UiO : Department of Mathematics
University of Oslo

The primitive ideal space of Deaconu-Renault groupoid C^*-algebras coming from $\mathbb{N}_0^k$-actions

Lars Andreas van Woensel Kooy Tveiten

Master's Thesis, Spring 2017

This master's thesis is submitted under the master's programme *Mathematics*, with programme option *Mathematics*, at the Department of Mathematics, University of Oslo. The scope of the thesis is 60 credits.

The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

The main goal of this thesis is to reproduce the article [SW16] by Sims and Williams, categorising the primitive ideal space of the class of Deaconu-Renault groupoid C^*-algebras generated by $\mathbb{N}_0^k$-actions. We go through the required groupoid prerequisites, from the definition of a groupoid to the construction of covariance C^*-algebras. We apply what we learn from [SW16] to study simplicity of the covariance C^*-algebras, and give an explicit application to the rotation algebra.

Contents

Acknowledgements

I would like to thank my supervisor, Adam Sørensen, for all his advice and guidance.
A thank is also in order to my partners in crime, Sissel and Paul (we finally made it!) – and all the friends and family who have kept my spirits up.

CHAPTER 1

Introduction

A dynamical system is, intuitively, a space in which the position of each point changes over time, and this change can be described by a function. There are numerous mathematical formulations of this umbrella term, and their applications extend widely. Chaos theory, fluid mechanics and statistical mechanics are a few examples of fields that have benefited from, or indeed have their foundations built on, the theory of dynamical systems. There are also real-life applications to engineering, biology and medicine (among others). Needless to say, dynamical systems are important. In this thesis, we will study dynamical systems by looking at C^*-algebras associated to groupoids. Explicitly, dynamical systems coming from $\mathbb{N}_0^k$-actions of commuting local homeomorphisms on a locally compact Hausdorff space. Let (X, T) denote such a dynamical system, where T is a function describing the position of points in the space X over time (as in our intuitive definition above). Then one can associate to (X, T) a groupoid G_T, called a *Deaconu-Renault groupoid*, and thereafter a C^*-algebra $C^*(G_T)$. The C^*-algebra $C^*(G_T)$ is a much larger object than the original dynamical system. Indeed, the groupoid G_T itself can be viewed as mashing the space X and the function (or action) T together to form a single object containing all information of both X and T. Afterwards, we form $C^*(G_T)$, which contains among other things the set of certain continuous functions on G_T. The fact that $C^*(G_T)$ is larger than the original dynamical system has both positive and negative aspects. On one hand, one loses some intuition and "hands-on" properties of the system. On the other hand, the added structure gives us more to work with, and in particular, it enables us to apply the machinery that is developed in both groupoid and C^*-algebra theory.

To get some hands-on experience with dynamical systems, we go through part of the paper [Pol12] in Appendix A. That article is about the growth rate of fixed points of hyperbolic toral automorphisms. The dynamical systems built out of commuting toral automorphisms are closely connected to Deaconu-Renault groupoids.

We build the theory of groupoid C^*-algebras by following the first few chapters of [Pat99]. The construction is very similar to the construction of group C^*-algebras; their difference lies in generalising known notions of the group case (such as locally compactness, the Haar measure, representations and so on) to the groupoid case. After these basics are completed, we move on to the so-called *covariance C^*-algebras*, which are basically a way to characterise groupoid C^*-algebras. The construction of these are more similar to that of crossed products, of which the groupoid C^*-algebras are also a generalisation. About half the thesis is devoted to reproducing [SW16]. The goal is to characterise the primitive ideal space of Deaconu-Renault groupoid C^*-algebras. We first do this in the easier case of when T is an irreducible action, and then generalise to non-irreducible actions. Finally, we apply the result from [SW16] to say something about simplicity for Deaconu-Renault C^*-algebras. As an application of what we arrive at, we give an alternative proof that the rotation algebra A_θ is simple if and only if θ is irrational.

The theory in [Pat99] is very general: Nice properties such as étale and Hausdorff are mostly not assumed. Deaconu-Renault groupoids are both Hausdorff and étale, so I have spent some time deciphering the general results of [Pat99] to the easier case. This has not always been successful, so some of the proofs may be more technical than needed. We also go quite a lot into the details in our proofs. In particular, [SW16] sketches (or skips entirely) parts of the proofs that are clear to someone who is already well versed with this field of research. I have spent quite a lot of time writing those arguments out fully.

CHAPTER 2

Preliminaries

General Topology

As one might expect, we will need various results from general topology throughout the thesis. This subsection will be the point of reference for those results.

Definition 2.0.1. A function $f\colon X \to Y$ between topological spaces is called *proper* if, for every compact set $K \subseteq Y$, the preimage $f^{-1}(K)$ is compact.

Lemma 2.0.2. *Let $f\colon X \to Y$ be a bijective function between topological spaces X and Y. If for all $S \subseteq X$ we have*

$$x \in \overline{S} \iff f(x) \in \overline{f(S)},$$

then f is a homeomorphism.

Proof. If $S \subseteq X$ is closed, then $x \in S$ if and only if $f(x) \in \overline{f(S)}$, so $f(S)$ is closed. If S is not closed, then $f(S) \neq \overline{f(S)}$ so $f(S)$ is not closed. Hence S is closed if and only if $f(S)$ is closed, and f is a homeomorphism. ■

Lemma 2.0.3. *Let $(a_\lambda)_{\lambda\in\Lambda}$ be a net in a topological space A, and let $a \in A$. Then $\lim_{\lambda\to\infty} a_\lambda = a$ if every subnet of $(a_\lambda)_\lambda$ has a subnet converging to a.*

Proof. Suppose (contrapositively) that $a_\lambda \not\to a$. Then for every neighbourhood U of b and every $\lambda \in \Lambda$, there is some $\beta_{U,\lambda} \geq \lambda$ with $a_{\beta_{U,\lambda}} \notin U$. Let U vary over all neighbourhoods of b, and let λ vary over Λ. Define a partial order by setting $\beta_{U,\alpha} \preceq \beta_{U',\alpha'}$ if $\alpha \leq \alpha'$ and $U' \subseteq U$. Then $(a_{\beta_{U,\lambda}})$ is a subnet of (a_λ) with no subnet converging to b. ■

Proposition 2.0.4. *If $f\colon K \to B$ is a continuous bijection, where K is compact and B is Hausdorff, then f is a homeomorphism.*

Lemma 2.0.5. *Let X be a topological space, $Y \subseteq X$ a subspace and $\chi_Y\colon X \to \{0, 1\}$ the indicator function of Y. Then χ_Y is continuous if and only if Y is closed in X as well as open.*

Definition 2.0.6. A topological space X is called a *Baire space* if, given a countable collection of open dense subsets $\{U_n\}_{n\in\mathbb{N}}$, their intersection $\bigcap_{n\in\mathbb{N}} U_n$ is also dense. If every locally closed subset of X has the Baire property, X is called *totally Baire.* (A *locally closed subset* of X is any intersection $C \cap U$ where C is closed and U is open.)

Proposition 2.0.7. *If X is a totally Baire space, Y is a topological space and $p\colon X \to Y$ is an open continuous map, then $p(X)$ is totally Baire.*

Theorem 2.0.8 (Baire category theorem)**.** *Every locally compact Hausdorff space is Baire.*

Remark 2.0.9. If X is locally compact Hausdorff, then every closed subspace $C \subseteq X$is also locally compact Hausdorff, and therefore Baire. If $U \subseteq X$ is open, then $C \cap U$ is open in U. Since open subsets of Baire spaces are Baire, $C \cap U$ is Baire. Hence X is totally Baire.

Group Theory and Algebra

In this thesis, we generalise the notion of a group C^*-algebra to a groupoid C^*-algebra. It is therefore appropriate to recall some basic definitions in group theory.

Definition 2.0.10. A *group* is a set G together with an associative binary operation $G \times G \to G$, denoted $(g, h) \mapsto gh$ for $g, h \in G$, such that the following hold:

(i) G has a unit e, such that $ge = eg = g$ for all $g \in G$, and

(ii) every element g has an inverse g^{-1} such that $gg^{-1} = g^{-1}g = e$.

If we skip requirement (ii), G is called a *monoid.* If G is a group in which the multiplication map and inversion map are continuous, G is called a *topological group.* Similarly, if G is a monoid in which the multiplication map is continuous, it is called a *topological monoid.*

For example, $\mathbb{Z}$ is a groupoid (and a monoid) under addition. The subset $\mathbb{N}_0 \subseteq \mathbb{Z}$ is a monoid, but not a group. The set $\mathbb{N}$ (without the unit 0) is neither a group nor a monoid under addition. Any group (monoid) can be looked upon as a topological group when endowed with the discrete topology. The set $\mathbb{R}$ under addition and with the standard topology is an example of a topological group that is not discrete.

The notion of an *action* will be central when working with Deaconu-Renault groupoids. The definition of group actions is given below, and we will generalise it to groupoid actions in a later chapter.

Definition 2.0.11. Let G be a group (monoid) with unit e and let X be any set. A *left group (monoid) action* of G on X is a function $G \times X \to X$, denoted $(g, x) \mapsto g \cdot x$ for $g \in G, x \in X$, such that

(i) $e \cdot x = x$ for all $x \in X$, and

(ii) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$ and $g \in G$.

The action is called

- *free* if $g \cdot x = x$ for some $x \in X$ and $g \in G$, then $g = e$.

If G is a topological group (monoid), then the action is called

- *proper* if the map $G \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is proper, and
- *strongly continuous* if, for all $g \in G$, the map $(g, x) \mapsto g \cdot x$ is continuous.

Definition 2.0.12. Suppose $(g, x) \mapsto g \cdot x$ is a left group (monoid) action of G on a set X. A subset $A \subseteq X$ is called *G-invariant* if $G \cdot A \subseteq A$, and it is called *G-irreducible* if it cannot be written as the union of two proper closed invariant subsets.

Definition 2.0.13. Suppose R is a ring (i.e. a group with a distributive multiplication operator) with multiplicative identity 1. A *left R-module* is an abelian group $(M, +)$ with a multiplication operator $\cdot \colon R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$, we have

(i) $r \cdot (x + y) = r \cdot x + r \cdot y$,

(ii) $(r + s) \cdot x = r \cdot x + s \cdot x$,

(iii) $(rs) \cdot x = r \cdot (s \cdot x)$, and

(iv) $1 \cdot x = x$.

C^*-algebras

The reader is assumed to have basic knowledge of C^*-algebras: Properties of positive elements, representation theory, characters and so on. It will be an advantage to be familiar with the construction of group C^*-algebras and crossed-product C^*-algebras. We also assume knowledge in various themes in functional analysis, such as the Radon-Nikodym derivative and Riesz' representation theorem. Certain additional C^*-algebra related notions are needed, in particular that of a primitive ideal.

Definition 2.0.14. Let A be a C^*-algebra. A *primitive ideal* in A is the kernel of any irreducible representation of A. We let $\mathrm{Prim}(A)$ denote the set of primitive ideals in A.

Definition 2.0.15. Let A be a C^*-algebra. If the ideal $\{0\}$ is primitive, then A is called *primitive*. This is the same as saying that A has a faithful irreducible representation.

Definition 2.0.16. Let A be a C^*-algebra with associated primitive ideal space $\mathrm{Prim}(A)$. Let $\mathcal{J}$ be a collection of ideals i A. The *kernel* of $\mathcal{J}$ is the ideal

$$\ker \mathcal{J} := \bigcap_{J \in \mathcal{J}} J.$$

The *hull* of an ideal $J \subseteq A$ is the set

$$\mathrm{hull}(J) := \{I \in \mathrm{Prim}(A) : J \subseteq I\}.$$

The *hull-kernel topology* on $\mathrm{Prim}(A)$ is the topology in which the closure of a set $\mathcal{J} \subseteq \mathrm{Prim}(A)$ is $\overline{\mathcal{J}} = \mathrm{hull}(\ker(\mathcal{J}))$.

Another notion we will need is that of a hereditary subalgebra of a C^*-algebra.

Definition 2.0.17. Let A be a C^*-algebra. A C^*-subalgebra $B \subseteq A$ is called *hereditary* if, given two positive elements $a \in A$ and $b \in B$, the inequality $a \leq b$ implies $a \in B$.

We have a nice characterisation of hereditary C^*-subalgebras found in [Mur90, Theorem 3.2.2].

Theorem 2.0.18. *Let B be a C^*-subalgebra of a C^*-algebra A. Then B is hereditary if and only if $bab' \in B$ for all $b, b' \in B$ and $a \in A$.*

Definition 2.0.19. Suppose J, A and B are C^*-algebras. We say that the sequence

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{\pi} B \longrightarrow 0$$

is a *short exact sequence* if $j\colon J \to A$ is an injective $*$-homomorphism, $\pi\colon A \to B$ is a surjective $*$-homomorphism, and that the image of j equals the kernel of π.

Remark 2.0.20. Let A, B and C be C^*-algebras, and let $\varphi\colon A \to B$ and $\psi\colon A \to C$ be linear functions (homomorphisms) with $\ker\psi \subseteq \ker\varphi$. Then there is an induced linear function (homomorphism) $\gamma\colon C \to B$ (defined by $c \mapsto \varphi(\psi^{-1}(c))$) such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \\ C \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Indeed, to see that γ is well-defined, we must show that elements $a, a' \in A$ with $\psi(a) = \psi(a')$ also has $\varphi(a) = \varphi(a')$. Since ψ and φ are linear, this is the same as saying that $a - a' \in \ker\psi \implies a - a' \in \ker\varphi$, which we know is the case.

CHAPTER 3

Introduction to Groupoids

3.1 Definition and Basic Properties

This thesis is about groupoid C^*-algebras, so we will of course need to know what a groupoid is and what their basic properties are. Our construction of the theory follows that of [Pat99], but with certain additions having [SW16] in mind.

In essence, a groupoid is just like a group, except that it is not necessarily possible to multiply two arbitrary elements. This will imply, for instance, that groupoids have several unit elements. In fact, a groupoid is a group if and only if it has precisely one unit. In many ways, the increased complexity we gain from multiple unit elements is mirrored in the ensuing theory. For instance, while we represent groups on unitary operators on some Hilbert space, we will need one Hilbert space per unit to represent a groupoid. But we are getting ahead of ourselves; let us first give a proper definition of groupoids.

Definition 3.1.1. A *groupoid* is a set G together with a subset $G^2 \subseteq G \times G$ of *composable pairs*, a product map $(a, b) \mapsto ab$ from G^2 to G, and an inverse map $a \mapsto a^{-1}$ from G to itself such that

(i) $(a^{-1})^{-1} = a$,

(ii) if $(a, b), (b, c) \in G^2$, then $(ab, c), (a, bc) \in G^2$ and

$$(ab)c = a(bc), \tag{3.1}$$

(iii) $(b, b^{-1}) \in G^2$ for all $b \in G$, and

(iv) if $(a, b) \in G^2$ then

$$a^{-1}(ab) = b \text{ and } (ab)b^{-1} = a. \tag{3.2}$$

We will denote the groupoid simply by G. If G is equipped with a topology for which products and inversion are continuous, we call it a *topological groupoid.*

In Proposition 3.1.7, we reformulate the definition of a groupoid in terms of categories, which in many ways is more intuitive. The reader will not necessarily benefit from scrutinising the above definition too much before reading that reformulation. Before we state it, however, it will be benefitial to introduce some more basic concepts. For instance, every groupoid is tightly connected to its range and domain maps, together with its its unit space.

Definition 3.1.2. Let G be a groupoid. Then the maps $d, r\colon G \to G$ defined by

$$\begin{aligned} d(x) &\coloneqq x^{-1}x, \\ r(x) &\coloneqq xx^{-1} \end{aligned}$$

are called the *domain map* (sometimes called the *source map*) and *range map* of G, respectively. For any subset $A \subseteq G$, we let r_A and d_A denote their restrictions to A, respectively. Their (common) image $r(G) = d(G)$ is called the *unit space* of G and is denoted G^0. The elements of G^0 are called *units*. For every unit $u \in G^0$, we denote

$$\begin{aligned} G_u &\coloneqq d^{-1}(u), \\ G^u &\coloneqq r^{-1}(u), \text{ and} \\ G_u^u &\coloneqq G^u \cap G_u. \end{aligned}$$

The set G_u^u is in fact a group, and is called the *isotropy group* at u.

Groupoids "stick together" to form new groupoids in a very natural way. Spesifically, every disjoint union of groupoids is a groupoid with the natural inherited maps. We give an example of this which we will be interested in later on.

Example 3.1.3. Let G be a groupoid. We define the *isotropy subgroupoid* of G as

$$\mathrm{Iso}(G) = \{x \in G : r(x) = d(x)\} = \bigcup_{u \in G^0} G_u^u.$$

This is a disjoint union of groups in G, and is therefore a subgroupoid of G.

Another useful subgroupoid of a groupoid is obtained by removing elements connected to certain units.

Example 3.1.4. Let G be a groupoid, and suppose $K \subseteq G^0$. We can restrict G to a subgroupoid $G|_K$, defined by

$$G|_K \coloneqq \{x \in G : r(x), d(x) \in K\}.$$

As mentioned, one can also define a groupoid (quite elegantly) in terms of categories, making certain parts of the theory more straightforward.

Definition 3.1.5. A *small category* $\mathcal{C}$ consists of

(i) a **set** of *objects*, $\mathrm{ob}(\mathcal{C})$;

(ii) a **set** of *morphisms* or *arrows*, $\hom(\mathcal{C})$, such that if $f \in \hom(\mathcal{C})$ then f has a *source object* and *range object* in $\mathrm{ob}(\mathcal{C})$. We let $\hom(a, b)$ be the set of morphisms with source a and range b; and

(iii) for every three objects a, b and c, there is a binary operation from $\circ\colon \hom(a, b) \times \hom(b, c)$ to $\hom(a, c)$.

Furthermore, we require that the following holds:

(iv) the binary operation above is associative, and

(iiv) for every $x \in \mathrm{ob}(\mathcal{C})$, there exists an *identity morphism* $id_x \in \hom(x,x)$, such that for every morphism $f \in \hom(a,x)$ and $g \in \hom(x,b)$ for $a,b \in \mathrm{ob}(\mathcal{C})$, we have

$$id_x \circ f = f \text{ and } g \circ id_x = g.$$

If $f \in \hom(a,b)$ for some $a,b \in \mathrm{ob}(\mathcal{C})$, then $g \in \hom(b,a)$ is the *inverse* of f if $f \circ g = id_a$.

Proposition 3.1.6. *A set G is a groupoid if and only if has the structure of a small gategory where every morphism has an inverse.*

Proof. We prove one direction; the construction should be clear enough to make the proof of the converse redundant. Suppose therefore that G is a groupoid. Define $\mathrm{ob}\, G \coloneqq G^0$, and identify $\hom G$ with G in the natural way, in the sense that $x \in G$ is a morphism from $d(x)$ to $r(x)$. Thus we may also identify $\hom(u,v)$ with $G_u \cap G^v$ for $u,v \in \mathrm{ob}\, G$. For $u,v,w \in \mathrm{ob}\, G$, we let the required binary operation $\hom(u,v) \times \hom(v,w) \to \hom(u,w)$ be the groupoid multiplication. By part (ii) of Definition 3.1.1, this operation is associative and is defined on the entire set $\hom(u,v) \times \hom(v,w)$. Parts (iii) and (iv) ensures that every $x \in \mathrm{ob}\, G$ has a left identity $r(x)$ and a right identity $d(x)$. Thus G has the structure of a small category, and parts (i) and (iii) of Definition 3.1.1 makes sure that every morphism has an inverse. ∎

The advantage of the category approach to groupoids is that many basic facts become more intuitive. The following properties will be seen as obvious when thinking of a groupoid as a category, while using the set theoretic definition requires nontrivial proofs. As illustration, we have included proofs of the first statement from both viewpoints.

Proposition 3.1.7. *Let G be a groupoid, and $x,y,z \in G$. Then the following statements hold:*

(i) $(x,y) \in G^2$ *if and only if* $d(x) = r(y)$*,*

(ii) $xd(x) = x = r(x)x$*,*

(iii) $d(x^{-1}) = r(x)$ *and* $r(x^{-1}) = d(x)$*,*

(iv) $x^{-1} = x$ *if x is a unit,*

(v) if both x and y are units and $(x,y) \in G^2$, then $x = y$,

(vi) $x^2 = x$ *if and only if x is a unit, and*

(vii) if $z = xy$, then $(y^{-1}, x^{-1}) \in G^2$, $z^{-1} = y^{-1}x^{-1}$, $r(x) = r(z)$ and $d(y) = d(z)$.

Proof. We will prove (i). Suppose first that $(x,y) \in G^2$. By (iii) of Definition 3.1.1, we have $x^{-1}(xy) = y$. By (ii) of the definition, we have $(x^{-1}(xy))y^{-1} = yy^{-1} = r(y)$. Similarly, we have $x^{-1}((xy)y^{-1}) = x^{-1}x = d(x)$. By (i) of the definition, we may change the placement of the parentheses for these expressions and write

$$r(y) = (x^{-1}(xy))y^{-1} = (x^{-1}x)(yy^{-1}) = x^{-1}((xy)y^{-1}) = d(x),$$

as we wanted.

Suppose next that $r(y) = d(x)$. By (ii), we have $(x, x^{-1}), (x^{-1}, x) \in G^2$. Thus $(x, x^{-1}x) \in G^2$ by (i). But then $(x, yy^{-1}) \in G^2$ by assumption. We have $(yy^{-1}, y) \in G^2$, so by (i) we have $(x, yy^{-1}y) \in G^2$. We have that $yy^{-1}y = y$ by (iii), and hence $(x, y) \in G^2$.

To prove (i) using the category approach, just note that $(x, y) \in G^2$ if and only if the (category) range of y is equal to the (category) source of x, in other words $r(y) = d(x)$. ■

3.2 Locally Compact Groupoids

When constructing a group C^*-algebra from a group G, one requires G to have a locally compact topology; then one defines $C^*(G)$ as the completion of $C_c(G)$. The same is the case for groupoid C^*-algebras. For non-Hausdorff groupoids, the definition of *locally compact groupoid* differs from the purely topological one. We will also require our locally compact groupoids to have a left Haar system, which is a generalisation of the Haar measure.

Local Compactness

Definition 3.2.1. A *locally compact groupoid* is a topological groupoid G such that

(i) G^0 is locally compact and Hausdorff in the relative topology from G,

(ii) there is a countable family $\mathcal{C}$ of subsets of G which are compact and Hausdorff, and whose interiors $\{C^\circ : C \in \mathcal{C}\}$ form a basis for the topology on G,

(iii) the sets G^u are locally compact and Hausdorff in the relative topology from G for every u, and

(iv) G admits a left Haar system $\{\lambda^u\}$ as defined below.

We will only be interested in locally compact groupoids which are Hausdorff. Under this assumption, note that part (ii) in the above definition reduces to the existence of a countable basis of relatively compact sets. In other words, condition (ii) means that G is a second-countable, locally compact set. Note also that parts (i) and (iii) then becomes implicit. We summarise this in the following remark.

Remark 3.2.2. A Hausdorff locally compact groupoid is a topological groupoid, which is second countable and locally compact in the usual sense, and which admits a left Haar system.

In the non-Hausdorff cases of locally compact groupoids, the usual definition of $C_c(G)$ as the set of complex-valued, compactly supported continuous functions on G will no longer be adequate. The space will simply not be large enough to have the desired properties. Instead, for a general locally compact groupoid G, one defines $\hat{C}_c(G)$ to be the span of those complex-valued functions f that are continuous with compact support on an open Hausdorff subset, each of the functions f being defined to be zero outside of that set. This set will take the place of $C_c(G)$. In the Hausdorff case, however, $\hat{C}_c(G) = C_c(G)$, as we prove below. The requirement of a left Haar system is irrelevant in the following proposition.

Proposition 3.2.3. *Suppose G is a Hausdorff locally compact groupoid. Then $\hat{C}_c(G) = C_c(G)$.*

Proof. We clearly have $C_c(G) \subseteq \hat{C}_c(G)$. To prove the other inclusion, suppose first that $f \in \hat{C}_c(G)$ has compact support in the open Hausdorff subset $U \subseteq G$, with f zero outside U. Since U is open, we have that open covers of $\operatorname{supp} f$ in U corresponds to open covers in G. Thus f has compact support. To show that

f is continuous, let $V \subseteq \mathbb{C}$ be open. If $0 \notin V$, then $f^{-1}(V)$ is open in U and hence in G. If $0 \in V$, then

$$f^{-1}(V) = f^{-1}(V \setminus \{0\}) \cup K^c.$$

The set $V \setminus \{0\}$ is open in $\mathbb{C}$, so the first part of the above union is open in G. Since G is Hausdorff and K is compact, K is closed, so K^c is open. Thus $f^{-1}(V)$ is open and f is continuous.

Next, let $f \in \hat{C}_c(G)$ be any element, so that $f = \sum_{i=1}^n \alpha_i f_i$ for nonzero constants $\{\alpha_i\}_{i=1}^n \subseteq \mathbb{C} \subseteq \{0\}$, and each f_i has compact support K_i. Then $\operatorname{supp} f \subseteq \cup_{i=1}^n K_i$, which is compact – hence $\operatorname{supp} f$ is compact. Furthermore, f is the composition of continuous functions

$$x \mapsto (f_1(x), \ldots, f_n(x)) \mapsto \sum_{i=1}^n \alpha_i f_i(x),$$

so it is continuous with compact support. Thus $\hat{C}_c(G) \subseteq C_c(G)$. ■

From here on we will assume that all groupoids are Hausdorff. For the definition of left Haar systems, recall that if μ is a measure on the measure space (X, τ), then we define $\operatorname{supp}(\mu)$ to be the set of all elements $x \in X$ which have the property that if U is an open neighbourhood of x, then $\mu(U) > 0$.

Definition 3.2.4. A *left Haar system* for a locally compact groupoid G is a family $\{\lambda^u : u \in G^0\}$ where each λ^u is a positive regular Borel measure on the locally compact Hausdorff space G^u, such that the following holds:

(i) the support of λ^u is precisely G^u,

(ii) for every $g \in C_c(G)$ we have $g^0 \in C_c(G^0)$, where

$$g^0(u) = \int_{G^u} g d\lambda^u,$$

and

(iii) for every $x \in G$ and $f \in C_c(G)$, we have

$$\int_{G^{d(x)}} f(xz) d\lambda^{d(x)}(z) = \int_{G^{r(x)}} f(y) d\lambda^{r(x)}(y).$$

A left Haar system gives us measures on the sets G^u. However, to each set G_u, there is also a canonical positive regular Borel measure λ_u.

Proposition 3.2.5. *If $\{\lambda^u : u \in G^0\}$ is a left Haar system for a locally compact groupoid G, then for each λ^u we can associate a positive regular Borel measure λ_u defined by*

$$\lambda_u(E) := \lambda^u(E^{-1})$$

for all Borel subsets $E \subseteq G_u$.

Proof. For the definition of λ_u to make sense, we must have that E^{-1} is a Borel subset of G^u whenever E is a Borel subset of G_u. This is indeed the case; suppose $E \subseteq G_u$ is Borel. Since $G_u^{-1} = G^u$, we have $E^{-1} \subseteq G^u$, and since $x \mapsto x^{-1}$ is a homeomorphism, E^{-1} is indeed a Borel set.

To see that λ_u is regular, let $E \subseteq G_u$ be Borel. Since λ^u is regular, there is a sequence $\{(O_n, K_n)\}_{n=1}^{\infty}$ of pairs of sets such that O_n and K_n are Borel subsets of G^u, O_n is open and K_n is compact with $K_n \subseteq E^{-1} \subseteq O_n$, and

$$\lim_{n\to\infty} \lambda^u(O_n \setminus K_n) = 0.$$

Since inversion is a homeomorphism, O_n^{-1} is open and K_n^{-1} is compact with $K_n^{-1} \subseteq E \subseteq O_n^{-1}$, and we have

$$\lim_{n\to\infty} \lambda_u(O_n^{-1} \setminus K_n^{-1}) = \lim_{n\to\infty} \lambda^u(O_n \setminus K_n) = 0.$$

Thus λ_u is regular. ∎

The study of general locally compact groupoids is of course very interesting, but for understanding [SW16], it will suffice to restrict to the so-called étale case. We will define this below. Recall that if G is a locally compact groupoid and $A \subseteq G$ is any subset, then we let r_A, d_A denote the restrictions of r and d to A, respectively.

Definition 3.2.6. Suppose G is a locally compact groupoid, and let G^{op} denote the family of open subsets A of G such that r_A and d_A are homeomorphisms onto open subsets of G. The groupoid G is called *étale* (sometimes referred to as *r-discrete*) if G^{op} is a basis for the topology on G.

When stating that some groupoid G is étale, we will implicitly mean that G is a locally compact groupoid. If G is étale, then every r-fibre G^u for $u \in G^0$ is discrete (hence the terminology r-discrete). Indeed, suppose $x \in G^u$. Sice G^{op} is a basis, there is some $U \in G^{\mathrm{op}}$ with $x \in U$. The set $U \cap G^u$ is open in the subspace topology of G^u, but since r is injective on U, we have $U \cap G^u = \{x\}$. Thus every subset of G^u is open.

Note also that the multiplication map and the maps r and d are open for étale groupoids. The fact that r and d are open follows from the fact that the image of every basis element (in G^{op}) is open; the prior requires some more argumentation.

Proposition 3.2.7. *If G is étale, then the multiplication map on G is open.*

Proof. Recall that r and d are open maps when G is étale. Let A and B be open subsets of G, and suppose $(a, b) \in A \times B$ is a composable pair. (If no such pair exists, then $AB = \emptyset$ is open.) Suppose further that $\{x_\lambda\}_{\lambda \in \Lambda}$ is a net converging to ab. Then $r(x_\lambda) \to r(ab) = r(a)$, and since r is an open map, $r(x_\lambda)$ is eventually in $r(A)$. Suppose therefore that we eventually have $r(x_\lambda) = r(a_\lambda)$ for $a_\lambda \in A$. In fact, since G^{op} is a basis for the topology on G, we may assume also that a_λ is eventually in some fixed basis element $U \subseteq A$. Since r is a homeomorphism on U, we have $a_\lambda \to a$. Thus $a_\lambda^{-1} x_\lambda \to a^{-1}ab = b$. Since B is open, $a_\lambda^{-1} x_\lambda$ eventually belongs to B, so $x_\lambda = a_\lambda(a_\lambda^{-1} x_\lambda)$ eventually belongs to AB. Thus AB is open. ∎

Remark 3.2.8. If G is étale, then G^0 is closed as well as open. Indeed, $G^0 \in G^{\mathrm{op}}$, so it is clearly open. In fact, one can take this to be the definition of an étale groupoid, and this is the approach taken by [Ren80]. To see that it is closed, it suffices to observe that $G^0 = r(G)$ is precisely the set of fixed points for the continuous function r. The set of fixed points of a continuous function on a Hausdorff space is always closed.

Quite a few things are simplified when restricting to the étale case. For instance, one can relatively easily characterise the left Haar systems of such a groupoid.

Proposition 3.2.9. *Let G be an étale groupoid, and set $P_+(G)$ to be the set of continuous functions $\alpha\colon G^0 \to (0,\infty)$. Every $\alpha \in P_+(G)$ defines a left Haar system $\{\Gamma^u(\alpha)\}$ where for each $u \in G^0$,*

$$\Gamma^u(\alpha) = \sum_{x\in G^u} \alpha(d(x))\delta_x.$$

Conversely, every left Haar system $\{\lambda^u\}$ is of the form $\{\Gamma^u(\alpha)\}$ for some $\alpha \in P_+(G)$.

Proof. First, let $\alpha \in P_+(G)$ and $\lambda^u = \Gamma^u(\alpha)$. We must check the properties of Definition 3.2.4. We start with (i). If $U \subseteq G^u$ is open, then $(\Gamma^u(\alpha))(U) > 0$ since α is strictly positive. Thus $G^u \subseteq \operatorname{supp}(\Gamma^u(\alpha))$. Similarly, $(\Gamma^u(\alpha))(V) = 0$ if $V \cap G^u = \emptyset$ and V is measurable, since $\Gamma^u(\alpha)$ is a sum of Dirac measures of elements in G^u. Thus $\operatorname{supp}(\Gamma^u(\alpha)) = G^u$.

For (ii), let $g \in C_c(G)$. Since G^{op} and $\{C^\circ : c \in \mathcal{C}\}$ from part (ii) of Definition 3.2.1 are bases for the topology on G, the family $\{V \cap C^\circ : V \in G^{\mathrm{op}}, C \in \mathcal{C}\}$ is an open cover of $\operatorname{supp} g$, which has a finite subcover $\{A_i\}_{i=1}^n$. (Note that each A_i has compact closure and is in G^{op}.) Let $\{\varphi_i\}_{i=1}^n$ be a partition of unity for the cover; such a partition of unity exists since G is locally compact and $\operatorname{supp} g$ is compact. Then each $f_i \coloneqq \varphi_i g$ has compact support inside A_i for $i = 1, \dots, n$. We will prove that $f_i^0 \in C_c(G^0)$ for all i, and hence their sum $\sum_{i=1}^n (\varphi_i g)^0 = g^0$ is. For any $u \in G^0$, we have

$$f_i^0(u) = \int_{G^u} f_i \, d\lambda^u = \sum_{x\in G^u} f_i(x)\alpha(d(x)).$$

However, f_i is zero outside A_i, so the only term in the above sum that is potentially nonzero, is the one coming from the point $r_i^{-1}(u) \in G^u$, where r_i is the range map restricted to A_i. Thus

$$f_i^0(u) = f_i(r_i^{-1}(u))\alpha(d(r_i^{-1}(u))).$$

This is a composition of continuous functions, so $f_i^0 \in C(G^0)$. Furthermore, f_i^0 is nonzero only inside $r(A_i) \subseteq r(\overline{A_i})$. We have that $r(\overline{A_i})$ is compact (hence closed) since $\overline{A_i}$ is compact and r is continuous, so $\operatorname{supp} f_i^0$ is a closed subset of a compact set and therefore compact.

For (iii), let $x \in G$ and $f \in C_c(G)$. We want

$$\int_{G^{d(x)}} f(xz) d\lambda^{d(x)}(z) = \int_{G^{r(x)}} f(y) d\lambda^{r(x)}(y).$$

By definition, we have

$$\int_{G^{r(x)}} f(y)d\lambda^{r(x)}(y) = \sum_{y\in G^{r(x)}} f(y)\alpha(d(y));$$

on the other hand, letting $y = xz$ yields

$$\begin{aligned}\int_{G^{d(x)}} f(xz)d\lambda^{d(x)}(z) &= \sum_{z\in G^{d(x)}} f(xz)\alpha(d(z)) \\ &= \sum_{z\in G^{d(x)}} f(y)\alpha(d(y)) \\ &= \sum_{y\in G^{r(x)}} f(y)\alpha(d(y)),\end{aligned}$$

which is precisely what we wanted. It might seem as if we sum over more elements in the last sum than in the previous one. This is not the case. Indeed, if $y \in G^{r(x)}$, then $z \in G^{d(x)}$ defined by $z = x^{-1}y$ is such that $y = xz$, so every term in the last sum occurs in the prior.

Conversely, let $\{\lambda^u\}$ be a left Haar system for G and let $u \in G^0$. We first prove that every point $x \in G^u$ has strictly positive λ^u-measure. Indeed, since $\operatorname{supp}\lambda^u = G^u$, every open neighbourhood of x has positive λ^u-measure, and every measurable set outside G^u has λ^u-measure zero. Since G^u is discrete in the subspace topology, there is some neighborhood U of x with $U \cap G^u = \{x\}$; thus $\lambda^u(\{x\}) = \lambda^u(U) > 0$.

Now we define $\alpha\colon G^0 \to \mathbb{R}$ by $\alpha = \chi^0_{G^0}$, so that

$$\alpha(u) = \int_{G^u} \chi_{G^0}\ d\lambda^u = \int_{\{u\}} 1\ d\lambda^u = \lambda^u(\{u\}).$$

This function is positive by the discussion above. We want to show that it is continuous. If $K \subseteq G^0$ is compact, then by Urysohn's lemma there exists some $\varphi_K \in C_c(G^0)$ with $\varphi_K = 1$ on K. Since G^0 is clopen by Remark 3.2.8, the function χ_{G^0} is continuous by Lemma 2.0.5, so $\varphi_k\chi_{G^0} \in C_c(G^0)$. By the definition of a left Haar system, the function

$$\alpha_K(u) \coloneqq \int_{G^u} \varphi_K\chi_{G^0}\ d\lambda^u$$

is continuous (with compact support). We have $\alpha_K(u) = \alpha(u)$ if $u \in K$. Suppose now that $(u_\lambda)_{\lambda\in\Lambda}$ is a net in G^0 converging to $u \in G^0$, and let K be a compact neighborhood of u. Since α_K is continuous, $\alpha_K(u_\lambda) \to \alpha_K(u)$. Since u_λ is eventually in K, the sequences $\alpha_K(u_\lambda)$ and $\alpha(u_\lambda)$ must eventually agree, so $\alpha(u_\lambda) \to \alpha(\lambda)$. Hence α is continuous, and we have $\alpha \in P_+(G)$. Finally, for any measurable $E \subseteq G$, we have

$$\lambda^u(E) = \sum_{x\in E\cap G^u} \lambda(\{x\}) = \sum_{x\in G^u} \alpha(x)\delta_x(E),$$

so $\lambda^u = \Gamma^u(\alpha)$.

■

Taking α to be identically equal to 1 on G^0, we obtain the left Haar system $\{\lambda^u\}$ where λ^u is the counting measure. **We will always take this to be the canonical left Haar system for an étale groupoid.**

The boldmath$*$-Algebra Structure of $C_c(G)$

We would like to construct a C^*-algebra out of $C_c(G)$. To do so, we will first see that $C_c(G)$ is a $*$-algebra, so we must first define a suitable norm, involution and multiplication (convolution). These are defined analogously to the group case.

Definition 3.2.10. For $f, g \in C_c(G)$, define their convolution product as

$$(f * g)(x) := \int_{G^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y). \tag{3.3}$$

For étale groupoids, the required discussion of involution and convolution is greatly simplified. The reason is that by the proof of Proposition 3.2.9, each $f \in C_c(G)$ is a linear combination of functions $g \in C_c(A)$ for some $A \in G^{\mathrm{op}}$. We illustrate how this works in the next proposition.

Proposition 3.2.11. *Suppose that $f \in C_c(A)$ and $g \in C_c(B)$ for $A, B \in G^{op}$, extended to be zero outside of A and B respectively. Then $f * g$ is zero outside AB, and for each $x \in AB$ there are unique $a \in A$ and $b \in B$ with $x = ab$ such that*

$$(f * g)(x) = f(a)g(b). \tag{3.4}$$

Thus the set $C_c(G)$ is closed under convolution.

Proof. Let $x \in G$; we will calculate $(f * g)(x)$. Since f is zero outside A and g is zero outside B, it suffices to integrate over the set of y's such that $y \in A$ and $y^{-1}x \in B$ – in other words $y \in G^{r(x)} \cap A \cap xB^{-1}$. (Note that this implies that $f * g$ is zero outside AB.) We calculate that

$$\begin{aligned} (f * g)(x) &= \int_{G^{r(x)}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y) \\ &= \int_{G^{r(x)} \cap A \cap xB^{-1}} f(y)g(y^{-1}x)d\lambda^{r(x)}(y). \end{aligned}$$

Since r is injective on A, the set $G^{r(x)} \cap A \cap xB^{-1}$ is either the empty set, or it consists of a single point a. Suppose the latter is the case, which it is if and only if $x \in AB$. Note that there must be a unique $b \in B$ such that $a = xb^{-1}$ – in other words, $x = ab$. Since the left Haar system on G is just counting measures, we get

$$(f * g)(x) = f(a)g(a^{-1}x) = f(a)g(b).$$

Each step in the sequence of functions

$$x = ab \mapsto (a, b) \mapsto (f(a), g(b)) \mapsto f(a)g(b)$$

is continuous, so $f * g$ is continuous; lastly, the multiplication at the end ensures that $f * g$ is compactly supported. The last assertion in the proposition follows by splitting $f, g \in C_c(G)$ into a sum of compactly supported, continuous functions on sets in G^{op}, and then using that the convolution is bilinear. ∎

Remark 3.2.12. Let $f \in C_c(G^0)$ and $g \in C_c(A)$ for some $A \in G^{\mathrm{op}}$, and let $x = r(x)x \in G^0 A = A$. Then by Proposition 3.2.11,

$$(f * g)(x) = f(r(x))g(x).$$

This identity holds for general elements $g \in C_c(G)$. In particular, for functions in $C_c(G^0)$, the convolution takes the even nicer form of pointwise multiplication.

We continue our quest of finding a suitable $*$-algebra structure on $C_c(G)$.

Proposition 3.2.13. *Let G be an étale groupoid. For each function $f\colon G \to \mathbb{C}$, define $f^*\colon G \to \mathbb{C}$ by*

$$f^*(x) = \overline{f(x^{-1})}. \tag{3.5}$$

When restricted to $C_c(G)$, the map $f \mapsto f^*$ is an involution.

Proof. We clearly have $f^{**} = f$ for all $f \in C_c(G)$, and that $f \mapsto f^*$ is linear. To see antimultiplicativity, suppose first that $f \in C_c(A)$ and $g \in C_c(B)$ for some $A, B \in G^{\mathrm{op}}$, and let $x = ab \in AB$. By Proposition 3.2.11, we see that

$$(f * g)^*(x) = \overline{(f * g)((ab)^{-1})} = \overline{f(b^{-1})g(a^{-1})} = \overline{g(a^{-1})f(b^{-1})} = (g^* * f^*)(x).$$

General antimultiplicativity then follows since the convolution is bilinear, and each function in $C_c(G)$ is a sum of functions f, g as above. Lastly, if f is in $C_c(G)$ then f^* is too. Indeed, f^* is a composition of continuous functions (inversion, f and complex conjugation), so it is continuous. Further, its support is $(\operatorname{supp} f)^{-1}$, the image of a compact set under a continuous function. ■

The only thing missing in our $*$-algebra structure is a norm. We will endow $C_c(G)$ with the so-called I-norm $\|\cdot\|_I$, which is associated with two other norms $\|\cdot\|_{I,r}$ and $\|\cdot\|_{I,d}$. The latter norms are defined by

$$\|f\|_{I,r} = \sup_{u \in G^0} \int_{G^u} |f(t)| d\lambda^u(t) \text{ and}$$
$$\|f\|_{I,d} = \sup_{u \in G^0} \int_{G_u} |f(t)| d\lambda_u(t),$$

for $f \in C_c(G)$. The I-norm is then given by

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,d}\}.$$

We have done most of the work to conclude with the below theorem, so the proof is quite short. We do not prove the separability, but refer the reader to [Pat99, p. 40].

Theorem 3.2.14. *Let G be an étale locally compact groupoid. Then $C_c(G)$ is a separable, normed $*$-algebra under convolution multiplication and the I-norm. Furthermore, the involution is isometric.*

Proof. Submultiplicativity of the I-norm follows from Cauchy-Schwarz' inequality. By this and the previous results, we know that $C_c(G)$ is a normed $*$-algebra under convolution. We now show that the involution is isometric. If $f \in C_c(G)$, then

$$\begin{aligned}
\|f\|_{I,r} &= \sup_{u \in G^0} \int_{G^u} |f(t)| d\lambda^u(t) \\
&= \sup_{u \in G^0} \int_{G_u} |f(t^{-1})| d\lambda_u(t) \\
&= \sup_{u \in G^0} \int_{G_u} |f^*(t)| d\lambda_u(t) \\
&= \|f^*\|_{I,d}.
\end{aligned}$$

Thus we also have $\|f^*\|_{I,r} = \|f^{**}\|_{I,d} = \|f\|_{I,d}$, so

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,d}\} = \max\{\|f^*\|_{I,d}, \|f^*_{I,r}\|\} = \|f^*\|_I.$$

■

3.3 Representation Theory for Locally Compact Groupoids

In order to properly define what a *representation* of a groupoid is, we must first get a few measure theoretic constructions in order. After that, it turns out that the notion of a *Hilbert bundle* has the required structure to form such a representation.

Measure Theory

Definition 3.3.1. Suppose G is a locally compact groupoid. A *positive Borel measure* on G is a $[0,\infty]$-valued measure on the Borel σ-algebra $\mathcal{B}(G)$. Regularity for positive Borel measures is defined as for the locally compact Hausdorff case.

If X is a locally compact Hausdorff space, the set of probability measures on X is denoted $P(X)$.

Proposition 3.3.2. *Suppose that G is an étale groupoid and that $\mu \in P(G^0)$. Then we will μ and the left Haar system for G determines a regular Borel positive measure ν on $\mathcal{B}(G)$, written*

$$\nu = \int_{G^0} \lambda^u d\mu(u). \tag{3.6}$$

Proof. Let G be an étale groupoid and define a linear functional φ on $C_c(G)$ by

$$\begin{aligned}\varphi(f) &= \int_{G^0}\int_{G^u} f(x)d\lambda^u(x)\ d\mu(u)\\ &= \int_{u\in G^0}\sum_{x\in G^u} f(x)\ d\mu(u),\end{aligned}$$

Since $f \in C_c(G)$, we have $|f| \in C_c(G)$ with compact support $K \subseteq G$. Then we can write

$$\begin{aligned}|\varphi(f)| &\leq \int_{G^0}\int_{G^u} |f(x)|\chi_K(x)d\lambda^u(x)d\mu(u)\\ &\leq \|f\|_\infty \int_{G^0}\int_{G^u} \chi_K(x)d\lambda^u(x)d\mu(u)\\ &= \|f\|_\infty \int_{G^0} \chi_K^0(u)d\mu(u)\\ &= \|f\|_\infty \cdot M,\end{aligned}$$

where $M = \int_{G^0} \chi_K^0(u)d\mu(u)$. Since $\chi_K^0(u) \in C_c(G^0)$, we must have that $\chi_K^0(u)$ is bounded; thus its integral must be finite since G^0 has finite μ-measure. Thus φ is continuous, and by the Riesz-Markov theorem, there is a unique regular Borel measure ν on G such that

$$\int_G f d\nu = \int_{G^0}\int_{G^u} f(x)d\lambda^u(x)d\mu(u). \tag{3.7}$$

■

We can associate to ν two regular Borel measures ν^{-1} and ν^2, the first one on G and the second on G^2. The first one is simply defined as $\nu^{-1}(W) = \nu(W^{-1})$ for all $W \in \mathcal{B}(G)$. We can reformulate this in terms of the measures λ_u in the following way: For each $f \in C_c(G)$, let $\hat{f}\colon G \to \mathbb{C}$ be defined by $\hat{f}(x) = f(x^{-1})$. Note that $\hat{f} \in C_c(G)$ if and only if $f \in C_c(G)$. Now we can write

$$\int_G f d\nu^{-1} = \int_G \hat{f} d\nu.$$

By the definition of ν and that $(\lambda^u)^{-1} = \lambda_u$, we have

$$\begin{aligned}\int_G \hat{f} d\nu &= \int_{G^0}\int_{G^u} f(x^{-1}) d\lambda^u(x) d\mu(u)\\ &= \int_{G^0}\int_{G_u} f(x) d\lambda_u(x) d\mu(u).\end{aligned}$$

This is similar to (3.7), and by notation parallel to that of (3.6) we write

$$\nu^{-1} = \int_{G^0} \lambda_u d\mu(u). \tag{3.8}$$

To define ν^2 on G^2, let $f \in C_c(G^2)$ and set

$$\nu^2(f) = \int_{G^0}\int_{G^u}\int_{G_u} f(x,y)\; d\lambda_u(x) d\lambda^u(y) d\mu(u). \tag{3.9}$$

For further discussion of this measure, see [Pat99, pp. 87–88].

Definition 3.3.3. Let $\mu \in P(G^0)$ for some locally compact groupoid G, and let ν be its associated measure. If ν is equivalent to ν^{-1} (they have the same zero sets), then μ is called *quasi-invariant*. If μ is quasi-invariant, then the Radon-Nikodym derivative

$$D = \frac{d\nu}{d\nu^{-1}}$$

is called the *modular function* of μ.

The modular function defined above is a generalisation of the modular function Δ in the group case, and our function D has similar properties. Spesifically, as noted in [Pat99, p.89], we have $D(xy) = D(x)D(y)$ for ν^2-a.e. $(x, y) \in G^2$. We prove another similar property below.

Proposition 3.3.4. *The function D from above satisfies $D(x)D(x^{-1}) = 1$ for ν-a.e. $x \in G$.*

Proof. If $g \in C_c(G)$, then for every Borel set E we have

$$\begin{aligned}\int_E g(x)\; d\nu(x) &= \int_{E^{-1}} g(x)D(x)\; d\nu^{-1}(x)\\ &= \int_E g(x^{-1})D(x^{-1})\; d\nu(x)\\ &= \int_{E^{-1}} g(x^{-1})D(x^{-1})D(x)\; d\nu^{-1}(x)\\ &= \int_E g(x)D(x)D(x^{-1})\; d\nu(x),\end{aligned}$$

so that $D(x)D(x^{-1}) = 1$ ν-a.e. as we wanted. ■

It turns out to be useful to have a measure that is equivalent to ν, but which is symmetrical with respect to inversion on G. To this end, let ν_0 be the measure on $\mathcal{B}(G)$ defined by

$$\nu_0(E) = \int_G \chi_E(x) D(x)^{-1/2} d\nu(x)$$

for all $E \in \mathcal{B}(E)$, so that $D^{-1/2} = \frac{d\nu_0}{d\nu}$.

Proposition 3.3.5. *The measure ν_0 as defined above is symmetrical with respect to inversion.*

Proof. First off, if $E \in \mathcal{B}(G)$, then

$$\begin{aligned}\nu_0(E) &= \int_G \chi_E(x) D^{-1/2}(x) d\nu(x) \\ &= \int_G \chi_E(x^{-1}) D^{-1/2}(x^{-1}) d\nu^{-1}(x) \\ &= \int_G \chi_{E^{-1}}(x) D^{1/2}(x) d\nu^{-1}(x),\end{aligned}$$

where we have used that $D(x^{-1}) = D(x)^{-1}$ ν-a.e. Since $D = \frac{d\nu}{d\nu^{-1}}$, we continue to find that

$$\begin{aligned}\nu_0(E) &= \int_G \chi_{E^{-1}}(x) D^{1/2}(x) D^{-1}(x) d\nu(x) \\ &= \int_G \chi_{E^{-1}}(x) D^{-1/2}(x) d\nu(x) \\ &= \nu_0(E^{-1}),\end{aligned}$$

just as we wanted. ■

Hilbert Bundles and Groupoid Representations

We leave the measure theory briefly to discuss Hilbert bundles. Their structure fits nicely together with the notion of a representation of a locally compact groupoid. Indeed, while we can represent a group as unitary on a single Hilbert space, the separated structure of a groupoid requires a more separated notion than a Hilbert space. We will realise the groupoid G as operators *between* Hilbert spaces in the Hilbert bundle.

Definition 3.3.6. A *Hilbert bundle* is a triple $(X, \mathcal{K}, \mu)$ where X is a second countable locally compact Hausdorff space, μ is a probability measure on X, and $\mathcal{K}$ is a collection $\{H_u : u \in X\}$ of Hilbert spaces.

Definition 3.3.7. A *section* of a Hilbert bundle $(X, \mathcal{K}, \mu)$ is a function $f\colon X \to \cup_{u\in X} H_u$ where $f(u) \in H_u$. A sequence of sections $\{f_n\}_{n\in\mathbb{N}}$ is *fundamental* if for each pair $m, n \in \mathbb{N}$, the function $u \mapsto \langle f_m(u), f_n(u)\rangle$ is μ-measurable on X, and for each $u \in X$, the set $\{f_n(u) : n \in \mathbb{N}\}$ spans a dense subspace of H_u.

Fundamental sections turn out to be helpful to the theory. **All Hilbert bundles we discuss are assumed to have a fundamental sequence.** The fundamental sequence defines a notion of *measurability* for sections.

Definition 3.3.8. Suppose $(X, \mathcal{K}, \mu)$ is a Hilbert bundle with fundamental sequence $\{f_n\}_{n\in\mathbb{N}}$. A section f on the Hilbert bundle is *measurable* if each function $u \mapsto \langle f(u), f_n(u)\rangle$ is measurable, where $u \in X$ and $n \in \mathbb{N}$.

The Hilbert space $L^2(X, \{H_u\}, \mu)$ is defined in the obvious way as the space of (equivalence classes of) measurable sections f for which the function $u \mapsto \|f(u)\|_2^2$ is μ-integrable. The norm of $f \in L^2(X, \{H_u\}, \mu)$ is given by

$$\|f\|_2 = \left[\int_X \|f(u)\|^2 d\mu(u)\right]^{\frac{1}{2}},$$

and the inner product of two sections $f, g \in L^2(X, \{H_u\}, \mu)$ is defined by

$$\langle f, g\rangle = \int_X \langle f(u), g(u)\rangle d\mu(u).$$

Now everything is in place, and we can define what a representation of a groupoid is.

Definition 3.3.9. Let G be a locally compact groupoid. A *representation* of G is a Hilbert bundle $(G^0, \{H_u\}, \mu)$, where μ is a quasi-invariant measure on G^0 (with associated measure ν), and for each $x \in G$, there is a unitary element $L(x) \in B(H_{d(x)}, H_{r(x)})$ such that:

(i) $L(u)$ is the identity map on H_u for all $u \in G^0$,

(ii) $L(x)L(y) = L(xy)$ for ν^2-a.e. $(x, y) \in G^2$,

(iii) $L(x)^{-1} = L(x^{-1})$ for ν-a.e. $x \in G$, and

(iv) for any $\xi, \eta \in L^2(G^0, \{H_u\}, \mu)$, the function

$$x \mapsto \langle L(x)\xi(d(x)), \eta(r(x))\rangle$$

is ν-measurable on G.

A representation of a locally compact groupoid G will be denoted $(\mu, \{H_u\}, L)$, or simply L if μ and $\{H_u\}$ are implicit. There are always representations of G.

Example 3.3.10. Let G be a locally compact groupoid, and let μ be any quasi-invariant measure on G^0. Define a Hilbert bundle by $G^0 \times \mathbb{C}$, so that each Hilbert space H_u is just $\mathbb{C}$. Then the *trivial representation* for μ is given by $L_{\text{triv}}(x) = id$ on $\mathbb{C} = H_{d(x)} = H_{r(x)}$.

The C^*-algebra of a Groupoid

Given a representation L of G, we can construct a representation π_L of the $*$-algebra $C_c(G)$. **We will require that all representations of $C_c(G)$ are I-norm continuous.**

Proposition 3.3.11. *Let G be a locally compact groupoid, and let $(\mu, \{H_u\}, L)$ be a representation of G. Then there is a representation $\pi_L : C_c(G) \to B(\mathcal{H})$ of $C_c(G)$ with $\|\pi_L\| \leq 1$, where $\mathcal{H} = L^2(G^0, \{H_u\}, \mu)$, given by*

$$\langle \pi_L(f)\xi, \eta\rangle = \int_G f(x)\langle L(x)(\xi(d(x))), \eta(r(x))\rangle d\nu_0(x). \tag{3.10}$$

Explicitly, we have

$$\pi_L(f)\xi(u) = \int_{G^u} f(x)L(x)(\xi(d(x)))D^{-1/2}(x)d\lambda^u(x). \tag{3.11}$$

Proof. The explicit formula (3.11) follows from (3.10) by writing it out, passing the integration through the inner product and using Riesz' representation theorem. The full argument can be found in [Pat99, p.94].

We start off by showing that the integrand of (3.10) does in fact belong to $L^1(G, \nu_0)$. Note that

$$|f(x)\langle L(x)(\xi(d(x))), \eta(r(x))\rangle| \leq |f(x)|\|(\xi(d(x)))\|\|\eta(r(x))\| \leq \left[|f(x)|^{1/2}\|\xi(d(x))\|D(x)^{-1/4}\right] \cdot \left[|f(x)|^{1/2}\|\eta(r(x))\|D(x)^{1/4}\right],$$

where the first inequality is due to the Cauchy-Schwarz inequality and that $L(x)$ is unitary. Integrating with respect to ν_0 and using the Cauchy-Schwarz inequality again yields

$$\int_G |f(x)\langle L(x)(\xi(d(x))), \eta(r(x))\rangle| d\nu_0(x) \leq$$

$$\left[\int_G |f(x)|\|\xi(d(x))\|^2 D(x)^{-1/2} d\nu_0(x)\right]^{1/2} \times \tag{3.12}$$

$$\left[\int_G |f(x)|\|\eta(r(x))\|^2 D(x)^{1/2} d\nu_0(x)\right]^{1/2}. \tag{3.13}$$

By properties of the Radon-Nikodym derivative, we find that $D^{1/2} = \left(\frac{d\nu_0}{d\nu}\right)^{-1} = \frac{d\nu}{d\nu_0}$ and $D^{-1/2} = D^{-1}D^{1/2} = \frac{d\nu^{-1}}{d\nu}\frac{d\nu}{d\nu_0} = \frac{d\nu^{-1}}{d\nu_0}$, so we can rewrite the right-hand side of (3.13) as

$$\left[\int_G |f(x)|\|\xi(d(x))\|^2 d\nu^{-1}(x)\right]^{1/2} \cdot \left[\int_G |f(x)|\|\eta(r(x))\|^2 d\nu(x)\right]^{1/2}.$$

By (3.8), we have

$$\begin{aligned}\int_G |f(x)|\|\xi(d(x))\|^2 d\nu^{-1}(x) &= \int_{G^0}\int_{G_u} |f(x)|\|\xi(d(x))\|^2 d\lambda_u(x)d\mu(u)\\ &= \int_{G^0} \|\xi(u)\|^2 \int_{G_u} |f(x)| d\lambda_u(x)d\mu(u)\\ &\leq \int_{G^0} \|\xi(u)\|^2 \|f\|_{I,d} d\mu(u)\\ &\leq \|f\|_{I,d}\|\xi\|_2^2.\end{aligned}$$

Similarly, we have

$$\int_G |f(x)| \|\eta(r(x))\|^2 d\nu(x) \leq \|f\|_{I,r} \|\eta\|_2^2,$$

so we can write

$$\begin{aligned}|f(x)\langle L(x)(\xi(d(x))), \eta(r(x))\rangle| &\leq [\|f\|_{I,d}\|\xi\|_2^2]^{1/2}[\|f\|_{I,r}\|\eta\|_2^2]^{1/2} \\ &\leq \|f\|_I \|\xi\|_2 \|\eta\|_2.\end{aligned}$$

Thus the map $\pi_L \colon C_c(G) \to B(L^2(G^0, \{H_u\}, \mu))$ is bounded with $\|\pi_L\| \leq 1$.

Next, we want to show that π_L is a homomorphism. Using (3.11), we will prove that

$$\pi_L(f * g)\xi(u) = \pi_L(f)(\pi_L(g)\xi(u))$$

for μ-a.e. $u \in G^0$, and we skip the technical argument for the general case taking null-sets into account. For μ-a.e. $u \in G^0$ we have

$$\begin{aligned}\pi_L(f * g)\xi(u) &= \int_{G^u} (f * g)(x) L(x)(\xi(d(x))) D^{1/2}(x)\ d\lambda^u(x) \\ &= \int_{G^u} \int_{G^{d(x)}} f(xy)g(y^{-1})\ d\lambda^{d(x)}(y) \cdot L(x)(\xi(d(x))) D^{-1/2}(x)\ d\lambda^u(x) \\ &= \sum_{x \in G^u} \sum_{y \in G^{d(x)}} f(xy)g(y^{-1}) L(x)(\xi(d(x))) D^{-1/2}(x),\end{aligned}$$

by the definition of the product and the fact that the Haar system for G is just counting measures. On the other hand, we have

$$\begin{aligned}&\pi_L(f)(\pi_L(g)\xi(u)) = \int_{G^u} f(x)L(x)\left[\pi_L(g)\xi(d(x))\right] D^{-1/2}(x)\ d\lambda^u(x) = \\ &\int_{G^u} f(x)L(x)\left[\int_{G^{d(x)}} g(y)L(y)(\xi(d(y)))D^{-1/2}(y)\ d\lambda^{d(x)}(y)\right] D^{-1/2}(x)\ d\lambda^u(x) \\ &= \int_{G^u} \int_{G^{d(x)}} f(x)g(y)L(xy)(\xi(d(y)))D^{-1/2}(xy)\ d\lambda^{d(x)}(y)\ d\lambda^u(x) \\ &= \sum_{x \in G^u} \sum_{y \in G^{d(x)}} f(x)g(y)L(xy)(\xi(d(y)))D^{-1/2}(xy),\end{aligned}$$

where we have just written out the definition and passed functions of x through the inner integral. Note that if $y \in G^{d(x)}$, then $r(y) = d(x)$ and $(x, y) \in G^2$. Substituting with $s = y^{-1}$ and $t = xy$, such that $y = s^{-1}$ and $x = ts$, yields

$$\begin{aligned}&\pi_L(f)(\pi_L(g)\xi(u)) = \\ &\int_{G^u} \int_{G^{d(ts)}} f(ts)g(s^{-1})L(t)(\xi(d(t)))D^{-1/2}(t)\ d\lambda^{d(ts)}(s^{-1})\ d\lambda^u(ts) \\ &= \sum_{ts \in G^u} \sum_{s^{-1} \in G^{d(ts)}} f(ts)g(s^{-1})L(t)(\xi(d(t)))D^{-1/2}(t),\end{aligned}$$

where we have used that $d(y) = d(s^{-1}) = d(xs^{-1}) = d(t)$. In the inner sum, we go through elements s^{-1} where $d(s) = r(s^{-1}) = d(ts) = d(s)$, which is obviously the case for every $s \in G^{d(t)}$. Given this, we can also reduce the outer summation to $r(ts) = r(t) \in G^u$. Thus

$$\begin{aligned}\pi_L(f)(\pi_L(g)\xi(u)) &= \sum_{t\in G^u}\sum_{s\in G^{d(t)}} f(ts)g(s^{-1})L(t)(\xi(d(t)))D^{-1/2}(t)\\ &= \pi_L(f*g)\xi(u).\end{aligned}$$

Next, we show that π_L is a $*$-map. For $\xi,\eta \in \mathcal{H}$ we have

$$\begin{aligned}\langle \pi_L(f^*)\xi,\eta\rangle &= \int_{G^u} \overline{f(x^{-1})}\langle L(x)\xi(d(x)),\eta(r(x))\rangle \, d\nu_0(x)\\ &= \int_{G^u} \overline{f(x)}\langle L(x^{-1})\xi(r(x)),\eta(d(x))\rangle \, d\nu_0(x)\end{aligned}$$

since ν_0 is invariant under inversion. By part (iii) in Definition 3.3.9 and since $L(x)$ is unitary, we can write

$$\begin{aligned}\langle L(x^{-1})\xi(r(x)),\eta(d(x))\rangle &= \langle \xi(r(x)), L(x^{-1})^{-1}\eta(d(x))\rangle\\ &= \langle \xi(r(x)), L(x)\eta(d(x))\rangle\\ &= \overline{\langle L(x)\eta(d(x)),\xi(r(x))\rangle},\end{aligned}$$

and hence we have

$$\begin{aligned}\langle \pi_L(f^*)\xi,\eta\rangle &= \int_{G^u} \overline{f(x)}\langle L(x^{-1})\xi(r(x)),\eta(d(x))\rangle \, d\nu_0(x)\\ &= \int_{G^u} \overline{f(x)\langle L(x)\eta(d(x)),\xi(r(x))\rangle} \, d\nu_0(x)\\ &= \overline{\langle \pi_L(f)\eta,\xi\rangle} = \langle \xi,\pi_L(f)\eta\rangle.\end{aligned}$$

The last thing we need to prove is nondegeneracy. Suppose for contradiction that $\overline{\pi_L(C_c(G))\mathcal{H}} \neq \mathcal{H}$, and let $\eta \in (\pi_L(C_c(G))\mathcal{H})^{\perp}$ be nonzero. This means that for all $f \in C_c(G)$ and $\xi \in \mathcal{H}$, we have $\langle \pi_L(f)\xi,\eta\rangle = 0$; in other words

$$\int_G f(x)\langle L(x)\xi(d(x)),\eta(r(x))\rangle \, d\nu_0(x) = 0.$$

Suppose U is any basic open set in G (with compact closure), and let $\{U_i\}_{i=1}^n$ be a finite open cover of the compact set $\overline{U}$ by basis elements. Then $K \coloneqq \cup_{i=1}^n \overline{U_i}$ is compact. Since G is normal, we can apply Urysohn's Lemma to find a function $f \in C_c(G)$ with $f \geq \chi_{\overline{U}}$. Assume in addition that f takes values in $[0,1]$. We can approximate the $L^1(G,\nu_0)$-function

$$x \mapsto |f(x)\langle L(x)\xi(d(x)),\eta(r(x))\rangle|$$

by functions of the form

$$x \mapsto g(x)f(x)\langle L(x)\xi(d(x)),\eta(r(x))\rangle$$

with $g \in C_c(U)$. Therefore we have

$$\begin{aligned}&\int_U |\langle L(x)\xi(d(x)),\eta(r(x))\rangle| \, d\nu_0\\ &\leq \int_G |f(x)\langle L(x)\xi(d(x)),\eta(r(x))\rangle| \, d\nu_0 = 0.\end{aligned}$$

By varying U, we find that $\langle L(x)\xi(d(x)), \eta(r(x))\rangle = 0$ ν_0-a.e. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a fundamental sequence for $(G^0, \{H_u\}, \mu)$. By the above, there exists a set $E \subseteq G$ with $\nu_0(E) = 0$ such that

$$\langle L(x)\xi_n(d(x)), \eta(r(x))\rangle$$

for all $n \in \mathbb{N}$ and $x \in G \setminus E$. For $x \in G \setminus E$, the vectors $\{\xi_n(d(x))\}_{n\in\mathbb{N}}$ span a dense subspace of $H_{d(x)}$. Since $L(x)\colon H_{d(x)} \to H_{r(x)}$ is unitary, $\{L(x)\xi_n(d(x))\}$ spans a dense subset of $H_{r(x)}$. Indeed, if $V \subseteq H_{r(x)}$ is open and nonempty, then so is $L(x)^{-1}(V)$; it contains some $\xi_n(d(x))$, so V contains $L(x)\xi_n(d(x))$. Since $\langle L(x)\xi(d(x)), \eta(r(x))\rangle = 0$ for all $x \in G \setminus E$, we must have $\eta(r(x)) = 0$ ν_0-a.e., and hence ν-a.e. since it is equivalent to ν_0. Thus

$$0 = \int_G \|\nu(r(x))\|\, d\nu = \int_{G^0}\int_{G^u} \|\eta(u)\|\, d\lambda^u\, d\mu(u) = \int_{G^u} \|\nu(u)\|\lambda^u(G^u)d\mu(u),$$

and since $\lambda^u(G^u) > 0$ for all $u \in G^0$, we must have $\eta(u) = 0$ μ-a.e. But this means that η is (in the equivalence class of) zero in $\mathcal{H}$, which is a contradiction. Hence π_L is nondegenerate, and the proof is complete. ■

The above proposition is part of the road to prove the fundamental theorem of analysis on locally compact groupoids, stated below. The proof can be found in [Pat99], but is omitted here since the material isn't very relevant for the rest of the thesis.

Theorem 3.3.12. *Let G be a locally compact groupoid. Then every representation of $C_c(G)$ is of the form π_L for some representation L of G, and the map $L \mapsto \pi_L$ preserves the natural equivalence relations on the representations of G and the representations of $C_c(G)$.*

We can now turn to one of the C^*-algebras associated to the étale groupoid G, namely the universal one.

Definition 3.3.13. Let G be an étale groupoid. The completion of $C_c(G)$ with respect to the C^*-norm

$$\|f\|_* = \sup\{\|\pi(f)\| : \pi \text{ is a representation of } C_c(G)\} \tag{3.14}$$

is called the *C^*-algebra of G*, denoted $C^*(G)$. The norm $\|\cdot\|_*$ is called the *universal norm* on $C_c(G)$.

We should clarify why (3.14) does indeed define a C^*-norm. First of all, by Proposition 3.3.11, each representation of G yields a representation of $C_c(G)$. There are always representations of G – for instance the trivial one – so the set we are taking the supremum of in (3.14) is nonempty. Secondly, Proposition 3.3.11 and Theorem 3.3.12 ensures that every representation of $C_c(G)$ has norm less than 1, so (3.14) is bounded by $\|f\|_I$. We clearly have the C^*-equality as

$$\|\pi(f^*f)\| = \|\pi(f)^*\pi(f)\| = \|\pi(f)\|^2,$$

so $\|\cdot\|_*$ is a C^*-seminorm. In fact, it is a norm since it dominates another norm – the *reduced C^*-norm* on $C_c(G)$. This is the norm on the *reduced C^*-algebra* associated to G, denoted $C_r^*(G)$. We can construct $C_r^*(G)$ by considering a certain representation $Ind\,\mu$, which is (not surprisingly) related to the left regular representation in the group case. The great majority of the argumentation in the rest of the thesis is based on the universal case, and we will therefore not go through the construction of the reduced algebra. This is why we will not go into too much detail in arguments involving the reduced case.

CHAPTER 4

Covariance C^*-algebras

The goal of this chapter is to prove that if G is an étale groupoid, then $C^*(G)$ is isomorphic to the *covariance C^*-algebra* $C_0(G^0)\times_\beta S$ for any additive (countable) inverse semigroup S of G^{op} with its natural localisation action on G^0. In order to prove this (rather copious) statement, we will of course need to know the meaning of every word contained in it. To do so, we must go through a bit of the theory on inverse semigroups. All proofs on this subject are omitted for brevity.

4.1 Inverse Semigroups

Definition 4.1.1. A *semigroup* is a set S with an associative binary operation $(a,b)\mapsto ab$ for $(a,b)\in S^2$. The semigroup S is called an *inverse semigroup* if for each $s\in S$, there is a unique element $s^*\in S$ such that

$$ss^*s = s \text{ and } s^*ss^* = s^*.$$

The map $s\mapsto s^*$ is called the *involution* on S.

A homomorphism between inverse semigroups always preserves involution.
The use of the terminology "involution" is in its place; we have $(st)^* = t^*s^*$ and $s^{**} = s$ for all elements s,t of an inverse semigroup. An important example of an inverse semigroup is this: Let X be set, and define

$$\mathcal{I}(X) \coloneqq \{\alpha\colon D\to R : D,R\subseteq X \text{ and } \alpha \text{ is bijective}\}.$$

Then $\mathcal{I}(X)$ is an inverse semigroup under composition, restricted to whereever it makese sense.

Given an inverse semigroup S, we denote its set of idempotents by $E(S) = \{s\in S : s^2 = s\}$. The elements of this set has several nice properties; for instance, all elements in $E(S)$ commute with each other, and we have $e^* = e$ for all $e\in E(S)$. We also have a partial order $\le$ on the elements of $E(S)$; we write $e_1\le e_2$ for $e_1,e_2\in E(S)$ if $e_1e_2 = e_1$.

There is also a notion of a *quotient inverse semigroup*. It differs from the notion of quotient groups in that normal subgroups are replaced with *congruences*. An equivalence relation $\sim$ on an inverse semigroup T is called a congruence if $as\sim at$ and $sa\sim ta$ whenever $s\sim t$ in T and $a\in T$. The set of equivalence classes $T/\sim$ is an inverse subsemigroup in the natural way. By reversing multiplication in $T/\sim$, it follows that congruences correspond to surjective antihomomorphisms. Every relation $\leq$ on T generates a congruence, namely the smallest congruence containing $\leq$.

Let G be a groupoid. A subset A of G is called a *G-set* if the restrictions d_A, r_A of the domain and source maps are injective on A. We let Σ denote the set of all G-sets. Note that $G^{\mathrm{op}} \subseteq \Sigma$. If G is an étale groupoid, then Σ is an inverse semiroup under set multiplication and with set inversion as involution. The set G^0 is the unit 1 and $\emptyset$ is the zero 0 for Σ. Furthermore, G^{op} is in fact an inverse subsemigroup of Σ. The inverse semigroup G^{op} (for étale G) may have many interesting inverse subsemigroups itself. An inverse subsemigroup S of G^{op} is called *additive* if it is a basis for the topology on G, *and* whenever $A, B \in S$ with $A \cup B \in G^{\mathrm{op}}$, then $A \cup B \in S$.

A *representation* of an inverse semigroup S on a separable Hilbert space $\mathcal{H}$ is a $*$-homomorphism from S into $B(\mathcal{H})$ such that

$$\overline{\operatorname{span}(\pi(S)\mathcal{H})} = \mathcal{H}.$$

Note that a representation of S maps S to an inverse semigroups of partial isometries.

Definition 4.1.2. Let S be an inverse semigroup, and let X be a set. A *right action* of S on X is an inverse semigroup antihomomorphism from S to $\mathcal{I}(X)$; in other words, a map $s \mapsto \alpha_s \in \mathcal{I}(X)$ such that

$$\alpha_{st} = \alpha_t \alpha_s \text{ and } \alpha_{s^*} = \alpha_s^*$$

for all $s, t \in S$. We will often denote the right action by $x \mapsto x \cdot s$, where $x \cdot s = \alpha_s(x)$ for all $s \in S, x \in X$.

This brings us back to étale groupoids again. It turns out that there is a canonical right action connected to each G^{op} for an étale groupoid G.

Example 4.1.3. Let G be an étale groupoid. For any $A \in G^{\mathrm{op}}$ and $u \in G^0$, set

$$\alpha_A(u) = A^{-1}uA.$$

We can reduce this to a simpler form. Indeed, $\alpha_A(u)$ is only defined when $u \in r(A)$, so $u = r(a)$ for some $a \in A$. Since r is injective on A, a is unique, and we have $uA = \{a\}$. Thus we have $A^{-1}uA = A^{-1}a$. If the product ba is defined for some $b \in A^{-1}$, we would have $d(b) = r(a)$, and since d is injective on A^{-1}, that b is unique and must be equal to a^{-1}. Then we can write

$$A^{-1}uA = A^{-1}a = \{a^{-1}a\} = \{d(a)\}.$$

Dropping the braces off $\{d(a)\}$, we have that α_A is a function from $r(A)$ to $d(A)$, given by

$$\alpha_A(r(a)) = d(a).$$

In fact, we can write $\alpha_A = d \circ r_A^{-1}$, which is a composition of homeomorphisms and is therefore a homeomorphism. This defines the *canonical right action* of G^{op} on G^0.

4.2 Localisations and Covariance C^*-algebras

Localisations and Covariant Systems

An essential part to most constructions in this section is the notion of a *localisation*.

Definition 4.2.1. Suppose that X is a locally compact Hausdorff space, G is an étale groupoid, and $S \subseteq G^{\mathrm{op}}$ is an additive inverse subsemigroup. Suppose further that there is a right action $x \mapsto x \cdot s = \alpha_s(x)$ of S on X, and we let D_s and R_s denote the domain and range of α_s, respectively. The pair (X, S) is called a *localisation* if

(i) the domain D_s for each map α_s is open,

(ii) each map α_s is a homeomorphism from D_s onto its range $R_s = D_{s^*}$, and

(iii) the family of such domains D_s is a basis for the topology on X.

Note that $\alpha_{ss^*} = \alpha_{s^*}\alpha_s$, which is the identity on D_s; hence $D_s = D_{ss^*}$. This means that the last condition in Definition 4.2.1 can be replaced by requiring $\{D_e : e \in E(S)\}$ to be a basis for the topology on X.

Example 4.2.2. Let G be an étale groupoid and S be an inverse subsemigroup of G^{op} which is a basis for the topology on G. Let $x \mapsto x \cdot s$ be the canonical right action on G^0 by S given in Example 4.1.3. Then the domain of α_s is $D_s = r(s)$ (as with α_s in Example 4.1.3). We will prove that (G^0, S) is a localisation by verifying the three parts of Definition 4.2.1.

We first note that D_s is open in G^0 since s is open and r_s is a homeomorphism, and as already noted in Example 4.1.3, each α_s is a homeomorphism. Now we must check that $\mathcal{C} := \{D_s : s \in S\}$ is a basis for the topology on G^0. For every $u \in G^0$ and basic neighbourhood $s \in S$ of u, we must check that there a member of $\mathcal{C}$ between u and s. Note that $D_s = r(s)$ is open in G and contains u. Since S is a basis for the topology on G, there is some $t \in S$ with $u \in t \subseteq r(s) \cap s$. Then $t \subseteq G^0$, so we have $D_t = r(t) = t$. But then $t \in \mathcal{C}$ and is between u and s, and $\mathcal{C}$ generates the topology on G^0.

Definition 4.2.3. A localisation (X, S) is called *extendible* if whenever $e_1, e_2 \in E(S)$, then there exists $e_3 \in E(S)$ such that $D_{e_3} = D_{e_1} \cup D_{e_2}$.

Proposition 4.2.4. *If G is an étale groupoid and S is a (countable) additive subsemigroup of G^{op} acting canonically on G^0, then the localisation (X, S) is extendible.*

Proof. If $e_1, e_2 \in E(S) \subseteq G^{\mathrm{op}}$, we must have $e_1, e_2 \in G^0$, so both e_1 and e_2 are singleton sets $\{u_1\}$ and $\{u_2\}$ respectively. Now we can write $D_{e_1} \cup D_{e_2} = r(\{u_1, u_2\}) = \{u_1, u_2\}$. Set $e_3 = \{u_1, u_2\}$. We clearly have $e_3 \in G^{\mathrm{op}}$, so $e_3 \in S$ since S is additive. But $D_{e_3} = r(e_3) = e_3$, so (X, S) is extendible. ■

Suppose X is a topological space and $U \subseteq X$ is open. Then we can regard $C_0(U)$ as a closed ideal in $C_0(X)$ by extending each $f \in C_0(U)$ to be zero outside of U. In particular, if (X, S) is a localisation, we can regard $C_0(D_s)$ as a

closed ideal of $C_0(X)$. For each $s \in S$, we define a map $\beta_s \colon C_0(D_{s^*}) \to C_0(D_s)$ by
$$\beta_s(F)(x) = F(x \cdot s) \tag{4.1}$$
for $x \in D_s$. We will sometimes denote $\beta_s(F)(x)$ by sF.

Proposition 4.2.5. *The function β_s defined above is an isomorphism between the closed ideals $C_0(D_{s^*})$ and $C_0(D_s)$ of $C_0(X)$, and $s \mapsto \beta_s$ is a homomorphism from S into $\mathcal{I}(C_0(X))$.*

Proof. We clearly have that β_s is a homomorphism. It also has an inverse, namely β_{s^*}. To show this, we need to know that $x \cdot ss^* = x$, or in other words that $\alpha_s\alpha_s^*(x) = x$, for all $s \in S$ and $x \in D_s$. From the definition of an inverse semigroup we have $\alpha_s\alpha_s^*\alpha_s = \alpha_s$ for all $s \in S$, so in particular we have for any such s that
$$\alpha_s(\alpha_s^*\alpha_s(x)) = \alpha_s(x)$$
for all $x \in D_s$. But then $\alpha_s^*\alpha_s(x) = x$ since α_s is injective. Now we can see that $\beta_s^{-1} = \beta_{s^*}$. Indeed, using the notation $\beta_s(F) = (x \mapsto F(x \cdot s))$ we have
$$\begin{aligned}\beta_s(\beta_{s^*}(F)) &= (x \mapsto \beta_{s^*}(F)(x \cdot s)) \\ &= (x \mapsto F(x \cdot ss^*)) \\ &= (x \mapsto F(x)) = F.\end{aligned}$$
Thus β_s is an isomorphism between closed ideals in $C_0(X)$.

We will now show that $s \mapsto \beta_s$ is a homomorphism. The first step towards proving that $\beta_{st} = \beta_s\beta_t$ for $s, t \in S$, is to prove that the two functions have the same domain. Recall that that $C_0(U) \cap C_0(V) = C_0(U \cap V)$ for any topological spaces U and V. The domain of $\alpha_{t^*s^*} = \alpha_{s^*}\alpha_{t^*}$ is $D_{t^*s^*}$, which equals
$$\alpha_{t^*}^{-1}(R_{t^*} \cap D_{s^*}) = \alpha_t(D_t \cap D_{s^*}).$$
On the other hand, the domain of $\beta_s\beta_t$ is
$$\beta_t^{-1}(C_0(D_t \cap D_{s^*})).$$
(Recall that multiplication in $\mathcal{I}(C_0(X))$ is defined as composition wherever it makes sense.) If $F \in \beta_t^{-1}(C_0(D_t \cap D_{s^*}))$, we would have $\beta_t(F) \in C_0(D_t \cap D_{s^*})$. This function is defined by $\beta_s(F)(x) = F(\alpha_t(x))$ for all $x \in D_t \cap D_{s^*}$, so $F \in C_0(\alpha_t(D_t \cap D_{s^*}))$. But then we have $F \in C_0(D_{t^*s^*})$, which is the domain of β_{st}. The other inclusion is similar, so the domain of β_{st} equals the domain of $\beta_s\beta_t$. Finally, for any F in the domain of β_{st}, we hav
$$\begin{aligned}\beta_s\beta_t(F) &= \beta_s(x \mapsto F(x \cdot t)) \\ &= (x \mapsto F((x \cdot t) \cdot s)) \\ &= (x \mapsto F(x \cdot st)) \\ &= \beta_{st}(F),\end{aligned}$$
so $s \mapsto \beta_s$ is a homomorphism. ∎

Definition 4.2.6. Suppose S is an inverse semigroup and A is a C^*-algebra. An *action* of S on A is defined to be a homomorphism β from S to $\mathcal{I}(A)$ such that

(i) for every $s \in S$, the domain E_{s^*} of every β_s is a closed ideal in A, and β_s is an isomorphism from the ideal E_{s^*} to the ideal E_s;

(ii) if $s, t \in S$ then there exists $w \in S$ such that $E_s \cup E_t \subseteq E_w$, and

(iii) the set $B = \cup_{s \in S} E_s$ is a dense subalgebra of A.

The triple (A, β, S) is said to *give* or even to *be* an (inverse semigroup) *covariant system.*

We will sometimes denote $\beta_s(a)$ by sa for $a \in E_{s^*}$. Combining several β_ts may require some care. If $s, t \in S$, then the domain of $\beta_s\beta_t = \beta_{st}$ is $\beta_t^{-1}(E_{s^*} \cap E_t) = \beta_{t^*}(E_{s^*}E_t)$ (and not E_{t^*}), since the composition $\beta_s\beta_t$ is only defined where it makes sense. Here we have used that $\beta_t^{-1} = \beta_{t^*}$, and that $I \cap J = IJ$ for any closed ideals I, J in the C^*-algebra A. (To see that $IJ = I \cap J$, just take an approximate identity $\{u_\lambda\}_{\lambda \in \Lambda}$ for J; then $x \in I \cap J$ can be written as $\lim_\lambda xu_\lambda$, which is an element of IJ since IJ is closed. The other inclusion is by the definition of ideal.) A useful property one can deduce from this is that $E_{st} = \beta_s(E_{s^*}E_t)$. Indeed, E_{st} is the range of $\beta_{ts} = \beta_s\beta_t$, so we have

$$E_{st} = \beta_s\beta_t(\beta_{t^*}(E_{s^*}E_t)) = \beta_s(E_{s^*}E_t).$$

We will be particularily interested in the covariant system $(C_0(X), \beta, S)$ where (X, S) is an extendible localisation. With $X = G^0$ for some étale groupoid G, it turns out that the induced C^*-algebra $C_0(G^0) \times_\beta S$ is isomorphic to $S^*(G)$, which is one of the main theorems of this section (Theorem 4.2.18).

Proposition 4.2.7. *If (X, S) is an extendible localisation, then the triple $(C_0(X), \beta, S)$ is a covariant system.*

Proof. We will check the three conditions of Definition 4.2.6. Note that the domain of β_s is $E_{s^*} = C_c(D_{s^*})$, so the first condition is just the first part of Proposition 4.2.5. For (ii), it suffices to consider $E(S)$, since if $s \in S$ then $E_s = E_{ss^*}$ (where $ss^* \in E(S)$). Suppose therefore that $e_1, e_2 \in E(S)$; since (X, S) is extendible we have another element $e_3 \in E(S)$ such that $D_{e_1} \cup D_{e_2} = D_{e_3}$. For (iii), note that since $\{D_e\}_{e \in E(S)}$ is a basis for the topology on X, each compact subset $K \subseteq X$ is covered by a finite number of the D_e's, say $\{D_{e_i}\}_{i=1}^n$. Suppose $f \in C_c(X)$ has compact support K. By the argument for (ii), there is some $w \in E(S)$ with

$$K \subseteq \cup_{i=1}^n E_{e_i} \subseteq E_w,$$

so we have $f \in B$ and thus $C_c(X) \subseteq B$. But $C_c(X)$ is dense in $C_0(X)$, so B is too. ■

To any covariant system (A, β, S), there is a related space of functions that will be essential. Define $\mathbf{C}(A, S)$ to be the set of functions $\theta \colon S \to A$ such that $\theta(s) \in E_s \subseteq A$ for all $s \in S$ and is zero on all but finitely many elements of S. The set $\mathbf{C}(A, S)$ is a vector space under pointwise operations. Next, let $V(A, S)$ be the set of elements $(a, s) \in \mathbf{C}(A, S)$ where $a \in E_s$ and $(a, s)(t) = \delta_s(t)a$ for $t \in S$. The map $a \mapsto (a, s)$ is linear from E_s to $V(A, S)$. It is clear that $V(A, S)$ spans $\mathbf{C}(A, S)$. Indeed, for $\theta \in \mathcal{B}(A, S)$ and $t \in S$, we have

$$\theta(t) = \sum_{s \in F} \delta_s(t)\theta(t) = \sum_{s \in F} (\theta(t), s)(t)$$

where F is the finite subset of S on which θ is nonzero.

Proposition 4.2.8. *Let (A, β, S) be a covariant system. The space $\mathbf{C}(A, S)$ described above is then a C^*-algebra with product and involution determined by the following product and involution on $V(A, S)$:*

$$(a, s)(b, t) = (s[(s^* a)b], st) \text{ and } (a, s)^* = (s^* a^*, s^*).$$

If (X, S) is an extendible localisation, we can simplify the associated product. For $(f, s), (g, t) \in V(A, S)$ we can define

$$(f, s)(g, t) = (f(sg), st), \tag{4.2}$$

where the function $sg\colon X \to [0, \infty)$ is defined by setting $sg(x) = g(x \cdot s)$ if $x \in D_s$ and zero otherwise. The involution on $\mathbf{C}(C_0(X), S)$ is given by

$$(f, s)^* = (s^* \overline{f}, s^*). \tag{4.3}$$

We define the norm $\|\cdot\|_1$ on $\mathbf{C}(A, S)$ by setting $\|\theta\|_1 = \sum_{s \in S} \|\theta(s)\|$. This makes $\mathbf{C}(A, S)$ into a normed algebra.

Covariant Representations

Definition 4.2.9. Let (A, β, S) be a covariant system. A *covariant representation* (or for (A, β, S) is a pair of representations φ of A and π of S on a Hilbert space $\mathcal{H}$ such that for all $s \in S$, the initial subspace H_s of $\pi(s)$ is $\varphi(E_{s^*})\mathcal{H}$, and for all $a \in E_s$ we have

$$\pi(s)\varphi(a)\pi(s^*) = \varphi(sa). \tag{4.4}$$

In the above definition, the set $\varphi(E_{s^*})\mathcal{H}$ is a closed linear subspace of $\mathcal{H}$ by Cohen's theorem [HR70, p.268]. Also, we can use (4.4) to see another useful property. Indeed, we have

$$\pi(s^* s)\varphi(a)\pi(s^* s) = \varphi(s^* sa),$$

and since $\beta_{s^* s}$ is the identity on E_{s^*}, this just equals $\varphi(a)$. Since $s^* s$ is idempotent and π is a homomorphism, we have

$$\varphi(a) = \pi(s^* s)\varphi(a)\pi(s^* s) = \pi(s^* s)\left[\pi(s^* s)\varphi(a)\pi(s^* s)\right] = \pi(s^* s)\varphi(a),$$

which again equals $\varphi(a)\pi(s^* s)$. Thus we have

$$\pi(s^* s)\varphi(a) = \varphi(a) = \varphi(a)\pi(s^* s) \tag{4.5}$$

for all $s \in S$ and $a \in E_{s^*}$.

Definition 4.2.10. Let (A, β, S) be a covariant system. A *representation* of $\mathbf{C}(A, S)$ on a Hilbert space $\mathcal{H}$ is a norm-continuous $*$-homomorphism $\Phi\colon \mathbf{C}(A, S) \to B(\mathcal{H})$ which is nondegenerate and satisfies the following property: For all $(a, e_1), (a, e_2) \in V(A, S)$ with $e_1, e_2 \in E(S)$, we have

$$\Phi((a, e_1)) = \Phi((a, e_2)). \tag{4.6}$$

Proposition 4.2.11. *Let (φ, π) be a covariant representation for the covariant system (A, β, S) on the Hilbert space $\mathcal{H}$. Then the map $\Phi\colon \mathbf{C}(A,S) \to B(\mathcal{H})$ given by*

$$\Phi((b,s)) = \varphi(b)\pi(s),$$

where $(b,s) \in V(A,S)$, is a representation of $\mathbf{C}(A,S)$ on $\mathcal{H}$.

Proof. We first show that Φ is a homomorphism on $V(A,S)$ (and hence on $\mathbf{C}(A,S)$). To this end, let $(b,s),(c,t) \in V(A,S)$. Since β_{ss^*} is the identity on E_{s^*}, we have $ss^*b = b$, so

$$\begin{aligned}\Phi((b,s))\Phi((c,t)) &= \varphi(b)\pi(s)\varphi(c)\pi(t)\\ &= \varphi(s(s^*b))\pi(s)\varphi(c)\pi(t).\end{aligned}$$

Now we can use (4.4) and the fact that π is a homomorphism to see that

$$\varphi(s(s^*b))\pi(s)\varphi(c)\pi(t) = \pi(s)\varphi(s^*b)\pi(s^*s)\varphi(c)\pi(t),$$

and (4.5) gives that this equals $\pi(s)\varphi(s^*b)\varphi(c)\pi(t)$. But φ is also a homomorphism, so

$$\begin{aligned}\pi(s)\varphi(s^*b)\varphi(c)\pi(t) &= \pi(s)\varphi([s^*b]c)\pi(t)\\ &= \pi(s)\varphi([s^*b]c)\pi(s^*s)\pi(t),\end{aligned}$$

where we have used (4.5) again. Note that this makes sense since $s^*b \in E_{s^*}$, so $[s^*b]c \in E_{s^*}$ since E_{s^*} is an ideal. Now we can write

$$\begin{aligned}\Phi((b,s))\Phi((c,t)) &= \pi(s)\varphi([s^*b]c)\pi(s^*s)\pi(t)\\ &= (\pi(s)\varphi([(s^*b)c])\pi(s^*))\pi(st)\\ &= \varphi(s[(s^*b)c])\pi(st)\\ &= \Phi((b,s)(c,t)).\end{aligned}$$

We can see that Φ is a $*$-homomorphism by applying (4.4) and (4.5). Indeed, for $s \in S$ and $b \in E_{s^*}$ we have

$$\begin{aligned}\Phi((b,s)^*) &= \Phi((s^*b^*, s^*)) = \varphi(s^*b^*)\pi(s^*)\\ &= \pi(s^*)\varphi(b^*)\pi(s)\pi(s^*) = \pi(s^*)\varphi(b^*)\pi(ss^*)\\ &= \pi(s^*)\varphi(b^*) = (\varphi(b)\pi(s))^*\\ &= \Phi((b,s))^*.\end{aligned}$$

For nondegeneracy, note that $\Phi((b,ss^*)) = \varphi(b)\pi(ss^*) = \varphi(b)$. But then

$$\overline{\varphi(A)} \subseteq \overline{\Phi(\mathbf{C}(A,S))} = \mathcal{H}$$

since φ is nondegenerate.

To see that Φ is continuous, simply note that

$$\begin{aligned}\|\Phi((b,s))\| &= \|\varphi(b)\pi(s)\| \leq \|\varphi(b)\|\|\pi(s)\| = \|\varphi(b)\|\\ &= \|b\| = \|(b,s)\|_1.\end{aligned}$$

Finally, to see (4.6), let $(b,e_1),(b,e_2) \in V(A,S)$ with $e_1, e_2 \in E(S)$; then

$$\Phi((b,e_1)) = \varphi(b)\pi(e_1) = \varphi(b) = \varphi(b)\pi(e_2) = \Phi((b,e_1)),$$

since the initial space of $\pi(e_i)$ is $\varphi(E_{e_i})$ for $i = 1, 2$. ∎

This result also has a converse. As the proof is quite long, we only give a (very rough) sketch of it. The full proof can be found in [Pat99, pp. 133–136].

Proposition 4.2.12. *Let (A, β, S) be a covariant system, and let Φ be a representation of $\mathbf{C}(A, S)$ on $\mathcal{H}$. Then there is a covariant representation (φ, π) for the system such that*

$$\Phi((b, s)) = \varphi(b)\pi(s) \tag{4.7}$$

for all $(b, s) \in V(A, S)$.

Proof (sketch). The function $\varphi\colon E_e \to A$ for $e \in E(S)$ defined by

$$\varphi(a) = \Phi((a, e))$$

extends to a homomorphism $\varphi\colon A \to B(\mathcal{H})$. The function is well defined by (4.6). For $s \in S$, let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a positive (bounded) approximate identity for E_s. Then the strong operator limit $\lim_{\lambda \to \infty} \Phi((p_\lambda, s))$ exists, and we set

$$\pi(s) = \lim_{\lambda \to \infty} \Phi((p_\lambda, s)).$$

It will turn out that (φ, π) is the desired covariant representation. Defining $\pi(s^*)$ the same way as $\pi(s)$ (replacing the approximate identity for E_s with an approximate identity for E_{s^*}), we find that

$$\pi(s^*) = \pi(s)^* \text{ and } \pi(s^*)\pi(s) = \pi(ss^*).$$

From this we can deduce that $\pi(s)$ is a partial isometry with initial subspace H_s and final subspace H_{s^*}. To show that $s \mapsto \pi(s)$ is a homomorphism, one first shows that $\pi(s)\pi(t) = \pi(st)$ on H_{st}, and then that $\pi(s)\pi(t)$ is zero on $H_{st}^\perp$. Verifying (4.7), and (4.4) to see that (φ, π) is indeed a covariant representation, is just straightforward calculation. Nondegeneracy of both φ and π follows from nondegeneracy for Φ. ∎

Corollary 4.2.13. *Let (A, β, S) be a covariant system. Then there is a one-to-one correspondance, given by (4.7), between the representations Φ of $\mathbf{C}(A, S)$ and the covariant representations (φ, π).*

Resolutions of the Identity and Covariance C^*-algebras

It turns out that a good way to connect groupoids to covariant systems is to express covariant representations through so-called *resolutions of the identity*. For any topological space X, we let $\mathcal{B}(X)$ denote its Borel algebra, and $B(X)$ the set of Borel functions on X.

Definition 4.2.14. Let X be a locally compact space and $\mathcal{H}$ a Hilbert space. A *resolution of the identity* on X with respect to $\mathcal{H}$ is a map P from $\mathcal{B}(X)$ into the self-adjoint projections in $B(\mathcal{H})$ such that:

(i) $P(\emptyset) = 0$ and $P(X) = 1$,

(ii) for all $E_1, E_2 \in \mathcal{B}(X)$, we have $P(E_1 \cap E_2) = P(E_1)P(E_2)$, and if $E_1 \cap E_2 = \emptyset$ then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$, and

(iii) for each $\xi, \eta \in \mathcal{H}$, the map $E \mapsto \langle P(E)\xi, \eta\rangle$ is a regular Borel (finite) measure $\mu_{\xi,\eta}$ on $\mathcal{B}(X)$.

If P is a resolution of the identity on X with respect to $\mathcal{H}$, then there is a representation Φ_P on $\mathcal{H}$ of $B(X)$ given by

$$\Phi_P(f) = \int_X f\ dP, \tag{4.8}$$

where $\langle \Phi_P(f)\xi, \eta\rangle = \int_X f\ d\mu_{\xi,\eta}$.

For a localisation (X, S), we set $x \leq s$ if $x \in D_s$. This relation turns out to have useful properties. The proofs of these properties are not difficult, but we leave them for brevity.

Proposition 4.2.15. *Let (X, S) be a localisation. Then*

(i) if $e \in E(S)$ then $ses^* \in E(S)$,

(ii) $x \leq s$ if and only if $x \leq ss^*$,

(iii) if $x \leq s$ and $e \in E(S)$, then $x \cdot s \leq e \iff x \cdot se = x \cdot s \iff x \cdot (ses^*) = x.ss^* = x$,

(iv) $x \cdot s \leq e \iff x \leq ses^*$, and

(v) $x \leq f \iff x \cdot s \leq s^* f s$.

Now we present the promised characterisation of covariant pairs.

Proposition 4.2.16. *Let (X, S) be an extendible localisation. Then the covariant representations of the covariant system $(C_0(X), \beta, S)$ on a Hilbert space $\mathcal{H}$ is given precisely by pairs (P, π), where P is a resolution of the identity on $\mathcal{B}(X)$ for $\mathcal{H}$, π is a representation of S on $\mathcal{H}$, and for all $e \in E(S)$ we have $P(D_e) = \pi(e)$.*

Proof (sketch). Suppose first that (P, π) is as stated above; we must prove that it yields a covariant representation. Define $\varphi' \colon B(X) \to B(\mathcal{H})$ by (4.8), i.e.

$$\varphi'(f) = \int_X f\ dP \tag{4.9}$$

for each Borel function f. Then φ', and its restriction φ to $C_0(X)$, are representations on $\mathcal{H}$. To see that (φ, π) is the required covariant representation, we must prove (4.4). To this end, let $s \in S$ and $e \in E(S)$ with $e \leq s^*s$. Then for all $\xi, \eta \in \mathcal{H}$, we have

$$\langle \int_X \chi_{D_e}\ dP\xi, \eta\rangle = \int_X \chi_{D_e}\ d\mu_{\xi,\eta} = \mu_{\xi,\eta}(D_e) = \langle P(D_e)\xi, \eta\rangle,$$

so $\varphi(\chi_{D_e}) = P(D_e) = \pi(e)$. Similarly, we have $\pi(ses^*) = \varphi(\chi_{D_{ses^*}})$, so

$$\pi(s)\varphi(\chi_{D_e})\pi(s^*) = \pi(s)\pi(e)\pi(s^*) = \pi(ses^*) = \varphi(\chi_{D_{ses^*}}).$$

By part (iv) of Proposition 4.2.15, we have $\chi_{D_{ses^*}}(x) = \chi_{D_e}(x \cdot s) = s\chi_{D_e}(x)$ (in the notation for (4.1), extended from $C_0(D_{s^*})$ to $B(D_{s^*})$, so we have

$$\pi(s)\varphi(\chi_{D_e})\pi(s^*) = \varphi(s\chi_{D_e}).$$

Now [Pat99] claims that by the Monotone Class Theorem and the fact that D_e's form a basis, we have the above equality for all Borel subsets W of D_{s^*}. Now consider the function $\sum_{i=1}^{n} \alpha_i \chi_{W_i}$, where $\alpha_i \in \mathbb{C}$ and each $W_i \subseteq D_{s^*}$ is a Borel set. Then we have

$$\begin{aligned}\pi(s)\varphi(\sum_{i=1}^{n} \alpha_i \chi_{W_i})\pi(s^*) &= \sum_{i=1}^{n} \alpha_i \pi(s)\varphi(\chi_{W_i})\pi(s^*) \\ &= \sum_{i=1}^{n} \alpha_i \varphi(s\chi_{W_i}) \\ &= \varphi(s \sum_{i=1}^{n} \alpha_i \chi_{W_i}).\end{aligned}$$

Any function in $f \in C_0(D_{s^*})$ can be approximated uniformly by a monotonely increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of such linear combinations. Since all such linear combinations are also bounded, we have by the Monotone Convergence Theorem that

$$\begin{aligned}\langle \varphi(\lim_{n\to\infty} f_n)\xi, \eta\rangle &= \int_X \lim_{n\to\infty} f_n \; d\mu_{\xi,\eta} \\ &= \lim_{n\to\infty} \int_X f_n \; d\mu_{\xi,\eta}\end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$, so $\varphi(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} \varphi(f_n)$. As multiplication is continuous in $B(\mathcal{H})$ and $f \mapsto sf$ is also continuous, we have

$$\begin{aligned}\pi(s)\varphi(f)\pi(s^*) &= \pi(s) \lim_{n\to\infty} \varphi(f_n)\pi(s^*) \\ &= \lim_{n\to\infty} \pi(s)\varphi(f_n)\pi(s^*) \\ &= \lim_{n\to\infty} \varphi(sf_n) \\ &= \varphi(s \lim_{n\to\infty} f_n) \\ &= \varphi(sf),\end{aligned}$$

so (φ, π) satisfies Equation (4.4). The initial space for the orthogonal projection $\varphi(\chi_{D_{s^*s}}) = \pi(s^*s)$ is $\varphi(C_0(D_{s^*}))\mathcal{H}$, so (φ, π) is a covariant representation.

Conversely, let (φ, π) be a covariant representation of some Hilbert space $\mathcal{H}$. By the Riesz-Markov theorem, φ determines a unique resolution of the identity, P, satisfying (4.9). Let $e \in E(S)$. We can extend φ by continuity to $B(D_e)$ (which is the pointwise completion of $C_0(D_e) \subseteq C_0(X)$). Now we again have that $P(D_e) = \varphi(\chi_{D_e})$ is the orthogonal projection onto $\varphi(C_0(D_e))\mathcal{H}$, so $P(D_e) = \pi(e)$, which is what we wanted. ∎

We are now ready to define the object naming this chapter, and finally the main theorem of the section. For a covariant system (A, β, S), define a seminorm on $\mathbf{C}(A, S)$ by setting

$$\|\theta\| = \sup\{\|\Phi(\theta)\| : \Phi \text{ is a representation of } \mathbf{C}(A, S)\}. \tag{4.10}$$

As in Definition 3.3.13, the seminorm above inherits the C^*-equality, so it is a C^*-seminorm.

Definition 4.2.17. Let (A, β, S) be a covariant system. The completion of the quotient of $\mathbf{C}(A, S)$ by the kernel of the seminorm (4.10) is a C^*-algebra denoted $A \times_\beta S$, and it is called the *covariance C^*-algebra* for (A, β, S).

The following result relates the covariance C^*-algebra to the groupoid algebra $C^*(G)$.

Theorem 4.2.18. *Let G be an étale groupoid and S an additive inverse subsemigroup of G^{op}. Then the triple $(C_0(G^0), \beta, S)$ is a covariant system, and $C^*(G)$ is canonically isomorphic to $C_0(G^0) \times_\beta S$.*

Proof. Set $X = G^0$; then (X, S) is an extendible localisation by Proposition 4.2.4, and $(C_0(X), \beta, S)$ is a covariant system by Proposition 4.2.7.

Our plan to construct the desired isomorphism is to first relate $V(X, S)$ to $C_c(G)$, and then extend to $\mathbf{C}(A, S)$.

For $(f, s) \in V(X, S)$, define $f_s \colon G \to \mathbb{C}$ by

$$f_s(x) = \begin{cases} f(r(x)) & \text{if } x \in s, \\ 0 & \text{if } x \in G \setminus s. \end{cases}$$

Note here that if $x \in s$, then $x \in D_s = r(s)$ so $r(x) \in D_s$. Since $f \in E_s = C_0(D_s)$, $f(r(x))$ makes sense. We claim that $(f, s) \mapsto f_s$ is a $*$-homomorphism from $V(A, S)$ to $C_c(G)$. We have $\operatorname{supp} f_s \in C_c(s)$ as

$$\operatorname{supp} f_s = r_s^{-1}(\operatorname{supp} f),$$

which is the image of a compact set under a continuous function, and is a subset of s. To see that $(f, s) \mapsto f_s$ is a homomorphism, let $(f, s), (g, t) \in V(A, S)$. We must show that

$$f_s * g_t = (f(sg))_{st},$$

using the notation from (4.2). We use (3.4) to see that $f_s * g_t \in C_c(st)$, and for $y = ab$ with $a \in s, b \in t$ we have

$$(f_s * g_t)(y) = f_s(a) g_t(b) = f(r(a)) g(r(b)).$$

On the other hand, since $r(x) \cdot s = d(x)$ for the canonical right action (see Example 4.1.3), we can write

$$\begin{aligned} (f(sg))_{st}(y) &= [f(r(ab))]\,[(sg(r(ab))] \\ &= [f(r(a))]\,[(sg(r(a))] \\ &= f(r(a)) g(d(a)) \\ &= f(r(a)) g(r(b)); \end{aligned}$$

thus $f_s * g_t = (f(sg))_{st}$. Showing that $(f, s) \mapsto f_s$ is a $*$-map is similar. Indeed, recalling the definition of the involution, (4.3), we must show that $(s^*\overline{f})_{s^*} = f_s^*$. Recall that $h^*(x) = \overline{h(x^{-1})}$ for $h \in C_c(G)$ and $x \in G$. For $x \in s^*$ we have

$$\begin{aligned} (s^*\overline{f})_{s^*}(x) &= (s^*\overline{f})(r(x)) = \overline{f(r(x) \cdot s^*)} \\ &= \overline{f(d(x))} = \overline{f(r(x^{-1}))} = f_s^*(x), \end{aligned}$$

so $(f, s) \mapsto f_s$ is a $*$-homomorphism. Note that it is also linear. Since $\mathbf{C}(A, S)$ is generated by $V(A, S)$, we can extend this map to a $*$-homomorphism $\Delta \colon \mathbf{C}(A, S) \to \mathbb{C}$.

The map Δ is surjective. Indeed, suppose first that $F \in C_c(s)$ for some $s \in S$. Then define $f \in V(A, S)$ by $f(x) = F(r_s^{-1}(x))$ for $x \in r(s)$. Then $f_s(x) = f(r(x)) = F(r(r_s^{-1}(x))) = F(x)$, so $F = \Delta((f, s))$. For a general $F \in C_c(G)$, we cover its (compact) support with finitely many elements $\{s_i\}_{i=1}^n$ from S. We may do this since S is a basis for the topology on G. This partition admits a partition of unity for F, so that we can write F as a sum of functions $f_{s_i}^{(i)}$ for $i = 1, \ldots, n$, where each $f^{(i)}$ is constructed as above. Thus $F = \Delta(\sum_{i=1}^n f^{(i)})$.

It turns out that Δ is isometric with respect to the C^*-norms on $\mathbf{C}(A, S)$ and $C_c(G)$. We will not prove this as it requires a few results we didn't include, but the argument can be found in [Pat99, pp. 138–139]. We only had a C^*-*seminorm* on $\mathbf{C}(A, S)$, but since Δ is isometric, it is actually a norm. Since Δ is an isometric homomorphism, it is injective, and hence an isomorphism between $\mathbf{C}(A, S)$ and $C_c(G)$. Therefore it extends to an isomorphism of the closures of these sets, so $C_0(G^0) \times_\beta S \cong C^*(G)$. ■

There is also a converse to the above theorem, in which we relate certain extendible localisations (X, S) to an étale groupoid G such that the conclusion of the above theorem holds. We will sketch how to prove this.

A problem with creating a converse to Theorem 4.2.18 is that there might different es in $E(S)$ with the same domains D_e. This cannot happen for an inverse subsemigroup of G^{op} acting on G^0. To relate an étale groupoid to (X, S), we therefore replace (X, S) with another extendible localisation (X, S_1) having the same covariance C^*-algebra, but for which $e \mapsto D_e$ on $E(S_1)$ is injective. Explicitly, let ρ be the relation on S defined by setting $s \rho t$ if $s, t \in E(S)$ and $D_s = D_t$. Then we set $S_1 = S/R$, where R is the congruence generated by ρ. Then (X, S_1) will be a localisation with corresponding action β_1. It turns out that $C_0(G^0) \times_\beta S$ is isomorphic to $C_0(G^0) \times_\beta S$, which solves the initial problem. This makes us able to identify e with D_e for $e \in S(E)$, and from now on we will assume that $e \mapsto D_e$ is an isomorphism for all localisations. Extendibility for S then assumes the form that if $e, f \in E(S)$, then $e \cup f \in E(S)$.

Next, we would like to relate the extendible localisation (X, S) to an étale groupoid $G(X, S)$. We start by constructing a "first approximation" to $G(X, S)$, denoted Ξ. Let $\Xi \subseteq X \times S$ be the subset given by

$$\Xi = \{(x, s) : x \in D_s, s \in S\}.$$

The set Ξ^2 of "composable pairs" is defined to be the set of pairs $((x, s), (x \cdot s, t))$ in $\Xi \times \Xi$. For such a pair, we define their product as $(x, s)(x \cdot s, t) = (x, st)$. We denote $\{(x, e) : x \in D_e, e \in E(S)\}$ by Ξ^0, the inverse map $a \mapsto a^{-1}$ is defined as $(x, s)^{-1} = (x \cdot s, s^*)$. Now Ξ looks like a groupoid, but the groupoid axioms do not hold in general. As usual when something doesn't look exactly like we want to, we just quotient out everything we don't want. To this end, set $(x, s) \sim (y, t)$ to mean that $x = y$ and there exists an element $e \in E(S)$ such that $x \leq e$ and $es = et$. Then $\sim$ is an equivalence relation, and

$$G(X, S) \coloneqq \Xi / \sim$$

turns out to be the desired étale groupoid. We summarise this and other useful properties in the following theorem. To this end, we let $\overline{(x,s)}$ denote the equivalence class of (x,s) in $\Xi/\sim$.

Theorem 4.2.19. *Let (X,S) be an extendible localisation. Then $G(X,S)$ is an étale groupoid with the following operations and topology. The composable pairs are those of the form $(\overline{(x,s)},\overline{(x\cdot s,t)})$, and the product and involution are given by*

$$\overline{(x,s)(x\cdot s,t)}\mapsto\overline{(x,st)}\ \text{and}\ \overline{(x,s)}\mapsto\overline{(x\cdot s,s^*)},$$

respectively. A basis for the topology on $G(X,S)$ is given by the family of sets of the form $D(U,s)$, where $s\in S$, U is an open subset of D_s, and

$$D(U,s)=\{\overline{(x,s)}:x\in U\}.$$

The unit space $G(X,S)^0$ is canonically identified with X. The map ψ_X, where $\psi_X(s)=\{\overline{(x,s)}:x\in D_s\}$, is a homomorphism from S into G^{op}, and $\psi_X(S)$ is a basis for $G(X,S)$.

Next, we want to formulate a condition on (X,S) that makes ψ_X in the above theorem is injective, and that $\psi_X(S)$ is additive in $G(X,S)^{\mathrm{op}}$. The condition will be appropriately called *additivity* of (X,S). We first define the notion of *compatibility* for partial homeomorphisms on X.

Suppose (X,S) is a localisation and $s,t\in S$. Recall that $e\in E(S)$ is identified with D_e. The pair (s,t) is r-compatible if for all $x\in ss^*\cap tt^*$, there exists $e\in E(S)$ with $x\le e$ and $es=et$. We say that (s,t) is *compatible* if both (s,t) and (s^*,t^*) are r-compatible.

We define a congruence on S by setting $s\sim t$ if $ss^*=tt^*$ and for every $x\in ss^*$, there exists $e\in E(S)$ such that $x\in e=D_e$ and $es=et$. A localisation (X,S) is called *additive* if

(i) if $s\sim t$ then $s=t$, and

(ii) if the pair (s,t) is compatible and $f=ss^*\cup tt^*\subseteq X$, then there exists $w\in S$ such that $f=ww^*$ and if $x\le ss^*$ (or $x\in tt^*$), then there exists $e\in E(S)$ such that $x\le e$ and $es=ew$ (or $et=ew$).

Any additive localisation is extendible. If (X,S) is an additive localisation, then the map ψ_X from Theorem 4.2.19 is an isomorphism from S to an inverse subsemigroup of $G(X,S)^{\mathrm{op}}$. Thus we can say that $C_0(X)\times_\beta S$ is isomorphic to $C^*(G(X,S))$, which is precisely what we want. We summarise the fruits of this chapter in the following corollary.

Corollary 4.2.20. *If G is an étale groupoid and S is an additive inverse subsemigroup of G^{op}, then $C^*(G)$ is canonically isomorphic to the covariance C^*-algebra $C_0(G^0)\times_\beta S$ for the localisation (G^0,S). Also, if (X,S) is an additive localisation, then $C_0(X)\times_\beta S$ is canonically isomorphic to $C^*(G(X,S))$.*

CHAPTER 5

Deaconu-Renault Groupoids

In this chapter, we pass from the land of general locally compact Hausdorff étale groupoids to one particular kind, namely the Deaconu-Renault groupoids. These groupoids are built from $\mathbb{N}_0^k$-actions T on some locally compact Hausdorff space X, and we denote them G_T. The space X will lie naturally inside G_T as the set of unit elements. To formulate the construction of these groupoids, we will need to generalise the notion of a group action. We will also be interested in further properties of the isotropy subgroupoid $\mathrm{Iso}(G)$. The first section in the chapter is devoted to investigate these basics, before we move onto the actual construction of Deaconu-Renault groupoids in Section 5.2. In the two following sections, we work on characterising the primitive ideal space of such groupoids, and that concludes the reproduction of [SW16]. In Section 5.5, we apply the theory to find a way of charcterising simple C^*-algebras coming from Deaconu-Renault groupoids, in terms directly related to their actions. We give an example by proving that the rotational algebra A_θ is simple if and only if θ is irrational.

One issue must be addressed before we start, and that is the notion of amenability for groupoids. This is a quite intricate subject, and there has not been enough time to go through that theory. Thus we will treat amenability as a black box throughout the chapter. However, we will rely on certain results from that theory, and we sum those up in Lemma 5.1.13.

5.1 Groupoid Actions and Isotropy

Definitions

Let G be a (locally compact Hausdorff) étale groupoid. Recall the definition of the isotropy subgroupoid $\mathrm{Iso}(G)$ from Example 3.1.3:

$$\mathrm{Iso}(G) = \{x \in G : r(x) = d(x)\} = \bigcup_{u \in G^0} G_x^x.$$

The isotropy subgroupoid will be essential when dealing with Deaconu-Renault groupoids. Note that, given some unit $u \in G^0$, there might not be any other groupoid element x with $r(x) = u$ and $d(x) = u$. In other words, the isotropy group $G_u^u = G^u \cap G_u$ at $u \in G^0$ is trivial. Groupoids with many such units behave nicely in certain ways; for instance, in many cases, their faithful representations have a nice representation (see Lemma 5.1.13).

Definition 5.1.1. A groupoid G is called *topologically principal* if the units with trivial isotropy group are dense in G^0. In other words, if

$$\overline{\{x \in G^0 : G_x^x = \{x\}\}} = G^0.$$

We now define the generalisation of a (left) group action.

Definition 5.1.2. Let G be a groupoid and X be any topological space. A *left action* of G on X is a pair (ρ, μ) of continuous maps $\rho\colon X \to G^0$ and $\mu\colon G \times_{d,\rho} X \to X$, where

$$G \times_{d,\rho} X := \{(a, x) \in G \times X : d(a) = \rho(x)\},$$

such that the the statements below hold. We let $a \cdot x$ denote $\mu(a, x)$ for all $(a, x) \in G \times_{d,\rho} X$.

(i) $\rho(a \cdot x) = r(a)$ for all $(a, x) \in G \times_{d,\rho} X$,

(ii) $a \cdot (b \cdot x) = (ab) \cdot x$ for $(a, b) \in G^2$ with $(b, x) \in G \times_{d,\rho} X$, and

(iii) $\rho(x) \cdot x = x$ for all $x \in G$.

The maps ρ and μ are called the *anchor map* and *action map*, respectively. If there is a left action of G on X, we say that X is a *left G-space*. Let $\Psi\colon G \times_{d,\rho} X \to X \times X$ be the map defined by $\Psi(a, x) = (a \cdot x, x)$. The action (ρ, μ) is called

- *free* if Ψ is injective, or equivalently if $a \cdot x = x$ implies $a = \rho(x)$, and
- *proper* if Ψ is a proper map, in other words if $\Psi^{-1}(K)$ is compact for all compact $K \subseteq X \times X$.

For a right action, replace $G \times_{d,\rho} X$ by $X \times_{\rho,r} G$. The conditions in the definition above are "reversed" in the obvious manner, including changing r to d.

Example 5.1.3. There is always a natural left action of G on G^0. Indeed, just let ρ be the identity. Then $G \times_{d,\rho} G^0 = \{(\gamma, d(\gamma)) : \gamma \in G\}$, and we define $\gamma \cdot d(\gamma) = r(\gamma)$ for all $\gamma \in G$.

Deaconu-Renault groupoids are closely related to dynamical systems, and they share many of the same notions. The next definitions should therefore not sound too foreign.

Definition 5.1.4. Let G be a groupoid, and let it act naturally on G^0. If $x \in G^0$, then $G \cdot x = r(G_x)$ is called the *orbit* of x and is denoted by $[x]$. A subset $A \subseteq G^0$ is called *invariant* if $G \cdot A \subseteq A$. The quotient space G/G^0 is called the *orbit space* of G. We can define an equivalence relation $\sim$ on G/G^0 by saying that two orbits are equivalent if they have the same closure. The quotient of G/G^0 by $\sim$ is called the *quasi-orbit space* of G.

Definition 5.1.5. Let G be a groupoid. An ideal $I \lhd C_0(G^0)$ is called *invariant* if the corresponding closed set

$$C_I := \{u \in G^0 : f(u) = 0 \text{ for all } f \in I\}$$

is invariant. If M is a representation of $C_0(G^0)$ with kernel I, then C_I is called the *support* of M. We say that C_I is *G-irreducible* if it is not the union of two proper closed invariant sets.

We have a nice characterisation of G-irreducible sets.

Lemma 5.1.6. *Let G be a locally compact groupoid. A closed invariant subset C of G^0 is G-irreducible if and only if there exists $u \in G^0$ such that $C = \overline{[u]}$.*

Proof. First, let $u \in G^0$ and suppose for contradiction that $\overline{[u]} = A \cup B$ for closed and invariant proper subsets $A, B \subseteq \overline{[u]}$. Suppose further that $a \in A$, so $[a] = G \cdot a \subseteq \overline{[u]}$. If $a \in [u]$, then there is some $x \in G$ with $d(x) = u$ and $r(x) = a$. If $v \in [u]$ by some $y \in G_u$, then $(y, x^{-1}) \in G^2$ since $r(x^{-1}) = d(x) = u = d(y)$. But then $d(yx^{-1}) = d(x^{-1}) = a$ and $r(yx^{-1}) = r(y) = v$, so $v \in [a]$. This implies $[a] = [u]$, contradicting the assumption that A was a proper subset. Hence $A \cap [u] = \emptyset$. But the exact same argument can be applied to B, so neither A or B intersects $[u]$. The set $[u]$ is not empty as it contains u itself, which contradicts that $A \cup B = \overline{[u]}$. Hence $\overline{[u]}$ is invariant.

For the converse, let $C \subseteq G^0$ be a closed, invariant and G-irreducible subset. Then $q(C) \subseteq G/G^0$, where q is the quotient map, is irreducible in the regular sense – i.e. it cannot be written as a union of two closed nonempty proper subsets. Indeed, suppose for contradiction that $q(C) = A_1 \cup A_2$ for such sets; then $q^{-1}(A_i)$ is a proper closed subset for $i = 1, 2$, and they cover C. Both A_i are invariant since they are unions of orbit closures. Proving that $C = \overline{[u]}$ then reduces to seeing that $q(C)$ has a dense point.

To prove this statement, we follow the proof of the lemma preceding [Gre78, Corollary 19]. Since G is locally compact and Hausdorff, it is totally Baire by Remark 2.0.9. We have that G/G^0 is the continuous open image of G, and hence totally Baire by Proposition 2.0.7. Since C is closed and G-irreducible, any nonempty open set of $q(C)$ is dense. Since G is second countable, so is $q(C)$, with basis (say) $\{U_n\}_{n \in \mathbb{N}}$. Since each of these sets are dense, we have by the Baire property that $\cap_{n \in \mathbb{N}} U_n$ is dense in $q(C)$. Hence it must be nonempty, containing a point $[u]$. But then $[u]$ is an element of *every* nonempty open subset of $q(C)$, and must therefore be dense.

■

Definition 5.1.7. Let G be a locally compact groupoid. We say that $C_0(G^0)$ is *G-simple* if it has no nonzero proper invariant ideals.

Lemma 5.1.6 implies that $C_0(G^0)$ is G-simple if and only if G^0 has a dense orbit.

Multiplier Algebras and $C_0(G^0)$

Let G be an étale groupoid. As we shall see, we can look at $C_0(G^0)$ as a C^*-subalgebra of $C^*(G)$. To do so, we need to take a small detour to discuss *multiplier algebras*. We follow [Mur90, pp.38–39]. To any C^*-algebra A, we can associate a large unital C^*-algebra $M(A)$ – the multiplier algebra of A – containing A as an ideal.

Definition 5.1.8. A *double centraliser* for a C^*-algebra A is a pair (L, R) of bounded linear maps on A, such that for all $a, b \in A$ we have

$$L(ab) = L(a)b, \; R(ab) = aR(b) \text{ and } R(a)b = aL(b).$$

If (L, R) is a double centraliser on a C^*-algebra, then $\|L\| = \|R\|$.

Definition 5.1.9. For a C^*-algebra A, we denote its set of double centralisers $M(A)$. This is a closed vector subspace of $\mathcal{B}(A) \oplus \mathcal{B}(A)$, where $\mathcal{B}(A)$ denotes the set of bounded linear operators on A. We define a norm on $M(A)$ by $\|(L, R)\| = \|L\| = \|R\|$. We define a product on $M(A)$ by

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1).$$

Finally, we define involution on $M(A)$ by $L^*(a) = (L(a^*))^*$. This makes $M(A)$ a C^*-algebra.

Example 5.1.10. Let A be a C^*-algebra. For any $x \in A$, there is an associated double centraliser (L_x, R_x) defined by $L_x(a) = xa$ and $R_x(a) = ax$ for all $a \in A$. Hence we can identify A in $M(A)$. From the left centralisers associated to elements of A, we can build all the right centralisers. Indeed, given $x \in A$ we have

$$R_x(a) = ax = (x^* a^*)^* = (L_{x^*}(a^*))^* = L^*_{x^*}(a).$$

There is a homomorphism $V\colon C_0(G^0) \to M(C^*(G))$ such that for $f \in C_c(G)$ and $\varphi \in C_0(G^0)$, we have

$$(V(\varphi)f)(x) = \varphi(r(x))f(x).$$

Since G is étale, V will take values in $C^*(G)$ and extend the inclusion $C_c(G^0) \hookrightarrow C^*(G)$. Hence we can regard $C_0(G^0)$ as a $*$-subalgebra of $C^*(G)$.

Let us check that $V(\varphi)$ does in fact give us an element of $C^*(G)$. If $\varphi \in C_c(G^0)$, then for any $f \in C_c(G_T)$ we have $L(\varphi)(f) = \varphi * f$ by Remark 3.2.12. Hence $V(\varphi)$ is just L_φ, which we can identify with φ by the discussion in Example 5.1.10. Since V is continuous and $C^*(G)$ is closed in $M(C^*(G))$, $V(G^0) \subseteq C^*(G)$.

If L is a nondegenerate representation of $C^*(G)$, we obtain an associated representation M of $C_0(G^0)$ by extension: $M(\varphi) = \overline{L}(V(\varphi))$. It turns out that the representation M factors in a useful way, through the C^*-algebra we get when we restrict the unit space of G to the support of M. This makes intuitive sense, since "ignoring" the kernel of the representation shouldn't change anything. We omit the proof of the first of the propositions.

Proposition 5.1.11. *Let G be a second-countable locally compact groupoid. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^0)$. Then $\ker M$ is invariant. If L is irreducible, then the support of M is G-irreducible.*

Proposition 5.1.12. *Let G be a second-countable locally compact groupoid. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^0)$. If $F = C_{\ker(M)}$ is the support of M, then L factors through $C^*(G|_F)$. In particular, if L is irreducible, then L factors through $C^*(G|_{\overline{[u]}})$ for some $u \in G^0$.*

Proof. First note that F is invariant by Proposition 5.1.11. It is also closed, since the set of zeros of a continuous function is closed and

$$\begin{aligned} F = C_{\ker(M)} &= \{u \in G^0 : f(u) = 0 \text{ for all } f \in \ker(M)\} \\ &= \bigcap_{f \in \ker(M)} \{u \in G^0 : f(u) = 0\} \end{aligned}$$

is an intersection of such sets. Thus $G^0 \setminus F$ is open. We claim that it is also invariant. Suppose for contradiction that there is some $u \in G^0 \setminus F$ with $G \cdot u = [u] \nsubseteq G^0 \setminus F$. Then there is some $v \in [u]$ with $v \in F$. But then $[u] = [v] \subseteq F$ since orbits partition G^0, which is a contradiction. Now by [MRW96, Lemma 2.10], we have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(G|_U) \xrightarrow{\iota} C^*(G) \xrightarrow{R} C^*(G|_F) \longrightarrow 0,$$

where ι is induced by extending functions in $C_c(G|_U)$ by 0, and R comes from restricting functions in $C_c(G)$. (Recall Definition 2.0.19.) To prove the desired factoring, we must prove that we do not lose any information by passing through R. In other words, we must prove that everything in the kernel of R is sent to zero by L, i.e. that the composition $L \circ \iota$ is zero. The following diagram commutes, where $i \colon C^*(G) \hookrightarrow M(C^*(G))$ is the natural inclusion.

$$\begin{tikzcd}
C_0(U) \arrow[d, hook] & & C_c(G|_U) \arrow[d, hook] \\
C_0(G^0) \arrow[r, "V"] \arrow[dr, "M"'] & M(C^*(G)) \arrow[d, "\overline{L}"] & C^*(G) \arrow[l, hook', "i"'] \arrow[dl, "L"] \\
& B(H) &
\end{tikzcd}$$

First, we shall see that $C_0(U)$ is in the kernel of M, when looked upon as a subset of $C_0(G^0)$ in the natural way. Suppose therefore that $\varphi \in C_0(U)$ is nonzero, and pick any $u \in U$ with $\varphi(u) \neq 0$. By the construction of U, there is some $\psi \in \ker(M)$ with $\psi(u) \neq 0$. By scaling, we may assume that $\psi(u) = \varphi(u)$. Set $\varphi_u \coloneqq \varphi - \psi$. We have $M(\varphi_u) = M(\varphi)$, while $\varphi_u(u) = 0$. For any $x \in G^u$ and $f \in C_c(G)$, we now have

$$(V(\varphi_u)f)(x) = \varphi_u(u)f(x) = 0.$$

Varying u would yield elements in $M(C^*(G))$ being zero on a cover of $\{G^u\}_{u \in U}$, and they are already zero on $\{G^u\}_{u \in F}$ by definition. Their images under $\overline{L}$ must be the same as $M(\varphi) = M(\varphi_u)$, so $M(\varphi) = 0$ and $\varphi \in \ker(M)$.

Suppose $f \in C_c(G|_U)$; we must prove that $L \circ \iota(f) = 0$. It suffices to prove that $\overline{L}(i \circ \iota(f)) = 0$. Since V is nondegenerate, there is a sequence of functions $\{\varphi_n\}_{n \in \mathbb{N}}$ in $C_0(G^0)$ with $V(\varphi_n) \to i \circ \iota(f)$. In fact, since $\operatorname{supp} f \subseteq G|_U$, we may assume that $\varphi_n \in C_0(U)$ for all n. But then

$$\overline{L}(i \circ \iota(f)) = \lim_{n \to \infty} \overline{L}(\varphi_n) = \lim_{n \to \infty} 0 = 0,$$

just as we wanted.

If L is irreducible, then F is G-irreducible by Proposition 5.1.11 and hence an orbit closure by Lemma 5.1.6. The last assertion follows. ■

The subgroupoid $\mathrm{Iso}(G)^\circ$

If G is an étale groupoid, then G^0 is open. Thus we have $G^0 \subseteq \mathrm{Iso}(G)^\circ$, and $\mathrm{Iso}(G)^\circ$ is an open étale subgroupoid of G. For Deaconu-Renault groupoids coming from $\mathbb{N}_0^k$-actions, $\mathrm{Iso}(G)^\circ$ is closed in G as well as open. This enables us to form the groupoid $G_T/\mathrm{Iso}(G_T)^\circ$, often called an *orbit space* since we mod out all the units. (This should not be confused with the orbit space G/G^0.) The groupoid $G_T/\mathrm{Iso}(G_T)^\circ$ will be a crucial part of the classification of the primitive ideal spaces. This subsection is about which properties of G, $G_T/\mathrm{Iso}(G_T)^\circ$, $C^*(G)$ and $C^*(G_T/\mathrm{Iso}(G_T)^\circ)$ we can deduce from the fact that $\mathrm{Iso}(G)^\circ$ is closed. We will by $\mathrm{Iso}(G)^\circ_u$ refer to the set $(\mathrm{Iso}(G)^\circ)_u = \{x \in \mathrm{Iso}(G)^\circ : d(x) = u\}$ for $u \in G^0$.

One of the useful properties of G when $\mathrm{Iso}(G)^\circ$ is closed, is that amenability behaves nicely when passing to the quotient groupoid defined below. As mentioned, we will be treating the notion of amenability for groupoids as a black box. The property of amenable groupoids which we will be using is stated below.

Lemma 5.1.13. *Suppose G is an amenable étale groupoid. By [BO08, Corollary 5.6.17], $C_r^*(G)$ and $C^*(G)$ are isomorphic. Thus, if G is topologically principal, then we have by [Exe11, Theorem 4.4] that any representation of $C^*(G)$ that is faithful on $C_0(G^0)$ is faithful on all of $C^*(G)$.*

If $u \in G^0$, then there is an associated left regular representation L^u of $C^*(G)$ on $\ell^2((G_T)_u)$ given by

$$L^u(f)\delta_x = \sum_{d(y)=r(x)} f(y)\delta_{yx}$$

for $f \in C_c(G)$. The reduced algebra $C_r^*(G)$ can be veiwed as the image of $C^*(G_T)$ under $\bigoplus_{u\in G^0} L^u$. By the above, this is an isomorphism if G is amenable.

Proposition 5.1.14. *Suppose that G is an étale groupoid such that $\mathrm{Iso}(G)^\circ$ is closed in G. Then the following statements hold.*

(i) The subgroupoid $\mathrm{Iso}(G)^\circ$ acts freely and properly on the right of G, and the orbit space $G/\mathrm{Iso}(G)^\circ$ is locally compact and Hausdorff.

(ii) For each $x \in G$, the map $y \mapsto xyx^{-1}$ is a bijection from $\mathrm{Iso}(G)^\circ_{d(x)}$ to $\mathrm{Iso}(G)^\circ_{r(x)}$.

(iii) For each $u \in G^0$, the set $\mathrm{Iso}(G)^\circ_u$ is a normal subgroup of G^u_u.

(iv) The set $G/\mathrm{Iso}(G)^\circ$ is a locally compact Hausdorff étale groupoid with respect to the operations $[x]^{-1} = [x^{-1}]$ for $x \in G$, and $[x][y] = [xy]$ for $(x, y) \in G^2$. The corresponding range map r' and domain map d' are given by $r'([x]) = r(x)$ and $d'([x]) = d(x)$.

(v) The groupoid $G/\mathrm{Iso}(G)^\circ$ is topologically principal.

(vi) If G is amenable, then so is $G/\mathrm{Iso}(G)^\circ$.

Proof. (i) We have that $\mathrm{Iso}(G)^\circ$ acts on the right of G by $(\rho, \cdot)$, where $\rho = d$ and $x \cdot y = xy$. Note that $G \times_{\rho,d} \mathrm{Iso}(G)^\circ$ is just the subset of G^2 where the second entry is an element of $\mathrm{Iso}(G)^\circ$. The orbit of $x \in G$ is

$$[x] = x \cdot \mathrm{Iso}(G)^\circ = \{x \cdot y : (x, y) \in G^2, y \in \mathrm{Iso}(G)^\circ\}.$$

Orbits partition G, and we may form the orbit space $G/\mathrm{Iso}(G)^\circ$ with the quotient topology.

If $xy = x$, then $y = d(x)$, so the action is free. The function Ψ in continuous, with inverse (defined on its image) given by $(a, x) \mapsto (a, a^{-1}x)$. This function is also continuous, so Ψ is a homeomorphism onto its image. Since $\mathrm{Iso}(G)^\circ$ is closed in G, the image of Ψ is closed in $G \times G$, so Ψ maps compact sets to compact sets. Hence the action is proper.

By the above discussion and [MW95, Corollary 2.3], the orbit space $G/\mathrm{Iso}(G)^\circ$ is locally compact and Hausdorff.

(ii) Let $\alpha_x \colon \mathrm{Iso}(G)_{d(x)} \to \mathrm{Iso}(G)_{r(x)}$ denote the map $y \mapsto xyx^{-1}$. We note that α_x is a bijection from $\mathrm{Iso}(G)_{d(x)}$ to $\mathrm{Iso}(G)_{r(x)}$. Indeed, if $z \in \mathrm{Iso}(G)^\circ_{r(x)}$, then $z = x(x^{-1}yx)x^{-1}$ where $x^{-1}yx \in \mathrm{Iso}(G)^\circ_{d(x)}$, so α_x is surjective. It is clearly injective with inverse $\alpha_{x^{-1}}$, taking z to $x^{-1}zx$.

If we can show that

$$x\,\mathrm{Iso}(G)^\circ x^{-1} \subseteq \mathrm{Iso}(G)^\circ \tag{5.1}$$

for all $x \in G$, then α_x would restrict to an injection $\mathrm{Iso}(G)_{d(x)} \to \mathrm{Iso}(G)_{r(x)}$. Its inverse $\alpha_{x^{-1}}$ would restrict to an injection in the other direction, thus proving the desired bijectivity.

To prove (5.1), it suffices to show that $x\,\mathrm{Iso}(G)^\circ x^{-1}$ is open. Suppose $y \in \mathrm{Iso}(G)^\circ$ with $r(y) = d(x)$ and let $U \in G^{\mathrm{op}}$ be an open neighbourhood of y in $\mathrm{Iso}(G)^\circ$. Let $V \in G^{\mathrm{op}}$ be an open neighbourhood of x. Then $y \in d(V) \cap r(U)$, so there is some open set in $W \subseteq d(V) \cap r(U)$ since G^{op} is a basis. Since the product of open sets are open, VUV^{-1} is an open neighbourhood of xyx^{-1}. If we have $u \in U, v \in V$ with $vuv^{-1} \in VUV^{-1}$, then $d(vuv^{-1}) = d(v^{-1}) = r(v) = r(vuv^{-1})$, so $VUV^{-1} \subseteq \mathrm{Iso}(G)$. Thus VUV^{-1} is an open neighbourhood of xyx^{-1} in $\mathrm{Iso}(G)^\circ$.

(iii) We must check that $\mathrm{Iso}(G)^\circ_u$ is closed under multiplication and inversion. If $x, y \in \mathrm{Iso}(G)^\circ_u$ have neighbourhoods U and V in $\mathrm{Iso}(G)_u$ (and hence in $\mathrm{Iso}(G)^\circ_u$), then UV is an open neighbourhood of xy in $\mathrm{Iso}(G)^\circ_u$, so $xy \in \mathrm{Iso}(G)^\circ_u$. Inversion is similar: If $x \in \mathrm{Iso}(G)^\circ_u$ has neighbourhood $U \subseteq \mathrm{Iso}(G)^\circ_u$, then U^{-1} is a neighbourhood of x^{-1} since inversion is a homeomorphism (and hence an open map). Finally, $\mathrm{Iso}(G)^\circ_u$ contains the identity u of G^u_u, so it is a subgroup.

From (ii) we know that $\mathrm{Iso}(G)^\circ_u$ is closed under conjugation by any element of G^u_u. Hence $\mathrm{Iso}(G)^\circ_u$ is normal.

(iv) We first show that the maps in question are well-defined. Multiplying $x \in G$ by any feasible element from $\mathrm{Iso}(G)^\circ$ doesn't change the range or source of x, so r' and d' are well-defined. Suppose next that $(x, y) \in G^2$ and $x' = xa, y' = yb$ for $a, b \in \mathrm{Iso}(G)^\circ$. Then $x'y' = xayb = (xy)(y^{-1}ayb)$. By (ii) we have $y^{-1}ay \in \mathrm{Iso}(G)^\circ$, and since $\mathrm{Iso}(G)^\circ$ is closed under multiplication as proved in (iii) we have $y^{-1}ayb \in \operatorname{Iso}(G)^\circ$. This means that $[x'y'] = [xy]$, so multiplication is well-defined. Similarly, if $x \in G$ with $x' = ax$ for some $a \in \operatorname{Iso}(G)^\circ$, we have $(x')^{-1} = x^{-1}(xa^{-1}x^{-1})$, so $[x^{-1}] = [(x')^{-1}]$.

The next step is to prove that the maps above are continuous. We will need the fact that the quotient map $q\colon G \to G/\operatorname{Iso}(G)^\circ$ is open. We follow the proof in [MW95, Lemma 2.1]. Suppose that $V \subseteq G$ is open; we must show that $q(V)$ is open in $G/\operatorname{Iso}(G)^\circ$. It suffices to prove that $q^{-1}(q(V)) = V \cdot \operatorname{Iso}(G)^\circ$ is open in G. This is the same as saying that any net converging to a point in $V \cdot \operatorname{Iso}(G)^\circ$ eventually lies in $V \cdot \operatorname{Iso}(G)^\circ$. Suppose therefore that $(a_\lambda)_{\lambda\in\Lambda}$ is a net in G converging to $vx = v \cdot x \in V \cdot \operatorname{Iso}(G)^\circ$. Then $d(a_\lambda) \to d(vx) = d(x)$. Since d is open, we may by [Wil07, Proposition 1.15] pass to a subnet and find $(x_\lambda) \to x$ with $d(x_\lambda) = d(a_\lambda)$. Then $a_\lambda x_\lambda^{-1} \to vxx^{-1} = v$, so $(a_\lambda x_\lambda^{-1})$ is eventually in V since V is open. But then $a_\lambda = a_\lambda x_\lambda^{-1} x_\lambda = (a_\lambda x_\lambda^{-1}) \cdot x_\lambda$ is eventually in $V \cdot \operatorname{Iso}(G)^\circ$, as we wanted.

We will prove that multiplication is continuous; the proof for inversion is similar. Suppose $[x_\lambda] \to [x]$ and $[y_\lambda] \to [y]$ where $(x_\lambda, y_\lambda) \in G^2$ for all λ in the indexing set Λ; we want to show that $[x_\lambda y_\lambda] \to [xy]$. By Lemma 2.0.3, it suffices to show that every subnet of $([x_\lambda y_\lambda])_{\lambda\in\Lambda}$ has a subnet converging to $[xy]$. By passing to any subnet and relabeling, we assume that $([x_\lambda y_\lambda])$ is a subnet. Since the quotient map $q\colon G \to G/\operatorname{Iso}(G)^\circ$ is open and surjective, we have by [Wil07, Proposition 1.15] that there are subnets $([x_{\lambda_\beta}])$ and $([y_{\lambda_\beta}])$ such that there exists nets (a_{λ_β}) and (b_{λ_β}) in $\operatorname{Iso}(G)^\circ$ with

$$x_{\lambda_\beta} a_{\lambda_\beta} \to x \text{ and } y_{\lambda_\beta} b_{\lambda_\beta} \to y.$$

Then $x_{\lambda_\beta} a_{\lambda_\beta} y_{\lambda_\beta} b_{\lambda_\beta} \to xy$ since multiplication is continuous, and hence $[x_{\lambda_\beta} y_{\lambda_\beta}] \to [xy]$ since q is continuous.

Lastly, we must show that $G/\operatorname{Iso}(G)^\circ$ is étale. It suffices to show that r' is a local homeomorphism. Let $[x] \in G/\operatorname{Iso}(G)^\circ$, and suppose K is a compact neighbourhood of x such that $r|_K$ is a homeomorphism. Since q is continuous, $q(K)$ is a compact neighbourhood of $[x]$. Since $r'|_{q(K)}$ is a continuous bijection onto its image, it is a homeomorphism by Proposition 2.0.4.

(v) By [Bro+14, Lemma 3.3] it suffices to prove that the interior of $\operatorname{Iso}(G/\operatorname{Iso}(G)^\circ)$ equals $q(G^0)$. The set $q(G^0)$ is clearly contained in this interior, since G^0 is open and q is an open map. Next, take some $b \in \operatorname{Iso}(G/\operatorname{Iso}(G)^\circ) \setminus q(G^0)$, so that $r'(b) = d'(b)$ but $b \neq r'(b)$. We must check that this isn't an interior point. We have $b = q(x)$ for some $x \in \operatorname{Iso}(G) \setminus \operatorname{Iso}(G)^\circ$. (We can't have $x \in \operatorname{Iso}(G)^\circ$ since then $b = q(x)$ must be a unit, but $b \neq r'(b)$.) Take an open neighbourhood U of b. Then $q^{-1}(U)$ is an open neighbourhood of x. Since x lies on the border of $\operatorname{Iso}(G)$, $q^{-1}(U)$ must intersect $G \setminus \operatorname{Iso}(G)$. Take some element $y \in q^{-1}(U) \cap \operatorname{Iso}(G)$, so that $s(y) \neq d(y)$. Then $q(y) \in U$ and $r'((y)) \neq d'(q(y))$. Thus $q(y) \notin \operatorname{Iso}(G/\operatorname{Iso}(G)^\circ)$, and b does not belong to the interior of $\operatorname{Iso}(G/\operatorname{Iso}(G)^\circ)$. Thus the interior of $\operatorname{Iso}(G/\operatorname{Iso}(G)^\circ)$ is precisely the set $q(G^0)$.

■

Proposition 5.1.15. *Let G be an étale groupoid such that $\mathrm{Iso}(G)^\circ$ is closed in G. Then there is a C^*-homomorphism $\kappa\colon C^*(G) \to C^*(G/\operatorname{Iso}(G)^\circ)$ such that*

$$\kappa(f)(b) = \sum_{q(x)=b} f(x) \text{ for } f \in C_c(G) \text{ and } b \in G/\operatorname{Iso}(G)^\circ,$$

where $q\colon G \to G/\operatorname{Iso}(G)^\circ$ is the quotient map.

Proof. For ease of notation, we set $H \coloneqq G/\operatorname{Iso}(G)^\circ$. By [MRW87, Lemma 2.9(b)] we have that κ defines a surjection $C_c(G) \to C_c(G/\operatorname{Iso}(G)^\circ)$. To see that this surjection is a $*$-map, we calculate

$$\begin{aligned}
\kappa(f^*)(b) &= \sum_{q(x)=b} \overline{f(x^{-1})} \\
&= \overline{\sum_{q(x)=b^{-1}} f(x)} \\
&= \kappa(f)^*(b),
\end{aligned}$$

where we have used that the equality of the sets $\{x^{-1} : q(x) = b\}$ and $\{x : q(x) = b^{-1}\}$. (This equality follows from the fact that $b^{-1} = q(x^{-1})$, as in part (iv) of Proposition 5.1.14.) To see that κ is a homomorphism, we first write

$$\begin{aligned}
\left(\kappa(f) * \kappa(g)\right)(b) &= \sum_{d'(a)=r'(b)} [\kappa(f)(a^{-1})][\kappa(g)(ab)] \\
&= \sum_{d'(a)=r'(b)} \left[\sum_{q(x)=a^{-1}} f(x)\right]\left[\sum_{q(y)=ab} g(y)\right] \\
&= \sum_{d'(a)=r'(b)} \left[\sum_{q(x)=a} f(x^{-1})\right]\left[\sum_{q(y)=ab} g(y)\right] \\
&= \sum_{d'(a)=r'(b)} \left[\sum_{q(x)=a}\sum_{q(y)=ab} f(x^{-1})g(y)\right].
\end{aligned}$$

We can rewrite this into

$$\begin{aligned}
&\sum_{d'(a)=r'(b)} \left[\sum_{q(x)=a}\sum_{q(x^{-1}y)=b} f(x^{-1})g(y)\right] \\
&= \sum_{d'(a)=r'(b)} \left[\sum_{q(x)=a}\sum_{q(z)=b} f(x^{-1})g(xz)\right],
\end{aligned}$$

since $q(y) = ab$ if and only if $q(x^{-1}y) = b$ as multiplication is well-defined. We can pull out the inner sum to get

$$\begin{aligned}
&\sum_{q(z)=b}\left[\sum_{d'(a)=r'(b)}\sum_{q(x)=a} f(x^{-1})g(xz)\right]\\
&= \sum_{q(z)=b}\sum_{d(x)=r(z)} f(x^{-1})g(xz)\\
&= \sum_{q(z)=b} (g*f)(z) = \kappa(f*g)(b),
\end{aligned}$$

where we have used that $d'(q(x)) = r'(q(y))$ if and only if $d(x) = r(y)$. Hence κ is a homomorphism.

We want to extend κ to the whole of $C^*(G)$, and to do so we have to show that it is continuous. Since κ is linear, it suffices to prove that it is a contraction. If π is a representation of $C_c(H)$, then $\pi \circ \kappa$ is a representation of $C_c(G)$ since κ is a $*$-homomorphism. If we can show that κ is an I-norm contraction, then the induced representation $\pi \circ \kappa$ is I-norm bounded since

$$\|\pi \circ \kappa(f)\| \leq \|\kappa(f)\|_I \leq \|f\|_I$$

for any $f \in C_c(G)$. Then we would have

$$\begin{aligned}
\|\kappa(f)\|_* &= \sup\{\|\pi(\kappa(f))\| : \pi \text{ is an } I\text{-norm bounded repr. of } C_c(H)\}\\
&\leq \sup\{\|\rho(f)\| : \rho \text{ is an } I\text{-norm bounded repr. of } C_c(G)\}\\
&= \|f\|_*.
\end{aligned}$$

This would prove that κ is continuous and extends like we wanted.

We look at the sum

$$\sum_{x\in H^u} |\kappa(f)(x)| \leq \sum_{x\in H^u}\sum_{q(y)=x} |f(y)|.$$

In the double sum to the right, we sum over all $y \in G$ such that $q(y) \in H^u$ for a given $u \in H^0$. This is precisely the same as the set $q^{-1}(H^u)$. Since we can identify H^0 with G^0 and q preserves the range map, we have $q^{-1}(H^u) \subseteq G^u$. Hence we can write

$$\sum_{x\in H^u}\sum_{q(y)=x} |f(y)| = \sum_{y\in q^{-1}(H^u)} |f(y)| \leq \sum_{y\in G^u} |f(y)|,$$

and similarly we have

$$\sum_{x\in H_u} |\kappa(f)(x)| \leq \sum_{y\in G_u} |f(y)|.$$

We plug this into the definition of the I-norm to get

$$\begin{aligned}
\|\kappa(f)\|_I &= \sup_{u\in H^0} \max\{\sum_{x\in H_u} |\kappa(f)(x)|, \sum_{x\in H^u} |\kappa(f)(x)|\}\\
&\leq \sup_{u\in G^0} \max\{\sum_{y\in G_u} |f(y)|, \sum_{y\in G^u} |f(y)|\}\\
&= \|f\|_I,
\end{aligned}$$

and we are done. ■

5.2 Deaconu-Renault Groupoids

We will now properly define Deaconu-Renault groupoids and prove some of their basic properties. This is also the time for us to change the notation of general groupoid elements. From now on, instead of letting u and v be general unit elements, we denote unit elements by x and y. This notation coincides well when the unit space of our groupoid is a locally compact Hausdorff space X. We will use greek letters such as γ to denote general groupoid elements.

Definition 5.2.1. Let X be a locally compact Hausdorff space. Given k commuting local homeomorphisms $\{T_i\}_{i=1}^k$ on X (which are not necessarily surjective), we obtain an action of $\mathbb{N}_0^k$ on X written $n \mapsto T^n = T_1^{n_1} \dots T_k^{n_k}$ for $n = (n_1, \dots, n_k) \in \mathbb{N}_0^k$. The corresponding *Deaconu-Renault groupoid* is the set

$$G_T \coloneqq \bigcup_{m,n \in \mathbb{N}^k} \{(x, m-n, y) \in X \times \mathbb{Z}^k \times X : T^m x = T^n y\}. \tag{5.2}$$

Its unit space is $G_T^0 = \{(x, 0, x) : x \in X\}$, which is canonically identified with X. The range and domain maps are $r(x,n,y) = x$ and $d(x,n,y) = y$, respectively. Multiplication is defined by $(x,n,y)(y,m,z) = (x, n+m, y)$, and inversion is given by $(x,n,y)^{-1} = (y, -n, x)$.

Below, we prove that there is a natural topology on G_T making it locally compact. All through the thesis, we have restricted our attention to special classes of groupoids, always excused by the fact that this is all we need to understand Deaconu-Renault groupoids. Spesifically, we have looked at Hausdorff étale groupoids. The reader would be cheated if we did not include the proof that this is the case.

Proposition 5.2.2. *Let X be a locally compact Hausdorff space with a Deaconu-Renault groupoid G_T. For open sets $U, V \subseteq X$ and for $m, n \in \mathbb{N}_0^k$, define*

$$Z(U, m, n, V) \coloneqq \{(x, m-n, y) : x \in U, y \in V, T^m x = T^n y\}. \tag{5.3}$$

These sets form a basis for a locally compact Hausdorff topology on G_T. The sets $Z(U,m,n,V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m(U) = T^n(V)$ are a basis for the same topology. With this topology, G_T is a (locally compact Hausdorff) étale groupoid.

Proof. We split the proof into sections. We let $\mathcal{B}$ denote the family of sets in question. The first step is to prove that $\mathcal{B}$ does indeed form a basis. After that, we show that the generated topology is locally compact (in the usual sense), and that the groupoid operations are continuous. The final part is to prove the existence of a left Haar system. This turns out to be the counting measures. The restricted topology will make sure that G_T is étale.

The set $\mathcal{B}$ is a basis: This is a modification of the proof in [ER07, Proposition 3.2]. It is clear that $\mathcal{B}$ covers G_T, so it remains to prove that if $x \in A \cap B$ for $A, B \in \mathcal{B}$, then we have $x \in C$ for some $C \in \mathcal{B}$ with $C \subseteq A \cap B$. Suppose therefore that

$$(x, r, y) \in Z(U_1, m_1, n_1, V_1) \cap Z(U_2, m_2, n_2, V_2).$$

We want $U \subseteq U_1 \cap U_2$, $V \subseteq V_1 \cap V_1$, and $m, n \in \mathbb{N}_0^k$ such that if $T^m x = T^n y$, then $T^{m_1} x = T^{n_1} y$ and $T^{m_2} x = T^{n_2} y$.

Note that $m_1 - n_1 = m_2 - n_2$. Set $m = m_2 + m_1$ and $n = m_2 + n_1$. If we have $T^m x = T^n y$ for some $u \in U_1 \cap U_2$ and $v \in V_1 \cap V_2$, then we would have

$$T^{m_2}(T^{m_1} u) = T^{m_2}(T^{n_1} v).$$

To force $T^{m_1} u = T^{n_1} v$, it would therefore suffice to require that T^{m_2} is injective on some appropriately chosen set. This is the motivation for the picking the following sets. Let W_1 be some open neighbourhood of $T^{m_1} x = T^{n_1} y$ such that T^{m_2} is injective (or a homeomorphism!) on W_1. This is possible since T is a local homeomorphism. Similarly, pick W_2 with $T^{m_2} x = T^{n_2} x \in W_2$ so that T^{m_1} is injective (or a homeomorphism!) on W_2. Now we can define

$$\begin{aligned} U &= U_1 \cap U_2 \cap T^{-m_2}(W_1) \cap T^{-m_1}(W_2) \text{ and} \\ V &= V_1 \cap V_2 \cap T^{-n_2}(W_1) \cap T^{-n_1}(W_2). \end{aligned}$$

We claim that $Z(U, m, n, V)$ is an element of $\mathcal{B}$ satisfying the desired conditions. Note that $(x, r, y) \in Z(U, m, n, V)$, since $x \in U$ and $y \in V$ by construction, and $m - n = m_1 - n_1 = r$. We also have

$$T^m x = T^{m_2}(T^{m_1} x) = T^{m_2}(T^{n_1} y) = T^n y.$$

If $(u, m - n, v) \in Z(U, m, n, V)$, then

$$T^{m_2}(T^{m_1} u) = T^{m_2}(T^{n_1} v),$$

so $T^{m_1} u = T^{n_1} v$ by the choice of W_1. Similarly, since $m_1 = m_2 + n_1 - n_2$, we could write

$$T^{m_1}(T^{m_2} u) = T^{m_1}(T^{n_2} v),$$

so $T^{m_2} u = T^{n_2} v$ by the same reason. Hence

$$Z(U, m, n, V) \subseteq Z(U_1, m_1, n_1, V_1) \cap Z(U_2, m_2, n_2, V_2).$$

In the discussion above, we could have picked W_1 and W_2 in such a manner that $T^m|_U$ and $T^n|_V$ were homeomorphisms. (See the parentheses.) Note also that $T^m(U) = T^n(V)$. We can ensure that $T^m U = T^n V$ by intersecting U with $T^{-m}(T^m U \cap T^n V)$ and V with $T^{-n}(T^m U \cap T^n V)$. But this means that for every neighbourhood Z of $(x, r, y) \in G_T$, there is some neighbourhood $Z(U, m, n, V)$ between x and Z such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m U = T^n V$. Thus the family of these sets generate the same topology, as claimed.

The topology is locally compact: Let K_1 and K_2 be any compact subsets of X, and let $p, q \in \mathbb{N}_0^k$. Then the set

$$K := \{(x, y) \in K_1 \times K_2 : T^p x = T^q y\}$$

is compact. Indeed, first define $\varphi\colon (x, y) \mapsto (T^m x, T^n y)$ on $X \times X$, which is continuous. Since X is Hausdorff, its diagonal is closed in $X \times X$. This means that $\varphi^{-1}(\mathrm{Diag}(X))$ is closed, so $K = (K_1 \times K_2) \cap \varphi^{-1}(\mathrm{Diag}(X))$ is compact. We claim that the map $(x, y) \mapsto (x, p, q, y)$ is a continuous surjection onto $Z(K_1, p, q, K_2)$ (which is defined as if K_1 and K_2 were to be open). The map is clearly surjective, and continuous since $(U \cap K_1, V \cap K_2)$ is mapped into (U, p, q, V) for any open sets $U, V \subseteq X$. But this means that K is compact in G_T. By varying K_1 and K_2, we can force K to be a neighbourhood of any point in G_T; hence G_T is locally compact.

G_T is a topological groupoid: We need to prove that the groupoid operations are continuous in the topology described above. We start with inversion. Let $(x, r, y) \in G_T$ and suppose (U, m, n, V) be a basic open neighbourhood of (x, r, y). Then

$$Z(U, m, n, V)^{-1} = \{(y, n-m, x) : x \in U, y \in V, T^m x = T^n y\} = Z(V, n, m, U)$$

is an open neighbourhood of $(x, r, y)^{-1} = (y, -r, x)$.

Multiplication is slightly more technical. Suppose $(x, m - n, y), (y, m' - n', z) \in G_T$, and let $Z(U, m, n, V)$ and $Z(U', m', n', V')$ be basic open neighbourhoods of these points, respectively. We pick the basis elements from the restricted basis described above, in other words with $T^m(U) = T^n(V)$, $T^m|_U$ and $T^n|_V$ being homeomorphisms, et cetera. We will see that

$$Z(U, m, n, V)Z(U', m', n', V') = Z(U, m + m', n + n', V'),$$

which is enough to prove that multiplication is continuous. Suppose first that $(u, m - n, v)(v, m' - n', v')$ is in the product. This equals $(u, m + m' - n - n', v')$, and we have

$$T^{m+m'} u = T^{m'}(T^m u) = T^{m'}(T^n v) = T^n(T^{m'} v) = T^{n+n'} v',$$

so the element is in $Z(U, m + m', n + n', V')$. For the other inclusion, let $(u, m + m' - n - n', v') \in Z(U, m + m', n + n', V')$. We need some $v \in V \cap U'$ such that

$$T^m u = T^n v \text{ and } T^{m'} v = T^{n'} v'.$$

The obvious candidate is $v := T^{m-n} u = T^{n'-m'} v'$, which clearly satisfies the above conditions. It only remains to see that $v \in V \cap U'$. This is also clear since $v \in T^{m-n} U = V$ and $v \in T^{n'-m'} V' = U'$ by our choice of basis.

The counting measures form a left Haar system: The final step to showing that G_T is a locally compact groupoid, is to show that it has a left Haar system. If this were the case, it is easily seen that G_T is étale. Indeed, let $Z := Z(U, m, n, V)$ be an element of the restricted basis described above. If $y \in r(Z)$, then there exists a *unique* point $x \in U$ with $T^m x = T^n y$, so r is injective on Z. The only choice of x would be $T^{n-m} y$, so

$$y \mapsto (T^{n-m}, m - n, y)$$

is a continuous inverse of r. Hence the range map restricts to a homeomorphism on Z, and the same holds for the domain map. One could also prove that G_T is étale by noting that X is open. Thus, the only thing left to show is that the counting measures form a left Haar system for G_T. The proof for this is a special case of one direction of Proposition 3.2.9.

■

Example 5.2.3. Let $X \coloneqq \mathbb{R}^d/\mathbb{Z}^d$ be the d-torus. Suppose further that $T = (T_1, \ldots, T_k)$ are k commuting hyperbolic toral automorphisms on X. The T_i comes from k commuting $d \times d$-matrices, with integer matrices and determinant ± 1. Then there is a group action of $\mathbb{N}^k$ on X given by

$$n \mapsto T^n = T_1^{n_1} \ldots T_k^{n_k}$$

for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$. We can then form the associated Deaconu-Renault groupoid G_T. We study this dynamical system further in Appendix A.

Remark 5.2.4. Let T be an action of $\mathbb{N}_0^k$ on X by local homeomorphisms. Recall that the *orbit* of a point $x \in X$ in groupoid terms is defined as $[x] = r(G_x)$. However, when dealing with dynamical systems such as the one in Example 5.2.3, the orbit of a point $x \in X$ is usually defined as

$$[x] \coloneqq \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbb{N}_0^k\}.$$

We observe that this coincides the definition of orbit in Definition 5.1.4. Indeed, we have $y \in r(G_x)$ if and only if there exists a point $\gamma = (y, m-n, x) \in G_T$ with $T^m x = T^n y$, which is the same as saying that $y \in [x]$ as defined above.

Remark 5.2.5. We will be using the k-torus $\mathbb{T}^k$ further in this chapter, but viewed as the product of k unit circles in $\mathbb{C}$. Addition, multiplication and complex conjugation in $\mathbb{T}^k$ is defined component wise. If we have some tuple $r = (r_1, \ldots, r_k) \in \mathbb{R}^k$ and a point $z = (z_1, \ldots, z_k) \in \mathbb{T}^k$, we define z^c by

$$z^r \coloneqq z_1^{r_1} \ldots z_k^{r_k}.$$

These conventions enables us to use identities connected to the unit circle, such as $\overline{z^r} = z^{-r}$ and $\overline{z}z = (1, \ldots, 1) = 1$ for $z \in \mathbb{T}^d$ and $r \in \mathbb{R}^k$.

We state [SW16, Lemma 3.5], but skip the proof since it's just cite-chasing with amenability.

Lemma 5.2.6. *Let G_T be the locally compact Hausdorff étale groupoid arising from an action T of $\mathbb{N}_0^k$ on X by local homeomorphisms as above. Let $c\colon G_T \to \mathbb{Z}^k$ be the function defined by $c(x, k, y) = k$. Then both $c^{-1}(\{0\})$ and G_T are amenable.*

Given an open set $U \subseteq X$, it will be useful to know which pairs $(m, n) \in \mathbb{N}_0^k$ gives us that $T^m = T^n$ on U.

Lemma 5.2.7. *Let X be a locally compact Hausdorff space, and let T be an action of $\mathbb{N}_0^k$ on X by local homeomorphisms. For each open set $U \subseteq X$, let*

$$\Sigma_U \coloneqq \{(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 : T^m x = T^n x \text{ for all } x \in U\}. \tag{5.4}$$

Then

(i) Σ_U *is a submonoid of* $\mathbb{N}_0^k \times \mathbb{N}_0^k$*,*

(ii) Σ_U *is an equivalence relation on* $\mathbb{N}_0^k$*,*

(iii) if $U \subseteq V$*, then* $\Sigma_U \subseteq \Sigma_V$*, and*

(iv) if $p \in \mathbb{N}_0^k$*, then we have* $\Sigma_U \subseteq \Sigma_{T^p U}$*.*

Proof. (i) It is clear that the unit $(0, 0)$ is an element of Σ_U, so all that is left is to show that Σ_U is closed under addition. Suppose therefore that $(m, n), (p, q) \in \mathbb{N}_0^k$. Then for all $x \in U$, we have

$$T^{m+p}x = T^m(T^p x) = T^m(T^q x) = T^q(T^m x) = T^q(T^n x) = T^{n+q}x,$$

so $(m + p, n + q) \in \mathbb{N}_0^k$.

(ii) The relation Σ_U is reflexive since $T^n x = T^n x$, symmetric since $T^n x = T^m x$ if and only if $T^m x = T^n x$, and transitive since $T^n x = T^m x$ and $T^m x = T^p$ implies $T^n x = T^p x$.

(iii) If $U \subseteq V$, then for all (m, n) with $T^m x = T^n x$ for all $x \in V$, we certainly have $T^m x = T^n x$ for all $x \in U$. Hence $\Sigma_U \subseteq \Sigma_V$.

(iv) If $T^n x = T^m x$ for all $x \in U$, then as in part (i) with $q = p$, we have $T^n(T^p x) = T^m(T^p x)$. Hence $\Sigma_U \subseteq \Sigma_{T^p U}$.

■

In Section 5.3, we characterise the primitive ideal space of Deaconu-Renault groupoids coming from so-called *irreducible actions*. We define this next. With an irreducible action, we can describe the Σ_U-sets in greater detail.

Definition 5.2.8. Suppose X is a locally compact Hausdorff space with a Deaconu-Renault groupoid G_T. If $X = G_T^0$ is $\mathbb{N}_0^k$-irreducible, then we say that *T acts irreducibly* on X.

Note that if T acts irreducibly on X, then Lemma 5.1.6 implies that X is an orbit closure.

Lemma 5.2.9. *Let T be an $\mathbb{N}_0^k$-irreducible action on a locally compact Hausdorff space X by local homeomorphisms. For all open subsets $U, V \subseteq X$, there exists a nonempty open set W such that $\Sigma_U \cup \Sigma_V \subseteq \Sigma_W$.*

Proof. We may assume that U and V are both nonempty. Fix $x \in X$ with $\overline{[x]} = X$. Since $[x]$ is dense, it intersects both U and V, and thus there are points $y \in U$, $z \in V$ with $T^n y = T^m z$ for some $n, m \in \mathbb{N}_0^k$. Then $W := T^n U \cap T^m V \neq \emptyset$. Since T^n, T^m are local homeomorphisms and therefore open, W is open. By Lemma 5.2.7 we have

$$\Sigma_U \subseteq \Sigma_{T^m U} \subseteq \Sigma_W,$$

and similarly,

$$\Sigma_V \subseteq \Sigma_{T^n V} \subseteq \Sigma_W.$$

■

Given X and T as in Lemma 5.2.9, define

$$\Sigma := \bigcup\{\Sigma_U : U \subseteq X \text{ is nonempty and open}\}. \tag{5.5}$$

We give $\mathbb{N}_0^k \times \mathbb{N}_0^k$ a partial order $\leq$ by saying that $(n_1, n_2) \leq (m_1, m_2)$ if $n_1 \leq m_1$ and $n_2 \leq m_2$ component-wise. We let $\Sigma^{\min}$ denote the collection of minimal elements of $\Sigma \setminus \{(0,0)\}$ with respect to this order.

Lemma 5.2.10. *Let T be an irreducible action of $\mathbb{N}_0^k$ by local homeomorphisms on a locally compact Hausdorff space X, and let Σ and Σ^{min} be as above. Then Σ is a submonoid of $\mathbb{N}_0^k \times \mathbb{N}_0^k$ and an equivalence relation on $\mathbb{N}_0^k$. We have $\Sigma = (\Sigma - \Sigma) \cap (\mathbb{N}_0^k \times \mathbb{N}_0^k)$. Furthermore, Σ^{min} is finite and generates Σ as a monoid.*

Proof. We start by showing that Σ is a monoid. It has a unit since $(0,0) \in \Sigma_X \subseteq \Sigma$. Suppose next that $(m,n),(p,q) \in \Sigma$. Then there are open sets $U, V \subseteq X$ with $(m,n) \in \Sigma_U, (p,q) \in \Sigma_V$. By Lemma 5.2.9, there is some open set $W \subseteq X$ with $\Sigma_U, \Sigma_V \subseteq \Sigma_W$. Then $(m+p, n+q) \in \Sigma_W \subseteq \Sigma$ by part (i) of 5.2.7. Hence Σ is closed under addition.

To see that Σ is an equivalence relation, note first that it inherits reflexivity and symmetry from the Σ_U's. For transitivity, suppose $(m,n),(n,p) \in \Sigma$, with $(m,n) \in \Sigma_U$ and $(n,p) \in \Sigma_V$ for open sets $U, V \subseteq X$. By Lemma 5.2.9, there is an open set W with $(m,n),(n,p) \in \Sigma_W$, so Σ inherits transitivity also.

Next, we show $\Sigma = (\Sigma - \Sigma) \cap (\mathbb{N}_0^k \times \mathbb{N}_0^k)$. One inclusion is trivial, since $\Sigma = (\Sigma - 0) \cap (\mathbb{N}_0^k \times \mathbb{N}_0^k)$. For the other inclusion, suppose $(m,n),(p,q) \in \Sigma$ with $m-p, n-q \in \mathbb{N}_0^k$. We must show that $(m-p, n-q) \in \Sigma$. As before, we may choose an open set W with $(m,n),(p,q) \in \Sigma_W$. Fix some element in $T^{p+q}W$, say $x = T^{p+q}y$ for $y \in W$. Since Σ_W is symmetric, we have $(q,p) \in \Sigma_W$, and since it closed under addition we have $(m+q, n+p) \in \Sigma_W$. Then we have

$$T^{m-p}x = T^{m-p}(T^{p+q}y) = T^{m+q}y = T^{n+p}y = T^{n-q}(T^{q+p}y) = T^{n-q}x,$$

so that $(m-p, n-q) \in \Sigma_{T^{p+q}W} \subseteq \Sigma$.

Next on the agenda is showing that $\Sigma^{\min}$ is finite. In our favourite article, [SW16], they refer to Dickon's lemma; however, we will prove it in an alternative way. More generally, if M is any subset of $\mathbb{N}_0^r \setminus \{(0,\ldots,0)\}$ for any $r \in \mathbb{N}$, ordered in the same manner, then the set of minimal elements of M is finite. We prove this by induction. The result clearly holds for $r = 1$, as we can just pick the smallest number of M. Suppose the result holds for $r = k$, and let $M \subseteq \mathbb{N}_0^{k+1}$ be nonzero. Let $M_1 \subseteq \mathbb{N}_0^k$ be the set of elements in M where the last coordinate is deleted, and let M_2 be the set where the first coordinate is deleted. Then $n = (n_1, \ldots, n_{k+1}) \in M$ is minimal if and only if $(n_1, \ldots, n_k)$ and $(n_2, \ldots, n_{k+1})$ are minimal in M_1 and M_2, respectively. By assumption there, can only be a finite number of these.

To see that $\Sigma^{\min}$ generates Σ, we must show that each element of Σ can be written as a finite sum of elements of $\Sigma^{\min}$. We prove this by induction on $|(m,n)| = \sum_{i=1}^k (m_i + n_i)$. If $|(m,n)| = 0$, we have $m = n = 0$ and the statement is clear. Next, let $N \in \mathbb{N}$ and suppose that all pairs $(m', n') \in \Sigma$ with $|(m', n')| \leq N$ is a finite sum of elements in $\Sigma^{\min}$. Suppose we have some $(m,n) \in \Sigma$ with $|(m,n)| = N+1$. By definition, there must be some $(p,q) \in \Sigma^{\min}$ with $(p,q) \leq (m,n)$. Then $m-p, n-q \geq 0$, so $(m-p, n-q) \in$

$(\Sigma - \Sigma) \cap \mathbb{N}_0^k \times \mathbb{N}_0^k = \Sigma$ by the discussion above. We have $|(m-p, n-q)| \leq N$, so

$$(m, n) = (m-p, n-q) + (p, q)$$

is a sum of elements in $\Sigma^{\min}$ by the induction hypothesis. ■

For T as above, we let

$$H(T) \coloneqq \{m - n : (m, n) \in \Sigma\} \tag{5.6}$$

and

$$Y^{\max} \coloneqq \bigcup \{Y \subseteq X : Y \text{ is open and } \Sigma_Y = \Sigma\}.$$

Lemma 5.2.11. *Let T be an irreducible action of $\mathbb{N}_0^k$ by local homeomorphisms on a locally compact Hausdorff space X. With Σ as in* (5.5), *we have*

$$\Sigma = \{(m, n) \in \mathbb{N}_0^k \times \mathbb{N}_0^k : m - n \in H(T)\}. \tag{5.7}$$

The set Y^{max} is nonempty and open, and is the maximal open set in X such that $\Sigma_{Y^{max}} = \Sigma$. We have $T^m Y^{max} \subseteq Y^{max}$ for all $m \in \mathbb{N}_0^k$.

Proof. We have $\Sigma \subseteq \{(m, n) : m - n \in H(T)\}$ by definition. Conversely, suppose we have $m - n = p - q$ where $(p, q) \in \Sigma$. Set $g = m - p \in \mathbb{Z}^k$ (not necessarily $\mathbb{N}_0^k$), and fix $a, b \in \mathbb{N}_0^k$ such that $g = a - b$. Since Σ is reflexive, we have $(b, b) \in \Sigma$, and since it is a monoid we have $(p + a, q + a) \in \Sigma$. We have $(m, n) = (p + q, q + g)$, and so

$$(m, n) = (p + (a - b), q + (a - b)) = (p + a, q + a) - (b, b) \in (\Sigma - \Sigma) \cap (\mathbb{N}_0^k \times \mathbb{N}_0^k).$$

Hence $(m, n) \in \Sigma$ by Lemma 5.2.10.

For each $(m, n) \in \Sigma^{\min}$, pick a representative open subset $U \subseteq X$ with $(m, n) \in \Sigma_U$. Since $\Sigma^{\min}$ is finite, repeated use of Lemma 5.2.9 yields a nonempty open set $Y \subseteq X$ such that $\Sigma^{\min} \subseteq \Sigma_Y$. Since Σ_Y is a monoid by Lemma 5.2.7, and contains the generators of Σ by Lemma 5.2.10, we have $\Sigma_Y = \Sigma$. Hence $Y^{\max}$ is nonempty. It is open as it is a union of open sets. It is also maximal, since any other set Y satisfying $\Sigma_Y = \Sigma$ will be a subset of $Y^{\max}$ by definition. Lastly, if $m \in \mathbb{N}_0^k$, then $\Sigma_{Y^{\max}} \subseteq \Sigma_{T^m Y^{\max}} = \Sigma$ by Lemma 5.2.7. Then $T^m Y^{\max} \subseteq Y^{\max}$ since $Y^{\max}$ is maximal. ■

If $Y \subseteq X$ is open and satisfies $\Sigma_Y = \Sigma$ and $T^p Y \subseteq Y$ for all $p \in \mathbb{N}_0^k$, then in some sense $G_T|_Y$ contains a lot of infomation about T. As shown below, we have an easy characterisation of $\operatorname{Iso}(G_T|_Y)^\circ$ if this is the case, and that allows us to apply Proposition 5.1.14 to the groupoid $G_T|_Y$.

Proposition 5.2.12. *Let T be an irreducible action of $\mathbb{N}_0^k$ by local homeomorphisms of a locally compact Hausdorff space X, and let G_T be the associated Deaconu-Renault groupoid. The set $H(T)$ of* (5.6) *is a subgroup of $\mathbb{Z}^k$. Let Σ be as in* (5.5), *and let $Y \subseteq X$ be an open set such that $\Sigma_Y = \Sigma$ and $T^p Y \subseteq Y$ for all $p \in \mathbb{N}_0^k$. Then*

$$\operatorname{Iso}(G_T|_Y)^\circ = \{(x, u, x) : x \in Y \text{ and } u \in H(T)\},$$

and $\operatorname{Iso}(G_T|_Y)^\circ$ is closed in $G_T|_Y$.

Proof. We have $0 \in H(T)$ since Σ is nonempty and reflexive. Since Σ is symmetric, we have $p-q \in H(T)$ if and only if $q-p \in H(T)$, so $n \in H(T)$ implies $-n \in H(T)$. Suppose that $m,n \in H(T)$ with, say, $m = p_1 - q_1$ and $n = p_2 - q_2$ for $(p_i, q_i) \in \Sigma$. We have $(p_1+p_2, q_1+q_2) \in \Sigma$ since Σ is a monoid, and thus $m+n = p_1+p_2-q_1-q_2 \in H(T)$. Hence $H(T)$ is a subgroup of $\mathbb{Z}^k$.

Let $Y \subseteq X$ be as described above, and let $x \in Y$ and $n \in H(T)$. By Lemma 5.2.10, there exists $(p,q) \in \Sigma$ with $n = p-q$. Pick some open neighbourhood $U \subseteq Y$ of x such that T^p and T^q are homeomorphisms when restricted to U. We have $T^p y = T^q y$ for all $y \in U$ since $(p,q) \in \Sigma$. Now we have

$$\{(y,n,y) : y \in U\} = Z(U,p,q,U).$$

Indeed, $\{(y,n,y) : y \in U\} \subseteq Z(U,p,q,U)$ is clear by the above discussion. For the reverse inclusion, let $y_1, y_2 \in U$ be such that $T^p y_1 = T^q y_2$; then $y_1 = y_2$ since T^p and T^q are homeomorphisms on U. Now we have that $\{(y,n,y) : y \in U\}$ is an open neighbourhood of (x,n,x) contained in $\{(y,n,y) : y \in Y, n \in H(T)\}$. Thus every point in the latter set is interior, hence $\{(y,n,y) : y \in Y, n \in H(T)\}$ is an open set (in $\mathrm{Iso}(G_T)$). Now we have

$$\{(y,n,y) : y \in Y, n \in H(T)\} \subseteq \mathrm{Iso}(G_T|_Y)^\circ.$$

For the reverse inclusion, suppose that $(z,m,z) \in \mathrm{Iso}(G_T|_Y)^\circ$; we must show that $m \in H(T)$. By Proposition 5.2.2, there exist $r,s \in \mathbb{N}_0^k$ and open sets $U,V \subseteq Y$ such that $(z,m,z) \in Z(U,r,s,V) \subseteq \mathrm{Iso}(G_T|_Y)^\circ$, with $T^r(U) = T^s(V)$. Thus for every $x \in U$, there exists $x' \in V$ with $T^r x = T^s x'$, and we can write

$$\begin{aligned} Z(U,r,s,V) &= \{(x, r-s, y) : x \in U, y \in V, T^r x = T^s y\} \\ &= \{(x, r-s, x') : x \in U\}. \end{aligned}$$

In fact, since $Z(U,r,s,V) \subseteq \mathrm{Iso}(G_T|_Y)$, its elements must have the same domain and range. Hence

$$Z(U,r,s,V) = \{(x,r-s,x) : x \in U\},$$

so we have $T^r x = T^s x$ for all $x \in U$. Thus $(r,s) \in \Sigma_U \subseteq \Sigma$, and $m \in H(T)$ as required.

Lastly, we prove that $\mathrm{Iso}(G_T|_Y)^\circ$ is closed. Suppose $(x,n,y) \in G_T|_Y$. This element being in $\mathrm{Iso}(G_T|_Y)^\circ$ is, by the discussion above, the same as saying that $x = y (\in Y)$ and $n \in H(T)$. Removing the elements of $G_T|_Y$ not satisfying this therefore gives us $\mathrm{Iso}(G_T|_Y)^\circ$. Elements with $n \notin H(T)$ is covered precisely by basis elements $Z(U,r,s,V)$ with $r-s \notin H(T)$, and elements with $x \neq y$ is covered precisely by basis elements with $U \cap V = \emptyset$. Hence

$$\mathrm{Iso}(G_T|_Y)^\circ = G_T|_Y \setminus \left(\bigcup_{m-n \notin H(T)} Z(U,m,n,V) \cup \bigcup_{U \cap V = \emptyset} Z(U,m,n,V) \right).$$

We only remove open sets, so $\mathrm{Iso}(G_T|_Y)^\circ$ is closed. ■

We finish off this section by putting things into context with the more general literature. This yields a corollary that is key to Theorem 5.4.2.

Corollary 5.2.13. *Let T be an irreducible action of $\mathbb{N}_0^k$ by local homeomorphisms on a locally compact Hausdorff space X. Let Σ and $H(T)$ be as in* (5.5) *and* (5.6)*, respectively. Suppose that Y is an open subset of X such that $T^pY \subseteq Y$ for all $p \in \mathbb{N}_0^k$ and such that $\Sigma_Y = \Sigma$. Then the following statements are true.*

(i) Regard $C_c(G_T|_Y)$ as a subalgebra of $C_c(G_T)$. The identity map extends to an injective homomorphism $\iota\colon C^*(G_T|_Y) \to C^*(G)$, and $\iota(C^*(G_T|_Y))$ is a hereditary subalgebra of $C^*(G_T)$.

(ii) The map $\pi \mapsto \pi \circ \iota$ is a bijection from the collection of irreducible representations of $C^*(G_T)$ that are injective on $C_0(X)$ to the space of irreducible representations of $C^*(G_T|_Y)$ that are injective on $C_0(Y)$, up to unitary equivalence. Moreover, the map $\ker \pi \mapsto \ker(\pi \circ \iota)$ is a homeomorphism from $\{I \in \operatorname{Prim} C^*(G_T) : I \cap C_0(X) = \{0\}\}$ onto $\{J \in \operatorname{Prim} C^*(G_T|_Y) : J \cap C_0(Y) = \{0\}\}$.

Proof. (i) The inclusion $C_c(G_T|_Y) \hookrightarrow C_c(G_T)$ is a continuous $*$-homomorphism (in any topology), so it extends to a $*$-homomorphism $\iota\colon C^*(G_T|_Y) \to C^*(G_T)$. Fix $x \in Y$, and let L^x be the regular representation of $C^*(G_T)$ on $\ell^2((G_T)_x)$ from Lemma 5.1.13. Then the image of $L^x \circ \iota$ leaves the subspace $\ell^2(\{(y, r, x) \in G_T : y \in Y\})$ invariant. Indeed, if we have $\gamma = (y, r, x) \in G_T$ for some $y \in Y$, then for any $f \in C_c(G_T|_Y)$ we can write

$$(L^x \circ \iota)(f)\delta_\gamma = \sum_{\alpha \in (G_T|_Y)_y} f(\alpha)\delta_{\alpha\gamma}.$$

Since $d(\gamma) = x$ and $r(\alpha) \in Y$, the above function is an element of $\ell^2(\{(y, r, x) \in G_T : y \in Y\})$.

Let L_Y^x be the corresponding regular representation of $C^*(G_T|_Y)$ on $\ell^2((G_T|_Y)_x) = \ell^2(\{(y, r, x) \in G_T : y \in Y\})$. We get a representation $L_Y^x \oplus 0$ of $C^*(G_T|_Y)$ on $\ell^2((G_T)_x)$ if we compose with the natural inclusion $\ell^2((G_T|_Y)_x) \hookrightarrow \ell^2((G_T)_x)$. By the above discussion, the two representations $L^x \circ \iota$ and $L_Y^x \oplus 0$ are unitarily equivalent. Since G_T is amenable, ι must be injective. Indeed, suppose $a \in C^*(G_T|_Y)$ is nonzero; then there must be some $x \in X$ such that $L_Y^x(a) \neq 0$; since L_Y^x is equivalent to $L^x \circ \iota$, we must have $\iota(a) \neq 0$.

Now we show that $\iota(C^*(G_T|_Y))$ is hereditary. Let $(f_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $C_0(Y)$. We claim that

$$\iota(C^*(G_T|_Y)) = \overline{\bigcup_{\lambda \in \Lambda} f_\lambda * C^*(G_T) * f_\lambda}. \tag{5.8}$$

To prove so, note first that (f_λ) is also an approximate identity for $\iota(C^*(G_T|_Y))$. Indeed, suppose $f \in C_c(G_T|_Y)$. Then for $\gamma \in G_T|_Y$, we have

$$(f_\lambda * f)(\gamma) = f_\lambda(r(\gamma))f(\gamma) \to f(\gamma).$$

This extends by continuity to all of $\iota(C^*(G_T|_Y))$. (We let $f * g$ denote the product of $f, f \in C_c(G_T)$ to separate it from the complex multiplication

$f(\gamma)g(\gamma)$.) Thus we can conclude that $\iota(C^*(G_T|_Y))$ is contained in the closure of $\cup_{\lambda\in\Lambda} f_\lambda * C^*(G_T) * f_\lambda$. To see the other inclusion, it suffices to show that $f_\lambda * f \in C_c(G_T|_Y)$ for all $f \in C_c(G_T)$. This is clear by the equation $(f_\lambda * f)(\gamma) = f_\lambda(r(\gamma))f(\gamma)$, since $f_\lambda(r(\gamma))$ is zero whenever $\gamma \notin G_T|_Y$.

Now we can show that $\iota(C^*(G_T|_Y))$ is hereditary by using Theorem 2.0.18. Suppose that $b, b' \in \iota(C^*(G_T|_Y))$ and $a \in C^*(G_T)$. There is no longer any complex multiplication in the picture, so we will not use the $*$-notation to denote multiplication. We can write

$$b = \lim_{\lambda\to\infty} f_\lambda b_\lambda f_\lambda \text{ and } b' = \lim_{\lambda\to\infty} f_\lambda b'_\lambda f_\lambda$$

for $b_\lambda, b'_\lambda \in C^*(G_T|_Y)$. But then we have

$$\begin{aligned} bab' &= \lim_{\lambda\to\infty} (f_\lambda b_\lambda f_\lambda) a (f_\lambda b'_\lambda f_\lambda) \\ &= \lim_{\lambda\to\infty} f_\lambda (b_\lambda f_\lambda a f_\lambda b'_\lambda) f_\lambda, \end{aligned}$$

which is an element of $\iota(C^*(G_T|_Y))$ by (5.8). Hence $\iota(C^*(G_T|_Y))$ is hereditary.

(ii) Recall that we can regard $C_0(Y) \subseteq C_0(X)$ as a $*$-subalgebra of $C^*(G_T)$. Let I_Y be the ideal generated by $C_0(Y)$. We claim that $\iota(C^*(G_T|_Y))$ is Morita equivalent to I_Y. To show that, we will first see that $\iota(C^*(G_T|_Y)) \subseteq I_Y$. Suppose for contradiction that this is not the case; then $\iota(C^*(G_T|_Y)) \cap I_Y$ will be a proper ideal in $\iota(C^*(G_T|_Y))$ containing $C_0(Y)$. This would contradict that $C_0(Y)$ is full in $C^*(G_T|_Y)$, so that is what we want to show. Let $I' \subseteq C^*(G_T|_Y)$ be an ideal containing $C_0(Y)$. We will show that $C_c(G_T|_Y) * I = C_c(G_T|_Y)$, where $I = I' \cap C_c(G_T|_Y)$. Then $I' = \overline{I} = C^*(G_T|_Y)$, so $C_0(Y)$ will be full in $C^*(G_T|_Y)$ as we wanted. Suppose therefore that $f \in C_c(G_T|_Y)$. Since $G_T|_Y$ is étale, we may assume that $f \in C_c(A)$ where $A \in G_T|_Y^{\mathrm{op}}$, as in the proof of Proposition 3.2.11. Since the support of f is compact, $K := d(\mathrm{supp}(f))$ is compact in $G_T|_Y^0$. Cover K in a finite number of relatively compact open subsets of $G_T|_Y^0$, and let U be the union of these elements. Then by Urysohn's lemma there exist a function $g \in C_c(G_T|_Y^0)$ which equals 1 on K and 0 outside U. Let $x \in \mathrm{supp}(f)$; then $d(x) \in K$. We have

$$(f * g)(x) = f(x)g(d(x)) = f(x)$$

by Remark 3.2.12. For $x \notin \mathrm{supp}(f)$, we have $(f * g)(x) = 0$, so $f * g = f$. Hence $C_c(G_T|_Y) * I \subseteq I$.

To see that $\iota(C^*(G_T|_Y))$ is Morita equivalent to I_Y, it suffices by [Bro77, Theorem 2.8] to show that $\iota(C^*(G_T|_Y))$ is a full hereditary subalgebra of I_Y. Any ideal containing $\iota(C^*(G_T|_Y))$ also contains $C_0(Y)$, and hence contains I_Y. Thus there can be no ideal between the two, and $\iota(C^*(G_T|_Y))$ is full in I_Y. Since $\iota(C^*(G_T|_Y))$ is hereditary in $C^*(G_T)$, it is certainly hereditary in I_Y too.

Now we can get along with proving the actual statement. Let π be an irreducible representation of $C^*(G_T)$ that is injective on $C_0(X)$. In particular, π is nonzero on I_Y, so by [Arv76, Theorem 1.3.4], the representation

$\pi|_{I_Y}$ is irreducible. As noted in the proof of [SW16, Corollary 3.12], one can implement the so-called Rieffel induction from I_Y to $\iota(C^*(G_T|_Y))$ by restriction of fuctions. Since these two C^*-algebras are Morita equivalent, Rieffel induction takes irreducible representations of the prior to irreducible representations of the latter. Hence $\pi \circ \iota$ is an irreducible representation of $C^*(G_T|_Y)$ that is injective on $C_0(Y)$. Additionally, since the representation π is fully determined by how it behaves on the ideal I_Y by [Arv76, Theorem 1.3.4], the map $\pi \mapsto \pi \circ \iota$ is injective.

Next, we will see that $\pi \mapsto \pi \circ \iota$ is surjective. Suppose ρ is an irreducible representation of $C^*(G_T|_Y)$ on a Hilbert space K, such that ρ is injective on $C_0(Y)$. We can view it as a representation of $\iota(C^*(G_T|_Y))$. By [Ped79, Proposition 4.1.8], there is an irreducible representation ρ' of I_Y on a Hilbert space H, such that there is a subspace $H' \subseteq H$ that is isomorphic to K, and such that ρ' is unitarily equivalent to ρ when the elements in its image is restricted to H'. By [Arv76, Theorem 1.3.4], we can extend ρ' uniquely to a representation π of $C^*(G_T)$ (on H). Now $\pi \circ \iota$ is unitarily equivalent to ρ. We illustrate this in the below diagram, which commutes up to unitary equivalence.

$$\begin{tikzcd}
C^*(G_T) \arrow[drr, "\pi"] & & & \\
I_Y \arrow[u, hook, "\subseteq"] \arrow[rr, "\rho'"'] & & B(H) \arrow[r, "\cdot|_{H'}"] & B(H') \\
\iota(C^*(G_T|_Y)) \arrow[u, hook, "\subseteq"] \arrow[rr, "\rho"] & & B(K) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Next, [SW16] claim that $\pi|_{C_0(X)}$ has $\mathbb{N}_0^k$-invariant support. This is the same as saying that, for $f \in C_0(X)$, $\pi(f) = 0$ implies $\pi(f \circ T^n) = 0$ for all $n \in \mathbb{N}_0^k$. This is clear in the related case of crossed product C^*-algebras. For example, take a transformation group crossed product $C(Z) \rtimes_\beta H$ for a compact space Z and a discrete group H. Let $\varphi \rtimes u$ be a nondegenerate representation, and let $f \in C(Z) \subseteq C_c(H, C(Z))$ satisfy $\varphi \rtimes u(f) = 0$. Then we have $\varphi(\beta_g(f)) = u_g \varphi(f) u_g^* = 0$ for any $g \in H$, so

$$\begin{aligned}
\|\varphi \rtimes u(\alpha_g(f))\| &= \| \int_H \varphi(\alpha_g(f)) u_h \, dh \| \\
&\leq \int_H \|\varphi(\alpha_g(f))\| \, dh = 0.
\end{aligned}$$

By Corollary 4.2.20, one can write $C^*(G)$ as a (different kind of) crossed product $C_0(X) \times_\beta S$ for an additive localisation (X, S). One could probably use the discussion of the associated covariant representations in [Sie97], or the possibly the temptatious identity in Definition 4.2.9, to find a similar argument in the case of groupoid C^*-algebras. However, this isn't obvious to the author, and we settle with this handwavy discussion.

Since π is faithful on $C_0(Y)$, its kernel cannot be the whole of $C_0(X)$. The space $C_0(X)$ is G_T-simple since T acts irreducibly, so $\ker\left(\pi|_{C_0(X)}\right) = \{0\}$ as we wanted.

■

5.3 The Primitive Ideals of the C^*-algebra of an Irreducible Deaconu-Renault Groupoid

In this section, we will focus on the case when T is an irreducible action of $\mathbb{N}_0^k$ on Y, where Y is a locally compact Hausdorff space with $\Sigma_Y = \Sigma$ in the usual notation. The results from the previous section can be applied, and we sum up the implications in the following remark.

Remark 5.3.1. Let T and Y be as above. By part (i) of Lemma 5.2.7, we have $\Sigma_U = \Sigma$ for all nonempty open subsets $U \subseteq Y$. Lemma 5.2.11 says that if $m - n \in H(T)$, then $T^m x = T^n x$ for all $x \in Y$. Proposition 5.2.12 says that

$$\operatorname{Iso}(G_T)^\circ = \{(x, n, x) : x \in Y \text{ and } n \in H(T)\}.$$

Lemma 5.3.2. *Suppose that T is an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y with $\Sigma_Y = \Sigma$. Then the sets*

$$\underline{Z}(U, q(m), q(n), V) \coloneqq \{(x, q(m-n), y) : x \in U, y \in V, T^m x = T^n y\}$$

form a basis for the topology on $G_T / \operatorname{Iso}(G_T)^\circ$.

Proof. We need some preparation. Recall that $H(T)$ is a (normal) subgroup of $\mathbb{Z}^k$. Let $q \colon \mathbb{Z}^k \to \mathbb{Z}^k / H(T)$ be the quotient map. As noted in Remark 5.3.1, we have $\operatorname{Iso}(G_T)^\circ = \{(x, n, x) : x \in Y \text{ and } n \in H(T)$. By this, we can identify $G_T / \operatorname{Iso}(G_T)^\circ$ with

$$\{(x, q(r), y) : (x, r, y) \in G_T\} \subseteq Y \times (\mathbb{Z}^k / H(T)) \times Y.$$

Indeed, if $[(x, n, y)] = (x, n, y) \cdot \operatorname{Iso}(G_T)^\circ \in G_T / \operatorname{Iso}(G_T)^\circ$, then

$$\begin{aligned} [(x, n, y)] &= \{(x, n, y)(y, m, y) : m \in H(T)\} \\ &= \{(x, n + m, y) : m \in H(T)\} \\ &\sim (x, n + H(T), y) \\ &= (x, q(n), y), \end{aligned}$$

where we have used $\sim$ as the vaguely defined form of similarity that mathematicians call "identification".

Let $p \colon G_T \to G_T / \operatorname{Iso}(G_T)^\circ$ be the other quotient map (not to be confused with q above). We denote by $\mathcal{B}$ the collection of sets of the form $\underline{Z}(U, q(m), q(n), V)$. We shall now prove that $\mathcal{B}$ forms a basis for the topology on $G_T / \operatorname{Iso}(G_T)^\circ$. We first show that these sets are in fact open. Pick any $\underline{Z}(U, q(m), q(n), V) \in \mathcal{B}$. For each $(x, q(m-n), y) \in \underline{Z}(U, q(m), q(n), V)$, we have $p^{-1}((x, q(m-n), y)) = (x, m-n, y) \cdot \operatorname{Iso}(G_T)^\circ$ by the identification in the previous paragraph. Thus

$$\begin{aligned} &p^{-1}(\underline{Z}(U, q(m), q(n), V)) \\ &\quad = \bigcup \{(x, m-n, y) \cdot \operatorname{Iso}(G_T)^\circ : x \in U, y \in V, T^m x = T^n y\} \\ &\quad = \bigcup \{(x, m-n, y) \cdot \operatorname{Iso}(G_T)^\circ : (x, m-n, y) \in Z(U, m, n, V)\} \\ &\quad = Z(U, m, n, V) \cdot \operatorname{Iso}(G_T)^\circ. \end{aligned}$$

The action in question is just multiplication. Since G_T is étale, multiplication is an open map, so $p^{-1}(\underline{Z}(U, q(m), q(n), V))$ (and hence $\underline{Z}(U, q(m), q(n), V)$) is open.

To show that $\mathcal{B}$ is a basis generating the topology, we must prove that there exists an element from $\mathcal{B}$ between any point and open neighbourhood containing it. Suppose therefore that we have a point $(x, q(m-n), y) \in G_T/\operatorname{Iso}(G_T)^\circ$ with an open neighbourhood S. Then $p^{-1}(S)$ is open in G_T, and we can write it as a union of basis elements. So we have

$$p^{-1}(S) = \bigcup_{i \in I} Z(U_i, m_i, n_i, V_i),$$

where I is a countable index set, and for all $i \in I$ we have $U_i, V_i \subseteq Y$ and $q(m_i - n_i) = m - n$. For one of these basis elements, say $Z(U_t, m_t, n_t, V_t)$ for some $t \in I$, we have $x \in p(U_t)$ and $y \in p(V_t)$. (If not, the above union would not cover $p^{-1}(S)$.) But then $p(Z(U_t, m_t, n_t, V_t)) = \underline{Z}(U_t, q(m-n), V_t)$ is an open neighbourhood of $(x, q(m-n), y)$ inside S, so $\mathcal{B}$ generates the topology. ■

Let T act on X as above, and recall the group $H(T)$ as defined in Equation (5.6). Set

$$H(T)^\perp = \{z \in \mathbb{T}^k : z^r = 1 \text{ for all } r \in H(T)\}.$$

Lemma 5.3.3. *Suppose that T is an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y such that $\Sigma_Y = \Sigma$. Then*

(i) there is a strongly continuous action α of $\mathbb{T}^k$ on $C^*(G_T)$ such that $\alpha_z(f)(x, n, y) = z^n f(x, n, y)$ for $f \in C_c(G_T)$.

Let $\kappa\colon C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be the homomorphism from Proposition 5.1.15.

(ii) There is a strongly continuous action $\tilde{\alpha}$ of $H(T)^\perp$ on $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ such that $\tilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$ for all $z \in H(T)^\perp \subseteq \mathbb{T}^k$.

(iii) If $\overline{z}w \notin H(T)^\perp$, then $(\ker(\kappa \circ \alpha_z) + \ker(\kappa \circ \alpha_w)) \cap C_0(Y) \neq \{0\}$.

(iv) We have $\ker(\kappa \circ \alpha_z) = \ker(\kappa \circ \alpha_w)$ if and only if $\overline{z}w \in H(T)^\perp$.

Proof. (i) Let $c\colon G_T \to \mathbb{Z}^k$ be the map defined by $c(x, n, y) = n$, and let $z \in \mathbb{T}^k$. Note that $c(\gamma^{-1}) = -c(\gamma)$ and $c(\gamma\delta) = c(\gamma) + c(\delta)$. There is an induced $*$-homomorphism $\alpha_z\colon C_c(G_T) \to C_c(G_T)$, defined by

$$\alpha_z(f)(\gamma) = z^{c(\gamma)} f(\gamma)$$

for $f \in C_c(G_T)$ and $\gamma \in G_T$. Indeed, the map is clearly linear. To see that α_z is a $*$-map, note first that $c(\gamma^{-1}) = -c(\gamma)$, and $\overline{z^r} = z^{-r}$ for all $r \in \mathbb{Z}^k$ since $z \in \mathbb{T}^k$. Now we can write

$$\begin{aligned}\alpha_z(f)^*(\gamma) &= \overline{z^{c(\gamma^{-1})} f(\gamma^{-1})} = \overline{z^{-c(\gamma)} f(\gamma^{-1})} \\ &= z^{c(\gamma)} \overline{f(\gamma^{-1})} = \alpha_z(f^*)(\gamma).\end{aligned}$$

To see that α_z is a homomorphism, suppose that $f \in C_c(A)$ and $g \in C_c(B)$ for $A, B \in G_T^{\mathrm{op}}$, and let $\gamma = \delta\beta$ for $\delta \in A$ and $\beta \in B$. We see that $\alpha_z(f) \in C_c(A)$ and $\alpha_z(g) \in C_c(B)$. Then

$$\begin{aligned}\alpha_z(f * g)(\gamma) &= z^{c(\gamma)}(f * g)(\gamma) = z^{c(\delta)+c(\beta)} f(\delta)g(\beta) \\ &= \left(z^{c(\delta)} f(\delta)\right)\left(z^{c(\beta)} g(\beta)\right) \\ &= \alpha_z(f)(\delta) * \alpha_z(g)(\beta).\end{aligned}$$

As in the proof of Proposition 3.2.11, multiplicativity of α_z for general f and g follows from linearity.

We want to extend α_z to $C^*(G_T)$. As when extending κ in Proposition 5.1.15, it suffices to prove that α_z is an I-norm contraction. In fact, it is I-norm preserving, since $|\alpha_z(f)(\gamma)| = |f(\gamma)|$. Hence α_z extends to $\alpha_z \colon C^*(G_T) \to C^*(G_T)$. Since $\alpha_{\bar{z}}$ is an inverse of α_z on $C_c(G_T)$ and hence $C^*(G_T)$, we have $\alpha_z \in \mathrm{Aut}(C^*(G_T))$. The map $z \mapsto \alpha_z$ is a homomorphism, since for $f \in C_c(G_T)$ we have

$$\begin{aligned}\alpha_{zw}(f)(\gamma) &= (zw)^{c(\gamma)} f(\gamma) = z^{c(\gamma)} w^{c(\gamma)} f(\gamma) \\ &= z^{c(\gamma)}(\alpha_w(f)(\gamma)) = (\alpha_z \circ \alpha_w)(f)(\gamma).\end{aligned}$$

Now we shall see that $z \mapsto \alpha_z$ is strongly continuous. Suppose first that $f \in C_c(G_T)$ has support in $c^{-1}(\{r\})$ for some $r \in \mathbb{Z}^k$. Then $\alpha_z(f) = z^r f$, so $z \mapsto \alpha_z(f)$ is continuous. Indeed, if $z_\lambda \to z$ in $\mathbb{T}^k$, then $\|(z_\lambda^r - z^r)f\| \to 0$. If $f \in C_c(G_T)$ is any element, we have that $f|_{c^{-1}(\{r\})} = f\chi_{c^{-1}(\{r\})}$ is continuous. To see this, it suffices to see that $c^{-1}(\{r\})$ is clopen. It is closed since any net $((x_\lambda, r, y_\lambda))_\lambda$ with constant middle term can only converge to a point with the same middle term. Openness is also clear, as

$$\begin{aligned}c^{-1}(\{r\}) &= \{(x, r, y) \in G_T : \text{ there exist } m, n \in \mathbb{N}_0^k \text{ with } T^m x = T^n y\} \\ &= \bigcup_{m-n=r} Z(Y, m, n, Y).\end{aligned}$$

Since the sets $c^{-1}(\{r\})$ partition G_T when r varies, we have

$$f = \sum \{f|_{c^{-1}(\{r\})} : \mathrm{supp}(f) \cap c^{-1}(\{r\}) \neq \emptyset\}.$$

Consequently, the function $z \mapsto \alpha_z(f)$ is a finite sum of continuous functions, and is therefore continuous (for any $f \in C_c(G_T)$). To see that $z \mapsto \alpha_z$ is strongly continuous, suppose $z_\lambda \to z$ in $\mathbb{T}^k$. Then

$$\|(\alpha_{z_\lambda} - \alpha_z)f\| \to 0$$

since $w \mapsto \alpha_w(f)$ is continuous.

(ii) As in the proof of Lemma 5.3.2, we can identify $G_T/\operatorname{Iso}(G_T)^\circ$ with

$$\{(x, q(r), y) : (x, r, y) \in G_T\} \subseteq Y \times (\mathbb{Z}^k/H(T)) \times Y.$$

Using this and the basis we found in Lemma 5.3.2, we can argue precisely as we have done above to see that there is an action $\widetilde{\alpha}$ of $H(T)^\perp$ on $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ such that

$$\widetilde{\alpha}_z(f)(x, q(r), y) = z^r f(x, q(r), y)$$

for all $f \in C_c(G_T/\operatorname{Iso}(G_T)^\circ)$. Note that $q(r) \mapsto z^r$ is well-defined for $z \in H(T)^\perp$. Indeed, if $q(r) = q(s)$ for $r, s \in \mathbb{Z}^k$, then $q(r-s) = 0$, so $r - s \in H(T)$. But then $z^{r-s} = 1$, so $z^r = z^s$.

Now we will check the identity $\widetilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$ for $z \in H(T)^\perp$. We let $c\colon G_T/\operatorname{Iso}(G_T)^\circ \to \mathbb{Z}^k/H(T)$ be the function $c(x, q(r), y) = q(r)$, as with the function of the same name in the beginning of this proof. For $q(r) \in \mathbb{Z}^k/H(T)$, we will by $z^{q(r)}$ mean z^r, which works since $q(r) \mapsto z^r$ is well-defined. Let $p\colon G_T \to G_T/\operatorname{Iso}(G_T)^\circ$ be the quotient map. For $f \in C_c(G_T)$, $z \in H(T)^\perp$ and $b \in G_T/\operatorname{Iso}(G_T)^\circ$, we have

$$(\widetilde{\alpha}_z \circ \kappa(f))(b) = z^{c(b)} \sum_{p(\gamma)=b} f(\gamma) = \sum_{p(\gamma)=b} z^{c(b)} f(\gamma).$$

Here, we can use the fact that $z^{c(p(\gamma))} = z^{c(\gamma)}$ for $\gamma \in G_T$ and $z \in H(T)^\perp$. Consequently, the above equals

$$\begin{aligned}\sum_{p(\gamma)=b} z^{c(\gamma)} f(\gamma) &= \kappa(\gamma \mapsto z^{c(\gamma)} f(\gamma))(b) \\ &= (\kappa \circ \alpha_z(f))(b).\end{aligned}$$

The identity $\widetilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$ extends to $C^*(G_T)$ by continuity of $\widetilde{\alpha}_z, \alpha_z$ and κ.

(iii) Suppose $\overline{z}w \notin H(T)^\perp$. Then we can find some $n \in H(T)$ with $z^n \neq w^n$. Indeed, suppose for contradiction that $z^m = w^m$ for all $m \in H(T)$; then we would have $(\overline{z}w)^m = z^{-m}w^m = 1$, so $\overline{z}w \in H(T)^\perp$. Pick any nonzero function $f \in C_c(Y)$, and define a function $f_n \in C_c(\{(x,n,x) \in G_T : x \in Y\}) \subseteq C_c(G_T)$ by setting $f_n(x,n,x) = f(x,0,x)$ and zero otherwise. (This function is continuous and compactly supported since f is.) Now we have $w^n f - f_n \in \ker(\kappa \circ \alpha_w)$ and $-z^n f + f_n \in \ker(\widetilde{\alpha}_z \circ \kappa)$. We will show the first one; the second is analogous. We can write

$$\begin{aligned}(\kappa \circ \alpha_w)(w^n f - f_n)(b) &= \kappa(\gamma \mapsto w^{c(\gamma)}(w^n f(\gamma) - f_n(\gamma)))(b) \\ &= \sum_{p(\gamma)=b} w^{c(\gamma)}(w^n f(\gamma) - f_n(\gamma)).\end{aligned}$$

We split the sums into the supports of the functions to get

$$\begin{aligned}&\left[\sum_{p(\gamma)=b, c(\gamma)=0} w^{c(\gamma)} w^n f(\gamma)\right] - \left[\sum_{p(\gamma)=b, c(\gamma)=n} w^{c(\gamma)} f_n(\gamma)\right] \\ &= \left[\sum_{p(\gamma)=b, c(\gamma)=0} w^n f(\gamma)\right] - \left[\sum_{p(\gamma)=b, c(\gamma)=n} w^n f_n(\gamma)\right].\end{aligned}$$

Suppose $b = (x, 0, x)$ for some $x \in Y$. (We may assume that the middle term is zero since all other such b's would be sent to zero, as 0 and n are elements of $H(T)$.) Then there is only one element in both summation indeces above. Thus the above equals

$$w^n f(u, 0, u) - w^n f_n(x, n, x) = 0,$$

which is what we wanted. A key element here is the fact that $(x, n, x) \in G_T$; otherwise the sum would be nonzero. We know that this is the case since $(x, n, x) \in \operatorname{Iso}(G_T)^\circ \subseteq G_T$ by Remark 5.3.1. Hence $w^n f - f_n \in \ker(\kappa \circ \alpha_w)$. Since $w^n f - f_n$ and $-z^n f + f_n$ are elements of $C_c(Y)$, and

$$(w^n f - f_n) + (-z^n f + f_n) = (w^n - z^n) f \neq 0$$

by our choice of n, we have that

$$(w^n - z^n) f \in (\ker(\kappa \circ \alpha_z) + \ker(\kappa \circ \alpha_w)) \cap C_0(Y) \setminus \{0\}.$$

Note that even though $(\ker(\kappa \circ \alpha_z) + \ker(\kappa \circ \alpha_w)) \cap C_0(Y) \neq \{0\}$, we cannot necessarily conclude that $\kappa \circ \alpha_z$ and $\kappa \circ \alpha_w$ aren't injective on $C_0(Y)$. Indeed, the functions $w^n f - f_n$ and $-z^n f + f_n$ that we picked out were *not* elements of $C_0(Y)$, but they became so when added together. Below, we will prove that the functions in question in fact *are* injective on $C_0(Y)$.

(iv) We start with a contrapositive proof of the "only if"-statement. Suppose that $\overline{z}w \notin H(T)^\perp$. If we can prove that $\kappa \circ \alpha_z$ and $\kappa \circ \alpha_w$ are injective (have zero kernel) on $C_0(Y)$, then we are done. Indeed, suppose for contradiction that we then had $\ker(\kappa \circ \alpha_z) = \ker(\kappa \circ \alpha_w)$. By (iii), we could find some nonzero element $f \in \ker(\kappa \circ \alpha_z) \cap C_0(Y)$, which is a contradiction.

We first see that $\kappa \circ \alpha_z$ and $\kappa \circ \alpha_w$ are injective on $C_c(Y)$. Let $f \in C_c(Y)$ be nonzero; suppose $f(\gamma) \neq 0$ for some $\gamma \in Y$. Since α_z and α_w equal the identity on $C_c(X)$, it suffices to see that κ does not send f to zero. Set $b := p(\gamma)$. Since the quotient map p is injective on X by the identity in Remark 5.3.1, we have

$$\kappa(f)(b) = \sum_{p(\beta) = b} f(\beta) = f(\gamma) \neq 0,$$

so $\kappa(f) \neq 0$. Note also that $\kappa \circ \alpha_z$ and $\kappa \circ \alpha_w$ are injective on $C_0(Y)$.

Suppose that $\overline{z}w \in H(T)^\perp$. Since $\overline{z}z = 1$ and $v \mapsto \alpha_v$ is a homomorphism, we have $\alpha_w = \alpha_{\bar{z}w} \circ \alpha_z$. Thus

$$\ker(\kappa \circ \alpha_w) = \ker(\kappa \circ \alpha_{\bar{z}w} \circ \alpha_z) = \ker(\widetilde{\alpha}_{\bar{z}w} \circ \kappa \circ \alpha_z).$$

Since $\alpha_{\bar{z}w}$ is an automorphism, it is injective, so $\ker(\kappa \circ \alpha_w) = \ker(\kappa \circ \alpha_z)$ as we wanted.

■

Since $\widetilde{\alpha}$ is a strongly continuous action, we can form the induced algebra $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \widetilde{\alpha})$ as in [RW98, Chapter 6.3]. This is defined as

$$\{s \in C(\mathbb{T}^k, C^*(G_T/\operatorname{Iso}(G_T)^\circ) : \\ s(wz) = \widetilde{\alpha}_z(s(w)) \text{ for all } w \in \mathbb{T}^k \text{ and } z \in H(T)^\perp\}.$$

We will often denote the induced algebra by $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ to avoid overwhelming notation; the proofs involving it are technical enough without dealing with complicated names.

Lemma 5.3.4. *Suppose that T is an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y with $\Sigma_Y = \Sigma$. If $(x, r, y) \in G_T$, then $(x, r + s, y) \in G_T$ for all $s \in H(T)$.*

Proof. We may assume that $r = m - n$ with $T^m x = T^n y$. Let $s \in H(T)$ be any element. Then $s = s_+ - s_-$ for $(s_+, s_-) \in \Sigma = \Sigma_Y$. Then we have $T^{s_+} z = T^{s_-} z$ for all $z \in Y$, and we can write

$$T^{m+s_+} x = T^{s_+} T^m x = T^{s_+} T^n y = T^{s_-} T^n y = T^{n+s_-} y.$$

Hence we have $(x, r + s, y) = (x, (m + s_+) - (n + s_-), y) \in G_T$. ■

Let $C_c(H(T))$ denote the set of finitely supported functions $H(T) \to \mathbb{C}$. The preceding lemma enables us to define a left multiplication of $C_c(H(T))$ on $C_c(G_T)$ by

$$\varphi \diamond f(x, r, y) \coloneqq \sum_{s \in H(T)} \varphi(s) f(x, s - r, y). \tag{5.9}$$

(We need Lemma 5.3.4 to be able to write $f(x, s - r, y)$.)

Let us check that $(\varphi * \psi) \diamond f = \varphi \diamond (\psi \diamond f)$. We calculate:

$$((\varphi * \psi) \diamond f)(x, p, y) = \sum_{r \in H(T)} \sum_{s \in H(T)} \varphi(s) \psi(r - s) f(x, p - r, y).$$

On the other hand, we have

$$\begin{aligned}(\varphi \diamond (\psi \diamond f))(x, p, y) &= \sum_{s \in H(T)} \varphi(s) \sum_{r \in H(T)} \psi(r) f(x, p - s - r, y) \\ &= \sum_{s \in H(T)} \sum_{r \in H(T)} \varphi(s) \psi(r) f(x, p - s - r, y) \\ &= \sum_{r \in H(T)} \sum_{s \in H(T)} \varphi(s) \psi(r - s) f(x, p - r, y),\end{aligned}$$

where the last equality comes from substituting r with $r + s$. Thus $(\varphi * \psi) \diamond f = \varphi \diamond (\psi \diamond f)$.

Recall that if $\varphi \in C_c(H(T))$, then its Fourier transform $\hat{\varphi} \in C(\mathbb{T}^k)$ is given by

$$\hat{\varphi}(z) = \sum_{n \in H(T)} \varphi(n) z^n,$$

and is constant on $H(T)^\perp$ cosets. We may regard $\hat{\varphi}$ as an element of $C(\mathbb{T}^k / H(T)^\perp)$. We have that $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is a uniformly dense subalgebra of $C(\mathbb{T}^k / H(T)^\perp)$.

Lemma 5.3.5. *Let T be an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y by local homeomorphisms such that $\Sigma_Y = \Sigma$, and let $\kappa \colon C^*(G_T) \to C^*(\mathrm{Iso}(G_T)^\circ)$ be as in Proposition 5.1.15. Then*

$$\kappa(\alpha_z(\varphi \diamond f)) = \hat{\varphi}(z) \kappa(\alpha_z(f))$$

for all $f \in C_c(G_T)$, $z \in \mathbb{T}^k$ and $\varphi \in C_c(H(T))$.

Proof. This is again only computation. We have

$$\begin{aligned}
\kappa(\alpha_z(\varphi \diamond f))(x, q(r), y) &= \sum_{m \in H(T)} \alpha_z(\varphi \diamond f)(x, r+m, y) \\
&= \sum_{m \in H(T)} z^{r+m} (\varphi \diamond f)(x, r+m, y) \\
&= \sum_{m \in H(T)} z^{r+m} \sum_{n \in H(T)} \varphi(n) f(x, r+m-n, y) \\
&= \sum_{m \in H(T)} z^{r+m} \sum_{n \in H(T)} \varphi(n) f(x, r+m-n, y) \\
&= \sum_{n \in H(T)} \sum_{m \in H(T)} z^{r+m} \varphi(n) f(x, r+m-n, y).
\end{aligned}$$

Note that we switched the order of the sums. This enables us to substitute m for $m-n$, and we may continue:

$$\begin{aligned}
&= \sum_{n \in H(T)} \sum_{m \in H(T)} z^{r+m+n} \varphi(n) f(x, r+m, y) \\
&= \sum_{m \in H(T)} z^{r+m} f(x, r+m, y) \sum_{n \in H(T)} z^n \varphi(n) \\
&= \sum_{m \in H(T)} z^{r+m} f(x, r+m, y) \hat{\varphi}(z) \\
&= \hat{\varphi}(z) \sum_{m \in H(T)} z^{r+m} f(x, r+m, y) \\
&= \hat{\varphi}(z) \kappa(\alpha_z(f))(x, q(r), y).
\end{aligned}$$

■

Proposition 5.3.6. *Let T be an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. Let*

$$\alpha\colon \mathbb{T}^k \to \operatorname{Aut}(C^*(G_T)) \text{ and } \widetilde{\alpha}\colon H(T)^\perp \to \operatorname{Aut}(C^*(G_T/\operatorname{Iso}(G_T)^\circ))$$

be as in Lemma 5.3.3, and let $\kappa\colon C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be as in Proposition 5.1.15. There is a C^*-isomorphism

$$\Phi\colon C^*(G_T) \to \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \widetilde{\alpha})$$

such that $\Phi(a)(z) = \kappa(\alpha_z(a))$ for $a \in C^*(G_T)$ and all $z \in \mathbb{T}^k$.

Proof. This is quite a long proof, so we break it into sections.

Φ is a $*$-homomorphism into $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$. Since α is strongly continuous, $z \mapsto \alpha_z(a)$ is continuous for any $a \in C^*(G_T)$. Thus, the map $z \mapsto \kappa(\alpha_z(a)) = \Phi(a)(z)$ is continuous.

Suppose $f \in C_c(G_T)$, $w \in \mathbb{T}^k$ and $z \in H(T)^\perp$. Lemma 5.3.3 gives $\widetilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$, so

$$\Phi(f)(wz) = \kappa(\alpha_{wz}(f)) = \kappa(\alpha_z(\alpha_w(f))) = \widetilde{\alpha}_z \kappa(\alpha_w(f)) = \widetilde{\alpha}_z(\Phi(f)(w)).$$

Thus Φ takes values in $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \widetilde{\alpha})$.

To see that Φ is a homomorphism, let $a, b \in C*(G_T)$ and $z \in \mathbb{T}^k$. Then

$$\Phi(ab)(z) = \kappa(\alpha_z(ab)) = \kappa(\alpha_z(a))\kappa(\alpha_z(b)) = (\Phi(a)(z))\,(\Phi(b)(z))\,.$$

Lastly, Φ is a $*$-map since α and κ are. Indeed, for $a \in C^*(G_T)$ we have

$$\Phi(a^*)(z) = (\kappa(\alpha_z(a)))^* = \Phi(a)^*(z).$$

Φ is injective. There is an action lt of $\mathbb{T}^k$ on the left of $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ given by left translation: $\operatorname{lt}_x(s)(w) = s(\overline{z}w)$. We have $\Phi \circ \alpha_z = \operatorname{lt}_{\overline{z}} \circ \Phi$. To each of the actions α and lt, we can associate the so-called faithful *conditional expectations*, denoted ξ and ζ respectively. They are defined as

$$\xi(a) = \int_{\mathbb{T}^k} \alpha_z(a)\; dz \text{ and}$$
$$\zeta(s) = \int_{\mathbb{T}^k} \operatorname{lt}_z(s)\; dz,$$

for $f \in C^*(G_T)$ and $s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$, where we use vector-valued integration. For $f \in C_c(G_T)$ and $z \in \mathbb{T}^k$, we have

$$\begin{aligned}
(\zeta \circ \Phi)(f)(z) &= \int_{w \in \mathbb{T}^k} \operatorname{lt}_w \Phi(f)(z) \\
&= \int_{w \in \mathbb{T}^k} \Phi \circ \alpha_{\bar{w}}(f)(z) \\
&= \Phi\left(\int_{w \in \mathbb{T}^k} \alpha_w(f)(z)\right) \\
&= (\Phi \circ \xi)(f)(z)
\end{aligned}$$

Hence, by [SWW14, Lemma 3.14], it suffices to prove that Φ restricts to an injection on the fixed-point algebra $C^*(G_T)^\alpha$ for α, which is defined as

$$C^*(G_T)^\alpha := \{a \in C^*(G_T) : \alpha_z(a) = a \text{ for all } z \in \mathbb{T}^k\}.$$

We have $C^*(G_T)^\alpha = \xi(C^*(G_T))$. If $f \in C_c(G_T)$, then $\xi(f) \in C_c(G_T)$ as well, by arguing as in [Wil07, Lemma 1.108]. Thus, if $f \in C_c(G_T)$, then for any $\gamma \in G_T$ we have

$$\begin{aligned}
\xi(f)(\gamma) &= \left(\int_{\mathbb{T}^k} \alpha_z(f)\; dz\right)(\gamma) \\
&= \int_{\mathbb{T}^k} \alpha_z(f)(\gamma)\; dz = \int_{\mathbb{T}^k} z^{c(\gamma)} f(\gamma)\; dz \\
&= \int_{\mathbb{T}^k} z^{c(\gamma)} f(\gamma)\; dz = f(\gamma) \int_{\mathbb{T}^k} z^{c(\gamma)}\; dz \\
&= \begin{cases} f(\gamma) & \text{if } \gamma \in c^{-1}(0), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Here, we have "passed evaluation through the integral" as in [Wil07, Remark 1.109], and used the fact that integrating a polynomial with no constant term over the circle gives zero. This implies that $C_c(c^{-1}(0))$ is dense in $C^*(G_T)^\alpha$. Since $c^{-1}(0)$ and G_T are amenable, their C^*-algebras both come from the left-regular representation of G_T. Hence the inclusion $C_c(c^{-1}(0)) \hookrightarrow C_c(G_T)$ induces a monomorphism $\rho\colon C^*(c^{-1}(0)) \hookrightarrow C^*(G_T)$. The image of ρ is $\rho(C^*(c^{-1}(0)) = \overline{C_c(c^{-1}(0))} = C^*(G_T)^\alpha$, so proving that $\Phi|_{C^*(G_T)^\alpha}$ is injective is the same as proving that $\Phi \circ \rho$ is injective. Since $c^{-1}(0)$ is amenable by Lemma 5.2.6, and clearly topologically principal, we have by Lemma 5.1.13 that it suffices to check that $(\Phi \circ \rho)|_{C_0(Y)}$ is injective. Since ρ restricts to the canonical inclusion on $C_0(Y)$, all we need to prove is that Φ is injective on $C_0(Y)$. We know that $\kappa \circ \alpha_z$ restricts to the identity of $C_0(Y) \subseteq C^*(G_T)$ to the identity of $C_0(Y) \subseteq C^*(G_T/\operatorname{Iso}(G_T)^\circ)$. Hence for $f \in C_0(Y)$, $z \in \mathbb{T}^k$ and $b \in G_T/\operatorname{Iso}(G_T)^\circ$, we have

$$\Phi(f)(z)(b) = \kappa(\alpha_z(f))(b) = \begin{cases} f(x) & \text{if } b = (x, 0, x) \in (G_T/\operatorname{Iso}(G_T)^\circ)^0, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we see that if $\Phi(f) = 0$ then $f = 0$, so Φ is injective.

Φ is surjective. There is a canonical operation $C(\mathbb{T}^k/H(T)^\perp) \times \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k} \to \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ given by

$$(\hat{\varphi} \cdot s)(z) = \hat{\varphi}(z)s(z),$$

for $\hat{\varphi} \in C(\mathbb{T}^k/H(T)^\perp)$, $s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$, and $z \in \mathbb{T}^k$. This operation makes $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ a $C(\mathbb{T}^k/H(T))$-module. For instance, the operation is compatible in the sense that $(\hat{\varphi} \cdot \hat{\psi}) \cdot s = \hat{\varphi} \cdot (\hat{\psi} \cdot s)$, where $\hat{\varphi} \cdot \hat{\psi}$ denotes pointwise multiplication of functions in $C(\mathbb{T}^k/H(T))$.

We now prove that under the operation above, $\Phi(C^*(G_T))$ is a $C(\mathbb{T}^k/H(T)^\perp)$-submodule of $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$. We must show that

$$C(\mathbb{T}^k/H(T)^\perp) \cdot \Phi(C^*(G_T)) \subseteq \Phi(C^*(G_T)).$$

By Lemma 5.3.5, we have that $\hat{\varphi} \cdot \Phi(f) = \Phi(\varphi \diamond f)$ for all $\varphi \in C_c(H(T))$ and $f \in C_c(G_T)$. Indeed, we have

$$(\hat{\varphi} \cdot \Phi(f))(z) = \hat{\varphi}(z)\Phi(f)(z) = \hat{\varphi}(z)\kappa(\alpha_z(f)) = \kappa(\alpha_z(\varphi \diamond f)) = \Phi(\varphi \diamond f)$$

for all $z \in \mathbb{T}^k$. Suppose next that $\hat{\psi} \in C(\mathbb{T}^k/H(T)^\perp)$. Since $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is dense in $C(\mathbb{T}^k/H(T)^\perp)$, we can find a net $(\hat{\varphi}_\lambda)$ of such functions converging to $\hat{\psi}$. Then for any $f \in C_c(G_T)$, we have

$$\hat{\psi} \cdot \Phi(f) = \lim_{\lambda\to\infty} \hat{\varphi}_\lambda \cdot \Phi(f) = \lim_{\lambda\to\infty} \Phi(\varphi_\lambda \diamond f).$$

Since Φ is an injective $*$-homomorphism, its range is closed, so $\hat{\psi} \cdot \Phi(f)$ is in the image of Φ. This extends to elements of $C^*(G_T)$, so $C(\mathbb{T}^k/H(T)^\perp) \cdot \Phi(C^*(G_T)) \subseteq \Phi(C^*(G_T))$ and we are done.

We will show that the image of Φ is all of $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ by applying [PST15, Lemma 3.6]. First note that, in the notation of [PST15, Definition

3.3], $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ is a $C(\mathbb{T}^k/H(T)^\perp)$-algebra with structure map η given by $\eta(f)s = f \cdot s$, for $f \in C(\mathbb{T}^k/H(T)^\perp)$ and $s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$. (Indeed, since the module operation just comes from pointwise multiplication in $\mathbb{C}$, η takes values in $Z(M(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}))$. We also have $\eta(1)s = s$ for all $a \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$, so in particular we have that $\eta(C(\mathbb{T}^k)/H(T)^\perp) \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ is dense in $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$.) Since $\Phi(C^*(G_T))$ is a $C(\mathbb{T}^k/H(T)^\perp)$-module, the second condition of the lemma is satisfied. The only thing left to prove is that for each $z \in \mathbb{T}^k/H(T)^\perp$, the set

$$\varepsilon_z(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}) = \{s(z) : s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}\} \subseteq C^*(G_T/\operatorname{Iso}(G_T)^\circ)$$

is covered by the set $\{\Phi(a)(z) = \kappa(\alpha_z(a)) : a \in C^*(G_T)\}$. It suffices to prove that the range of $\kappa \circ \alpha_z$ contains $C_c(G_T/\operatorname{Iso}(G_T)^\circ)$.

Let $f \in C_c(G_T/\operatorname{Iso}(G_T)^\circ)$, and let $c\colon G_T/\operatorname{Iso}(G_T)^\circ \to \mathbb{Z}^k/H(T)$ pick out the middle element, as in the proof of Lemma 5.3.3. Given any $r \in \mathbb{Z}^k$, the set $U_r := \{(x, q(r), y) : (x, r, y) \in G_T\} = c^{-1}(q(r))$ is clopen by Lemma 5.3.2, and the U_r are disjoint for different $q(r)$. Hence $\chi_{U_r} f \in C_c(G_T/\operatorname{Iso}(G_T)^\circ)$, and we can write

$$f = \sum_{q(r) \in \mathbb{Z}^k/H(T)} \chi_{U_r} f.$$

Since f has compact support, the above sum must be finite. Thus, to show that f is in the image of $\kappa \circ \alpha_z$, we may without loss of generality assume that $f \in c^{-1}(q(r))$ for some r. Define $h \in C_c(G_T)$ by

$$h(\gamma) = \begin{cases} \overline{z}^r f(p(\gamma)) & \text{if } c(\gamma) = r \\ 0 & \text{otherwise} \end{cases},$$

where $p\colon C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ is the quotient map. We have that h is continuous since $c^{-1}(r)$ is clopen in G_T. We see that $\kappa(\alpha_z(h)) = f = 0$ on $\left(c^{-1}(q(r))\right)^c$, and for $c(\gamma) = q(r)$ we have

$$\begin{aligned} \kappa(\alpha_z(h))(b) &= \sum_{p(\gamma)=b} \alpha_z(h)(\gamma) = \sum_{p(\gamma)=b} z^{c(\gamma)} h(\gamma) \\ &= \sum_{\substack{p(\gamma)=b,\\ c(\gamma)=r}} z^{q(r)} \overline{z}^{q(r)} f(p(\gamma)) = \sum_{\substack{p(\gamma)=b,\\ c(\gamma)=r}} f(p(\gamma)). \end{aligned}$$

There is precisely one $\gamma \in G_T$ with both $p(\gamma) = b$ *and* $c(\gamma) = r$. Hence $(\kappa \circ \alpha_z)(h) = f$, and we are done.

■

For any $x \in X$, let $\ell^2(\overline{[x]}) = \overline{\operatorname{span}}\{\delta_y : y \in \overline{[x]}\}$. There has not been enough time to go through the material covered in all of the proofs. The next lemma is one example, and we omit its proof.

Lemma 5.3.7. *Let G be an étale groupoid, and fix $x \in G^0$. Then there is an irreducible representaion $\omega_{[x]} \colon C^*(G) \to \mathcal{B}(\ell^2([x]))$ satisfying*

$$\omega_{[x]}(f)\delta_y = \sum_{d(\gamma)=y} f(\gamma)\delta_{r(\gamma)}$$

for all $f \in C_c(G)$. If G is topologically principal and amenable, and $[x]$ is dense in G^0, then $\omega_{[x]}$ is faithful, and hence $C^*(G)$ is primitive.

We will now make use of the quasi-orbit space $\mathcal{Q}(G) = \{\overline{[x]} : x \in G^0\}$. Recall that $\mathcal{Q}(G)$ carries the quotient topology for the map $q \colon G^0 \to \mathcal{Q}(G)$ that identifies x and y precisely when $\overline{[x]} = \overline{[y]}$. Also, if $S \subseteq \mathcal{Q}(G)$, then

$$\overline{S} = \{q(u) : u \in \overline{q^{-1}(S)}\}. \tag{5.10}$$

Indeed, since $S \subseteq \mathcal{Q}(G)$ is closed if and only if $q^{-1}(S) \subseteq G^0$ is closed, we have

$$\begin{aligned}\overline{S} &= \bigcap\{R \subseteq \mathcal{Q}(G) : R \text{ closed}, S \subseteq R\}\\ &= \bigcap\{R \subseteq \mathcal{Q}(G) : q^{-1}(R) \text{ closed}, q^{-1}(S) \subseteq q^{-1}(R)\}\\ &= q\left(\bigcap\{q^{-1}(R) \subseteq G^0 : q^{-1}(R) \text{ closed}, q^{-1}(S) \subseteq q^{-1}(R)\}\right)\\ &= q\left(\overline{q^{-1}(S)}\right).\end{aligned}$$

We don't necessarily have $\overline{q^{-1}(S)} = \bigcap\{q^{-1}(R) \subseteq G^0 : q^{-1}(R) \text{ closed}, q^{-1}(S) \subseteq q^{-1}(R)\}$, but their images under q are the same, so the argument is valid.

Lemma 5.3.8. *Let G be an amenable étale groupoid such that $G|_U$ is topologically principal for every closed invariant subset $U \subset G^0$. For $u \in U$, let ω_u be the irreducible representation from Lemma 5.3.7. The map $u \mapsto \ker \omega_u$ from G^0 to $\mathrm{Prim}(C^*(G))$ induces to a homeomorphism of $\mathcal{Q}(G)$ onto $\mathrm{Prim}(C^*(G))$.*

Proof. For $u \in G^0$, we claim that $\ker \omega_u \cap C_0(G^0) = C_0(G^0 \setminus \overline{[u]})$. For $f \in C_c(G^0)$, we have $\omega_{\overline{[u]}}(f) = 0$ if and only if $\omega_{\overline{[u]}}(f)(\delta_y) = 0$ for all $y \in \overline{[u]}$. In other other words, if we for all such y have

$$\omega_{\overline{[u]}}(f)(\delta_y) = \sum_{d(\gamma)=y} f(\gamma)\delta_{r(\gamma)} = f(y)\delta_y = 0,$$

where we have used that $f = 0$ outside G^0. This happens if and only if $f(y) = 0$ for all $y \in \overline{[u]}$. Hence $C_c(G^0 \setminus \overline{[u]}) = \ker \omega_u \cap C_c(G^0)$. Since ω_u is continuous, we have $\omega_u(f) = 0$ for $f \in C_0(G^0)$ if and only if f is zero on $\overline{[u]}$, so $C_0(G^0 \setminus \overline{[u]}) = \ker \omega_u \cap C_0(G^0)$.

Now [Ren91, Corollary 4.9] implies that $\ker \omega_u = \ker \omega_v$ if and only if $\overline{[u]} = \overline{[v]}$. (Since amenability is a black box for us, we won't go into details on how the result works.) Hence the map $G^0 \to \mathrm{Prim}(C^*(G))$ defined by $u \mapsto \ker \omega_u$ induces a well-defined injection $\mathcal{Q}(G) \to \mathrm{Prim}(C^*(G))$ given by $\overline{[u]} \mapsto \ker \omega_u$.

Next we shall see that the induced map is surjective. Suppose π is an irreducible representation of $C^*(G)$. Since G is étale, the representation M that π induces on $C_0(G^0)$ is just restriction, so $M = \pi|_{C_0(G^0)}$. By Proposition 5.1.11,

$\operatorname{supp} M = \operatorname{supp} \pi \cap C_0(G^0)$ is G-irreducible, and hence equal to $\overline{[u]}$ for some $u \in G^0$ by Lemma 5.1.6. An irreducible representation of $C_0(G^0)$ with support $\overline{[u]}$ must have kernel $C_0(G^0 \setminus \overline{[u]})$. Indeed, as we define support in Definition 5.1.5, we have that $f(x) = 0$ for all $f \in \ker M$ if and only if $x \in \overline{[u]}$. The inclusion $C_0(G^0 \setminus \overline{[u]}) \subseteq \ker M$ comes from the "if"-part of the statement, and the reverse inclusion comes from the "only if"-part. So now we have

$$\ker \pi \cap C_0(G^0) = C_0(G^0 \setminus \overline{[x]}) = \ker \omega_x \cap C_0(G^0).$$

Another application of [Ren91, Corollary 4.9] implies that $\ker \pi = \ker \omega_x$.

We now show that $\overline{[u]} \mapsto \ker \omega_u$ is a homeomorphism. Since the map is bijective, it suffices by Lemma 2.0.2 to show that $S \subseteq \mathcal{Q}(G)$ is closed if and only if its image $\{\ker \omega_v : v \in S\}$ is closed in $\operatorname{Prim}(C^*(G))$. Hence, given $S \subseteq \mathcal{Q}(G)$ and some $u \in G^0$, we must show that

$$\overline{[u]} \in \overline{S} \text{ if and only if } \ker \omega_u \in \overline{\{\ker \omega_v : q(y) \in S\}}. \tag{5.11}$$

Fix $S \subseteq \mathcal{Q}$ and $u \in G^0$. Recall from Definition 2.0.16 that we have

$$\begin{aligned}\overline{\{\ker \omega_v : q(v) \in S\}} &= \operatorname{hull}\Big(\bigcap_{q(v) \in S} \ker \omega_v\Big) \\ &= \{\ker \omega_w : \bigcap_{q(v) \in S} \ker \omega_v \subseteq \ker \omega_w\},\end{aligned}$$

where we have used that $[w] \mapsto \ker \omega_w$ is surjective. Again, [Ren91, Corollary 4.9] implies that $\ker \omega_u \in \overline{\{\ker \omega_v : q(v) \in S\}}$ if and only if $\left(\bigcap_{q(v) \in S} \ker \omega_v\right) \cap C_0(G^0) \subseteq \ker \omega_u \cap C_0(G^0)$. We have

$$\begin{aligned}\left(\bigcap_{q(v) \in S} \ker \omega_v\right) \cap C_0(G^0) &= \bigcap_{q(v) \in S} (\ker \omega_v \cap C_0(G^0)) \\ &= \bigcap_{q(v) \in S} C_0(G^0 \setminus \overline{[v]}) \\ &= C_0(G^0 \setminus q^{-1}(S)) \\ &= C_0(G^0 \setminus \overline{q^{-1}(S)}),\end{aligned}$$

where the last equality comes from the fact that the inverse image of $\{0\}$ under $f \in C_0(G^0 \setminus q^{-1}(S))$ is closed. On the other hand, $\ker \omega_u \cap C_0(G^0) = C_0(G^0 \setminus \overline{[u]})$. Hence $\ker \omega_u \in \overline{\{\ker \omega_v : q(v) \in S\}}$ if and only if

$$C_0(G^0 \setminus \overline{q^{-1}(S)}) \subseteq C_0(G^0 \setminus \overline{[u]}),$$

which happens if and only if $\overline{[u]} \subseteq \overline{q^{-1}(S)}$. We claim that this again is equivalent to

$$q(u) \in \overline{S} = \{q(v) : v \in \overline{q^{-1}(S)}\},$$

where we have used (5.10). Indeed, if $u \in \overline{[u]} \subseteq \overline{q^{-1}(S)}$, then we clearly have $q(u) \in \overline{S}$.

Suppose next that $q(u) \in \overline{S}$. Then there exists some $v \in \overline{q^{-1}(S)}$ with $q(v) = q(u)$, or equivalently $\overline{[v]} = \overline{[u]}$. Since $\overline{q^{-1}(S)}$ is closed and invariant, it can be written as a union of orbit closures. One of these orbits must contain v (and must therefore equal $\overline{[v]}$), and hence $\overline{[u]} = \overline{[v]} \in \overline{q^{-1}(S)}$. Now we have proved (5.11), and we are done. ■

We will soon describe the primitive ideal space of $C^*(G_T)$. Before that, we need to make a few observations that will make the proof of Theorem 5.3.12 more to the point. It is needed, as the proof connects a lot of the results we have had so far, and is therefore quite technical.

Lemma 5.3.9. *Let G be an étale groupoid. The quotient map $q\colon G \to G/\operatorname{Iso}(G)^\circ$ induces a homeomorphism $\mathcal{Q}(G) \cong \mathcal{Q}(G/\operatorname{Iso}(G)^\circ)$.*

Proof. The quotient map q restricts to a homeomorphism on the unit spaces of G and $G/\operatorname{Iso}(G)^\circ$. Since q also preserves the range and domain maps, it maps G-orbits bijectively to $G/\operatorname{Iso}(G)^\circ$-orbits. Indeed, if $q(u) = a$ for some $u \in G^0$ and $a \in G/\operatorname{Iso}(G)^\circ$, we have

$$q([u]) = q(r(d^{-1}(u)) = r(d^{-1}(q(u))) = [a],$$

which is a bijection of the orbits. Since q is a homeomorphism on G^0, closure commutes with its images, so q takes orbit closures to orbit closures. Hence q induces a homeomorphism between $\mathcal{Q}(G)$ and $\mathcal{Q}(G/\operatorname{Iso}(G)^\circ)$. ■

Lemma 5.3.10. *Let G be an étale groupoid such that $\operatorname{Iso}(G)^\circ$ is closed in G. For $C \subseteq G^0$, we have*

$$(G/\operatorname{Iso}(G)^\circ)\,|_C = G|_C/(\operatorname{Iso}(G)^\circ|_C).$$

Proof. For $x \in G$, we let $[x]$ denote the orbit $x\cdot\operatorname{Iso}(G)^\circ$, as in Proposition 5.1.14. We have $[x] \in G/\operatorname{Iso}(G)^\circ|_C$ if and only if every $y \in [x]$ has range and domain in C. This is the same as saying that every $y \in [x]$ has $y \in G|_C$ and $[x] = x\cdot\operatorname{Iso}(G)^\circ = x\cdot\operatorname{Iso}(G)^\circ|_C$, so the conclusion follows. ■

Lemma 5.3.11. *Let T be an action on a locally compact Hausdorff space by local homeomorphisms. Then there is an isomorphism $\mathbb{T}^k/H(T)^\perp \cong H(T)^\wedge$.*

Proof. Let $\bar{z}$ be the class of $z \in \mathbb{T}^k$ in the quotient group. Then $\varphi_{\bar{z}}\colon r \mapsto z^r$ is an element of $\operatorname{Hom}(H(T), \mathbb{T}^k) = H(T)^\wedge$. Let $\Psi\colon \mathbb{T}^k/H(T)^\perp \to H(T)^\wedge$ be the homomorphism $\bar{z} \mapsto \varphi_{\bar{z}}$. If $\bar{z} = \bar{z}'$, say $z' = zw$ for some $w \in H(T)$, then

$$\varphi_{\bar{z}}(r) = z^r = z^r 1 = z^r w^r = (zw)^r = (z')^r = \varphi_{\bar{z}'} r$$

for all $r \in H(T)$, so Ψ is well-defined. Injectivity is clear since $z^r = 1$ for all $r \in H(T)$ if and only if $z \in H(T)^\perp$.

For surjectivity, choose generators $g_1, \ldots, g_n$ for $H(T)$ such that for $r = (r_1, \ldots, r_n)$ we have $rg_i = r_i$ for $i = 1, \ldots, n$. We may do this since $H(T)$ is a direct sum of subgroups of $\mathbb{Z}$. Set $z \coloneqq \varphi(g_1 + \ldots + g_n)$; then any element of $r \in H(T)$ can be written as $rg_1 + \ldots + rg_n$. Now we have

$$\varphi(r) = \varphi(g_1)^{r_1} \cdot \ldots \cdot \varphi(g_n)^{r_n} = z^r,$$

so $\varphi = \varphi_{\bar{z}}$ and Ψ is surjective. ■

We define

$$H(T)^\wedge := \operatorname{Hom}(H(T), \mathbb{Z}^k).$$

We let $C^*(G)^\wedge$ denote the *spectrum* of $C^*(G)$, consisting of all unitary equivalence classes of irreducible representations of $C^*(G)$. So the topology on $C^*(G)^\wedge$ is the pull-back of the one on $\operatorname{Prim}(C^*(G))$. We now describe the topology of $\operatorname{Prim}(C^*(G_T))$ for the particular $\mathbb{N}_0^k$-actions T we have been focusing on in this chapter.

Theorem 5.3.12. *Let T be an irreducible action of $\mathbb{N}_0^k$ on a locally compact space Y by local homeomorphisms such that $\Sigma_Y = \Sigma$, in the notation of* (5.5)*. Suppose that for every $y \in Y$, the set*

$$\Sigma_{\overline{[y]}} := \{(m,n) \in \mathbb{N}_0^k \times \mathbb{N}_0^k : T^m x = T^n y \text{ for all } x \in \overline{[y]}\}$$

satisfies $\Sigma_{\overline{[y]}} = \Sigma$. Let $\alpha\colon \mathbb{T}^k \to \operatorname{Aut}(C^*(G_T))$ be as in Lemma 5.3.3, and let $\kappa\colon C^*(G_T) \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be as in Proposition 5.1.15. For $y \in (G_T/\operatorname{Iso}(G_T)^\circ)^0$, let ω_y be as described in Lemma 5.3.7. Then the map $(y,z) \mapsto \ker(\omega_y \circ \alpha_z)$ from $Y \times \mathbb{T}^k$ to $\operatorname{Prim}(C^*(G_T))$ descends to a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\wedge \cong \operatorname{Prim}(C^*(G_T))$.

Proof. We have done most of the work to prove this theorem now, this proof is mostly about connecting the dots. It isn't the easiest proof to get an overview of, but we try to make it easier by breaking it into sections.

Part one. We want to prove that $(y,z) \mapsto \ker(\omega_y \circ \alpha_z)$ induces a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\wedge \cong \operatorname{Prim}(C^*(G_T))$. However, as we prove in this part of the proof, it suffices to show that $(y,z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$ induces a homeomorphism induces a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\perp \to \operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})$.

Let $\Phi\colon C^*(G_T) \to \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}(C^*(G_T/\operatorname{Iso}(G_T)^\circ), \widetilde{\alpha})$ be the isomorphism from Proposition 5.3.6. For each $y \in Y$, let $\tilde{\omega}_y$ and ω_y be the irreducible representations of $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ and $C^*(G_T)$, respectively, coming from Lemma 5.3.7. We claim that $\omega_y = \tilde{\omega}_y \circ \kappa$. Indeed, for $f \in C_c(G_T)$ and $x \in \overline{[y]}$, we have

$$\begin{aligned}
\tilde{\omega}_y(\kappa(f))(\delta_x) &= \sum_{d(b)=x} \kappa(f)(b)\delta_{r(b)} \\
&= \sum_{d(b)=x} \sum_{q(\gamma)=b} f(\gamma)\delta_{r(b)} \\
&= \sum_{d(b)=x} \sum_{q(\gamma)=b} f(\gamma)\delta_{r(\gamma)}.
\end{aligned}$$

We note that for $b \in G_T/\operatorname{Iso}(G_T)^\circ$ and $\gamma \in G_T$, we have $d(b) = x$ and $q(\gamma) = b$ if and only if $d(\gamma) = x$. Hence we continue to find that the above equals

$$\sum_{d(\gamma)=x} f(\gamma)\delta_{r(\gamma)} = \omega_y(f)(\delta_x),$$

which is what we wanted.

Let $z \in \mathbb{T}^k$. Our next claim is that

$$\Phi(\ker(\omega_y \circ \alpha_z)) = \{s \in \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k} : s(z) \in \ker \tilde{\omega}_y\}.$$

To see this, let $\varepsilon_z \colon \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k} \to C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ be the homomorphism defined by evaluation at $z \in \mathbb{T}^k$. By the definition of Φ and the fact that $\omega_y = \tilde{\omega}_y \circ \kappa$, the following diagram commutes.

$$\begin{tikzcd} C^*(G_T) \arrow[d, "\omega_y"] & C^*(G_T) \arrow[l, "\alpha_z"'] \arrow[r, "\Phi"] & \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k} \arrow[dl, "\varepsilon_z"] \\ \ell^1(\overline{[y]}) & C^*(G_T/\operatorname{Iso}(G_T)^\circ) \arrow[l, "\tilde{\omega}_y"] & \arrow[from=1-1, to=2-2, "\kappa"] \end{tikzcd}$$

Hence we have $a \in \ker(\omega_y \circ \alpha_z)$ if and only if $\Phi(a)(z) \in \ker(\widetilde{\omega}_y)$, which is precisely what we wanted.

Since Φ is an isomorphism, it maps primitive ideals in $C^*(G_T)$ to primitive ideals in $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$. Since the primitive ideals in $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ coming from the kernels of $\omega_y \circ \alpha_z$ are precisely the ones that are kernels of $\tilde{\omega}_y \circ \varepsilon_z$, it suffices to prove that

$$(y, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z) \tag{5.12}$$

induces a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\perp \to \operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})$.

Part two. To prove that (5.12) induces a homeomorphism, we will use that there is an induced homeomorphism $(C^*(G_T/\operatorname{Iso}(G_T)^\circ)^\wedge \times \mathbb{T}^k)/H(T)^\perp \to (\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})^\wedge$. This is what we prove in this part.

First, we claim that $\operatorname{Iso}(G_T)^\circ|_{\overline{[y]}} = \operatorname{Iso}(G_T|_{\overline{[y]}})^\circ$ for all $y \in Y$. One inclusion is clear: Since $G_T|_{\overline{[y]}}$ is a subgroupoid of G_T and $\operatorname{Iso}(G_T|_{\overline{[y]}})^\circ$ has unit space $\overline{[y]}$, we have $\operatorname{Iso}(G_T|_{\overline{[y]}})^\circ|_{\overline{[y]}} = \operatorname{Iso}(G_T|_{\overline{[y]}})^\circ$ and

$$\operatorname{Iso}(G_T|_{\overline{[y]}})^\circ \subseteq \operatorname{Iso}(G_T)^\circ|_{\overline{[y]}}.$$

For the other inclusion, we use that $\Sigma_{\overline{[y]}} = \Sigma$ and $T^p\overline{[y]} \subseteq \overline{[y]}$ for all $p \in \mathbb{N}_0^k$. We can use Proposition 5.2.12 to see that

$$\begin{aligned} \operatorname{Iso}(G_T)^\circ|_{\overline{[y]}} &= \{(x, n, x) \in \operatorname{Iso}(G)^\circ : x \in \overline{[y]}\} \\ &\subseteq \{(x, n, x) : n \in \mathbb{N}_0^k \text{ and } x \in \overline{[y]}\} \\ &= \operatorname{Iso}(G_T|_{\overline{[y]}})^\circ. \end{aligned}$$

Now by Lemma 5.3.10, we have

$$(G_T/\operatorname{Iso}(G_T)^\circ)\,|_{\overline{[y]}} = G_T|_{\overline{[y]}}/\operatorname{Iso}(G)^\circ|_{\overline{[y]}} = G_T|_{\overline{[y]}}/\operatorname{Iso}(G_T|_{\overline{[y]}})^\circ,$$

which is topologically principal by Proposition 5.1.14. The restriction of $(G_T/\operatorname{Iso}(G_T)^\circ)\,|_{\overline{[y]}}$ to *another* orbit closure, say $\overline{[x]}$, will still be topologically principal. Any closed invariant subset C of G_T^0 can be written as the union $C = \overline{[y]} \cup \overline{[x]}$ by varying x and y, so the restriction of $G_T/\operatorname{Iso}(G_T)^\circ$ to any closed invariant subset of the unit space is topologically principal.

Now we can apply Lemma 5.3.8. We have $\overline{[x]} = \overline{[y]}$ if and only if $\ker \tilde{\omega}_x = \ker \tilde{\omega}_y$, which happens if and only if $\ker(\tilde{\omega}_x \circ \varepsilon_z) = \ker(\tilde{\omega}_y \circ \varepsilon_z)$ since ε_z is surjective. Thus (5.12) induces a well-defined map $(\overline{[y]}, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$. Combining this map with the homeomorphism $\overline{[y]} \mapsto \ker \tilde{\omega}_y$ from Lemma 5.3.8 yields well-defined map $\operatorname{Prim}(C^*(G_T/\operatorname{Iso}(G_T)^\circ) \times \mathbb{T}^k \to \operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})$ defined by

$$M\colon (\ker \tilde{\omega}_y, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z).$$

We will now apply [RW98, Proposition 6.16]. To do so, we introduce the *diagonal action* of $H(T)^\perp$ on $C^*(G_T/\operatorname{Iso}(G_T)^\circ)^\wedge \times \mathbb{T}^k$, defined as follows:

$$(\pi, z) \cdot x := (\pi \circ \widetilde{\alpha}_x, z \cdot x).$$

Since $H(T)^\perp$ acts freely and properly on $\mathbb{T}^k$ by right translation ($z \cdot x = zx$ for $z \in \mathbb{T}^k, x \in H(T)^\perp$), we can apply [RW98, Proposition 6.16]. The proposition states that given $z \in \mathbb{T}^k$ and $\pi \in C^*(G_T/\operatorname{Iso}(G_T)^\circ)^\wedge$, the map $N(\pi, z)\colon f \mapsto \pi(f(z))$ is an irreducible represenation of $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$. Furthermore, $(\pi, z) \mapsto N(\pi, z)$ induces a homeomorphism of $(C^*(G_T/\operatorname{Iso}(G_T)^\circ)^\wedge \times \mathbb{T}^k)/H(T)^\perp$ onto $(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})^\wedge$. We observe that our map M is just the map corresponding N when passing from the spectrum to the primitive ideal space. Since the topology on the prior is the pull-back of the topology on the latter, M induces a homeomorphism of $(\operatorname{Prim} C^*(G_T/\operatorname{Iso}(G_T)^\circ) \times \mathbb{T}^k)/H(T)^\perp$ onto $\operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k})$.

Part three. Let us recap. What we *want to prove* is that (5.12) induces a homeomorphism

$$\mathcal{Q}(G_T) \times H(T)^\perp \cong \operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}). \tag{5.13}$$

What we *have proven* is that (5.12) induces a homeomorphism

$$(\operatorname{Prim}(C^*(G_T/\operatorname{Iso}(G_T)^\circ)) \times \mathbb{T}^k)/H(T)^\perp \cong \operatorname{Prim}(\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k});$$

by Lemma 5.3.11, we have an isomorphism $\mathbb{T}^k/H(T)^\perp \cong H(T)^\wedge$, and we have yet another homeomorphism

$$\operatorname{Prim}(C^*(G_T/\operatorname{Iso}(G_T)^\circ) \cong \mathcal{Q}(G_T).$$

To prove (5.13), it therefore suffices to show that $H(T)^\perp$ acts trivially on $\operatorname{Prim}(G_T/\operatorname{Iso}(G_T)^\circ)$ by the diagonal action described above. In other words, given some irreducible representation π of $C^*(G_T/\operatorname{Iso}(G_T)^\circ)$ and any $x \in H(T)^\perp$, we must show that $\ker(\pi \circ \widetilde{\alpha}_x) = \ker(\pi)$. It suffices to prove that $\widetilde{\alpha}_x$ maps ideals onto themselves. By the definition of $\widetilde{\alpha}_x$ in the proof of Lemma 5.3.3, we know that $\widetilde{\alpha}_x$ restricts to the identity on $C_0(G_T^0) \subseteq C^*(G_T/\operatorname{Iso}(G_T)^\circ)$. We also have $\widetilde{\alpha}_x(C_0(G_T^0)^c) \subseteq C_0(G_T^0)^c$ since $\widetilde{\alpha}_x$ is an automorphism. Therefore, for any ideal $I \subseteq C^*(G_T/\operatorname{Iso}(G_T)^\circ)$, we have

$$I \cap C_0(G_T^0) = \widetilde{\alpha}_x(I \cap C_0(G_T^0)) = \widetilde{\alpha}_x(I) \cap C_0(G_T^0).$$

Now [Ren91, Corollary 4.9] implies that $\widetilde{\alpha}_x(I) = I$, which is what we wanted to show.

■

5.4 The Primitive Ideals of the C^*-algebra of a Deaconu-Renault Groupoid

In this section, we will use Theorem 5.3.12 to characterise the primitive ideal space in the non-irreducible case. We have to refine our notation to accommodate actions which are not necessarily irreducible.

Let T be an action of $\mathbb{N}_0^k$ on a locally compact space X by local homeomorphisms. Recall the two equivalent definitions of orbits in Remark 5.2.4. For $x \in X$ and a (relatively) open subset $U \subseteq \overline{[x]}$, define

$$\Sigma(x)_U \coloneqq \{(m,n) \in \mathbb{N}_0^k \times \mathbb{N}_0^k : T^m y = T^n y \text{ for all } y \in U\},$$

and set

$$\Sigma(x) \coloneqq \bigcup_U \Sigma(x)_U.$$

Since $\overline{[x]}$ is irreducible, $T|_{\overline{[x]}}$ acts irreducibly on $\overline{[x]}$. Hence Lemma 5.2.11 implies that

$$Y(x) \coloneqq \bigcup \{Y \subseteq \overline{[x]} : Y \text{ is relatively open and } \Sigma(x)_Y = \Sigma(x)\}$$

is nonempty and is the maximal relatively open subset of $\overline{[x]}$ such that $\Sigma(x)_{Y(x)} = \Sigma(x)$, and that $T^m Y(x) \subseteq Y(x)$ for all $m \in \mathbb{N}_0^k$. Define $H(x) \coloneqq H(T|_{\overline{[x]}})$. Then Lemma 5.2.11 also gives us that

$$H(x) = \{(m,n) \in \mathbb{N}_0^k \times \mathbb{N}_0^k : m - n \in H(T)\},$$

which by Proposition 5.2.12 is a subgroup of $\mathbb{Z}^k$. The same proposition also gives us that

$$\mathcal{I}(x) \coloneqq \operatorname{Iso}(G_T|_{Y(x)})^\circ = \{(y,g,y) : y \in Y(x) \text{ and } g \in H(x)\},$$

and that this is a closed subset of $G_T|_{Y(x)}$.

Lemma 5.4.1. *Let T be an action of $\mathbb{N}_0^k$ on a locally compact Hausdorff space X by local homeomorphisms. For $x, y \in X$, we have $Y(x) = Y(y)$ if and only if $\overline{[x]} = \overline{[y]}$.*

Proof. If $\overline{[x]} = \overline{[y]}$, then $Y(x) = Y(y)$ by definition. Suppose conversely that $Y(x) = Y(y)$. Since orbits partition X, it suffices to show that $[y] \cap \overline{[x]} \neq \emptyset$. Since $Y(x) = Y(y)$ is open and nonempty in $\overline{[y]}$, we have $Y(x) \cap [y] \neq \emptyset$. But we have $Y(x) \subseteq \overline{[x]}$, so $[y] \cap \overline{[x]} \neq \emptyset$. ∎

We need one final result before proving the main theorem.

Theorem 5.4.2. *Let T be an action of $\mathbb{N}_0^k$ on a locally compact Hausdorff space X by local homeomorphisms. Let $x \in X$ and $z \in \mathbb{T}^k$. Suppose ρ is a faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. Let $\iota\colon C^*(G_T|_{Y(x)})) \to C^*(G_T)$ be the monomorphism from Corollary 5.2.13, let*

$$\Phi\colon C^*(G_T|_{Y(x)}) \to \operatorname{Ind}_{H(x)^\perp}^{\mathbb{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \widetilde{\alpha})$$

be the isomorphism from Proposition 5.3.6, and let

$$\varepsilon_z\colon \operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k} \to C^*(G_T|_{Y(x)}/\mathcal{I}(x))$$

denote evaluation at z. Furthermore, let $R_x\colon C^*(G_T) \to C^*(G_T|_{\overline{[x]}})$ be the homomorphism induced by restriction of compactly supported functions. There is a unique irreducible representation $\pi_{x,z,\rho}$ of $C^*(G_T)$ such that

(i) $\pi_{x,z,\rho}$ factors through R_x, and

(ii) the representation $\pi^0_{x,z,\rho}$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi_{x,z,\rho} = \pi^0_{x,z,\rho} \circ R_x$ satisfies $\pi_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi$.

Every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$ for some x, z, ρ.

Proof. First off, we have that $\rho \circ \varepsilon_z \circ \Phi$ is an irreducible representation of $C^*(G_T|_{Y(x)})$. It is injective on $C_0(Y(x))$ since ρ is faithful and $\varepsilon_z \circ \Phi$ is injective on $\Phi(C_0(Y(x)))$. To see the last part, note first that $\varepsilon_z \circ \Phi = \kappa \circ \widetilde{\alpha}_z$ on $C_c(G_T|^0_{Y(x)}) = C_c(Y(x))$. Since $\widetilde{\alpha}_z$ and κ restricts to the identity on $C_c(Y(x))$, the map $\varepsilon_z \circ \Phi$ extends injectively to $C_0(Y(x))$. Now we can apply the second part of Corollary 5.2.13 to $Y(x) \subseteq \overline{[x]}$ to obtain a unique representation $\pi^0_{x,z,\rho}$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi^0_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. As in the proof of Proposition 5.1.12, there is homomorphism $R_x\colon C^*(G_T) \to C^*(G_T|_{\overline{[x]}})$ induced by restriction of functions. Now $\pi_{x,z,\rho} \coloneqq \pi^0_{x,z,\rho} \circ R_x$ satisfies (i) and (ii).

For uniqueness, suppose φ is another irreducible representation of $C^*(G_T)$ on $\mathcal{B}(H)$ for some Hilbert space H, satisfying (i) and (ii). Let φ^0 be the representation from (ii) such that $\varphi = \varphi^0 \circ R_x$. Since φ is irreducible, we have for every $\xi \in H$ that

$$\varphi(C^*(G_T))\xi = \varphi^0(R_x(C^*(G_T)))\xi \subseteq \varphi^0(G_T|_{\overline{[x]}})\xi$$

is dense in H. Hence φ^0 is an irreducible representation of $C^*(G_T|_{\overline{[x]}})$ satisfying $\varphi^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. But as we noted above, that representation is unique, so $\varphi^0 = \pi^0_{x,z,\rho}$ and $\varphi = \pi_{x,z,\rho}$.

Now we will prove that every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$. Pick any irreducible representation φ of $C^*(G_T)$. By Proposition 5.1.12, φ factors through $C^*(G_T|_{\overline{[x]}})$ for some $x \in X$. Thus $\varphi = \varphi^0 \circ R_x$ for some irreducible representation φ^0 of $C^*(G_T|_{\overline{[x]}})$ that is faithful on $C_0(\overline{[x]})$. The second part of Corollary 5.2.13 implies that φ^0 is uniquely determined by the irreducible representation $\varphi^0 \circ \iota$ which is faithful on $C_0(Y(x))$. In fact, since Φ is an isomorphism we can uniquely determine φ^0 by $\varphi^0 \circ \iota \circ \Phi^{-1}$, which will then be an irreducible representation of $\operatorname{Ind}_{H(T)^\perp}^{\mathbb{T}^k}$ that is faithful on $C_0(Y(x))$. By the last part of Theorem 5.3.12, we know that $\ker(\varphi^0 \circ \iota \circ \Phi^{-1})$ is of the form $\ker(\omega_y \circ \varepsilon_z)$ for some $y \in \mathcal{Q}(G_T|_{\overline{[x]}})$ and $z \in H(T)^\wedge \subseteq \mathbb{T}^k$. Since $\ker(\varepsilon_z) \subseteq \ker(\omega_y \circ \varepsilon_z)$, we have

$$\ker(\varepsilon_z) \subseteq \ker(\varphi^0 \circ \iota \circ \Phi^{-1}).$$

Now $\varphi \circ \iota \circ \Phi^{-1}$ induces an irreducible representation ρ of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$, with $\varphi \circ \iota \circ \Phi^{-1} = \rho \circ \varepsilon_z$. (See Remark 2.0.20.) Composing with Φ on both sides yields $\varphi^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$, as we wanted.

All that's left to show is that ρ is faithful. Since φ^0 is faithful on $C_0(\overline{[x]})$ and ι is injective, $\rho \circ \varepsilon_z = \varphi^0 \circ \iota \circ \Phi^{-1}$ is injective on $\Phi(C_0(Y(x)))$. Hence ρ is faithful on $C_0(Y(x)) = C_0((G_T|_{Y(x)}/\mathcal{I})^0)$. Lemma 5.2.6 says that $G_T|_{Y(x)}$ is amenable, so $G_T|_{Y(x)}/\mathcal{I}(x)$ is also amenable by part (vi) of Proposition 5.1.14. By (v) in the same proposition, $G_T|_{Y(x)}/\mathcal{I}(x)$ is topologically principal. By [Exe11, Theorem 4.4], ρ is faithful on all of $G_T|_{Y(x)}/\mathcal{I}(x)$. ■

Finally, we present the main theorem.

Theorem 5.4.3. *Suppose that T is an action of $\mathbb{N}^k$ on a locally compact Hausdorff space X by local homeomorphisms. For each $x \in X$ and $z \in \mathbb{T}^k$, there is an irreducible representation $\pi_{x,z}$ of $C^*(G_T)$ on $\ell^2([x])$ such that*

$$\pi_{x,z}(f)\delta_y = \sum_{(u,g,y)\in G_T} z^g f(u,g,y)\delta u \text{ for all } f \in C_c(G_T). \tag{5.14}$$

The relation on $X \times \mathbb{T}^k$ given by

$$(x,z) \sim (y,w) \text{ if and only if } \overline{[x]} = \overline{[y]} \text{ and } \overline{z}w \in H(x)^{\perp}$$

is an equivalence relation, and $\ker(\pi_{x,z}) = \ker(\pi_{y,w})$ if and only if $(x,z) \sim (y,w)$. The map $(x,z) \mapsto \ker(\pi_{x,z})$ induces a bijection from $(X \times \mathbb{T}^k)/\sim$ to $\mathrm{Prim}(C^*(G_T))$.

Proof. We split the proof into sections to make it easier to follow. First, we prove the existence of $\pi_{x,z}$ and that it satisfies (5.14). Then we prove that

$$\ker \pi_{x,z} = \ker \pi_{x,w} \text{ if and only if } (x,z) \sim (y,w) \tag{5.15}$$

in two parts, and lastly we show that $(x,z) \mapsto \pi_{x,z}$ induces a bijection.

Existence of $\boldsymbol{\pi_{x,z}}$: Let $x \in X$ and $z \in \mathbb{T}^k$, let $\alpha_z \in \mathrm{Aut}(C^*(G_T))$ be the automorphism from Lemma 5.3.3, and let $\omega'_{[x]}$ be the irreducible representation of $C^*(G_T)$ from Lemma 5.3.7. Define the irreducible representation $\pi_{x,z} \coloneqq \omega'_{[x]} \circ \alpha_z$. For $f \in C_c(G_T)$, we have

$$\begin{aligned} \pi_{x,z}(f)\delta_y &= \sum_{d(\gamma)=y} \alpha_z(f)(\gamma)\delta_{r(\gamma)} \\ &= \sum_{d(\gamma)=y} \alpha_z(f)(\gamma)\delta_{r(\gamma)} \\ &= \sum_{(u,r,y)\in G_T} z^r f(u,r,y)\delta_u, \end{aligned}$$

so $\pi_{x,z}$ satisfies (5.14). Furthermore, as in the end of the proof of Proposition 5.1.12, the support of $\pi_{x,z}|_{C_0(G_T^0)}$ is $\overline{[x]}$. By Proposition 5.1.12, $\pi_{x,z}$ factors through R_x.

It is clear that $\sim$ is an equivalence relation. The next step is to prove the equivalence (5.15). We first prove the contrapositive of " $\Longrightarrow$ ", and then move on to " $\Longleftarrow$ ".

Proving (5.15)**, part one:** Suppose first that $\overline{[x]} \neq \overline{[y]}$. Then $\pi_{x,z}$ and $\pi_{y,w}$ have different supports when restricted to $C_0(G_T^0)$, which by definition implies different kernels. We will prove the contrapositive of " $\Longleftarrow$ ", which requires two steps.

Suppose next that $\overline{[x]} = \overline{[y]}$ but $\overline{z}w \notin H(T)^\perp$. Then both $\pi_{x,z}$ and $\pi_{y,w}$ induces representations $\pi^0_{x,z}$ and $\pi^0_{y,w}$ of $C^*(G_T|_{\overline{[x]}})$. (For later, note that $\pi_{x,z} = \pi^0_{x,z} \circ R_x$.) Let $\iota\colon C^*(G_T|_{Y(x)}) \to C^*(G_T|_{\overline{[x]}})$ be the monomorphism from part (i) of Corollary 5.2.13. Then part (ii) of the same corollary implies that $\ker \pi^0_{x,z} = \ker \pi^0_{y,w}$ if and only if $\ker(\pi^0_{x,z} \circ \iota) = \ker(\pi^0_{y,w} \circ \iota)$. By Lemma 5.4.1 we have $Y(x) = Y(y)$, and for $f \in C_c(G_T|_{Y(x)}) = C_c(G_T|_{Y(x)})$ we have

$$(\pi^0_{x,z} \circ \iota)(f)\delta_y = \sum_{(u,r,y)\in G_T|_{Y(x)}} z^r f(u,r,y)\delta_u.$$

Note that, given $n \in H(x)$, we have $(u,r,y) \in G_T|_{Y(x)}$ if and only if $(u, r+n, y) \in G_T|_{Y(x)}$ by Lemma 5.3.4. Hence we can write

$$\sum_{(u,r,y)\in G_T|_{Y(x)}} z^r f(u,r,y)\delta_u = \sum_{(u,r+n,y)\in G_T|_{Y(x)}} z^r f(u,r,y)\delta_u$$

for any $n \in H(x)$. As in Lemma 5.3.5 we have for $f \in C_c(G_T|_{Y(x)})$ that $\pi^0_{x,z} \circ \iota(\varphi \diamond f) = \hat{\varphi}(z)(\pi^0_{x,z} \circ \iota)(f)$ and $\pi^0_{y,w} \circ \iota(\varphi \diamond f) = \hat{\varphi}(w)(\pi^0_{y,w} \circ \iota)(f)$. Indeed, we calculate to find that

$$\begin{aligned}
(\pi^0_{x,z} \circ \iota)(\varphi \diamond f)\delta_y &= \sum_{(u,r,y)\in G_T|_{y(x)}} z^r (\varphi \diamond f)(u,r,y)\delta_u \\
&= \sum_{(u,r,y)\in G_T|_{y(x)}} z^r \sum_{s\in H(x)} \varphi(s) f(u, r-s, y)\delta_u \\
&= \sum_{(u,r,y)\in G_T|_{y(x)}} \sum_{s\in H(x)} z^r \varphi(s) f(u, r-s, y)\delta_u.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\hat{\varphi}(z)(\pi^0_{x,z} \circ \iota)(f)\delta_y &= \sum_{s\in H(x)} \varphi(s) z^s \sum_{(u,r,y)\in G_T|_{y(x)}} z^r f(u,r,y)\delta_u \\
&= \sum_{(u,r,y)\in G_T|_{y(x)}} \sum_{s\in H(x)} \varphi(s) z^{s+r} f(u,r,y)\delta_u \\
&= \sum_{(u,r,y)\in G_T|_{y(x)}} \sum_{t=s+r\in H(x)} \varphi(s) z^t f(u, t-s, y)\delta_u \\
&= (\pi^0_{x,z} \circ \iota)(\varphi \diamond f)\delta_y,
\end{aligned}$$

as we wanted.

Pick some $\varphi \in C_c(H(x))$ with $\hat{\varphi}(w) = 0$ and $\hat{\varphi}(z) \neq 0$. For instance, choose $r \in H(x)$ with $z^r \neq w^r$, and set $\varphi := \chi_{\{r\}} - w^{-r}\chi_{\{2r\}}$. Then $\hat{\varphi}(w) = w^r - w^{2r-r} = 0$, while $\hat{\varphi}(z) = z^r - z^{2r}w^{-r} = z^r(1 - \overline{w}z) \neq 0$. We may also pick an $f \in C_c(Y(x))$ with $f(x) = 1$. Then

$$\pi^0_{y,w} \circ \iota(\varphi \diamond f) = \hat{\varphi}(w)(\pi^0_{y,w} \circ \iota)(f) = 0,$$

but

$$\begin{aligned}\pi^0_{x,z} \circ \iota(\varphi \diamond f)\delta_x &= \hat{\varphi}(z)(\pi^0_{x,z} \circ \iota)(f)\delta_x \\ &= \hat{\varphi}(z) \sum_{(u,r,x)\in Y(x)} z^r f(u,r,x)\delta_u \\ &= \hat{\varphi}(z) z^0 f(x)\delta_x \\ &= \hat{\varphi}(z)\delta_x \neq 0.\end{aligned}$$

Hence the kernels are different.

Proving (5.15)**, part two:** Suppose that $\overline{[x]} = \overline{[y]}$ and $\overline{z}w \in H(T)^\perp$. We still have $Y(x) = Y(y)$ by Lemma 5.4.1. Let $\omega_{[x]}$ and $\omega_{[y]}$ be the irreducible representations of $C^*(G_T|_{Y(x)}/\mathcal{I}(x)) = C^*(G_T|_{Y(y)}/\mathcal{I}(y))$ from Lemma 5.3.7. Now we claim that $\pi^0_{x,z} \circ \iota = \omega_{[x]} \circ \varepsilon_z \circ \Phi$. The calculations to prove so are done by just looking at the definitions of the functions. Indeed, let $\kappa\colon C^*(G_T|_{Y(x)}) \to C^*(G_T|_{Y(x)}/\mathcal{I}(x))$ be as in Proposition 5.1.15, and let α act on $C^*(G_T|_{Y(x)})$ as in Lemma 5.3.3. If $f \in C_c(G_T|_{Y(x)})$ and $u \in \overline{[x]}$, we have

$$\begin{aligned}(\omega_{[x]} \circ \varepsilon_z \circ \Phi)(f)(\delta_u) &= \sum_{d(\gamma)=u} (\varepsilon_z \circ \Phi)(f)(\gamma)\delta_{r(\gamma)} \\ &= \sum_{\substack{(v,r,u)\in \\ G_T|_{Y(x)}/\mathcal{I}(x)}} (\varepsilon_z \circ \Phi)(f)(v,r,u)\delta_v \\ &= \sum_{\substack{(v,r,u)\in \\ G_T|_{Y(x)}/\mathcal{I}(x)}} (\Phi(f))(z)(v,r,u)\delta_v \\ &= \sum_{\substack{(v,r,u)\in \\ G_T|_{Y(x)}/\mathcal{I}(x)}} \kappa(\alpha_z(f))(v,r,u)\delta_v \\ &= \sum_{\substack{(v,r,u)\in \\ G_T|_{Y(x)}/\mathcal{I}(x)}} \sum_{\substack{(v,s,u)\in G_T|_{Y(x)} \\ \text{with } q(s)=r}} \alpha_z(f)(v,s,u)\delta_v \\ &= \sum_{\substack{(v,r,u)\in \\ G_T|_{Y(x)}/\mathcal{I}(x)}} \sum_{\substack{(v,s,u)\in G_T|_{Y(x)} \\ \text{with } q(s)=r}} z^s f(v,s,u)\delta_v \\ &= \sum_{(v,s,u)\in G_T|_{Y(x)}} z^s f(v,s,u)\delta_v \\ &= (\pi^\circ_{x,z} \circ \iota)(f)\delta_u.\end{aligned} \tag{5.16}$$

The same calculations can be done to see that $\pi^\circ_{y,w} \circ \iota = \omega_{[y]} \circ \varepsilon_w \circ \Phi$. By the definition of Φ, we have $\varepsilon_z \circ \Phi = \kappa \circ \alpha_z$. Hence

$$\widetilde{\alpha}_{\bar{z}w} \circ \varepsilon_z \circ \Phi = \widetilde{\alpha}_{\bar{z}w} \circ \kappa \circ \alpha_z = \widetilde{\alpha}_{\bar{z}w}\widetilde{\alpha}_z \circ \kappa = \widetilde{\alpha}_w \circ \kappa = \varepsilon_w \circ \Phi,$$

so we have

$$\omega_{[x]} \circ \widetilde{\alpha}_{\bar{z}w} \circ \varepsilon_z \circ \Phi = \omega_{[x]} \circ \varepsilon_w \circ \Phi.$$

Since $\omega_{[x]}$ and $\omega_{[y]}$ are injective and $\widetilde{\alpha}_{\bar{z}w}$ is an automorphism, we have

$$\begin{aligned}\ker(\omega_{[y]} \circ \varepsilon_w \circ \Phi) &= \ker(\omega_{[x]} \circ \varepsilon_w \circ \Phi) \\ &= \ker(\omega_{[x]} \circ \widetilde{\alpha}_{\bar{z}w} \circ \varepsilon_z \circ \Phi) \\ &= \ker(\varepsilon_z \circ \Phi) \\ &= \ker(\omega_{[x]} \circ \varepsilon_z \circ \Phi).\end{aligned}$$

Hence $\ker(\pi^0_{x,z} \circ \iota) = \ker(\pi^0_{y,w} \circ \iota)$. Part (ii) of Corollary 5.2.13 implies that $\ker \pi^0_{x,z} = \ker \pi^0_{y,w}$. Since $\overline{[x]} = \overline{[y]}$, we have $R_x = R_y$, so

$$\ker \pi_{x,z} = R_x^{-1}(\ker \pi^0_{x,z}) = R_x^{-1}(\ker \pi^0_{x,z}) = \ker \pi_{y,w},$$

which is what we wanted.

The map $(x, z) \mapsto \ker \pi_{x,z}$ induces a bijection: It suffices to prove that $(x, z) \mapsto \ker \pi_{x,z}$ is surjective. Suppose $I \in \mathrm{Prim}(C^*(G_T))$. Theorem 5.4.2 gives us that $I = \ker \pi_{x,z,\rho}$ for some $x \in X$, $z \in \mathbb{T}^k$ and faithful irreducible representation ρ of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. Pick an element $y \in [x] \cap Y(x)$, and let $\omega_{[y]}$ be the corresponding faithful representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$ from Lemma 5.3.7. Since ρ and $\omega_{[y]}$ are faithful, we have

$$\ker(\omega_{[y]} \circ \varepsilon_z \circ \Phi) = \ker(\varepsilon_z \circ \Phi) = \ker(\rho \circ \varepsilon_z \circ \Phi),$$

so from part (ii) in Theorem 5.4.2, we can write

$$\ker(\pi_{x,z,\omega_{[y]}}) = \ker(\omega_{[y]} \circ \varepsilon_z \circ \Phi) = \ker(\rho \circ \varepsilon_z \circ \Phi) = \ker(\pi_{x,z,\rho}).$$

Now, all we need to prove is that that $\pi_{x,z,\omega_{[y]}} = \pi_{x,z}$. It may be useful to have in mind that the following diagram commutes.

$$\begin{tikzcd} C^*(G_T) \arrow[r, "\pi_{x,z,\omega_{[y]}}"] \arrow[dr, "R_x"'] & \ell^1(\overline{[x]}) & C^*(G_T|_{Y(x)}/\mathcal{I}(x)) \arrow[l, "\omega_{[y]}"'] \\ & C^*(G_T|_{\overline{[x]}}) \arrow[u, "\pi^0_{x,z,\omega_{[y]}}"] & C^*(G_T|_{Y(x)}) \arrow[l, hook', "\iota"'] \arrow[u, "\varepsilon_z \circ \Phi"'] \end{tikzcd}$$

Since both $\pi_{x,z,\omega_{[y]}}$ and $\pi_{x,z}$ factor through R_x, it suffices to show that $\pi^0_{x,z,\omega_{[y]}} = \pi^0_{x,z}$. We will first see that $\pi^0_{x,z}$ and $\pi^0_{x,z,\omega_{[y]}}$ agree on $\iota(C_c(G_T|_{Y(x)}))$. So let

$$f \in \iota(C_c(G_T|_{Y(x)})) \subseteq C_c(G_T|_{\overline{[x]}}).$$

Then $f \in C_c(G_T|_{Y(x)})$ since ι extends the inclusion $C_c(G_T|_{Y(x)}) \hookrightarrow C_c(G_T|_{\overline{[x]}})$. We can write

$$\pi^0_{x,z,\omega_{[y]}}(f) = (\pi_{x,z,\omega_{[y]}} \circ \iota)(f) = (\omega_{[y]} \circ \varepsilon_z \circ \Phi)(f).$$

As with the calculations in (5.16), we have for any basis element $\delta_u \in \ell^1(\overline{[x]})$ that

$$(\omega_{[y]} \circ \varepsilon_z \circ \Phi)(f)\delta_u = \sum_{(v,r,u) \in G_T|_{Y(x)}} z^r f(v, r, u)\delta_u,$$

where we have used that $Y(x) = Y(y)$. For $f \in C_c(G_T|_{Y(x)})$ and $u \in \overline{[x]}$, we have

$$\begin{aligned}
\pi_{x,z}(f)\delta_u &= \omega_{[x]}(\alpha_z(f))\delta_u \\
&= \sum_{s(\gamma)=u} \alpha_z(f)(\gamma)\delta_{r(\gamma)} \\
&= \sum_{(v,r,u)\in G_T|_{Y(x)}} \alpha_z(f)(v,r,u)\delta_v \\
&= \sum_{(v,r,u)\in G_T|_{Y(x)}} z^r f(v,r,u)\delta_v \\
&= (\omega_{[y]} \circ \varepsilon_z \circ \Phi)(f)\delta_u \\
&= \pi_{x,z,\omega_{[y]}}(f)\delta_u.
\end{aligned}$$

We can extend this identity by continuity, so that $\pi^0_{x,z} \circ \iota = \pi^0_{x,z,\omega_{[y]}} \circ \iota$. By part (ii) of Corollary 5.2.13, $\pi^0_{x,z} = \pi^0_{x,z,\omega_{[y]}}$, and we are done.

■

5.5 An Application to Simplicity

For this section, let $T = (T_1, \ldots, T_k)$ be an action of $\mathbb{N}_0^k$ on a locally compact Hausdorff space X by local homeomorphisms, and let G_T be the corresponding Deaconu-Renault groupoid. We will investigate when $C^*(G_T)$ is simple. This happens if and only if the equivalence relation $\sim$ from Theorem 5.4.3 degenerates to one element. It turns out that this has something to do with the fixed points of the functions $T_1, \ldots, T_k$. It might not come as a big surprise that fixed points turn up in questions like these, as they are quite central in the study of dynamical systems. The fact that we end this chapter with fixed points also creates a nice bridge to Appendix A, where we discuss the growth rate of periodic points for certain toral automorphisms.

Lemma 5.5.1. *We have that $C^*(G_T)$ is simple if and only if every orbit in X is dense and $\Sigma = \{(n, n) : n \in \mathbb{N}_0^k\} \approx \mathbb{N}_0^k$, in the notation of* (5.5).

Proof. If $C^*(G_T)$ is simple, then $(x, z) \sim (y, w)$ for all $x, y \in X, z, w \in \mathbb{T}^k$. In particular, orbit closures $\overline{[x]}$ are equal for all $x \in X$. Fix some $x \in X$, and pick another element $y \in X$. Then $y \in [y] \subseteq \overline{[y]} = \overline{[x]}$, so $[x]$ is dense. We also have $\overline{z}w \in H(T)^\perp$ for *all* $z, w \in \mathbb{Z}^k$. The multiplication map on $\mathbb{T}^k$ is surjective, so this can only happen when $H(T)^\perp = \mathbb{T}^k$. This is the same as saying that $H(T) = \{0\}$, which again is the same as saying that all elements of Σ is of the form (n, n) for some $n \in \mathbb{N}_0^k$. We have $T^n x = T^n x$ for all $x \in X$ and $n \in \mathbb{N}_0^k$, so $\{(n, n) : n \in \mathbb{N}_0^k\} \subseteq \Sigma_X \subseteq \Sigma$.

The converse statement is clear. ∎

We use the notation $\Sigma \approx \mathbb{N}_0^k$ to mean precisely that $\Sigma = \{(n, n) \in \mathbb{N}_0^k\}$. We will investigate further what entails $\Sigma \approx \mathbb{N}_0^k$. By definition, this is the case if and only if for all open subset $U \subseteq X$ and all distinct $m, n \in \mathbb{N}_0^k$, there exists some $y \in U$ with $T^m y \neq T^n y$. This again is precisely the same as saying that for any distinct $m, n \in \mathbb{N}_0^k$, the set on which T^m and T^n disagree is dense in X. Instead of using that phrase over and over, we give it a name. So, if $f, g \colon X \to X$ are any functions, we let $f \# g$ denote the fact that f and g disagree on a dense subset of X. Hence $\Sigma \approx \mathbb{N}_0^k$ if and only if $T^m \# T^n$ for all distinct $m, n \in \mathbb{N}_0^k$. For continuous f and g, we can reformulate $f \# g$ into a statement about where they *agree* instead.

Lemma 5.5.2. *Suppose $f, g \colon X \to Y$ are continuous for locally compact Hausdorff spaces X, Y. Then $f \# g$ if and only if the set on which f and g agree is nowhere dense.*

Proof. Let x be such that $f(x) \neq g(x)$; it suffices to prove that there is a neighbourhood of x on which f and g do not agree. Suppose for contradiction that this is not the case. Then there is a net $x_\lambda \to x$ with $f(x_\lambda) = g(x_\lambda)$, so

$$f(x) = f(\lim_{\lambda \to \infty} x_\lambda) = \lim_{\lambda \to \infty} f(x_\lambda) = \lim_{\lambda \to \infty} g(x_\lambda) = g(\lim_{\lambda \to \infty} x_\lambda) = g(x),$$

which is a contradiction. The converse is clear. ∎

If $\Sigma \approx \mathbb{N}_0^k$, then in particular we have $T_i \# T_j$ for distinct $i, j \in \{1, \ldots, k\}$. The converse is not true, however. Suppose for instance that T_1 is any local homeomorphism whose set of fixed points is nowhere dense, and T_2 is the identity;

then $T_1 \# T_2$, but we do not have $T_1 \# T_1 T_2 = T_1$. Thus we are interested in knowing when $f \# g$ implies $f \#(f \circ g)$.

Lemma 5.5.3. *Let $f, g\colon X \to X$ be commuting local homeomorphisms. Then $f\#(f \circ g)$ if and only if the subset of $f(X)$ fixed by g is nowhere dense. In particular, if the sets of fixed points for f and g are nowhere dense (i.e. $f, g \# \mathrm{id}$), we have $f, g \#(f \circ g)$.*

Proof. We have $f\#(f \circ g) = (g \circ f)$ if and only if

$$\{x \in X : f(x) = g(f(x))\} = \{x \in f(X) : g(x) = x\}$$

is nowhere dense. The last assertion follows. ■

We want to put a condition on the action T ensuring that $T^m \# T^n$ for all distinct $m, n \in \mathbb{N}_0^k$. It looks like this might be achieved if the fixed points for each T^n are nowhere dense. It turns out that we must also require $T_i \# T_j$ for distinct i, j. Before proving this is the case, we need two lemmas.

Lemma 5.5.4. *Let $f\colon X \to X$ be a local homeomorphism. If $S \subseteq X$ is nowhere dense, then $f^{-1}(S)$ is nowhere dense.*

Proof. Let $U \subseteq X$ be any nonempty open set. We may assume that f is a homeomorphism on U. Then $f^{-1}(S) \cap U$ contains at most one point. If it doesn't, we are done. If it contains a point x, then $U \setminus \{x\}$ is an open subset of U not intersecting $f^{-1}(S)$, since X is Hausdorff. Note that $U \setminus \{x\}$ is nonempty, since if not, then $\{f(x)\} \subseteq S$ is an open set without a nonempty subset not intersecting S. So we are done. ■

Lemma 5.5.5. *If every point in X has dense orbit, then the union of the ranges of the T_i is dense in X. In particular, the set*

$$X \setminus \bigcup_{i=1}^{k} T_i(K)$$

is nowhere dense in X.

Proof. We may assume that X consists of more than one point. Let x be any point in X, and let U be any open neighborhood of x. Since $[x]$ is dense, the set $[x] \setminus \{x\}$ is also dense, so there is a sequence of tuples $n_k \in \mathbb{N}_0^k \setminus \{0\}$ such that $x_k := T^{n_k} x \to x$. Then in particular, there is some $l \in \mathbb{N}$ with $x_l \in U$. The tuple n_l must have a nonzero entry, say the i-th entry; let n_l' be the tuple where we subtract 1 from that entry in n_l. Then T_i applied to the point $T^{n_l'} x \in X$ equals $T^{n_l} x = x_l$, which is in U. The first assertion follows.

To prove the last part, just observe that since local homeomorphisms are open maps, the set $\cup_{i=1}^k T_i(K)$ is open. Since it is also dense, its complement is contained in its boundary. The boundary of open sets are nowhere dense, and the conclusion follows. ■

Proposition 5.5.6. *If $T_i \# T_j$ for distinct $i, j \in \{1, \ldots, k\}$ and $T^n \# id$ for all $n \in \mathbb{N}_0^k \setminus \{0\}$, then $\Sigma = \{(0,0)\}$. If every point of X has dense orbit, the converse also holds.*

Proof. Recall that $\Sigma \approx \mathbb{N}_0^k$ if and only if $T^m \# T^n$ for all distinct $m, n \in \mathbb{N}_0^k$.

We prove the first statement by a weird kind of induction argument. For the first step, let i, j be distinct. We know that the set F on which T_i and T_j agree is nowhere dense. The set on which $T_j T_i = T_i^2$ is precisely the set $T_i^{-1}(F)$, which is nowhere dense by Lemma 5.5.4. Hence $T_i T_j \# T_i^2$. We also have $T_i, T_j \# T_i T_j, T_i^2$ by Lemma 5.5.3. We now have $T^m \# T^n$ for (m, n) in the set $N_2 \coloneqq \{(m, n) \in \mathbb{N}_0^k : 0 < |m|, |n| \leq 2\}$.

For the "induction step", let $\{S_i\}_{i \in N_2}$ denote the set of commuting homeomorphisms that we proved were different on a dense set above. But then we may apply the exact same argument as above to see that all distinct combinations of at most two of the S_i disagree on a dense set. This is precisely the same as saying that $T^m \# T^n$ for all (m, n) in the set $N_4 \coloneqq \{(m, n) \in \mathbb{N}_0^k : 0 < |m|, |n| \leq 4\}$. Applying the argument again will yield the result for N_8, then for N_{16} and so on. This process does not stop, so eventually we have $T^m \# T^n$ for any $m, n \in \mathbb{N}_0^k$.

Now suppose every orbit in X is dense. Pick any $n \in \mathbb{N}_0^k$ with $|n| > 1$. By assumption, we have $T_i \# T_i T^n$ for all $i \in \{1, \ldots, k\}$. Now Lemma 5.5.3 gives us that, for each i, the subset of $T_i(X)$ fixed by T^n is nowhere dense. Hence the set of fixed points on $\cup_{i=1}^k T_i(X)$ is nowhere dense, so $T^n \# \mathrm{id}$ by Lemma 5.5.5. Lastly, to see that $T_i \# \mathrm{id}$ for all i, note that every fixed point for T_i must certainly be a fixed point for T_i^2. The set of fixed points for T_i^2 is nowhere dense by the discussion above, and the conclusion follows.

■

We may not remove the demand that $T_i \# T_j$ for distinct i, j. Indeed, suppose $T_1 = T_2$ is the function $x \mapsto x^2$ on the unit interval $X = [0, 1]$. Then $T^n \# \mathrm{id}$ for all $n \in \mathbb{N}_0^2$, but $T_1 T_2 = T_1^2$ is the same function $x \mapsto x^2$.

We now sum up what we have discovered in the following corollary of Theorem 5.4.3.

Corollary 5.5.7. *Let $T = (T_1, \ldots, T_k)$ be an action of $\mathbb{N}_0^k$ on a locally compact Hausdorff space X by local homeomorphisms, and let G_T be the corresponding Deaconu-Renault groupoid. Then $C^*(G_T)$ is simple if and only the following conditions hold.*

(i) Every orbit in X is dense.

(ii) For each distinct $0 < i, j \leq k$, the set on which the functions T_i and T_j agree is nowhere dense in X. ($T_i \# T_j$)

(iii) For each $n \in \mathbb{N}_0^k$, the set of fixed points for T^n is nowhere dense in X. ($T^n \# \mathrm{id}$)

Example 5.5.8. Let $\theta \in \mathbb{R}$ be any number, and consider the C^*-dynamical system $(C(\mathbb{T}), \mathbb{Z}, \alpha^\theta)$ where α^θ is the circle rotation by θ, given by

$$\alpha_\theta(z) = e^{2\pi i \theta}$$

for $z \in \mathbb{T}$. The resulting crossed-product C^*-algebra, $A_\theta = C(\mathbb{T}) \rtimes_{\alpha^\theta} \mathbb{Z}$, is simple if and only if θ is irrational. We will formulate this in terms of Deaconu-Renault groupoids to give an alternative way proof that this is the case. The proof itself is almost trivial, but we must do some work to see that we can actually form A_θ as a Deaconu-Renault groupoid.

Note that α_θ is a homeomorphism $\mathbb{T} \to \mathbb{T}$. We let α_θ also denote the resulting $\mathbb{N}_0$-action also, so that

$$\alpha_\theta^n(z) = e^{2\pi i n\theta} z$$

for all $n \in \mathbb{N}_0$. Thus we can form the corresponding Deaconu-Renault groupoid $G_\theta \coloneqq G_{\alpha_\theta}$. It is easy to see that the associated C^*-algebra $C^*(G_\theta)$ is simple if and only if θ is irrational. Indeed, the orbit of every point in $\mathbb{T}$ is dense if and only if θ is irrational, so if $\theta \in \mathbb{Q}$, then A_θ is not simple by Corollary 5.5.7. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then α_θ^n has no fixed points for any $n \in \mathbb{N}_0$. We apply Corollary 5.5.7 again to see that $C^*(G_{T_\theta})$ is simple.

We will show that there is an isomorphism $C^*(G_\theta) \cong A_\theta$. We will first need to recall some basic facts about crossed products algebras. Recall that the crossed product $C(\mathbb{T}) \rtimes_{\alpha^\theta} \mathbb{Z}$ is defined as the completion of $C_c(\mathbb{Z}, C(\mathbb{T}))$ in the universal norm given by

$$\begin{aligned}\|f\|_*^c &\coloneqq \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ is a covariant representation of } (C(\mathbb{T}), \mathbb{Z}, \alpha_\theta)\} \\ &= \sup\{\|\pi(f)\| : \pi \text{ is an } \ell^1\text{-norm bounded representation of } C_c(\mathbb{Z}, C(\mathbb{T}))\},\end{aligned}$$

where we have used [Wil07, Corollary 2.46]. The convolution on $C_c(\mathbb{Z}, C(\mathbb{T}))$ is given by

$$(f * g)(r)(z) = \sum_{s \in \mathbb{Z}} f(s)(z) g(r-s)(\alpha_\theta^{-s}(z))$$

for $f, g \in C(\mathbb{Z}, C(\mathbb{T}))$, and involution is given by

$$f^*(r)(z) = f(-r)^*(\alpha_\theta^{-r}(z)) = \overline{f(-r)(\alpha_\theta^{-r}(z))}.$$

For more on the crossed product, see (for instance) [Wil07, Section 2.3].

We now return to our groupoid G_θ. Say we have $(z, m-n, w) \in G_\theta$, so that $e^{2\pi i m\theta} z = e^{2\pi i n\theta} w$; this is the same as saying that $w = e^{2\pi i(m-n)\theta} z$. Hence we can write each element of G_θ uniquely as $(z, -r, \alpha_\theta^{-r}(z))$ for some $z \in \mathbb{T}$ and $r \in \mathbb{Z}$, and this defines a bijection

$$G_\theta \to \{(z, -r, \alpha_\theta^{-r}(z))\}.$$

For any $f \in C_c(G_\theta)$, there is a unique element $\Phi(f) \in C_c(\mathbb{Z}, C(\mathbb{T}))$ given by

$$\Phi(f)(r)(z) = f(z, -r, \alpha_\theta^{-r}(z)).$$

The bijection $\Phi\colon C_c(G_\theta) \to C_c(\mathbb{Z}, C(\mathbb{T}))$ is a $*$-isomorphism. Indeed, it is clearly linear, and we have

$$\begin{aligned}\Phi(f^*)(r)(z) &= f^*(z, -r, \alpha_\theta^{-r}(z)) \\ &= \overline{f((z, -r, \alpha_\theta^{-r}(z))^{-1})} \\ &= \overline{f(\alpha_\theta^{-r}(z), r, z)} \\ &= \overline{\Phi(f)(-r)(\alpha_\theta^{-r}(z))} \\ &= \Phi(f)^*(r)(z)\end{aligned}$$

so Φ is a $*$-map. To see that it is a homomorphism, note that for $\gamma = (z, -r, \alpha_\theta^{-r}(z)) \in G_\theta$, we have

$$G_\theta^{r(\gamma)} = \{(z, -s, \alpha_\theta^{-s}(z)) : s \in \mathbb{Z}\}.$$

Now we have, by the convolution in $C_c(G_\theta)$ defined in Definition 3.2.10, that

$$\begin{aligned}\Phi(f*g)(r)(z) &= (f*g)(z,-r,\alpha_\theta^{-r}(z)) \\ &= \sum_{s\in\mathbb{Z}} f(z,-r,\alpha_\theta^{-r}(z))g(\alpha_\theta^{-s}(z),s-r,\alpha_\theta^{-r}(z)) \\ &= \sum_{s\in\mathbb{Z}} \Phi(f)(r)(z)\Phi(g)(r-s)(\alpha_\theta^{-s}(z)) \\ &= (\Phi(f)*\Phi(g))\,(r)(z),\end{aligned}$$

as we wanted.

To show that Φ extends to an isomorphism between A_θ and $C^*(G_\theta)$, all we need is for Φ to be isometric with respect to the universal norms on $C_c(G_\theta)$ and $C_c(\mathbb{Z},C(\mathbb{T}))$. If we can show that Φ is isometric when $C_c(G_\theta)$ has the I-norm and $C_c(\mathbb{Z},C(\mathbb{T}))$ has the ℓ^1-norm, the latter will follow. Indeed, if that were the case, then π is an I-norm bounded representation of $C_c(G_\theta)$ if and only if π', given by $\pi'(\Phi(f))=\pi(f)$, is an ℓ^1-norm bounded representation π' of $C_c(\mathbb{Z},C(\mathbb{T}))$. Hence

$$\begin{aligned}\|f\|_* &= \sup\{\|\pi(f)\| : \pi \text{ is } I\text{-norm bounded}\} \\ &= \sup\{\|\pi'(\Phi(f))\| : \pi' \text{ is } \ell^1\text{-norm bounded}\} \\ &= \|\Phi(f)\|_*^c,\end{aligned}$$

and we would be done.

To see that Φ is isometric in this way, we first see that the I-norm can be simplified. Indeed, in the notation preceding Theorem 3.2.14, we have for all $f\in C_c(G_\theta)$ that

$$\begin{aligned}\|f\|_{I,r} &= \sup_{z\in\mathbb{T}} \sum_{\gamma\in G^z} |f(\gamma)| \\ &= \sup_{z\in\mathbb{T}} \sum_{r\in\mathbb{Z}} |f(z,-r,\alpha_\theta^{-r}(z))| \\ &= \sum_{r\in\mathbb{Z}} \sup_{z\in\mathbb{T}} |f(z,-r,\alpha_\theta^{-r}(z))| \\ &= \sum_{r\in\mathbb{Z}} \sup_{w=\alpha_\theta^{r}(z)\in\mathbb{T}} |f(\alpha_\theta^{-r}(w),-r,w)| \\ &= \sup_{w\in\mathbb{T}} \sum_{r\in\mathbb{Z}} |f(\alpha_\theta^{-r}(w),-r,w)| \\ &= \|f\|_{I,d}.\end{aligned}$$

All sums above are finite, so there is no trouble in interchanging the order with the supremum. Hence $\|\cdot\|_I = \|\cdot\|_{I,r}$. Now we can write

$$\begin{aligned}\|\Phi(f)\|_1 = \sum_{r\in\mathbb{Z}} \|\Phi(f)(r)\|_{\sup} &= \sum_{r\in\mathbb{Z}} \sup_{z\in\mathbb{T}} |\Phi(f)(r)(z)| \\ = \sup_{z\in\mathbb{T}} \sum_{r\in\mathbb{Z}} |f(z,-r,\alpha_\theta^{-r}(z))| &= \|f\|_I,\end{aligned}$$

which is what we wanted. Hence $A_\theta \cong C^*(G_\theta)$, which is simple if and only if θ is irrational.

We finish off the chapter with an example connected to Appendix A.

Example 5.5.9. Recall the Deaconu-Renault G_T groupoid from Example 5.2.3, coming from k commuting $d \times d$-matrices with determinant ± 1. We never have that $C^*(G_T)$ is simple, since 0 always has the trivial orbit.

APPENDIX A

The Growth Rate of Periodic Points for Hyperbolic Toral Automorphisms

In this appendix, we get hands-on experience of working explicitly with dynamical dynamical systems. Spesifically, we will be working with commuting hyperbolic automorphisms $\mathbb{T}^d \to \mathbb{T}^d$. We introduced these dynamical systems in Example 5.2.3, and we could study them via Deaconu-Renault groupoids. We will go thorugh most of [Pol12] by Pollicott, where we study the growth rate of the number of periodic points for the dynamical system. We will assume general knowledge in linear algebra, and some basics about dynamical systems. We follow the conventions of terminology used in [BS15].

A.1 Introduction

We define the d-dimensional torus as $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. We will be looking at linear maps $T_A \colon \mathbb{T}^d \to \mathbb{T}^d$, represented by an invertible $d \times d$ integer matrix A. The columns of such a matrix necessarily span the whole of $\mathbb{R}^d$, so T_A is surjective. Furthermore, if $T_A(x + \mathbb{Z}^d) = 0$ for some $x \in \mathbb{R}^d$, then Ax is some integer vector. Since A is an integer matrix, x must also be a vector of integers, so the class of x in $\mathbb{T}^d$ is 0. Hence T_A is both surjective and injective, and since it is linear, T_A is an automorphism. This is our motivation for the next definition.

Definition A.1.1. A **hyperbolic toral automorphism** is a linear map $T_A \colon \mathbb{T}^d \to \mathbb{T}^d$, represented by the $d \times d$ integer matrix A, such that A is invertible and has no eigenvalues on the unit circle.

Note that the integer matrix A is invertible if and only if $|\det(A)| = 1$. We will only consider orientation-preserving automorphisms, where $\det(A) = 1$; in other words where $A \in \mathrm{SL}(d, \mathbb{Z})$. Each T_A is linear and therefore continuous, so $(\mathbb{T}^d, T_A)$ is a discrete-time dynamical system. Suppose we had k of these, represented by matrices $A_1, A_2, \ldots, A_k$, and that they all commute. Then we may associate a group action $\mathcal{A} \colon \mathbb{Z}^k \times \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathcal{A}(n_1, \ldots, n_k, x) = A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k} x + \mathbb{Z}^d,$$

for every $x \in \mathbb{T}^d$. This is indeed an action: Suppose $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, and $m = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. Then

$$\begin{aligned}\mathcal{A}(n+m, x) &= A_1^{n_1+m_1} \ldots A_k^{n_k+m_k} x + \mathbb{Z}^d \\ &= (A_1^{n_1} \ldots A_k^{n_k})(A_1^{m_1} \ldots A_k^{m_k}) x + \mathbb{Z}^d \\ &= \mathcal{A}(n, \mathcal{A}(m, x)),\end{aligned}$$

since we assume that the matrices commute. Furthermore, $\mathcal{A}(0, x) = Ix + \mathbb{Z}^d$, so the identity in $\mathbb{Z}^d$ gives the identity in $\mathbb{T}^d$.

As mentioned, we consider the growth of the number of *periodic points* for these commuting hyperbolic toral automorphisms. A point $x \in \mathbb{T}^d$ is a periodic point for the dynamical system if $\mathcal{A}(n, x) = x$ for some $n \in \mathbb{Z}^k$. We shall be interested in the cardinality of the set of periodic points, given a set of integers $n \in \mathbb{Z}^k$. We will denote this cardinality by

$$N(n) = |\{x \in \mathbb{T}^d : \mathcal{A}(n, x) = x\}|.$$

If we had $A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k} = I$ for some $(n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k$, $N(n)$ would be infinite, so we will only be interested in matrices that do not satisfy this. This condition is the same as saying that the action $\mathcal{A}$ is nondegenerate.

We start out by considering $\mathbb{T}^3$ with two commuting hyperbolic toral automorphisms, and generalise to $\mathbb{T}^d$ and k automorphisms for $d \geq 3, k \geq 2$.

A.2 The case where $d = 3, k = 2$

Suppose, for this entire section, that $A_1, A_2 \in \mathrm{SL}(3, \mathbb{Z})$ are commuting hyperbolic matrices. Then the function $\mathcal{A}\colon \mathbb{Z}^2 \times \mathbb{T}^3 \to \mathbb{T}^3$ defined by

$$\mathcal{A}(n_1, n_2, x) = A_1^{n_1} A_2^{n_2} x + \mathbb{Z}^3$$

is a group action of $\mathbb{Z}^2$ on $\mathbb{T}^2$. We also assume that the action is nondegenerate.

The purpose of this section is to give upper and lower bounds for the growth of the number of periodic points of the action associated to (n_1, n_2) as the Euclidean norm $\|(n_1, n_2)\|$ tends to infinity. We start off with a few lemmas.

Lemma A.2.1. *Suppose A_1 and A_2 are as specified above, that A_1 has eigenvalues $\alpha_1, \alpha_2, \alpha_3$ and A_2 has eigenvalues $\beta_1, \beta_2, \beta_3$. Then we have that*

1. *the eigenvalues of A_1 and A_2 are real,*
2. *each of the common eigenvectors v_1, v_2, v_3 for A_1 and A_2 have irrational slopes, and*
3. *each of the numbers $\log|\alpha_i|/\log|\beta_i|$ for $i = 1, 2, 3$ are irrational.*

Proof. For (1) and (2), see [Pol12].

For the last part, let i be 1, 2 or 3 and suppose $\frac{\log(\alpha_i)}{\log(\beta_i)} = \frac{p}{q}$ for numbers $p, q \in \mathbb{Z}$. We have $p\log(\alpha_i) = q\log(\beta_i)$, so

$$\alpha_i^p \beta_i^q = e^{p\log(\alpha_i) + q\log(\beta_i)} = e^{2p\log(\alpha_i)} = \alpha_i^{2p},$$

hence $\alpha_i^p = \beta_i^q$. If v_i is the eigenvector corresponding to α_i and β_i, then we have

$$A_1^p v_i = \alpha_i^p v_i = \beta_i^q v_i = A_2^q v_i,$$

so $A_1^p A_2^{-q} = I$ when we restrict to the eigenspace $\mathbb{R}v_i$. But that would mean that $A_1^p A_2^{-q} = I$ on the *dense* subset $\mathbb{R}v_i + \mathbb{Z}^3$ of $\mathbb{T}^3$. Since $A_1^p A_2^{-q}$ is continuous, it must in fact act as the identity on the whole of $\mathbb{T}^3$. This means that for every $x \in \mathbb{R}^3$, there is an integer vector $n_x \in \mathbb{Z}^3$ such that $A_1^p A_2^{-q} x = x + n_x$. Since $A_1^p A_2^{-q}$ is continuous, n_x must be equal for all x, and since $A_1^p A_2^{-q} 0 = 0$, we have $n_x = 0$. This means that $A_1^p A_2^{-q} = I$, which contradicts our nondegenerative assumption. ■

Lemma A.2.2. *For each* $(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, *we have*

$$N(n_1, n_2) = |\det(I - A_1^{n_1} A_2^{n_2})|.$$

Proof. See [FM13, p. 171]. ■

The previous lemma has a useful corollary.

Lemma A.2.3. *For each* $(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, *we have*

$$N(n_1, n_2) = \\ |(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_3^{n_1}\beta_3^{n_2}) + (\alpha_1^{-n_1}\beta_1^{-n_2} + \alpha_2^{-n_1}\beta_2^{-n_2} + \alpha_3^{-n_1}\beta_3^{-n_2})|.$$

Proof. The matrix $A_1^{n_1} A_2^{n_2}$ has eigenvalues $1 - \alpha_i^{n_1}\beta_i^{n_2}$ for $i = 1, 2, 3$, and the determinant is the product of these eigenvalues. Thus by the previous lemma, we have

$$\begin{aligned} N(n_1, n_2) =& |(1 - \alpha_1^{n_1}\beta_1^{n_2})(1 - \alpha_2^{n_1}\beta_2^{n_2})(1 - \alpha_3^{n_1}\beta_3^{n_2})| \\ =& |1 - (\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_3^{n_1}\beta_3^{n_2}) \\ &+ ((\alpha_1\alpha_2)^{n_1}(\beta_1\beta_2)^{n_2} + (\alpha_1\alpha_3)^{n_1}(\beta_1\beta_3)^{n_2} + (\alpha_2\alpha_3)^{n_1}(\beta_2\beta_3)^{n_2}) \\ &- (\alpha_1\alpha_2\alpha_3)^{n_1}(\beta_1\beta_2\beta_3)^{n_2}|. \end{aligned}$$

We have $\det(A_1) = \det(A_2) = 1$, so $\alpha_1\alpha_2\alpha_3 = \beta_1\beta_2\beta_3 = 1$. Thus we may rewrite the last line as

$$N(n_1, n_2) = \\ |(\alpha_1^{n_1}\beta_1^{n_2} + \alpha_2^{n_1}\beta_2^{n_2} + \alpha_3^{n_1}\beta_3^{n_2}) + (\alpha_1^{-n_1}\beta_1^{-n_2} + \alpha_2^{-n_1}\beta_2^{-n_2} + \alpha_3^{-n_1}\beta_3^{-n_2})|,$$

as desired. ■

We will use this lemma to estimate the growth of $N(n_1, n_2)$, and it will be convenient to introduce the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} \log|\alpha_1| \\ \log|\beta_1| \end{pmatrix}, u_2 = \begin{pmatrix} \log|\alpha_2| \\ \log|\beta_2| \end{pmatrix}, \text{ and } u_3 = \begin{pmatrix} \log|\alpha_3| \\ \log|\beta_3| \end{pmatrix}.$$

Each of these vectors have irrational slopes. Indeed, from Lemma A.2.1 we know that $\frac{\log|\alpha_i|}{\log|\beta_i|}$ is irrational for all i. Thus for each i, the line $\mathbb{R}u_i$ given by the equation $y = \frac{\log|\beta_i|}{\log|\alpha_i|}x$ has irrational slope.

Lemma A.2.4. *The vectors u_1, u_2 and u_3 are nonzero and satisfy $u_1+u_2+u_3 = 0$.*

Proof. For the first part, suppose that $u_i = 0$ for some i. Then $|\alpha_i| = 1$ and $|\beta_i| = 1$, both of which are contradictions. For the second part, we use that $\det(A_1) = \alpha_1\alpha_2\alpha_3 = 1$ and $\det(A_2) = \beta_1\beta_2\beta_3 = 1$. Thus

$$\log|\alpha_1\alpha_2\alpha_3| = \log|\alpha_1| + \log|\alpha_2| + \log|\alpha_3| = 0,$$

and similarly, $\log|\beta_1| + \log|\beta_2| + \log|\beta_3| = 0$. Now we have

$$u_1 + u_2 + u_3 = \begin{pmatrix} \log|\alpha_1| + \log|\alpha_2| + \log|\alpha_3| \\ \log|\beta_1| + \log|\beta_2| + \log|\beta_3| \end{pmatrix} = 0,$$

as desired. ∎

Our next step is to parameterise all unit vectors in $\mathbb{R}^2$ by

$$w_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

for $0 \le \theta < 2\pi$. Then we have

$$\langle u_i, w_\theta \rangle = \cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|.$$

Given $(n_1, n_2) \in \mathbb{Z}^2$, let $R \coloneqq \|(n_1, n_2)\|_2$. Then we can write $(n_1, n_2) = (R\cos\theta, R\sin\theta)$. Thus we have

$$\begin{aligned} |\alpha_i^{n_1}\beta_i^{n_2}| &= \exp(n_1 \log|\alpha_i| + n_2 \log|\beta_i|) \\ &= \exp(R(\cos\theta \log|\alpha_i| + \sin\theta \log|\beta_i|)) \\ &= e^{R\langle w_\theta, u_i \rangle}. \end{aligned}$$

Lemma A.2.5. *For every $\theta \in [0, 2\pi)$, there is an i such that $\langle u_i, w_\theta \rangle > 0$.*

Proof. We claim that it suffices to prove that the u_i are not collinear. Indeed, if they are not collinear, we have $\langle u_i, w_\theta \rangle = \|u_i\|\|w_\theta\| \cos\phi_i = \|u_i\| \cos\phi_i$ where ϕ_i is the angle between u_i and w_θ. If $\|u_i\| \cos\phi_i \le 0$ for all i, then all the vectors u_i would lie in the half plane defined by the line to which w_θ is orthogonal to; this is impossible, since the u_i are nonzero and sums up to 0.

Now suppose for contradiction that the u_i are collinear. Then we may choose a $\delta \ne 0$ such that

$$\delta = \frac{\log|\alpha_1|}{\log|\beta_1|} = \frac{\log|\alpha_2|}{\log|\beta_2|} = \frac{\log|\alpha_3|}{\log|\beta_3|}.$$

From the last part of Lemma A.2.1, δ must be irrational; hence the set $\{n \log|\alpha_1| + m \log|\beta_1| : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Choose sequences $\{n_k\}, \{m_k\}$ in $\mathbb{Z} \setminus \{0\}$ so that $n_k \log|\alpha_1| + m_k \log|\beta_1| \to 0$. Then for all $r \in \mathbb{R}^+$,

$$A^{n_k} B^{m_k} r v_1 + \mathbb{Z}^3 = e^{n_k \log|\alpha_1| + m_k \log|\beta_1|} r v_1 + \mathbb{Z}^3 \to r v_1 + \mathbb{Z}^3.$$

Thus $A^{n_k} B^{m_k} \to I$ on the set $\mathbb{R}^+ v_1 + \mathbb{Z}^3$. This set is dense in $\mathbb{T}^3$ by the second part of Lemma A.2.1, so $A^{n_k} B^{m_k} \to I$. Each $A^{n_k} B^{m_k}$ is unequal to I because of our nondegenerative assumtion, and since we are dealing with integer matrices, the convergence $A^{n_k} B^{m_k} \to I$ is impossible. ∎

Lastly, we denote

$$\begin{aligned}\overline{\lambda} &:= \sup_{0\le\theta<2\pi} \left\{ \max_{i=1,2,3} \{\langle u_i, w_\theta\rangle\} \right\}, \text{ and} \\ \underline{\lambda} &:= \inf_{0\le\theta<2\pi} \left\{ \max_{i=1,2,3} \{\langle u_i, w_\theta\rangle\} \right\}.\end{aligned}$$

We are now ready to state and prove our main theorem.

Theorem A.2.6. *Let $A_1, A_2 \in \mathrm{SL}(3,\mathbb{Z})$ be commuting independent hyperbolic matrices. The growth rates of the periodic points satisfy*

$$\begin{aligned}\limsup_{\|(n_1,n_2)\|_2\to\infty} \frac{\log N(n_1,n_2)}{\|(n_1,n_2)\|_2} &= \overline{\lambda} \text{ and} \\ \liminf_{\|(n_1,n_2)\|_2\to\infty} \frac{\log N(n_1,n_2)}{\|(n_1,n_2)\|_2} &= \underline{\lambda}.\end{aligned}$$

Furthermore, $0 < \underline{\lambda} < \overline{\lambda} < \infty$.

Proof. We will prove the first equality in the theorem; the other is similar.

From Lemma A.2.3 and the discussion above, we can write

$$\begin{aligned}N(n_1,n_2) = |&(e^{R\langle u_1,w_\theta\rangle} + e^{R\langle u_2,w_\theta\rangle} + e^{R\langle u_3,w_\theta\rangle}) \\ &+ (e^{R\langle u_1,w_{-\theta}\rangle} + e^{R\langle u_2,w_{-\theta}\rangle} + e^{R\langle u_3,w_{-\theta}\rangle})|. \end{aligned} \tag{A.1}$$

From Lemma A.2.5, there is at least one i such that $\langle u_i, w_\theta\rangle > 0$; hence $N(n_1,n_2) \to \infty$ as $R \to \infty$. Thus, when $R \to \infty$, the term or terms with largest positive exponent will dominate the sum in (A.1). Suppose that there is always one dominating term; the general argument is similar, but requires much detail. Writing R instead of $\|(n_1,n_2)\|_2$, we get that

$$\begin{aligned}\limsup_{R\to\infty} \frac{\log N(n_1,n_2)}{R} &= \lim_{R\to\infty} \sup_{x\ge R} \left\{ \max_{i=1,2,3} \{ \frac{\log(e^{\langle u_i,w_\theta\rangle})}{x} \} : 0 \le \theta < 2\pi \right\} \\ &= \lim_{R\to\infty} \sup_{x\ge R} \left\{ \max_{i=1,2,3} \{\langle u_i, w_\theta\rangle\} : 0 \le \theta < 2\pi \right\}.\end{aligned}$$

Note that the terms with $\langle u_i, w_{-\theta}\rangle$ are covered since we are taking the supremum when θ varies from 0 to 2π. The last term does not depend on R, so we may simply write

$$\limsup_{R\to\infty} \frac{\log N(n_1,n_2)}{R} = \sup_{0\le\theta<2\pi} \left\{ \max_{i=1,2,3} \{\langle u_i, w_\theta\rangle\} \right\}.$$

We recognise the right-hand side as $\overline{\lambda}$, which is clearly less than ∞, so we are done. ∎

A.3 Generalisations to $\mathbb{Z}^k$-actions

We would like to generalise Theorem A.2.6 to higher-dimensional actions. For the rest of this section, we make the following assumptions.

1. We assume that $A_1, \ldots, A_k \in \mathrm{SL}(d, \mathbb{Z})$ commute. This gives rise to an action $\mathcal{A}\colon \mathbb{Z}^k \times \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$\mathcal{A}(n_1, \ldots, n_k, x) = A_1^{n_1} \ldots A_k^{n_k} x + \mathbb{Z}^d,$$

   as discussed in the beginning of this article.

2. We assume that each matrix $A_1^{n_1} \ldots A_k^{n_k}$ such that $n(n_1, \ldots, n_k) \in \mathbb{Z}^k$ is ergodic (i.e. they do not have eigenvalues which are roots of unity).

3. We assume that the action is nondegenerate, so that if there are integers $n_1, \ldots, n_k \in \mathbb{Z}$ such that $A_1^{n_1} \ldots A_k^{n_k} = I$ then $n_1, \ldots, n_k = 0$.

4. We assume that the action is irreducible, that is, no $\mathcal{A}(n_1, \ldots, n_k)\colon \mathbb{T}^d \to \mathbb{T}^d$ preserves a proper invariant toral subgroup of $\mathbb{T}^d$.

5. We assume that the matrices A_i are semisimple (that is, they diagonalise over the complex numbers), and that A_i has complex eigenvalues $\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}$ for $i = 1, \ldots, k$.

As before, we let

$$N(n_1, \ldots, n_k) = |\{x \in \mathbb{T}^d : \mathcal{A}(n_1, \ldots, n_k, x) = x\}|$$

denote the number of periodic points for the function $\mathcal{A}(n_1, \ldots, n_k)$. Furthermore, we denote

$$\underline{\lambda} = \inf_w \left\{ \langle w, \sum_{j : \langle w, v_j \rangle \geq 0} v_j \rangle \right\},$$

$$\overline{\lambda} = \sup_w \left\{ \langle w, \sum_{j : \langle w, v_j \rangle \geq 0} v_j \rangle \right\}.$$

The generalisation of Theorem A.2.6 in this setting will be the following.

Theorem A.3.1. *The growth rates of the number of periodic points satisfy*

$$\limsup_{\|(n_1, \ldots, n_k)\|_2 \to \infty} \frac{\log N(n_1, \ldots, n_k)}{\|(n_1, \ldots, n_k)\|_2} \leq \overline{\lambda}, \text{ and}$$

$$\liminf_{\|(n_1, \ldots, n_k)\|_2 \to \infty} \frac{\log N(n_1, \ldots, n_k)}{\|(n_1, \ldots, n_k)\|_2} \geq \underline{\lambda}.$$

Furthermore, $0 < \underline{\lambda} < \overline{\lambda} < \infty$.

To prove this, we will need two lemmas; the first one is a generalisation of Lemma A.2.2.

Lemma A.3.2. *For each* $(n_1, \ldots, n_k) \in \mathbb{Z}^k \setminus \{0, \ldots, 0\}$*, we may write*

$$N(n_1, \ldots, n_k) = |\det(I - A_1^{n_1} \ldots A_k^{n_k})|.$$

Proof. See [FM13, p. 171]. ∎

Lemma A.3.3. *Given* $d \geq 2$*, there exists* $\epsilon > 0$ *such that if* α *is an algebraic integer of degree* d *whose conjugate roots do not lie on the unit circle, then the conjugate values* $\alpha = \alpha_1, \ldots, \alpha_d$ *cannot all be contained in the annulus*

$$A(\epsilon) = \{z \in \mathbb{C} : 1 - \epsilon \leq |z| \leq 1 + \epsilon\}.$$

Proof. Assume for contradiction that for some $d \geq 2$ we can find an infinite sequence of monomials

$$P_n(x) = x^d + c_{d-1}^{(n)} x^{d-1} + \ldots c_k^{(n)} x^k + \ldots c_1^{(n)} x^1 + c_0^{(n)} \in \mathbb{Z}[x]$$

whose roots $\alpha_1^{(n)}, \ldots, \alpha_d^{(n)} \in A(\frac{1}{n})$ do not lie on the unit circle. Since $P_n(x) = \prod_{i=1}^d (x - \alpha_i^{(n)})$, we see that

$$\begin{aligned}
\left|c_k^{(n)}\right| &= \left| \sum_{i_1 < \ldots < i_{d-k}} \alpha_{i_1}^{(n)} \ldots \alpha_{i_{d-k}}^{(n)} \right| \\
&\leq \sum_{i=1}^{d} \left|\alpha_i^{(k)} \ldots \alpha_d^{(n)}\right| \\
&\leq d\left(1 + \frac{1}{n}\right)^d \leq d2^d
\end{aligned}$$

for $k = 1, \ldots, d$ and $n \geq 1$. In particular, we have $c_k^{(n)} \in \mathbb{Z} \cap [-d2^d, d2^d]$ for all $k = 1, \ldots, d$ and $n \geq 1$. By the pigeonhole principle, we may choose a subsequence $P(x) \coloneqq P_{n_1}(x) = P_{n_2}(x) = P_{n_3}(x) = \ldots$ for which the coefficients all agree. But then the roots of $P(x)$ are all arbitrarily close to 1 in modulus, so they must lie on the unit circle. This is a contradiction. ∎

Proof of Theorem A.3.1. Since each A_i is diagonalisable, the product $A \coloneqq A_1^{n_1} \ldots A_k^{n_k}$ is diagonalisable with diagonalisation $A = PDP^{-1}$, say; thus $I - A = PP^{-1} - PDP^{-1} = P(I - A)P^{-1}$ is also diagonalisable. Therefore we may write $\det(I - A_1^{n_1} \ldots A_k^{n_k})$ as the product of the eigenvalues of the matrix $I - A_1^{n_1} \ldots A_k^{n_k}$, namely the complex numbers $1 - \prod_{i=1}^k (\alpha_j^{(i)})^{n_i}$. Hence we have

$$\det(I - A_1^{n_1} \ldots A_k^{n_k}) = \prod_{j=1}^{d} \left| 1 - \prod_{i=1}^{k} (\alpha_j^{(i)})^{n_i} \right|. \tag{A.2}$$

We will parametrise elements $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ as $(p_1 R, \ldots, p_k R)$, where

1. $p_i \in [-1, 1]$ for $i = 1, \ldots, k$,
2. $\|(p_1, \ldots, p_k)\|_2 = 1$ and
3. $R = \|(n_1, \ldots, n_k)\|_2$.

We introduce the notation $v_j = (\log|\alpha_j^{(1)}|, \ldots, \log|\alpha_j^{(k)}|)$ for $j = 1, \ldots, k$, and $w = (p_1, \ldots, p_k)$. For each $j = 1, \ldots, k$, we have

$$\prod_{i=1}^{k} (\alpha_j^{(i)})^{Rp_i} = e^{R\sum_{i=1}^{k} p_i \log|\alpha_j^{(i)}|} = e^{R\langle w, v_j\rangle};$$

hence by (A.2) we may write

$$N(n_1, \ldots, n_k) = \prod_{j:\langle w,v_j\rangle \geq 0} (e^{R\langle w,v_j\rangle} - 1) \prod_{j:\langle w,v_j\rangle < 0} (1 - e^{R\langle w,v_j\rangle}).$$

The second product is always between 0 and 1, so

$$N(n_1, \ldots, n_k) \geq \prod_{j:\langle w,v_j\rangle \geq 0} (e^{R\langle w,v_j\rangle} - 1).$$

The right-hand side above is a polynomial in $e^{R\langle w,v_j\rangle}$ of some degree m with leading coefficient 1. So is $\prod_{j:\langle w,v_j\rangle\geq 0} e^{R\langle w,v_j\rangle}$, and thus

$$\lim_{R\to\infty} \frac{\prod_{j:\langle w,v_j\rangle\geq 0}(e^{R\langle w,v_j\rangle} - 1)}{\prod_{j:\langle w,v_j\rangle\geq 0} e^{R\langle w,v_j\rangle}} = 1.$$

This means that there exists a number $R_0 \in \mathbb{R}$ so that if $R \geq R_0$, then

$$\begin{aligned}\prod_{j:\langle w,v_j\rangle \geq 0} (e^{R\langle w,v_j\rangle} - 1) &\geq \frac{1}{2} \prod_{j:\langle w,v_j\rangle \geq 0} (e^{R\langle w,v_j\rangle}) \\ &= \frac{1}{2} e^{R\langle w, \sum_{j:\langle w,v_j\rangle\geq 0} v_j\rangle}.\end{aligned}$$

We recognise the inner product in the exponent as $\underline{\lambda}$, so

$$N(n_1, \ldots, n_k) \geq \frac{1}{2} e^{\underline{\lambda}\|(n_1,\ldots,n_k)\|_2}.$$

This gives us that

$$\frac{\log N(n_1, \ldots, n_k)}{\|N(n_1, \ldots, n_k)\|_2} \geq \frac{\log\frac{1}{2} + \underline{\lambda}R}{R},$$

which in turn yields that the growth rate satisfies

$$\begin{aligned}&\liminf_{\|(n_1,\ldots,n_k)\|_2\to\infty} \frac{\log N(n_1, \ldots, n_k)}{\|(n_1, \ldots, n_k)\|_2} \\ &\qquad \geq \liminf_{R\to\infty} \frac{\log\frac{1}{2} + \underline{\lambda}R}{R} = \underline{\lambda}.\end{aligned}$$

Similarly, we have

$$N(n_1, \ldots, n_k) \leq 2e^{\overline{\lambda}\|(n_1,\ldots,n_k)\|_2},$$

which gives us

$$\limsup_{\|(n_1,\ldots,n_k)\|_2\to\infty} \frac{\log N(n_1,\ldots,n_k)}{\|(n_1,\ldots,n_k)\|_2} \leq \overline{\lambda}.$$

It remains to show that $0 < \underline{\lambda}$ and $\overline{\lambda} < \infty$. We have by definition that $\underline{\lambda} \geq 0$. To show that $\underline{\lambda} \neq 0$ and $\overline{\lambda} < \infty$, we first note that the function

$$\begin{aligned} w \mapsto \langle w, &\sum_{j:\langle w,v_j\rangle\geq 0} v_j\rangle \\ &= \sum_{j=1}^{k} \max\left(\langle w, v_j\rangle, 0\right) \end{aligned}$$

is a continuous map on the compact unit circle in $\mathbb{R}^k$. That means that it will attain its infimum $\underline{\lambda}$ and supremum $\overline{\lambda}$. This implies that $\overline{\lambda} < \infty$. In addition, to prove that $\lambda \neq 0$, we only have to show that there is no w such that $\langle w, v_j\rangle = 0$ for all $j = 1, \ldots, k$.

Suppose for contradiction that there is such a w. Fix a $j \in \{1, \ldots, d\}$. By Dirichlet's theorem of simultaneous diophantine appriximation, for any $\epsilon > 0$, we may choose $1 \leq q \leq (\frac{1}{\epsilon} + 1)^k$ and $(n_1, \ldots, n_k) \in \mathbb{Z}^k$ such that $|n_l - qp_l| \leq \epsilon$ for $l = 1, \ldots, k$. Thus

$$|n_l \log|\alpha_j^{(l)}| - qp_l \log|\alpha_j^{(l)}|| \leq \epsilon|\log|\alpha_j^{(l)}|| \tag{A.3}$$

for $l = 1, \ldots, k$. Summing the inequations (A.3) for $l = 1, \ldots, k$ and applying the reverse triangle inequality yields

$$\begin{aligned} \left|\left|\sum_{l=1}^{k} n_l \log|\alpha_j^{(l)}|\right| - \left|\sum_{l=1}^{k} qp_l \log|\alpha_j^{(l)}|\right|\right| &\leq \sum_{l=1}^{k} \left|n_l \log|\alpha_j^{(l)}| - qp_l \log|\alpha_j^{(l)}|\right| \\ &\leq \epsilon \sum_{l=1}^{k} |\log|\alpha_j^{(l)}|| = \epsilon L, \end{aligned}$$

where $L := \sum_{l=1}^{k} |\log|\alpha_j^{(l)}||$. In particular, we have

$$\begin{aligned} \left|\sum_{l=1}^{k} n_l \log|\alpha_j^{(l)}|\right| &\leq \epsilon L + q\left|\sum_{l=1}^{k} p_l \log|\alpha_j^{(l)}|\right| \\ &= \epsilon L + q\langle w, v_j\rangle = \epsilon L, \end{aligned}$$

since we assume that $\langle w, v_j\rangle = 0$. We restate this as

$$\left|\log\left|\left(\alpha_j^{(1)}\right)^{n_1} \ldots \left(\alpha_j^{(k)}\right)^{n_k}\right|\right| \leq \epsilon L. \tag{A.4}$$

We recognise $\left(\alpha_j^{(1)}\right)^{n_1} \ldots \left(\alpha_j^{(k)}\right)^{n_k}$ for $j = 1, \ldots, d$ as the eigenvalues of the matrix $A_1^{n_1} \ldots A_k^{n_k}$, so (A.4) implies that the eigenvalues of this matrix can be arbitrarily close to 1 in modulus. In other words, the algebraic integers, and its conjugates, occuring as zeros of the polynomial $\det(zI - A_1^{n_1} \ldots A_k^{n_k}) = 0$ can be arbitrarily close to 1 in modulus. By Lemma A.3.2, this is impossible, finishing our proof. ■

Bibliography

[Arv76] William Arveson. *An invitation to C^*-algebras*. Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976, pp. x+106.

[BO08] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*. Vol. 88. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008, pp. xvi+509. ISBN: 978-0-8218-4381-9; 0-8218-4381-8. DOI: 10.1090/gsm/088. URL: http://dx.doi.org/10.1090/gsm/088.

[Bro+14] Jonathan Brown et al. "Simplicity of algebras associated to étale groupoids". In: vol. 88. 2. 2014, pp. 433–452. DOI: 10.1007/s00233-013-9546-z. URL: http://dx.doi.org/10.1007/s00233-013-9546-z.

[Bro77] Lawrence G. Brown. "Stable isomorphism of hereditary subalgebras of C^*-algebras". In: *Pacific J. Math.* 71.2 (1977), pp. 335–348. ISSN: 0030-8730. URL: http://projecteuclid.org/euclid.pjm/1102811431.

[BS15] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Corrected paper back edition of the 2002 original [MR1963683]. Cambridge University Press, Cambridge, 2015, pp. xii+247. ISBN: 978-1-107-53894-8; 978-0-521-80841-5.

[ER07] R. Exel and J. Renault. "Semigroups of local homeomorphisms and interaction groups". In: *Ergodic Theory Dynam. Systems* 27.6 (2007), pp. 1737–1771. ISSN: 0143-3857. DOI: 10.1017/S0143385707000193. URL: http://dx.doi.org/10.1017/S0143385707000193.

[Exe11] R. Exel. "Non-Hausdorff étale groupoids". In: *Proc. Amer. Math. Soc.* 139.3 (2011), pp. 897–907. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-2010-10477-X. URL: http://dx.doi.org/10.1090/S0002-9939-2010-10477-X.

[FM13] Ioannis Farmakis and Martin Moskowitz. *Fixed point theorems and their applications*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013, pp. xii+234. ISBN: 978-981-4458-91-7. DOI: 10.1142/8748. URL: http://dx.doi.org/10.1142/8748.

[Gre78] Philip Green. "The local structure of twisted covariance algebras". In: *Acta Math.* 140.3-4 (1978), pp. 191–250. ISSN: 0001-5962. DOI: 10.1007/BF02392308. URL: http://dx.doi.org/10.1007/BF02392308.

[HR70] Edwin Hewitt and Kenneth A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups.* Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York-Berlin, 1970, pp. ix+771.

[MRW87] Paul S. Muhly, Jean N. Renault, and Dana P. Williams. "Equivalence and isomorphism for groupoid C^*-algebras". In: *J. Operator Theory* 17.1 (1987), pp. 3–22. ISSN: 0379-4024.

[MRW96] Paul S. Muhly, Jean N. Renault, and Dana P. Williams. "Continuous-trace groupoid C^*-algebras. III". In: *Trans. Amer. Math. Soc.* 348.9 (1996), pp. 3621–3641. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-96-01610-8. URL: http://dx.doi.org/10.1090/S0002-9947-96-01610-8.

[Mur90] Gerard J. Murphy. *C^*-algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990, pp. x+286. ISBN: 0-12-511360-9.

[MW95] Paul S. Muhly and Dana P. Williams. "Groupoid cohomology and the Dixmier-Douady class". In: *Proc. London Math. Soc. (3)* 71.1 (1995), pp. 109–134. ISSN: 0024-6115. DOI: 10.1112/plms/s3-71.1.109. URL: http://dx.doi.org/10.1112/plms/s3-71.1.109.

[Pat99] Alan L. T. Paterson. *Groupoids, inverse semigroups, and their operator algebras.* Vol. 170. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1999, pp. xvi+274. ISBN: 0-8176-4051-7. DOI: 10.1007/978-1-4612-1774-9. URL: http://dx.doi.org/10.1007/978-1-4612-1774-9.

[Ped79] Gert K. Pedersen. "C^*-algebras and their automorphism groups". In: London Mathematical Society Monographs 14 (1979), pp. ix+416.

[Pol12] Mark Pollicott. "A note on the growth of periodic points for commuting toral automorphisms". In: *ISRN Geometry* 2012 (2012).

[PST15] N. Christopher Phillips, Adam P. W. Sørensen, and Hannes Thiel. "Semiprojectivity with and without a group action". In: *J. Funct. Anal.* 268.4 (2015), pp. 929–973. ISSN: 0022-1236. DOI: 10.1016/j.jfa.2014.11.005. URL: http://dx.doi.org/10.1016/j.jfa.2014.11.005.

[Ren80] Jean Renault. *A groupoid approach to C^*-algebras.* Vol. 793. Lecture Notes in Mathematics. Springer, Berlin, 1980, pp. ii+160. ISBN: 3-540-09977-8.

[Ren91] Jean Renault. "The ideal structure of groupoid crossed product C^*-algebras". In: *J. Operator Theory* 25.1 (1991). With an appendix by Georges Skandalis, pp. 3–36. ISSN: 0379-4024.

[RW98] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^*-algebras.* Vol. 60. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xiv+327. ISBN: 0-8218-0860-5. DOI: 10.1090/surv/060. URL: http://dx.doi.org/10.1090/surv/060.

[Sie97] Nándor Sieben. "C^*-crossed products by partial actions and actions of inverse semigroups". In: *J. Austral. Math. Soc. Ser. A* 63.1 (1997), pp. 32–46. ISSN: 0263-6115.

[SW16] Aidan Sims and Dana P. Williams. "The primitive ideals of some étale groupoid C^*-algebras". In: *Algebr. Represent. Theory* 19.2 (2016), pp. 255–276. ISSN: 1386-923X. DOI: 10.1007/s10468-015-9573-4. URL: http://dx.doi.org/10.1007/s10468-015-9573-4.

[SWW14] Aidan Sims, Benjamin Whitehead, and Michael F. Whittaker. "Twisted C^*-algebras associated to finitely aligned higher-rank graphs". In: *Doc. Math.* 19 (2014), pp. 831–866. ISSN: 1431-0635.

[Wil07] Dana P. Williams. *Crossed products of C^*-algebras*. Vol. 134. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007, pp. xvi+528. ISBN: 978-0-8218-4242-3; 0-8218-4242-0. DOI: 10.1090/surv/134. URL: http://dx.doi.org/10.1090/surv/134.