 1 + 2 + 3 - 3 \\
&= 8 \\
(C.H)_{p_2} &= 6 \cdot 2 + 2 + 4 - 3 \\
&= 15 \\
(C.H)_{p_3} &= 6 \cdot 0 + 1 + 3 - 3 \\
&= 1.
\end{aligned}$$

This agrees with the intersections in Table A.7 calculated using the Hessian curve.

3.5 Inflection points on rational cuspidal curves

For rational cuspidal curves, we get the following corollary to Theorem 3.3.3.

Corollary 3.5.1. *For a rational, cuspidal curve, that is a cuspidal curve of genus $g = 0$, the number of inflection points are given by*

$$v = 3(d-2) - \sum_{j=1}^{n} (m_j + l_j - 3). \tag{3.7}$$

Let Q denote the complete linear system of lines, and assume that C is a rational cuspidal curve, with parametrisation

$$\phi = \big(\phi_0(s,t) : \phi_1(s,t) : \phi_2(s,t)\big).$$

By [Fis01, p. 76], the tangent to C at a point $\phi(s,t)$ is given by

$$\begin{vmatrix} x & y & z \\ \frac{\partial \phi_0}{\partial s} & \frac{\partial \phi_1}{\partial s} & \frac{\partial \phi_2}{\partial s} \\ \frac{\partial \phi_0}{\partial t} & \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_2}{\partial t} \end{vmatrix}.$$

Let $\xi(s,t)$ be the homogeneous polynomial in s,t of degree $3(d-2)$ that is given by the *Wronskian* of the functions ϕ_0, ϕ_1 and ϕ_2:

$$\begin{vmatrix} \frac{\partial^2\phi_0}{\partial s^2} & \frac{\partial^2\phi_1}{\partial s^2} & \frac{\partial^2\phi_2}{\partial s^2} \\ \frac{\partial^2\phi_0}{\partial s\partial t} & \frac{\partial^2\phi_1}{\partial s\partial t} & \frac{\partial^2\phi_2}{\partial s\partial t} \\ \frac{\partial^2\phi_0}{\partial t^2} & \frac{\partial^2\phi_1}{\partial t^2} & \frac{\partial^2\phi_2}{\partial t^2} \end{vmatrix}.$$

By [KS03, p. 950], the singular points and the inflection points of C occur as the roots of ξ, and the Q-Weierstrass weight of a point on C is the multiplicity of the corresponding zero of ξ. Hence, Corollary 3.5.1 has a nice interpretation. Because the Wronskian is of degree $3(d-2)$, we can think of formula (3.7) as counting the total sum of the Q-Weierstrass weights and subtracting the contributions from inflections and singular points.

Let us look at an example.

Example 3.5.2. Let $C = C_8$ be the rational cuspidal quintic as given in Table 2.2, with parametrisation

$$\phi = (s^4t : s^2t^3 - s^5 : t^5 + 2s^3t^2).$$

The Wronskian of this parametrisation is given by

$$\begin{vmatrix} 12s^2t & 2t^3 - 20s^3 & 12st^2 \\ 4s^3 & 6st^2 & 12s^2t \\ 0 & 6s^2t & 4s^3 + 20t^3 \end{vmatrix},$$

so $\xi(s,t) = 320s^3(s^3 + 2t^3)^2$. Using Table A.19, let $p_1 = (0 : 0 : 1)$ and let p_2, p_3, p_4 denote the three simple cusps on C. According to the Wronskian, the Q-Weierstrass weight of p_1, which corresponds to $s = 0$ in ϕ, should be 3, while the Q-Weierstrass weights of the simple cusps, that correspond to the zeros of $s^3 + 2t^3$, are all 2. Calculating the Q-Weierstrass weight by formula (3.4) in the proof of Theorem 3.3.3 gives

$$\begin{aligned} w_{p_1}(Q) &= 2 + 4 - 3 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} w_{p_i}(Q) &= 2 + 3 - 3 \\ &= 2 \end{aligned}$$

for $i = 2, 3, 4$, exactly as we found with the Wronskian.

CHAPTER 4

Sextactic points

In this chapter we investigate and discuss the main subject of the thesis; sextactic points. First, we introduce some notation and central notions, such as the osculating conic. We then study Cayley's 2-Hessian curve and show that it does not have the properties it should, before we correct Cayley's mistake and find the correct defining polynomial of the 2-Hessian curve. Further, we look at formulas for the number of sextactic points on various curves, and present a new formula for cuspidal curves.

4.1 Sextactic points

The general theory of five pointic contact between a plane curve and a conic at a given point was established by Cayley in his 1859 article *On the Conic of Five-pointic Contact at any point of a Plane Curve* [Cay59]. The later article *On the Sextactic Points of a Plane Curve* [Cay65] investigates the subject further, and it is here Cayley presents the main tool to develop the theory of sextactic points on a plane curve, the 2-Hessian. To understand the results of these articles we first need to introduce some notation. Note that in the following, we use both $\partial_x F$ and F_x to denote the partial derivative of F with respect to x, depending on what is convenient when considering indices and other notation.

Let C be a plane, irreducible curve given by the equation $F(x, y, z) = 0$, and let $p = (x_1 : y_1 : z_1)$ be a point on the curve. For shortness, Cayley defines

$$\begin{aligned} DF_p(x, y, z) &= x\partial_x F(p) + y\partial_y F(p) + z\partial_z F(p), \\ D^2F_p(x, y, z) &= x^2\partial_x^2 F(p) + y^2\partial_y^2 F(p) + z^2\partial_z^2 F(p) \\ &\quad + 2xy\partial_x\partial_y F(p) + 2xz\partial_x\partial_z F(p) + 2yz\partial_y\partial_z F(p). \end{aligned}$$

Moreover, Cayley denotes the mixed second order partial derivatives of F by

$$a = F_{xx}, b = F_{yy}, c = F_{zz}, f = F_{yz}, g = F_{xz}, h = F_{xy},$$

and so

$$H_F = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

is the determinant of the Hessian-matrix of F. Furthermore, Cayley puts

$$H_{H_F} = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

Indeed, we have

$a' = (H_F)_{xx}, b' = (H_F)_{yy}, c' = (H_F)_{zz}, f' = (H_F)_{yz}, g' = (H_F)_{xz}, h' = (H_F)_{xy}.$

With this notation Cayley defines the following:

$$\begin{aligned}
\mathcal{A} &= bc - f^2, \mathcal{B} = ac - g^2, \mathcal{C} = ab - h^2,\\
\mathcal{F} &= hg - af, \mathcal{G} = hf - bg, \mathcal{H} = fg - hc,\\
\Omega &= (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{G}, \mathcal{H}) \cdot (a', b', c', 2f', 2g', 2h'),\\
\partial_x \Omega_{\bar{H}} &= (\partial_x \mathcal{A}, \partial_x \mathcal{B}, \partial_x \mathcal{C}, \partial_x \mathcal{F}, \partial_x \mathcal{G}, \partial_x \mathcal{H}) \cdot (a', b', c', 2f', 2g', 2h'),\\
\partial_x \Omega_{\bar{F}} &= (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{G}, \mathcal{H}) \cdot (\partial_x a', \partial_x b', \partial_x c', 2\partial_x f', 2\partial_x g', 2\partial_x h').
\end{aligned}$$

Similarly, we obtain $\partial_y \Omega_{\bar{H}}, \partial_z \Omega_{\bar{H}}, \partial_y \Omega_{\bar{F}}$ and $\partial_z \Omega_{\bar{F}}$ by replacing x with y and z respectively in the derivation in each of the expressions $\partial_x \Omega_{\bar{H}}$ and $\partial_x \Omega_{\bar{F}}$. Lastly, Ψ denotes the determinant

$$\Psi = - \begin{vmatrix} 0 & \partial_x H_F & \partial_y H_F & \partial_z H_F \\ \partial_x H_F & a & h & g \\ \partial_y H_F & h & b & f \\ \partial_z H_F & g & f & c \end{vmatrix}.$$

4.1.1 The osculating conic

We are now able to state the results by Cayley that form the basis of our quest for sextactic points.

Theorem 4.1.1 ([Cay59, p. 377])**.** *Let C be a plane curve of degree d given by a polynomial F. If p is a point on C that is neither singular nor an inflection point and $\Lambda = -3\Omega H_F + 4\Psi$, then*

$$O_p := V\left(D^2 F_p - \left(\frac{2}{3}\frac{1}{H_F(p)}(DH_F)_p + \Lambda(p) DF_p\right) DF_p\right)$$

is the conic that intersects C at p with intersection multiplicity at least 5.

The conic O_p given in Theorem 4.1.1 is called the *osculating conic of C at the point p*. Beware that this should not be mistaken for the local ring of germs of regular functions on a variety near a point p, $\mathcal{O}_p$, although the notations are similar.

Now, we may finally give the formal definition of sextactic points.

Definition 4.1.2. Let p be a smooth point on a curve C that is not an inflection point. Then p is called a *sextactic point* if

$$(C.O_p)_p \geq 6,$$

where O_p is the osculating conic of C at p. A sextactic point p is said to be of type s, or *s-sextactic*, when $s = (O_p.C)_p - 5$.

The connection to the case of tangent lines and inflection points is evident. An inflection point on a curve is a smooth point where the tangent intersects the curve with multiplicity higher than expected, while a sextactic point is a point where the osculating conic intersects the curve with multiplicity higher than expected. The following figure illustrates the nodal curve and its osculating conic at a sextactic point.

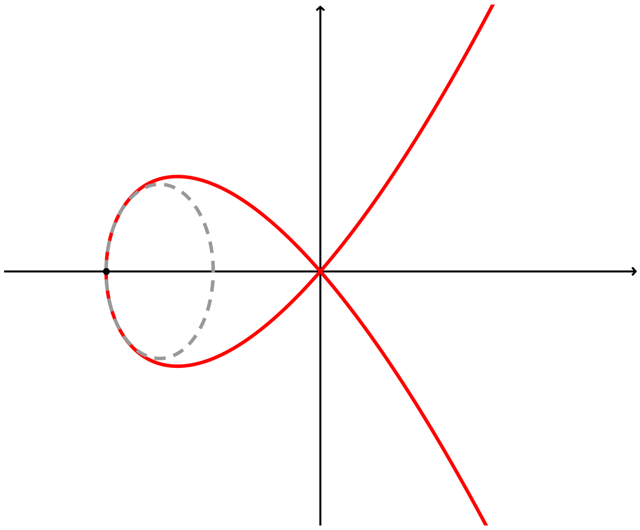

Figure 4.1: The nodal cubic with an osculating conic.

4.2 The 2-Hessian curve

In this section we investigate a result by Cayley claiming that to given a curve, there exist another curve that intersects the original in sextactic points. His formula turns out to be wrong, and we correct it.

In the following, the determinant of the Jacobian matrix with respect to three polynomials F, G, H is denoted by $\mathrm{Jac}(F, G, H)$. That is,

$$\mathrm{Jac}(F, G, H) = \begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix}.$$

4.2.1 Cayley's 2-Hessian

We now look at Cayley's 2-Hessian curve from [Cay65]. The defining polynomial of this curve does not, however, give the sextactic points.

Theorem 4.2.1 ([Cay65, p. 545])**.** *Let $C = V(F)$ be a plane curve of degree d with $H = \det H_F$. Then there exists a curve of degree $12d - 27$ given as the zero set of*

$$\begin{aligned} &(12d^2 - 54d + 57)H\,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) \\ +&(d-2)(12d-27)H\,\mathrm{Jac}(F, H, \Omega_{\bar{F}}) \\ +&40(d-2)^2\,\mathrm{Jac}(F, H, \Psi) \end{aligned}$$

such that the intersection points between C and this curve are the singular points, the sextactic points, and higher order inflection points of C.

Remark 4.2.2. It should be remarked that it is unclear whether Cayley assumes that the curve is smooth or not. This makes no difference, however, as all considerations Cayley makes are local and thus works for all smooth points on singular curves as well. Furthermore, note that if C is singular, then Cayley's 2-Hessian intersects C in the singular points because each term in the defining polynomial of the 2-Hessian has a Jacobian as a factor, where F appears as one of the arguments of the Jacobian. Hence, we get the theorem above, without restrictions on the smoothness of the curve.

As mentioned, the theorem is not true, as the following example shows.

Example 4.2.3. Let $C_{1\mathrm{B}}$ be the quartic curve given in Table 2.1. Using Program B.5, we find that its defining polynomial is given by

$$F = x^4 - x^3y + y^3z.$$

Using [Maple] and Program B.4, we obtain the following polynomial for Cayley's 2-Hessian:

$$129699947520x^3y^{18} - 97274960640x^2y^{19} + 28910390400xy^{20} - 3174474240y^{21}.$$

Finding the intersection points between the curve defined by this polynomial and $C_{1\mathrm{B}}$ is done using the `intersectcurves` command in [Maple] and gives the output

$$\Big[[81, [x, y, 1]], [1, [x - \tfrac{64}{3}, y - \tfrac{256}{3}, 1]], \\ [1, [-\tfrac{5572}{20449}y + x - 8, y^2 + \tfrac{12535237}{1251264}y + \tfrac{418161601}{234612}, 1]]\Big].$$

This means that the intersection points are

$$\begin{aligned} p_1 &= (0 : 0 : 1), \\ p_2 &= \left(\tfrac{64}{3} : \tfrac{256}{3} : 1\right), \\ p_3 &= \left(\tfrac{593021}{89376} + \tfrac{28457}{29792}i\sqrt{143} : \tfrac{-12535237}{2502528} + \tfrac{2924207}{834176}i\sqrt{143} : 1\right), \\ p_4 &= \left(\tfrac{593021}{89376} - \tfrac{28457}{29792}i\sqrt{143} : \tfrac{-12535237}{2502528} - \tfrac{2924207}{834176}i\sqrt{143} : 1\right). \end{aligned}$$

The first point is the singular point, and a computation of the tangential intersection at the other three points using Program B.1 and `singularities` reveals that they are not higher order inflection points. We would therefore expect the last three points to be sextactic. Starting with p_2, we find the osculating conic O_{p_2} and then calculate the intersection multiplicity between C and O_{p_2} using the `intersectcurves` command. This gives

$$\left[\left[6, \left[x - \tfrac{64}{3}, y - \tfrac{256}{3}, 1\right]\right], \left[1, \left[-\tfrac{77}{32}y + x - 8, y^2 + \tfrac{11776}{1331}y + \tfrac{65536}{3993}, 1\right]\right]\right]$$

and thus shows that the intersection multiplicity is 6 at p_2 as expected. Since p_3 and p_4 have complex coordinates, we use the [Maple] command `singularities` to calculate the intersection multiplicities in these points. We get the following output for the intersection $(O_{p_3}.C)_{p_3}$:

$$\left[\left[\tfrac{28457}{29792}i\sqrt{143}z + \tfrac{593021}{89376}z, \tfrac{2924207}{834176}i\sqrt{143}z - \tfrac{12535237}{2502528}z, 1\right], 2, 5, 2\right].$$

This means that $(O_{p_3}.C)_{p_3} = 5$, and so p_3 is not a sextactic point on C despite being one of the intersection points between the curve and Cayley's 2-Hessian. A similar calculation for p_4 gives

$$\left[\left[-\tfrac{28457}{29792}i\sqrt{143}z + \tfrac{593021}{89376}z, -\tfrac{2924207}{834176}i\sqrt{143}z - \tfrac{12535237}{2502528}z, 1\right], 2, 5, 2\right],$$

showing that also $(O_{p_4}.C)_{p_4} = 5$ and that consequently p_4 is not sextactic as well.

Clearly, this means that Cayley's 2-Hessian curve is not correct. What is true, and this is probably why Cayley did not realise his mistake, is that the equation above gives a curve that fulfils the desired properties in the case of cubic curves and some simple quartics. To see this we study the following examples.

Example 4.2.4. Let $F = -x^3 - x^2z + y^2z$ be the defining polynomial of the nodal cubic. Again using Program B.4, we obtain a polynomial

$$39813120x^8y - 106168320x^6y^3 + 79626240x^4y^5 - 13271040y^9.$$

Using the maple command `intersectcurves` now produces the following output

$$\left[[24, [x, y, 1]], [1, [x + 1, y, 1]], \left[1, \left[x + 4, y^2 + 48, 1\right]\right]\right],$$

which means that there are four points of intersection between the curves, $(0 : 0 : 1), (-1 : 0 : 1), (-4 : 4i\sqrt{3} : 1)$ and $(-4 : -4i\sqrt{3} : 1)$. We recognise $(0 : 0 : 1)$ as the singular point on the curve, which means that the remaining points should be sextactic or higher order inflection points. Using Program B.2, we can find the osculating conic at each of these points, and then determine if the points are in fact sextactic. Doing so with the `singularities` command, we obtain the following output

$$\{[[0, 0, 1], 2, 1, 2], [[-z, 0, 1], 2, 6, 2]\}$$
$$\{[[0, 0, 1], 2, 1, 2], \left[\left[-4z, 4iz\sqrt{3}, 1\right], 2, 6, 2\right]\}$$
$$\{[[0, 0, 1], 2, 1, 2], \left[\left[-4z, -4iz\sqrt{3}, 1\right], 2, 6, 2\right]\},$$

showing that each of the points are indeed the sextactic points on this curve.

Example 4.2.5. Now, let $F = x^3 - xyz + y^3 + z^3$ be the defining polynomial of the elliptic curve. Then Program B.4 gives

$$\begin{aligned} &-116625899520x^6y^3 + 116625899520x^6z^3 + 116625899520x^3y^6 \\ &\quad - 116625899520x^3z^6 - 116625899520y^6z^3 + 116625899520y^3z^6, \end{aligned}$$

making the output from `intersectcurves` command in [Maple]

$$\begin{aligned} \Big[&\left[1, \left[x^3 - x + 2, y - 1, 1\right]\right], \left[1, \left[x^3 - xy + 2, y^2 + y + 1, 1\right]\right], \\ &\left[1, \left[-1 + x, y^3 - y + 2, 1\right]\right], \left[1, \left[x - y, y^3 - \tfrac{1}{2}y^2 + \tfrac{1}{2}, 1\right]\right], \\ &\left[1, \left[x + \tfrac{1}{2}y^5 + \tfrac{1}{2}y^3 + y^2 + \tfrac{1}{2}y + 1, y^6 + y^4 + 4y^3 + y^2 + 2y + 4, 1\right]\right], \\ &\left[1, \left[4y^5 + 2y^4 + y^3 + 2y^2 + x + y, y^6 + \tfrac{1}{2}y^5 + \tfrac{1}{4}y^4 + y^3 + \tfrac{1}{4}y^2 + \tfrac{1}{4}, 1\right]\right] \Big]. \end{aligned}$$

This output corresponds to 27 points. Taking any of these points, finding the osculating conic O_p with Program B.2 and then calculating the intersection $(O_p.C)_p$ with the elliptic curve will produce a local intersection multiplicity of 6 as wanted.

These examples give no reason for doubting Cayley's 2-Hessian, but as we saw in Example 4.2.3, Cayley's 2-Hessian failed to give all sextactic points when we tried to use it on a curve of higher degree.

Remark 4.2.6. Although the article [Cay65] is not heavily cited, we find it somewhat surprising that no one has noticed his mistake. This is probably because, as Pereira put it in his article [Per01, p. 1385], "the formula obtained by Cayley is not very simple." Of the articles we have found citing [Cay65] directly, only Cukierman, in [Cuk97], bothers to give the defining polynomial when he comments upon the 2-Hessian. Hence, we would not expect that anyone besides Cayley has ever tested to see if Theorem 4.2.1 was indeed correct.

4.2.2 The Correct 2-Hessian

Although Cayley's 2-Hessian curve does not have the desired properties, the idea in his proof provides the right clues to find such a curve. Indeed, we only have to correct two mistakes in Cayley's proof in order to find the correct defining polynomial for a 2-Hessian with the desired properties. We provide the following theorem:

Theorem 4.2.7. *Let $C = V(F)$ be a plane curve of degree d with $H = \det H_F$. Then there exists a curve of degree $12d - 27$ given as the zero set of*

$$\begin{aligned} &(12d^2 - 54d + 57)H \operatorname{Jac}(F, H, \Omega_{\bar{H}}) \\ &+(d-2)(12d-27)H \operatorname{Jac}(F, H, \Omega_{\bar{F}}) \\ &-20(d-2)^2 \operatorname{Jac}(F, H, \Psi) \end{aligned}$$

such that the intersection points between C and this curve are the singular points, higher order inflection points, and the sextactic points of C.

This curve is referred to as the *2-Hessian of C* and is denoted by H_2^C or simply H_2 if there is no ambiguity concerning the curve in question.

Proof of Theorem 4.2.7. There are two mistakes in Cayley's deduction of the curve, the most prominent of which occur in section 19 on p. 553 of [Cay65]. To see this, first observe that in section 17 on p. 552 Cayley has correctly arrived at a condition for the 2-Hessian on the form

$$\begin{aligned} &(15d^2 - 54d + 51)H \operatorname{Jac}(F, \nabla, H)H \\ &+(30d - 54)(d-2)H \operatorname{Jac}(F, \overline{\nabla} H, H) \\ &+(d-2)^2\{9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H\} = 0. \end{aligned} \tag{4.1}$$

Cayley's mistake occurs as he attempts to simplify the last term of this equation. Although we will not need the notation further, we have that in equation (4.1) $\vartheta = \lambda x + \mu y + \nu z$ and $\partial = (B\nu - C\mu)\partial_x + (C\lambda - A\nu)\partial_y + (A\mu - B\lambda)\partial_z$, with λ, μ and ν arbitrary constants and $A = F_x, B = F_y, C = F_z$. Furthermore, ∇ is not the usual gradient, but rather a function defined similarly to Ψ by Cayley in section 7 of [Cay65], while $\overline{\nabla}H = (\partial_x \nabla H, \partial_y \nabla H, \partial_z \nabla H)$. Also, it should be noted that Cayley uses slightly different notation, as he has $m := d$ and $U := F$ in his article.

To simplify the last term of equation (4.1), Cayley introduces a variable W in section 18 and correctly states that

$$W := H\partial\Omega - 5\Omega\partial H = \frac{-3}{4d-9}\vartheta \operatorname{Jac}(F,\Omega,H) - \frac{5d-9}{4d-9}\partial(\Omega H). \tag{4.2}$$

Then, in section 19, he also correctly states that

$$\Psi\partial H = \frac{\frac{1}{2}}{4d-9}\vartheta \operatorname{Jac}(F,\Psi,H) + \frac{\frac{3}{2}(d-2)}{4d-9}H\partial\Psi. \tag{4.3}$$

Observing that $9H^2\partial\Omega - 45H\Omega\partial H + 40\Psi\partial H = 9HW + 40\Psi\partial H$, he wants to use these expressions to simplify the last term of equation (4.1). Doing this, Cayley obtains

$$\begin{aligned} 9HW + 40\Psi\partial H = &-\frac{9(5d-9)}{4d-9}H\partial(\Omega H) + \frac{60(d-2)}{4d-9}H\partial\Psi \\ &+ \frac{\vartheta}{4d-9}\left[-27H \operatorname{Jac}(F,\Omega,H) + 40 \operatorname{Jac}(F,\Psi,H)\right], \end{aligned} \tag{4.4}$$

which is not correct.

Looking at this more closely, we see that multiplying equation (4.2) with $9H$ gives

$$9HW = \frac{-27H}{4d-9}\vartheta \operatorname{Jac}(F,\Omega,H) - \frac{9(5d-9)}{4d-9}H\partial(\Omega H),$$

while multiplying equation (4.3) with 40 gives

$$\begin{aligned} 40\Psi\partial H &= \frac{40\cdot\frac{1}{2}}{4d-9}\vartheta \operatorname{Jac}(F,\Psi,H) + \frac{40\cdot\frac{3}{2}(d-2)}{4d-9}H\partial\Psi \\ &= \frac{\vartheta}{4d-9}20 \operatorname{Jac}(F,\Psi,H) + \frac{60(d-2)}{4d-9}H\partial\Psi. \end{aligned}$$

Thus, adding these together, we get

$$\begin{aligned} 9HW + 40\Psi\partial H =& \frac{-27H}{4d-9}\vartheta \operatorname{Jac}(F,\Omega,H) - \frac{9(5d-9)}{4d-9}H\partial(\Omega H) \\ &+ \frac{\vartheta}{4d-9}20 \operatorname{Jac}(F,\Psi,H) + \frac{60(d-2)}{4d-9}H\partial\Psi. \end{aligned}$$

If we now combine the first and third term and factor out their common factor, we obtain

$$\begin{aligned} 9HW + 40\Psi\partial H = &-\frac{9(5d-9)}{4d-9}H\partial(\Omega H) + \frac{60(d-2)}{4d-9}H\partial\Psi \\ &+ \frac{\vartheta}{4d-9}\left[-27H \operatorname{Jac}(F,\Omega,H) + 20 \operatorname{Jac}(F,\Psi,H)\right], \end{aligned} \tag{4.5}$$

where we now see that the coefficient in front of $\mathrm{Jac}(F, \Psi, H)$ in the last parenthesis is 20 as opposed to 40 in equation (4.4).

Using the incorrect result from equation (4.4), Cayley rewrites equation (4.1) in section 20 such that the condition of the 2-Hessian becomes

$$\begin{aligned}
&(15d^2 - 54d + 51)(4d - 9)H \,\mathrm{Jac}(F, \nabla, H)H \\
&+ 6(5d - 9)(d - 2)(4d - 9)H \,\mathrm{Jac}(F, \overline{\nabla} H, H) \\
&+ 3(d - 2)\{-3(5d - 9)(d - 2)H\partial(\Omega H) + 20(d - 2)^2 H\partial\Psi\} \\
&+ (d - 2)^2\{-27H \,\mathrm{Jac}(F, \Omega, H) + 40 \,\mathrm{Jac}(U, \Psi, H)\} = 0,
\end{aligned} \tag{4.6}$$

which he then represents as

$$3H \,\mathrm{II} + (d - 2)^2\vartheta\{-27H \,\mathrm{Jac}(F, \Omega, H) + 40 \,\mathrm{Jac}(F, \Psi, H)\} = 0. \tag{4.7}$$

Using the correct result from equation (4.5), we instead obtain the expression

$$3H \,\mathrm{II} + (d - 2)^2\vartheta\{-27H \,\mathrm{Jac}(F, \Omega, H) + 20 \,\mathrm{Jac}(F, \Psi, H)\} = 0. \tag{4.8}$$

In sections 21–25 Cayley does several calculations to simplify the expression II, without doing any mistakes. The subsequent result is that

$$\begin{aligned}
\mathrm{II} = &- (5d^2 - 18d + 17)\vartheta \,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) \\
&- (5d - 9)(d - 2)\vartheta \,\mathrm{Jac}(F, H, \Omega_{\bar{F}}),
\end{aligned}$$

and thus equation (4.8) becomes

$$\begin{aligned}
3H\{&- (5d^2 - 18d + 17) \,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) - (5d - 9)(d - 2) \,\mathrm{Jac}(F, H, \Omega_{\bar{F}})\} \\
&+ (d - 2)^2\{-27H \,\mathrm{Jac}(F, \Omega, H) + 20 \,\mathrm{Jac}(F, \Psi, H)\} = 0
\end{aligned} \tag{4.9}$$

after throwing out the common factor ϑ. Interchanging the last two rows of the determinants of the Jacobian matrices in the last term changes their signs, and gives

$$\begin{aligned}
3H\{&- (5d^2 - 18d + 17) \,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) - (5d - 9)(d - 2) \,\mathrm{Jac}(F, H, \Omega_{\bar{F}})\} \\
&+ (d - 2)^2\{27H \,\mathrm{Jac}(F, H, \Omega) - 20 \,\mathrm{Jac}(F, H, \Psi)\} = 0.
\end{aligned} \tag{4.10}$$

By the product rule $\partial_x\Omega = \partial_x\Omega_{\bar{H}} + \partial_x\Omega_{\bar{U}}$ and likewise for y and z, so

$$\mathrm{Jac}(F, H, \Omega) = \mathrm{Jac}(F, H, \Omega_{\bar{H}}) + \mathrm{Jac}(F, H, \Omega_{\bar{F}}).$$

Using this, gathering terms and simplifying, we get

$$\begin{aligned}
&(12d^2 - 54d + 51)H \,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) \\
&+(d - 2)(12d - 27)H \,\mathrm{Jac}(F, H, \Omega_{\bar{F}}) \\
&-20(d - 2)^2 \,\mathrm{Jac}(F, H, \Psi) = 0.
\end{aligned} \tag{4.11}$$

When Cayley does the same calculations using equation (4.7), he instead ends up with the following formula on p. 556:

$$\begin{aligned}
&(12d^2 - 54d + 51)H \,\mathrm{Jac}(F, H, \Omega_{\bar{H}}) \\
&+(d - 2)(12d - 27)H \,\mathrm{Jac}(F, H, \Omega_{\bar{F}}) \\
&-40(d - 2)^2 \,\mathrm{Jac}(F, H, \Psi) = 0.
\end{aligned}$$

We see that he has obtained $-40(d-2)^2\,\mathrm{Jac}(F,H,\Psi)$ as the last term instead of the correct $-20(d-2)^2\,\mathrm{Jac}(F,H,\Psi)$ in equation (4.11).

The other mistake is that the formula presented by Cayley on p. 545 in [Cay65], as stated in Theorem 4.2.1, is different from the formula he actually obtains on p. 556 of [Cay65]. As noted in the previous paragraph, the last term of formula that Cayley obtains is $-40(d-2)^2\,\mathrm{Jac}(F,H,\Psi)$, but in Theorem 4.2.1 the last term is $+40(d-2)^2\,\mathrm{Jac}(F,H,\Psi)$.

To be absolutely clear, we see that the reason for the negative sign in front of this term is the interchanging of the second and third row of $\mathrm{Jac}(F,\Psi,H)$ in equation (4.9) done to obtain equation (4.10). ■

Let us try to find the sextactic points of the curve $C_{1\mathrm{B}}$ again, this time with the correct 2-Hessian.

Example 4.2.8. Let $C = C_{1\mathrm{B}}$ be the curve from Table 2.1. Using Program B.3, we obtain

$$-44442639360x^3y^{18} + 33331979520x^2y^{19} - 11904278400xy^{20} + 1587237120y^{21}$$

as the defining polynomial for the 2-Hessian H_2. Finding the intersection points of H_2 and C using `intersectcurves` gives the following points:

$$\begin{aligned} p_1 &= (0:0:1), \\ p_2 &= \left(\tfrac{64}{3} : \tfrac{256}{3} : 1\right), \\ p_3 &= \left(\tfrac{49}{24} + i\tfrac{77\sqrt{7}}{24} : \tfrac{-637}{48} + i\tfrac{343\sqrt{7}}{48} : 1\right), \\ p_4 &= \left(\tfrac{49}{24} - i\tfrac{77\sqrt{7}}{24} : \tfrac{-637}{48} - i\tfrac{343\sqrt{7}}{48} : 1\right). \end{aligned}$$

Observe that of these, p_1 and p_2 are unchanged from Example 4.2.3 while p_2 and p_3 are different, as we would expect. Using Program B.2, we find the osculating conics for p_3 and p_4, O_{p_3} and O_{p_4}. Then computing the intersections $(O_{p_3}.C)$ and $(O_{p_4}.C)$ in [Maple] respectively gives

$$\left[\left[\tfrac{77}{24}i\sqrt{7}z + \tfrac{49}{24}z, \tfrac{343}{48}i\sqrt{7}z - \tfrac{637}{48}z, 1\right], 2, 6, 2\right]$$

and

$$\left[\left[-\tfrac{77}{24}i\sqrt{7}z + \tfrac{49}{24}z, -\tfrac{343}{48}i\sqrt{7}z - \tfrac{637}{48}z, 1\right], 2, 6, 2\right].$$

Hence, $(O_{p_3}.C) = (O_{p_4}.C) = 6$ and H_2 gives only singular and sextactic points when intersected with C.

Next, we consider a curve that is not a rational cuspidal curve.

Example 4.2.9. In [AS09], Alwaleed and Sakai consider a family of smooth plane quartics C_a defined by the polynomial

$$F = x^4 + y^4z^4 + a(x^2y^2 + x^2z^2 + y^2z^2),$$

where $a \neq -1, \pm 2$. For $a = 14$, they claim that there are a total of 68 sextactic points on C_{14}, eight 3-sextactic points and 60 1-sextactic points. Using [Maple] and Program B.3 to compute the 2-Hessian H_2, and then using `intersectcurves` to find all intersectcurves points and intersectcurves multiplicities between C_{14} and H_2 indeed gives 60 sextactic points of type 1, in addition to 8 sextactic points of type 3. We do not give all 68 intersection points, but the 8 points with intersection multiplicity $(C_{14}.H_2)_{p_i} = 3$ are

$$\begin{aligned}
p_1 &= \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : 1\right), \quad & p_2 &= \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : 1\right),\\
p_3 &= \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : 1\right), \quad & p_4 &= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : 1\right),\\
p_5 &= \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : 1\right), & p_6 &= \left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : 1\right),\\
p_7 &= \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : 1\right), \quad & p_8 &= \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i : \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i : 1\right).
\end{aligned}$$

Taking any p_i and computing the osculating conic with Program B.2 confirms that $(O_{p_i}.C) = 8$, so that all p_i are in fact sextactic points of type 3.

4.3 Sextactic point formulas

After Cayley's articles [Cay59] and [Cay65], other mathematicians have made some contributions to the theory regarding sextactic points. A recent effort is part of the work of G. Thorbergsson and M. Umehara, where they give a formula for the number of sextactic points on a smooth curve.

Theorem 4.3.1 ([TU02, p. 90])**.** *Let C be a smooth algebraic curve of degree d in $P^2(\mathbb{C})$. Then C has exactly $3d(4d-9)$ sextactic points counted with multiplicities if all inflection points of C are simple. If C has k inflection points with multiplicities $v_1, \dots, v_k$ respectively, then C has*

$$3d(5d-11) - \sum_{i=1}^{k}(4v_i - 3)$$

sextactic points counted with multiplicities.

The next result is an attempt to find a formula for cuspidal curves that gives the number of sextactic points on the curve.

Theorem 4.3.2 (Sextactic point formula)**.** *Let C be a cuspidal curve of genus g and degree d. Suppose that there are k inflection points of type v_i, $i = 1, \dots, k$, and that there are n cusps. Let p be a point on C. Then m_p will denote its multiplicity, T_p its tangent and $l_p = (T_p.C)_p$. Assume that for the cusps $l_j \neq 2m_j$ for $j = 1, \dots, n$. Then the number of sextactic points s on C, counted with multiplicity, is given by*

$$s = 6(2d + 5g - 5) - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15). \tag{4.12}$$

Proof. Let C be a cuspidal curve, as described in the theorem, and let Q be the complete linear system on C of conics, where the divisor to each conic is given by intersection with the curve. By Bézout's theorem Q is a g^5_{2d}. To find the Q-Weierstrass points on C, we will use the Puiseux parametrisation of C at a point to calculate the intersection multiplicities of curves in Q.

Suppose first that p is a singular point of multiplicity m and tangential intersection l. Then the local parametrisation can be expressed as

$$(t^m : ct^l + \cdots : 1)$$

after a linear change of coordinates. We choose the standard basis for Q given by all degree 2 monomials in x, y, z, i.e.

$$x^2, y^2, z^2, yz, xz, xy,$$

and substitute the local parametrisation into this basis. This gives

$$t^{2m}, c^2t^{2l} + \cdots, 1, ct^l + \cdots, t^m, ct^{m+l} + \cdots.$$

Since $l \neq 2m$ by assumption, this means that all of the basis elements give curves with distinct intersection multiplicity at p. The multiplicities are given by

$$h_0 = 0, h_1 = m, h_2 = l, h_3 = 2m, h_4 = m + l, h_5 = 2l. \tag{4.13}$$

This means that the Q-Weierstrass weight of a singular point can be calculated directly as

$$\begin{aligned} w_p(Q) &= \sum_{i=0}^{5}(h_i - i) \\ &= 4m + 4l - 15. \end{aligned} \tag{4.14}$$

Suppose next that the point p is an inflection point. Again, possibly after a linear change of coordinates, we may represent the Puiseux parametrisation of C at the point p as

$$(t : ct^l + \cdots : 1),$$

where $l > 2$. As then $l \neq 2m = 2$, it is clear that we can do the exact same analysis as for the singular point. This means that the Q-Weierstrass weight is given by

$$\begin{aligned} w_p(Q) &= 4 + 4l - 15 \\ &= 4l - 11 \\ &= 4(l-2) - 3, \end{aligned} \tag{4.15}$$

where we in the last equality have represented the Q-Weierstrass weight of p given its type as an inflection point on C.

Suppose now that p is a smooth point on C that is not an inflection point. As before, we may assume that we have a Puiseux parametrisation for C locally at p, now on the form

$$(t : ct^2 + \cdots : 1). \tag{4.16}$$

If we continue as done above and substitute this into the standard basis for Q, we obtain

$$t^2, c^2t^4 + \cdots, 1, ct^2 + \cdots, t^1, ct^3 + \cdots.$$

As $l = 2$ and $m = 1$, we have that $l = 2m$ and we see that both x^2 and yz give curves that intersect C with multiplicity 2 at p. We must therefore find another conic that intersects C with a multiplicity distinct from the other five.

Let O_p denote the osculating conic of C at p. Then $(O_p.C)_p = \rho$, and by Theorem 2.3.1 this means that when we substitute the parametrisation (4.16) into the defining equation for O_p, we obtain a power series

$$et^\rho + \cdots,$$

where $e \neq 0$. Since $\rho \geq 5$, this means that we can use O_p to obtain distinct intersection multiplicities and hence find the Weierstrass weight of p with respect to Q. So, using instead

$$x^2, y^2, z^2, xz, xy, O_p$$

and substituting the parametrisation (4.16) into this gives

$$t^2, c^2t^4 + \cdots, 1, t^1, ct^3 + \cdots, et^\rho + \cdots.$$

Hence, we have the following distinct intersection multiplicities:

$$h_0 = 0, h_1 = 1, h_2 = 2, h_3 = 3, h_4 = 4, h_5 = \rho \tag{4.17}$$

and the Q-Weierstrass weight is given by

$$\begin{aligned} w_p(Q) &= \sum_{i=0}^{5}(h_i - i) \\ &= \rho - 5. \end{aligned}$$

Assume now that p is not a sextactic point. Then $\rho = 5$ and thus $w_p(Q) = 0$, so that p is not a Q-Weierstrass point. If, however, p is a sextactic point on C, then $\rho \geq 6$, and thus $w_p(Q) > 0$ so that p is a Weierstrass point with respect to Q, with weight equal to its sextactic type. Denoting the sextactic type of a sextactic point p by s_p, we get that the total number of sextactic points on C, counted with multiplicity, is

$$\begin{aligned} s &= \sum_{p \in \operatorname{Sext} C} s_p \\ &= \sum_{p \in \operatorname{Sext} C} w_p(C). \end{aligned}$$

We are now in a position to apply Proposition 2.4.4. Recalling that Q is a g^5_{2d}, we get that

$$\begin{aligned} \sum_{p \in C} w_p(Q) &= 6(2d) + 30(g_a - \delta - 1) \\ &= 6(2d + 5(g - 1)). \end{aligned} \tag{4.18}$$

The left-hand side of this equation can be split up into three pieces, according to the arguments above. We have

$$\begin{aligned} \sum_{p \in C} w_p(Q) &= \sum_{p \in \operatorname{Sing} C} w_p(Q) \quad + \sum_{p \in \operatorname{Infl} C} w_p(Q) + \sum_{p \in \operatorname{Sext} C} w_p(Q) \\ &= \sum_{j=1}^{n}(4m_j + 4l_j - 15) + \sum_{i=1}^{k}(4v_i - 3) \quad + s. \end{aligned} \tag{4.19}$$

Putting all this together in equation (4.18) and isolating s, we get

$$s = 6(2d + 5g - 5) - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15),$$

which is exactly the formula we wanted. ■

Remark 4.3.3. It is worth noting how similar this proof is to the proof of Theorem 3.3.3. The fact that sextactic points are a very natural generalisation of inflection points is mirrored in the fact that we can use known theory and methods that work for inflection points to prove new things for sextactic points with minor changes to the arguments themselves.

Let us look at an example.

Example 4.3.4. Consider the rational cuspidal quartic C_4, as given in Table 2.1. By Table A.8 it has three simple cusps and no inflection points. Hence $l = 3$ and $m = 2$ for all the cusps and we are in a position to apply Theorem 4.3.2. Thus, by formula (4.12), the number of sextactic points on C_4 is

$$\begin{aligned} s &= 6(2 \cdot 4 + 5 \cdot 0 - 5) - 0 - \sum_{j=1}^{3}(4 \cdot 2 + 4 \cdot 3 - 15) \\ &= 18 - 15 \\ &= 3, \end{aligned}$$

which agrees with Table A.8, where the sextactic points are found using the 2-Hessian curve.

4.3.1 The case $l = 2m$

In general, there is no procedure to calculate the Weierstrass weight of a singular point p with respect to Q if $l = 2m$. We can, however, in certain cases still determine the weight for a singular point when $l = 2m$. Recall that for a smooth point that is not an inflection point $l = 2m$, but in the proof of Theorem 4.3.2 we were still able to calculate its Q-Weierstrass weight. We did this by using the osculating conic at the point, which intersects C at p with intersection multiplicity higher than any other conic. For a singular point p, we do not have an osculating conic to rely on, but we can still use this method if we can determine what possible intersection multiplicities γ that can occur between C and conic sections at p.

Observe that from (4.13), we can determine five distinct intersection multiplicities for curves from Q at the singular point p. Since $l = 2m$, they can be given as

$$h_0 = 0, h_1 = m, h_2 = 2m, h_3 = 3m, h_4 = 4m. \tag{4.20}$$

Looking for possible values of γ, other than these five, we have a lower bound of $2m$ because $l = 2m$. Bézout's theorem gives a natural upper bound on γ since a conic intersects a curve of degree d at p with intersection multiplicity at most $2d$. Hence,

$$2m < \gamma \leq 2d. \tag{4.21}$$

Lastly, since our curve C is cuspidal we can deploy Lemma 2.2.7 to further restrict the possible values of γ. Let $\overline{m}_p$ be the multiplicity sequence of p with an infinite number of ones appended. By Lemma 2.2.7 γ must be on the form

$$\gamma = km_0 + m_k, \tag{4.22}$$

for some k with $m_0 = m_1 = \ldots = m_{k-1}$.

In some cases these restrictions provide enough information to determine the intersection multiplicity needed to calculate the Weierstrass weight of p with respect to Q. Let us look at an example.

Example 4.3.5. Let C be the quartic curve C_2 from Table 2.1. By Table A.6, C has a single cusp p and no more singular points. Moreover, the tangential intersection multiplicity at p is $l = 4$, while the multiplicity of p is given by $m = 2$. Hence, we are not in a position to apply Theorem 4.3.2, because $l = 2m$, but we can attempt the method described above to find the Weierstrass weight of p with respect to the complete linear system of conics Q.

Using equation (4.12), without expressing the Q-Weierstrass weight of a singular point as $4m + 4l - 15$, gives the following formula for the number of sextactic points:

$$s = 6(2d + 5g - 5) - \sum_{i=1}^{k}(4v_i - 3) - \sum_{q \in \operatorname{Sing} C} w_q(Q).$$

In this example, we get

$$s = 18 - 3 - w_p(Q) \tag{4.23}$$

because the curve is rational and has three simple inflection points. From (4.20), we already have five distinct intersection multiplicities, namely

$$h_0 = 0, h_1 = 2, h_2 = 4, h_3 = 6, h_4 = 8. \tag{4.24}$$

Furthermore, from the bound (4.21), we have that the possible intersections γ are restricted by

$$4 < \gamma \leq 8,$$

so that we are left with only 5 and 7 as possible values of gamma distinct from the intersection multiplicities (4.24). From Table A.6, the multiplicity sequence of p is $[2_3]$ and thus we can use (4.22) to conclude that $h_5 = 7$ because the only numbers less than or equal to 8 that occur on the form $km_0 + m_k$ are 4, 6 and 7. Hence, the Weierstrass weight of p with respect to Q is given by

$$\begin{aligned} w_p(Q) &= \sum_{i=0}^{5}(h_i - i) \\ &= 27 - 15 \\ &= 12. \end{aligned}$$

Putting this into equation (4.23) gives the number of sextactic points on C as

$$s = 18 - 3 - 12 = 3,$$

which is indeed the number of sextactic points on C found using the 2-Hessian, as presented in Table A.6.

Example 4.3.6. Let $C = C_2$ be the rational cuspidal quintic as given in Table 2.2. By Table A.12, C has a single cusp at $p_1 = (0 : 0 : 1)$, with multiplicity sequence $[2_6]$ and tangential intersection $(T_p.C)_p = 4$, and 6 simple inflection points. Hence, we cannot apply formula (4.12) from Theorem 4.3.2.

To find the number of sextactic points, we need to determine the Weierstrass weight of p_1 with respect to the complete linear system Q of conics on C. From (4.20) $0, 2, 4, 6$ and 8 are 5 possible distinct intersection multiplicities between conics and C at p. Moreover, by (4.21), the only other possible intersection multiplicities are $5, 7, 9$ and 10. From the multiplicity sequence of p_1 and (4.22), the only possible intersection multiplicities are $4, 6, 8$ and 10. Thus, $h_5 = 10$ and

$$\begin{aligned} w_{p_1}(Q) &= \sum_{i=0}^{5}(h_i - i) \\ &= 15. \end{aligned}$$

Hence, using formula (4.12), without expressing the Q-Weierstrass weight of a singular point using (4.14), we get

$$\begin{aligned} s &= 6(2 \cdot 5 + 5 \cdot 0 - 5) - \sum_{i=1}^{6}(4 \cdot 1 - 3) - w_{p_1}(Q) \\ &= 30 - 6 - 15 \\ &= 9, \end{aligned}$$

which is indeed the number of sextactic points on C_2 found in Table A.12.

4.3.2 Coolidge's formula

Another contribution concerning sextactic points comes from J. L. Coolidge, which in [Coo31] presents a formula for the number of sextactic points on curves with restricted types of singularities. In our notation, the theorem is as follows:

Theorem 4.3.7 ([Coo31, Theorem 4, p. 280])**.** *If a curve C of degree d and genus g has no singular points but n ordinary ones and k simple cusps, and its dual $C^\vee$ has no singular points but ordinary ones and cusps, the number of sextactic points is*

$$3(d^2 - 2n - 3k + 6(g - 1)).$$

Remark 4.3.8. Coolidge claims in a footnote on the same page that Cayley proved this result in his article [Cay68]. However, after inspecting the article we cannot see that Cayley proved this for other than cubic curves.

It is worth noting that our formula and the formula which Coolidge found for simple cusps and nodes in Theorem 4.3.7 coincide in the case that all singularities are simple cusps. To show this, suppose that C is a cuspidal curve with only simple cusps, i.e. cusps with $m = 2$ and $l = 3$. First, we observe that this means that $\delta = 1$ for all cusps, and so if there are k cusps on the curve, we can express the genus as

$$g = \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{k} \delta_j = \frac{(d-1)(d-2)}{2} - k.$$

Hence, the formula in Theorem 4.3.7 gives

$$\begin{aligned} s &= 3(d^2 - 3k - 6(g-1)) \\ &= 3\left(d^2 - 3k + 6\left(\tfrac{(d-1)(d-2)}{2} - k\right) - 6\right) \\ &= 3(d^2 - 3k + 3(d-1)(d-2) - 6k - 6) \\ &= 3(4d^2 - 9d - 9k). \end{aligned}$$

Note that since we also assume that the dual of C has only simple singularities, all inflection points of C are simple. This means that we need only count the number of inflection points in our formula in Corollary 4.4.1. In other words, we have

$$\begin{aligned} s &= 3d(5d-11) - \sum_{i=1}^{v}(4v_i - 3) - \sum_{i=1}^{k}(30\delta_j + 4m_j + 4l_j - 15) \\ &= 3d(5d-11) - v - \sum_{i=1}^{k}(30 + 8 + 12 - 15) \\ &= 3d(5d-11) - v - 35k. \end{aligned}$$

Now, using Theorem 3.3.3 to give an expression for the number of inflection points gives

$$\begin{aligned} s &= 3d(5d-11) - \left(3d(d-2) - \sum_{j=1}^{k}(6\delta_j + m_j + l_j - 3)\right) - 35k \\ &= 3d(5d-11) - 3d(d-2) + \sum_{j=1}^{k}(6 + 2 + 3 - 3) - 35k \\ &= 15d^2 - 11d - 3d^2 + 6d + 8k - 35k \\ &= 12d^2 - 9d - 27k \\ &= 3(4d^2 - 9d - 9k). \end{aligned}$$

Hence, the two formulas coincide as claimed.

4.4 Local intersections

Expressing the genus using Proposition 2.2.5, gives an interesting corollary of Theorem 4.3.2.

Corollary 4.4.1. *Let C be a cuspidal curve of genus g and degree d. Suppose that there are k inflection points of type v_i, $i = 1, \dots, k$, and that there are n cusps with delta invariant δ_j, $j = 1, \dots, n$. Let p be a point on C. Then m_p will denote its multiplicity, T_p its tangent and $l_p = (T_p.C)_p$. Assume that for the cusps $l_j \neq 2m_j$ for $j = 1, \dots, n$. Then the number of sextactic points s on C, counted with multiplicity, is given by*

$$s = 3d(5d-11) - \sum_{i=1}^{k}(4v_i - 3) - 30\sum_{j=1}^{n}\delta_j - \sum_{j=1}^{n}(4m_j + 4l_j - 15). \qquad (4.25)$$

Proof. Using Proposition 2.2.5 we can rewrite formula (4.12) to

$$\begin{aligned}
s &= 6(2d + 5\Big(\tfrac{(d-1)(d-2)}{2} - \sum_{j=1}^{n} \delta_j\Big) - 5) - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15) \\
&= 12d + 15(d-1)(d-2) - 30\sum_{j=1}^{n} \delta_j - 30 - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15) \\
&= 3d(5d - 11) - 30\sum_{j=1}^{n} \delta_j - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15),
\end{aligned}$$

which gives the desired formula. ■

Corollary 4.4.1 is more than a simple rewriting of Theorem 4.3.2. It has an an important geometric interpretation in form of the intersection with the Hessian and 2-Hessian curve. To see this observe that from Bézout's theorem, we have that

$$\begin{aligned}
C.H_2 + C.H &= d(12d - 27) + 3d(d-2) \\
&= 3d(5d - 11),
\end{aligned}$$

and so (4.25) can be written as

$$s = C.H_2 + C.H - \sum_{i=1}^{k}(4v_i - 3) - 30\sum_{j=1}^{n} \delta_j - \sum_{j=1}^{n}(4m_j + 4l_j - 15).$$

From (4.15) in the proof of Theorem 4.3.2, we have that the Q-Weierstrass weight of an inflection point has can be expressed on the same form as the weight of a singular point. Thus, we can include the Q-Weierstrass weight of the inflection points into the sum of the weights for the singular points. Furthermore, $\delta_p = 0$ for a smooth point, so we can express the number of sextactic points as

$$s = C.H_2 + C.H - \sum_{j=1}^{n+k}(30\delta_j + 4m_j + 4l_j - 15).$$

By Theorem 3.4.1, the intersection multiplicity between H and C at a point $p \in C$ is given by $(C.H)_p = 6\delta_p + m_p + l_p - 3$. Hence, we can extract this from the formula above and remove all dependency on the Hessian curve. We get

$$s = C.H_2 - \sum_{j=1}^{n+k}(24\delta_j + 3m_j + 3l_j - 12).$$

Thus, if p is either a singular point where $l \neq 2m$ or an inflection point, the local contribution when intersected with the 2-Hessian is equal to

$$(C.H_2)_p = 24\delta_p + 3m_p + 3l_p - 12. \tag{4.26}$$

In particular, we see that if p is a simple inflection point, then $(C.H_2)_p = 0$, which agrees with Theorem 4.2.7.

If p is a singular point where $l = 2m$, we can still obtain an expression for the local intersection by not expressing the Q-Weierstrass weight of p explicitly as $4m + 4l - 15$. With the same arguments as above, we will then obtain

$$(C.H_2)_p = 24\delta_p + w_p(Q) - (m_p + l_p - 3). \tag{4.27}$$

Let us look at some examples to see if this agrees with the calculations in Appendix A.

Example 4.4.2. Let $C = C_{1\mathrm{B}}$ be the quintic curve given in Table 2.2. Using the information in Table A.10, we can attempt to calculate the local intersections with the 2-Hessian curve for the singular and inflection points by formula (4.26). Let $p_1 = (1:0:0), p_2 = (0:0:1), p_3 = (-\frac{162}{3125} : \frac{3}{5} : 1)$ and observe that p_3 is a simple inflection point. By (4.26)

$$\begin{aligned} (C.H_2)_{p_1} &= 24 \cdot 6 + 3 \cdot 4 + 3 \cdot 5 - 12 \\ &= 159, \\ (C.H_2)_{p_2} &= 24 \cdot 0 + 3 \cdot 1 + 3 \cdot 4 - 12 \\ &= 3, \\ (C.H_2)_{p_3} &= 0. \end{aligned}$$

Hence, we see that formula (4.26) agrees with the expressions in Table A.10 found calculating the intersection multiplicities directly.

Next, we consider an example where $l = 2m$ at a singular point.

Example 4.4.3. Let C be the quintic curve C_4 given in Table 2.2. Again, we would like to use the information from the table in Appendix A corresponding to C, to calculate the local intersection multiplicity. From Table A.15 we see that the two inflection points on C are simple, and thus do not contribute to the intersection multiplicity with H_2. Let p_1 and p_2 denote the two cusps $(0:0:1)$ and $(1:0:0)$ respectively. For p_1, $l \neq 2m$ and so we can use formula (4.26) as before. This gives

$$\begin{aligned} (C.H_2)_{p_1} &= 24 \cdot 3 + 3 \cdot 3 + 3 \cdot 4 - 12 \\ &= 81, \end{aligned}$$

which agrees with the result in Table A.15. For p_2, $l = 2m$ and thus we must find the Weierstrass weight of p_2 with respect to the linear system Q of conics on C and use the alternative formula (4.27). The Q-Weierstrass weight is found using the method described in Section 4.3.1. Hence, we must determine what intersection multiplicity γ a conic can have with C at p_2 other than

$$h_0 = 0, h_1 = 2, h_2 = 4, h_3 = 6, h_4 = 8.$$

The multiplicity sequence of p_2 is $[2_3]$, so since $4 < \gamma \leq 10$ and must be on the form (4.22), the only possible value of γ is 7. Hence, the Q-Weierstrass weight of p_2 is

$$\begin{aligned} w_p(Q) &= \sum_{i=0}^{5} (h_i - i) \\ &= 12. \end{aligned}$$

Using (4.27) now gives

$$\begin{aligned}(C.H_2)_{p_2} &= 24 \cdot 3 + 12 - (2 + 4 - 3)\\ &= 81,\end{aligned}$$

which matches the result in Table A.15. Out of curiosity, we see that if we had used formula (4.26), we would have obtained

$$24 \cdot 3 + 3 \cdot 2 + 3 \cdot 4 - 12 = 78.$$

4.5 Sextactic points on rational cuspidal curves

For rational cuspidal curves, we have the following corollary to Theorem 4.3.2, which is obtained by setting $g = 0$ in (4.12).

Corollary 4.5.1. *Let C be a rational cuspidal curve of degree d. Suppose that there are k inflection points of type v_i, $i = 1, \ldots, k$, and that there are n cusps with delta invariant δ_j, $j = 1, \ldots, n$. Let p be a point on C. Then m_p will denote its multiplicity, T_p its tangent and $l_p = (T_p.C)_p$. Assume that for the cusps $l_j \neq 2m_j$ for $j = 1, \ldots, n$. Then the number of sextactic points s on C, counted with multiplicity, is given by*

$$s = 6(2d-5) - \sum_{i=1}^{k}(4v_i - 3) - \sum_{j=1}^{n}(4m_j + 4l_j - 15).$$

4.5.1 Osculating conic

In [Cay59], Cayley considers "consecutive" points on a curve to obtain a defining polynomial for the osculating conic to a curve C at a point p. Cayley does this by considering a point on the curve as given by a parametrisation and using the fourth order derivatives of this parametrisation to obtain a condition for five pointic contact. Assuming that C is a rational cuspidal curve with parametrisation $\phi(s,t) = (\phi_0(s,t) : \phi_1(s,t) : \phi_2(s,t))$, we could therefore expect that the osculating conic to C at a point $\phi(s,t)$ is given by

$$\begin{vmatrix} x^2 & y^2 & z^2 & yz & xz & xy \\ \frac{\partial^4(\phi_0^2)}{\partial s^4} & \frac{\partial^4(\phi_1^2)}{\partial s^4} & \frac{\partial^4(\phi_2^2)}{\partial s^4} & \frac{\partial^4(\phi_1\phi_2)}{\partial s^4} & \frac{\partial^4(\phi_0\phi_2)}{\partial s^4} & \frac{\partial^4(\phi_0\phi_1)}{\partial s^4} \\ \frac{\partial^4(\phi_0^2)}{\partial s^3\partial t} & \frac{\partial^4(\phi_1^2)}{\partial s^3\partial t} & \frac{\partial^4(\phi_2^2)}{\partial s^3\partial t} & \frac{\partial^4(\phi_1\phi_2)}{\partial s^3\partial t} & \frac{\partial^4(\phi_0\phi_2)}{\partial s^3\partial t} & \frac{\partial^4(\phi_0\phi_1)}{\partial s^3\partial t} \\ \frac{\partial^4(\phi_0^2)}{\partial s^2\partial t^2} & \frac{\partial^4(\phi_1^2)}{\partial s^2\partial t^2} & \frac{\partial^4(\phi_2^2)}{\partial s^2\partial t^2} & \frac{\partial^4(\phi_1\phi_2)}{\partial s^2\partial t^2} & \frac{\partial^4(\phi_0\phi_2)}{\partial s^2\partial t^2} & \frac{\partial^4(\phi_0\phi_1)}{\partial s^2\partial t^2} \\ \frac{\partial^4(\phi_0^2)}{\partial s\partial t^3} & \frac{\partial^4(\phi_1^2)}{\partial s\partial t^3} & \frac{\partial^4(\phi_2^2)}{\partial s\partial t^3} & \frac{\partial^4(\phi_1\phi_2)}{\partial s\partial t^3} & \frac{\partial^4(\phi_0\phi_2)}{\partial s\partial t^3} & \frac{\partial^4(\phi_0\phi_1)}{\partial s\partial t^3} \\ \frac{\partial^4(\phi_0^2)}{\partial t^4} & \frac{\partial^4(\phi_1^2)}{\partial t^4} & \frac{\partial^4(\phi_2^2)}{\partial t^4} & \frac{\partial^4(\phi_1\phi_2)}{\partial t^4} & \frac{\partial^4(\phi_0\phi_2)}{\partial t^4} & \frac{\partial^4(\phi_0\phi_1)}{\partial t^4} \end{vmatrix}.$$

Let us look at an example to see if this gives the same osculating conic as Cayley's formula from Theorem 4.1.1 in a point on the curve.

Example 4.5.2. Consider the quintic curve C_6 from Table 2.2. C_6 is parametrised by $\phi = (s^4t - \frac{1}{2}s^5 : s^3t^2 : -\frac{3}{2}st^4 + t^5)$, and so calculating the osculating conic using the determinant above in [Maple] gives

$$
\begin{aligned}
cs^{10}t^7(&15s^{13}z^2 - 70s^{12}tz^2 + 270s^{11}t^2yz + 130s^{11}t^2z^2 - 432s^{10}t^3xz \\
&+ 324s^{10}t^3y^2 - 1584s^{10}t^3yz - 120s^{10}t^3z^2 + 2610s^9t^4xz - 2565s^9t^4y^2 \\
&+ 3860s^9t^4yz + 55s^9t^4z^2 - 6540s^8t^5xz + 9090s^8t^5y^2 - 5000s^8t^5yz \\
&- 10s^8t^5z^2 - 270s^7t^6xy + 8700s^7t^6xz - 18870s^7t^6y^2 + 3630s^7t^6yz \\
&+ 1800s^6t^7xy - 6480s^6t^7xz + 25100s^6t^7y^2 - 1400s^6t^7yz - 81s^5t^8x^2 \\
&- 4980s^5t^8xy + 2562s^5t^8xz - 21869s^5t^8y^2 + 224s^5t^8yz + 450s^4t^9x^2 \\
&+ 7320s^4t^9xy - 420s^4t^9xz + 12150s^4t^9y^2 - 990s^3t^{10}x^2 - 6030s^3t^{10}xy \\
&- 3920s^3t^{10}y^2 + 1080s^2t^{11}x^2 + 2640s^2t^{11}xy + 560s^2t^11y^2 - 585st^{12}x^2 \\
&- 480st^12xy + 126t^{13}x^2),
\end{aligned}
$$

where $c = -286773903360$. Let us test whether this expression gives the correct osculating conic at the point on C_6 given by the parametrisation. Let $p = \phi(2,1) = (0 : -4 : 1)$. For $s = 2, t = 1$, the expression above becomes

$$d(\tfrac{256}{3}x^2 + \tfrac{131072}{9}xz - \tfrac{131072}{27}y^2 - \tfrac{606208}{27}yz + \tfrac{5120}{9}xy - \tfrac{327680}{27}z^2),$$

where $d = 123886326251520$. Finding the defining polynomial of O_p by using Theorem 4.1.1, Program B.2 gives

$$\tfrac{256}{3}x^2 + \tfrac{131072}{9}xz - \tfrac{131072}{27}y^2 - \tfrac{606208}{27}yz + \tfrac{5120}{9}xy - \tfrac{327680}{27}z^2,$$

hence, the two polynomials differ by a constant factor and thus give us the same conic as hoped.

4.5.2 Weierstrass weight

The Wronskian of the Veronese embedding of ϕ into $\mathbb{P}^5$ is

$$
\begin{vmatrix}
\frac{\partial^5(\phi_0^2)}{\partial s^5} & \frac{\partial^5(\phi_1^2)}{\partial s^5} & \frac{\partial^5(\phi_2^2)}{\partial s^5} & \frac{\partial^5(\phi_1\phi_2)}{\partial s^5} & \frac{\partial^5(\phi_0\phi_2)}{\partial s^5} & \frac{\partial^5(\phi_0\phi_1)}{\partial s^5} \\
\frac{\partial^5(\phi_0^2)}{\partial s^4\partial t} & \frac{\partial^5(\phi_1^2)}{\partial s^4\partial t} & \frac{\partial^5(\phi_2^2)}{\partial s^4\partial t} & \frac{\partial^5(\phi_1\phi_2)}{\partial s^4\partial t} & \frac{\partial^5(\phi_0\phi_2)}{\partial s^4\partial t} & \frac{\partial^5(\phi_0\phi_1)}{\partial s^4\partial t} \\
\frac{\partial^5(\phi_0^2)}{\partial s^3\partial t^2} & \frac{\partial^5(\phi_1^2)}{\partial s^3\partial t^2} & \frac{\partial^5(\phi_2^2)}{\partial s^3\partial t^2} & \frac{\partial^5(\phi_1\phi_2)}{\partial s^3\partial t^2} & \frac{\partial^5(\phi_0\phi_2)}{\partial s^3\partial t^2} & \frac{\partial^5(\phi_0\phi_1)}{\partial s^3\partial t^2} \\
\frac{\partial^5(\phi_0^2)}{\partial s^2\partial t^3} & \frac{\partial^5(\phi_1^2)}{\partial s^2\partial t^3} & \frac{\partial^5(\phi_2^2)}{\partial s^2\partial t^3} & \frac{\partial^5(\phi_1\phi_2)}{\partial s^2\partial t^3} & \frac{\partial^5(\phi_0\phi_2)}{\partial s^2\partial t^3} & \frac{\partial^5(\phi_0\phi_1)}{\partial s^2\partial t^3} \\
\frac{\partial^5(\phi_0^2)}{\partial s\partial t^4} & \frac{\partial^5(\phi_1^2)}{\partial s\partial t^4} & \frac{\partial^5(\phi_2^2)}{\partial s\partial t^4} & \frac{\partial^5(\phi_1\phi_2)}{\partial s\partial t^4} & \frac{\partial^5(\phi_0\phi_2)}{\partial s\partial t^4} & \frac{\partial^5(\phi_0\phi_1)}{\partial s\partial t^4} \\
\frac{\partial^5(\phi_0^2)}{\partial t^5} & \frac{\partial^5(\phi_1^2)}{\partial t^5} & \frac{\partial^5(\phi_2^2)}{\partial t^5} & \frac{\partial^5(\phi_1\phi_2)}{\partial t^5} & \frac{\partial^5(\phi_0\phi_2)}{\partial t^5} & \frac{\partial^5(\phi_0\phi_1)}{\partial t^5}
\end{vmatrix}
$$

and gives a homogeneous polynomial in s, t of degree $6(2d - 5)$. For a smooth curve, [Mir95, Chapter VII, Section 4, pp. 233–246] ensures that the multiplicity of a zeros of ξ equals the Weierstrass weight of that point with respect to the complete linear system on C of conics. Note that this is the same as the flattening points of the Veronese embedding, as described in [Arn96, p. 15]. Our computations with the curves in Table 2.1 and Table 2.2 indicate that this also holds for rational cuspidal curves. Let us consider a couple examples.

Example 4.5.3. Let C be the quintic $C_{3\mathrm{A}}$ as described in Table 2.2, and let Q be the complete linear system of conics on C. It has parametrisation $(s^5 : s^3t^2 : t^5)$, and so using [Maple] to compute its Wronskian gives

$$\xi = -2809741879384473600000t^{13}s^{17}.$$

From Table A.13, there are only two Q-Weierstrass points $p_1 = (0 : 0 : 1)$ and $p_2 = (1 : 0 : 0)$, both corresponding to the zeros of ξ. To calculate their Q-Weierstrass weight, we observe that by Table A.13, $l \neq 2m$ for both points, and thus the Q-Weierstrass weights are given by $4m + 4l - 15$. This means that

$$\begin{aligned} w_{p_1}(Q) &= 4 \cdot 3 + 4 \cdot 5 - 15 \\ &= 17 \\ w_{p_2}(Q) &= 4 \cdot 2 + 4 \cdot 5 - 15 \\ &= 13, \end{aligned}$$

which agrees with the order of their corresponding zeros in ξ.

Next, we consider the curve $C_{3\mathrm{B}}$, which has the exact same type of cusps as $C_{3\mathrm{A}}$, and see that the Wronskian picks up on the nuances between these curves and identifies the correct Q-Weierstrass weight for all points, even when $l = 2m$.

Example 4.5.4. Let C be the quintic $C_{3\mathrm{B}}$ from Table 2.2 with parametrisation $(s^5 : s^3t^2 : st^4 + t^5)$. With [Maple], we find that the Wronskian is

$$\xi = cs^{17}t^{10}(192s^3 + 1680s^2t + 5275st^2 + 5250t^3),$$

where $c = -535188929406566400$. Following Table A.14, let $p_1 = (0 : 0 : 1)$ and $p_2 = (1 : 0 : 0)$. Furthermore, let p_3 denote the inflection point, while p_4 and p_5 denote the sextactic points. These are all the Weierstrass points for the complete linear system Q of conics on C. The cusps p_1 and p_2 correspond to $s = 0$ and $t = 0$, while p_3, p_4 and p_5 correspond to the zeros of $192s^3+1680s^2t+5275st^2+5250t^3$. The inflection point p_3 is simple, and by the proof of Theorem 4.3.2, the sextactic points have Q-Weierstrass weight equal to their sextactic type. Hence, $w_{p_1}(Q) = w_{p_2}(Q) = w_{p_3}(Q) = 1$, which agrees with the multiplicities of the corresponding zeros of ξ. We can compute the Q-Weierstrass weight of p_1 directly as $4m + 4l - 15$ because $l_{p_1} \neq 2m_{p_1}$. Thus $w_{p_1}(Q) = 17$, which also agrees with the multiplicity of the zero at s in ξ. For p_2, $l_{p_2} = 2m_{p_2}$, and so we must use the method of Section 4.3.1 to find the Q-Weierstrass weight. We want to determine what possible intersection multiplicities that can occur between $C_{3\mathrm{B}}$ and a conic at p_2. From (4.20) we already have the intersection multiplicities $0, 2, 4, 6$ and 8. Using (4.21), the only possible values for an intersection multiplicity γ is $5, 7, 9$ and 10. Now, because the multiplicity sequence of p_2 is $[2_2]$, the only possible value of γ by Lemma 2.2.7 is 4 or 5. Hence, $\gamma = 5$ and the Q-Weierstrass weight is

$$\begin{aligned} w_{p_2}(Q) &= 10 \cdot 2 + 5 - 15 \\ &= 10. \end{aligned}$$

Consequently, the Wronskian gave all Q-Weierstrass points with the correct Q-Weierstrass weight.

Given that the Wronskian can be used to find the Q-Weierstrass weight of the points on a rational cuspidal curve, the formula in Corollary 4.5.1 is very natural. The degree of ξ is $6(2d-5)$, and we subtract the contributions from inflection and singular points to obtain the number of sextactic points.

CHAPTER 5

Binomial curves

In this chapter, we consider binomial curves and consider sextactic points on them. We take this investigation further and consider hyperosculating points, using the same theory of Weierstrass points as for the inflection points and sextactic points.

5.1 Defining properties

A *binomial curve* is a curve $C_{m,l}$ given by a defining polynomial of the form

$$F = y^m z^{l-m} - x^l,$$

where $l > m \geq 1$ and $\gcd(m, l) = 1$. By computing its partial derivatives, we get

$$\frac{\partial F}{\partial x} = -lx^{l-1}, \quad \frac{\partial F}{\partial y} = my^{m-1}z^{l-m}, \quad \frac{\partial F}{\partial z} = (l-m)y^m z^{l-m-1}. \tag{5.1}$$

Let first $m = 1$. Equating all partial derivatives in (5.1) to 0 then gives that $C_{1,l}$ is a family of curves, each with a single singular point, namely $(0 : 1 : 0)$. This is a cusp with multiplicity $l - 1$. By symmetry, a similar result holds when $l - m = 1$, i.e. $m = l - 1$.

Using (5.1) and assuming that m and l are such that neither $m = 1$ nor $m = l - 1$, we get a family of curves, each with two cusps, $p_1 = (0 : 0 : 1)$ and $p_2 = (0 : 1 : 0)$. From F we see that p_1 and p_2 have multiplicities m and $l - m$ respectively.

The curve $C_{m,l}$ is rational and can be given a parametrisation on the form

$$(s^{l-m}t^m : t^l : s^l) \quad \text{for } (s : t) \in \mathbb{P}^1.$$

Note that with this parametrisation, p_1 corresponds to the point given by $t = 0$, while p_2 corresponds to $s = 0$.

5.2 Inflection points

To investigate possible inflection points on a binomial curve, we compute the defining polynomial of the Hessian curve using [Maple]; this gives

$$\det H_F = (l-1)^2(l-m)mlx^{l-2}y^{2m-2}z^{2l-2m-2}.$$

The Puiseux parametrisation of $C_{m,l}$ at p_1 and p_2 is and

$$(t^m : t^l : 1) \text{ and } (s^{l-m} : 1 : s^l) \tag{5.2}$$

respectively, so we we can use Theorem 2.3.1 to compute the intersection multiplicities between $C_{m,l}$ and its Hessian curve H at these points. We get

$$\begin{aligned}(C_{m,l}.H)_{p_1} &= \operatorname{ord}_t \left((l-1)^2(l-m)ml(t^m)^{l-2}(t^l)^{2m-2}1^{2l-2m-2}\right) \\ &= \operatorname{ord}_t \left((l-1)^2(l-m)mlt^{ml-2m}t^{2ml-2l}\right) \\ &= 3ml-2m-2l\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}(C_{m,l}.H)_{p_2} &= \operatorname{ord}_s \left((l-1)^2(l-m)ml(s^{l-m})^{l-2}1^{2m-2}(s^l)^{2l-2m-2}\right) \\ &= \operatorname{ord}_s \left((l-1)^2(l-m)mls^{2l^2-2ml-2l}s^{l^2-2l-ml+2m}\right) \\ &= 3l^2-4l-3ml+2m.\end{aligned} \tag{5.4}$$

Hence, the sum is

$$\begin{aligned}(C_{m,l}.H)_{p_1} + (C_{m,l}.H)_{p_2} &= 3l^2-6l \\ &= 3l(l-2).\end{aligned}$$

By Bézout's theorem $3l(l-2)$ is the total intersection multiplicity between the curves, so there are no other intersection points between $C_{m,l}$ and H. Note that if we are in the case that either p_1 or p_2 is a smooth point, then these computations show that it is an inflection point. Moreover, we may determine their inflection types by considering the intersection multiplicities with the Hessian. From (5.5), we get that if $m=1$,

$$\begin{aligned}(C_{m,l}.H)_{p_1} &= 3l-2-2l \\ &= l-2,\end{aligned}$$

and likewise for $m=l-1$, (5.6) gives

$$\begin{aligned}(C_{m,l}.H)_{p_2} &= 3l^2-4l-3(l-1)l+2(l-1) \\ &= l-2.\end{aligned}$$

In other words, all binomial curves where $m \neq 1$ and $m \neq l-1$ are bicuspidal with no inflection points, while the curves where either $m=1$ or $m=l-1$ are unicuspidal and have a single inflection point of type $l-2$.

5.3 Sextactic points

We now consider sextactic points on binomial curves. Using the same procedure as for the inflection points, we compute the defining polynomial of the 2-Hessian curve H_2 from Theorem 4.2.7 with Program B.3 in [Maple]. This gives

$$\begin{aligned}K(m,l)\cdot \big(&6l^2(m-1)(l-m-1)x^{5l-9}y^{7m-9}z^{7l-7m-9} \\ &+ 6l^2(m-1)(l-m-1)x^{4l-9}y^{8m-9}z^{8l-8m-9} \\ &+ m(l-2)(l-3)(l-m)x^{3l-9}y^{9m-9}z^{9l-9m-9}\big),\end{aligned}$$

where $K(m,l) = 6l^3(l-1)^7(l-2)m^3(l-2m)(l-m)^3$. By Theorem 2.3.1, substituting the parametrisations (5.2) into this expression gives the intersection multiplicities between $C_{m,l}$ and the 2-Hessian curve H_2 at p_1 and p_2. Hence,

$$\begin{aligned}(C_{m,l}.H_2)_{p_1} &= \text{ord}_t \left(c_0 t^{-9m-9l}\big(c_1 t^{9ml} t^{3ml} + c_2(t^{4ml}t^{8ml} + t^{5ml}t^{7ml})\big)\right) \\ &= \text{ord}_t \left(c_0(c_1+2c_2)t^{12ml-9m-9l}\right) \\ &= 12ml - 9m - 9l,\end{aligned} \tag{5.5}$$

where $c_0 = -K(m,l)\cdot(l-m-1)$, $c_1 = m(l-2)(l-3)(l-m)$ and $c_2 = 6l^2(m-1)$. For p_2, we get

$$\begin{aligned}(C_{m,l}.H_2)_{p_2} &= \text{ord}_s \Big(c_0 s^{-18l+9m}\big(c_1 s^{9l^2-9ml} s^{3l^2-3ml} \\ &\qquad + c_2(s^{4l^2-4ml}s^{8l^2-8ml} + s^{5l^2-5ml}s^{7l^2-7ml})\big)\Big) \\ &= \text{ord}_s \left(c_0(c_1+2c_2)s^{12l^2-18l-12ml+9m}\right) \\ &= 12l^2 - 18l - 12ml + 9m.\end{aligned} \tag{5.6}$$

Thus, the sum becomes

$$\begin{aligned}(C_{m,l}.H_2)_{p_1} + (C_{m,l}.H_2)_{p_2} &= 12l^2 - 27 \\ &= l(12l-27) \\ &= C_{m,l}.H_2\end{aligned}$$

by Bézout's theorem. Hence, there are no other intersections between $C_{m,l}$ and H_2 than p_1 and p_2, so there are no sextactic points on $C_{m,l}$.

We could have prove this by using the sextactic point formula from Theorem 4.3.2 instead of using the 2-Hessian directly. We state the result as a theorem, and then use Corollary 4.5.1 to give an alternative proof.

Theorem 5.3.1. *There are no sextactic points on a binomial curve $C_{m,l}$.*

Proof. Suppose that $C_{m,l}$ is a binomial curve given by

$$F = y^m z^{l-m} - x^l,$$

where $l > m \geq 1$ and $\gcd(m,l) = 1$. Observe that the multiplicities of p_1 and p_2 are given by $m_1 = m$ and $m_2 = l - m$ respectively, and that $(T_{p_i}.C_{m,l}) = l$ for both p_1 and p_2. Furthermore, $l \neq 2m_i$ because $\gcd(m,l) = 1$, so by Corollary 4.5.1 we get that the number of sextactic points s on $C_{m,l}$ is

$$\begin{aligned}s &= 6(2l-5) - (4m_1 + 4l - 15) - (4m_2 + 4l - 15) \\ &= 12l - 30 - (4m + 4l - 15) - (4(l-m) + 4l - 15) \\ &= 12l - 30 - 12l + 30 \\ &= 0.\end{aligned}$$

∎

Remark 5.3.2. Note that this argument also holds when $m = 1$ or $m = l - 1$ because an inflection point contributes with the same Weierstrass weight with respect to the complete linear system of conics as a singular point where $l \neq 2m$. See equation (4.15) on p. 37.

Let us look at an example where we use the 2-Hessian curve to check if there are any sextactic points on a specific binomial curve.

Example 5.3.3. Consider the binomial curve $C_{4,7}$, i.e. the curve given by

$$F = z^3y^4 - x^7.$$

Using [Maple] and Program B.3, we find that the defining polynomial of the 2-Hessian curve H_2 is

$$-19910302433280x^{12}y^{19}z^{12}(7301x^{14} - 2646x^7y^4z^3 - 9880y^8z^6).$$

Intersecting C and H_2 using `intersectcurves` in [Maple] gives

$$\big[[237, [x, y, 1]], [162, [x, 1, 0]]\big],$$

and so only the two cusps $(0 : 0 : 1)$ and $(0 : 1 : 0)$ are intersection points. Hence, there are no sextactic points on $C_{4,7}$ as expected.

5.4 Weierstrass points of complete linear systems of higher degree

For binomial curves we look beyond inflection points and sextactic points, and consider Weierstrass points with respect to a complete linear system consisting of curves of higher degree.

Following [Arn96], we introduce the following notation. Let Q_n be the complete linear system consisting of curves of degree n on C. A smooth Q_n-Weierstrass point on a curve C that is not a Q_i-Weierstrass point for any $i < n$ is called an *n-inflection* on C. In particular, this means that 1-inflections are the same as inflection points and 2-inflections are the same as sextactic points. Hence, the discussion in Section 5.2 shows that there are either 0 or a single 1-inflection on a binomial curve $C_{m,l}$, while Theorem 5.3.1 shows that there are no 2-inflections on $C_{m,l}$.

Let us now consider the case where we look at the complete linear system of cubic curves on a binomial curve and the corresponding 3-inflections.

Theorem 5.4.1. *Let $C_{m,l}$ be a binomial curve of degree $l > 3$. Then there are no 3-inflections on C.*

Proof. Consider the complete linear system Q on $C_{m,l}$ consisting of cubic curves. There are 10 monomials of degree 3 in x, y, z, and so, by Theorem 2.1.2, Q is a g^9_{3r}. We want to use Proposition 2.4.4 to show that $p_1 = (0 : 0 : 1)$ and $p_2 = (0 : 1 : 0)$ are the only Q-Weierstrass points on $C_{m,l}$. In this setting, equation (2.4) on p. 13 is

$$\begin{aligned}\sum_{p\in C} w_p(Q) &= 10 \cdot 3r + 10 \cdot 9(g - 1) \\ &= 30r - 90,\end{aligned} \tag{5.7}$$

because $C_{m,l}$ is rational. To calculate the Q-Weierstrass weight of the points $p_1, p_2 \in C$, we consider the basis of monomials for Q, i.e.

$$x^3, y^3, z^3, x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, xyz. \tag{5.8}$$

To find the possible intersection multiplicities of cubic curves at p_1 and p_2, we want as usual to apply Theorem 2.3.1 to the curves given by the polynomials in (5.8). Recall that the Puiseux parametrisation of $C_{m,l}$ at p_1 is

$$(t^m : t^l : 1),$$

so substituting this into (5.8) gives

$$t^{3m}, t^{3l}, 1, t^{2m+l}, t^{2m}, t^{m+2l}, t^m, t^{2l}, t^l, t^{m+l}. \tag{5.9}$$

Because $\gcd(m, l) = 1$, all of these orders are distinct, and so the intersections we are looking for are given by

$$\begin{aligned} h_0 = 0, h_1 = m, h_2 = l, h_3 = 2m, h_4 = m + l, h_5 = 2l, \\ h_6 = 3m, h_7 = 2m + l, h_8 = m + 2l, h_9 = 3l. \end{aligned}$$

This means that the Q-Weierstrass weight of p_1 is

$$\begin{aligned} w_{p_1}(Q) &= \sum_{i=0}^{9} (h_i - i) \\ &= 10m + 10l - 45. \end{aligned}$$

The Puiseux parametrisation of $C_{m,l}$ at p_2 is given by

$$(s^{l-m} : 1 : s^l),$$

and doing the same substitution for this parametrisation gives

$$s^{3l-3m}, 1, s^{3l}, s^{2l-2m}, s^{3l-2m}, s^{l-m}, s^{3l-m}, s^l, s^{2l}, s^{2l-m}.$$

All of these orders are also distinct and, using the same notation as above, we get that

$$\begin{aligned} h_0 = 0, h_1 = l - m, h_2 = l, h_3 = 2l - 2m, h_4 = 2l - m, h_5 = 2l, \\ h_6 = 3l - 3m, h_7 = 3l - 2m, h_8 = 3l - m, h_9 = 3l. \end{aligned}$$

Hence, the Q-Weierstrass weight for p_2 is

$$\begin{aligned} w_{p_2}(Q) &= \sum_{i=0}^{9} (h_i - i) \\ &= 20l - 10m - 45 \\ &= 10(l - m) + 10l - 45. \end{aligned}$$

The sum of the two is

$$\begin{aligned} w_{p_1}(Q) + w_{p_2}(Q) &= 10m + 10l - 45 + 10(l - m) + 10l - 45 \\ &= 30l - 90, \end{aligned}$$

which is exactly the right-hand side of equation (5.7). Thus all other points on C are of Q-Weierstrass weight zero, meaning that there are no other Weierstrass points on C with respect to Q and thus no 3-inflections on C. ■

Remark 5.4.2. Note that the limitation on l in the theorem comes in naturally after substituting the Puiseux parametrisation into the basis (5.8). If $l \leq 3$, there would be a linear dependence between the functions in (5.9), making the dimension drop such that we would not get distinct intersections.

There is nothing special about cubic curves in the proof above, so by generalising this theorem, we obtain the following result.

Theorem 5.4.3. *Assume that $C_{m,l}$ is a binomial curve of degree l, then there are no n-inflections on $C_{m,l}$ for $n \in \mathbb{N}$ such that $2 \leq n < l$.*

Remark 5.4.4. We have to assume $n \geq 2$, as $C_{m,l}$ for $m = 1$ or $m = l - 1$ has an inflection point.

Proof of Theorem 5.4.3. Let Q be the complete linear system on C of curves of degree n. By Section 2.4, Q is of dimension $n(n+3)/2$ and degree nl. Recalling that $C_{m,l}$ is rational, equation (2.4) from Remark 2.4.5 now gives that

$$\begin{aligned} \sum_{p \in C_{m,l}} w_p(Q) &= \frac{(n+1)(n+2)}{2} nr + (g-1) \frac{(n+1)(n+2)}{2} \frac{n(n+3)}{2} \\ &= \frac{n(n+1)(n+2)}{2} r - \frac{n(n+1)(n+2)(n+3)}{4}. \end{aligned} \tag{5.10}$$

To find the Weierstrass weight with respect to Q of the singular points on $C_{m,l}$, we want as usual to calculate the possible intersection multiplicities of curves from Q and $C_{m,l}$ at the points. We know that the Puiseux parametrisation of $C_{m,l}$ for $p_1 = (0:0:1)$ and $p_2 = (0:1:0)$ is given by

$$(t^m : t^l : 1) \tag{5.11}$$

and

$$(s^{l-m} : 1 : s^l) \tag{5.12}$$

respectively. To find the possible intersections of $C_{m,l}$ at p_i we use Theorem 2.3.1 and substitute the Puiseux parametrisations for p_1 (5.11) and p_2 (5.12) into the basis for Q of all monomials in the variables x, y, z of degree n.

Let us first consider p_1. When substituting (5.11) into this basis, we obtain terms in t of order $am + bl$ for $a, b \in \mathbb{N} \cup \{0\}$ such that $a + b \leq n$. In other words, we get the orders

$$0, m, l, 2m, m+l, 2l, \ldots, nm, (n-1)m + l, \ldots, m + (n-1)l, nl. \tag{5.13}$$

Now, we need to show that no two of these orders can be the same. Indeed, because $\gcd(m, l) = 1$ and $n < l$,

$$am + bl = cm + dl \tag{5.14}$$

if and only if $a = c$ and $b = d$. To see this, let $0 \leq a, b, c, d \leq n$, and assume for contradiction that $b \neq d$ and that (5.14) holds. This implies that also $a \neq c$, so because all coefficients are positive in (5.14) we can without loss of generality assume that $a > c$ and $d > b$. equation (5.14) is then equivalent to

$$(a-c)m = (d-b)l.$$

Because $\gcd(m, l) = 1$, this means that there exists a $k \in \mathbb{N}$ such that

$$m \cdot k = d - b \text{ and } l \cdot k = a - c, \tag{5.15}$$

for the same k. But by assumption $l > n \geq a, c \geq 0$ and hence $l \cdot k > a - c$, which contradicts (5.15). Thus $b = d$ and $a = c$, which means that all elements of (5.13) are unique, and gives the possible intersection multiplicities at p_1. Here it should be noted that we would not be able to achieve a contradiction if $n \geq l$, as then l and a, c could not be compared without further information.

To find the Q-Weierstrass weight, we need to compute the sum

$$w_{p_1}(Q) = \sum_{i=0}^{n(n+3)/2} (h_i - i).$$

To keep track of our calculations, we handle each part of the sum individually, starting with the sum of the intersection multiplicities. From (5.13) the intersection multiplicities can be represented as the sequence $jm + (d - j)l$, where $d = 1, \dots, n$ and $j = 0, \dots, d$. Hence,

$$\begin{aligned}
\sum_{i=0}^{n(n+3)/2} h_i &= \sum_{d=1}^{n} \sum_{j=0}^{d} (jm + (d-j)l) \\
&= m \sum_{d=1}^{n} \sum_{j=0}^{d} j + l \left(\sum_{d=1}^{n} \sum_{j=0}^{d} d - \sum_{d=1}^{n} \sum_{j=0}^{d} j \right) \\
&= \frac{m}{2} \sum_{d=1}^{n} d(d+1) + l \left(\sum_{d=1}^{n} d(d+1) - \frac{1}{2} \sum_{d=1}^{n} d(d+1) \right) \qquad (5.16) \\
&= \frac{m}{2} \cdot \frac{1}{3} n(n+1)(n+2) + l \left(\frac{1}{3} n(n+1)(n+2) - \frac{1}{2} \cdot \frac{1}{3} n(n+1)(n+2) \right) \\
&= \frac{n(n+1)(n+2)}{6} (m + l).
\end{aligned}$$

In the calculations above, we have used the well-known summation formula for the sum of the d first integers,

$$\sum_{j=0}^{d} j = \frac{d(d+1)}{2},$$

as well as the formula

$$\sum_{d=1}^{n} d(d+1) = \frac{1}{3} n(n+1)(n+2),$$

that can be proved by induction on n. The sum of the $\frac{n(n+3)}{2}$ first integers is computed and simplified as follows.

$$\begin{aligned}\sum_{i=0}^{n(n+3)/2} i &= \frac{1}{2}\left(\frac{n(n+3)}{2}\left(\frac{n(n+3)}{2}+1\right)\right)\\ &= \frac{1}{8}\left(n^2(n+3)^2+2n(n+3)\right)\\ &= \frac{1}{8}\left(n^4+6n^3+9n^2+2n^2+6n\right)\\ &= \frac{1}{8}n\left(n^3+6n^2+11n+6\right)\\ &= \frac{1}{8}n(n+1)(n+2)(n+3).\end{aligned} \tag{5.17}$$

Thus, combining (5.16) and (5.17) gives the following Q-Weierstrass weight for p_1:

$$w_{p_1}(Q) = \frac{n(n+1)(n+2)}{6}(m+l) - \frac{1}{8}n(n+1)(n+2)(n+3). \tag{5.18}$$

To calculate the Q-Weierstrass weight for the other point p_2 let $\hat{m}$ be the multiplicity of p_2 and $\hat{l}$ be the tangential intersection at p_2. We know, from the Puiseux parametrisation of C at p_2, that $\hat{m} = l - m$ and $\hat{l} = l$. Since $\gcd(m, l) = 1$, also $\gcd(\hat{m}, \hat{l}) = 1$, and so, by using the exact same argument as for p_1, we get that

$$\begin{aligned}w_{p_2}(Q) &= \frac{n(n+1)(n+2)}{6}(\hat{m}+\hat{l}) - \frac{1}{8}n(n+1)(n+2)(n+3)\\ &= \frac{n(n+1)(n+2)}{6}(2l-m) - \frac{1}{8}n(n+1)(n+2)(n+3).\end{aligned} \tag{5.19}$$

Combining (5.18) and (5.19), we get that the sum of the Q-Weierstrass weights of p_1 and p_2 is

$$\begin{aligned}w_{p_1}(Q) + w_{p_2}(Q) &= \frac{n(n+1)(n+2)}{6}(m+l+\hat{m}+\hat{l}) - \frac{2}{8}n(n+1)(n+2)(n+3)\\ &= \frac{n(n+1)(n+2)}{6}(m+l+2l-m) - \frac{1}{4}n(n+1)(n+2)(n+3)\\ &= \frac{n(n+1)(n+2)}{2}l - \frac{1}{4}n(n+1)(n+2)(n+3),\end{aligned}$$

which is exactly the right-hand side of equation (5.10). Hence, there are no other points of non-zero Q-Weierstrass weight, so there are no n-inflection points on $C_{m,l}$. ■

Appendices

APPENDIX A

Tables

This appendix is a collection of tables that contain information about the singular points, the inflection points, and the sextactic points of the cubic curves from Example 2.5.1, 2.5.2 and 2.5.3, and the rational cuspidal curves from Table 2.1 and Table 2.2. For each such point p, we give its multiplicity sequence, $\overline{m}_p$, and its delta invariant, δ_p. Furthermore, we have calculated the intersection multiplicity between the example curves and the following curves at p: the tangent at p; the osculating conic at p, where it exists; the Hessian curve; and the 2-Hessian curve. These curves are denoted by T_p, O_p, H, and H_2, respectively.

To perform the calculations, we used [Maple] and the [Maple] commandsa and programs from Appendix B. In particular, for the rational cuspidal curves, we used [Macaulay2] and the code from Program B.5 to find the defining polynomial from the parametrisations given in Table 2.1 and Table 2.2.

We give the explicit coordinates of the points if they are not too complicated. If the coordinate of a point is very elaborate, we simply denote the point by p_i, where i corresponds to the row of the point in the table.

A.1 Cubics

Nodal cubic

Table for the nodal cubic given by the polynomial $F = -x^3 - x^2z + y^2z$:

Point p	$\boldsymbol{m_p}$	$\boldsymbol{\delta_p}$	$\boldsymbol{(T_p.C)_p}$	$\boldsymbol{(O_p.C)_p}$	$\boldsymbol{(H.C)_p}$	$\boldsymbol{(H_2.C)_p}$
$(0:0:1)$	2	1	3+3	-	6	24
$(0:1:0)$	1	0	3	-	1	0
$(-\frac{4}{3}:\frac{4}{9}i\sqrt{3}:1)$	1	0	3	-	1	0
$(-\frac{4}{3}:-\frac{4}{9}i\sqrt{3}:1)$	1	0	3	-	1	0
$(-1:0:1)$	1	0	2	6	0	1
$(-4:4i\sqrt{3}:1)$	1	0	2	6	0	1
$(-4:-4i\sqrt{3}:1)$	1	0	2	6	0	1

Table A.1: Intersections for the nodal cubic.

Cuspidal cubic

Table for the cuspidal cubic given by the polynomial $F = zy^2 - x^3$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[2]	1	3	-	8	27
$(0:1:0)$	[1]	0	3	-	1	0

Table A.2: Intersections for cuspidal cubic.

Elliptic curve

Table for the elliptic curve given by the polynomial $F = y^2z - xz^2 - z^3 - x^3$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
p_i, $i = 1, \ldots, 9$	[1]	0	3	-	1	0
p_i $i = 10, \ldots, 36$	[1]	0	2	6	0	1

Table A.3: Intersections for the elliptic curve.

A.2 Quartics

Quartic C_{1A}

Table for the quartic curve C_{1A} with parametrisation $(t^3s : t^4 : s^4)$ and defining polynomial $F = x^4 - y^3z$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3]	3	4	-	22	81
$(0:1:0)$	[1]	0	4	-	2	3

Table A.4: Intersections for the quartic C_{1A}.

Quartic C_{1B}

Table for the quartic curve C_{1B} with parametrisation $(ts^3 : s^4 : st^3 - t^4)$ and defining polynomial $F = x^4 - x^3y + y^3z$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3]	3	4	-	22	81
$(8:16:1)$	[1]	0	3	-	1	0
$(0:1:0)$	[1]	0	3	-	1	0
$(\frac{64}{3} : \frac{256}{3} : 1)$	[1]	0	2	6	0	1
$(\frac{49}{24} + i\frac{77\sqrt{7}}{24} : \frac{-637}{48} + i\frac{343\sqrt{7}}{48} : 1)$	[1]	0	2	6	0	1
$(\frac{49}{24} - i\frac{77\sqrt{7}}{24} : \frac{-637}{48} - i\frac{343\sqrt{7}}{48} : 1)$	[1]	0	2	6	0	1

Table A.5: Intersections for the quartic C_{1B}.

Quartic C_2

Table for the quartic curve C_2 with parametrisation $(t^2s^2 : t^4 : s^4 - t^3s)$ and defining polynomial $F = (-x^2 + yz)^2 - xy^3$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	$[2_3]$	3	4	-	21	81
$(\frac{4}{9} : \frac{16}{9} : 1)$	[1]	0	3	-	1	0
$(\frac{-2}{9} - \frac{2i\sqrt{3}}{9} : \frac{-8}{9} + \frac{8i\sqrt{3}}{9} : 1)$	[1]	0	3	-	1	0
$(\frac{-2}{9} + \frac{2i\sqrt{3}}{9} : \frac{-8}{9} - \frac{8i\sqrt{3}}{9} : 1)$	[1]	0	3	-	1	0
$(-\frac{4}{55} \cdot 7^{2/3} : -\frac{112}{55} \cdot 7^{1/3} : 1)$	[1]	0	2	6	0	1
p_6	[1]	0	2	6	0	1
p_7	[1]	0	2	6	0	1

Table A.6: Intersections for the quartic C_2.

Quartic C_3

Table for the quartic C_3 with parametrisation $(s^4 + ts^3 : t^2s^2 : t^4)$ and defining polynomial $F = x^2z^2 - 2xy^2z + y^4 - y^3z$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[2]	1	3	-	8	27
$(1:0:0)$	$[2_2]$	2	4	-	15	55
$(\frac{-135}{4096} : \frac{9}{64} : 1)$	[1]	0	3	-	1	0
$(\frac{-223}{73728} - \frac{119i\sqrt{15}}{73728} : \frac{11}{384} + \frac{i\sqrt{15}}{128} : 1)$	[1]	0	2	6	0	1
$(\frac{-223}{73728} + \frac{119i\sqrt{15}}{73728} : \frac{11}{384} - \frac{i\sqrt{15}}{128} : 1)$	[1]	0	2	6	0	1

Table A.7: Intersections for the quartic C_3.

Quartic C_4

Table for the quartic curve C_4 with parametrisation $(ts^3 - \frac{1}{2}s^4 : t^2s^2 : t^4 - 2t^3s)$ and defining polynomial $F = 4x^2z^2 + 8xy^3 + 12xy^2z - 3y^4 - 4y^3z$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[2]	1	3	-	8	27
$(1:0:0)$	[2]	1	3	-	8	27
$(-\frac{1}{2}:-1:1)$	[2]	1	3	-	8	27
$(\frac{3}{8}:1:0)$	[1]	0	2	6	0	1
$(\frac{-1}{2}:\frac{1}{3}:1)$	[1]	0	2	6	0	1
$(0:\frac{-4}{3}:1)$	[1]	0	2	6	0	1

Table A.8: Intersections for the quartic C_4.

A.3 Quintics

Quintic C_{1A}

Table for the quintic curve C_{1A} with parametrisation $(s^5 : st^4 : t^5)$ and defining polynomial $F = y^5 - xz^4$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(1:0:0)$	[4]	6	5	-	42	159
$(0:0:1)$	[1]	0	5	-	3	6

Table A.9: Intersections for the quintic C_{1A}.

Quintic C_{1B}

Table for the quintic curve C_{1B} with parametrisation $(s^5 - s^4t : st^4 : t^5)$ and defining polynomial $F = y^5 - y^4z - xz^4$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(1:0:0)$	[4]	6	5	-	42	159
$(0:0:1)$	[1]	0	4	-	2	3
$(-\frac{162}{3125}:\frac{3}{5}:1)$	[1]	0	3	-	1	0
p_4	[1]	0	2	6	0	1
p_5	[1]	0	2	6	0	1
p_6	[1]	0	2	6	0	1

Table A.10: Intersections for the quintic C_{1B}.

Quintic C_{1C}

Table for the quintic curve C_{1C}, where $a = 1$, with parametrisation $(s^5 + as^4t - (1+a)s^2t^3 : st^4 : t^5),\ a \neq -1$ and defining polynomial $F = y^5 + y^4z - 2y^2z^3 - xz^4$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(1:0:0)$	[4]	6	5	-	42	159
p_2	[1]	0	3	-	1	0
p_3	[1]	0	3	-	1	0
p_4	[1]	0	3	-	1	0
p_5	[1]	0	2	6	0	1
p_6	[1]	0	2	6	0	1
p_7	[1]	0	2	6	0	1
p_8	[1]	0	2	6	0	1
p_9	[1]	0	2	6	0	1
p_{10}	[1]	0	2	6	0	1

Table A.11: Intersections for the quintic C_{1C}.

Quintic C_2

Table for the quintic curve C_2 with parametrisation $(s^4t : s^2t^3 - s^5 : t^5 - 2s^3t^2)$ and defining polynomial $F = x^5 - 2x^2y^3 + 2x^3yz + y^4z - 2xy^2z^2 + x^2z^3$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	$[2_6]$	6	4	-	39	156
p_i $i = 2, \cdots, 7$	[1]	0	3	-	1	0
p_i $i = 8, \cdots, 16$	[1]	0	2	6	0	1

Table A.12: Intersections for quintic C_2.

Quintic C_{3A}

Table for the quintic curve $C_{3\mathrm{A}}$ with parametrisation $(s^5 : s^3t^2 : t^5)$ and defining polynomial $F = y^5 - x^3z^2$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3,2]	4	5	-	29	108
$(1:0:0)$	$[2_2]$	2	5	-	16	57

Table A.13: Intersections for quintic $C_{3\mathrm{A}}$.

Quintic C_{3B}

Table for the quintic curve $C_{3\mathrm{B}}$ with parametrisation $(s^5 : s^3t^2 : st^4 + t^5)$ and defining polynomial $F = y^5 + 2x^2y^2z - x^3z^2 - xy^4$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3,2]	4	5	-	29	108
$(1:0:0)$	$[2_2]$	2	4	-	15	55
$(\frac{759375}{28672} : \frac{3375}{448} : 1)$	[1]	0	3	-	1	0
p_4	[1]	0	2	6	0	1
p_5	[1]	0	2	6	0	1

Table A.14: Intersections for quintic $C_{3\mathrm{B}}$.

Quintic C_4

Table for the quintic curve C_4 with parametrisation $(s^4t - \frac{1}{2}s^5 : s^3t^2 : \frac{1}{2}st^4 + t^5)$ and defining polynomial $F = xy^4 + 4y^5 + 8x^2y^2z + 16xy^3z - 16y^4z + 16x^3z^2$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3]	3	4	-	22	81
$(1:0:0)$	$[2_3]$	3	4	-	21	81
$(\frac{-3339}{10} - \frac{729\sqrt{21}}{10} : \frac{153}{5} + \frac{33\sqrt{21}}{5} : 1)$	[1]	0	3	-	1	0
$(\frac{-3339}{10} + \frac{729\sqrt{21}}{10} : \frac{153}{5} - \frac{33\sqrt{21}}{5} : 1)$	[1]	0	3	-	1	0
p_5	[1]	0	2	6	0	1
p_6	[1]	0	2	6	0	1
p_7	[1]	0	2	6	0	1

Table A.15: Intersections for the quintic C_4.

Quintic C_5

Table for the quintic curve C_5 with parametrisation

$$(s^4t - s^5 : s^2t^3 - \tfrac{5}{32}s^5 : -\tfrac{47}{128}s^5 + \tfrac{11}{16}s^3t^2 + st^4 + t^5)$$

and defining polynomial

$$\begin{aligned} F = {} & 14359x^5 + 274368x^4y - 1536256x^3y^2 + 1927168x^2y^3 \\ & - 3538944xy^4 + 2883584y^5 - 231168x^4z - 485376x^3yz \\ & + 1703936x^2y^2z - 3670016xy^3z - 1048576y^4z + 786432x^3z^2 \\ & + 786432x^2yz^2 + 2097152xy^2z^2 - 1048576x^2z^3 : \end{aligned}$$

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	$[2_4]$	4	4	-	27	107
$(\frac{256}{109} : \frac{48}{109} : 1)$	$[2_2]$	2	4	-	15	55
p_3	[1]	0	3	-	1	0
p_4	[1]	0	3	-	1	0
p_5	[1]	0	3	-	1	0
p_6	[1]	0	2	6	0	1
p_7	[1]	0	2	6	0	1
p_8	[1]	0	2	6	0	1

Table A.16: Intersections for quintic C_5.

Quintic C_6

Table for the quintic curve C_6 with parametrisation $(s^4t - \frac{1}{2}s^5 : s^3t^2 : -\frac{3}{2}st^4 + t^5)$ and defining polynomial $F = 9xy^4 - 4y^5 - 24x^2y^2z + 48xy^3z - 16y^4z + 16x^3z^2$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	[3]	3	4	-	22	81
$(1:0:0)$	$[2_2]$	2	4	-	15	55
$(-1:-2:1)$	[2]	1	3	-	8	27
$(\frac{-245}{4374} - \frac{343\sqrt{105}}{4374} : -\frac{49}{81} + \frac{7\sqrt{105}}{27} : 1)$	[1]	0	2	6	0	1
$(\frac{245}{4374} - \frac{343\sqrt{105}}{4374} : -\frac{49}{81} - \frac{7\sqrt{105}}{27} : 1)$	[1]	0	2	6	0	1

Table A.17: Intersections for the quintic C_6.

Quintic C_7

Table for the quintic curve C_7 with parametrisation

$$(s^4t - s^5 : s^2t^3 - \frac{5}{32}s^5 : -\frac{125}{128}s^5 - \frac{25}{16}s^3t^2 - 5st^4 + t^5)$$

and defining polynomial

$$\begin{aligned} F = 709375x^5 &+ 4800000x^4y + 54560000x^3y^2 - 199424000x^2y^3 \\ &+ 265420800xy^4 - 8126464y^5 - 3360000x^4z - 17664000x^3yz \\ &+ 18022400x^2y^2z + 49807360xy^3z - 1048576y^4z + 4915200x^3z^2 \\ &- 3932160x^2yz^2 + 2097152xy^2z^2 - 1048576x^2z^3 : \end{aligned}$$

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	$[2_2]$	2	4	-	15	55
$(\frac{11392}{20275} + \frac{26496\sqrt{5}}{101375} : \frac{24}{4055} + \frac{1368\sqrt{5}}{20275} : 1)$	$[2_2]$	2	4	-	15	55
$(\frac{11392}{20275} - \frac{26496\sqrt{5}}{101375} : \frac{24}{4055} - \frac{1368\sqrt{5}}{20275} : 1)$	$[2_2]$	2	4	-	15	55

Table A.18: Intersections for the quintic C_7.

Quintic C_8

Table for the quintic curve C_8 with parametrisation $(s^4t : s^2t^3 - s^5 : t^5 + 2s^3t^2)$ and defining polynomial $F = 27x^5 - 2x^2y^3 + 18x^3yz - y^4z + 2xy^2z^2 - x^2z^3$:

Point p	$\overline{m}_p$	δ_p	$(T_p.C)_p$	$(O_p.C)_p$	$(H.C)_p$	$(H_2.C)_p$
$(0:0:1)$	$[2_3]$	3	4	-	21	81
$(\frac{\sqrt[3]{2}}{3} - i\frac{\sqrt[3]{2}\sqrt{3}}{3} : \frac{1}{\sqrt[3]{2}} + i\frac{\sqrt{3}}{\sqrt[3]{2}} : 1)$	$[2]$	1	3	-	8	27
$(-\frac{2\sqrt[3]{2}}{3} : -\sqrt[3]{4} : 1)$	$[2]$	1	3	-	8	27
$(\frac{\sqrt[3]{2}}{3} + i\frac{\sqrt[3]{2}\sqrt{3}}{3} : \frac{1}{\sqrt[3]{2}} - i\frac{\sqrt{3}}{\sqrt[3]{2}} : 1)$	$[2]$	1	3	-	8	27
$(\frac{14}{29}14^{1/3} : -\frac{13}{29}14^{2/3} : 1)$	$[1]$	0	2	6	0	1
$(\frac{28\cdot 14^{1/3}}{(29i)\sqrt{3}-29} : -\frac{13}{58}14^{2/3}(i\sqrt{3}-1) : 1)$	$[1]$	0	2	6	0	1
$(-\frac{28\cdot 14^{1/3}}{(29i)\sqrt{3}+29} : \frac{13}{58}14^{2/3}(i\sqrt{3}+1) : 1)$	$[1]$	0	2	6	0	1

Table A.19: Intersections for the quintic C_8.

APPENDIX B

Programming and code

The first section of this appendix consists of [Maple] code that enables us to calculate the 2-Hessian and the osculating conic. We have also included a program for the tangent at a smooth point of a curve and the implementation of Cayley's 2-Hessian, as well as a brief explanation of the [Maple] commands `intersectcurves`, `singularities` and `Hessian` in the way we use them. The second section is the [Macaulay2] code that returns the defining polynomial of a rational curve, given its parametrisation.

B.1 Maple

The tangent

A [Maple] program for the defining polynomial of the tangent of a curve $C = V(F)$ at a smooth point p:

Program B.1: Tang

```
Tang := proc (F, p)
local dx, dy, dz;
dx := eval(diff(F, x), [x = p[1], y = p[2], z = p[3]]);
dy := eval(diff(F, y), [x = p[1], y = p[2], z = p[3]]);
dz := eval(diff(F, z), [x = p[1], y = p[2], z = p[3]]);
return expand(eval(X*dx+Y*dy+Z*dz, [X = x, Y = y, Z = z]))
end proc;
```

The osculating conic

A [Maple] program for calculating the defining polynomial of the osculating conic of the curve $V(F)$ at the point p:

Program B.2: OscCon

```
OscCon := proc (F, p)
local Hes, H, Om, Phsi, c_A, c_B, c_C, c_F, c_G, c_H,
HesHes, JacUHPsi, m, a_p, b_p, c_p, f_p, g_p, h_p,
a, b, c, f, g, h, D2F, DH, DF, A, Lam, H_eval,
x_1, y_1, z_1; x_1 := p[1]; y_1 := p[2]; z_1 := p[3];
```

```
Hes, H := Hessian(F, [x, y, z], determinant);
HesHes := Hessian(H, [x, y, z]);
a, b, c, f, g, h := Hes[1, 1], Hes[2, 2], Hes[3, 3],
                    Hes[2, 3], Hes[1, 3], Hes[1, 2];
a_p, b_p, c_p, f_p, g_p, h_p := HesHes[1, 1], HesHes[2, 2],
                                HesHes[3, 3], HesHes[2, 3],
                                HesHes[1, 3], HesHes[1, 2];
c_A, c_B, c_C := b*c-f^2, a*c-g^2, a*b-h^2;
c_F, c_G, c_H := h*g-a*f, h*f-b*g, f*g-h*c;

Phsi := (diff(H, x))^2*c_A + (diff(H, y))^2*c_B
      + (diff(H, z))^2*c_C + 2*(diff(H, y))*(diff(H, z))*c_F
      + 2*(diff(H, x))*(diff(H, z))*c_G
      + 2*(diff(H, x))*(diff(H, y))*c_H;

Om := `<,>`(c_A, c_B, c_C, c_F, c_G, c_H)
    . `<,>`(a_p, b_p, c_p, 2*f_p, 2*g_p, 2*h_p);

Lam := expand((1/9)*(-3*H*Om+4*Phsi)/H^3);

A := eval(Lam, [x = x_1, y = y_1, z = z_1]);

D2F := X^2*(eval(diff(F, x$2), [x = x_1, y = y_1, z = z_1]))
    + Y^2*(eval(diff(F, y$2), [x = x_1, y = y_1, z = z_1]))
    + Z^2*(eval(diff(F, z$2), [x = x_1, y = y_1, z = z_1]))
    + 2*X*Y*(eval(diff(F, x, y), [x = x_1, y = y_1, z = z_1]))
    + 2*X*Z*(eval(diff(F, x, z), [x = x_1, y = y_1, z = z_1]))
    + 2*Y*Z*(eval(diff(F, y, z), [x = x_1, y = y_1, z = z_1]));

DF := X*(eval(diff(F, x), [x = x_1, y = y_1, z = z_1]))
    + Y*(eval(diff(F, y), [x = x_1, y = y_1, z = z_1]))
    + Z*(eval(diff(F, z), [x = x_1, y = y_1, z = z_1]));

DH := X*(eval(diff(H, x), [x = x_1, y = y_1, z = z_1]))
    + Y*(eval(diff(H, y), [x = x_1, y = y_1, z = z_1]))
    + Z*(eval(diff(H, z), [x = x_1, y = y_1, z = z_1]));

H_eval := eval(H, [x = x_1, y = y_1, z = z_1]);

return eval(expand(D2F-((2/3)*DH/H_eval+A*DF)*DF),
           [X = x, Y = y, Z = z])
end proc:
```

The 2-Hessian

A [Maple] program for the defining polynomial of the 2-Hessian curve of the curve $C = V(F)$:

Program B.3: Hessian2

```
Hessian2 := proc (F, degF:=degree(F))
local Hes, H, JacH, JacF, Phsi,
c_A, c_B, c_C, c_F, c_G, c_H, HesHes, JacPsi,
m, a_p, b_p, c_p, f_p, g_p, h_p, a, b, c, f, g, h;

m := degF;
Hes, H := Hessian(F, [x, y, z], determinant);
HesHes := Hessian(H, [x, y, z]);
a, b, c, f, g, h := Hes[1, 1], Hes[2, 2], Hes[3, 3],
                    Hes[2, 3], Hes[1, 3], Hes[1, 2];
a_p, b_p, c_p, f_p, g_p, h_p := HesHes[1, 1], HesHes[2, 2],
                                HesHes[3, 3], HesHes[2, 3],
                                HesHes[1, 3], HesHes[1, 2];

c_A, c_B, c_C := b*c-f^2, a*c-g^2, a*b-h^2;
c_F, c_G, c_H := h*g-a*f, h*f-b*g, f*g-h*c;

Phsi := (diff(H, x))^2*c_A + (diff(H, y))^2*c_B
      + (diff(H, z))^2*c_C + 2*(diff(H, y))*(diff(H, z))*c_F
      + 2*(diff(H, x))*(diff(H, z))*c_G
      + 2*(diff(H, x))*(diff(H, y))*c_H;

JacH := Determinant(simplify(Matrix([[Jacobian([F, H], [x, y, z])],
[Matrix(1, 3, [(diff(c_A, x))*a_p + (diff(c_B, x))*b_p
+ (diff(c_C, x))*c_p + 2*(diff(c_F, x))*f_p
+ 2*(diff(c_G, x))*g_p + 2*(diff(c_H, x))*h_p,
(diff(c_A, y))*a_p + (diff(c_B, y))*b_p + (diff(c_C, y))*c_p
+ 2*(diff(c_F, y))*f_p + 2*(diff(c_G, y))*g_p
+ 2*(diff(c_H, y))*h_p, (diff(c_A, z))*a_p + (diff(c_B, z))*b_p
+ (diff(c_C, z))*c_p + 2*(diff(c_F, z))*f_p
+ 2*(diff(c_G, z))*g_p + 2*(diff(c_H, z))*h_p])]])),
method = multivar);

JacF := Determinant(simplify(Matrix([[Jacobian([F, H], [x, y, z])],
[Matrix(1, 3, [(diff(a_p, x))*c_A + (diff(b_p, x))*c_B
+ (diff(c_p, x))*c_C + (diff(2*f_p, x))*c_F
+ (diff(2*g_p, x))*c_G + (diff(2*h_p, x))*c_H,
(diff(a_p, y))*c_A + (diff(b_p, y))*c_B + (diff(c_p, y))*c_C
+ (diff(2*f_p, y))*c_F + (diff(2*g_p, y))*c_G
+ (diff(2*h_p, y))*c_H, (diff(a_p, z))*c_A + (diff(b_p, z))*c_B
+ (diff(c_p, z))*c_C + (diff(2*f_p, z))*c_F
+ (diff(2*g_p, z))*c_G + (diff(2*h_p, z))*c_H])]])),
method = multivar);
```

```
JacPsi := Determinant(simplify(Jacobian([F, H, Phsi], [x, y, z])),
method = multivar);

return expand((12*m^2-54*m+57)*H*JacH
            + (m-2)*(12*m-27)*H*JacF - 20*(m-2)^2*JacPsi)
end proc;
```

Cayley's 2-Hessian

A [Maple] program for the defining polynomial of Cayley's 2-Hessian curve of the curve $C = V(F)$:

Program B.4: CayH2

```
CayH2 := proc (F)
local Hes, H, JacH, JacU, Phsi,
c_A, c_B, c_C, c_F, c_G, c_H, HesHes, JacPsi,
m, a_p, b_p, c_p, f_p, g_p, h_p, a, b, c, f, g, h;

m := degree(F);
Hes, H := Hessian(F, [x, y, z], determinant);
HesHes := Hessian(H, [x, y, z]);
a, b, c, f, g, h := Hes[1, 1], Hes[2, 2], Hes[3, 3],
                    Hes[2, 3], Hes[1, 3], Hes[1, 2];
a_p, b_p, c_p, f_p, g_p, h_p := HesHes[1, 1], HesHes[2, 2],
                                HesHes[3, 3], HesHes[2, 3],
                                HesHes[1, 3], HesHes[1, 2];

c_A, c_B, c_C := b*c-f^2, a*c-g^2, a*b-h^2;
c_F, c_G, c_H := h*g-a*f, h*f-b*g, f*g-h*c;

Phsi := (diff(H, x))^2*c_A + (diff(H, y))^2*c_B
      + (diff(H, z))^2*c_C + 2*(diff(H, y))*(diff(H, z))*c_F
      + 2*(diff(H, x))*(diff(H, z))*c_G
      + 2*(diff(H, x))*(diff(H, y))*c_H;

JacH := Determinant(Matrix([[Jacobian([F, H], [x, y, z])],
[Matrix(1, 3, [(diff(c_A, x))*a_p + (diff(c_B, x))*b_p
+ (diff(c_C, x))*c_p + 2*(diff(c_F, x))*f_p
+ 2*(diff(c_G, x))*g_p + 2*(diff(c_H, x))*h_p,
(diff(c_A, y))*a_p + (diff(c_B, y))*b_p + (diff(c_C, y))*c_p
+ 2*(diff(c_F, y))*f_p + 2*(diff(c_G, y))*g_p
+ 2*(diff(c_H, y))*h_p, (diff(c_A, z))*a_p + (diff(c_B, z))*b_p
+ (diff(c_C, z))*c_p + 2*(diff(c_F, z))*f_p
+ 2*(diff(c_G, z))*g_p + 2*(diff(c_H, z))*h_p])]]),
method = multivar);

JacU := Determinant(Matrix([[Jacobian([F, H], [x, y, z])],
[Matrix(1, 3, [(diff(a_p, x))*c_A + (diff(b_p, x))*c_B
+ (diff(c_p, x))*c_C + (diff(2*f_p, x))*c_F
```

```
+ (diff(2*g_p, x))*c_G + (diff(2*h_p, x))*c_H,
(diff(a_p, y))*c_A + (diff(b_p, y))*c_B + (diff(c_p, y))*c_C
+ (diff(2*f_p, y))*c_F + (diff(2*g_p, y))*c_G
+ (diff(2*h_p, y))*c_H, (diff(a_p, z))*c_A + (diff(b_p, z))*c_B
+ (diff(c_p, z))*c_C + (diff(2*f_p, z))*c_F
+ (diff(2*g_p, z))*c_G + (diff(2*h_p, z))*c_H])]]),
method = multivar);

JacPsi := Determinant(Jacobian([F, H, Phsi], [x, y, z]),
method = multivar);

return expand((12*m^2-54*m+57)*H*JacH
           + (m-2)*(12*m-27)*H*JacU + 40*(m-2)^2*JacPsi)
end proc;
```

Maple commands

Intersectcurves

The `intersectcurves(F,G,x,y,z)` command is found in the *algcurves* package in [Maple]. It takes polynomials $F(x, y, z)$ and $G(x, y, z)$, with rational coefficients, and returns a list on the form

$$\big[[m_1, [a_1(x, y), b_1(y), c_1]], \dots, [m_n, [a_n(x, y), b_n(y), c_n]\big],$$

where $[a_i(x, y), b_i(y), c_i]$ describes the set of intersection points between the curves $C = V(F)$ and $D = V(G)$ with intersection multiplicity $m_i > 0$. The intersection points are given by three possible cases:

(i) $c_i = 1$: The y-coordinates of the intersection points are given by the roots β of the irreducible polynomial $b_i(y) = 0$, and the x-coordinates are the roots of the irreducible polynomial $a_i(x, \beta) = 0$.

(ii) $b_i = 1, c_i = 0$: The x-coordinates of the intersection points are given by the irreducibe polynomial $a_i(x) = 0$.

(iii) $b_i = c_i = 0$: The point $(1 : 0 : 0)$ is the only intersection point with intersection multiplicity m_i.

Singularities

The `singularities(F,x,y,z)` command is part of the *algcurves* package in [Maple]. It takes a square free polynomial $F(x, y, z)$ and returns a list

$$\big\{[p_1, m_1, \delta_1, r_1], \dots, [p_n, m_n, \delta_n, r_n]\big\},$$

where p_i is a singular point on $C = V(F)$ in homogeneous coordinates. The numbers m_i, δ_i and r_i denote the multiplicity of p_i, the delta invariant of p_i, and number of branches of C through p_i respectively.

Hessian

The `Hessian(F,[x,y,z], determinant)` command is part of the *VectorCalculus* subpackage of the package *Student* in [Maple]. It takes the function F and returns the Hessian matrix of F as well as the determinant of the Hessian matrix.

B.2 Macaulay2

The [Macaulay2] code for finding the defining polynomial of a rational curve, given its parametrisation $\phi = (f(s,t) : g(s,t) : h(s,t))$:

Program B.5: Macaulay2 code

```
i1 : R = QQ[s,t];
i2 : S = QQ[x,y,z];
i3 : phi =  map(R,S,{f,g,h});
i4 : ker phi
```

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