INFLECTIONAL LOCI OF QUADRIC FIBRATIONS

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1. Introduction

The osculatory behavior of scrolls in $\mathbb{P}^N$ has been investigated in many papers. The fact that they contain many lines implies that the dimension of every k-th osculating space is considerably smaller than what is expected for a general smooth projective variety. In particular, for n-dimensional scrolls over curves this dimension does not exceed kn and assuming that equality holds at a general point one can describe the k-th inflectional locus and its cohomology class [17]. Recently, an analogous investigation has been carried out also for scrolls over projective varieties of dimension ≥ 2 [18].

A natural question arising from these studies (which we have also been asked when presenting our results) is: what about other special varieties, in particular quadric fibrations over a curve? The aim of this paper is to approach this question. Actually, we focus on case $k = 2$ and we analyze the relationships between the osculatory behavior of such varieties and several other aspects of their geometry, like linear normality, embedding in a scroll, ampleness, etc.

The case of quadric fibrations looks particularly nice since any quadric fibration $X \subset \mathbb{P}^N$ over a smooth curve C is naturally contained as a divisor inside a projective bundle P over C, and the embedding of X in $\mathbb{P}^N$ extends to a morphism $\varphi : P \to \mathbb{P}^N$ to the same projective space, which maps every linear fiber of P isomorphically to the linear span of the corresponding fiber of X in $\mathbb{P}^N$. This map φ, however, is not always an embedding, which turns out to be equivalent to the fact that its image $R = \varphi(P)$ may not be a scroll over C.

In a sense this contrasts the naive expectation that the inflectional locus of X should be determined by that of R. On the one hand, when φ is an embedding, looking at the pair (P, X) one can compare the osculatory behavior of X with respect to that of R along X. In particular, letting Φ_2 denote the second inflectional locus, we have that $\Phi_2(X) \supseteq X \cap \Phi_2(R)$ and we have examples showing that this is not an equality in general. On the other hand, $\Phi_2(X)$ always contains the set S of singular points of singular fibers of X if $n \geq 3$ and all singular fibers if $n = 2$, facts which are not evident if we simply look at the pair (X, P), because the linear span of a fiber F of X is a linear $\mathbb{P}^n$ inside R regardless of the fact that F is a smooth or a singular fiber.

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We realize the role of S via another, more direct geometric approach, looking at $X \subset \mathbb{P}^N$ by itself, and at the linear subsystem of hyperplane sections of X having a given point $x \in X$ as a singular point of multiplicity 3 (Theorem 12).

We want to emphasize that everywhere we work without the assumption that X is linearly normally embedded. This allows us to put in evidence flexes arising from isomorphic projections or even hypo-osculation phenomena deriving from them (Section 7).

Among the results, we mention the upper bound we obtain for the highest dimension σ_k of a k-th osculating space to an n-dimensional quadric fibration X: we have $\sigma_k \leq k(n+1)-1$ (Corollary 15). Observe that for $k = n = 2$ this is the same natural bound occurring for any smooth surface, while this is not the case for $k = 2$ in higher dimension and for $k \geq 3$ even for $n = 2$. As a consequence, conic fibrations have no special role among surfaces from the point of view of osculation for $k = 2$. However, we include this case in our discussion not only for sake of completeness but also for the library of examples they offer to illustrate concretely the various situations arising in our study.

In some instances (maximal dimension of the generic osculating space and appropriate codimension of the inflectional locus) we determine the cohomology class of $\Phi_2(X)$ by means of Porteous' formula (Theorem 16). In particular for a conic bundle in $\mathbb{P}^6$ with finitely many flexes we obtain an explicit formula for their number (Proposition 23), which gives rise to some speculation for the smallest number of flexes occurring in this context.

Special attention is devoted to the case of rational quadric fibrations, i.e., $C = \mathbb{P}^1$, since in this case the vector bundle $\mathcal{V}$ giving rise to P is decomposable, which allows us to make explicit all the integers $(a_0, \ldots, a_n)$ and b determining P, and X inside P. For instance, when $a_0 \geq 2$ we show that $\Phi_2(X) = S$ if $n \geq 3$ and the union of singular fibers if $n = 2$ (Corollary 13). Another relevant point fitting into this setting is the study of quadric fibrations $X \subset \mathbb{P}^{2n+1}$ (Proposition 17) and the related analysis of the projective embeddings of $\mathbb{Q}^{n-1} \times \mathbb{P}^1$ and the corresponding inflectional loci (Proposition 18 and Proposition19).

The paper is organized as follows. In Section 2 we relate the kth jet map of a smooth projective variety to that of any of its smooth divisors and we produce some Chern class computations. In Sections 3 and 4 we present quadric fibrations over curves in general and over $\mathbb{P}^1$ respectively, focusing the discussion on properties which are relevant for the sequel. Section 5 is devoted to the osculating spaces of quadric fibrations: we determine the appropriate upper bound for their dimensions for any k and we compute the cohomology class of the inflectional locus for $k = 2$. In Section 6 we consider quadric fibrations whose enveloping scroll has small codimension in the ambient projective space, producing also some explicit computations in the rational case. In Sections 7 and 8 we focus on the case of conic fibrations, i.e., $n = 2$, in projective spaces of dimension 6, 5 and 4, specializing our formulas and exhibiting several examples which illustrate a variety of interesting phenomena.

2. Background material

Let $M \subset \mathbb{P}^N = \mathbb{P}(V)$ be a non-degenerate smooth projective variety of dimension m, let $\mathcal{L}$ be the hyperplane bundle and identify V with a subspace of $H^0(M, \mathcal{L})$. Let $\mathcal{P}^k_M(\mathcal{L})$ be the kth principal parts bundle of $\mathcal{L}$ and let $j^M_k : V_M = V \otimes \mathcal{O}_M \to \mathcal{P}^k_M(\mathcal{L})$ be the sheaf homomorphism associating to every section of $\mathcal{L}$ in V its kth jet evaluated at x, for every $x \in M$. We simply write j_k instead of j^M_k when there is no ambiguity for the variety M we are dealing with. We recall that for every $x \in M$ and for every $k \geq 2$ the kth osculating space to M at x is defined as $\mathrm{Osc}^k_x(M) := \mathbb{P}(\mathrm{Im} j_{k,x})$.

Let s_k denote the maximum rank of $j_{k,x}$ on M. The kth inflectional locus of M is defined as follows:

$$\Phi_k(M) := \{x \in M \mid \mathrm{rk}(j_{k,x}) < s_k\}.$$

For instance, if $(M, \mathcal{L})$ is a scroll over a smooth curve, then $s_k \leq km + 1$, with equality in general (e.g., see [17]). We will evaluate s_k for quadric fibrations over curves in Section 5. Clearly, the dimension σ_k mentioned in the Introduction is simply $s_k - 1$.

Now let $X \subset M$ be a smooth hypersurface and let $\mathcal{L}_X$ be the restriction of $\mathcal{L}$ to X. Let Ω_X and Ω_M denote the locally free sheaves corresponding to the cotangent bundles of X and M respectively and let S^k denote the kth symmetric power operation. To relate the kth principal part bundles of $\mathcal{L}$ on M and of $\mathcal{L}_X$ on X, consider the exact sequence

$$0 \to \mathcal{O}_X(-X) \to \Omega_M|_X \to \Omega_X \to 0 \tag{1}$$

twisted by $\mathcal{L}$, the standard exact sequence on M

$$0 \to S^k\Omega_M \otimes \mathcal{L} \to \mathcal{P}^k_M(\mathcal{L}) \to \mathcal{P}^{k-1}_M(\mathcal{L}) \to 0, \tag{2}$$

restricted to X, and the exact sequence on X

$$0 \to S^k\Omega_X \otimes \mathcal{L}_X \to \mathcal{P}^k_X(\mathcal{L}_X) \to \mathcal{P}^{k-1}_X(\mathcal{L}_X) \to 0.$$

For $k = 1$, combining these three exact sequences we get a surjective map $\phi_1 : \mathcal{P}^1_M(\mathcal{L})|_X \to \mathcal{P}^1_X(\mathcal{L}_X)$ giving rise, thanks to the snake lemma, to the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}_X(-X) \otimes \mathcal{L}_X & \to & \mathcal{K}_1 & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_M|_X \otimes \mathcal{L}_X & \to & \mathcal{P}^1_M(\mathcal{L})|_X & \to & \mathcal{L}_X & \to & 0 \\
 & & \downarrow & & \phi_1 \downarrow & & \mathrm{id} \downarrow & & \\
0 & \to & \Omega_X \otimes \mathcal{L}_X & \to & \mathcal{P}^1_X(\mathcal{L}_X) & \to & \mathcal{L}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

from which we see that the kernel of ϕ_1 is the line bundle $\mathcal{K}_1 = \mathcal{O}_X(-X) \otimes \mathcal{L}_X$. Similarly, for $k = 2$ we have a surjective map ϕ_2 as in the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{K} \otimes \mathcal{L}_X & \to & \mathcal{K}_2 & \to & \mathcal{O}_X(-X) \otimes \mathcal{L}_X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S^2\Omega_M|_X \otimes \mathcal{L}_X & \to & \mathcal{P}^2_M(\mathcal{L})|_X & \to & \mathcal{P}^1_M(\mathcal{L})|_X & \to & 0 \\ & & \downarrow & & \phi_2 \downarrow & & \phi_1 \downarrow & & \\ 0 & \to & S^2\Omega_X \otimes \mathcal{L}_X & \to & \mathcal{P}^2_X(\mathcal{L}_X) & \to & \mathcal{P}^1_X(\mathcal{L}_X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & , \end{array}$$

where $\mathcal{K}$ is the kernel of the map $S^2\Omega_M|_X \to S^2\Omega_X$. This is a locally free sheaf of rank $\binom{m+1}{2} - \binom{m}{2} = m$. On the other hand, taking into account the exact sequence (1), we see that $\mathcal{K} \cong \Omega_M|_X \otimes \mathcal{O}_X(-X)$. Once $\mathcal{K}$ is known, we can describe $\mathcal{K}_2$ via the first exact row in the latter diagram. The obvious relation between $\mathcal{P}^k_M(\mathcal{L})$ and its restriction to X is provided by the exact sequence

$$0 \to \mathcal{P}^k_M(\mathcal{L}) \otimes \mathcal{O}_M(-X) \to \mathcal{P}^k_M(\mathcal{L}) \to \mathcal{P}^k_M(\mathcal{L})|_X \to 0.$$

In particular, $\left(\mathcal{P}^k_M(\mathcal{L})\right)_x \cong \left(\mathcal{P}^k_M(\mathcal{L})|_X\right)_x$ for every $x \in X$. Now, let $\langle X \rangle$ be the linear span of X in $\mathbb{P}^N$, and let W be the quotient of V defined by the inclusion $\langle X \rangle = \mathbb{P}(W) \subseteq \mathbb{P}(V)$. For $k \leq 2$, taking into account the isomorphism above, we can consider the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{j^M_{k,x}} & \left(\mathcal{P}^k_M(\mathcal{L})\right)_x \\ \rho \downarrow & & \downarrow (\phi_k)_x \\ W & \xrightarrow{j^X_{k,x}} & \left(\mathcal{P}^k_X(\mathcal{L}_X)\right)_x, \end{array}$$

where $\rho : V \to W$ is the obvious surjection. Set $j'_{k,x} = j^X_{k,x} \circ \rho$; clearly, $\mathrm{rk}(j'_{k,x}) = \mathrm{rk}(j^X_{k,x})$. So, letting $k = 2$, we have a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{id}} & V \\ j^M_{2,x} \downarrow & & \downarrow j'_{2,x} \\ \left(\mathcal{P}^2_M(\mathcal{L})\right)_x & \xrightarrow{(\phi_2)_x} & \left(\mathcal{P}^2_X(\mathcal{L}_X)\right)_x, \end{array} \tag{3}$$

where the lower horizontal homomorphism is surjective, being induced by the sheaf homomorphism ϕ_2 introduced in the second big diagram.

Lemma 1. *For every $x \in X$, we have*

$$\mathrm{rk}(j^M_{2,x}) - (m+1) \leq \mathrm{rk}(j^X_{2,x}) \leq \mathrm{rk}(j^M_{2,x}) - 1. \tag{4}$$

Proof. According to (2), we have

$$\left(\mathcal{P}^2_M(\mathcal{L})\right)_x \cong \left(S^2\Omega_M \otimes \mathcal{L}\right)_x \oplus \left(\mathcal{P}^1_M(\mathcal{L})\right)_x. \tag{5}$$

at every point $x \in M$, and the same holds for $\left(\mathcal{P}^2_X(\mathcal{L}_X)\right)_x$ at every point $x \in X$. Now fix $x \in X$. With the local coordinates around x chosen before, the matrix representing $j'_{1,x}$ is

$$M_1 = \begin{pmatrix} s \\ s_{u_1} \\ \cdot \\ \cdot \\ s_{u_{m-1}} \end{pmatrix},$$

s ranging over a basis of V. Similarly, the matrix representing $j^M_{1,x}$ is

$$\widetilde{M_1} = \begin{pmatrix} M_1 \\ s_v \end{pmatrix}.$$

Clearly,

$$\mathrm{rk}(j'_{1,x}) = \mathrm{rk}(M_1) = m$$

and

$$\mathrm{rk}(j^M_{1,x}) = \mathrm{rk}(\widetilde{M_1}) = m + 1. \tag{6}$$

Now, $j'_{2,x}$ is represented by the matrix

$$M_2 = \begin{pmatrix} M_1 \\ N \end{pmatrix},$$

where N is the matrix whose rows are the second derivatives $s_{u_i u_j}$. Note that N defines an element of $\left(S^2\Omega_X \otimes \mathcal{L}_X\right)_x$. Hence, due to the direct sum in the analog of (5) rewritten for $\left(\mathcal{P}^2_X(\mathcal{L}_X)\right)_x$, we have

$$\mathrm{rk}(M_2) = \mathrm{rk}(M_1) + \mathrm{rk}(N) = m + \mathrm{rk}(N). \tag{7}$$

Similarly, $j^M_{2,x}$ is represented by the matrix

$$\widetilde{M_2} = \begin{pmatrix} \widetilde{M_1} \\ \widetilde{N} \end{pmatrix} = \begin{pmatrix} M_1 \\ s_v \\ N \\ N' \end{pmatrix},$$

where N' is the matrix whose rows are the derivatives of s_v. Since $\widetilde{N}$ defines an element of $\left(S^2\Omega_M \otimes \mathcal{L}\right)_x$, due to the direct sum in (5) we have

$$\mathrm{rk}(\widetilde{M_2}) = \mathrm{rk}(\widetilde{M_1}) + \mathrm{rk}(\widetilde{N}) \geq \mathrm{rk}(\widetilde{M_1}) + \mathrm{rk}(N).$$

Therefore, recalling (6) and (7),

$$\mathrm{rk}(\widetilde{M_2}) \geq m + 1 + \mathrm{rk}(N) = \mathrm{rk}(M_2) + 1.$$

This shows that

$$\mathrm{rk}(j^X_{2,x}) = \mathrm{rk}(j'_{2,x}) \leq \mathrm{rk}(j^M_{2,x}) - 1.$$

So the right inequality in the statement is proved. Next, note that

$$\mathrm{rk}(j^M_{2,x}) - \mathrm{rk}(j^X_{2,x}) = \mathrm{rk}(j^M_{2,x}) - \mathrm{rk}(j'_{2,x}) \leq \dim(\mathrm{Ker}(\phi_2)_x),$$

due to the commutative diagram (3). But the kernel of $(\phi_2)_x$ is $\left(\mathcal{K}_2\right)_x$, as we see from the second big diagram, and, clearly, $\mathrm{rk}(\mathcal{K}_2) = \mathrm{rk}(\mathcal{K}) + 1 = m + 1$. This proves the left inequality. □

Example. Suppose that $(M, \mathcal{L})$ is a scroll over a smooth curve C, with projection $p : M \to C$. Then $\mathrm{rk}(j^M_{2,x}) \leq 2m + 1$ for every $x \in M$. Suppose that equality holds at the general point, and let $\Phi_2(M) = \{x \in M \mid \mathrm{rk}(j^M_{2,x}) \leq 2m\}$ be the inflectional locus of M.

(a) Let $X \subset M$ be any smooth hypersurface not contained in $\Phi_2(M)$. According to (4) we have that $m \leq \mathrm{rk}(j^X_{2,x}) \leq 2m$ at every point $x \in X \setminus \Phi_2(M)$. In Section 5 we will see that in general, if p makes X a quadric fibration then equality on the right holds at the general point.

(b) Let X be a fiber of p. Clearly $\mathrm{rk}(j^X_{2,x}) = m$ at every point $x \in X$ since $(X, \mathcal{L}_X) = (\mathbb{P}^{m-1}, \mathcal{O}_{\mathbb{P}^{m-1}}(1))$. This situation is covered by Lemma 1 and (4) holds with equality on the left at the general point of a general fiber.

(c) Consider a general element $X \in |\mathcal{L}|$, so that $(X, \mathcal{L}_X)$ itself is a scroll over the same base curve C as M. Then $\mathrm{rk}(j^X_{2,x}) \leq 2(m-1) + 1$ at every point $x \in X$. This situation too fits into Lemma 1, but now both inequalities in (4) are strict.

We conclude this section collecting some Chern class computations we need in the sequel.

Lemma 2. *Let $M \subset \mathbb{P}^N$ be any smooth projective variety of dimension m, let $\mathcal{L}$ be the hyperplane bundle and set $L := c_1(\mathcal{L})$.*

(i) $c_1(\mathcal{P}^2_M(\mathcal{L})) = (m+2)K_M + \binom{m+2}{2}L$.

(ii) *If $m = 2$, then* $c_2(\mathcal{P}^2_M(\mathcal{L})) = 5c_2(M) + 5K_M^2 + 20K_M L + 15L^2$.

(iii) *If $m = 3$, then* $c_3(\mathcal{P}^2_M(\mathcal{L})) = 7K_M^3 + 20K_M c_2(M) - 8c_3(M) + 72K_M^2 L + 48c_2(M)L + 180K_M L^2 + 120L^3$.

Here $c_i(M)$ stands for the i-th Chern class of the tangent bundle.

Proof. Standard computations, using (2) recursively. In particular, (ii) can be found in [14] and (iii) in [5]. □

3. Generalities on quadric fibrations over curves

Let $X \subset \mathbb{P}^N$ be a smooth complex projective variety of dimension n, and let $\mathcal{L} := \mathcal{O}_{\mathbb{P}^N}(1)|_X$. As in [6], we say that X or $(X, \mathcal{L})$ is a *quadric fibration* (a conic fibration if $n = 2$) over a curve if there exists a surjective morphism $\pi : X \to C$ onto a smooth curve C such that any general fiber F of π is a smooth quadric hypersurface $\mathbb{Q}^{n-1} \subset \mathbb{P}^n$ and $\mathcal{L}|_F = \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$. We point out that this definition is slightly more general than that frequently adopted in adjunction theory [3, p. 81]. Actually, in our context it is true that $K_X + (n-1)\mathcal{L} = \pi^*\mathcal{A}$ for some line bundle $\mathcal{A} \in \mathrm{Pic}(C)$, but $\mathcal{A}$ is not necessarily ample. For instance, the del Pezzo threefold $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1))$ is a quadric fibration over $\mathbb{P}^1$ via any of the projections onto the three factors in our sense, but not in the adjunction theoretic one.

So, let X be a quadric fibration over C. We know that every fiber of π is reduced for any $n \geq 2$, and irreducible if $n \geq 3$. Moreover, singular fibers, if any, are quadric cones with an isolated singular point and $\mathcal{L}$ induces the hyperplane bundle on each of them. Actually, X has Picard number ≥ 2; moreover, $(X, \mathcal{L})$ cannot be a scroll over a curve at the same time (except $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,1))$, for which, however, the assertion above is obviously true). Then $K_X + (n-1)\mathcal{L}$ is nef [7, (11.7)]. Thus the assertion follows from [7, (11.8.5), argument in (5-ii) at pp. 100–101]. In particular, for $n = 2$, any singular fiber has the form $e_1 + e_2$, where e_1, e_2 are two distinct (-1)-curves in X with $e_i \cdot \mathcal{L} = e_1 \cdot e_2 = 1$.

Note that our X can be embedded fiberwise into a projective bundle over C (see e.g. [6, Section 4]). Actually, $\mathcal{L}$ embeds every fiber $F_u = \pi^{-1}(u)$, $u \in C$, of π as a quadric hypersurface in $\mathbb{P}^n$, hence $h^0(\mathcal{L}_{F_u}) = n+1$. Therefore $\mathcal{V} := \pi_*\mathcal{L}$ is a vector bundle of rank $n+1$ on C. Moreover, $\mathcal{V}$ is globally generated. To see this, for every $u \in C$, consider the diagram

$$\begin{array}{ccc} H^0(X,\mathcal{L}) & \longrightarrow & H^0(F_u, \mathcal{L}_{F_u}) \\ \downarrow & & \downarrow \\ H^0(C,\mathcal{V}) & \longrightarrow & \mathcal{V}_u, \end{array}$$

where the vertical arrows are isomorphisms. Since $\mathcal{L}$ is very ample and embeds F_u as a quadric hypersurface $Q \subset \mathbb{P}^n$ (smooth or a cone with vertex a point), the restriction homomorphism $H^0(X,\mathcal{L}) \to H^0(F_u, \mathcal{L}_{F_u})$ is surjective and then so is also the homomorphism $H^0(C,\mathcal{V}) \to \mathcal{V}_u$.

Set $P := \mathbb{P}(\mathcal{V})$, let ξ be the tautological line bundle on P, and let $\widetilde{\pi} : P \to C$ be the projection. Then X embeds fiberwise into P, i.e., $\widetilde{\pi}|_X = \pi$; moreover, $\xi|_X = \mathcal{L}$, and X can be regarded as a divisor in the linear system $|2\xi + \widetilde{\pi}^*\mathcal{B}|$ for some line bundle $\mathcal{B}$ on C (here the additive notation is used for the tensor product of line bundles). Set $\widetilde{\mathcal{L}} := \xi$. Clearly $\widetilde{\mathcal{L}}$ is a spanned line bundle and the pair $(P, \widetilde{\mathcal{L}})$ is a scroll over C. We emphasize, however, that $\widetilde{\mathcal{L}}$ is not necessarily very ample, nor even ample (for an example see [20, Theorem 1.5, 7th case in the table]). A relevant point is the following.

Lemma 3. *The inclusion $X \subset P$ induces an isomorphism $H^0(P, \widetilde{\mathcal{L}}) \cong H^0(X, \mathcal{L})$.*

Proof. Consider the exact sequence

$$0 \to -\widetilde{\mathcal{L}} - \widetilde{\pi}^*\mathcal{B} = \widetilde{\mathcal{L}} - X \to \widetilde{\mathcal{L}} \to \mathcal{L} \to 0.$$

Put $\mathcal{F} := -\widetilde{\mathcal{L}} - \widetilde{\pi}^*\mathcal{B}$. We have $\widetilde{\pi}_*\mathcal{F} = \widetilde{\pi}_*(-\widetilde{\mathcal{L}}) \otimes (-\mathcal{B}) = 0$, since $\widetilde{\pi}_*(-\widetilde{\mathcal{L}}) = 0$. Since also $R^1\widetilde{\pi}_*\mathcal{F} = 0$ by Grauert's theorem (see e.g. [9, Ch. III, Corollary

12.9, p. 288]), we get $h^i(P, \mathcal{F}) = 0$ for $i = 0, 1$. Then the assertion follows from the long cohomology sequence. □

It follows from Lemma 3 that we can identify V with a vector subspace of both $H^0(X, \mathcal{L})$ and $H^0(P, \widetilde{\mathcal{L}})$. Let $\varphi : P \to \mathbb{P}^N$ be the map defined by V as a subspace of the latter. Note that the embedding of X in $\mathbb{P}^N$ is given by $\varphi|_X$ Clearly, φ is a morphism if $V = H^0(P, \widetilde{\mathcal{L}})$ since $\widetilde{\mathcal{L}}$ is spanned. In principle, however, a proper subspace $V \subset H^0(P, \widetilde{\mathcal{L}})$ need not span $\widetilde{\mathcal{L}}$, so that φ could be only a rational map. We show that this is not the case.

Proposition 4. *The map $\varphi : P \to \mathbb{P}(V)$ is a morphism. Moreover, φ restricted to every fiber of P is an embedding.*

Proof. Suppose that $V \subset H^0(P, \widetilde{\mathcal{L}})$ does not span $\widetilde{\mathcal{L}}$ at a point $p \in P$. Let $\widetilde{F_0} \cong \mathbb{P}^n$ be the fiber of P containing p and let $F_0 = Q$ ($\mathbb{Q}^{n-1}$ or a cone with vertex a point) be the fiber of X inside $\widetilde{F_0} = \mathbb{P}^n$. Consider the commutative diagram

$$\begin{array}{ccc} H^0(P, \widetilde{\mathcal{L}}) & \xrightarrow{\text{iso}} & H^0(X, \mathcal{L}) \\ \widetilde{\rho} \downarrow & & \downarrow \rho \\ H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \xrightarrow{r} & H^0(Q, \mathcal{O}_Q(1)), \end{array}$$

where the horizontal homomorphisms are defined by restriction to X and the vertical ones by restriction to the fibres $\widetilde{F_0}$ and F_0 respectively. By what we said, $s(p) = 0$ for every $s \in V \subset H^0(P, \widetilde{\mathcal{L}})$, so that $\widetilde{\rho}(V) \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is a strict inclusion. On the other hand, $\rho(V) = H^0(Q, \mathcal{O}_Q(1))$, because $\varphi|_X$ is an embedding. This is a contradiction, since r in the diagram above is an isomorphism. Thus V spans $\widetilde{\mathcal{L}}$. Moreover, the above diagram shows that $\widetilde{\rho}(V) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ for every fiber $\widetilde{F}$ of P, hence $\varphi|_{\widetilde{F}}$ is an embedding. □

We set

$$R := \varphi(P) = \bigcup_{u \in C} \langle F_u \rangle,$$

so that R is the $(n+1)$-dimensional algebraic subvariety of $\mathbb{P}^N$ swept out by the linear spans $\langle F_u \rangle$ of all fibers F_u ($u \in C$) of X. We refer to R as the *enveloping ruled variety* (or *scroll*, when R is so) of our quadric fibration $X \subset \mathbb{P}^N$. Note that X being non-degenerate is equivalent to R having the same property.

Proposition 5. *R is a scroll (over C) if and only if φ is an embedding.*

Proof. If φ is an embedding, then R is obviously a scroll. To prove the converse, assume $R \subset \mathbb{P}^N$ is a scroll over C, with $\pi' : R \to C$, $\pi : P \to C$, and $\pi = \pi' \circ \varphi$. Since R is a scroll, $R = \mathbb{P}(\pi'_* \mathcal{O}_R(1))$. Recall that $\mathcal{V} := \pi_* \mathcal{O}_P(1)$, and set $\mathcal{U} =: \pi'_* \mathcal{O}_R(1)$. The map $\mathcal{O}_R(1) \to \varphi_* \varphi^* \mathcal{O}_R(1)$, equal to the adjoint of the identity $\varphi^* \mathcal{O}_R(1) = \mathcal{O}_P(1) \to \mathcal{O}_P(1)$, gives a natural map $\alpha \colon \mathcal{U} \to \mathcal{V}$

of locally free sheaves on C, of the same rank. Since φ is an embedding for each fiber of C, it follows that α gives isomorphisms $\mathcal{U}\otimes k(x) \to \mathcal{V}\otimes k(x)$ for each point $x \in C$. But then (by Nakayama), $\alpha\colon \mathcal{U} \to \mathcal{V}$ is an isomorphism. The morphism $P \to \mathbb{P}^N = \mathbb{P}(V)$ is obtained from $V_C \to \mathcal{V} = \pi_*\mathcal{O}_P(1)$ via $P = \mathbb{P}(\mathcal{V}) \to C \times \mathbb{P}(V) \to \mathbb{P}(V)$, and similarly $R = \mathbb{P}(\mathcal{U}) \to \mathbb{P}^N$ is obtained by $R = \mathbb{P}(\mathcal{U}) \to C \times \mathbb{P}(V) \to \mathbb{P}(V)$. Since $P \to \mathbb{P}^N$ factors via $R \to \mathbb{P}^N$, $V_C \to \mathcal{V}$ factors via the isomorphism $\alpha\colon \mathcal{U} \to \mathcal{V}$. This implies that P is isomorphic to R, so φ is an embedding. □

Suppose that φ is an embedding and consider the following diagram

$$\begin{array}{ccc} P & \overset{\varphi}{\to} & R = \varphi(P) \\ \cup & & \cap \\ X & \hookrightarrow & \mathbb{P}^N. \end{array}$$

A consequence of Lemma 3 is that X is linearly normally embedded in $\mathbb{P}^N$ if and only if so is its enveloping scroll R.

Let $X \subset \mathbb{P}^N$ be a quadric fibration over C, and consider $R = \varphi(P)$. If $C = \mathbb{P}^1$, then $|\widetilde{\mathcal{L}}|$ maps P to either a scroll or a cone, as it will be clear from Section 4. If, in addition, X is linearly normal, then $V = H^0(P, \widetilde{\mathcal{L}})$ by Lemma 3, hence R itself is necessarily a scroll or a cone. However, this is no longer true in general if either

a) $C \neq \mathbb{P}^1$, even if X is linearly normal, or
b) $C = \mathbb{P}^1$, but X is not linearly normal.

As to a), R is neither a scroll nor a cone for the elliptic conic bundle in $\mathbb{P}^5$ discussed before Proposition 22, as we will see in Section 7. On the other hand, R is a cone for the elliptic conic fibration in $\mathbb{P}^4$ mentioned in Section 8. As to b), consider that by projecting a scroll into a lower dimensional space one can produce singularities, e. g., a double locus. Set $\Delta = \mathrm{Sing}(R)$. Clearly, $\Delta \cap X \subseteq \mathrm{Sing}(X) = \emptyset$, since X is smooth. Therefore X does not meet $\varphi^{-1}(\Delta)$. If $\dim(\Delta) > 0$, this means that X, as a divisor inside P, cannot be ample. For an example see Section 7 (Togliatti's example).

In particular we have

Corollary 6. *If R is a developable or a singular ruled variety (but not a cone), then X is either irrational or not linearly normal.*

To conclude this section, we compute some numerical characters of our quadric fibration X in terms of $a := \deg \mathcal{V}$, $b := \deg \mathcal{B}$ and the genus $q := g(C)$ of the base curve C.

Proposition 7. *Let μ, $g = g(X, \mathcal{L})$ and $d = d(X, \mathcal{L})$ be the number of singular fibers of $\pi : X \to C$, the sectional genus and the degree of $X \subset \mathbb{P}^N$, respectively. Then*

(1) $\mu = 2a + (n+1)b$;
(2) $g = a + b + 2q - 1$;
(3) $d = 2a + b$.

Proof. According to [6, (3.3)], the singular fibers of π correspond to the zeroes of a section of the line bundle $2\det\mathcal{V} + (n+1)\mathcal{B}$ on C. Hence their number is given by (1). To prove (2), recall that $K_P = -(n+1)\widetilde{\mathcal{L}} + \widetilde{\pi}^*(K_C + \det\mathcal{V})$, by the canonical bundle formula for projective bundles. As $X \in |2\widetilde{\mathcal{L}} + \widetilde{\pi}^*\mathcal{B}|$, by adjunction we get $K_X + (n-1)\mathcal{L} = \pi^*\mathcal{G}$, where $\mathcal{G}$ is a line bundle on C of degree $\deg\mathcal{G} = 2q-2+a+b$. Hence the genus formula shows that $g = \deg\mathcal{G} + 1$, which gives (2). Finally, taking into account the Chern–Wu relation, we have $d = \mathcal{L}^n = \widetilde{\mathcal{L}}^n \cdot X = \widetilde{\mathcal{L}}^n \cdot (2\widetilde{\mathcal{L}} + \widetilde{\pi}^*\mathcal{B}) = 2\widetilde{\mathcal{L}}^{n+1} + \widetilde{\pi}^*\mathcal{B}\cdot\widetilde{\mathcal{L}}^n = 2a + b$. □

Note that when $\mu = 0$, $\pi : X \to C$ is in fact a fiber bundle, hence we refer to X as a *quadric bundle*. In particular, if X is a quadric bundle, it follows from Proposition 7 (1), that $b = -\frac{2a}{n+1}$.

4. Rational quadric fibrations

Here we collect some material on rational quadric fibrations, which will be useful in Section 6.

Quadric fibrations over $\mathbb{P}^1$ are discussed in [20, Sect. 1 in general, and Sect. 4 for $n = 3$]. First of all, as we said, $\mathcal{V} = \pi_*\mathcal{L}$ is a spanned vector bundle of rank $n+1$ on $\mathbb{P}^1$, hence we can write

$$\mathcal{V} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{where} \quad 0 \le a_0 \le a_1 \le \cdots \le a_n.$$

Then $a = \sum_{i=0}^n a_i$. Moreover, $\mathcal{B} = \mathcal{O}_{\mathbb{P}^1}(b)$. Let $\widetilde{F}$ be a fiber of $\widetilde{\pi} : P \to \mathbb{P}^1$ and let $z_0, \dots, z_n$ be the homogeneous coordinates on $\widetilde{F}$ corresponding to the summands of $\mathcal{V}$. As X is the zero locus of a section $s \in H^0(P, 2\widetilde{\mathcal{L}} + b\widetilde{F})$ we can describe X fiberwise by an equation

$$(8) \qquad \begin{pmatrix} z_0 & \dots & z_n \end{pmatrix} A \begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix} = 0,$$

where $A = [\alpha_{ij}(u)]$ is a symmetric $(n+1)\times(n+1)$ matrix whose entries depend on $u \in \mathbb{P}^1$. The map sending s to the matrix A is just the composite isomorphism

$$H^0(P, 2\widetilde{\mathcal{L}} + b\widetilde{F}) \cong H^0(\mathbb{P}^1, S^2\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(b)) \cong \bigoplus_{i \le j} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i + a_j + b)),$$

where S^2 stands for the second symmetric power, from which we see that $\alpha_{ij} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i + a_j + b))$. In particular,

$$\det A \in H^0(\mathbb{P}^1, 2\det\mathcal{V} + (n+1)\mathcal{O}_{\mathbb{P}^1}(b)),$$

so that the number of singular fibers of X is $\mu = 2a + (n+1)b$, according to Proposition 7 (1). Notice that for $n = 3$ [20, Theorem 4.2] gives explicit conditions for the existence of a smooth X in the linear system $|2\widetilde{\mathcal{L}} + b\widetilde{F}|$ in terms of the multidegree (a_0, a_1, a_2, a_3) of $\mathcal{V}$ and b.

Now let C_i be the section of $P \to C$ corresponding to the surjection $\mathcal{V} \to \mathcal{O}_{\mathbb{P}^1}(a_i)$. Note that $\widetilde{\mathcal{L}} \cdot C_i = a_i$. Thus the map $\varphi : P \to \mathbb{P}^N$ defined by V as a vector subspace of $H^0(P, \widetilde{\mathcal{L}})$ sends C_i to a rational curve of degree a_i in $\mathbb{P}^N$ (to a point if $a_i = 0$). Computing the intersection (in P), we have

$$X \cdot C_i = (2\widetilde{\mathcal{L}} + b\widetilde{F}) \cdot C_i = 2a_i + b. \tag{9}$$

Hence, if $2a_i + b < 0$ for some i, then $X \supset C_i$. In terms of the matrix A, this means that $\alpha_{ii}(u) = 0$ for all $u \in \mathbb{P}^1$, since $h^0(\mathcal{O}_{\mathbb{P}^1}(2a_i + b)) = 0$. In other words, X contains the point $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ (with 1 in ith position) on every fiber $\widetilde{F}$. On the other hand, if $X \not\supset C_i$, then the above relation says that X meets the section C_i at a finite set of points: in particular, if $X \not\supset C_i$, then

$$X \cap C_i = \emptyset \ \text{ if and only if } \ 2a_i + b = 0. \tag{10}$$

So we have

Remark. Suppose that $a_i = 0$. Then $\varphi(C_i)$ is a point. This prevents C_i from being contained in X, since $\varphi|_X$ is an embedding. Therefore X meets C_i at b points (in particular, $b \geq 0$).

In fact the possibility for some a_i to be zero is somehow restricted, as shown by the following proposition.

Proposition 8. *Let $a_0 = 0$. Then*

1) *$a_2 \geq 1$ and $b = 0$ or 1;*
2) *if, in addition, $a_1 = 0$, then X meets both C_0 and C_1 at a single point. In particular, $b = 1$.*

Proof. If $a_2 = 0$ then $a_0 = a_1 = a_2 = 0$. Let Σ be the subscroll ($\mathbb{P}^2$-subbundle) of P generated by C_0, C_1, and C_2, consider the conic fibration $X \cap \Sigma$ and let γ_u denote its fiber inside $\widetilde{F_u} := \widetilde{\pi}^{-1}(u)$, where $u \in \mathbb{P}^1$. Clearly φ maps each fiber of Σ isomorphically to a fixed plane $\Lambda \subset \mathbb{P}^N$, but it cannot map every fiber γ_u to the same conic inside Λ, since $\varphi|_X$ is an embedding. Then φ maps two distinct fibers γ_u, $\gamma_{u'}$ $(u \neq u')$ to two distinct conics inside Λ. As these conics intersect, we argue that there is a point on γ_u and a point on $\gamma_{u'}$ having the same image via φ. But this is impossible since $\varphi|_X$ is an embedding. This proves the first assertion in 1). The second follows from [20, Lemma 1.1]. Now suppose that $a_0 = a_1 = 0$. In every fiber $\widetilde{F_u}$ consider the line ℓ_u joining the points where the fiber meets C_0 and C_1. Let $\Sigma = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ be the subscroll ($\mathbb{P}^1$-subbundle) of P generated by C_0 and C_1, whose fibers are the lines ℓ_u. Note that φ maps every line ℓ_u isomorphically to a fixed line $\lambda \subset \mathbb{P}^N$. Clearly $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$, and by considering the two projections $\pi|_\Sigma$, $\varphi|_\Sigma$ of Σ, we can regard C_0 and $\ell = \ell_u$ as the generators of $\mathrm{Pic}(\Sigma)$. Consider $D := X \cap \Sigma$, which is a divisor on Σ. Thus $D = \alpha C_0 + \beta\ell$, up to linear equivalence. We have

$$\alpha = (\alpha C_0 + \beta\ell) \cdot \ell = D \cdot \ell = X \cdot \ell = 2,$$

since ℓ_u and $X \cap \widetilde{F_u}$ are a line and a quadric hypersurface respectively inside $\widetilde{F_u} = \mathbb{P}^n$. On the other hand, C_0 and C_1 cannot be contained in X, by the Remark above. Moreover, $X \cap C_0$ can consist of one point at most, since $\varphi|_X$ is an embedding. Therefore

$$\beta = (2C_0 + \beta\ell) \cdot C_0 = D \cdot C_0 = 0 \text{ or } 1.$$

In conclusion, either: i) $D = 2C_0$, or ii) $D = 2C_0 + \ell$, up to linear equivalence. In case i) D consists of two sections in the same linear equivalence class as C_0. But then φ would map each of them to a single point of λ, which is impossible since $\varphi|_X$ is an embedding. In case ii) D is a section of the projection $\varphi|_\Sigma : \Sigma \to \lambda = \mathbb{P}^1$, whose fiber is C_0. Thus

$$1 = D \cdot C_0 = X \cdot C_0 = (2\widetilde{\mathcal{L}} + b\widetilde{F}) \cdot C_0 = 2a_0 + b = b.$$

The same is true for C_1, since $a_1 = 0$ too. This gives 2). □

Finally, we point out that in the rational case the positivity properties of X as a divisor inside P can be easily expressed.

Proposition 9. *Consider the line bundle $\mathcal{M} := 2\widetilde{\mathcal{L}} + b\widetilde{F} \in \mathrm{Pic}(P)$. Then*

(1) *$\mathcal{M}$ is nef if and only if $2a_0 + b \geq 0$.*
(2) *$\mathcal{M}$ is ample if and only if it is very ample, if and only if $2a_0 + b > 0$.*

Proof. Set $\mathcal{W} = \mathcal{V}(-a_0)$. Then $\mathcal{W}$ is normalized as in [3, p. 74]. Moreover, $P = \mathbb{P}(\mathcal{W})$, with tautological line bundle $\zeta := \widetilde{\mathcal{L}} - a_0\widetilde{F}$. Therefore $\mathcal{M} = 2\zeta + (2a_0 + b)\widetilde{F}$. Then the assertion follows from [3, Lemma 3.2.4]. □

5. Second osculating spaces and flexes

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a quadric fibration over a smooth curve C with $\dim(X) = n$ as in Section 3, and consider P and $\widetilde{\mathcal{L}}$. If φ is an embedding (in particular $\widetilde{\mathcal{L}}$ is very ample) then the situation fits into the general setting considered in Section 2, with $(M, L) = (P, \widetilde{\mathcal{L}})$. Moreover, things are easier than there, because $j'_{2,x} = j^X_{2,x}$ in the present situation, thanks to Lemma 3. In particular, according to (4), at every point $x \in X$ such that $\mathrm{rk}(j^P_{2,x}) = 2n + 3$ (i.e., $x \notin \Phi_2(R)$; recall that $R = \varphi(P)$) we have

$$n + 1 \leq \mathrm{rk}(j^X_{2,x}) \leq 2n + 2.$$

In a moment we will see that in many cases equality on the right occurs at the general point of X. Anyhow, an obvious consequence of Lemma 1 is the following

Corollary 10. *Suppose that φ is an embedding, so that we can consider $\Phi_2(R)$. Then $\Phi_2(R) \cap X \subseteq \Phi_2(X)$. In particular, if R is hypo-osculating along X (i.e. $\mathrm{rk} j^R_{2,x} < 2n+3$ at every point $x \in X$), then X is hypo-osculating.*

In fact we have examples of strict inclusion, e.g. see the remark after Theorem 12. Moreover, we will see that all singular points of singular fibers of X belong to $\Phi_2(X)$, even though the approach used in Section 2 does not make evident their special role. So, here we study the dimensions of the second (and higher) osculating spaces by a different approach, relying on the size of the linear subsystem of osculating hyperplanes to X at a given point. This will make evident the role of singular points of singular fibres. Another advantage of this approach is that it does not require φ to be an embedding.

From now on in this section we put $j_k = j^X_k$, since there is no ambiguity. We know that X has only finitely many singular fibers, and each of them is a quadric cone with vertex a point. Let S be the set of these vertices. The following lemma is inspired by [13]. For every hypersurface Z of X, every point $y \in X$ and every integer $r \geq 0$ we denote by $|V - Z - ry|$ the linear system defined by the vector subspace of sections $s \in V$ vanishing along Z and having a point of multiplicity $\geq r$ at y. We use the same notation with $\mathcal{L}$ in place of V whenever $V = H^0(X, \mathcal{L})$.

Lemma 11. *Let $x \in X$ and let F be the fiber containing x.*

(i) *If $n \geq 3$ then F is a fixed component of $|V - 3x|$, the moving part being either $|V - F - 2x|$ or $|V - F - x|$ according to whether $x \notin S$ or $x \in S$ respectively.*

(ii) *If $n = 2$ then the same conclusion holds provided that x is not a smooth point of a singular fiber.*

(iii) *Let $n = 2$, let $F = e_1 + e_2$ be a singular fiber, where e_1 and e_2 are two lines, and let $x \in e_i \setminus e_j$. Then e_i is a fixed component of $|V - 3x|$, the moving part being $|V - e_i - 2x|$.*

Proof. First suppose that $x \in F_{\mathrm{sm}}$ is a smooth point of F and let $D \in |V - 3x|$. In cases (i) and (ii) we can take an irreducible conic γ in F passing through x. If γ were not contained in D, then we would get

$$2 = \mathcal{L} \cdot \gamma = D \cdot \gamma \geq \mathrm{mult}_x(D) \geq 3,$$

a contradiction. Therefore $\gamma \subset D$. This occurs for all γ and since the closure of the locus swept out by the various γ's is the whole fiber F we have that $F \subset D$. Moreover, this happening for any $D \in |V - 3x|$, we conclude that F is a fixed component of $|V - 3x|$. Write $D = F + E$, where the residual part E is a hypersurface. The fact that F is smooth at x implies that E must have a double point at x. This shows that the moving part of $|V - 3x|$ is $|V - F - 2x|$. Now suppose that F is singular, i.e., a quadric cone, and let x be its vertex. We can repeat the argument with a line $\ell \in F$ through x in place of a conic. Since $\mathcal{L}_F$ is the hyperplane bundle, if ℓ is not contained in $D \in |V - 3x|$ we have

$$1 = \mathcal{L} \cdot \ell = D \cdot \ell \geq \mathrm{mult}_x(D) \geq 3,$$

a contradiction. Arguing as before we thus conclude that F is a fixed component of $|V - 3x|$. Write $D = F + E$ as before. Since F has a double point at x, the only condition E has to satisfy it that of passing through x. Therefore the moving part of $|V - 3x|$ is $|V - F - x|$. This proves (i) and (ii). The same argument applied to the line e_i proves (iii). □

Lemma 11 shows that $\dim(|V - 3x|)$ is equal to either $\dim(|V - F - 2x|)$ or $\dim(|V - F - x|)$ according to whether $x \notin S$ or $x \in S$ respectively in cases (i) and (ii), while $\dim(|V - 3x|) = \dim(|V - e_i - 2x|)$ in case (iii). This allows us to compute (or estimate) the dimension of $|V - 3x|$. Thus, from the obvious equality

$$N = \dim(|V - 3x|) + \dim(\mathrm{Osc}^2_x(X)) + 1,$$

noting that $\dim(|V - F|) = N - (n+1)$ since $\langle F \rangle = \mathbb{P}^n$, and $\dim(|V - e_i|) = 2$, we get the following results.

a1) If $x \notin S$, and x is not a smooth point of a singular fiber in case $n = 2$, then $\mathrm{rk}(j_{2,x}) = n + h + 1$, where h is the number of linearly independent linear conditions to be imposed to the elements in $|V - F|$ in order to have a double point at x; in particular, $n + 1 \leq \mathrm{rk}(j_{2,x}) \leq 2n + 2$ (since $0 \leq h \leq n + 1$). Moreover, let

$$W_F := \{s \in V | s = 0 \text{ on } F\}.$$

Clearly we can regard W_F as a vector subspace of $H^0(X, \mathcal{L} - F)$. Suppose that

$$\mathcal{L} - F \text{ is ample and spanned by } W_F.$$

Then the right inequality above is strict if and only if $x \in \mathcal{J}_1(W_F)$, the first jumping set of W_F (see [19]). In fact the higher jumping sets allow us to say more, see below.

a2) If $n = 2$ and x is a smooth point of a singular fiber then $\mathrm{rk}(j_{2,x}) = 2 + h$, where h is the number of linearly independent linear conditions to be imposed to the elements in $|V - e_i|$ in order to have a double point at x; in particular, $3 \leq \mathrm{rk}(j_{2,x}) \leq 5$ (since $j_{1,x}$ has rank 3 everywhere and $h \leq 3$). An additional comment as in a1) can be repeated referring to the vector subspace W_{e_i}.

b) If $x \in S$, then $n + 1 \leq \mathrm{rk}(j_{2,x}) \leq n + 2$, with equality on the left if and only if x is a base point of $|W_F|$.

As a consequence, we obtain

Theorem 12. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a quadric fibration with* $\dim(X) = n$ *over a smooth curve, and let $\mathcal{L}$, S and W_F for every fiber F be as before.*

(1) *We have* $n+1 \leq \mathrm{rk}(j_{2,x}) \leq 2n+2$ *for every* $x \in X$*, and* $\mathrm{rk}(j_{2,x}) \leq n+2$ *for every* $x \in S$*.*
(2) *If* $n = 2$*, then* $\mathrm{rk}(j_{2,x}) \leq 5$ *at any smooth point of any singular fiber; in particular,* $\Phi_2(X)$ *contains all singular fibers.*
(3) *If* W_F *spans* $\mathcal{L} - F$ *for every singular fiber* F*, then* $\mathrm{rk}(j_{2,x}) = n+2$ *for all* $x \in S$*.*
(4) *If* $\mathcal{L}-F$ *is ample and spanned by* W_F *for every fiber* F*, then* $\mathrm{rk}(j_{2,x}) = 2n+2$ *for all* $x \in (X \setminus S) \setminus \bigcup_F \mathcal{J}_1(W_F)$ *if* $n \geq 3$ *(for all* x *not lying on singular fibers and* $\bigcup_F \mathcal{J}_1(W_F)$ *if* $n = 2$*). In particular, if in addition* $|W_F|$ *defines an immersion for any fiber* F*, then* $\mathrm{rk}(j_{2,x}) = 2n+2$ *at all points* $x \notin S$*, if* $n \geq 3$ *(all points not lying on singular fibers if* $n = 2$*), and* $\Phi_2(X) = S$ *(with* $\mathrm{rk}(j_{2,x}) = n+2$ *for all* $x \in S$*) if* $n \geq 3$*.*

In fact, looking at a single fiber F_u of X we can say even more. Assume for simplicity that $n \geq 3$. If $\mathcal{L} - F_u$ is ample and spanned by W_{F_u}, then

$$\Phi_2(X) \cap F_u = \big(\mathcal{J}_1(W_{F_u}) \cap F_u\big) \cup \mathrm{Sing}(F_u).$$

Moreover, for any smooth point x of F_u lying on $\Phi_2(X)$ we have that $\mathrm{rk}(j_{2,x}) = 2n+2-r$ if $x \in \mathcal{J}_r(W_{F_u})$.

Remark. Suppose that $\mathcal{L}-F_u$ is ample and spanned by W_{F_u} for every $u \in C$. Recall that the first jumping set $\mathcal{J}_1(W_{F_u})$ is just the ramification divisor of the morphism defined by W_{F_u}. If $h^0(\mathcal{L} - F_u) = n+1$ for every $u \in C$, then we have a finite morphism $\psi_u : X \to \mathbb{P}^n$ and the ramification formula says that $K_X = \psi_u^* K_{\mathbb{P}^n} + R_u$, where R_u is the ramification divisor. Therefore

$$R_u = K_X + (n+1)(\mathcal{L} - F_u).$$

In particular, as $(K_X)_{F_u} = K_{F_u} = -(n-1)\mathcal{L}_{F_u}$ since $(X, \mathcal{L})$ is a quadric fibration, we have that $(R_u)_{F_u} \in |2\mathcal{L}_{F_u}|$. Moreover, since all fibers are numerically equivalent, we see that the numerical equivalence class of R_u is the same for all $u \in C$: call it $\mathcal{R}$. Thus both $\mathcal{R}$ and the numerical equivalence class of the closure of $\Phi_2(X) \setminus S$, when restricted to any fiber, are the same. So, under the assumptions above, $\overline{\Phi_2(X) \setminus S}$ cuts every fiber F_u along a quadric section. Note however, that even if $S = \emptyset$, this does not at all imply that the cohomology class of $\Phi_2(X)$ is $\mathcal{R}$. For an explicit example see the discussion after Proposition 22.

The case when $C = \mathbb{P}^1$ is of particular interest because then all fibers are linearly equivalent.

Let $C = \mathbb{P}^1$, let $(a_0, \ldots, a_n)$ be the multidegree of $\mathcal{V} = \pi_* \mathcal{L}$, with $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, and let $V = H^0(X, \mathcal{L})$ (i. e., assume that X is linearly normally embedded). Consider the vector bundle

$$\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i - 1),$$

whose tautological line bundle on P is $\widetilde{\mathcal{L}} - \widetilde{F}$. Clearly, $\widetilde{\mathcal{L}} - \widetilde{F}$ is spanned for $a_0 \geq 1$, and very ample for $a_0 \geq 2$. In particular,

i) if $a_0 \geq 1$ then W_F spans $\mathcal{L} - F$ for every fiber F;
ii) if $a_0 \geq 2$, then $\mathcal{L} - F$ is very ample for every fiber F.

So, in the rational case we have the following.

Corollary 13. *Let X be a linearly normally embedded rational quadric fibration and suppose that $a_0 \geq 2$. Then*

(1) *$\Phi_2(X) = S$ and $\mathrm{rk}(j_{2,x}) = n + 2$ at every point $x \in S$ if $n \geq 3$;*
(2) *$\Phi_2(X)$ is the union of singular fibers if $n = 2$.*

In particular, X is uninflected if and only if X has no singular fibers.

Note that if $a_0 \geq 2$, then $\Phi_2(R)$ is empty [16, Corollary 2.3]. For instance, let $(a_0, \dots, a_n) = (2, \dots, 2)$. Here the emptiness of $\Phi_2(R)$ also follows from the fact that $\varphi(P) = R \subset \mathbb{P}^{3n+2}$ is a balanced rational normal scroll, fitting in the range considered in [17, Corollary 2]. Since $\widetilde{\mathcal{L}}$ is very ample, so is $2\widetilde{\mathcal{L}}$, and then any general element $X \in |2\widetilde{\mathcal{L}}|$ is a smooth hypersurface. By restricting the projection $P \to \mathbb{P}^1$ to X we thus get a quadric fibration. Since $a = 2(n+1)$ and $b = 0$, there are $\mu = 4n + 4$ singular fibers, and then Corollary 13 says that $\Phi_2(X) = S$ consists of $4n+4$ points if $n \geq 3$. So here we have the strict inclusion $\emptyset = \Phi_2(P) \cap X \subset \Phi_2(X) = S$ (compare with Corollary 10).

Notice that conditions i) and ii) before Corollary 13 are only sufficient. Actually the spannedness or very ampleness of a line bundle on X does not imply that the same property holds for the line bundle on P inducing it.

The geometric approach developed before for $k = 2$ can be extended to higher values, leading to the determination of an upper bound for the generic rank of j_k. As a byproduct, it will also exhibit an unexpected role played by the parity of k. As before, let S be the set of singular points of the singular fibers of X. Then Lemma 11 can be generalized as follows.

Lemma 14. *Let F be the fiber containing x.*

(i) *Let $x \in X \setminus S$. Then the linear system $|V - (k+1)x|$ has*
  (i-a) *$(k-1)F$ as fixed component, the moving part being $|V - (k-1)F - 2x|$, if either $n \geq 3$ or $n = 2$ and F is a non-singular fiber.*
  (i-b) *te_i as fixed component, the moving part being $|V - te_i - (k+1-t)x|$, if $n = 2$, $F = e_1 + e_2$ and $x \in e_i \setminus e_j$, where $t = \frac{k}{2}$ or $\frac{k+1}{2}$ according to whether k is even or odd.*

(ii) *Let $x \in S$ and set $k' = [k/2]$, where $[\ ]$ stands for the least integer function. Then $|V-(k+1)x|$ has $k'F$ as fixed component, the moving part being either $|V - k'F - x|$ or $|V - k'F - 2x|$ according to whether k is even or odd.*

Proof. The proof proceeds by induction (on k) from Lemma 11. □

This allows us to estimate the rank of $j_{k,x}$ at every point $x \in X$. For example, for $x \in X \setminus S$ if $n \geq 3$ and for x outside the singular fibers if $n = 2$, we have that $\dim(|V - (k+1)x|) = \dim(|V - (k-1)F - 2x|) = \dim(|V - (k-1)F|) - h$, where h is the number of linearly independent linear conditions needed to impose on elements of $|V - (k-1)F|$ in order to have a double point at x. Clearly, $h \leq n + 1$. Note that for any integer s we have $\dim(|V - sF|) \geq \dim(|V|) - s(n+1)$ since $\langle F \rangle = \mathbb{P}^n$ (with equality for $s = 1$, because $|V|$ is very ample). This gives

$$\dim(|V - (k+1)x|) \geq N - (k-1)(n+1) - h.$$

Therefore, from the obvious equality

$$N = \dim(|V - (k+1)x|) + \dim(\mathrm{Osc}^k_x(X)) + 1,$$

we get the bound

$$\mathrm{rk}(j_{k,x}) \leq (k-1)(n+1) + h.$$

This shows that $\mathrm{rk}(j_{k,x}) \leq k(n+1)$ for every $x \in X \setminus S$ if $n \geq 3$ and for every x outside the singular fibers if $n = 2$. In any case, this inequality holds at the general point, hence at any point of X. Similarly, for $x \in S$ we get $\mathrm{rk}(j_{k,x}) \leq k'(n+1) + 1$ or $\leq (k'+1)(n+1)$ according to whether k is even or odd. In particular we obtain

Corollary 15. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a quadric fibration of dimension n over a smooth curve, and let s_k denote the maximal rank of $j_{k,x}$ on X. Then*

$$s_k \leq k(n+1).$$

Remark. Let $M \subset \mathbb{P}^N$ be any smooth surface and let $\mathcal{L}$ be its hyperplane bundle. Then $\mathrm{rk}\mathcal{P}^2_M(\mathcal{L}) = 6$. So, the upper bound for s_2 provided by Theorem 12 when $n = 2$ is the same holding for any smooth surface. It turns out that conic fibrations over curves do not play any special role among surfaces from the point of view of osculation if we confine to $k = 2$. This is not the case, however, for $k \geq 3$. Actually, for $k = 3$, Corollary 15 shows that $s_3 \leq 9$, while $\mathrm{rk}\mathcal{P}^3_M(\mathcal{L}) = 10$.

We shall now come back to the case $k = 2$. Under suitable assumptions we determine the cohomology class of the inflectional locus.

Theorem 16. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a quadric fibration of dimension $n \geq 2$ over a smooth curve, for which the generic rank of $j_{2,x}$ is $2n+2$ (i.e., the maximum). Suppose that $\Phi_2(X)$ is empty or of the expected codimension $(N-2n)\binom{n}{2}$. Then the cohomology class of $\Phi_2(X)$ is given by*

$$[\Phi_2(X)] = \det[c_{\binom{n}{2}-i+j}(\mathcal{P}^2_X(\mathcal{L}))], \qquad 1 \leq i, j \leq N - 2n.$$

Proof. Consider the vector bundle map

$$j_2 : V_X \to \mathcal{P}^2_X(\mathcal{L}).$$

Note that V_X and $\mathcal{P}^2_X(\mathcal{L})$ have ranks $\dim(V) = N+1$ and $\binom{n+2}{2}$ respectively. Hence, due to the assumption on the rank of $j_{2,x}$, the expected codimension of the degeneracy locus $D_{2n+1}(j_2)$, which is equal to $\Phi_2(X)$, is

$$\ell := \big(N+1-(2n+1)\big)\left(\binom{n+2}{2} - (2n+1)\right) = (N-2n)\binom{n}{2}.$$

Thus, if $\Phi_2(X)$ has codimension ℓ, then Porteous formula [8, Thm. 14.4, p. 254] says that its class is given by

$$\det[c_{\binom{n+2}{n}-(2n+1)-i+j}(\mathcal{P}^2_X(\mathcal{L}) \otimes V_X^\vee)], \qquad 1 \leq i, j \leq N+1-(2n+1).$$

This is the same expression as in the statement, since V_X is the trivial bundle. □

Remark. The condition that the generic rank of $j_{2,x}$ is maximal implies that $\mathbb{P}^{2n+1} = \mathrm{Osc}^2_x(X) \subseteq \mathbb{P}^N$, hence

$$N \geq 2n+1. \tag{11}$$

On the other hand, the fact that $\ell \leq n$ is equivalent to $N - 2n \leq \frac{2}{n-1}$, which in order to be compatible with (11) requires that

$$n \leq 3. \tag{12}$$

Therefore, conditions (11) and (12) fix the range of validity of the formula in Theorem 16. It works for $(n, N, \ell) = (2,5,1), (2,6,2)$ and $(3,7,3)$. In these cases, the explicit expression of $[\Phi_2(X)]$ is given by c_1, $c_1^2 - c_2$ and c_3 respectively, where $c_i = c_i(\mathcal{P}^2_X(\mathcal{L}))$, and hence can be computed using Lemma 2.

6. Quadric fibrations and enveloping scrolls of low codimension

Let $X \subset \mathbb{P}^N$ be a quadric fibration over a smooth curve C, and consider again $\mathcal{V}$, $P := \mathbb{P}(\mathcal{V})$, the morphism $\varphi : P \to \mathbb{P}^N$ defined by V, regarded as a subspace of $H^0(P, \widetilde{\mathcal{L}})$, and the enveloping ruled variety $R := \varphi(P)$. As noted in Section 3, $R \subset \mathbb{P}^N$ is an $(n+1)$-dimensional variety swept out by a 1-dimensional family of $\mathbb{P}^n$s parameterized by C. Clearly, X can be embedded in a $\mathbb{P}^N$ for any $N \geq 2n+1$, and, usually, one says that X has *small codimension* to mean that $\mathrm{codim}_{\mathbb{P}^N}(X) \leq \dim(X)$, i.e., $N \leq 2n$.

Remark. If X has small codimension, then φ is not an embedding.

Otherwise, according to Proposition 5, R would be a scroll of dimension $(n+1)$ over C contained in $\mathbb{P}^{2(n+1)-2} = \mathbb{P}^{2n}$, but this is impossible.

Therefore, the smallest value of N for which φ can be an embedding, is $2n+1$. So, let $N = 2n+1$ and suppose that φ is an embedding. Then, by Proposition 5, R is a scroll over C. But the only $(n+1)$-dimensional scroll over a curve in $\mathbb{P}^{2n+1}$ is the Segre product $\mathbb{P}^n \times \mathbb{P}^1$. Then $C = \mathbb{P}^1$, and $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n+1)}$. In particular, $V = H^0(P, \widetilde{\mathcal{L}})$, $a = \deg \mathcal{V} = n+1$ and $\widetilde{\mathcal{L}} = \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(1,1)$; then $X \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(2, 2+b)|$ for some integer b.

Furthermore, since R is $\mathbb{P}^n \times \mathbb{P}^1 \subset \mathbb{P}^{2n+1}$ Segre-embedded by φ, we have $h^0(\mathcal{L}) = h^0(\widetilde{\mathcal{L}}) = 2n+2$, so that X itself is linearly normally embedded in $\mathbb{P}^{2n+1}$ (this also follows from a general fact pointed out in Section 3). Finally, the number of singular fibers of X is $\mu = (n+1)(2+b)$, hence $b \geq -2$, and equality occurs if and only if X is a quadric bundle. Moreover, in this special case, from the fact that $X \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(2,0)|$ we argue that X itself is the product of a smooth $(n-1)$-dimensional quadric with $\mathbb{P}^1$. Recall that $P = \mathbb{P}^n \times \mathbb{P}^1 \subset \mathbb{P}^{2n+1}$ is perfectly hypo-osculating, because $\mathrm{rk}(j^P_{2,x}) = 2n+2 < 2\dim(P)+1$ at every point. Thus X is hypo-osculating by Corollary 10. If $\mu > 0$, then $\Phi_2(X) \supseteq S \neq \emptyset$, so that X cannot be perfectly hypo-osculating. On the other hand, if $\mu = 0$, i.e. X is $\mathbb{Q}^{n-1} \times \mathbb{P}^1$ embedded in $\mathbb{P}^{2n+1}$ via the Segre embedding, a direct check shows that $j_{2,x}$ has rank $2n+1$ at every point (e. g., see the comment concerning the case $\alpha = 1$ before Proposition 19; see also [23, Theorem 0.5] for $n = 2$). In conclusion, this gives the following result.

Proposition 17. *Let $X \subset \mathbb{P}^{2n+1}$ be a quadric fibration, with $\dim(X) = n$, hyperplane bundle $\mathcal{L}$, and consider P and φ. If φ is an embedding, then $P = \mathbb{P}^n \times \mathbb{P}^1$, φ is the Segre embedding, $X \in |\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(2, \frac{\mu}{n+1})|$, where μ is the number of singular fibers of X, and X is linearly normally embedded. In particular, $(n+1)|\mu$; $\mu = 0$ if and only if $X = \mathbb{Q}^{n-1} \times \mathbb{P}^1$; moreover, X is hypo-osculating, and it is perfectly hypo-osculating if and only if $\mu = 0$.*

A corollary of Proposition 17 is that if $X \subset \mathbb{P}^{2n+1}$ is a quadric fibration not linearly normally embedded, then φ is not an embedding (or, equivalently, R is not a scroll by Proposition 5). The example of the Togliatti–del Pezzo surface in Section 7 fits exactly in this situation.

As for the Segre product $\mathbb{Q}^{n-1} \times \mathbb{P}^1$, we note that there is essentially only one way to embed it in a projective space as a quadric bundle.

Proposition 18. *Let $X \subset \mathbb{P}^N$ be a quadric fibration. If $X \cong \mathbb{Q}^{n-1} \times \mathbb{P}^1$, then $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(\alpha)^{\oplus(n+1)}$ with $\alpha > 0$.*

Proof. Clearly $P = \mathbb{P}(\mathcal{V})$ with $\mathcal{V}$ as in Section 4. The assumption on X implies that $\mathcal{L} = \mathcal{O}_X(1,\alpha)$ for some integer $\alpha > 0$. Moreover, $K_X = \mathcal{O}_X(1-n,-2)$. On the other hand, by adjunction,

$$K_X = (K_P + X)_X = (1-n)\mathcal{L} + (a+b-2)F = \mathcal{O}_X(1-n, \alpha(1-n)+a+b-2),$$

and by comparing these two expressions we get

$$\alpha(1-n) + a + b = 0.$$

Since X has no singular fibers we know that $b = -\frac{2a}{n+1}$. This, combined with the above relation, gives

$$(a,b) = \big((n+1)\alpha, -2\alpha\big). \tag{13}$$

Keeping the notation as in Section 4, let C_i be the section of P corresponding to the summand $\mathcal{O}_{\mathbb{P}^1}(a_i)$ of $\mathcal{V}$. If $C_i \subset X$ we have

$$a_i = \deg(C_i) = C_i \cdot \widetilde{\mathcal{L}} = C_i \cdot \mathcal{O}_X(1, \alpha) \geq \alpha.$$

On the other hand, if $C_i \not\subset X$, then (9) and (13) show that

$$0 \leq C_i \cdot X = 2a_i + b = 2(a_i - \alpha),$$

hence $a_i \geq \alpha$ also in this case. Hence

$$a_i \geq \alpha \quad \text{for every } i = 0, \dots, n.$$

But $\sum_{i=0}^{n} a_i = a = (n+1)\alpha$ by (13), and therefore $a_i = \alpha$ for every i. □

It turns out that for $X = \mathbb{Q}^{n-1} \times \mathbb{P}^1$, any inclusion $X \subset \mathbb{P}^N$ as a quadric bundle, not necessarily linearly normal, factors through the inclusion $X \subset \mathbb{P}^n \times \Gamma$, where $\Gamma \subset \mathbb{P}^r$ is a smooth rational curve of some degree $\alpha (\geq r)$ and the Segre embedding $\mathbb{P}^n \times \mathbb{P}^r \subset \mathbb{P}^N$. Clearly, $r = \frac{N-n}{n+1}$.

We can explicitly describe $\Phi_2(X)$ for such X, at least when $\Gamma \subset \mathbb{P}^r$ can be described by homogeneous coordinates

$$(1 : u : \cdots : u^k : \cdots : u^{\alpha-1} : u^\alpha) \tag{14}$$

for k taking $r-3$ values in the range $1 < k < \alpha - 1$. We insist on this case, since the role these X play for quadric fibrations is analog to that of balanced rational scrolls in the study of the osculatory behavior of scrolls. First of all, we point out the following fact:

$$u \in \Phi_2(\Gamma) \text{ if and only if } u^2 \text{ does not appear in (14).} \tag{15}$$

Next, note that X, due to its special feature, is fiberwise defined in P by a homogeneous equation of degree 2 not depending on the fibers. In terms of the matrix appearing in (8), this says that $A = [\alpha_{ij}]$ is a constant matrix (with respect to $u \in \mathbb{P}^1$), and we can assume that A has the form:

$$A = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & I & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix},$$

I standing for the identity matrix of order $(n-1)$. In other words, X is described in terms of the homogeneous coordinates $z_0, \dots, z_n$ in the fiber $\widetilde{F_u}$ for any $u \in C$, by the following equation:

$$z_0 z_n = \sum_{i=1}^{n-1} z_i^2.$$

Every section C_i of P $(i = 0, \dots, n)$ is a copy of Γ mapped by φ to a smooth rational curve of degree α in a $\mathbb{P}_i^r \subset \mathbb{P}^N$, and we can regard $R = \varphi(P)$ as the decomposable scroll, in the sense of [16, p. 1550], generated by them. Clearly, $C_0 \subset X$, while $C_i \cap X = \emptyset$ for $i = 1, \dots, n$, due to the equation of X. Set

$$c_i(u) = C_i \cap \widetilde{F_u}$$

for every $u \in \mathbb{P}^1$, consider the affine coordinates $v_i = z_i/z_0$ on $\widetilde{F_u}$ and use $(v_1, \dots, v_n, u)$ as local coordinates on P. Identifying V with a subspace of $H^0(P, \widetilde{\mathcal{L}})$, every $s \in V$ can be written locally as

$$s = c_0(u) + \sum_{i=1}^{n} v_i c_i(u).$$

Thus, regarding V with a subspace of $H^0(X, \mathcal{L})$ and using $(v_1, \dots, v_{n-1}, u)$ as local coordinates on X, the same section s can be written as

$$s = c_0(u) + \sum_{i=1}^{n-1} v_i c_i(u) + \psi c_n(u),$$

where $\psi = \psi(v_1, \dots, v_{n-1}) = \sum_{i=1}^{n-1} v_i^2$. Describing $R \subset \mathbb{P}^N$ as a decomposable scroll we can assume that $c_i(u)$ has homogeneous coordinates given by (14) in a linear subspace $\mathbb{P}^r \subset \mathbb{P}^N$; moreover condition (15) can be rephrased for every C_i (embedded by φ). Thus points $p \in X$ are described in $\mathbb{P}^N$ by the following homogeneous coordinates

$$(1 : u : .. : u^k : .. : u^\alpha : v_1 : v_1 u : .. : v_1 u^k : .. : v_1 u^\alpha : .. : \psi : \psi u : .. : \psi u^\alpha),$$

in terms of their local coordinates $(u, v_1, \dots, v_{n-1})$ on X. Hence the matrix representing the homomorphism $j_{2,p} = j_{2,p}^X$ at p is

$$M_2(p) = \begin{pmatrix} s \\ s_{v_1} \\ \dots \\ s_{v_{n-1}} \\ s_u \\ \dots \\ s_{v_i v_j} \\ \dots \\ s_{u v_1} \\ \dots \\ s_{u v_{n-1}} \\ s_{uu} \end{pmatrix} (p) \quad ,$$

its first row s corresponding to the $(N+1)$-tuple displayed above. Consider the submatrix T of $M_2(p)$, of type $\binom{n+2}{2} \times (2n+2)$, whose first row is

$$(1, u, v_1, v_1 u, \dots, v_{n-1}, v_{n-1} u, \psi, \psi u),$$

i. e., the $2n+2$ columns of T are the derivatives of those components of s whose degree with respect to u is ≤ 1. It is immediate to check that T has rank $(1+n) + 1 + (n-1) = 2n+1$. The first summand counts the first $n+1$ independent rows; the second corresponds to the block of rows $s_{v_i v_j}$: note that $\psi_{v_i v_j} = \delta_{ij}$, hence all such rows are zero except those of type $s_{v_i v_i}$, which are all proportional to each other; finally, the last summand counts the independent rows corresponding to the mixed derivatives with respect to u and a v_i. Note that the last row of T, corresponding to s_{uu}, is zero.

As $\mathrm{rk}(j_{2,p}) \leq 2n+2$ by Theorem 12, the inflectional locus $\Phi_2(X)$ is described by the $(2n+2)\times(2n+2)$-minors obtained by augmenting a submatrix of rank $2n+1$ of T with a new column of $M_2(p)$ and the corresponding piece of the row s_{uu}, the only further row which is not completely zero a priori unless $\alpha = 1$. In particular, for $\alpha = 1$ we see that X is perfectly hypo-osculating, the rank of $j_{2,p}$ being computed at every point by that of T, which is $2n+1$.

If $\alpha \geq 2$ the terms on the row s_{uu} which are not a priori zero are monomials of type $2k(k-1)u^{k-2}$, $2k(k-1)v_i u^{k-2}$ for $i = 1, \ldots, n-1$ and $2k(k-1)\psi u^{k-2}$. So, taking into account (15) referring to C_0 embedded by φ, we see that if $c_0(0) \notin \Phi_2(C_0)$ then $\mathrm{rk}(M_2(p)) = \mathrm{rk}(T) + 1 = 2n+2$, at every point p, so that $\Phi_2(X) = \emptyset$, while if $c_0(0) \in \Phi_2(C_0)$, then $\Phi_2(X)$ contains the fiber F_0.

The previous discussion can be summarized as follows.

Proposition 19. *Let $\Gamma \subset \mathbb{P}^r$ be a non-degenerate smooth rational curve of degree α described by (14) and let $X \subset \mathbb{P}^N$ ($N = nr + n + r$) be the image of $\mathbb{Q}^{n-1} \times \Gamma$ in the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^r \subset \mathbb{P}^N$.*

(1) *If $\alpha = 1$, then X is perfectly hypo-osculating.*
(2) *If $\alpha \geq 2$, then $\Phi_2(X) = \bigcup_{u_j \in \Phi_2(\Gamma)} F_{u_j}$. In particular, if Γ is linearly normal, then $\Phi_2(\Gamma) = \emptyset$, hence X is uninflected.*

In accordance with [16, Theorem 2.2] we can note that $\Phi_2(X) = \Phi_2(R) \cap X$ in both cases.

Now consider again a quadric fibration over a smooth curve C of genus q, $X \subset \mathbb{P}^N$, with $\dim(X) = n \geq 2$. Suppose that X is linearly normal and that $\mathcal{V}$ is very ample. Hence $R = \varphi(P)$ is the enveloping scroll of X and it is linearly normally embedded in the same projective space. Assume furthermore that $N = 2n+2$. Then a conjecture of Ionescu and Toma [12, p. 642] (which is in fact a theorem for $n \leq 3$) says that $q \leq 1$.

First suppose that $q = 1$. Then, according to [17, final remark], $\mathcal{V} = \mathcal{F}_{n+1}(p_1 + p_2)$, where $\mathcal{F}_{i+1}$ is defined inductively by a non-split exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{F}_{i+1} \to \mathcal{F}_i \to 0,$$

starting from $\mathcal{F}_1 = \mathcal{O}_C(p)$ for some point $p \in C$, and $p_1, p_2 \in C$. In this case $a = \deg \mathcal{V} = \deg \mathcal{F}_{n+1} + 2\ \mathrm{rk}(\mathcal{F}_{n+1}) = 2n+3$. Note that this is also the degree of R. The numerical characters of X, according to Proposition 7, are

$$d = 4n + 6 + b, \quad g = 2n + 4 + b, \quad \mu = (n+1)(b+4) + 2.$$

As to b we can observe the following. The line bundle $2\widetilde{\mathcal{L}} + \widetilde{\pi}^*\mathcal{B}$ must be effective since its linear system has to contain X. Hence $h^0(2\widetilde{\mathcal{L}} + \widetilde{\pi}^*\mathcal{B}) = h^0(S^2\mathcal{V} \otimes \mathcal{B}) > 0$, which implies that $\deg(S^2\mathcal{V} \otimes \mathcal{B}) > 0$. We have $\mathrm{rk}(S^2\mathcal{V}) = \binom{n+2}{2}$ and $\deg(S^2\mathcal{V}) = (n+2) \deg \mathcal{V} = (n+2)(2n+3)$, hence $\deg(S^2\mathcal{V} \otimes \mathcal{B}) =$

$\binom{n+2}{2}(4+\frac{2}{n+1}+b)$. Therefore, the positivity condition gives $b \geq -4$. In particular, this shows that X cannot be a quadric bundle.

Next suppose $q = 0$. Then our assumptions say that $a_0 \geq 1$, $h^0(\mathcal{V}) = N+1 = 2n+3$. Since $\mathcal{V}$ has rank $n+1$, by the Riemann–Roch theorem this implies $a = \deg \mathcal{V} = n+2$. Therefore $(a_0, \dots, a_n) = (1, \dots, 1, 2)$. In this case R has degree $n+2$. According to Proposition 7, the numerical characters of X are

$$d = 2(n+2) + b, \quad g = n + b + 1, \quad \mu = (n+1)(b+2) + 2.$$

Since we are in the rational case, normalizing the vector bundle in the form $\mathcal{V}(-1)$ and applying [3, Lemma 3.2.4, p. 74] we see that $2\widetilde{\mathcal{L}} + b\widetilde{F}$ is very ample provided that $b \geq -1$. Note that this condition is stronger than the effectivity we used for $q = 1$. In fact, for any such b our R does really contain a quadric fibration $X \in |2\widetilde{\mathcal{L}} + b\widetilde{F}|$, which is linearly normally embedded in $\mathbb{P}^{2n+2}$ by $\Gamma(\widetilde{\mathcal{L}}_X)$, with the above invariants. Even in this case we note that X cannot be a quadric bundle.

Relying on the conjecture of Ionescu and Toma, the two situations described before are the only possibilities for $n \leq 3$, and at least conjecturally for $n \geq 4$, occurring for a quadric fibration X linearly normally embedded in $\mathbb{P}^{2n+2}$ with $\mathcal{V}$ very ample.

As to the inflectional locus of R, in the rational case we know that $\Phi_2(R)$ is the decomposable subscroll of R generated by the lines $\varphi(C_0), \dots, \varphi(C_{n-1})$. In the elliptic case, the class of $\Phi_2(R)$ can be described provided that appropriate assumptions formulated in [17] are satisfied. In both cases, according to Corollary 10, X inherits an inflectional locus $\Phi_2(X)$ which cuts a hyperplane section on every fiber.

7. CONIC FIBRATIONS IN $\mathbb{P}^5$ AND $\mathbb{P}^6$

Let $n = 2$. Any conic fibration embedded in a projective space can be mapped isomorphically to a conic fibration $X \subset \mathbb{P}^5$ via a generic projection. Clearly, however, this projection may enlarge the inflectional locus or even produce hypo-osculation (see the Togliatti example below). In the following, we shall use the notation introduced in Section 3.

Proposition 20. *Let $X \subset \mathbb{P}^5$ be a conic fibration over a smooth curve C. If* $\mathrm{rk}(j_{2,x}) = 6$ *at the general point $x \in X$, then the cohomology class of $\Phi_2(X)$ is that of*

$$2\mathcal{L} + 4\pi^*(K_C + \det \mathcal{V} + \mathcal{B}).$$

In particular, if X is not hypo-osculating, then $\Phi_2(X)$ cuts out a 0-cycle of degree 4 on every fiber of X.

Proof. According to Theorem 16, $[\Phi_2(X)] = c_1(\mathcal{P}^2_X(\mathcal{L}))$, since $\ell = 1$. Thus the class of $\Phi_2(X)$ is that of $4K_X + 6\mathcal{L}$ by Lemma 2 (i) (see also [23, Theorem 0.3]). Now, the canonical bundle formula for projective bundles gives $K_P =$

$-3\widetilde{\mathcal{L}} + \widetilde{\pi}^*(K_C + \det \mathcal{V})$, hence, recalling that $X \in |2\widetilde{\mathcal{L}} + \widetilde{\pi}^*\mathcal{B}|$, by adjunction we get

$$K_X = (K_P + X)_X = -\mathcal{L} + \pi^*(K_C + \det \mathcal{V} + \mathcal{B}).$$

This proves the assertion. □

Note that there are conic fibrations $X \subset \mathbb{P}^5$ not satisfying the assumption of Proposition 20. As a consequence they may have finitely many points where $\mathrm{rk}(j_{2,x})$ is strictly less than the maximum. In particular, it could happen that $\Phi_2(X)$ is finite. Here is an example.

Example (Togliatti's example). Let $Y \subset \mathbb{P}^6$ be the del Pezzo surface of degree 6, and let $\mathcal{L} = -K_Y$. According to a well-known result of Togliatti [25] (see also [15, Corollary 4.2]) there exists a 6-dimensional vector subspace V of $H^0(Y, \mathcal{L})$ such that: (a) Y can be projected isomorphically into $\mathbb{P}(V)$, and (b) for its image $X \subset \mathbb{P}^5$ we have $\mathrm{rk}(j_{2,x}) < 6$ for every $x \in X$. Abstractly, X ($= Y$, by (a)) is the blow-up of $\mathbb{P}^2$ at three not collinear points p_0, p_1, p_2 and $\mathcal{L} = -K_X = \eta^*\mathcal{O}_{\mathbb{P}^2}(3) - e_0 - e_1 - e_2$, where $\eta : X \to \mathbb{P}^2$ is the blowing up and e_i is the exceptional curve $\eta^{-1}(p_i)$ $(i = 0, 1, 2)$. As a consequence, our $X \subset \mathbb{P}^5$ is a conic fibration over $\mathbb{P}^1$ in three different ways, each being defined by the pencil of lines of $\mathbb{P}^2$ passing through one of the points p_i. Each of these fibrations has exactly two singular fibers: for instance, if $\pi : X \to \mathbb{P}^1$ is given by the pencil through p_0, then the two singular fibers of π are $\widetilde{\ell}_i + e_i$ $(i = 1, 2)$, where $\widetilde{\ell}_i$ is the proper transform of the line $\ell_i \subset \mathbb{P}^2$ joining p_0 and p_i. According to [15, Proposition 4.3] the rank of $j_{2,x}$ is 5 except at the six points of the finite set $T = \cup_{i=0}^2 S_i$, where S_i is the set of singular points of the singular fibers in the conic fibration defined by the pencil of lines through p_i. Hence $\Phi_2(X) = T$. A natural question is what are P and R in this case. First of all, note that R cannot be a scroll. Otherwise, lying in $\mathbb{P}^5$ it would be the Segre product $\mathbb{P}^2 \times \mathbb{P}^1$, with $X \in |\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2, \alpha)|$ for some integer α. Then $K_X = (K_{\mathbb{P}^2 \times \mathbb{P}^1} + X)_X = (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, \alpha - 2))_X$, hence

$$6 = K_X^2 = (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, \alpha - 2))^2 \cdot (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2, \alpha)) = -3\alpha + 8,$$

a contradiction. Since $\mathcal{V}$ is spanned, we can write $\mathcal{V} = \oplus_{i=0}^2 \mathcal{O}_{\mathbb{P}^1}(a_i)$, with $0 \le a_0 \le a_1 \le a_2$. Then $P = \mathbb{P}(\mathcal{V})$ and $X \in |2\widetilde{\mathcal{L}} + b\widetilde{F}|$. Note that $h^0(\widetilde{\mathcal{L}}) = h^0(\mathcal{L}) = 7$ by Lemma 3. Hence $a = 4$, and so $2 = \mu = 2a + 3b = 8 + 3b$, since π has two singular fibers. Therefore $b = -2$. This in turn implies $a_0 > 0$ by Proposition 8. Then $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. In particular, $2a_0 + b = 0$. According to Proposition 9, this shows that X is not an ample divisor inside P. Note that P embedded in $\mathbb{P}^6$ by $|\widetilde{\mathcal{L}}|$ is the scroll containing the original surface Y. Let $c \in \mathbb{P}^6$ be the center of the projection $\mathbb{P}^6 \dashrightarrow \mathbb{P}(V) = \mathbb{P}^5$ giving rise to X. Clearly $c \notin P$: otherwise, projecting Y from c, the conic F such that $c \in \widetilde{F}$ would map 2 to 1 to a line, contradicting the fact that the projection maps Y to X isomorphically.

As to R, note that the general hyperplane section of P, projected from c gives rise to a quartic surface of $\mathbb{P}^4$ having a single double point. Since these surfaces are hyperplane sections of R we argue that R is a ruled variety of degree 4 with a line of singular points: R is a quartic 3-fold of the second kind according to [26, Theorem 6,(ii)] (see also [24, §3]).

The discussion in Section 6 includes rational conic fibrations in $\mathbb{P}^5$ such that φ is an embedding. Now, let $C = \mathbb{P}^1$ again, but suppose that φ is not an embedding. If X is linearly normal, then R is a cone, as pointed out in Section 3. Moreover, $a = 3$ because $6 = h^0(\mathcal{L}) = h^0(\mathcal{V}) = 3 + a$. Let us focus on this case. There are "a priori" two possibilities, namely

(1) $(0, 0, 3)$;
(2) $(0, 1, 2)$.

In case (1) R is the cone with vertex a line, say ℓ, over a smooth twisted cubic Γ. Let Π be a general hyperplane containing ℓ, and let q_i, $i = 1, 2, 3$, denote the points where Π intersects Γ. Then cutting X with Π, we see that $\Pi \cap X$ consists of ℓ itself (note that $\ell \subset X$, because, being contained in every plane of R, it intersects every fiber F of X) plus the three fibers F_i of X contained in the planes $\langle l, q_i\rangle$, for $i = 1, 2, 3$. This says that X has degree 7, hence Proposition 7 implies $b = 1$, $g = 3$, and $\mu = 9$. For an explicit description of X see [10, Theorem 4.1, ii), case $k = 9$]. We claim that $\mathrm{rk}(j_{2,x}) = 6$ at the general point $x \in X$. Equivalently we show that for a general fiber F of X, $|\mathcal{L} - 3x| = \emptyset$ for a general $x \in F$. According to Lemma 11 any element of $|\mathcal{L} - 3x|$ has the form $F + E$, where E is an element of $|\mathcal{L} - F|$ having a double point at x. We have $h^0(\mathcal{L} - F) = h^0(\mathcal{V}(-1)) = h^0\big(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)\big) = 3$. Hence $|\mathcal{L} - F|$ is a net, and therefore one cannot impose conditions on an element of $|\mathcal{L} - F|$ to have a double point at a general point $x \in F$ unless F itself is a fixed component of $|\mathcal{L} - F|$. But this is impossible. Otherwise $|\mathcal{L} - 2F|$ would be a net, while $h^0(\mathcal{L} - 2F) = h^0(\mathcal{V}(-2)) = 2$. Thus Proposition 20 applies and we conclude that $\Phi_2(X) = 2\mathcal{L} + 8F$.

Lemma 21. *Case* (2) *does not occur. In other words, the surface corresponding to* (1) *is the only rational conic fibration, linearly normally embedded in* $\mathbb{P}^5$*, with* $\mathcal{V}$ *not very ample.*

Proof. In case (2) R is generated by a point, say v, a line ℓ and a conic γ. It can be regarded as the cone of vertex v over the rational normal cubic scroll Σ in the hyperplane $\Pi := \langle \ell, \gamma\rangle$ generated by ℓ and γ. Note that ℓ is the minimal section of Σ, hence $\ell^2 = -1$ on Σ. Call f the fibers of Σ. Note that the hyperplane Π cuts X along a curve G, which must lie on Σ, and, moreover, is a bisecant, i.e., $G \in |2\ell + \beta f|$ for some nonnegative integer β. This is due to the fact that every fiber f of Σ is a line inside the plane $\langle v, f\rangle$ of R, which contains a conic F. Now, since G is a hyperplane section of X, the degree of X is $d = G^2 = (2\ell + \beta f)^2 = 4(\beta - 1)$, a positive multiple of 4. Taking into account the expression of d provided by Proposition 7 (recall that here $q = 0$, $a = 3$), we argue that $b \geq 2$. But then Proposition 8, 1) prevents the linear system $|2\widetilde{\mathcal{L}} + b\widetilde{F}|$ from containing a smooth element as X has to be, a contradiction. □

Note that in case (2) the invariants of X would be $(d, g, \mu) = (8, 4, 12)$. We want to emphasize that a rational conic fibration $X \subset \mathbb{P}^5$ with these invariants really does exist (e.g., see [11, (6.2)]). However, the corresponding projective bundle $(P, \widetilde{\mathcal{L}})$ is the Segre product $\mathbb{P}^2 \times \mathbb{P}^1$ in which X fits as an element of $|\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(2, 4)|$ (see Proposition 17).

When C is an irrational curve, φ is not an embedding by Proposition 17. Here is an example.

Example (an elliptic conic bundle in $\mathbb{P}^5$). Let C be a smooth curve of genus 1 and let $\mathcal{U}$ be the rank-2 vector bundle on C arising as a non split extension

$$0 \to \mathcal{O}_C \to \mathcal{U} \to \mathcal{O}_C(z) \to 0, \tag{16}$$

for some $z \in C$. Recall that $\mathcal{U}$ is ample. Consider the $\mathbb{P}^1$-bundle over C, $X := \mathbb{P}(\mathcal{U})$, with projection π, denote by σ the tautological section and set $F_y = \pi^{-1}(y)$ for any $y \in C$. The line bundle $\mathcal{L} := 2\sigma + F_y$ is very ample and $|\mathcal{L}|$ embeds X in $\mathbb{P}^5$ as a conic bundle of degree 8 (such $(X, \mathcal{L})$ is a rather known conic bundle, e. g., it is mentioned in [3, Thm 11.4.2, case 3, p. 288]). Now consider the rank 3 vector bundle $\mathcal{V} := \pi_*\mathcal{L}$. Note that $\mathcal{V} = S^2\mathcal{U} \otimes \mathcal{O}_C(y)$, due to the expression for $\mathcal{L}$. In particular $\mathcal{V}$ is ample and its degree is 6. Set $P := \mathbb{P}(\mathcal{V})$, let $\widetilde{\pi} : P \to C$ be the projection, and let $\widetilde{\mathcal{L}}$ be the tautological line bundle on P. According to our general setting, X embeds fiberwise in P as a divisor. In fact, recalling that X has no singular fibers, we have that $X \in |2\widetilde{\mathcal{L}} - \widetilde{\pi}^*\mathcal{D}|$, where $\mathcal{D} \in \mathrm{Pic}(C)$ is a line bundle of degree 4. On the other hand, $\mathcal{V}$ is not very ample. This follows from the previous general discussion: in particular, it is equivalent to the assertion that φ is not an embedding, or to the fact that $R := \varphi(P)$ is not a scroll, since in the present case $V = H^0(X, \mathcal{L})$. Moreover, R is not even a cone, since the ampleness of $\mathcal{V}$ prevents φ from contracting anything. In conclusion, R is neither a scroll, nor a cone. As to the osculating spaces and the inflectional locus of X we can complete the description as follows.

Proposition 22. *Let $(X, \mathcal{L})$ be as in the Example. Then the generic rank of $j_{2,x}$ is 6; moreover, the class of the inflectional locus $\Phi_2(X)$ is $2\mathcal{L} + 8F$ (or, equivalently, $4\sigma + 10F$).*

Proof. Let F_w be any fiber of X. We claim that $|\mathcal{L} - 3x| = \emptyset$ for a general $x \in F_w$. This is equivalent to saying that $\mathrm{Osc}^2_p(X) = \mathbb{P}^5$ at a general point $p \in X$. Since X has no singular fibers, according to Lemma 11 any element of $|\mathcal{L} - 3x|$ has the form $F_w + E$, where E is an element of $|\mathcal{L} - F_w|$ having a double point at x. Note that $S^2\mathcal{U} \otimes \mathcal{M}$ is an ample vector bundle of degree 3 for any line bundle $\mathcal{M} \in \mathrm{Pic}(C)$ of degree zero. Hence

$$h^0(\mathcal{L} - F_w) = h^0(\mathcal{V} \otimes \mathcal{O}_C(-w)) = h^0(S^2\mathcal{U} \otimes \mathcal{O}_C(y - w)) = 3.$$

Thus $|\mathcal{L} - F_w|$ is a net and then we cannot impose conditions on an element of $|\mathcal{L} - F_w|$ to have a double point at a general point $x \in F_w$ unless F_w is a fixed component of $|\mathcal{L} - F_w|$. In this case, however, we would get $|\mathcal{L} - F_w| = F_w + |T|$, where $|T|$ is a net of divisors linearly equivalent to $2\sigma + F_y - 2F_w$. Note that the linear series $|2w|$ on C contains a divisor of the form $y + y'$ for some $y' \in C$. Then T is linearly equivalent to $2\sigma - F_{y'}$. Now consider the exact sequence

$$0 \to \sigma - F_{y'} \to T \to T|_\sigma \to 0. \tag{17}$$

Twisting (16) by $\mathcal{O}_C(-y')$ we see that $h^0(\sigma - F_{y'}) = h^0(\mathcal{U} \otimes \mathcal{O}_C(-y')) \leq 1$. On the other hand, $h^0(T|_\sigma) \leq 1$ since $T|_\sigma$ is a line bundle of degree 1. Then the cohomology sequence of (17) shows that $h^0(T) \leq 2$, contradicting the fact that $|T|$ is a net. This proves the first assertion in the statement, which allows us to apply Proposition 20 and get the second assertion. □

Remark. In connection with the remark after Theorem 12 we point out that here $\mathcal{L} - F_w$ is ample and spanned by $W_{F_w} = H^0(X, \mathcal{L} - F_w)$ for every $w \in C$. The ampleness comes from the fact that $\mathcal{L} - F_w$ is numerically equivalent to 2σ, and σ itself is ample on X. As to the spannedness, write $\mathcal{L} - F_w = K_X + M_w$, where $M_w = 4\sigma + F_y - F_u - F_w$ is numerically equivalent to $M := 4\sigma - F$, for every $w \in C$. Note that M is nef and $M^2 > 5$. Moreover, $M = 2\sigma + (2\sigma - F)$ is the sum of the double of an ample divisor and a nef divisor, hence $MD \geq 2$ for any effective divisor D on X. Then the claimed spannedness follows from Reider's theorem (see e.g. [3, Theorem 8.5.1]). As $h^0(\mathcal{L} - F_w) = 3$, as shown in the proof of Proposition 22, the morphism defined by W_{F_w} represents X as a double cover $\psi_w : X \to \mathbb{P}^2$, whose ramification divisor is $R_w = K_X + 3(\mathcal{L} - F_w)$. An immediate check shows that its cohomology class is different from that of $\Phi_2(X)$ established in Proposition 22. However, clearly, both classes cut out a 0-cycle of degree 4 on every fiber.

We conclude this Section considering conic fibrations in $\mathbb{P}^6$. Actually, in this case, according to the final remark in Section 5, Theorem 16 applies to count the flexes, provided that there are finitely many.

Proposition 23. *Let $X \subset \mathbb{P}^6$ be a conic fibration over a smooth curve such that $\mathrm{rk}(j_{2,x})$ is generically 6 and suppose that $\Phi_2(X)$ is finite. Then X is a conic bundle and the degree of $\Phi_2(X)$ is given by*

$$\iota = 11K_X^2 - 5c_2(X) + 56(g-1) - 7d,$$

where d and g are the degree and the sectional genus of X.

Proof. X cannot have singular fibers by Theorem 12 (2). Next use Lemma 2, (i) and (ii), to compute $c_1^2 - c_2$. □

From the abstract point of view, X is a $\mathbb{P}^1$-bundle over C. Thus expressing K_X^2 and $c_2(X)$ in terms of the genus q of the base curve C, the above formula becomes

$$\iota = 68(1-q) + 56(g-1) - 7d. \tag{18}$$

Clearly, when φ is an embedding and the scroll R has a 2-dimensional inflectional locus, then $\Phi_2(X)$ cannot be 0-dimensional as required in Proposition 23, since it has to contain $X \cap \Phi_2(R)$. However, the situation depicted by Proposition 23 can occur, as the following example shows.

Example. Let $Z \subset \mathbb{P}^8$ be a conic bundle such that $j_{2,z}^Z$ is surjective at every point $z \in Z$; for instance we can take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(2,2)$. Then projecting Z from a general line ℓ produces a smooth surface $X \subset \mathbb{P}^6$ satisfying all the assumptions in Proposition 23. To see this, note that every osculating space to Z is a $\mathbb{P}^5$, hence the second osculating developable of Z is a hypersurface, say W, in $\mathbb{P}^8$. Clearly X is a conic bundle isomorphic to Z since ℓ is general; moreover, as W intersects ℓ at $\deg W$ points, we argue that X has finitely many ($\deg W$) inflection points. Hence the formula giving ι applies to X. An alternative way to compute ι is the following. Note that W is the image of the projectivization of $\mathcal{P}_Z^2(\mathcal{L})$, where $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^8}(1))_Z$, via a linear subsystem of the complete linear system defined by the tautological line bundle, say ξ. Therefore $\iota = \deg W = \xi^7$ and the Chern-Wu relation gives $\xi^7 = c_1^2 - c_2$, where $c_i = c_i(\mathcal{P}_Z^2(\mathcal{L}))$. Notice that this computation more generally holds for the general projection $X \subset \mathbb{P}^6$ of any 2-regular surface $Z \subset \mathbb{P}^8$, regardless of whether it is a conic bundle or not.

Here is an easy application of Proposition 23.

Proposition 24. *Let $X \subset \mathbb{P}^6$ be a non-degenerate rational conic bundle satisfying the assumptions of Proposition 23. Then $\iota \geq 12$ and equality occurs if and only if X is the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^8$ by $\mathcal{O}(2,2)$, from a general line. Apart from this case, $\iota \geq 40$.*

Proof. We have $q = 0$ by assumption. Moreover, the genus formula immediately shows that $d = 4g + 4$, since X is a Segre–Hirzebruch surface embedded as a conic bundle. Thus (18) gives $\iota = 28g - 16$. If $g = 0$, then X is a rational scroll and a conic bundle at the same time. But this implies that $(X, \mathcal{L}) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2))$, which is impossible since $X \subset \mathbb{P}^6$ is non-degenerate. Therefore $g \geq 1$, which implies $\iota \geq 12$ and equality is equivalent to $g = 1$. As $q = 0$, this in turn is equivalent to $\mathcal{L} = -K_X$. Then the assertion follows from the classification of del Pezzo surfaces. Note that the effectiveness follows from the discussion in the Example above. □

Notice that for the rational conic bundle $X \subset \mathbb{P}^6$ with $\iota = 12$ characterized by Proposition 24, expressing the invariants d, g in terms of a and b and recalling that $b = -\frac{2}{3}a$, (18) gives $a = 6$. So, recalling Proposition 18 and (13) we argue that $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}$ and $X \in |2\widetilde{\mathcal{L}} - 4\widetilde{F}|$.

8. CONIC FIBRATIONS IN $\mathbb{P}^4$

Clearly, any conic fibration $X \subset \mathbb{P}^4$ is hypo-osculating. Also, $\Phi_2(X) \supseteq S$, by Theorem 12 (1). Recall moreover that X is linearly normal, since it is not the isomorphic projection of the Veronese surface.

It is known that there exist conic fibrations over a smooth curve of genus 1 embedded in $\mathbb{P}^4$ [1]. They have degree 8 and contain exactly 8 singular fibers. Moreover, in [4] and [2] it is proved, independently, that every irrational conic fibration $X \subset \mathbb{P}^4$ is of this type. Clearly, in this case $\mathcal{V}$ cannot be very ample. In fact in [22, Proposition 1.2] it is shown that for the corresponding $\mathbb{P}^2$-bundle P, the image $R := \varphi(P)$ is a cone.

Now, let $X \subset \mathbb{P}^4$ be a rational conic fibration. From the relation $5 = h^0(\mathcal{L}) = h^0(\widetilde{\mathcal{L}}) = h^0(\mathcal{V}) = 3 + a$, we see that the multidegree (a_0, a_1, a_2) of $\mathcal{V}$ can only be either

(1) $(0, 0, 2)$, or
(2) $(0, 1, 1)$.

In case (1) R is the cone with vertex a line, say l, over the conic C_2. Let Π be a general hyperplane containing l, and call q_1 and q_2 the points where Π intersects C_2. Then cutting X with Π, we see that $\Pi \cap X$ consists of l itself (note that $l \subset X$ because, being contained in every plane of R, it intersects at two points every fiber of X) plus the two fibers F_1, F_2 of X contained in the planes $\langle l, q_1\rangle$ and $\langle l, q_2\rangle$ respectively. This says that X has degree 5. In case (2) R is the cone with vertex a point over the smooth quadric surface Q generated by the two directrix lines C_1, C_2. Let Π be the hyperplane generated by the two lines C_1, C_2. Clearly, it cuts X along a curve, say G, lying on the quadric Q and intersecting in two points each of the two rulings. Then, the degree of X, which coincides with the self-intersection G^2 of this curve on Q is 4.

Having this in mind, recall that the rational conic fibrations in $\mathbb{P}^4$ are completely classified [4]. In accordance with this classification, our X is one of the following:

i) a del Pezzo surface of degree 4: i.e., X is the blow-up of $\mathbb{P}^2$ at 5 points $p_0, \dots, p_4$ in general position, and $\mathcal{L} = \eta^*\mathcal{O}_{\mathbb{P}^2}(3) - e_0 - \cdots - e_4$, where $\eta : X \to \mathbb{P}^2$ is the blowing up and e_i is the exceptional curve $\eta^{-1}(p_i)$ $(i = 0, \dots, 4)$;
ii) a Castelnuovo surface of degree 5: i.e., X is the blow-up of $\mathbb{P}^2$ at 8 points $p_0, \dots, p_7$ in general position, and $\mathcal{L} = \eta^*\mathcal{O}_{\mathbb{P}^2}(4) - 2e_0 - e_1 - \cdots - e_7$, where $\eta : X \to \mathbb{P}^2$ is the blowing up and e_i is the exceptional curve $\eta^{-1}(p_i)$ $(i = 0, \dots, 7)$.

According to the above discussion, it follows that (1) corresponds to ii), while (2) corresponds to i).

Note that in case i), X is a conic fibration over $\mathbb{P}^1$ in 5 different ways, each being defined by the pencil of lines of $\mathbb{P}^2$ passing through one of the points p_i. We can observe that each of these fibrations has exactly 4 singular fibers:

for instance, if $\pi : X \to \mathbb{P}^1$ is given by the pencil through p_0, then the 4 singular fibers of π are $\widetilde{\ell}_i + e_i$ $(i = 1, \dots, 4)$, where $\widetilde{\ell}_i$ is the proper transform of the line ℓ_i joining p_0 and p_i. On the other hand, in case ii) there is a unique structure of conic fibration $\pi : X \to \mathbb{P}^1$, defined by the pencil of lines through p_0. Here there are 7 singular fibers, all of them having the form $\widetilde{\ell}_i + e_i$ $(i = 1, \dots, 7)$, where, as before, $\widetilde{\ell}_i$ is the proper transform of the line ℓ_i joining p_0 and p_i. From Proposition 7 (1) and (3) we get $\mu = 2a + 3b = d + 2b$ and this allows us to determine the class of X inside P. In case i) we have that $X \in |2\widetilde{\mathcal{L}}|$ (because $4 = \mu = 4 + 2b$), while in case ii) $X \in |2\widetilde{\mathcal{L}} + \widetilde{F}|$ (because $7 = \mu = 5 + 3b$).

To complete the discussion, let us describe the inflectional locus $\Phi_2(X)$ in both cases.

In case i), according to [15, Theorem 2.1] the rank of $j_{2,x}$ is 5 at the general point $x \in X$, and 4 at any point of the finite set $T = \cup_{i=0}^4 S_i$ where S_i is the set of singular points of the 4 singular fibers in the conic fibration defined by the pencil of lines through p_i. Note that T consists of 20 points, since a straightforward verification shows that $S_i \cap S_j = \emptyset$ if $i \neq j$. Therefore, $\Phi_2(X) = T$. In case ii), let $x \in X$. Relying on the plane model of X, we can easily see that $|\mathcal{L} - 3x| = \emptyset$ unless $x \in S$, the set of singular points of singular fibers of X. Moreover, if $x = \widetilde{\ell}_i \cap e_i$, then $|\mathcal{L} - 3x|$ is a single element corresponding to the reducible plane quartic consisting of the line ℓ_i and the cubic passing through $p_0, p_1, p_2, \dots, p_7$ and tangent in p_i to the direction represented by x. In other words, $\mathrm{rk} j_{2,x} = 5$ for all $x \in X \setminus S$ and $\mathrm{rk} j_{2,x} = 4$ at any point $x \in S$. Therefore, $\Phi_2(X) = S$.

References

[1] H. Abo, W. Decker, N. Sasakura, *An elliptic conic bundle in $\mathbb{P}^4$ arising from a stable rank-3 vector bundle*, Math. Z. **229** (1998), 725–741.

[2] R. Brown, K. Ranestad, *Conic bundles in projective fourspace*, in "Algebraic Geometry (Catania, 1993/ Barcelona, 1994)", Lecture Notes in Pure and Appl. Math., vol. 200, Dekker, New York, 1998, 331–339.

[3] M. C. Beltrametti, A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, de Gruyter Expositions in Math., Vol. 16, de Gruyter, 1995.

[4] Ph. Ellia, G. Sacchiero, *Smooth surfaces of $\mathbb{P}^4$ ruled in conics*, in "Algebraic Geometry (Catania, 1993/ Barcelona, 1994)", Lecture Notes in Pure and Appl. Math., vol. 200, Dekker, New York, 1998, 49–62.

[5] A, Dickenstein, S. Di Rocco, R. Piene, *Higher order duality and toric embeddings*, arXiv:1111.4641. To appear in Ann. Inst. Fourier **63** (2013).

[6] T. Fujita, *Classification of polarized manifolds of sectional genus two*, Algebraic Geometry and Commutative Algebra, in Honor of Masayoshi Nagata Vol. I, Kinokuniya, 1987, 73–98.

[7] T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series **155**, Cambridge Univ. Press, 1990.

[8] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1984.

[9] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York–Heidelberg–Berlin 1977.

[10] P. Ionescu, *Embedded projective varieties of small invariants*, in "Proceedings of the 1982 Week of Algebraic Geometry, Bucharest", Lecture Notes in Math. **1056**, Springer-Verlag 1984, pp. 142–187.

[11] P. Ionescu, *Embedded projective varieties of small invariants, III*, in "Algebraic Geometry, Proceedings, L'Aquila 1988", Lecture Notes in Math. **1417**, Springer-Verlag 1990, pp. 138–154.

[12] P. Ionescu, M. Toma, *On very ample vector bundles on curves*, Internat. J. Math. **8** (1997), 633–643.

[13] A. Lanteri, *On the osculatory behavior of surface scrolls*, Le Matematiche **55** (2000), 447–458.

[14] A. Lanteri, R. Mallavibarrena, *Higher order dual varieties of projective surfaces*, Comm. Algebra **27** (1999), 4827–4851.

[15] A. Lanteri, R. Mallavibarrena, *Osculatory behavior and second dual varieties of Del Pezzo surfaces*, Adv. Geom. **1** (2001), 345–363.

[16] A. Lanteri, R. Mallavibarrena. *Osculating properties of decomposable scrolls*, Math. Nachr. **282** (2009), 1548–1566.

[17] A. Lanteri, R. Mallavibarrena, R. Piene, *Inflectional loci of scrolls*, Math. Z. **258** (2008), 557–564.

[18] A. Lanteri, R. Mallavibarrena, R. Piene, *Inflectional loci of scrolls over smooth, projective varieties*, Indiana Univ. Math. J. **61** (2012), 2:77–750.

[19] A. Lanteri, M. Palleschi, A. J. Sommese, *Discriminant loci of ample and spanned line bundles*, J. reine angew. Math. **477** (1996), 199–219.

[20] A. Lanteri, M. Palleschi, A. J. Sommese, *Del Pezzo surfaces as hyperplane sections*, J. Math. Soc. Japan **49** (1997), 501–529.

[21] R. Piene, G. Sacchiero, *Duality for rational normal scrolls*, Comm. Algebra **12** (1984), 1041–1066.

[22] K. Ranestad, *A geometric construction of elliptic conic bundles in* $\mathbb{P}^4$, Math. Z. **231** (1999), 771–781.

[23] T. Shifrin, *The osculatory behavior of surfaces in* $\mathbb{P}^5$, Pacif. J. Math. **123** (1986), 227–256.

[24] H. P. F. Swinnerton-Dyer, *An enumeration of all varieties of degree* 4, Amer. J. Math. **95** (1973), 403–418.

[25] E. Togliatti, *Alcuni esempi di superfici algebriche degli iperspazi che rappresentano un'equazione di Laplace*, Comm. Math. Helv. **1** (1929), 255–272.

[26] S. Xambo, *Scrolls and quartics*, Collect. Math. **33** (1982), 89–101.

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